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The fact that measuring a quantum system reduces it to apparently classical behavior, eliminating the interference patterns that are a hallmark of quantumness, cries out for an explanation. That explanation has been provided by the recognition of decoherence, whereby the interference is destroyed by the very interaction that acquires information. We begin by showing how this scenario plays out in a simple analytical model—designed to be pedagogical—of measurement in a double-slit experiment. This model illustrates the continuous trade-off between information and interference in a concrete mathematical framework that makes explicit the role of measurement-induced decoherence in the process, thereby providing a natural stepping stone to a discussion of environmentally-induced decoherence and the real target of this work: the origin and accessibility of classical reality.

Quantum Darwinism, building on the decoherence program, provides an explanation for the emergence of classicality within what we have come to recognize as a fundamentally quantum universe; that is, how some physical observables take on robust, objective, verifiable values despite the intrinsically fragile and subjective nature of observables in quantum theory. It does so by acknowledging the role of the environment as a witness, continuously monitoring certain "pointer observables" of the system, and as a communication channel, amplifying information about those observables and distributing many copies of it throughout the world. Past work in this area focused on the ability of the environment to perform this amplification and the amount of information potentially available in fragments of the environment, but little focus has been placed on the observer's ability to actually collect that information. Here we show that redundant information is available to observers even when locality prohibits them from accessing quantum correlations within their fragment; a "bit-by-bit" measurement will suffice with only a slight increase in the required fragment size. Moreover, we show that, except in the case of a very low entropy environment, the decoherence process that produces these objective, classical states gives rise also to a convenient classical measurement toward which local observers can evolve. Together, these results demonstrate that even local observers can, and indeed almost certainly will if given time to adapt, share in the objective, classical reality that emerges when living in a large quantum universe, thereby providing another stepping stone in the bridge that quantum Darwinism is building between our quantum universe and classical experience.

Local Accessibility of Objective, Classical Reality

by

Joshua James Kincaid

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APPROVED:

Major Professor, representing Physics

Chair of the Department of Mathematics

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Joshua James Kincaid, Author

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1: Introduction

The most startling insight of the last century of physics has been that, at the most fundamental level, its constituent parts behave in ways that are entirely alien to our everyday experience. The "quantum" nature of reality—the intrinsically probabilistic outcome of experiments, the existence of arbitrary superpositions of states, the fact that measuring one property of a system can "reset" others to unpredictable values, and so on—is, somehow, "washed out" in the world we casually observe, manifesting only when experimenters carefully isolate the system of interest. But the naive formalism of quantum mechanics, as established in the first half of the last century, suggests just the opposite should be true: combining quantum systems should amplify the quantum "weirdness", as states that exhibit "unintuitive" behavior occupy exponentially larger portions of the available space of states.

This has been illustrated by the thought experiment of Schrödinger [36], in which a most unfortunate cat has been locked in a container with a radioactive isotope and a bottle of poisonous gas: if the isotope decays, the bottle breaks and the cat dies; if the isotope does not decay, the bottle remains intact and the cat lives. According to the principles of quantum mechanics, the isotope may decay or not, and therefore its state will be, until measured, in a superposition of the two possibilities. The uncomfortable conclusion is that the *cat* is also in such a superposition, being in some sense both alive and dead up until the container is opened and a measurement is made to determine what has happened. Moreover, a bit of consideration demonstrates that neither the isotope, nor the gas, nor even the container, are necessary to frame this problem. Merely the existence of a cat, composed as it is of numerous interacting atoms, all of which should adhere to the laws of physics, is sufficient. In this case, the question becomes not "is the cat alive or dead", but simply "is there a cat?"

Of course, we know, empirically, that there is a cat[1], so even "is there a cat?" is the wrong question: a more compelling question is "*why* is there a cat?" Or, to be a bit more general, why do we perceive objects as persisting in apparently robust states that seem unaffected by how many people observe them or how often such observations take place, when the lessons of quantum mechanics suggest that the state of a system should be quite fragile, changing when measured and therefore precluding an objective, verifiable determination of those properties?

In this chapter, I give a brief review of some aspects and tools of quantum theory that will play a significant role in the sequel[2]: Section 1.1 discusses the general theory of quantum measurement and the problem of preferred bases, Section 1.3 introduces quantum mutual information and its decomposition into the Holevo quantity (essentially classical correlations) and quantum discord (inherently quantum correlations), and Section 1.2 outlines the problem of the quantum-to-classical transition, including the problems of the absence of interference and the observation of outcomes and a working definition of "objectivity" as used in the quantum Darwinism.

Chapter 2 is devoted to decoherence[44, 48], beginning with a general overview of measurement-induced decoherence, followed by a model for such in the context of a double-slit experiment, and concluding with a short discussion of environmentally-

[1] In fact, there appear to be many cats, each distinguishable from the others, and each, apparently, persistently stable in its localized state.

[2] Several standard results from quantum mechanics are provided here without derivation; these may be found in standard pedagogical treatments such as Refs. [38, 26]

induced decoherence. This leads naturally to Chapter 3, which summarizes the core ideas of quantum Darwinism [50] and some of its recent results, including the explicit demonstration of amplification and redundancy in the case of a photon system and a spin-system embedded in a spin-environment.

The primary result of this work is presented in Chapter 4, in which we show that the amplified information is largely available even to observers who are constrained to make local measurements on very small portions of the environment, such as single-qubit measurements on a spin-environment. Moreover, we identify a unique, optimal measurement axis that such observers can identify and converge to by employing an adaptive, iterative measurement scheme, thereby ensuring that many local observers can come to a consensus even when they have no initial information about the system. Finally, we observe that individuals can perform subsequent measurements on other parts of the environment in order to verify their initial findings, ensuring that this information is robust, objective, and verifiable.

1.1 Quantum measurement

The postulates of quantum mechanics, however stated, can be broken up into two kinds. On the one hand are the dynamical laws that govern the evolution over time of a quantum system and its observables. On the other hand are the postulates governing how we are to extract information about the system—what a measurement is and what happens when you make one. The fundamental assumption that underlies the decoherence program, generally, and quantum Darwinism in particular, is that this second set of postulates is superfluous. However one defines a measurement, the apparatus used to perform that measurement should itself be ultimately described as quantum mechanical systems, in which case the evolution of the joint system-apparatus should follow the first set of postulates. This is most clearly identified by the idea of the Von-Neumann chain [40].

1.1.1 The von Neumann chain

Suppose that a system, $\mathcal{S}$, is in a well-defined state, $|\Psi\rangle_{\mathcal{S}} = \sum_n c_n |n\rangle_{\mathcal{S}}$, where the states $\{|n\rangle\}$ are eigenstates of some complete observable. If one wishes to make a measurement on $\mathcal{S}$, it is necessary first to identify an appropriate apparatus, $\mathcal{A}$, which will then interact with $\mathcal{S}$ in such a way that a "measurement" results. If the system and apparatus are initially independent, and if we are free to neglect their interactions with any other systems (such as the environment), then we can suppose that the apparatus is likewise in some well-defined initialized state $|0\rangle_{\mathcal{A}}$. Hence, following the usual Hilbert space formalism, the initial states of the system and apparatus can together be described by a single ket, $|\Psi, 0\rangle_{\mathcal{SA}} = \sum_n c_n |n, 0\rangle_{\mathcal{SA}}$ in the product Hilbert space. Then this state will evolve unitarily, and in calling it a "measurement" we expect that it will do so according to the rule $|n, 0\rangle_{\mathcal{SA}} \to |n, A_n\rangle_{\mathcal{SA}}$, where $|A_n\rangle_{\mathcal{A}}$ is the state of the measurement apparatus that indicates an outcome associated to the state $|n\rangle_{\mathcal{S}}$ and we assume these states are mutually orthogonal. Hence the joint state becomes $\sum_n c_n |n, A_n\rangle_{\mathcal{SA}}$, and we see that the system and apparatus have become entangled to an extent that precludes decomposition into pure states in their respective Hilbert spaces.

Of course, this unitary evolution simply tells us how the measuring device and the system evolved, but it doesn't address how an observer is to determine the outcome of the experiment. We may suppose that the observer is also initially represented by a state, $|i\rangle_{\mathcal{O}}$, and that after the measurement interaction, the observer will "measure" the apparatus to determine it's state. This should follow the same basic outline as the apparatus-system interaction, leading to a final state $\sum_n c_n |n, A_n, O_n\rangle_{\mathcal{SAO}}$, where the states $\{|O_n\rangle\}$ represent an observer who has found the apparatus to be in state $|A_n\rangle_{\mathcal{A}}$.

Now the observer may want to share that information, effectively being themselves measured by some other observer. Or perhaps the "observer" in the previous paragraph should have really been an environment upon which some observer would then perform a subsequent measurement. In any case, we see that this process can be continued indefinitely, entangling more and more initially independent entities in such a way that each state of the system corresponding to a particular outcome is correlated to a specific state of each of the other entities in a consistent way. At first glance, this seems to provide a model of measurement that is independent of any additional postulates, but there are a number of difficulties that must be overcome before we can interpret it as such. These are the "measurement-problems", which are ultimately tied to the larger question of the quantum-to-classical transition. These problems are laid out in the remainder of this chapter, and one partial solution to them is presented in the rest of this paper.

1.2 The quantum-to-classical transition

Quantum mechanics has a reputation for oddity, in that quantum systems are known to exhibit a range of features not apparently present in our everyday classical world. But, as previously noted, this classical world in fact arises from a fundamentally quantum substrate; i.e., all the classical "things" with which we are familiar are ultimately composed of large numbers of quantum "things" (i.e., particles). The strangeness of quantum mechanics can be traced, ultimately, to the superposition principle, which allows that *any* superposition of possible states is itself a possible state, and the absence of such arbitrary superpositions is the essence of our classical world. But the nature of quantum theory is that as the number of interacting particles increases, the size of the Hilbert space, and therefore the possibilities for exotic superpositions, grows exponentially. Hence the question: Why does our quantum world appear classical when the vast majority of possible states are highly non-classical? This question can be broken down into (at least) three key components as outlined below.

1.2.1 Preferred bases

From the discussion of the von Neumann chain, we have a model of "measurement" that explains how certain observable outcomes become correlated to particular states of measuring devices, observers, and the environment. But the superposition principle tells us that we may consider arbitrary superpositions of our initial states for both the apparatus and the system. Consider, for example, a spin-system, which is represented in some basis as $|\Psi\rangle_{\mathcal{S}} = a\,|\uparrow\rangle + b\,|\downarrow\rangle$ and a corresponding measurement device that interacts with this according to the rule $|\updownarrow, 0\rangle_{\mathcal{SA}} \to \left|\updownarrow, A_{\updownarrow}\right\rangle_{\mathcal{SA}}$. By hypothesis, the states $|A_\uparrow\rangle_{\mathcal{A}}$ and $|A_\downarrow\rangle_{\mathcal{A}}$ are orthogonal, as are $|\uparrow\rangle_{\mathcal{S}}$ and $|\downarrow\rangle_{\mathcal{S}}$. Hence, let us consider the complementary bases $|\pm\rangle_{\mathcal{S}} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_{\mathcal{S}} \pm |\downarrow\rangle_{\mathcal{S}})$ and $|A_\pm\rangle_{\mathcal{S}} = \frac{1}{\sqrt{2}}(|A_\uparrow\rangle_{\mathcal{S}} \pm |A_\downarrow\rangle_{\mathcal{S}})$. Then we have, for example,

$$\begin{aligned}
|+,0\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow,0\rangle_{\mathcal{SA}} + |\downarrow,0\rangle_{\mathcal{SA}}\right) \\
&\to \frac{1}{\sqrt{2}}\left(|\uparrow,A_\uparrow\rangle_{\mathcal{SA}} + |\downarrow,A_\downarrow\rangle_{\mathcal{SA}}\right) \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|\uparrow,A_+\rangle_{\mathcal{SA}} + |\uparrow,A_-\rangle_{\mathcal{SA}}\right) + \frac{1}{\sqrt{2}}\left(|\downarrow,A_+\rangle_{\mathcal{SA}} - |\downarrow,A_-\rangle_{\mathcal{SA}}\right)\right] \\
&= \frac{1}{\sqrt{2}}\left(|+,A_+\rangle_{\mathcal{SA}} + |-,A_-\rangle_{\mathcal{SA}}\right)
\end{aligned} \tag{1.1}$$

Hence, we see that the interaction appears to create correlations between the system and apparatus in multiple, complementary bases. If we interpret the state $|\uparrow\rangle_{\mathcal{S}}$ as being spin-up along the z-axis and $|+\rangle_{\mathcal{S}}$ as being spin up along the x-axis, the spin-apparatus interaction appears to describe a measurement along *both* axes.

There are two problems presented by this observation. First, quantum mechanics precludes the simultaneous measurement of spin along two orthogonal axes, but it seems here that once the measurement has been made, an observer could determine these two values by simply reinterpreting the measurement output; even worse, an appropriate change of basis would reveal that the correlations are established for spin along *every* axis. Second, our experience rejects the notion that there is a "universal" measuring apparatus, capable of performing arbitrary measurements simply by adjusting our interpretation of its result. Rather, we expect that once an observable of interest has been established, one must select an appropriate apparatus, dictated by the physical nature of the observable, in order to measure it; one configures their Stern-Gerlach device in a specific orientation when measuring spin along a specific axis and then expects to gain no information regarding spin along other axes without first reconfiguring the device. Hence, it seems that to each apparatus there is a preferred basis in which it is able to measure, and that measurements in other bases entail the choice of a different apparatus. A resolution to this problem of the preferred basis is provided in Section 3.1. Briefly, the existence of extra degrees of freedom, either within the apparatus or within the environment in which it is embedded, selects out a particular basis (i.e., a pointer *observable* of the apparatus), and these pointer states are those that can be used to infer the result of a measurement.

1.2.2 Absence of interference

Having identified the problem of the preferred basis and deferred its resolution, we can move on from measurement in particular to the more general question of the rise of classicality. The first question that needs answering[3] is where is all the interference? As illustrated beautifully in double slit experiments, quantum systems exhibit a form of self-interference, whereby a superposition of states gives rise to interference patterns when the corresponding observable is measured. But such interference effects are notably absent from our everyday experience, as evidenced by just how unexpected the patterns demonstrated by double slit experiments are.

The solution to this problem requires a shift in the traditional approach to fundamental physics. When treating classical problems of physics, particularly outside of thermodynamics, one often neglects complicating features of the system and the role of the environment—frictionless surfaces in vacuums and all that. This is justified by the assumption that the interaction between the system and the environment is much weaker than the dynamics of the system itself and should, therefore, have only a minor effect on the observed outcome.

But this assumption fails utterly in our everyday experience, where we *rely* on the interactions between systems and their environment as a proxy for performing actual measurements. A casual observer does not *measure*, for example, the position of a book when looking at it. Rather, the observer interacts with light scattered from the book and then *infers* the position as a result of that interaction. Hence, the environment plays a critical role in our experience of the classical world, so it should come as no surprise that the emergence of classicality requires us to incorporate the environment explicitly into

[3]Following here the treatment of "measurement problems" in Ref. [34]

our models. This is demonstrated also by the fact that experiments intending to showcase quantum interference invariably require some effort to isolate the system from external factors (i.e., the environment) in order to preserve coherent quantum superpositions.

The process by which the environment suppresses interference—"decoherence"—has been much studied over the last several decades. In Chapter 2, we review this process in two forms: first, measurement induced decoherence is introduced and used to explain the loss of interference in our double-slit model; second, we discuss the formal aspects of environmental decoherence and the resulting suppression of interference.

1.3 Mutual information: The Holevo quantity and discord

The amount of information shared between two quantum systems, $\mathcal{S}$ and $\mathcal{A}$, is quantified by their mutual information, $I(\mathcal{S}:\mathcal{A}) = H_{\mathcal{S}} + H_{\mathcal{A}} - H_{\mathcal{S}\mathcal{A}}$, where

$$H_{\mathcal{K}} = -\mathrm{tr}\rho_{\mathcal{K}} \log_2 \rho_{\mathcal{K}}$$

is the von Neumann entropy of $\mathcal{K}$ (representing the amount of "missing" information regarding the state of $\mathcal{K}$), and the subsystem entropies are computed using the reduced density operator, $\rho_{\mathcal{S}} = \mathrm{tr}_A \rho_{\mathcal{S}\mathcal{A}}$. The mutual information effectively summarizes the degree of correlation between the two systems, characterizing the total information that is shared between them, but it fails to characterize the nature of those correlations or the accessibility of that information. The reason for this is clear: the mutual information depends only on the interaction between the two systems, but the accessibility of that information depends on the subsequent measurement one performs. In particular, for an arbitrary POVM on $\mathcal{S}$, $\hat{\Pi}_{\mathcal{S}}$, the mutual information can be split into two parts [30, 55],

$$I\left(\mathcal{S}:\mathcal{A}\right)=\chi\left(\hat{\Pi}_{\mathcal{S}}:\mathcal{A}\right)+\mathcal{D}\left(\hat{\Pi}_{\mathcal{S}}:\mathcal{A}\right), \tag{1.2}$$

where the Holevo quantity,

$$\chi\left(\hat{\Pi}_{\mathcal{S}}:\mathcal{F}\right)=H\left(\sum_{\hat{s}}p_{\hat{s}}\rho_{\mathcal{F}|\hat{s}}\right)-\sum_{\hat{s}}p_{\hat{s}}H(\rho_{\mathcal{F}|\hat{s}}), \tag{1.3}$$

provides a bound on the amount of classical information about $\mathcal{S}$ that can be transmitted by $\mathcal{A}$, and the discord, $\mathcal{D}\left(\hat{\Pi}_{\mathcal{S}}:\mathcal{A}\right)$, characterizes the "quantumness" of the correlations between $\mathcal{S}$ and $\mathcal{A}$.

In other words, for any choice of observable, the mutual information can be cleanly separated into the classical information about that observable that can be acquired from $\mathcal{A}$ (without any direct measurement on $\mathcal{S}$) and the information associated with (perhaps inaccessible) quantum correlations between $\mathcal{S}$ and $\mathcal{A}$. The essential realization of the quantum Darwinism program with respect to these quantities arises when we take $\mathcal{A}$ to be some fragment, $\mathcal{A}=\mathcal{F}$, of the environment, $\mathcal{E}$. In this case, one finds [55] that the Holevo quantity very rapidly approaches the entropy of the system $H_{\mathcal{S}}$ as the fragment size increases when $\hat{\Pi}_{\mathcal{S}}$ is the pointer-observable POVM, indicating that nearly complete information about this system observable can be acquired with only a small portion of the environment; for other observables, however, the information is stored in global correlations, necessitating access to a much larger portion of the environment.

1.3.1 "Objective" reality

At this point, we can introduce a working definition of objectivity [29]: information about a system observable is "objective" if it can be independently acquired by many different observers in a way that doesn't alter the value of the observable. quantum mechanics precludes any direct measurement of the system from yielding information in a way that satisfies this criteria: the act of measuring the system can "reset" it, eliminating the information gained from previous measurements. Hence, only information acquired indirectly can achieve objectivity. In light of the previous discussion, it's clear that this occurs precisely for the pointer observables of the system, as these are the observables about which significant information is accessible from only a small fragment of the environment; hence, the information is redundantly proliferated into the environment, and many observers can acquire fragments of sufficiently large size to access that information. Other observables, on the other hand, require much larger fragments, thereby limiting the number of copies to a much smaller number (in fact, only one in the case where the required fragment size is greater than half the environment). A precise bound on the redundancy of the information transmitted in this way is provided by the quantum Chernoff information, which we discuss in Section 3.2.1.

2: Decoherence

In this chapter, we present a model of the double-slit experiment that demonstrates the inverse relationship between the visibility of quantum interference and the acquisition of information, highlighting the role of decoherence in the process.

2.1 The double slit

In his Bakerian Lecture of 1801 [43], Thomas Young put forth several arguments supporting the claim that light propagates as a wave rather than as a particle, as had been previously proposed by Newton. More importantly, he proposed a relatively simple experiment that could unambiguously demonstrate this fact: by passing the light from a single source simultaneously through two nearby apertures and then observing it's incidence on a distant plate, an interference pattern analogous to those previously displayed by water and sounds waves may be observed. The successful implementation of this experiment was achieved shortly thereafter, and the wave-like nature of free light became firmly established in the scientific literature.

A hundred years later, several experimental observations, most notably the photoelectric effect, led Einstein to propose a return to a model of light that was inherently particle-like [16]. This picture of light as being composed of "quanta"—discrete elements of fixed energy—successfully explained a number of observed phenomena, such as the fact that the photoelectric effect depended strongly on the frequency, but not the intensity, of the incident light, but it raised two important questions. First, if light behaves as a particle, how do we explain the interference patterns observed in the double slit experiment? Second, if light, as a particle, were able to exhibit such interference, might not other particulate matter do so as well [12]?

The resolution to both of these questions was provided by the development of quantum mechanics, in which the classical picture of things being either particles *or* waves was supplanted by a more robust model in which a system may exhibit wave-like or particle-like behavior depending on the details of the experiment performed.

Quantum mechanics has been applied to explain a number of phenomena, but the nature of quantum interference in double-slit experiments has remained an active area of research. The first demonstration of interference in matter systems was finally achieved in 1961 when interference between electrons was successfully observed [21]. From there, researchers have continued to demonstrate evidence of interference between ever larger objects and in exotic situations such as Bose-Einstein condensates [2] or with topological defects [14]. Finally, research in the last few decades [4, 15, 3] has demonstrated unambiguously that interference can be observed even when only a single "particle" is passing through the slits at any one time, thereby demonstrating that the interference must come from the intrinsic nature of the particle rather than arising from the statistics or interactions between large numbers of particles.

Perhaps more interesting than the observation that matter systems can exhibit self-interference is the subsequent observation that the interference can be *lost* if one attempts to track the particles through the slits, thereby determining "which path" the particle took. A schematic of this behavior, which has been demonstrated experimentally by using circular polarizers at the slits to "mark" each photon,[41] is shown in Fig. 2.1. More striking still is the fact that giving up the which-path information by, for example, passing the circularly polarized light through a linear polarizer, restores the interference pattern.

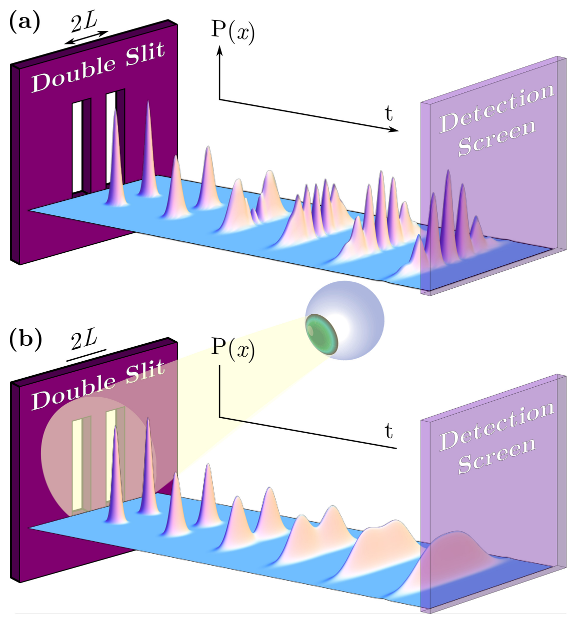

Figure 2.1: Schematic representation of the one dimensional version of the double slit experiment. In this setup, a coherent superposition of two Gaussian wavepackets of width Δ emerge from the two slits separated by a distance $2L$. (a) When no measurement is made and coherence is otherwise preserved, the probability density $P(x)$ shows that as time progresses the two packets begin to interfere, ultimately resulting in a well-defined interference pattern. (b) In the case of a perfect measurement, each particle takes either the left or the right path. In this case $P(x)$ observed at the detection screen will be an incoherent sum of the two spreading Gaussian wavepackets, i.e., no interference will be present.

In short, the results of various double-slit experiment provide the paradigmatic example of quantum mechanical "weirdness"; as Richard Feynman put it, this experiment contains "the only mystery [of quantum mechanics]" where the particles—whatever they may be, electrons, photons, etc.—behave "sometimes like a particle and sometimes like a wave." [18].

We know, then, that determining which path a particle took eliminates the appearance of interference. If however, one is willing to forgo *perfect* determination of the path, partial interference may yet be preserved. This relationship has previously been demonstrated in theoretical analyses and proposed experimental realizations of the double-slit experiment for light [42, 31], and a general treatment of imperfect two-state discrimination from basic quantum mechanical principles can be found in textbooks.[1] In short, when the apparatus/observer acquires information about the system the interference pattern disappears.

This deep relationship between interference and information can be understood in the context of decoherence and entanglement, and plays a significant role in understanding the quantum-to-classical transition [28, 29, 7, 50, 55, 53, 51]. While there exists a multitude of papers [46, 47, 48, 1, 39, 37, 49] and books [22, 26, 34] on the more general subjects of quantum information and decoherence, the explicit application of these ideas to specific physical systems is lacking. To that end, we examine a model of double-slit interference[2] in the presence of measurement that allows the measurement precision to be tuned and thus allows us to examine the interplay between path information gained and the loss of interference: When one distinguishes between a particle at the left and right slits, then the interference is destroyed; when no information is gained, then in-

[1] See Chapter 2.6.2 of Ref.[34]

[2] The remainder of this section is derived from Ref. [23]

terference is manifest. This model makes use of the standard wave-function approach to quantum mechanics and helps make the core concepts of interference, measurement, distinguishability, decoherence, and dephasing more clear.

In order to keep the discussion concise, we assume that the reader is familiar with certain mathematical and conceptual tools not necessarily presented in introductory courses, the foremost of which are density operators,[3] the evaluation of Gaussian integrals,[38] and the partial trace.[4] We also provide only a brief introduction to quantum entropy and mutual information.[5] It is also convenient to work with dimensionless parameters, but we wish to retain the traditional symbols for readability. To that end, we fix a length scale Δ (the slit width) and denote physical quantities that carry a dimension with an overbar. The unbarred version is then the natural dimensionless parameter determined by Δ and physical constants. Hence we have dimensionless position $x = \overline{x}/\Delta$, time $t = \hbar\bar{t}/m\Delta^2$, momentum $p = \overline{p}\Delta/\hbar$, and so on. Similarly, we use a dimensionless Hamiltonian, $\mathbf{H} = (m\Delta^2/\hbar^2)\overline{\mathbf{H}}$, momentum operator, $\mathbf{P} = (\Delta/\hbar)\overline{\mathbf{P}}$, and position operator, $\mathbf{X} = \overline{\mathbf{X}}/\Delta$.

2.2 Interference without measurement

In order to establish the basic framework, we first consider the case where no measurement is attempted. The prototypical version of the double slit experiment is to have particles impinge on a barrier one by one. The barrier has two slits that let particles through, where they then continue to travel until striking the detection screen. The latter will reveal the interference pattern—or lack thereof—that emerges after many repetitions of

[3] See Chapter 2.4 of Ref.[26]
[4] See Page 105 of Ref.[26]
[5] See Chapter 11 of Ref.[26]

the experiment. We will consider a simplified version of this scenario where, as indicated in Fig. 2.1, there is just one spatial dimension and the evolution starts after the single particle exits the double slit.

When the particle—the system $\mathcal{S}$—exits the slits in a superposition of two Gaussian states,[6] i.e., its state $|\Psi\rangle_{\mathcal{S}}$ is given by the wavefunction

$$\langle x|\Psi\rangle_{\mathcal{S}} = \Psi(x) = A\left[e^{-(x+L)^2/2} + e^{-(x-L)^2/2}\right], \tag{2.1}$$

the two Gaussian components will begin to spread. Here, as throughout, x and L are dimensionless parameters that correspond to position and slit-spacing ($2L$), respectively, and which depend implicitly on the slit-width. The normalization is

$$A = \sqrt{\frac{1}{2\sqrt{\pi}\left[1+\exp\left(-L^2\right)\right]}}. \tag{2.2}$$

Note that time also represents the role of a second spatial dimension—the one in which the source, barrier, and screen are separated. As time moves on, one can imagine the particle moving from the barrier to the detection screen. More formally, one could include the additional spatial dimensions and integrate them out, as they do not play an important role.

Making use of the free-particle Hamiltonian $\mathbf{H} = \mathbf{P}^2/2$, we find the time-evolved state

[6] The exact wavefunction emerging from the slits in an experimental set-up will be complicated, depending on the exact slit shape geometry, the incoming state, particle type, etc. A superposition of Gaussian wavepackets, which we consider, leads to a clean presentation of interference and measurement. A square wavepacket, $\psi(x) = \Theta(x \pm L + \Delta) - \Theta(x \pm L - \Delta)$, with Θ the Heaviside step-function at each slit, which matches the ideal geometry of the slit, is also tractable, though we don't consider it here. The actual wavefunction, however, will be more complicated. Indeed, a reasonable approximate form is $\psi = \exp\left(-1/[\alpha((\Delta/2)^2 - (x \mp L)^2)]\right)$ (and zero for $x > \pm L + \Delta/2$ and $x < \pm L - \Delta/2$) at each slit of width Δ. This function, while only approximate, is already non-analytic, but can both match the geometry and have smooth boundaries.

by integrating

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ip^2t/2}e^{ip(x-x')}\Psi(x')\,dx'\,dp. \tag{2.3}$$

Using the initial wavefunction and evaluating the resulting Gaussian integral, one finds that the time-dependent wavefunction is given by

$$\begin{aligned}\Psi\left(x,t\right) = \frac{A}{\sqrt{1+it}}&\left\{\exp\left[-\frac{\left(x+L\right)^2}{2(1+it)}\right]\right.\\ &\left.+\exp\left[-\frac{\left(x-L\right)^2}{2(1+it)}\right]\right\}.\end{aligned} \tag{2.4}$$

The associated probability density is then

$$P\left(x,t\right) = \Gamma^{xt}\left[\cosh\left(\frac{2xL}{1+t^2}\right)+\cos\left(\frac{2txL}{1+t^2}\right)\right], \tag{2.5}$$

where the factor

$$\Gamma^{xt} = \frac{2A^2}{\sqrt{1+t^2}}\exp\left(\frac{-x^2-L^2}{1+t^2}\right) \tag{2.6}$$

has been introduced for readability. The first term in Eq. (2.5), the one with the hyperbolic cosine, is just a sum of two Gaussian wavepackets, which represent particles coming from the left or right slit, respectively. The second term, the one with the cosine, describes the interference between these two sources. Figure 2.2 shows the probability density, Eq. (2.5), for various times—i.e., the separation between the slits and detection screen—for a slit spacing $2L \gg 1$.

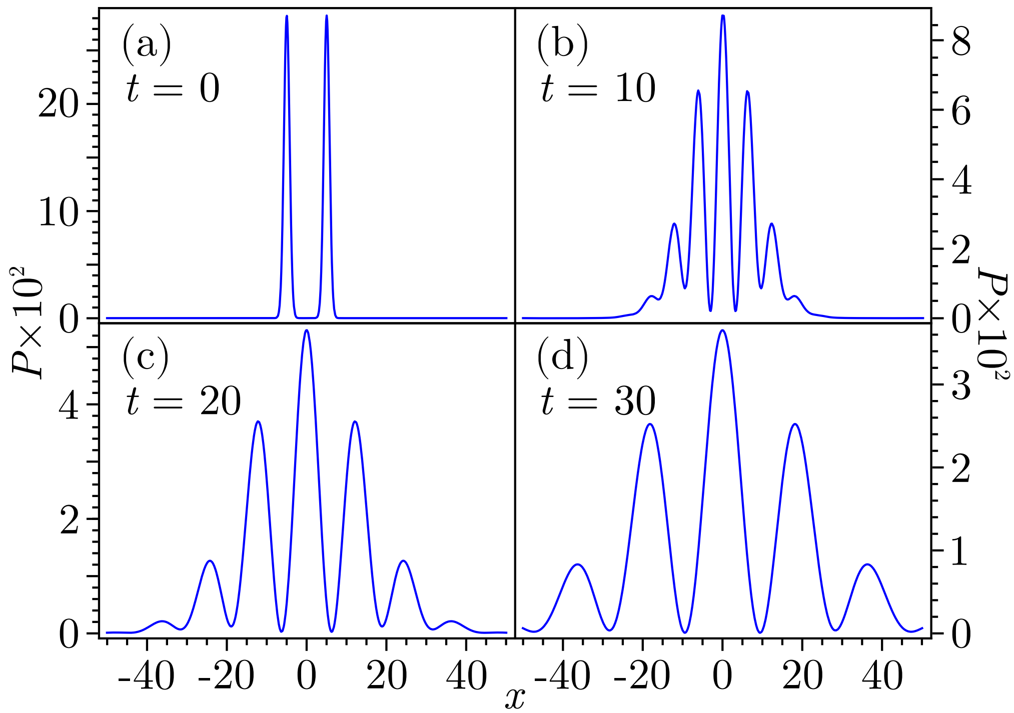

Figure 2.2: Probability density of a free particle with $L = 5$ at different times after passing through two slits. Note that this corresponds to the time of flight from the slit to the detection screen. (a) Initially, the two packets are well separated. (b) As they spread, they will start to interfere. (c) Eventually, a well-defined interference pattern develops, which (d) begins to spread out.

2.3 The effect of measurement

We now want to consider how measurement affects the appearance of interference. Specifically, we are interested in how the interference pattern is lost as the amount of information gained increases. In an introductory quantum course, one would discuss, for example, double-slit interference of electrons and measurement via photons. In this case, the observation is performed by measuring scattered light, which, with a suitably short wavelength, distinguishes the position of the electron. As the wavelength is increased, the scattered light no longer is imprinted with the left/right position of the electron. Calculating this scattering process, however, requires a large amount of background material, unsuitable for introductory courses.

Instead, we consider an idealized measurement process that nevertheless admits variable precision ranging continuously from perfect measurement to no measurement at all. Following the idea of a von Neumann chain,[40] this process will make use of an auxiliary quantum system, the apparatus $\mathcal{A}$. When the apparatus, in an appropriately initialized state $|0\rangle_{\mathcal{A}}$, interacts with a particle in the state $\left|\psi^L\right\rangle_{\mathcal{S}}$ with wavefunction $\psi^L(x)$ that is completely localized around the left slit (where we use the less strict condition, $\psi^L(x) = 0$ for $x > 0$), a perfect measurement would bring the composite state $\left|\psi^L, 0\right\rangle_{\mathcal{SA}}$ to $\left|\psi^L, L\right\rangle_{\mathcal{SA}}$. Similarly, when the apparatus interacts with a particle in a state $\left|\psi^R\right\rangle_{\mathcal{S}}$ with a wavefunction $\psi^R(x)$ that is completely localized around the right slit, a perfect measurement would bring the state $\left|\psi^R, 0\right\rangle_{\mathcal{SA}}$ to $\left|\psi^R, R\right\rangle_{\mathcal{SA}}$. This process transfers information about the particle's state into $\mathcal{A}$, encoding the outcome of the left/right-measurement in a subspace of dimension two spanned by the basis states $|L\rangle_{\mathcal{A}}$ and $|R\rangle_{\mathcal{A}}$. This left/right information is accessible to observers who can "read" the apparatus state. If one has a limited resolution measurement, or wavefunctions $\psi^L(x)$ and $\psi^R(x)$ that have overlap, then this information transfer cannot be perfect.

The measurement interaction In general, after a measuring apparatus $\mathcal{A}$ interacts with a system $\mathcal{S}$, observers can infer the state of the system by interacting with (and amplifying information from) the apparatus through the standard measurement process; i.e., by measuring a non-degenerate observable of $\mathcal{A}$ corresponding to the possible measurement outcomes. Of course, such a subsequent measurement could be treated similarly, requiring yet another measuring apparatus, and so on, leading one ultimately to the von Neumann chain. We are here concerned only with the first step in such a chain, considering only the interaction between $\mathcal{S}$ and $\mathcal{A}$. Later on, we briefly discuss the observer as an additional link in the von Neumann chain.

In our case, the relevant (non-degenerate) eigenstates of $\mathcal{A}$ are $|L\rangle_{\mathcal{A}}$ and $|R\rangle_{\mathcal{A}}$. We assume that the apparatus and system interact immediately after the particle passes through the slit, so that the particle wavefunction, Eq. (2.1), does not have time to evolve on its own before the measurement is made. As usual, the interaction between the apparatus and the system results in a unitary transformation of the joint state. Specifically, in keeping with the above discussion, we require that during the measurement process the joint state, initially $|\Psi, 0\rangle_{\mathcal{SA}}$, evolves as

$$|\Psi, 0\rangle_{\mathcal{SA}} \mapsto \left|\mathbf{M}_\sigma^L \Psi, L\right\rangle_{\mathcal{SA}} + \left|\mathbf{M}_\sigma^R \Psi, R\right\rangle_{\mathcal{SA}} \equiv |\Phi\rangle_{\mathcal{SA}} \tag{2.7}$$

during the interaction (here, as elsewhere, we use Ψ for our specific system state in distinction to the ψ used for generic system states in the introduction to Sec. 2.3). When the apparatus registers "L", the system will be in a state $\left|\Psi^L\right\rangle \propto \mathbf{M}_\sigma^L |\Psi\rangle_{\mathcal{S}}$ that is localized— to a precision σ—around the left slit due to the act of measurement itself. The conditional state $\left|\Psi^L\right\rangle$ depends on both the initial system state and the measurement operator (similarly for $\left|\Psi^R\right\rangle \propto \mathbf{M}_\sigma^R |\Psi\rangle_{\mathcal{S}}$). The initial state and the states $\left|\Psi^L\right\rangle$ and $\left|\Psi^R\right\rangle$ resulting from such an interaction are shown in Fig. 2.3 (a) using an explicit form of the measurement operator to be derived later in Eq. (2.16). The right-hand side of Eq. (2.7) cannot, in general, be written as a simple product of system and apparatus states. Thus, the interaction has caused the two to become *entangled* (except, of course, in the limiting case of no discrimination).

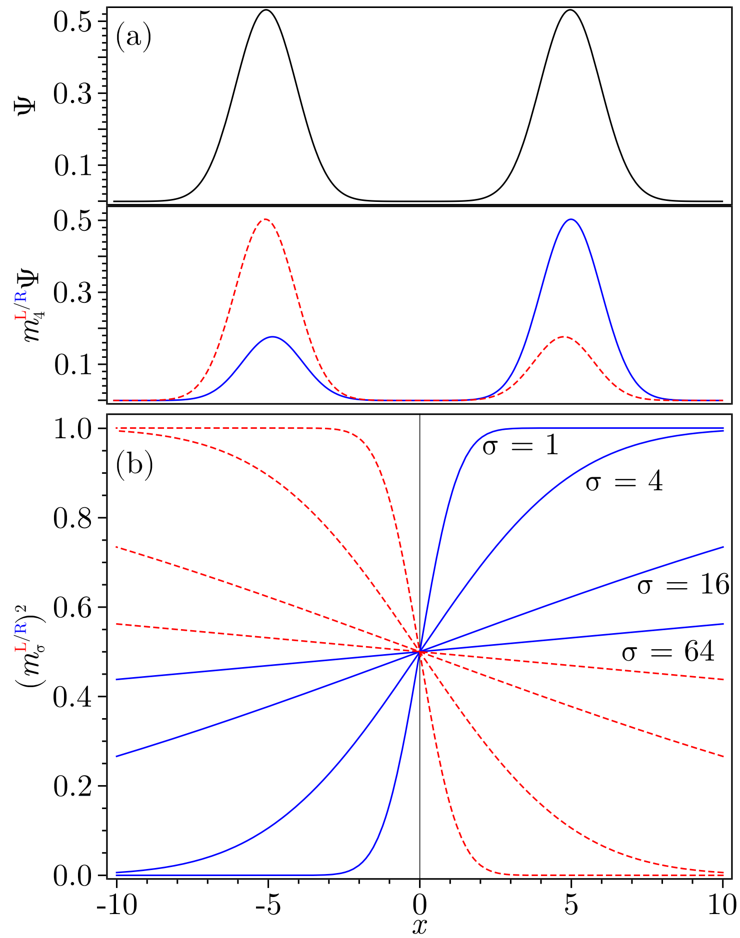

Figure 2.3: The effect of measurement. (a) The initial wavefunction Ψ with peaks at $L = \pm 5$ and the conditional states of the post-measurement wavefunction with $\sigma = 4$, $\Psi^L = m_4^L \Psi$ (dashed) and $\Psi^R = m_4^R \Psi$ (solid). (b) The measurement functions squared, $(m_\sigma^L)^2$ (dashed) and $(m_\sigma^R)^2$ (solid), versus position. As σ increases and the measurement becomes less precise, they tend toward a common constant value; in the opposite limit, they become complementary step-functions.

The operators $\mathbf{M}_\sigma^{L/R}$ appearing in Eq. (2.7) are called "measurement operators" and are written inside the ket in order to make clear the fact that they act only on the system $\mathcal{S}$. As noted, these operators determine the state of the particle after the measurement. Since the left/right measurement distinguishes the position of the particle, it suffices to take $\mathbf{M}_\sigma^{L/R}$ diagonal in the position basis, giving rise to a pair of "measurement functions"

$$\begin{aligned} \mathbf{M}_\sigma^L \left|x\right\rangle &= m_\sigma^L(x) \left|x\right\rangle, \\ \mathbf{M}_\sigma^R \left|x\right\rangle &= m_\sigma^R(x) \left|x\right\rangle. \end{aligned} \tag{2.8}$$

With this choice, the action of $\mathbf{M}_\sigma^{L/R}$ on the system wavefunction is purely multiplicative, $\langle x| \mathbf{M}_\sigma^{L/R} |\Psi\rangle = m_\sigma^{L/R}(x)\Psi(x)$. Since the apparatus states $|L\rangle_\mathcal{A}$ and $|R\rangle_\mathcal{A}$ are orthogonal, the requirement that Eq. (2.7) constitutes a *unitary* transformation will be satisfied whenever

$$\mathbf{M}_\sigma^{L\dagger}\mathbf{M}_\sigma^L + \mathbf{M}_\sigma^{R\dagger}\mathbf{M}_\sigma^R = \mathbf{I}, \tag{2.9}$$

which, from Eq. (2.8), is equivalent to

$$\left|m_\sigma^L(x)\right|^2 + \left|m_\sigma^R(x)\right|^2 = 1. \tag{2.10}$$

Equation (2.7) can then be extended to a unitary transformation defined on the whole joint Hilbert space. We note that, while Eq. (2.7) does not uniquely determine this unitary transformation on the whole space, it suffices as a description of the interaction when the apparatus is initialized to $|0\rangle_\mathcal{A}$.

While the composite state evolves unitarily, we are also interested in the states of the system and apparatus separately. In particular, we are interested in whether the system state exhibits interference and how the apparatus state encodes information about the system. For this reason we will examine the density operator $\rho_{\mathcal{SA}} = |\Phi\rangle_{\mathcal{SA}}\langle\Phi|$ and the reduced states of the system and apparatus. Taking the partial trace[7] gives

$$\begin{aligned}\rho_{\mathcal{S}} &= \mathrm{tr}_{\mathcal{A}}\rho_{\mathcal{SA}} = {}_{\mathcal{A}}\langle L|\rho_{\mathcal{SA}}\,|L\rangle_{\mathcal{A}} + {}_{\mathcal{A}}\langle R|\rho_{\mathcal{SA}}\,|R\rangle_{\mathcal{A}} \\ &= \mathbf{M}_\sigma^L\,|\Psi\rangle_{\mathcal{S}}\langle\Psi|\,\mathbf{M}_\sigma^{L\dagger} + \mathbf{M}_\sigma^R\,|\Psi\rangle_{\mathcal{S}}\langle\Psi|\,\mathbf{M}_\sigma^{R\dagger} \\ &\equiv \frac{1}{2}\left|\Psi^L\right\rangle_{\mathcal{S}}\!\left\langle\Psi^L\right| + \frac{1}{2}\left|\Psi^R\right\rangle_{\mathcal{S}}\!\left\langle\Psi^R\right|,\end{aligned} \tag{2.11}$$

which is a mixture of the states corresponding to distinct detection outcomes. Note that the degree of overlap between the states $\left|\Psi^L\right\rangle$ and $\left|\Psi^R\right\rangle$ depends on the measurement precision; they are orthogonal when the measurement perfectly distinguishes left from right ($\sigma = 0$) but identical in the opposite limit of $\sigma \to \infty$ (in which case $\mathbf{M}_\sigma^L = \mathbf{M}_\sigma^R = \mathbf{I}/\sqrt{2}$ and no actual measurement is made). Hence, except in this latter case, the system transitions from a coherent superposition or pure state (i.e., one representable by a ket) to a mixed state (one which cannot be represented by a ket): it has been *decohered*,[49] to an extent that depends on the measurement precision, through its interaction with the apparatus.

The partial trace over the system can be evaluated in the position basis with the help

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[7] Recall that the partial trace is defined by

$$\mathrm{tr}_{\mathcal{A}}(\rho_{\mathcal{SA}}) = \sum_k {}_{\mathcal{A}}\langle k|\rho_{\mathcal{SA}}\,|k\rangle_{\mathcal{A}},$$

where $\{|s_i\rangle_{\mathcal{A}}\}$ is any basis for the apparatus Hilbert space, resulting in an operator that acts only on the system. For example, one has

$$\mathrm{tr}_{\mathcal{A}}\left(\left|\mathbf{M}_\sigma^L\Psi, L\right\rangle_{\mathcal{SA}}\left\langle\mathbf{M}_\sigma^R\Psi, R\right|\right) = 0,$$

as the left hand side is equal to $\mathbf{M}_\sigma^L\,|\Psi\rangle_{\mathcal{S}}\langle\Psi|\,\mathbf{M}_\sigma^{R\dagger}\mathrm{tr}(|L\rangle_{\mathcal{A}}\langle R|)$. Note that the states $|L\rangle_{\mathcal{A}}$ and $|R\rangle_{\mathcal{A}}$ in Eq. (2.11) do not constitute a complete basis for our apparatus state, but they do give the only nonzero terms in the trace here.

of Eq. (2.8), leading to the apparatus state

$$\rho_{\mathcal{A}} = \begin{pmatrix} \int \left|m_\sigma^L\right|^2 |\Psi|^2 \, d\, x & \int m_\sigma^R m_\sigma^L \, |\Psi|^2 \, d\, x \\ \int m_\sigma^L m_\sigma^R \, |\Psi|^2 \, d\, x & \int \left|m_\sigma^R\right|^2 |\Psi|^2 \, d\, x \end{pmatrix}, \tag{2.12}$$

when represented in the $\{|L\rangle_{\mathcal{A}}\,, |R\rangle_{\mathcal{A}}\}$ subspace.

Further evaluation requires a definite choice of $m_\sigma^{L/R}(x)$. While there are many possibilities, a physically meaningful choice can be made by considering first a continuous position measurement,[9, 13] out of which we can build a coarse-grained, binary measurement. To that end, consider the position-indexed, commuting set of operators $\mathbf{F}_\sigma(x')$, defined by

$$\mathbf{F}_\sigma(x')\,|x\rangle = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-x')^2}{2\sigma^2}\right)|x\rangle\,, \tag{2.13}$$

which represent a smooth analog of the projection operator $|x'\rangle\langle x'|$ (to which $\mathbf{F}_\sigma(x')$ tends as $\sigma \to 0$).

By integrating separately over the positive and negative domains of $\mathbf{F}_\sigma$, we arrive at a pair of coarse-grained operators acting on the position basis as

$$\begin{aligned} \mathbf{F}_\sigma^L\,|x\rangle &= \int_{-\infty}^{0} \mathbf{F}_\sigma(x')\,|x\rangle \; dx' \\ &= \left(\frac{1}{\sqrt{\pi}}\int_{x/\sqrt{2\sigma^2}}^{\infty} e^{-u^2}\,du\right)|x\rangle\,, \\ &= \frac{1}{2}\mathrm{Erfc}\left(\frac{x}{\sigma\sqrt{2}}\right)|x\rangle\,, \end{aligned} \tag{2.14}$$

$$\mathbf{F}_\sigma^R\,|x\rangle = \int_{0}^{\infty} \mathbf{F}_\sigma(x')\,|x\rangle \; dx' = \frac{1}{2}\mathrm{Erfc}\left(\frac{-x}{\sigma\sqrt{2}}\right)|x\rangle\,, \tag{2.15}$$

where we made use of the complementary error function Erfc. These operators correspond to left and right positions with precision σ.

Comparing to Eqs. (2.9) and (2.10), we see that taking

$$m_\sigma^L(-x) = m_\sigma^R(x) \equiv m_\sigma(x) = \left[\frac{1}{2}\text{Erfc}\left(\frac{-x}{\sigma\sqrt{2}}\right)\right]^{1/2} \tag{2.16}$$

yields measurement operators satisfying $\mathbf{M}_\sigma^{L/R\dagger}\mathbf{M}_\sigma^{L/R} = \mathbf{F}_\sigma^{L/R}$. One can check directly that the pair of operators $\mathbf{F}_\sigma^{L/R}$ satisfy $\mathbf{F}_\sigma^L + \mathbf{F}_\sigma^R = \mathbf{I}$, so we conclude that Eq. (2.16) provides a physically meaningful function that satisfies our criteria. Figure 2.3 shows how the function m_σ^2 changes as one varies σ.

Returning to Eq. (2.12) and inserting Eq. (2.16), one finds that the diagonal terms evaluate to Gaussian integrals, which can be computed exactly. The off-diagonal terms contain the product $m_\sigma^L m_\sigma^R$, which does not result in a simple closed-form expression (although it is easily evaluated numerically for specific values of σ). In order to obtain an analytic expression for arbitrary σ, some approximation will be necessary. To that end, recall that Ψ is a superposition of two Gaussians centered at L and $-L$, respectively. For $L \gg 1$, the value of m_σ changes little over the regions in which Ψ is non-negligible. This can be seen qualitatively by considering the curves in Fig. 2.3 or analytically by expanding m_σ in a Taylor series about $x = \pm L$. For example, expanding around $x = L$, we find

$$m_\sigma(x) \approx m_\sigma(L) + \frac{e^{-L^2/2\sigma^2}}{\sigma\sqrt{8\pi}m_\sigma(L)}(x - L). \tag{2.17}$$

Then for $\sigma > 0$, $m_\sigma(L)$ is bounded below by $1/\sqrt{2}$. Substituting this lower bound for m and the maximizing the derivative with respect to σ shows that, for fixed L,

$$\frac{e^{-L^2/2\sigma^2}}{\sigma\sqrt{8\pi}m_\sigma(L)} \leq \frac{1}{2L\sqrt{\pi e}} < \frac{1}{5L}. \tag{2.18}$$

Hence, the linear coefficient here is certainly smaller than $0.2/L$ for all σ, i.e., over the width of the Gaussian (1 in the dimensionless units employed here). The first-order change to $m_\sigma(x)$ near L is at most $0.2/L$ (this is a worst case estimate, and for $L = 5$ gives a bound of 0.04). Higher order terms are likewise suppressed. Similarly, the first-order approximation about $x = -L$ is given in Eq. (2.17) with the replacement $L \mapsto -L$. The linear coefficient can easily be seen to be bounded graphically, but in this case the denominator includes a factor of $m_\sigma(-L)$, which cannot be bounded below by a positive constant. A numerical calculation provides a bound of approximately $0.379/L$ on the actual coefficient, and there are many functions that can be used to find bounds analytically. For example, inserting the bound

$$\mathrm{Erfc}\left(x/\sqrt{2}\right) > \sqrt{2/\pi}(x/(x^2+1))e^{-x^2/2}$$

into the linear coefficient and maximizing yields a bound slightly lower than $0.4/L$

term near $x = -L$ is bounded by $0.4/L$.[8]

We therefore consider $m_\sigma(x)e^{-(x\pm L)^2/2} \approx m_\sigma(\mp L)e^{-(x\pm L)^2/2}$ and $m_\sigma(-x)e^{-(x\pm L)^2/2} \approx m_\sigma(\pm L)e^{-(x\pm L)^2/2}$; i.e., the functions $m_\sigma(x)$ are approximated, but not the Gaussian

[8]The first-order approximation about $x = -L$ is given by Eq. (2.17) with the replacement $L \mapsto -L$. The linear coefficient can easily be seen to be bounded graphically, but in this case the denominator includes a factor of $m_\sigma(-L)$, which cannot be bounded below by a positive constant. A numerical calculation provides a bound of approximately $0.379/L$ on the actual coefficient, and there are many functions that can be used to find bounds analytically. For example, inserting the bound

$$\mathrm{Erfc}\left(x/\sqrt{2}\right) > \sqrt{2/\pi}(x/(x^2+1))e^{-x^2/2}$$

into the linear coefficient and maximizing yields a bound slightly lower than the $0.4/L$ used in the main text.

envelopes. This results in the approximations

$$\begin{aligned}\Psi^L(x) = \langle x \big| \Psi^L \rangle &= \sqrt{2} m_\sigma^L(x)\Psi(x) \\ &= A\sqrt{2}[m_\sigma(-x)e^{-(x+L)^2/2} + m_\sigma(-x)e^{-(x-L)^2/2}] \\ &\approx B[m_\sigma(L)e^{-(x+L)^2/2} + m_\sigma(-L)e^{-(x-L)^2/2}],\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\Psi^R(x) = \langle x \big| \Psi^R \rangle &= \sqrt{2} m_\sigma^R(x)\Psi(x) \\ &= A\sqrt{2}[m_\sigma(x)e^{-(x+L)^2/2} + m_\sigma(x)e^{-(x-L)^2/2}] \\ &\approx B[m_\sigma(-L)e^{-(x+L)^2/2} + m_\sigma(L)e^{-(x-L)^2/2}].\end{aligned} \tag{2.20}$$

Note that in both $\Psi^L(x)$ and $\Psi^R(x)$, each term in the superposition is approximated separately.

A straightforward integration shows that the normalization constant of the approximate states should be

$$B^{-2} = \sqrt{\pi}\left(1 + \beta_\sigma e^{-L^2}\right) \approx \sqrt{\pi}, \tag{2.21}$$

where the approximation is for $L \gg 1$ (i.e., $e^{-L^2} \ll 1$), and

$$\beta_\sigma = 2m_\sigma(-L)m_\sigma(L). \tag{2.22}$$

As we will show, β_σ, which ultimately depends on both σ and L through the ratio σ/L, is the parameter that relates the measurement precision to the visibility of interference fringes and the information acquired by the measurement apparatus. In some sense, one can think of β_σ as the relevant quantification of the overlap between the left and right measurements.

Returning again to Eq. (2.12), it is clear that the approximation of Eqs. (2.19) and (2.20), does not affect the diagonal terms. It does, however, allow us to evaluate the off-diagonal terms as intended, which are now also just Gaussian integrals. Doing so, we find

$$\rho_{\mathcal{A}} = \frac{1}{2}\begin{pmatrix} 1 & \frac{\beta_\sigma + e^{-L^2}}{1+\beta_\sigma e^{-L^2}} \\ \frac{\beta_\sigma + e^{-L^2}}{1+\beta_\sigma e^{-L^2}} & 1 \end{pmatrix} \approx \frac{1}{2}\begin{pmatrix} 1 & \beta_\sigma \\ \beta_\sigma & 1 \end{pmatrix}. \tag{2.23}$$

This has trace 1, as expected, and the eigenvalues are

$$\lambda_\pm = \frac{1}{2}\left(1 \pm \frac{\beta_\sigma + e^{-L^2}}{1+\beta_\sigma e^{-L^2}}\right) \approx \frac{1}{2}(1 \pm \beta_\sigma). \tag{2.24}$$

As in Eq. (2.21), the approximate expressions are for $\exp(-L^2) \ll 1$, but we retain a finite σ/L in β_σ in order to investigate the full range of measurement precision.

Post-measurement evolution

It is well known that the act of measuring exactly which path a particle takes in passing through the slits prevents the appearance of interference effects. Having determined the immediate effect of measurement on the particle's state, we must now evaluate the subsequent evolution in order to determine how the interference is affected.

After the measurement, the system evolves as a free particle while the apparatus remains unchanged. The joint evolution is

$$\begin{aligned} |\Psi, 0\rangle &\mapsto \left|\mathbf{M}_\sigma^L \Psi, L\right\rangle_{\mathcal{SA}} + \left|\mathbf{M}_\sigma^R \Psi, R\right\rangle_{\mathcal{SA}} \\ &\mapsto \frac{1}{\sqrt{2}}\left|\mathcal{U}_t \Psi^L, L\right\rangle + \frac{1}{\sqrt{2}}\left|\mathcal{U}_t \Psi^R, R\right\rangle, \end{aligned} \tag{2.25}$$

with $\mathcal{U}_t = e^{-i\mathbf{P}^2 t/2}$ and $\left|\Psi^{L/R}\right\rangle$ given in Eq. (2.11). We thus have a system state comprising an equal mixture of the wavefunctions

$$\begin{aligned}\Psi^{L/R}(x,t) &= \langle x|\,\mathcal{U}_t \left|\Psi^{L/R}\right\rangle \\ &= \int e^{-\frac{i}{2}p^2 t} \langle x|p\rangle \int \langle p|y\rangle\!\langle y\left|\Psi^{L/R}\right\rangle \, dy\, dp. \end{aligned} \tag{2.26}$$

Evaluation of the inner integral can be done with approximations (2.19) and (2.20) and $\langle x|p\rangle = e^{ixp}/\sqrt{2\pi}$, which reduces the integrals appearing in Eq. (2.26) to a sum of Gaussian integrals. These give

$$\begin{aligned}\Psi^{L}(x,t) = \left(\frac{B^2}{1+it}\right)^{1/2} &\left\{ m_\sigma(-L)\exp\left[\frac{-\left(x+L\right)^2}{2\left(1+it\right)}\right] \right. \\ &\left. + m_\sigma(L)\exp\left[\frac{-\left(x-L\right)^2}{2\left(1+it\right)}\right]\right\}, \end{aligned} \tag{2.27}$$

and

$$\begin{aligned}\Psi^{R}(x,t) = \left(\frac{B^2}{1+it}\right)^{1/2} &\left\{ m_\sigma(L)\exp\left[\frac{-\left(x+L\right)^2}{2\left(1+it\right)}\right] \right. \\ &\left. + m_\sigma(-L)\exp\left[\frac{-\left(x-L\right)^2}{2\left(1+it\right)}\right]\right\}. \end{aligned} \tag{2.28}$$

Recalling that our particle is in an equal mixture of these two states, the probability density associated with detecting the particle at position x is given by

$$\begin{aligned}P_\sigma(x,t) &= \frac{1}{2}\left|\Psi^{L}(x,t)\right|^2 + \frac{1}{2}\left|\Psi^{R}(x,t)\right|^2 \\ &= \Gamma_\sigma^{xt}\left[\cosh\left(\frac{2xL}{1+t^2}\right) + \beta_\sigma \cos\left(\frac{2txL}{1+t^2}\right)\right], \end{aligned} \tag{2.29}$$

where we have reintroduced and generalized

$$\Gamma_\sigma^{xt} = \frac{\exp\left(\frac{-x^2-L^2}{1+t^2}\right)}{\sqrt{\pi+\pi t^2}\left(1+\beta_\sigma e^{-L^2}\right)} \approx \frac{\exp\left(\frac{-x^2-L^2}{1+t^2}\right)}{\sqrt{\pi}\sqrt{1+t^2}}, \tag{2.30}$$

for readability. As for the unmeasured free particle, the first term in the brackets (the hyperbolic cosine) describes the spread of the two incoherent Gaussian wavepackets with time. The second term (with the cosine) gives rise to the interference pattern and is also the same as in Eq. (2.5), except for a factor of β_σ. Hence, we recover the unmeasured case in the limit where the measurement is not at all precise, for $\sigma \to \infty$ ($\beta_\sigma \to 1$), and the interference is suppressed for smaller values of σ. In the limit of a perfectly precise measurement, $\sigma \to 0$ ($\beta_\sigma \to 0$), the interference term vanishes and we find

$$P_0(x,t) = \Gamma_0^{xt} \cosh\left(\frac{2xL}{1+t^2}\right) \propto \exp\left[\frac{-(x+L)^2}{1+t^2}\right] + \exp\left[\frac{-(x-L)^2}{1+t^2}\right], \tag{2.31}$$

which describes an incoherent sum of the particle coming either from the left or right slit as shown in Fig. 2.1b.

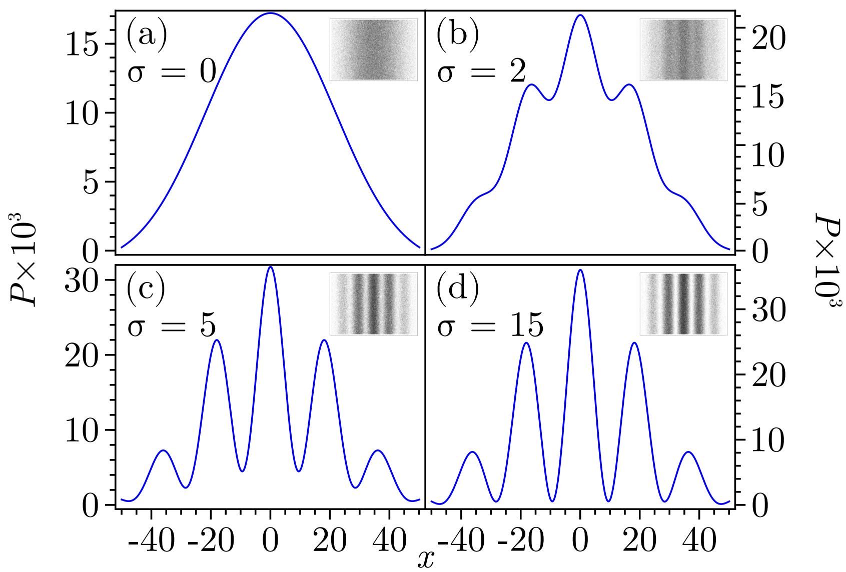

Figure 2.4: Probability density with $L = 5$ at $t = 30$ for various values of σ. Insets show simulated detection screens. (a) At $\sigma = 0$, a perfect measurement has been made; only a single, broad fringe appears. (b) As σ passes $\sigma^* \approx 3L/10$, interference begins to appear. Increasing σ further, (c) the difference between constructively and destructively interfering regions is clearly visible, so that (d) by $\sigma = 3L$ the interference is nearly total.

Figure 2.4 demonstrates how the distribution varies as σ increases.[9] Beyond this qualitative demonstration, a quantitative description is provided by the interferometric visibility, which relates the amplitude of a wave to its average value. At sufficiently late times, when a maximum appears at $x = 0$, this visibility may be expressed as

$$\mathcal{V} = \frac{P(0,t) - P(x^*,t)}{P(0,t) + P(x^*,t)}, \tag{2.32}$$

[9]See supplementary material at http://dx.doi.org/10.1119/1.4943585 for a calculation providing dynamic visualization of the distribution for arbitrary parameters.

where $x^* = \pi(1 + t^2)/2tL$ corresponds to the first minimum of the oscillation term. Evaluating Eq. (2.32) then yields the expression

$$\mathcal{V} = \frac{\Gamma^{\sigma}_{0t}\left[1+\beta_\sigma\right] - \Gamma^{\sigma}_{x^*t}\left[\cosh\left(1/t\right) - \beta_\sigma\right]}{\Gamma^{\sigma}_{0t}\left[1+\beta_\sigma\right] + \Gamma^{\sigma}_{x^*t}\left[\cosh\left(1/t\right) - \beta_\sigma\right]}, \tag{2.33}$$

which, after canceling common factors and taking the limit $t \to \infty$, reduces to

$$\mathcal{V} = \frac{(1+\beta_\sigma) - e^{-\pi^2/4L^2}\left(1-\beta_\sigma\right)}{(1+\beta_\sigma) + e^{-\pi^2/4L^2}\left(1-\beta_\sigma\right)} \approx \beta_\sigma. \tag{2.34}$$

Hence, for large L, we have $\mathcal{V} \approx \beta_\sigma$. Thus, $\beta_\sigma = 2m_\sigma(L)m_\sigma(-L)$, which is a quantification of the measurement precision with respect to the slit width. It further has a direct physical meaning as the visibility of the post measurement interference fringes.

2.4 Information

We have called the interaction determined in Eq. (2.7) a measurement interaction on the grounds that a subsequent projective measurement of the apparatus alone will allow an observer to infer (or attempt to infer, in the case of an imperfect measurement) the state of the system. This interaction is just a particular example of a positive-operator valued measure (POVM).[10] The key idea is that the apparatus acquires information about the state of the system due to this interaction. To make this statement quantitative, we make use of two key ideas from the theory of quantum information: entropy and mutual information.

For any state represented by a density operator ρ with eigenvalues $\{\lambda_i\}$, the von

[10] See Chapter 2.2.6 of Ref.[26]

Neumann entropy is defined by

$$H(\rho) = -\mathrm{tr}[\rho \log_2 \rho] = -\sum_i \lambda_i \log_2 \lambda_i, \tag{2.35}$$

in which we take $0 \log 0 = 0$ whenever it arises. In particular, $H(\rho) \geq 0$, with equality if and only if the state is pure (i.e., $\lambda_i = 1$ for one i and $\lambda_i = 0$ otherwise). Hence, the entropy is a measure of our state's "mixedness" and quantifies our uncertainty about the state of the system. It thus also quantifies the amount of information we gain about the system when a measurement is made.[11]

If our system is composed of two subsystems in a state $\rho_{\mathcal{SA}}$ and with reduced states $\rho_{\mathcal{S}}$ and $\rho_{\mathcal{A}}$, the quantum mutual information between $\mathcal{S}$ and $\mathcal{A}$ is defined by

$$I\left(\mathcal{S} : \mathcal{A}\right) = H(\rho_{\mathcal{S}}) + H(\rho_{\mathcal{A}}) - H(\rho_{\mathcal{SA}}). \tag{2.36}$$

This quantifies the amount of information about system $\mathcal{S}$ that is in $\mathcal{A}$.

In the case of a measurement implemented by some apparatus as described previously, we have

$$\begin{aligned} I\left(\mathcal{S} : \mathcal{A}\right) &= H(\rho_{\mathcal{S}}) + H(\rho_{\mathcal{A}}) - H(\rho_{\mathcal{SA}}) & (2.37) \\ &= H(\rho_{\mathcal{S}}) + H(\rho_{\mathcal{A}}), & (2.38) \end{aligned}$$

since the joint-state is pure (having evolved unitarily from a pure product state). Moreover, when the joint state is pure, the Schmidt decomposition [35, 17][12] ensures that we can use $|\Phi\rangle_{\mathcal{SA}} = \sum_i a_i \, |i\rangle_{\mathcal{S}} \otimes |i\rangle_{\mathcal{A}}$, where $\{|i\rangle_{\mathcal{S}}\}$ and $\{|i\rangle_{\mathcal{A}}\}$ are orthonormal bases for the

[11] We note that this statement assumes the system state is initially pure.

[12] See also chapter 2.5 of Ref.[26]

two subsystems, in writing the joint density matrix $\rho_{\mathcal{SA}} = |\Phi\rangle_{\mathcal{SA}}\langle\Phi|$. Taking the partial traces, one can see that the values $\{a_i\}$ will be the eigenvalues for *both* $\rho_{\mathcal{S}}$ and $\rho_{\mathcal{A}}$, so that their entropies will be the same. In particular, we have

$$I\left(\mathcal{S}:\mathcal{A}\right) = 2H(\rho_{\mathcal{A}}) = 2H(\rho_{\mathcal{S}}). \tag{2.39}$$

Again, the entropy $H(\rho_{\mathcal{S}})$ gives a measure of the mixedness (the degree of decoherence) of our system state after the measurement. This measurement-induced decoherence of the system is associated with information acquisition by the measurement apparatus, which is reflected in this generation of entropy. That is, the system goes from a pure state initially, with entropy of zero, to a mixed state with nonzero entropy.

Following Eq. (2.39), we see that determining the mutual information between the system and the apparatus amounts to finding the entropy for the reduced state of either the system or the apparatus. In a general measurement scheme, these may depend on time, but when the system and apparatus states undergo *independent* unitary evolution after the measurement process, we need only consider the states immediately after the measurement. In our case, for example, the particles passing through the slits evolve as free particles after the interaction, while the apparatus state remains unchanged. Hence, we may compute the mutual information from the state of the apparatus immediately after the measurement, without involving the more complicated time-dependent state of the particle.

We have previously already found the eigenvalues of the state $\rho_{\mathcal{A}}$ after measurement has occurred, which are given in Eq. (2.24). These eigenvalues give the mutual information

$$I\left(\mathcal{S}:\mathcal{A}\right) = 2H(\rho_{\mathcal{A}}) = 2\left(-\lambda_+\log_2\lambda_+ - \lambda_-\log_2\lambda_-\right).$$

In the limit of no measurement, $\sigma \to \infty$, the eigenvalues are 1 and 0, so the mutual information is zero: the apparatus stores no information about the state of the system. This limit is precisely that in which the standard interference pattern is observed. On the other hand, $\sigma \to 0$ corresponds to the complete absence of interference and the eigenvalues monotonically approach $(1 \pm e^{-L^2})/2 \approx 1/2$, in which case the mutual information approaches $I(\mathcal{S} : \mathcal{A}) = 2$. The dependence of the mutual information and visibility on the precision σ, is shown in Fig. 2.5, demonstrating that as the information gained by the apparatus decreases, the visibility of the interference increases.

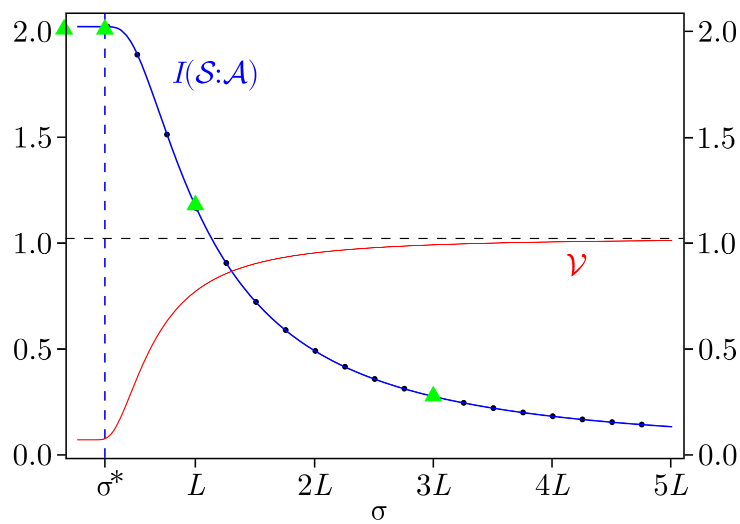

Figure 2.5: The dependence of mutual information $I(\mathcal{S} : \mathcal{A})$ and visibility $\mathcal{V}$ on σ. Extremely precise measurements, corresponding to small values of σ, result in the apparatus acquiring significant information about the system while the interference is negligible. As σ increases, the measurement is less capable of distinguishing the particle's path, and the apparatus fails to decohere the system state. Hence, less information is transferred into the apparatus and the interference becomes significant. Visibility is calculated according to the exact expression in Eq. (2.34) with $L = 5$. The vertical dashed line at $\sigma^\star \approx 3L/10$ indicates the approximate precision at which this transition begins to become apparent. Dots indicate values calculated numerically from the exact apparatus state, Eq. (2.12), for $L = 5$. Triangles correspond to σ values considered in Fig. 2.4.

As is shown in Fig. 2.5, both $I(\mathcal{S}:\mathcal{A})$ and $\mathcal{V}$ are initially flat as σ increases from 0, as at small σ the measurement is extremely precise and the particle at the left and right slits can be effectively distinguished. Beyond a threshold value, $\sigma = \sigma^\star \approx 3L/10$,[13] the apparatus rapidly loses the ability to distinguish between the two paths, and interference begins to emerge.

Considering the above discussion, one notes that the *quantum* mutual information indicates that, in the case of perfect measurement, we get *two* bits of information, despite the fact that the information in which we are interested appears to be a simple binary statement regarding the particle's path, or one bit. This is a peculiarity of quantum information, corresponding to the existence of non-classical correlations (entanglement) between the apparatus and the system. If there is a third link in the von Neumann chain—e.g., an observer making measurements on *the apparatus*—we will find that there is only one bit of information between the system and apparatus or between the system and observer. Indeed, that there are many links in the von Neumann chain, including not just the apparatus and observer, but also the large surrounding environment, e.g., photons, is why quantum correlations are so hard to detect.[55] The presence of many such links is reflected in the redundant acquisition of information by the environment,

[13]The estimate for $\sigma^\star$ is found by observing that small values of σ correspond to large values in the argument of Erfc. Hence, the small-σ case can be analyzed using an asymptotic expansion, leading to

$$\beta_\sigma \approx \sqrt{\frac{2\sigma}{L}\sqrt{\frac{2}{\pi}}}\exp\left(-L^2/4\sigma^2\right).$$

Further expanding this expression as a Taylor series about $\sigma = L/2$ and locating the x-intercept then gives

$$\beta_{\sigma^\star} \approx \left(\frac{2}{\pi}\right)^{1/4} e^{-1}\left[1+\frac{5}{L}\left(\sigma^\star - \frac{L}{2}\right)\right] = 0,$$

so we find

$$\sigma^\star \approx 3L/10,$$

which is the value stated in the main text.

which is the quantum Darwinian process responsible for the emergence of the classical, objective world.[51]. To see this, consider an observer $\mathcal{O}$ that perfectly measures the apparatus immediately after the particle has been measured. In other words, if the observer is initially in the state $|0\rangle_{\mathcal{O}}$, the observer and the apparatus evolve according to

$$\begin{aligned} |L,0\rangle_{\mathcal{AO}} &\mapsto |L,L\rangle_{\mathcal{AO}}\,, \\ |R,0\rangle_{\mathcal{AO}} &\mapsto |R,R\rangle_{\mathcal{AO}}\,. \end{aligned} \tag{2.40}$$

Then we should replace Eq. (2.7) with

$$|\Psi,0,0\rangle_{\mathcal{SAO}} \mapsto \frac{1}{2}\left|\Psi^L,L,L\right\rangle_{\mathcal{SAO}} + \frac{1}{2}\left|\Psi^R,R,R\right\rangle_{\mathcal{SAO}}\,, \tag{2.41}$$

where for a perfect measurement the system states $\left|\Psi^L\right\rangle$ and $\left|\Psi^R\right\rangle$ would be orthogonal.

If we now compute the partial trace over the apparatus as done in Eq. (2.11), we find that the joint system-*observer* state is

$$\rho_{\mathcal{SO}} = \frac{1}{2}\left|\Psi^L,L\right\rangle_{\mathcal{SO}}\!\left\langle\Psi^L,L\right| + \frac{1}{2}\left|\Psi^R,R\right\rangle_{\mathcal{SO}}\!\left\langle\Psi^R,R\right|. \tag{2.42}$$

A second partial trace over the observer will recover Eq. (2.11), showing that the entropy of the particle is unaffected by the observer's measurement of the apparatus. The entropy of the observer, though, is $H(\rho_{\mathcal{O}}) = 1$. In contrast to the previous discussion, the joint state $\rho_{\mathcal{SO}}$ is not pure, so its entropy does not vanish. Rather, the orthogonality of $\left|\Psi^L,L\right\rangle_{\mathcal{SO}}$ and $\left|\Psi^R,R\right\rangle_{\mathcal{SO}}$, due to the presence of the left/right record in the observer's state, indicate that this, too, has $H(\rho_{\mathcal{SO}}) = 1$. Hence, the mutual information is

$$I\left(\mathcal{S}:\mathcal{O}\right) = H(\rho_{\mathcal{S}}) + H(\rho_{\mathcal{O}}) - H(\rho_{\mathcal{SO}}) \tag{2.43}$$

$$= H(\rho_{\mathcal{S}}) + 1 - 1 = H(\rho_{\mathcal{S}}), \tag{2.44}$$

indicating that the observer acquires an amount of information equal to what is available about the path of the particle, $H(\rho_{\mathcal{S}})$. If path information is present, or the measurement precision is $\sigma \to 0$ yielding $H(\rho_{\mathcal{S}}) \to 1$, then the observer will acquire 1 bit of information. If not, $H(\rho_{\mathcal{S}}) \approx 0$, the observer will learn nothing about the path of the particle and interference will be observed at the screen. When the observer is present and the global state is Eq. (2.41), the apparatus will also have mutual information given by Eq. (2.44), as entanglement with $\mathcal{S}$ is "locked up" in joint correlations between $\mathcal{S}$ and $\mathcal{AO}$.

Finally, we note that in the large L approximation, Eqs. (2.24) and (2.34) together allow us to write

$$I\left(\mathcal{S}:\mathcal{O}\right) = H_{bin}\left(\frac{1+\mathcal{V}}{2}\right). \tag{2.45}$$

This explicitly connects the information gain by the observer with the loss of visibility of the interference fringes through the binary entropy

$$H_{bin}(x) = -x\log_2 x - (1-x)\log_2{(1-x)}, \tag{2.46}$$

which characterizes the uncertainty regarding the outcome of a classical event that could result in one of two outcomes with probabilities x and $1-x$, respectively. When the fringes are readily apparent $\mathcal{V} \approx 1$ and the information acquired by the observer (or apparatus) is $I\left(\mathcal{S}:\mathcal{O}\right) \approx 0$. On the other extreme, when the fringes are not visible $\mathcal{V} \approx 0$

and the information gain is $I\left(\mathcal{S}:\mathcal{O}\right)\approx 1$. Note also that in the intermediate regimes one can have quite high visibilities even for $I\left(\mathcal{S}:\mathcal{O}\right)$ near 1, but for $\sigma\approx L/2$ there are still visible interference fringes despite gaining nearly complete information about the system's path. This fact was also noted in the case of interference of photons.[42]

It bears mentioning that while we have focused exclusively on how the act of measurement can cause a loss of interference, this is by no means the only reason interference may not be observed. Another cause for interference loss is *dephasing*, which occurs when, for example, the relative phase between the two Gaussians in the superposition varies from trial to trial. In this case, the absence of interference is a statistical result arising from the oscillation term acquiring a different phase in each trial, which causes the probability density to be shifted. If the phase is Gaussian distributed with a width γ (and mean 0), then the expected probability density for a free particle ($\beta_\sigma=1$) becomes

$$\langle P\rangle\approx\Gamma_{xt}\left[\cosh\left(\frac{2xL}{1+t^2}\right)+e^{-\frac{1}{2}\gamma^2}\cos\left(\frac{2xLt}{1+t^2}\right)\right], \tag{2.47}$$

when $L\gg 1$ (this calculation is similar to that in Sect. 2.2). Hence, a sharply-peaked distribution of phases will exhibit interference that becomes washed out as the distribution widens. While this "dephasing" process produces a similar experimental outcome (namely, the loss of interference), it is important to note that the physical process is quite different than that of measurement-induced decoherence.[34] In particular, decoherence removes interference from the wavefunction for every trial, whereas the loss of interference due to dephasing is found only as a result of averaging over many different trials.

3: Quantum Darwinism

While the decoherence program may provide an explanation for the absence of interference in our classical world (interference is suppressed by interactions with an environment), it fails to resolve the problem of the preferred basis or, more generally, to explain the origin of an apparently objective classical reality. A more complete resolution is provided by quantum Darwinism [50, 51], which recognizes that the environment is a communication channel, selectively amplifying and proliferating information in a way that admits access to many different, independent observers. In this chapter, we review the main results of the quantum Darwinism program, highlighting the means by which quantum Darwinism attempts to explain the emergence of an objective, classical reality from a quantum foundation [22, 49, 34].

3.1 Solving the basis problems

By considering a specific model of double-slit interference, we have shown in Chapter 2 how the precision with which one determines the path of particles passing through two slits is directly correlated with the loss of observed interference effects in the subsequent evolution of the particles. In particular, we have shown how the absence of interference can be attributed to an apparatus gaining maximal information. Notably, this discussion didn't require any concrete identification of the "apparatus"; i.e., it may be a physical device, an observer, or, most importantly, the environment.

As part of that previous discussion, we looked at the situation in which a third participant, an observer, "checks" the apparatus after it has measured the particle, thereby acquiring some of the which-path information indirectly. But this leads to a problem we have so far avoided mentioning: that of the preferred measurement basis. In essence, how does the observer know which variable was measured by the apparatus, and therefore which measurement to *perform* on the apparatus? If they are incorrect and measure the apparatus in the "wrong" basis, they may destroy the information it acquired, effectively resetting it to an initialized state. But this is not our usual experience; it seems that our measurement devices are typically found in states that correspond to some natural basis, and that this basis is the one we should measure if we want to acquire information from the apparatus. The question, then, is why this should be the case.

The answer comes in two flavors, the first of which, provided by Zurek [45, 46], is rooted in the observation that real measuring devices are embedded in a typically large environment with which they are consistently interacting. This interaction results in decoherence between certain pointer states, so that an apparatus initially in a superposition of these states should rapidly decohere into an improper mixture of them, effectively transferring information about those states, and hence any correlated information about the system, into the environment. In other words, the environment acts as a communication channel, selectively transmitting information about the apparatus to our third-party observer. In the event that the observer chooses the "wrong" basis, the information may be lost from their fragment, but the large number of such fragments permits them to "try again".

Hence, the preferred basis for the apparatus is the basis that survives this environmental monitoring process, but, on its own, this is not entirely satisfactory. For example, one expects that isolating a measuring device should not significantly alter its effectiveness. Moreover, the "apparatus" is not always distinct from the environment (as in the case of our everyday experience using a photon environment to acquire information about the systems around us) or the system (as we can, in general, simply consider their joint state). To handle these difficulties, we note that the environment accomplishes its role as a communication channel by bringing a large number of independent degrees of freedom to the table; the pointer states are those about which information can be accessed from a small subset of these degrees of freedom, without resorting to a measurement on the whole environment. But our experience of measuring devices is that, in contrast to the two-dimensional apparatus discussed in the above model, they are typically macroscopic, and hence have themselves a large number of degrees of freedom, only some of which will be directly accessible to the third-party observer at any one time, and only some of which will be involved in the interaction with the system. In particular, the preferred basis is selected by form of the measurement interaction and the internal configuration of the apparatus/environment.

So we've seen that the preferred basis problem may be resolved by the large number of degrees of freedom available in the apparatus or, more generally, the environment. Of course, the environment is typically *huge*, which leads to the next question: just how much of the environment does an observer need in order to gain "enough" information about the system? The answer, it turns out, is "not much".

3.2 Amplification and redundancy: The origin of objectivity

In order to answer the question of how much of the environment is needed, we first have to decide what constitutes "enough" information. The standard is to introduce an information deficit, δ, which represents the amount of information that observers are willing to forgo. Then we can say that a fragment, $\mathcal{F}$, of the environment carries "enough" information about the system when the mutual information between them satisfies $I(\mathcal{F}:\mathcal{S}) \geq (1-\delta)H_{\mathcal{S}}$. With this characterization, we define the minimum fragment size, ${}^\sharp\mathcal{F}_\delta$, needed to acquire this much information.[1]

Knowing ${}^\sharp\mathcal{F}_\delta$, we can define the *redundancy* of information, $R_\delta = 1/{}^\sharp\mathcal{F}_\delta$, this being the number of distinct fragments of the environment that transmit the necessary amount of classical information. This redundancy is the origin of objectivity: because observers are making measurements on their individual fragments, they alter neither the state of the system nor the information acquired by other observers, and each such observer acquires the same, nearly complete information regarding the state of the system.

Of course, the recognition that redundancy *can* provide access to objective information is itself insufficient; one still needs to show that the redundancy of typical situations is sufficient to explain our classical experience. This has since been demonstrated in a number of situations, including that of a photon [32] environment in which, for example, a 1μm grain of dust subjected to 1μs of Solar illumination yields a redundancy on the order $R_\delta \approx 10^7$ [32]; i.e., tens of millions of independent observers can each acquire information about the grain's position without disturbing it. The redundancy remains large even when the environment subsystems are initially in mixed states; in the case of a spin environment, for example, the redundancy simply acquires a scale factor of $1-h$, where h is the initial entropy of the individual environment spins[2] [52].

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[1] More specifically, ${}^\sharp\mathcal{F}_\delta$ should be the minimum fragment size such that the *average* mutual information between a fragment of size ${}^\sharp\mathcal{F}_\delta$ and the system $\mathcal{S}$ is equal to $(1-\delta)H_{\mathcal{S}}$.

[2] An important feature of the models considered here is the assumed homogeneity of the environment, in which each environment subsystem begins in the same initial state; future work will be needed to clarify the exact degree to which this assumption can be relaxed.

3.2.1 The quantum Chernoff information

Early studies of redundancy were often complicated by the need to explicitly compute the mutual information between the system and an arbitrary fragment, which in turn requires an explicit calculation, or at least a strong approximation, for the fragment's entropy. These complications were ameliorated by the recognition of the quantum Chernoff information [53]: the redundancy is characterized by the quantum Chernoff bound, $\overline{\xi}_{QCB}$, which provides a bound on the distinguishability between the states of the environment subsystems conditioned on the state of the system [5, 25]. Explicitly, the redundancy is given approximately[3] by

$$R_\delta \approx \sharp\mathcal{E} \frac{\overline{\xi}_{QCB}}{\ln 1/\delta}$$

with

$$\overline{\xi}_{QCB} = \max_{0\le c\le 1} \left(- \ln \left\langle \mathrm{tr} \left[\rho_{k|\uparrow}^{c} \rho_{k|\downarrow}^{1-c} \right] \right\rangle_{k\in\mathcal{E}} \right), \tag{3.1}$$

the typical quantum Chernoff information, where $\rho_{k|\hat{s}}$ is the state of an environment subsystem conditioned on the system being in state $\hat{s}$ and the trace is averaged over the environment. Hence, we see that the redundancy depends only weakly on δ while scaling linearly with the environment size. Moreover, a lower bound can be achieved by skipping the maximization and just setting c to a computationally convenient value such as $c = 1/2$.

The key feature of the quantum Chernoff information, made immediately clear by Eq. 3.1, is that it allows us to characterizes the redundancy, and therefore the amplification, of information in terms of (an average over) individual *environment* subsystems, removing the need to consider arbitrary states in the (exponentially large) Hilbert space

[3]For spins systems, the approximation has been explicitly shown to be within a factor of 2 in the asymptotic limit, and solved examples suggest it to be asymptotically exact

of the environment/fragment. A less immediately obvious, though equally important, fact is that outside of the extreme case where the system is in a pure $|\uparrow\rangle$ or $|\downarrow\rangle$ state, the *system*'s probabilities entail only minor corrections to Eq. 3.1, so that this *typical* quantum Chernoff information becomes the sole quantity of interest in the appearance of an objective, classical reality.

3.2.2 A central spin in a spin environment

Following Ref. [54], let us demonstrate the use of the quantum Chernoff information by considering a spin-spin system in which a central qubit, $\mathcal{S}$, interacts with a spin-environment, $\mathcal{E}$ according to the Hamiltonian,

$$\mathbf{H} = \sigma_{\mathcal{S}}^z \sum_{k=1}^{\sharp\mathcal{E}} g_k \sigma_k^z, \tag{3.2}$$

where the g_k are coupling constants between the system and each of the environment spins, indexed by k. Assuming the initial joint system-environment state is

$$\rho = \rho_{\mathcal{S}}(0) \otimes \left[\bigotimes_{k=1}^{\sharp\mathcal{E}} \rho_k(0) \right]$$

Referring to Eq. 3.1 we need to identify the conditional states of the environment spins, $\rho_{k|\hat{s}}$ after the k-th environment spin has interacted with the system for a time, t. This is straightforward, and one finds

$$\rho_{k|\hat{s}} = \mathcal{V}_{\hat{s}} \rho_k(0) \mathcal{V}_{\hat{s}}^{\dagger} \tag{3.3}$$

with

$$\mathcal{V}_{\hat{s}} = \exp\left(\pm i g_k t \sigma_k^z\right),$$

where the sign of the argument is determined by whether $\hat{s} = \uparrow$ or $\downarrow$. Hence, the interaction has the effect of rotating the environment spin's Bloch vector about the z-axis, and we see that $\rho_{k|\uparrow} = \rho_{k|\downarrow}^{\dagger}$; i.e., the two conditional states are rotated by equal amounts but in opposite directions, resulting in a separation by an angle $\Theta = 4 g_k t \sin^2\theta$ where θ is the angle between the original Bloch vector and the z axis.

In this case, $\mathrm{tr}\left[\rho_{k|\uparrow}^{c}\rho_{k|\downarrow}^{1-c}\right]$ is symmetric about $c = 1/2$ and obtains a minimum there [54], so we end up with

$$\begin{aligned}\overline{\xi}_{QCB} &= -\ln\left[\left\langle 1 - \left(1 - \sqrt{1-a^2}\right)\sin^2\Theta/2\right\rangle_1\right] \\ &= -\ln\left[\left\langle 1 - \left(1 - \sqrt{1-a^2}\right)\sin^2\left(2 g_k t \sin^2\theta\right)\right\rangle_1\right],\end{aligned}$$

where a is the magnitude of the Bloch vector—i.e., a measure of the mixedness of the environment spins—and the average is over fragments of size 1. In the special case of a *pure* initial state, we have $a = 1$ and so

$$\overline{\xi}_{QCB} = -\ln\left[\left\langle\cos^2\left(2 g_k t \sin^2\theta\right)\right\rangle_1\right].$$

Observe that, except in the extreme case $\theta = 0$ (i.e., for an initial state aligned with the z-axis, which will be unaffected by the interaction and therefore insensitive to the state of the system), the QCB is nonzero. Hence, redundant information is proliferated redundantly into the environment in almost all cases.

4: Local observers with limited tools

We live in a quantum Universe, and quantum systems change their state when observed; they can be re-prepared by measurements. Observers—from human beings down to microscopic organisms—have to learn, adapt, and respond within this fundamentally unpredictable medium. In this chapter, we demonstrate that in our world, where observers learn about quantum systems indirectly via the environment, there is no substantial advantage conferred to quantum observers. For instance, observers that can simultaneously access and manipulate the quantum states of large environment fragments have their ignorance about the external world decrease rapidly as they accumulate "the evidence" (a larger share of the environment). However, as we will show in Section 4.3, this decrease is only slightly faster than a local, classical observer. These results suggest that observers would be unlikely to evolve quantum hardware, as the slight advantage gained comes at a tremendous cost. Moreover, they show that all observers—whatever their innate capabilities—can access objective information about the effectively classical world, producing consistency and consensus in an unpredictable quantum Universe.

We begin with a review of the essential elements of quantum Darwinism before proceeding to the primary result of this research: environmentally-induced decoherence gives rise both to classical objective states of quantum systems and to a natural, local measurement by which observers can acquire information about these states. In other words, the environment stores the classical information not only redundantly, but also *locally*.

4.1 Quantum Darwinism

The conflict between the quantum and classical worlds has been a perplexing feature since the inception of quantum physics. As outlined in Chapter 3, quantum Darwinism provides a framework for quantifying and understanding the absence of quantum effects (i.e., Schrödinger's cat) and the emergence of classical reality in our everyday experience that goes beyond decoherence: Decoherence amplifies information—information about certain, select states of the system $\mathcal{S}$—and banishes what remains—complementary information about quantum coherences—to global correlations with the environment $\mathcal{E}$, simultaneously making some information accessible to all and the rest accessible to none [55]. Indeed, in our everyday experience, we bask in the photon (and other) environment [53, 32, 33, 24], which for all practical purposes selects position—the location of objects—as the preferred observable, simultaneously preventing access to the "weird" superpositions of objects in different locations. Position is the preferred pointer observable because interactions depend on distances [47].

The environment is thus not just a source of decoherence. It is a communication channel, acquiring and transmitting information about quantum systems to many observers (see Fig. 4.1). Information in this context is given by the Holevo quantity $\chi\left(\hat{\Pi}_{\mathcal{S}} : \mathcal{F}\right)$, which quantifies the amount of information about the system's effectively classical pointer basis (e.g., location). The corresponding pointer observable, $\hat{\Pi}_{\mathcal{S}}$, is transmitted by a quantum communication channel—the environment fragment $\mathcal{F}$, the "piece" available to an observer. The pointer states are thus special: They survive the interaction with the environment and "live on" to proliferate copies of themselves, giving rise to objectivity in the sense that many observers can identify the state of $\mathcal{S}$ independently without perturbing the system itself [29].

This amplification is captured by the redundancy of information [28, 7]. When an observer needs a fragment of size $^\sharp\mathcal{F}_\delta$ to acquire $\chi \simeq (1-\delta)\, H_\mathcal{S}$ bits of information about the system's pointer states, the redundancy is

$$R_\delta = {}^\sharp\mathcal{E}/{}^\sharp\mathcal{F}_\delta,$$

where $^\sharp\mathcal{E}$ is the size of the environment and $H_\mathcal{S} = H(\hat{\Pi}_\mathcal{S})$ is the entropy of the pointer observable, i.e., the missing information about $\mathcal{S}$. The information deficit δ quantifies the information about the system that observers are prepared to forgo.

For this transmission of information to occur, the environment must decohere the system. That is, photons, phonons, spins, etc., "incoming" from some source (i.e., out of equilibrium with the system), must scatter/interact with the system, decohering it and acquiring information. When decoherence is strong compared to the system's inherent dynamics (which is the case for all but the most isolated and microscopic systems), the initial state, at time $t=0$, of an environment component k, $\rho_k\,(t=0)$, is rotated into the conditional state $\rho_{k|\hat{s}}$ depending on what pointer state $\hat{s}$ it interacted with. These states are the "evidence"—imprints of the system's state—carried by the environment. For incoming photons scattering off a superposition $\sim |x_1\rangle + |x_2\rangle$ of, e.g., a dust grain, these conditional states will be the photon interacting with the particle at x_1 and at x_2, respectively. For spin environments, the conditional states are ones that interacted with, e.g., the $\hat{s} = \uparrow, \downarrow$ components of the superposition.

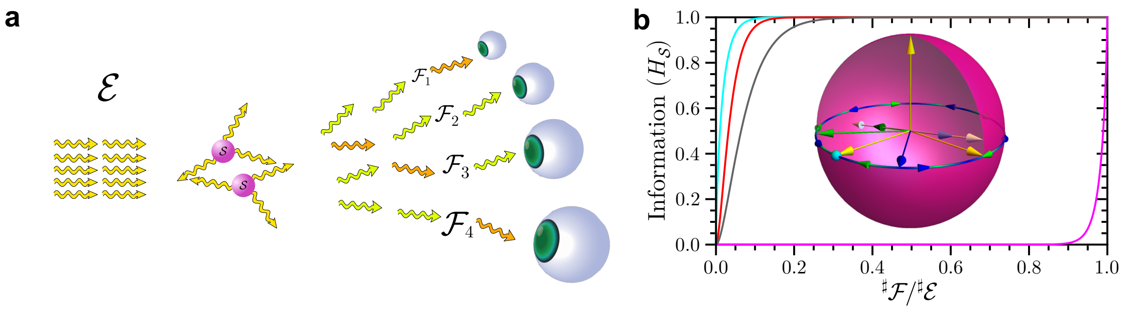

Figure 4.1: **The transmission of information by the environment.** (a) Schematic of the environment as a communication channel, where observers intercept components of the environment, bit-by-bit, and then locally process them (i.e., the chain in Eq. (4.1) for a representative environment spin). (b) Information versus the size of the fragment. The cyan and magenta curves show the classical and quantum components, respectively, of the mutual information (see Section 4.3). All the quantum information is pushed into correlations with nearly the whole environment, making it inaccessible to observers. The antisymmetry between classical and quantum information is evident by the 180 degree rotational correspondence of their respective curves [55]. This is the quantitative reflection of the statement that when redundant information is present, i.e., $^\sharp\mathcal{F}_\delta$ is small compared to $^\sharp\mathcal{E}$, then access to quantum correlations requires the ability to measure at least $^\sharp\mathcal{E} - {}^\sharp\mathcal{F}_\delta$ components. This is clearly prohibited: The photon environment is large (10^{20} or more photons!) and we need only a handful of them to learn about the location of objects. Recovering all, or nearly all, of the photons that have scattered and observing quantum effects is a practical impossibility. However, observers can get some information. The red line indicates the classical information actually available to an observer free to make arbitrary measurements; i.e., the quantum Chernoff information [53, 54], while the dark grey line shows the locally accessible information by using a measurement like that depicted by the grey arrows in the inset Bloch sphere (the Bloch sphere represents the conditional rotation – green and blue counter-rotating arrows – and then local measurement – projected grey arrows – used by the observer). This information rapidly plateaus at the same value of information, the total information about the effectively classical pointer states. Indeed, this is represented by the fast decay of ignorance, $P_e \sim \exp\left(-{}^\sharp\mathcal{F}\xi\right)$, as the observer acquires more and more bits of the environment. Therefore, even under tight constraints on the observer's capabilities, they still can easily access the redundant, objective information present in—and communicated by—the environment.

4.2 Local observers

When redundant information is present, i.e., $^\sharp\mathcal{F}_\delta$ is small compared to $^\sharp\mathcal{E}$, then access to quantum correlations requires the ability to measure at least $^\sharp\mathcal{E} - {}^\sharp\mathcal{F}_\delta$ components simultaneously (see Fig. 4.1). Observers, however, are inherently limited, e.g., they are local in space and time, in addition to limitations of memory (classicality and capacity). For instance, a local observer would intercept, e.g., photons from the environment one at a time and in one location. In the case of this photon environment of our everyday experience, there is a particular basis that we measure in, corresponding to angular direction and momentum (color). In other words, there seems to exist a *fixed* basis that allows observers to efficiently access classical, objective reality.

To acquire information "bit-by-bit", observers must extract information from the $\rho_{k|\hat{s}}$ by making a local measurement, converting $\rho_{k|\hat{s}}$ from a quantum to classical state $\tilde{\rho}_{k|\hat{s}}$, which can be represented by the chain of events

$$\rho_k\left(t=0\right) \xrightarrow{\hat{s}\,\text{imprint}} \rho_{k|\hat{s}} \xrightarrow{\text{local observation}} \tilde{\rho}_{k|\hat{s}}\,. \tag{4.1}$$

From there, the observer can continue to accumulate more bits of the environment, ultimately acquiring a string of outcomes in the local basis they are measuring in. Their (average) ability to successfully deduce the state $\hat{s}$ will be quantified by an exponential decay of the error probability to distinguish the *classical* states of the fragment. The exponent of this error decay is the Chernoff Information ξ, i.e., $P_e \sim \exp\left(-{}^\sharp\mathcal{F}\xi\right)$, where $^\sharp\mathcal{F}$ is the size of the fragment (the number of bit-by-bit measurements the observer has performed).

The inverse Chernoff information essentially gives the characteristic size of an environment fragment needed to make a correct assessment of the system state. The amplification, or redundancy, of locally accessible information is thus

$$\tilde{R}_\delta \simeq {}^\sharp\mathcal{E} \frac{\xi}{\ln 1/\delta}, \tag{4.2}$$

where the ratio $\xi / \ln(1/\delta)$ is a measure of the efficiency of the amplification (see Section 4.3 for details).

Already Eq. (4.2) shows that amplification is macroscopic (i.e., extensive in the environment size) and universal (as long as there is some sliver of orthogonality in the local conditional states $\tilde{\rho}_{k|\hat{s}}$, then ξ will be nonzero. That is, observers have to work hard—i.e., choose *exactly* the wrong measurement basis—not to find out this information). Figure 4.2 shows the Chernoff Information ξ—the unattenuated efficiency of amplification—versus the bit-by-bit measurement basis for a spin environment (both pure and mixed).

A surprising feature emerges for low entropy environments— the sharp peak in ξ that appears out of place. For exactly zero entropy environments (pure initial states and no randomness in the interactions), the maximum efficiency occurs for a local measurement axis that is aligned with one of the conditional states (either will do). As the environment entropy increases (but considering a fixed direction of the conditional states on the Bloch sphere), the peak diminishes, finally resulting in the optimal measurement transitioning from the "aligned" axis to the y-axis (see Section 4.3 for more details). The latter axis is selected out by the initial condition (x-axis state) and the interaction (z-axis), i.e., it is the axis complementary to the plane defined by the initial state and interaction. Any initial state in the xz-plane will have the same optimal measurement, so long as the environment has a moderate amount of entropy ($h \gtrsim 0.05$ for the example shown).

This is remarkable: Except for (very) low entropy environments, there is a single natural basis to measure in. This is analogous to photons in our Universe, where we measure in a single basis, momentum (i.e., direction they come from). *In essence, this indicates that not only do classical, objective states of the system emerge from decoherence, but a convenient classical measurement results as well.*

These results are captured by Fig. 4.3, which shows the (unattenuated) amplification efficiency for three modes of observation. The regime of this diagram that observers will fall into depends on both their inherent capabilities and properties of the external world. Observers that can globally store and manipulate the whole fragment state can use this ability to most efficiently acquire information about the system. Observers that are local—i.e., can only manipulate $\mathcal{E}$ bit-by-bit—but can otherwise make arbitrary quantum measurements, can also gain an advantage *so long as the environment has a low enough entropy.* This will likely only occur at low temperatures and with very regular interactions (irregularities add ignorance, as without prior knowledge the conditional states are averages over the random interactions). Local observers that are constrained in the measurements they can make still have a lot of power at their fingertips: As shown by Fig. 4.2 and the amount of locally accessible amplified information in Fig. 4.3, essentially any measurement will allow a (highly) constrained observer to acquire the objective information about the system. Moreover, the convenient basis allows them to develop effective means of acquiring objective information, i.e., it serves as a fixed point to evolve to. We note that in some restricted settings, a small quantum memory can be used to acquire and store the relevant part of quantum states to be distinguished [6]. However, having such a quantum resource is still costly, as it requires a highly isolated quantum degree of freedom (or, for mixed states, more).

Our universe is indeed low entropy and thus highly regular. Most of our information

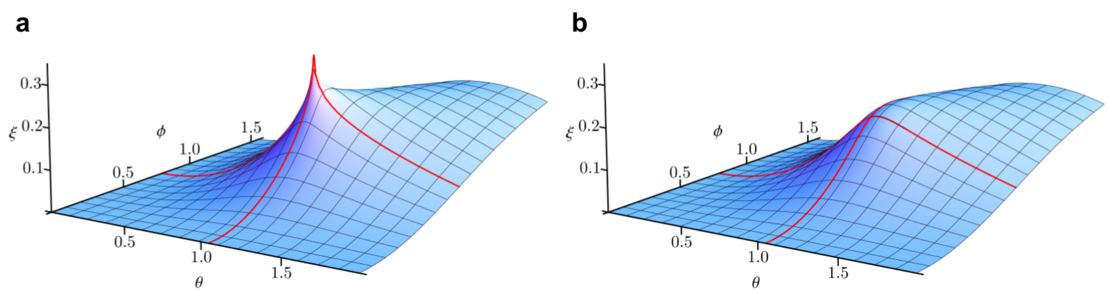

Figure 4.2: **The unattenuated efficiency of amplification, ξ, of locally accessible information.** Observers measure the environment locally—bit-by-bit—in some basis characterized by the Bloch angles θ and ϕ, and for time $t = \pi/3$ (and per component interaction Hamiltonian $\sigma^z_{\mathcal{S}}\sigma^z_k$). (a) When the environment is low entropy, the optimal measurement basis is aligned with one of the two conditional states (either will do), as highlighted by the intersection of the two solid, red lines. This manifests itself as a peak in ξ, which is a remnant of the diverging ξ when the conditional states become orthogonal. This peak only exist for low entropy conditional states, i.e., below threshold of $h \lesssim 0.05$. (b) For a mixed environment, the peak is washed out (here, $h \approx 0.08$, which is already sufficiently high for the flatness to be visually apparent). Indeed, above the threshold entropy, a basis emerges that gives the maximal amount of locally available information regardless of the time of interaction or the entropy. This basis is complementary to the initial environment state and the interaction axis with the system. Here, these are along the x-axis and z-axis, respectively, giving the convenient basis σ^y ($\theta = \phi = \pi/2$). Just as with the photon environment, there is thus a natural, classical basis—a basis for which we do not need to worry about superpositions thereof—to measure in. Hence, the optimal basis is dependent on the conditional state *only for very low entropy environments*. We expect the photon environment has richer behavior, as we can change the incoming photon direction, for instance, without changing this objective basis. This flexibility is permitted since the dimension of the photon Hilbert space is much larger. For the spin case, we only have the "classical" freedom of changing the mixedness and direction within a single plane of the Bloch sphere, any other change (in the initial state) will make use of complementarity (superposition).

comes from the photon environment (not, e.g., air molecules, which could be out of equilibrium with some superposition of a dust grain and will decohere it, but do not lead to locally accessible information). The above results suggest, though, that observers in our world do not have a strong selection to exploit quantum effects, as the cost is high (i.e., quantum memory and the ability to make arbitrary measurements) and/or the window of regularity (for a local quantum advantage) is small, and the practical gain is low.

However, the regularity present in our world is sufficient for observers to find out about quantum systems (or classical, e.g., already decohered quantum systems), including various basic properties, without prior knowledge (i.e., without knowing that it exists, its dimensionality, its interaction with the environment, or the environments' initial state). To do so, an observer only needs to intercept fragments of the environment. As we show in Section 4.3, the information shared between fragments of the environment is close—to within $\epsilon \sim \exp\left(-^\sharp\mathcal{F}\xi\right)$—to the classical information about the system deposited in the fragment, $\chi\left(\hat{\Pi}_{\mathcal{S}} : \mathcal{F}'\right)$. Moreover, to acquire this information, confirm it, and determine, e.g., the dimensionality of $\mathcal{S}$, requires only bit-by-bit measurements on components of the environment. These measurements allow an observer to characterize the classical conditional states, $\tilde{\rho}_{k|\hat{s}}$. The number of such states they find yields the (minimum) dimensionality of the quantum system. These considerations, of course, require a promise by Nature, that they will be guaranteed that the initial environment state and its dynamics and interaction with the system has at least some regularity. Only then can observers learn indirectly via the environment. If observers had the ability to adapt their measurement basis, they could exploit this indirect acquisition of information to "intercept", but not destroy, the regularity present in the Universe.

Observers, though, are constrained by the hardware that they are—or can be— built

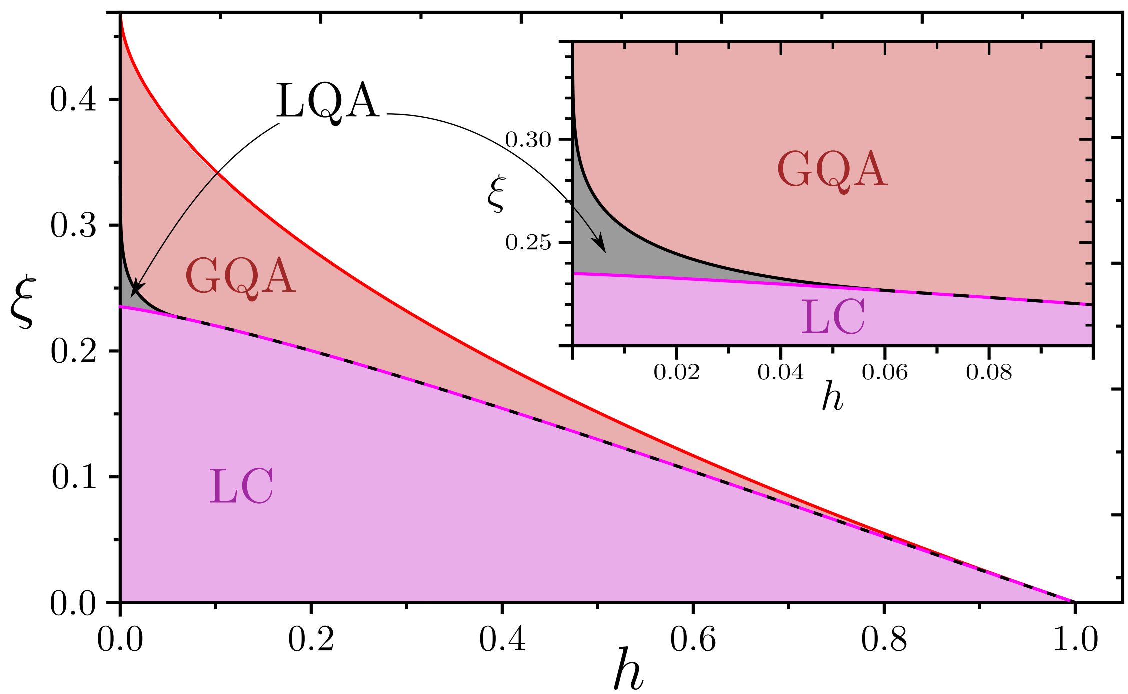

Figure 4.3: **The phases of observation: Globally quantum advantaged (GQA, red), Locally quantum advantaged (LQA, black), and classical (LC, magenta).** The unattenuated efficiency, ξ, of amplification versus the entropy of the post-scattering conditional state, h, of an environment component with typical parameters for three modes of observation: Fully quantum observers—ones that can acquire and manipulate large fragments of the environment—have access to the most information, i.e., they can be in the GQA regime. Their ignorance about the quantum system decays more rapidly, providing them near certainty with less "evidence" (the size of the fragment). Locally quantum observers—ones that can manipulate the environment bit-by-bit but otherwise arbitrarily—also gain an advantage *so long as the environment has a very low entropy*. Local, classical observers are ones that are "stuck" with some basis that is given to them. Nevertheless, for a broad range of conditions, there is a single, optimal basis to locally extract information about the system from the environment components. To make use of the quantum advantage requires prior knowledge of the initial conditions and interactions, or a stringent amount of regularity in the Universe (i.e., only for very pure states and constrained interactions, giving a low entropy post-scattering/interaction conditional state). However, this advantage is not substantial and this suggests that it is unlikely that observers have evolved to exploit quantum effects in observation.

with, i.e., they can not make arbitrary measurements. What the results above together with this discussion show is that there exists a pathway—a pathway that can be taken randomly where natural selection provides a biasing force—that can lead observers to good, local measurements to make. As a starting point, just about any local basis will do and this can—but does not have to—evolve to a more efficient acquisition of information. In other words, objective, classical reality is inescapable [28, 53, 8] and redundancy leads to robust observation. Thus, even in the unpredictable quantum Universe, observers can find objective properties in the world, ones on which they can base subsequent actions, surviving Darwinism by exploiting quantum Darwinism.

4.3 Methods

The total amount of information about a system that an environment fragment, $\mathcal{F}$, has about a system, $\mathcal{S}$, is quantified by the quantum mutual information,

$$I\left(\mathcal{S}:\mathcal{A}\right) = H_{\mathcal{S}} + H_{\mathcal{F}} - H_{\mathcal{S}\mathcal{F}},$$

where $H_{\mathcal{A}} = -\rho_{\mathcal{A}} \log_2 \rho_{\mathcal{A}}$ is the von Neumann entropy for some subsystem $\mathcal{A}$. This mutual information is divided naturally into classical and quantum components [55], $I\left(\mathcal{S}:\mathcal{A}\right) = \chi\left(\hat{\Pi}_{\mathcal{S}}:\mathcal{F}\right) + \mathcal{D}\left(\hat{\Pi}_{\mathcal{S}}:\mathcal{F}\right)$, with the Holevo quantity [20],

$$\chi\left(\hat{\Pi}_{\mathcal{S}}:\mathcal{F}\right) = H\left(\sum_{\hat{s}} p_{\hat{s}}\rho_{\mathcal{F}|\hat{s}}\right) - \sum_{\hat{s}} p_{\hat{s}} H(\rho_{\mathcal{F}|\hat{s}}),$$

providing an upper bound on the amount of *classical* information (i.e., information about the eigenstates of the system's pointer observable $\hat{\Pi}_{\mathcal{S}} = \sum_{\hat{s}} \pi_{\hat{s}} |\hat{s}\rangle\langle\hat{s}|$) that can be transmitted by the fragment. The remaining piece—the quantum discord $\mathcal{D}\left(\hat{\Pi}_{\mathcal{S}} : \mathcal{F}\right)$—quantifies the coherences in the basis $\hat{\Pi}_{\mathcal{S}}$ [45, 30, 19, 55].

This information is stored redundantly in the environment, with redundancy $R_\delta = \sharp\mathcal{E}/\sharp\mathcal{F}_\delta$, when a fragment of size $\sharp\mathcal{F}_\delta$ achieves $\chi\left(\hat{\Pi}_{\mathcal{S}} : \mathcal{F}\right) \approx (1-\delta)H_{\mathcal{S}}$ bits of information about $\mathcal{S}$. For a two-dimensional system, Fano's inequality [26, 11] provides the lower-bound,

$$\chi\left(\hat{\Pi}_{\mathcal{S}} : \mathcal{F}\right) \geq H_{\mathcal{S}} - H(P_e),$$

where $H(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is the binary entropy and P_e is the error probability for distinguishing between the (two) conditional states $\rho_{\mathcal{F}|\hat{s}}$. Hence, for a fragment of size $\sharp\mathcal{F}_\delta$, we have an upper bound on the information deficit given by

$$H(P_e) \geq \delta H_{\mathcal{S}}$$

In the limit $\sharp\mathcal{F}_\delta \to \infty$, the error probability decays exponentially, $P_e \propto e^{-\sharp\mathcal{F}_\delta \xi}$, with

$$\overline{\xi}_{QCB} = -\ln\left\langle \min_{0\leq c\leq 1} \mathrm{tr}\left[\rho_{k|\uparrow}^{c}\rho_{k|\downarrow}^{1-c}\right]\right\rangle \tag{4.3}$$

giving the "typical" Chernoff Information, $\overline{\xi}_{QCB}$ [53], an extension to non-i.i.d. (not independent and identically distributed) states of the quantum Chernoff Bound [5, 27, 25]. This calculation is readily extended to higher dimensional systems.

4.3.1 Conditional states

For spin environments and without loss of generality, a single environment spin has an initial state

$$\rho_k(0) = \frac{1}{2}\left(\mathbf{I} + a_x\sigma_x + a_z\sigma_z\right).$$

We have chosen the basis so that the initial environment state is in the xz-plane. For the Hamiltonian we consider, this assignment can be done without limiting the class of initial states. After interacting with the system according to the Hamiltonian $\mathbf{H} = \sigma_{\mathcal{S}}^z \otimes \sum_k g_k \sigma_k^z$ for a time t, the conditional states evolve to

$$\rho_{k|\hat{s}} \doteq \frac{1}{2}\begin{pmatrix} 1 + a_z & a_x e^{-2\imath g_k h_{\hat{s}} t} \\ a_x e^{2\imath g_k h_{\hat{s}} t} & 1 - a_z \end{pmatrix}, \tag{4.4}$$

where $h_{\hat{s}} = 1, -1$ are the eigenvalues corresponding to $\hat{s} = \uparrow, \downarrow$, respectively; i.e., the Bloch-vectors rotate in opposite directions about the z-axis, $\vec{a} \mapsto \vec{a}_{\hat{s}}$ as depicted in Fig. 4.1b insert.

If the observer now fixes a measurement basis by choosing an axis $\hat{m}$ on the Bloch sphere, there are two measurement operators

$$|+\rangle\langle+| = \frac{1}{2}(\hat{I} \pm \hat{m} \cdot \vec{\sigma}) \tag{4.5}$$

and $|-\rangle\langle-| = \mathrm{I} - |+\rangle\langle+|$. The measurement interaction then transforms the conditional states according to

$$\rho_{k|\hat{s}} \mapsto \frac{1}{2}\left[\hat{I} + (2p_{\hat{s}} - 1)\hat{m} \cdot \vec{\sigma}\right] \tag{4.6}$$

with $p_{\hat{s}} = \langle +| \rho_{k|\hat{s}} |+\rangle$ the probability of finding $\rho_{k|\hat{s}}$ in the state $|+\rangle$. Specifically,

$$p_{\hat{s}} = \frac{1}{2}(1 + \vec{a}_{\hat{s}} \cdot \hat{m}), \tag{4.7}$$

or $2p_{\hat{s}} - 1 = \vec{a}_{\hat{s}} \cdot \hat{m}$. Thus, we can write the measurement transformation as

$$\rho_{k|\hat{s}} \mapsto \frac{1}{2}\left[\hat{I} + (\vec{a}_{\hat{s}} \cdot \hat{m})\hat{m} \cdot \vec{\sigma}\right] \equiv \mathrm{Diag}[p_{\hat{s}}, 1 - p_{\hat{s}}]. \tag{4.8}$$

The diagonal form is in the basis defined by $\hat{m}$ (i.e., the post-measurement conditional states commute, allowing them to be simultaneously diagonalized in that basis).

4.3.2 The local Chernoff information

When one is restricted to local measurements, ξ is determined by applying Eq. (4.3) to the conditional states $\rho_{\hat{s}}$ defined in Eq. (4.8). Simultaneous diagonalization then allows us to write

$$\xi = -\ln \min_{0 \leq c \leq 1} \left(p_{+|\uparrow}^{c} p_{+|\downarrow}^{1-c} + p_{-|\uparrow}^{c} {p_{-|\downarrow}}^{1-c} \right), \tag{4.9}$$

which is just the classical Chernoff Information associated with a Bernoulli random variable [10]. Evaluating Eq. (4.9) explicitly requires determining the value of c. Note first that $p_{+|\uparrow} = p_{+|\downarrow}$ is obtained only when the two conditional states coincide, in which case no distinction is possible, i.e., the observer would not be able to infer the system's pointer state. Thus, without loss of generality, we will take $p_{+|\uparrow} > p_{+|\downarrow}$ and $p_{+|\uparrow} > 1/2$.

Given this, the simplest case is that of $p_{+|\uparrow} = 1$, which occurs only for an aligned measurement of a pure initial state. In this case, Eq. (4.9) simplifies to

$$\xi = -\ln\left[\min_{0\leq c\leq 1} p_{+|\downarrow}^{1-c}\right]. \tag{4.10}$$

The minimum occurs at $c = 0$, from which we conclude $\xi = -\ln p_{\downarrow}$. This is expected, as a misidentification can occur only if the received conditional states are all $\rho_{k|\downarrow}$, but every measurement yields outcome $|+\rangle$; hence, the error probability is just $p_{+|\downarrow}^{\sharp\mathcal{F}}$. A similar simplification occurs in the case $p_{+|\downarrow} = 0$, so that $\xi = -\ln p_{-|\uparrow}$, corresponding to the likelihood of finding all $|-\rangle$ outcomes when the received states are $\rho_{k|\uparrow}$.

For other, less extreme cases, the value of c that minimizes Eq. (4.9) is

$$c = \frac{\log\left[\frac{p_{+|\downarrow}}{p_{-|\uparrow}}\frac{\log\left(\frac{p_{+|\uparrow}}{p_{+|\downarrow}}\right)}{\log\left(\frac{p_{-|\downarrow}}{p_{-|\uparrow}}\right)}\right]}{\log\left[\frac{p_{+|\downarrow}p_{-|\uparrow}}{p_{+|\uparrow}p_{-|\downarrow}}\right]}. \tag{4.11}$$

This simplifies greatly in the special case $p_{+|\downarrow} = p_{-|\uparrow}$, which holds whenever the two conditional Bloch-vectors have equal and opposite projections onto the measurement axis (as in the case for a y-axis measurement when we set the initial state to be in the xz-plane). In such cases, Eq. (4.11) gives $c = 1/2$, yielding

$$\xi = -\log\left[2p_{+|\downarrow}^{1/2}{p_{-|\downarrow}}^{1/2}\right] = -\frac{1}{2}\log\left[4p_{+|\uparrow}p_{+|\downarrow}\right]. \tag{4.12}$$

An exact simplification of Eq. (4.11) is not generally possible.

4.3.3 Optimized local measurements

Knowing how ξ depends on the probabilities allows us to determine the optimal measurement axis. Starting from Eq. (4.7), we see that the probabilities depend only on the two conditional Bloch vectors, $\vec{a}_\uparrow$ and $\vec{a}_\downarrow$, and their projections onto the measurement axis. Hence, we work now with coordinates chosen so that the conditional states lie in the xy-plane at angles $\pm\alpha$ from the $\hat{x}$-axis, and the measurement axis is determined by the usual Bloch angles (θ, ϕ). The optimum measurement axis is then found by expanding Eq. (4.7) in these variables and then setting the derivatives of Eq. (4.9) with respect to θ and ϕ equal to zero. The actual computation is simplified by the fact that c is defined so as to optimize ξ, so the derivatives of Eq. (4.9) can be computed by skipping the minimization and treating c as a constant defined by Eq. (4.11).

For the θ-derivative, one finds that

$$d\xi/d\theta \propto \cos\theta.$$

Hence, this derivative vanishes at $\theta = \pi/2$, which corresponds to a measurement in the plane of the conditional states. The ϕ-derivative admits a similar proportionality, $\frac{d\xi}{d\phi} \propto \sin\theta$, but this yields a minimum (i.e., $\theta = 0$, in which case the measurement axis is orthogonal to both conditional states and so yields no information). The proportionality term, though, depends on ϕ, α, and c in rather complicated ways, but one can verify by inspection that it vanishes if $p_{+|\uparrow} = 1 - p_{+|\downarrow}$; i.e., when the measurement axis lies in the yz-plane. We therefore see that a critical point exists whenever the measurement axis is in the plane of the conditional states and orthogonal to their bisector. In terms of the original coordinates, in which the conditional states are allowed z-components, this corresponds to a measurement along the y-axis.

However, whether or not a critical point is a local maximum, minimum, or neither cannot be determined by the first derivative. In fact, Fig. 4.2 shows that the y-axis measurement is optimal only when entropy is past the threshold value. At very low entropy, it transitions into a saddle-point, and the true optimum moves toward an aligned measurement, as shown by the deviation of the blue curve (optimal measurement) away from the green curve (y-axis measurement) in Fig. 4.3. In this regime, we compute the optimum measurement numerically.

In Fig. 4.2a, the sharp peak in ξ is unusual. The rapid increase in ξ as the entropy nears zero can be understood by looking at the information available to an observer making an aligned measurement at a calculationally convenient time. Specifically, at sufficiently low entropy, there will always be a time at which $p_{+|\downarrow} = 1/2$. This simplifies the expressions above sufficiently to approximate ξ, and we find

$$\xi \approx \frac{A + \ln(1/B)\ln[2\ln(1/B)] - \ln(2/B)\ln[\ln(2/B)]}{\ln 2/B}$$

where $A = \ln(2)\ln(2\ln 2)$ has been introduced for readability and $B = 1-a$ characterizes the Bloch vector's impurity. This has derivative

$$\frac{d\xi}{dB} = \frac{A - \ln(2)\ln\ln(1/B)}{\varepsilon \ln^2 2/B},$$

which diverges as $B \to 0$ (i.e, in the limit of zero entropy).

4.4 The environment has it all

Quantum Darwinism recognizes the role of the environment as a communication channel, transmitting information about a system to the broader world. Indeed, the environment is the way observers learn about the world, which is possible only because it carries all the information that observers need in order to acquire that knowledge. To see this, we began by a specific calculation where a system is purely decohered by the environment and where the global state is pure. We then examine the more general case.

For a pure $\mathcal{SE}$ state undergoing pure decoherence, the mutual information between fragments of the environment, $I\left(\mathcal{F}:\mathcal{F}'\right)$, can be calculated following results in Ref.[52]. It is

$$\begin{aligned} I\left(\mathcal{F}:\mathcal{F}'\right) &= H_{\mathcal{F}} + H_{\mathcal{F}'} - H_{\mathcal{F}\mathcal{F}'} \\ &= H_{\mathcal{S}d\mathcal{F}} + H_{\mathcal{S}d\mathcal{F}'} - H_{\mathcal{S}d\mathcal{F}\mathcal{F}'} \\ &= H\left(\kappa_{\mathcal{F}}\right) + H\left(\kappa_{\mathcal{F}'}\right) - H\left(\kappa_{\mathcal{F}\mathcal{F}'}\right), \end{aligned} \tag{4.13}$$

where $H_{\mathcal{S}d\mathcal{A}}$ indicates the entropy of the system decohered by the subset $\mathcal{A}$ of the environment and $H\left(\kappa_{\mathcal{A}}\right)$ is the binary entropy of $\kappa_{\mathcal{A}} = \left(1+\gamma^{\sharp\mathcal{A}}\right)/2$ (for clarity, we have implicitly assumed throughout that the diagonal elements of the system's density matrix are $1/2$ in its pointer basis, significantly simplifying the expression for $\kappa_{\mathcal{A}}$ from Ref.[52]). Here, γ is the decoherence factor from one environment spin (we assume it is the same for all spins, an assumption that is easily relaxed). Taking the leading order terms in

$\gamma^{2\sharp\mathcal{A}}$ gives

$$I\left(\mathcal{F}:\mathcal{F}'\right) \approx 1-\frac{1}{\ln 4}\left(\gamma^{2\sharp\mathcal{F}}+\gamma^{2\sharp\mathcal{F}'}-\gamma^{2(\sharp\mathcal{F}+\mathcal{F}')}\right)$$
$$= 1-\frac{1}{\ln 4}\left(e^{-\bar{\xi}_{QCB}\sharp\mathcal{F}}+e^{-\bar{\xi}_{QCB}\sharp\mathcal{F}'}-e^{-\bar{\xi}_{QCB}(\sharp\mathcal{F}+\mathcal{F}')}\right) \tag{4.14}$$

where the second line follows from the relationship between the decoherence factor and the quantum Chernoff Bound for pure environments [54]. This demonstrates that each of the environment fragments holds the same information. They can thus be used not only to learn about the system, but also to confirm an initial observation by repeating the measurement on a different fragment, as Eq. (4.14) rapidly approaches $H_{\mathcal{S}}=1$.

More generally, we can show that when $I\left(\mathcal{S}:\mathcal{A}\right)$ grows rapidly and then plateaus, observers can both determine the missing information about $\mathcal{S}$ and confirm that information using fragments of the environment only. Considering a system with "surplus decoherence" (that the environment even without some small fragments is sufficient to decohere the system), then we have a system and two fragment state of the form

$$\rho_{\mathcal{S}\mathcal{F}\mathcal{F}'}=\sum_{\hat{s}}p_{\hat{s}}|\hat{s}\rangle\langle\hat{s}|\otimes\rho_{\mathcal{F}|\hat{s}}\otimes\rho_{\mathcal{F}'|\hat{s}},$$

where we have also taken the environment fragments to be independent. This gives the $\mathcal{F}\mathcal{F}'$ state

$$\rho_{\mathcal{F}\mathcal{F}'}=\sum_{\hat{s}}p_{\hat{s}}\rho_{\mathcal{F}|\hat{s}}\otimes\rho_{\mathcal{F}'|\hat{s}}.$$

To bound the information in $\mathcal{F}'$ that can confirm the information already determined from $\mathcal{F}$, we will show that $I\left(\mathcal{F}:\mathcal{F}'\right)$ is close to $\chi\left(\hat{\Pi}_{\mathcal{S}}:\mathcal{F}'\right)$, and specifically the pointer information communicated by $\mathcal{F}'$. Essentially, this will make use of

$$\begin{aligned} I\left(\mathcal{F}:\mathcal{F}'\right)_{\rho_{\mathcal{F}\mathcal{F}'}} &= I\left(\mathcal{F}\mathcal{A}:\mathcal{F}'\right)_{\rho_{\mathcal{F}\mathcal{F}'}\otimes|0\rangle_{\mathcal{A}}\langle 0|} \\ &= I\left(\mathcal{F}\mathcal{A}:\mathcal{F}'\right)_{\tilde{\rho}_{\mathcal{F}\mathcal{F}'\mathcal{A}}} \\ &\geq I\left(\mathcal{A}:\mathcal{F}'\right)_{\tilde{\rho}_{\mathcal{F}'\mathcal{A}}} \\ &\geq \chi\left(\hat{\Pi}_{\mathcal{A}}:\mathcal{F}'\right)_{\sigma_{\mathcal{F}'\mathcal{A}}}, \end{aligned}$$

where, for the first equality, we tacked on an auxiliary system $\mathcal{A}$. The second equality uses a state that has a unitary rotation on $\mathcal{F}\mathcal{A}$ only,

$$\tilde{\rho}_{\mathcal{F}\mathcal{F}'\mathcal{A}} = \mathcal{U}_{\mathcal{F}\mathcal{A}}\rho_{\mathcal{F}\mathcal{F}'\mathcal{A}}\mathcal{U}^{\dagger}_{\mathcal{F}\mathcal{A}}.$$

The two inequalities make use of the data processing inequality [26] (first ignoring $\mathcal{F}$ and then making a measurement on $\mathcal{A}$ in the basis corresponding to the pointer basis of $\mathcal{S}$, yielding a final state $\sigma_{\mathcal{F}'\mathcal{A}}$).

When $\mathcal{F}$ acquires information about $\mathcal{S}$, this means that there exists a POVM with elements $\Lambda_{\hat{s}}$ that have the following properties: $\mathrm{tr}\Lambda_{\hat{s}}\rho_{\mathcal{F}|\hat{s}} \geq 1-\epsilon$ and $\mathrm{tr}\Lambda_{\hat{s}'}\rho_{\mathcal{F}|\hat{s}} \leq \epsilon$ for $\hat{s}' \neq \hat{s}$, where ϵ is the exponentially decaying error probability for distinguishing the states (i.e., the Chernoff bounds computed above). Thus, the unitary operator defined by $\mathcal{U}_{\mathcal{F}\mathcal{A}}\left|\psi\right\rangle_{\mathcal{F}}\left|0\right\rangle_{\mathcal{A}} = \sum_{\hat{s}}\sqrt{\Lambda_{\hat{s}}}\left|\psi\right\rangle_{\mathcal{F}}\left|\hat{s}\right\rangle_{\mathcal{A}}$ will transfer the information in $\mathcal{F}$ into $\mathcal{A}$.

After performing this unitary and measuring $\mathcal{A}$ in the basis $\hat{\Pi}_{\mathcal{A}}$, we have the state

$$\sigma_{\mathcal{F}'\mathcal{A}} = \sum_{\hat{a}} q_{\hat{a}}\sigma_{\mathcal{F}'|\hat{a}} \otimes |\hat{a}\rangle\langle\hat{a}|,$$

where the "pointer states" $\hat{a}$ on $\mathcal{A}$ correspond to $\hat{s}$ on $\mathcal{S}$ and

$$\sigma_{\mathcal{F}'|\hat{a}} = \frac{1}{q_{\hat{a}}} \sum_{\hat{s}} \rho_{\mathcal{F}'|\hat{s}} p_{\hat{s}} \mathrm{tr} \Lambda_{\hat{a}} \rho_{\mathcal{F}|\hat{s}}$$

and

$$q_{\hat{a}} = \sum_{\hat{s}} p_{\hat{s}} \mathrm{tr} \Lambda_{\hat{a}} \rho_{\mathcal{F}|\hat{s}}.$$

This gives the Holevo quantity

$$\chi\left(\hat{\Pi}_{\mathcal{A}} : \mathcal{F}'\right)_{\sigma_{\mathcal{F}'\mathcal{A}}} = H\left(\sum_{\hat{s}} p_{\hat{s}} \rho_{\mathcal{F}'|\hat{s}}\right) - \sum_{\hat{a}} q_{\hat{a}} H\left(\sigma_{\mathcal{F}'|\hat{a}}\right).$$

We note that the first term is the same as that in $\chi\left(\hat{\Pi}_{\mathcal{S}} : \mathcal{F}'\right)$, i.e., in the information communicated about $\mathcal{S}$ by $\mathcal{F}'$. The second term is the sum over entropies of the conditional states $\sigma_{\mathcal{F}'|\hat{a}}$ which are close to, but not the same as, $\rho_{\mathcal{F}'|\hat{s}}$ for $\hat{s} = \hat{a}$. Due to the properties of the POVM elements, the probabilities and states obey

$$|q_{\hat{s}} - p_{\hat{s}}| \leq \epsilon$$

and

$$\frac{1}{2} \mathrm{tr} \left|\sigma_{\mathcal{F}'|\hat{s}} - \rho_{\mathcal{F}'|\hat{s}}\right| \leq 2\epsilon.$$

The latter is proven by

$$\begin{aligned}
\operatorname{tr}\left|\sigma_{\mathcal{F}'|\hat{s}} - \rho_{\mathcal{F}'|\hat{s}}\right| &= \operatorname{tr}\left|\frac{1}{q_{\hat{s}}}\sum_{\hat{a}} \rho_{\mathcal{F}'|\hat{a}} p_{\hat{a}} \operatorname{tr}\Lambda_{\hat{s}}\rho_{\mathcal{F}|\hat{a}} - \rho_{\mathcal{F}'|\hat{s}}\right| \\
&= \operatorname{tr}\left|\rho_{\mathcal{F}'|\hat{s}}\left(\frac{p_{\hat{s}}}{q_{\hat{s}}}\operatorname{tr}\Lambda_{\hat{s}}\rho_{\mathcal{F}|\hat{s}} - 1\right) + \frac{1}{q_{\hat{s}}}\sum_{\hat{a}\neq\hat{s}} \rho_{\mathcal{F}'|\hat{a}} p_{\hat{a}} \operatorname{tr}\Lambda_{\hat{s}}\rho_{\mathcal{F}|\hat{a}}\right| \\
&\leq \left|\frac{p_{\hat{s}}}{q_{\hat{s}}}\operatorname{tr}\Lambda_{\hat{s}}\rho_{\mathcal{F}|\hat{s}} - 1\right| + \frac{1}{q_{\hat{s}}}\sum_{\hat{a}\neq\hat{s}} p_{\hat{a}} \operatorname{tr}\Lambda_{\hat{s}}\rho_{\mathcal{F}|\hat{a}} \\
&\leq 4\epsilon.
\end{aligned}$$

Moreover, by the continuity of the entropy [26],

$$\left|H\left(\rho_{\mathcal{F}'|\hat{s}}\right) - H\left(\sigma_{\mathcal{F}'|\hat{s}}\right)\right| \leq 2\epsilon \sharp\mathcal{F}' \log_2 D + \eta\left(2\epsilon\right),$$

with $\eta\left(x\right) = -x\log_2 x$ and D the local Hilbert space dimension of each component of $\mathcal{F}'$.

Thus, we can bound the common information about $\mathcal{S}$ that is in $\mathcal{F}$ and $\mathcal{F}'$ by the following:

$$\begin{aligned}
&\left|\chi\left(\hat{\Pi}_{\mathcal{A}} : \mathcal{F}'\right)_{\sigma_{\mathcal{F}'\mathcal{A}}} - \chi\left(\hat{\Pi}_{\mathcal{S}} : \mathcal{F}'\right)\right| \\
&= \left|\sum_{\hat{s}}\left[p_{\hat{s}} H\left(\rho_{\mathcal{F}'|\hat{s}}\right) - q_{\hat{s}} H\left(\sigma_{\mathcal{F}'|\hat{s}}\right)\right]\right| \\
&\leq \sum_{\hat{s}} p_{\hat{s}}\left|H\left(\rho_{\mathcal{F}'|\hat{s}}\right) - \frac{q_{\hat{s}}}{p_{\hat{s}}} H\left(\sigma_{\mathcal{F}'|\hat{s}}\right)\right| \\
&\leq 4\epsilon \sharp\mathcal{F}' \log_2 D + 2\eta\left(2\epsilon\right).
\end{aligned}$$

Therefore, to within the exponentially decaying error probability ϵ (with exponent given by the Chernoff bound), the information that can be confirmed by $\mathcal{F}'$ is the information it has about $\mathcal{S}$ (we note that $\chi\left(\hat{\Pi}_{\mathcal{A}} : \mathcal{F}'\right)_{\sigma_{\mathcal{F}'\mathcal{A}}}$, which is also close to $I(\mathcal{F} : \mathcal{F}')$ since the latter does not grow linearly with the size of $\mathcal{F}$ or $\mathcal{F}'$, but rather is bounded by the missing information regarding the system, i.e., when $^\sharp\mathcal{F} + {}^\sharp\mathcal{F}' \leq {}^\sharp\mathcal{E}/2$ then $H_{\mathcal{S}} \geq I(\mathcal{S}\mathcal{F} : \mathcal{F}') \geq I(\mathcal{F} : \mathcal{F}')$). Hence, one can use the other fragments as a proxy for $\mathcal{S}$, confirming information they have received in a fragment $\mathcal{F}$ without ever having to interact with or consider the system $\mathcal{S}$.

5: Conclusions

The question of how an apparently classical world arises within a fundamentally quantum universe is complicated, and its ultimate resolution uncertain. However, as shown here and in the references, the quantum Darwinism program provides one framework for finding such a resolution. In particular, we have shown how environmental decoherence allows observers, even those bound to make local measurements, to acquire objective, verifiable information about the states of certain systems. Moreover, we have shown that except in special cases, unlikely to be encountered in the every-day world, the optimal measurement for acquiring that information is essentially independent of the state of the system in question, thereby (together with the regularity of our environment) allowing observers to adaptively identify the optimal measurement without any prior knowledge regarding the state, or even existence, of the system.

Stepping back to take a broader view, the implications of quantum Darwinism can be summarized thus: a large quantum universe almost necessarily appears classical to all but the most careful observers, conspiring to hide quantum correlations in inaccessible degrees of freedom within the environment while simultaneously using that environment to amplify and distribute information about select observables. Observers immersed in the environment can therefore easily come to a consensus about those observables from just their local piece of the environment, but must struggle against the global environment in order to find evidence of quantum "weirdness". Hence, we experience our world as classical not in spite of, but *because* of its underlying quantum nature. The quantum does not transition into the classical; it merely hides behind it.

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