AN ABSTRACT OF THE THESIS OF

Ralph Leland James for the M.S. in Mathematics
(Name) (Degree) (Major)

Date thesis is presented Feb. 11, 1965

Title THE SOLUTION OF SINGULAR VOLTERRA INTEGRAL EQUATIONS BY SUCCESSIVE APPROXIMATIONS

Abstract approved [redacted]

The integral equation $tR(t) = \int_0^t K(t-u)R(u)du$, where the kernel function $K(t)$ satisfies certain conditions, has a unique solution $R(t)$ which satisfies appropriate auxiliary conditions. Successive approximations to $R(t)$ are derived by means of a trapezoidal numerical integration scheme. They converge uniformly and in the mean to $R(t)$ for $0 \leq t < \infty$. A number of properties of $R(t)$ and the approximations are derived.

THE SOLUTION OF SINGULAR VOLTERRA INTEGRAL EQUATIONS BY SUCCESSIVE APPROXIMATIONS

by

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A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

June 1965

APPROVAL:

[redacted]

Professor of Mathematics

In Charge of Major

[redacted]

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[redacted]

Dean of Graduate School

Date thesis is presented Feb 11, 1965

Typed by Carol Baker

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THE SOLUTION OF SINGULAR VOLTERRA INTEGRAL EQUATIONS BY SUCCESSIVE APPROXIMATIONS

CHAPTER I

INTRODUCTION

In this paper we shall study the integral equation

$$tR(t) = \int_0^t K(t-u)R(u)du, \qquad 0 \le t < \infty . \tag{1.1}$$

The kernel function $K(t)$, $0 < t < \infty$, satisfies the following conditions:

$$K(t) \ge 0 \quad \text{for} \quad 0 < t < \infty , \tag{1.2}$$

$$K(t) \quad \text{is continuous for} \quad 0 < t < \infty , \tag{1.3}$$

$$\lim_{t \to 0} K(t) = +\infty , \tag{1.4}$$

$$\int_0^1 K(u)du < +\infty , \tag{1.5}$$

$$\int_1^\infty \frac{K(u)}{u} du < +\infty , \tag{1.6}$$

$$K(t) \quad \text{is monotone non-increasing for} \quad 0 < t < \infty . \tag{1.7}$$

These conditions imply

$$K(t) \to 0, \quad \frac{1}{t}\int_0^t K(u)du \to 0 \quad \text{as} \quad t \to \infty . \tag{1.8}$$

There are several ways in which (1. 1) can be classified. For example it is a Volterra integral equation. In Fredholm's classification, (1. 1) is an integral equation of the third kind. Moreover, (1. 1) is homogeneous and the kernel has a weak singularity at zero.

In general it is usually difficult to construct non-trivial solutions to homogeneous integral equations. Moreover, a solution is not unique since any constant multiple of it is also a solution.

We shall show that (1. 1) has a real solution $R(t)$, $0 \leq t < \infty$, unique a. e. to within a constant factor, such that $e^{-xt}R(t)$ is integrable (Lebesgue) for some $x > 0$; moreover, such a solution $R(t)$ will be shown to be bounded, continuous, non-negative and integrable on $(0, \infty)$. The existence of $R(t)$ will be established by constructing successive approximations and proving that they converge uniformly to $R(t)$. The uniqueness will be established by Laplace transform analysis.

Since a unique solution is desired it is essential that $R(t)$ satisfy some normalization condition. It is convenient to require that

(1. 9) $$\max_{0 \leq t < \infty} R(t) = 1 .$$

A different normalization condition may be desired for some purposes. Thus consider the same integral equation, but with the unknown function $P(t)$:

$$tP(t) = \int_0^t K(t-u)P(u)du, \qquad 0 \leq t < \infty \quad , \tag{1.10}$$

where

$$\int_0^\infty P(t)dt = 1 \, . \tag{1.11}$$

The functions $P(t)$ and $R(t)$ are related by

$$P(t) = \frac{R(t)}{\int_0^\infty R(t)dt} \quad , \tag{1.12}$$

$$R(t) = \frac{P(t)}{\max\limits_{0 \leq t < \infty} P(t)} \quad . \tag{1.13}$$

A special case of (1.1) was studied in (1). In that paper $K(t) = \mu t^{-\lambda}$, $\mu > 0$, $0 < \lambda < 1$. All of the relevant results obtained in (1) are established here in the more general case considered. Further generalizations can be obtained by slightly weakening some of the conditions (1.2) - (1.7) on $K(t)$. However, the analysis then becomes more complicated.

CHAPTER II

UNIQUENESS AND OTHER PROPERTIES OF SOLUTIONS

We proceed to prove a uniqueness theorem and to establish other properties of a solution $R(t)$ of (1.1).

Suppose that $R(t)$ is a solution of (1.1). Assume that $e^{-xt}R(t)$ is integrable for $x > x_0$. Then the Laplace transform of $R(t)$,

$$\hat{R}(z) = \int_0^\infty e^{-zt} R(t)dt, \quad x > x_0, \quad (z = x + iy),$$

is defined at least in the indicated half-plane. By (1.5) and (1.6),

$$\hat{K}(z) = \int_0^\infty e^{-zt} K(t)dt, \quad x > 0 .$$

Transform (1.1) and use the convolution theorem to obtain

$$-\hat{R}'(z) = \hat{K}(z)\hat{R}(z), \quad x > x_1 ,$$

where $x_1 = \max(x_0, 0)$. Integration yields

$$\text{(2.1)} \quad \hat{R}(z) = \hat{R}(\xi) e^{-\int_\xi^z \hat{K}(w)dw} = \hat{R}(\xi) e^{-\int_0^\infty \frac{e^{-zt} - e^{-\xi t}}{t} K(t)dt}, \quad \xi, x > x_1 ,$$

with any path of integration in the half-plane for the w-integral.

The standard results from Laplace transform analysis that we have used can be found in (4).

Theorem 2.1. If $R(t)$ is a solution of (1.1) with the property that $e^{-xt}R(t)$ is integrable for x sufficiently large, then $R(t)$ is unique a.e. to within a constant factor.

Proof. Equation (2.1) determines $\hat{R}(z)$ to within a constant factor, so that $R(t)$ is determined a.e. to within a constant factor.

Theorem 2.2. Let $R(t)$ be a non-negative solution of (1.1) such that $e^{-xt}R(t)$ is integrable for x sufficiently large. Then $R(t)$ is integrable on $(0,\infty)$ and

$$\hat{R}(z) = \hat{R}(0)e^{\int_0^\infty \frac{1-e^{-zt}}{t} K(t)dt} \tag{2.2}$$

Proof. It follows from (1.5), (1.6) and (2.1) that $\hat{R}(z)$ exists at least for $x > 0$. Since $K(t) \geq 0$, Lebesgue's monotone convergence theorem (cf. (7, p. 72)) yields

$$\lim_{x \to 0} \hat{R}(x) = \hat{R}(\xi)e^{-\int_0^\infty \frac{1-e^{-\xi t}}{t} K(t)dt} \qquad (\xi > x_1) ,$$

which is finite. Since $R(t) \geq 0$, another application of the same theorem yields

$$\lim_{x \to 0} \hat{R}(x) = \lim_{x \to 0} \int_0^\infty e^{-xt}R(t)dt = \int_0^\infty R(t)dt = \hat{R}(0) .$$

Thus, $R(t)$ is integrable. Solve for $\hat{R}(\xi)$ in terms of $\hat{R}(0)$ and substitute into (2.1) to obtain (2.2).

For convenience below, let

$$M(t) = \sup_{0 \le u \le t} |R(u)|, \quad t \geq 0, \tag{2.3}$$

for any solution of (1.1) which is bounded on each finite interval.

<u>Theorem 2.3.</u> If $R(t)$ is a solution of (1.1) which is bounded on each finite interval, then $R(t)$ is continuous on $(0, \infty)$ and

$$|R(t) - R(s)| \leq \frac{M(t_0)}{t} \left[2\int_0^{|t-s|} K(u)du + |t-s| \right], \quad 0 < s,\ t \leq t_0 \ . \tag{2.4}$$

<u>Proof.</u> Let $0 < s \leq t \leq t_0$. Then by (1.1) and (1.7),

$$|tR(t) - sR(s)| = \left|\int_0^t K(t-u)R(u)du - \int_0^s K(s-u)R(u)du\right|$$

$$= \left|\int_s^t K(t-u)R(u)du + \int_0^s [K(t-u) - K(s-u)]\, R(u)du\right|$$

$$\leq M(t_0) \left[\int_0^{t-s} K(u)du + \int_0^s K(u)du - \int_{t-s}^t K(u)du\right]$$

$$= M(t_0) \left[\int_0^{t-s} K(u)du + \int_0^{t-s} K(u)du - \int_s^t K(u)du\right]$$

$$\leq 2M(t_0) \int_0^{t-s} K(u)du \ .$$

It follows that

$$|tR(t) - tR(s)| \le |tR(t)-sR(s)| + |sR(s)-tR(s)|$$

$$\le 2M(t_0)\int_0^{|t-s|} K(u)du + M(t_0)|t-s|, \qquad 0 < s,t \le t_0 \quad .$$

Hence, (2. 4) holds and $R(t)$ is continuous on $(0,\infty)$.

By (1. 6), for a sufficiently large we have

(2. 5) $$\int_a^\infty \frac{K(u)}{u} du \le \frac{1}{2} \quad .$$

Fix a such that (2. 5) is satisfied and define

(2. 6) $$A = 2\int_0^a K(u)du \quad .$$

Theorem 2. 4. If $R(t)$ is a solution of (1. 1) which is bounded on each finite interval and $R(t) \not\equiv 0$, then

(i) $R(t)$ is bounded on $[0,\infty)$,

(ii) $R(t) < M(A)$, $t > A$,

(iii) $R(t) \to 0$ as $t \to \infty$.

Proof. By (1. 1), (2. 3) and Theorem 2. 3,

$$|R(t)| \le \frac{M(t)}{t}\int_0^t K(u)du, \quad t > 0 ;$$

and $R(t)$ is continuous on $(0,\infty)$. By (2. 5) and (2. 6),

$$(2.8) \qquad \frac{1}{t}\int_0^t K(u)\,du \leqq \frac{1}{t}\int_0^a K(u)\,du + \int_a^\infty \frac{K(u)}{u}\,du < 1, \; t > A.$$

By (2.3), $|R(t)|$ cannot attain a maximum at $t > A$. So $M(t) = M(A)$ for $t > A$ and (1.8), (2.7) imply the theorem.

Theorem 2.5. Let $R(t)$ be a solution of (1.1) such that $R(t)$ is bounded on every finite interval, and $R(0+)$ exists. Then $R(0+) = 0$.

Proof. Assume $|R(0+)| > 0$. Then, by Theorem 2.3, there exists $\delta > 0$ and $\epsilon > 0$ such that

$$|R(t)| \geq \delta > 0 \quad \text{for} \quad 0 < t < \epsilon \quad .$$

By (1.1) and (1.7),

$$|R(t)| \geq \frac{\delta K(t)}{t} \quad , \quad 0 < t < \epsilon \quad .$$

In view of (1.4), this contradicts the boundedness of $R(t)$ on every finite interval.

We have shown that if $R(t)$ is a solution of (1.1) such that $e^{-xt}R(t)$ is integrable for some x, then $R(t)$ is unique a.e. to within a multiplicative factor and if, in addition, $R(t)$ is non-negative, then $R(t)$ is integrable on $(0, \infty)$. Moreover, if $R(t)$ is bounded, then $R(t)$ is continuous on $(0, \infty)$ and tends to zero as t tends to infinity. Finally, if $R(t)$ is continuous on $[0, \infty)$, then $R(0) = 0$.

CHAPTER III

THE METHOD OF SUCCESSIVE APPROXIMATIONS

For $h > 0$, let $R_h(t)$, $0 \le t < \infty$, denote a non-trivial, continuous, non-negative, piecewise-linear function with possible changes in slope only at the points $t = nh$, $n = 1, 2, \ldots$. Then $R_h(t)$, $t \geq 0$, is determined in terms of the values $R_h(nh)$, $n \geq 0$, by linear interpolation :

$$(3.1) \qquad R_h(t) = \frac{(n+1)h-t}{h} R_h(nh) + \frac{t-nh}{h} R_h((n+1)h) ,$$

$$nh \le t \le (n+1)h, \qquad n \ge 0 .$$

Assume that $R_h(t)$ satisfies equation (1.1) at the points $t = nh$:

$$(3.2) \qquad nhR_h(nh) = \int_0^{nh} K(nh-u)R_h(u)du, \qquad n \ge 0 .$$

Ultimately, we shall let $h \to 0$.

By (3.1) and (3.2),

$$nhR_h(nh) = \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} K(nh-u)R_h(u)du$$

$$= \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} K(nh-u) \frac{(k+1)h-u}{h} R_h(kh)du +$$

$$+ \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} K(nh-u) \frac{u-kh}{h} R_h((k+1)h)\, du .$$

Replace k by k-1 in the second sum, collect terms and simplify to obtain

$$(3.3)\quad (n-n_h)R_h(nh) = -\frac{1}{h^2}b_nR_h(0) + \frac{1}{h^2}\sum_{k=0}^{n-1} c_{n-k}R_h(kh), \qquad n \geq 1,$$

where

$$(3.4)\quad b_n = b_n(h) = \int_{nh}^{(n+1)h} K(u)[(n+1)h-u]\,du, \quad n \geq 1,$$

$$(3.5)\quad c_n = c_n(h) = \int_{(n-1)h}^{nh} K(u)[u-(n-1)h]\,du + \int_{nh}^{(n+1)h} K(u)[(n+1)h-u]\,du, \quad n \geq 1,$$

and

$$(3.6)\quad n_h = \frac{1}{h^2}\int_0^h (h-u)K(u)\,du = \int_0^1 (1-t)K(ht)\,dt .$$

We shall need some properties of these quantities. It follows from (1.2), (1.7) and (3.6) that n_h is a non-negative, monotone decreasing function of h and

$$\frac{1}{2}K(h) \leq n_h \leq \frac{1}{h}\int_0^h K(u)\,du .$$

Hence, by (1.4), (1.5) and (1.8),

$$(3.7)\quad \begin{aligned} &n_h \to +\infty \quad \text{as} \quad h \to 0, \\ &n_h \to 0 \quad \text{as} \quad h \to \infty, \end{aligned}$$

and

(3.8) $$hn_h \to 0 \quad \text{as} \quad h \to 0 .$$

It follows from (1.2), (1.4), (3.4) and (3.5) that

(3.9) $$c_1 > b_1 \geqq 0,$$

and from (3.5)

(3.10) $$\frac{c_n}{n} \leq 2h^2 \int_{(n-1)h}^{(n+1)h} \frac{K(u)}{u}\, du, \quad n \geq 2 .$$

By (3.3),

(3.11) $$(1-n_h)R_h(h) = \frac{1}{h^2} R_h(0)(c_1 - b_1) .$$

First assume that n_h is not an integer. Then (3.3) determines $R_h(nh)$, $n \geq 1$, inductively in terms of $R_h(0)$, and (3.1) yields a solution $R_h(t)$ of (3.2). If $n_h > 1$ and $R_h(0) \gtrless 0$, then by (3.11) and (3.9), $R_h(h) \lessgtr 0$. Since $R_h(t) \geq 0$ by hypothesis, it follows that $R_h(0) = 0$ and, hence $R_h(t) \equiv 0$ if h is so small that $n_h > 1$ by (3.7).

Since we desire $R_h(t) \not\equiv 0$ and intend to let $h \to 0$, we must assume that n_h is an integer. We shall, henceforth, restrict h to the bounded and countable set

(3.12) $$H = \{h : n_h = 1, 2, \ldots\} .$$

Now (3.3) determines $R_h(nh)$ for $1 \le n < n_h$, but not for $n \ge n_h$, in terms of $R_h(0)$. As before, (3.7), (3.9), (3.11) and $R_h(t) \ge 0$ imply that $R_h(0) = 0$ for h sufficiently small. Let $R_h(0) = 0$ for all $h \in H$. Then by (3.3),

$$R_h(nh) = 0, \quad 0 \le n < n_h \quad . \tag{3.13}$$

We now assume that

$$R_h(n_h h) > 0 \quad . \tag{3.14}$$

Then (3.3) reduces to

$$(n-n_h)R_h(nh) = \frac{1}{h^2} \sum_{k=n_h}^{n-1} c_{n-k} R_h(kh), \qquad n > n_h \; , \tag{3.15}$$

which determines $R_h(nh)$, $n > n_h$, inductively in terms of $R_h(n_h h)$. Then (3.1) yields a solution $R_h(t)$ of (3.2). Since (3.1) and (3.15) are linear relations,

$$R_h(t) = R_h(n_h h) R_h^1(t) \; , \tag{3.16}$$

where $R_h^1(t)$ is the particular solution with $R_h^1(n_h h) = 1$.

Equations (3.1) and (3.13) yield

$$R_h(t) = 0, \quad 0 \le t \le (n_h - 1)h \; , \tag{3.17}$$

where $(n_h - 1)h \to 0$ as $h \to 0$ by (3.8). We have by (3.5), (3.9) and (1.2) that

$$c_1 > 0; \quad c_n \geq 0, \quad n \geq 1; \tag{3.18}$$

so that by (3.1), (3.14) and (3.15),

$$R_h(t) > 0, \quad t > (n_h - 1)h . \tag{3.19}$$

Hence, the hypothesis that $R_h(t) \geq 0$ is satisfied.

Theorem 3.1. $R_h(t)$ is bounded and attains its maximum at some point $t = kh \leq A$, where A is defined by (2.6).

Proof. The proof is based on (3.1) and (3.2). It is analogous to that of Theorem 2.4, parts (i) and (ii).

Thus far, $R_h(t)$ is the general non-trivial, non-negative solution of (3.1) and (3.2). Since by Theorem 3.1, $R_h(t)$ is bounded and attains its maximum, we will assume that

$$\max_{0 \leq t < \infty} R_h(t) = 1, \quad h \in H . \tag{3.20}$$

Then $R_h(t)$ is determined completely and is given explicitly by

$$R_h(t) = \frac{R_h^1(t)}{\max\limits_{0 \leq u < \infty} R_h^1(u)} . \tag{3.21}$$

Theorem 3.2. As $t \to \infty$, $R_h(t) \to 0$ uniformly in h.

Proof. The proof is based on (3.1), (3.2) and Theorem 3.1.

It is analogous to that of Theorem 2.4, part (iii).

Theorem 3.3. For each $h \in H$, $R_h(t)$ is integrable on $(0, \infty)$.

Proof. The proof is analogous to that of Theorem 2.2, with generating functions used in place of Laplace transforms. Since the function $R_h(t)$ is piecewise-linear and $R_h(0) = 0$,

$$\int_0^\infty R_h(t)dt = h \sum_{n=0}^{\infty} R_h(nh) \tag{3.22}$$

if the sum is finite. Let

$$A_h(w) = \sum_{n=0}^{\infty} R_h(nh)w^n, \qquad w \geq 0, \tag{3.23}$$

when the series converges. Since $0 \leq R_h(nh) \leq 1$, $A_h(w)$ is defined at least for $0 \leq w < 1$. Note that $R_h(t)$ is integrable if $A_h(1)$ exists, in which case

$$\int_0^\infty R_h(t)dt = hA_h(1) \quad . \tag{3.24}$$

Let $c_0 = 0$ and use (3.5) to define

$$B_h(w) = \sum_{n=0}^{\infty} c_n w^n, \qquad 0 \leq w < 1 \quad . \tag{3.25}$$

By (3.10) and (1.6), $B_h(w)$ exists. Since $c_0 = 0$ and $R_h(0) = 0$,

(3. 3) yields

$$\sum_{n=0}^{\infty} (n-n_h)R_h(nh)w^n = \frac{1}{h^2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_{n-k} R_h(kh)w^n , \qquad 0 \le w < 1 .$$

By (3. 23), (3. 25) and the convolution theorem for sums (cf.(6,p.179)),

$$wA_h'(w) - n_h A_h(w) = \frac{1}{h^2} A_h(w)B_h(w), \qquad 0 \le w < 1 .$$

Integration yields, with the use of (3. 25),

$$(3.26) \qquad A_h(w) = Cw^{n_h} \exp \{\frac{1}{h^2} \sum_{n=1}^{\infty} \frac{c_n}{n} w^n\}, \qquad 0 \le w < 1,$$

where C is a constant of integration.

By (3. 10) and (1. 6), the series $\sum_{n=1}^{\infty} \frac{c_n}{n}$ converges. Hence, by Abel's theorem (cf. (6, p. 177)), applied to $\sum_{n=1}^{\infty} \frac{c_n}{n} w^n$,

$$A_h(1-) = C \exp \{\frac{1}{h^2} \sum_{n=1}^{\infty} \frac{c_n}{n} \},$$

which is finite. Since $R_h(nh) \ge 0$, an elementary Tauberian theorem (cf. (6, p. 189)) applied to (3. 23) yields $A_h(1) = A_h(1-)$. Thus, $A_h(1)$ exists, so that $R_h(t)$ is integrable and the theorem is proved.

We also obtained

$$A_h(1) = C \exp\left\{\frac{1}{h^2}\sum_{n=1}^{\infty}\frac{c_n}{n}\right\}.$$

This result and (3.26) yield

(3.27) $$A_h(w) = A_h(1)w^{n_h}\exp\left\{\frac{1}{h^2}\sum_{n=1}^{\infty}\frac{c_n}{n}(w^n-1)\right\}, \quad 0 \le w \le 1 .$$

Define

(3.28) $$P_h(t) = \frac{R_h(t)}{\int_0^\infty R_h(u)du}, \quad t \ge 0 .$$

Then

(3.29) $$nhP_h(nh) = \int_0^{nh} K(nh-u)P_h(u)du, \quad n \ge 0,$$

and

(3.30) $$\int_0^\infty P_h(t)dt = h\sum_{n=0}^{\infty} P_h(nh) = 1, \quad h \in H .$$

Each function $P_h(t)$, $h \in H$, is a non-trivial, continuous, piecewise-linear, non-negative, approximation to the solution $P(t)$ of (1.10) and (1.11). $R_h(t)$ is determined in terms of $P_h(t)$ by

(3.31) $$R_h(t) = \frac{P_h(t)}{\max\limits_{0 \le u < \infty} P_h(u)} .$$

In this chapter we have constructed approximations to $R(t)$ and $P(t)$, and have derived some of their properties. Other properties are derived in the next chapter.

CHAPTER IV

PROPERTIES OF THE APPROXIMATIONS

We now establish some important properties of the functions $R_h(t)$. A property of the constants c_n, defined by (3.5), will be needed in the proof of the first theorem.

By (3.5), (1.7) and a change of variable we have,

$$c_{n+1} = \int_{(n-1)h}^{nh} K(u+h)[u-(n-1)h]\,du + \int_{nh}^{(n+1)h} K(u+h)[(n+1)h-u]\,du \leq c_n$$

for $n \geq 1$, whence

(4.1) $\{c_n\}$ is a monotone non-increasing sequence for each fixed h.

<u>Theorem 4.1.</u> For h sufficiently small, the functions $R_h(t)$, $h \in H$, are monotone non-decreasing in some interval $0 \leq t < t_0$ such that $K(t_0+h) \leq 1$.

<u>Proof.</u> There exists a unique integer $N_h \geq n_h$, $h \in H$, such that

$$(4.2) \quad \begin{cases} R_h(kh) < R_h((k+1)h), & n_h - 1 \leq k < N_h \\ R_h(N_h h) \geq R_h((N_h+1)h). \end{cases}$$

Then $R_h(t)$ is non-decreasing for $0 \le t \le N_h h$. By (3.15),

$$(N_h+1-n_h)R_h((N_h+1)h) - (N_h-n_h)R_h(N_h h)$$
$$= \frac{1}{h^2}\sum_{k=0}^{N_h} c_{N_h+1-k}\, R_h(kh) - \frac{1}{h^2}\sum_{k=0}^{N_h-1} c_{N_h-k} R_h(kh) \quad .$$

We replace k by $k+1$ in the first sum and use $R_h(0) = 0$ to obtain

$$(N_h+1-n_h)[R_h((N_h+1)h)-R_h(N_h h)] + R_h(N_h h)$$
$$= \frac{1}{h^2}\sum_{k=0}^{N_h-1} c_{N_h-k}[R_h((k+1)h) - R_h(kh)] \quad .$$

Hence, by (4.1), (4.2) and $R_h(0) = 0$,

$$R_h(N_h h) \ge \frac{1}{h^2} c_{N_h} \sum_{k=0}^{N_h-1} [R_h((k+1)h)-R_h(kh)] = \frac{1}{h^2} c_{N_h} R_h(N_h h) \quad .$$

Therefore, $c_{N_h} \le h^2$. By (3.5) and (1.7),

$$c_{N_h} \ge K((N_h+1)h)\left\{\int_{(N_h-1)h}^{N_h h}[u-(N_h-1)h]\,du + \int_{N_h h}^{(N_h+1)h}[(N_h+1)h-u]du\right\} = h^2 K((N_h+1)h).$$

Therefore, $K((N_h+1)h) \le 1$ and the theorem follows from (1.4) and (1.7).

Lemma 4.1. Let $0 \le a \le 1$ and $0 \le b \le 1$. Then

$$a \int_0^b K(u)du \le \int_0^{ab} K(u)du \quad .$$

Proof. The lemma is obvious if either $a = 0$ or $b = 0$. For $0 < a \le 1$ and $0 < b \le 1$, an equivalent assertion is

$$\frac{1}{t} \int_0^t K(u)du \le \frac{1}{s} \int_0^s K(u)du \ , \qquad t \ge s > 0 \ .$$

Define $f(x) = \frac{1}{x} \int_0^x K(u)du, \quad x > 0$. Then by (1.7),

$$f'(x) = \frac{1}{x} K(x) - \frac{1}{x^2} \int_0^x K(u)du \le 0$$ and the lemma follows.

Theorem 4.2. The functions $R_h(t)$, $h \in H$, are uniformly equicontinuous on each interval $t_1 \le t < \infty$ with $t_1 > 0$. Moreover,

$$|R_h(t) - R_h(s)| \le \frac{1}{t_1} \left\{ 6 \int_0^{|t-s|} K(u)du + |t-s| \right\}, \quad t, s \ge t_1, h \in H. \tag{4.3}$$

Proof. For $n > m \ge 0$ it follows from (3.2), (3.20), (1.7) and (1.2) that

$$|nhR_h(nh) - mhR_h(mh)| = \left|\int_0^{nh} K(nh-u)R_h(u)du - \int_0^{mh} K(mh-u)R_h(u)du\right|$$

$$\le \int_{mh}^{nh} K(nh-u)du + \int_0^{mh} [K(mh-u) - K(nh-u)]\, du$$

$$= 2\int_0^{nh-mh} K(u)du - \int_{mh}^{nh} K(u)du \le 2\int_0^{nh-mh} K(u)du \quad .$$

By symmetry

$$(4.4) \qquad |nhR_h(nh) - mhR_h(mh)| \le 2\int_0^{|nh-mh|} K(u)du \quad , \qquad n, m \ge 0 \quad .$$

By (3.20) and (4.4),

$$|nhR_h(nh) - nhR_h(mh)| \le |nhR_h(nh) - mhR_h(mh)| + |mhR_h(mh) - nhR_h(mh)| ,$$

$$(4.5) \quad |R_h(nh) - R_h(mh)| \le \frac{1}{nh}\left[2\int_0^{|nh-mh|} K(u)du + |nh-mh|\right] , \qquad m \ge 0,\ n > 0 \ .$$

Let $nh \le s,\ t \le (n+1)h$. Then by (3.1), (4.5) and Lemma 4.1,

$$|R_h(t) - R_h(s)| = \frac{|t-s|}{h} |R_h((n+1)h) - R_h(nh)|$$

$$\le \frac{1}{(n+1)h} \left\{\frac{|t-s|}{h}\, 2\int_0^{h} K(u)du + |t-s|\right\} ,$$

$$(4.6) \quad |R_h(t) - R_h(s)| \le \frac{1}{(n+1)h}\left\{2\int_0^{|t-s|} K(u)du + |t-s|\right\} , \qquad t_1 \le s,\ t \le (n+1)h,$$

and (4.3) holds in this case.

Now assume that $s \leq nh \leq t$ for some n. Define m and n such that $(m-1)h \leq s \leq mh \leq nh \leq t \leq (n+1)h$. By (4.5), (4.6), we have for $s, t \geq t_1$,

$$|R_h(t)-R_h(s)| \leq |R_h(t)-R_h(nh)| + |R_h(nh)-R_h(mh)| + |R_h(mh)-R_h(s)|$$

$$\leq \frac{1}{t_1}\left\{2\int_0^{t-nh} K(u)du + (t-nh) + 2\int_0^{nh-mh} K(u)du + (nh-mh)\right.$$

$$\left. + \; 2\int_0^{mh-s} K(u)du + mh-s\right\} \leq \frac{1}{t_1}\left\{6\int_0^{t-s} K(u)du + t-s\right\},$$

and (4.3) is true in general.

Theorem 4.2 allows us to use in the next chapter, the powerful

Arzelà-Ascoli Theorem. Each nonvoid, bounded, equicontinuous family of real functions defined on a closed and bounded interval contains a uniformly convergent sequence.

For a proof, see (5, p. 59).

CHAPTER V

CONVERGENCE OF THE $R_h(t)$

We shall consider the convergence of the functions $R_h(t)$ as $h \to 0$ through H. We shall regard $\{R_h : h \in H\}$ as a <u>sequence</u> ordered by letting h decrease through H.

<u>Theorem 5.1.</u> Every infinite subsequence $\{R_h : h \in H' \subset H\}$ of $\{R_h : h \in H\}$ has a further subsequence which converges for all $t \geq 0$, uniformly on any interval $0 < t_1 \leq t \leq t_2 < \infty$.

<u>Proof.</u> Since $0 \leq R_h(t) \leq 1$, Theorem 4.2 implies that the Arzelà-Ascoli theorem is applicable, and by that theorem there exists successive (nested) subsequences of $\{R_h : h \in H'\}$ which, respectively converge uniformly on the intervals $[\frac{1}{m}, m]$, $m = 2, 3, \ldots$. Then the usual diagonal procedure yields a single subsequence which converges uniformly on each of the intervals $[\frac{1}{m}, m]$. Since $R_h(0) = 0$ for all $h \in H$, this subsequence converges pointwise for $t \geq 0$.

<u>Theorem 5.2.</u> Suppose that $R_{h'}(t) \to R(t)$ as $h' \to 0$ through some $H' \subset H$ uniformly on each interval $0 < t_1 \leq t \leq t_2 < \infty$. Then $R(t)$ is the unique solution of (1.1), (1.9) and

$$0 \leq R(t) \leq 1 \quad . \tag{5.1}$$

Proof. In (3.2), let $n \to \infty$, $h \to 0$ and $nh \to t$ with $h \in H'$. A theorem of Lebesgue (cf. (3, p. 22-23)) implies that

$$\int_0^{nh} K(nh-u)R_h(u)du \to \int_0^t K(t-u)R(u)du \quad .$$

Thus, $R(t)$ satisfies (1.1). Theorem 3.1, (3.20) and $R_h(t) \geq 0$ imply (1.9) and (5.1). By Theorem 2.1, $R(t)$ is unique.

Auxiliary Lemma. If a sequence of monotone functions converges pointwise to a continuous function on a closed and bounded interval, then the convergence is uniform.

The proof can be found in (2, p. 90).

Theorem 5.3. There exists a unique solution $R(t)$, $0 \leq t < \infty$, of (1.1) such that:

(i) $R(t)$ is bounded and $\max_{0 \leq t < \infty} R(t) = 1$,

(ii) $R(t)$ is non-negative,

(iii) $R(t)$ is continuous on $[0, \infty)$,

(iv) $R(t)$ is integrable on $(0, \infty)$.

Moreover, $R_h(t) \to R(t)$ uniformly as $h \to 0$ through H.

Proof. By Theorems 5.1 and 5.2 there exists a unique function $R(t)$, $0 < t < \infty$, which satisfies (1.1), (i) and (ii).

By Theorem 4.1, $R(t)$ is monotone in some neighborhood of $t = 0$, so that $R(0+)$ exists. Then Theorems 2.3 and 2.5 imply (iii). Theorem 2.2 implies (iv).

Fix $t \in [0, \infty)$ arbitrarily and let $r(t)$ be any accumulation point, as $h \to 0$ through H, of the numerical sequence $\{R_h(t) : h \in H\}$. Then by (3.20), the sequence of functions $\{R_h : h \in H\}$ has a subsequence which converges at t to the value $r(t)$. By Theorems 5.1 and 5.2, there is a further subsequence which converges to R on $[0, \infty)$. Therefore, $r(t) = R(t)$ and, hence, $R_h(t) \to R(t)$ as $h \to 0$ through H.

By Theorem 4.1, the Auxiliary Lemma and the continuity of $R(t)$, we have that $R_h(t) \to R(t)$ uniformly in some neighborhood of $t = 0$ and the pointwise convergence and uniform equicontinuity of the functions $R_h(t)$ imply uniform convergence on every finite interval. Then by Theorem 3.2, the convergence in uniform on $[0, \infty)$.

In this chapter we have shown that the approximations converge uniformly to a function $R(t)$ with the prescribed properties.

CHAPTER VI

CONVERGENCE OF THE $P_h(t)$

We shall investigate the convergence of the functions $P_h(t)$ as $h \to 0$ through H. We shall regard $\{P_h : h \in H\}$ as a sequence ordered by letting h decrease through H.

We define a function $\psi_h(w,t)$, $0 < w \leq 1$, $t \geq 0$, by

(6.1) $\psi_h(w,t) = w^t$ for $t = nh$, $n = 0, 1, \ldots$, and linear between.

<u>Lemma 6.1.</u> As $w \to 1-$, $\int_0^\infty P_h(t)\psi_h(w,t)dt \to \int_0^\infty P_h(t)dt = 1$ uniformly for $h \in H$.

<u>Proof.</u> Since $P_h(t)\psi_h(w,t)$ is piecewise-linear and $P_h(0)\psi_h(w,0) = 0$, we have by (3.28) and (3.23),

$$\int_0^\infty P_h(t)\psi_h(w,t)dt = h\sum_{n=0}^{\infty} P_h(nh)w^{nh} = hE_hA_h(w^h),$$

where E_h is a constant. Let $w = 1$ above to obtain

$$1 = \int_0^\infty P_h(t)dt = hE_hA_h(1), \quad \text{whence} \quad hE_h = \frac{1}{A_h(1)} .$$

Therefore, by (3.27),

$$(6.2)\quad \int_0^\infty P_h(t)\psi_h(w,t)dt = \frac{A_h(w^h)}{A_h(1)} = w^{hn_h} \exp\{-\frac{1}{h^2}\sum_{n=1}^{\infty} \frac{c_n}{n}(1-w^{nh})\} .$$

We want to show that this converges to 1 uniformly in h as $w \to 1-$. In view of (3.8), it suffices to show that

$$(6.3)\qquad \frac{1}{h^2}\sum_{n=2}^{\infty} \frac{c_n}{n}(1-w^{nh}) \to 0 \text{ uniformly in } h \text{ as } w \to 1- .$$

By (3.10),

$$0 \le \frac{1}{h^2}\,\frac{c_n}{n}(1-w^{nh}) \le 2\int_{(n-1)h}^{(n+1)h} \frac{K(u)}{u}(1-w^{nh})du \le 2\int_{(n-1)h}^{(n+1)h} \frac{K(u)}{u}(1-w^{2u})du, \quad n \ge 2.$$

Hence,

$$(6.4)\quad \frac{1}{h^2}\sum_{n=2}^{\infty} \frac{c_n}{n}(1-w^{nh}) \le 2\sum_{n=2}^{\infty}\int_{(n-1)h}^{(n+1)h} \frac{K(u)}{u}(1-w^{2u})du$$

$$\le 4\int_0^\infty \frac{K(u)}{u}(1-w^{2u})du .$$

Since $\lim_{u\to 0} \frac{1-w^{2u}}{u} = -2\log w$, which is finite, the last integral exists by (1.5) and (1.6), and does not depend on h. As $w \to 1-$, the integrand decreases everywhere to zero. Therefore, by Lebesgue's monotone convergence theorem,

$$\int_0^{\infty} \frac{k(u)}{u} (1-w^{2u})du \to 0 \quad \text{as} \quad w \to 1- \quad .$$

This result and (6.4) implies (6.3) which yields the desired result.

Lemma 6.2. There exists a constant M such that $P_h(t) \leq M$, $0 \leq t < \infty$, for all $h \in H$.

Proof. By (3.28), (3.20) and Theorem 3.1,

$$P_h(t) = \frac{R_h(t)}{\int_0^{\infty} R_h(u)du} \leq \frac{1}{\int_0^{A} R_h(u)du} < \infty, \quad 0 \leq t < \infty, \; h \in H.$$

By Theorem 2.4 and the uniform convergence of the $R_h(t)$, there exists $h_0 \in H$ such that

$$\int_0^A R_h(u)du \geq \frac{1}{2} \int_0^A R(u)du > 0, \quad h \leq h_0, \quad h \in H \; .$$

So $P_h(t) \leq M_0$ for $h \leq h_0$, $h \in H$, where $M_0 = \dfrac{2}{\int_0^A R(u)du}$.

Since each $P_h(t)$, $h \in H$, is bounded and the set $\{h \in H : h > h_0\}$ is finite, the lemma follows. (Another proof uses Fotou's lemma).

Lemma 6.3. Let w, $0 < w < 1$, be fixed. Then

$$\lim_{T\to\infty} \int_T^{\infty} P_h(t)\psi_h(w,t)dt = 0 \quad \text{uniformly for} \quad h \in H.$$

Proof. By Lemma 6.2 we have

$$\int_T^{\infty} P_h(t)\psi_h(w,t)dt \leq M \int_T^{\infty} \psi_h(w,t)dt, \quad h \in H \ .$$

By the definition of $\psi_h(w,t)$, we have $w^s \geq w^{nh} \geq \psi_h(w,t)$ for $s \leq nh \leq t$, so that $w^{t-h} \geq \psi_h(w,t)$ for all $t > 0$. Therefore $\psi_h(w,t) \leq w^{t-h} \leq w^{\frac{t}{2}}$ for all $h \leq \frac{t}{2}$. Let $T \geq 2\max\{h : h \in H\}$. Then, for all $h \in H$,

$$\int_T^{\infty} \psi_h(w,t)dt \leq \int_T^{\infty} w^{\frac{t}{2}} dt \to 0 \quad \text{as} \quad T \to \infty,$$

which implies the desired result.

Lemma 6.4. As $T \to \infty$,

$$\int_T^{\infty} P_h(t)dt \to 0 \qquad \text{and} \qquad \int_0^T P_h(t)dt \to 1$$

uniformly for $h \in H$.

Proof. We have

$$0 \leq \int_T^\infty P_h(t)dt = \int_T^\infty P_h(t)(1-\psi_h(w,t))dt + \int_T^\infty P_h(t)\psi_h(w,t)dt$$

$$\leq \int_0^\infty P_h(t)(1-\psi_h(w,t))dt + \int_T^\infty P_h(t)\psi_h(w,t)dt, \quad 0 \leq w \leq 1, \; h \in H .$$

Then the lemma follows from Lemmas 6.1 and 6.3, and (3.30).

Lemma 6.5. There exists a constant λ such that

$\max\limits_{0 \leq t < \infty} P_h(t) \geq \lambda > 0$ for all $h \in H$.

Proof. By Lemma 6.4, for T sufficiently large we have

$\int_0^T P_h(t)dt > \frac{1}{2}$ for all $h \in H$. Then $\frac{1}{2} < T \max\limits_{0 \leq t < \infty} P_h(t)$ and

$\max\limits_{0 \leq t \leq \infty} P_h(t) > \frac{1}{2T}$ for all $h \in H$.

Theorem 6.1. As $h \to 0$,

$$\int_0^\infty R_h(t)dt \to \int_0^\infty R(t)dt .$$

Proof.
$$\left| \int_0^\infty R_h(t)dt - \int_0^\infty R(t)dt \right| \leq \left| \int_0^T (R_h(t)-R(t))dt \right|$$

$$+ \int_T^\infty R_h(t)dt + \int_T^\infty R(t)dt .$$

By (3.31) and Lemma 6.5,

$$\int_T^\infty R_h(t)dt = \frac{1}{\max\limits_{0 \le u < \infty} P_h(u)} \int_T^\infty P_h(t)dt \le \frac{1}{\lambda} \int_T^\infty P_h(t)dt .$$

Therefore,

$$\left| \int_0^\infty R_h(t)dt - \int_0^\infty R(t)dt \right| \le \left| \int_0^T (R_h(t) - R(t))dt \right| + \frac{1}{\lambda} \int_T^\infty P_h(t)dt$$

$$+ \int_T^\infty R(t)dt.$$

The theorem follows from Lemma 6.4 and the uniform convergence of $R_h(t)$ to $R(t)$

Theorem 6.2. There exists a unique bounded, continuous, non-negative solution $P(t)$ of (1.10) and (1.11), and $P_h(t) \to P(t)$ uniformly for $0 \le t < \infty$ as $h \to 0$ through H.

Proof. The existence and uniqueness of $P(t)$ follow from (1.12) and the existence and uniqueness of $R(t)$. By Theorem 6.1, (1.12), (3.28) and the uniform convergence of $R_h(t)$ to $R(t)$, the functions $P_h(t)$ converge uniformly to $P(t)$ on $0 \le t < \infty$ as $h \to 0$ through H.

Theorem 6.3. The functions $\{P_h(t) : h \in H\}$ converge in the mean to $P(t)$.

Proof. We have

$$\int_0^\infty |P_h(t)-P(t)|dt \leq \int_0^T |P_h(t)-P(t)|dt + \int_T^\infty P_h(t)dt + \int_T^\infty P(t)dt .$$

The theorem follows from Lemma 6.4 and Theorem 6.2.

Corollary. As $h \to 0$,

$$\int_a^b P_h(t)dt \to \int_a^b P(t)dt, \quad 0 \leq a \leq b \leq \infty .$$

Proof.

$$0 \leq \left| \int_a^b P_h(t)dt - \int_a^b P(t)dt \right| \leq \int_a^b |P_h(t)-P(t)| dt \leq \int_0^\infty |P_h(t)-P(t)| dt \to 0$$

as $h \to 0$ by Theorem 6.3.

We have shown that the functions $P_h(t)$ converge uniformly and in the mean to a solution $P(t)$ of (1.10) with the prescribed properties.

We have by (1.12), (3,28) and Theorem 6.1 that $P_h(t) = \Omega_h R_h(t)$, $P(t) = \Omega R(t)$, where Ω and Ω_h are constants, and $\Omega_h \to \Omega$. In view of this, the two sets of approximations are very closely related. In fact, any property derived above for either set of approximations, P_h or R_h, will also be true for the other set with trivial modifications.

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