SOLUTION OF A NONLINEAR EQUATION ARISING IN A DISCONTINUOUS CONTROL PROBLEM

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CHAPTER I

INTRODUCTION

Continuously operating control elements have long been used in the automatic control of various types of physical systems and the resulting automatic control systems have been extensively treated in the literature. Systems of this type, however, have the disadvantage of not making full use of the power available in the driving element, resulting in many cases in a very uneconomical arrangement. In recent years, then, considerable effort has been made to replace these continuously operating elements by ones which approximate an "on-off" (relay) type element, for it has been recognized that although a linear system offers the advantage of ease of analysis, certain nonlinear systems may be employed which make full use of the power available and which also possess better control characteristics than would be possible by using a linear system. Also the relatively simple construction of the relay type system usually has the two advantages of higher reliability and lower cost.

In the present study several assumptions are made. It is assumed that the "uncontrolled" system possesses only one degree of freedom and that it is governed by a linear, second order differential equation with constant coefficients. It is also assumed that the values of the error and its first two time derivatives are measured precisely. These are the standard assumptions which are made when speaking of the idealized system (1, p. 11), (2, p. 3-17). In addition it is usually assumed that the on-off element can be characterized mathematically by a simple step function, i.e., properties of the mechanism such as inertia, hysteresis and dead-zone are ignored. In this study the on-off element is represented by a continuous function with continuous derivatives and an effort is thus made to include the inertia and the dead-zone of the element.

The resulting differential equation for the system involves a uniformly convergent series in powers of the unknown $y(t)$ and its first three time derivatives. This equation is then transformed into a nonlinear Volterra integral equation of the second kind for which the resolvent kernel is obtained. On applying the resolvent kernel a new equation is obtained, which is then solved by the methods of E. Schmidt for nonlinear integral equations (4, p. 370-399). This discussion includes a proof of uniqueness of the solution under certain restrictions. In the appendix is given a discussion of resolvent kernels for Volterra integral equations of the second kind. It is shown that if the kernel $K(s-t)$ is a polynomial in $s-t$, then the resolvent $\Gamma(s-t)$ can be obtained in closed form, making use of exponential functions.

Although nothing has been obtained thus far regarding limit cycles in such systems as have been described here, it is hoped that this approach to the problem will eventually yield some results in that direction. This very important problem is thus the next to be considered in these investigations.

CHAPTER II

THE DIFFERENTIAL EQUATION FOR THE ERROR

Denoting by y(t) the error (e.g., the heading error in the case of a missile), the differential equation in y for the uncontrolled system is written

(1) $$a_0\ddot{y} + a_1\dot{y} + a_2y = 0$$

where a_0, a_1, a_2 are constants and $\dot{y}$, $\ddot{y}$ are the first and second time derivatives, respectively, of the function y. When the control system is added the equation becomes

(2) $$a_0\ddot{y} + a_1\dot{y} + a_2y = f$$

where f denotes a force- or moment-producing term which is a function of the measured values of y and its derivatives. An example of the form which f may take in an idealized system is given by

(3) $$f = k\ \mathrm{sgn}(b_0\ddot{y} + b_1\dot{y} + b_2y)$$

where the "signum" function is defined as follows:

(4) $$\begin{aligned} \mathrm{sgn}(x) &= \frac{x}{|x|} \qquad (x \neq 0), \\ &= 0 \qquad (x = 0). \end{aligned}$$

The function

(5) $$F = b_0\ddot{y} + b_1\dot{y} + b_2y \qquad (b_0 \neq 0)$$

which determines the sign of k in the control equation is called the "control function". The coefficients b_0, b_1, b_2 are taken to be constants.

A more realistic form for f is obtained by taking into account the "dead zone" in the relay as well as the inertia in the element itself and in the control mechanism (e.g., the mechanism employed for control of the flaps in a missile). The effect of inertia is approximated by choosing a function f which does not change sign discontinuously but which changes sign in a continuous manner; the effect of the dead zone is approximated by requiring that f remain equal to zero in an interval containing F = 0. For this purpose one might employ the entire function

$$(6) \qquad \varphi(x) = e^{-e^{-\lambda(x-\mu)}} - e^{-e^{\lambda(x+\mu)}} .$$

By taking λ sufficiently large $\varphi(x)$ can be made to approximate arbitrarily closely the discontinuous function

$$1/2[\operatorname{sgn}(x+\mu) + \operatorname{sgn}(x-\mu)]$$

with a dead zone of length 2μ and centered about $x = 0$.

The exact form of equation (2) considered in this study is the following:

$$a_0\ddot{y} + a_1\dot{y} + a_2 y = k_0\ddot{\beta}(F) + k_1\dot{\beta}(F) + k_2\beta(F) \tag{7}$$

$$(k_j = \text{constant}, \ k_0 \neq 0)$$

in which $\dot{\beta}$ is some entire function which approximates the desired situation. It will be assumed further that $\dot{\beta}(x)$ is a strictly increasing function of x such that $\dot{\beta}(0) = 0$. The function $\ddot{\beta}(F)$ is the time derivative of $\dot{\beta}(F)$, which exists since $\dot{\beta}$ is an entire function; $\beta(F)$ is given by

$$\beta(F) = \int_0^t \dot{\beta}(F)\,d\tau + \beta_0. \tag{8}$$

It should be noted that since $F = b_0\ddot{y} + b_1\dot{y} + b_2 y$,

$$\ddot{\beta}(F) = \frac{d\dot{\beta}}{dF}\cdot(b_0\dddot{y} + b_1\ddot{y} + b_2\dot{y}), \tag{9}$$

so that equation (7) is an equation of third order. Thus it will be supposed that initial conditions will be given for y, $\dot{y}$ and $\ddot{y}$. Also, in order to completely determine β its initial value β_0 must be given. Moreover, from equations (7) and (9) and the assumption that $k_0 b_0 \neq 0$,

$$\dddot{y} = \frac{a_0\ddot{y} + a_1\dot{y} + a_2 y}{k_0 b_0 \frac{d\dot{\beta}}{dF}} - \frac{b_1\ddot{y} + b_2\dot{y}}{b_0} - \frac{k_1\dot{\beta} + k_2\beta}{k_0 b_0 \frac{d\dot{\beta}}{dF}} \tag{10}$$

so that $\dddot{y}$ is a continuous function since $\frac{d\dot{\beta}}{dF}$ is everywhere different from zero.

<u>Series representations for β, $\dot\beta$ and $\ddot\beta$.</u> Since $\dot\beta$ is an entire function and since F is a linear combination of differentiable (and hence bounded) functions it follows that the power series in powers of F for $\dot\beta(F)$ converges for all values of t in any interval $[0, T]$.

Denote the series for $\dot\beta(F)$ by

$$\dot\beta(F) = \sum_{n=1}^{\infty} c_n F^n. \tag{11}$$

That this series converges uniformly and absolutely for t in any closed interval $[0, T]$ follows from the Weierstrass "M test" since F is bounded on $[0, T]$ and $\dot\beta$ is an entire function. Thus the series for $\beta(F)$ can be obtained from (11) by integrating the series term by term from 0 to t and adding β_o.

The series obtained from (11) by differentiating term by term with respect to t is

$$\sum_{n=1}^{\infty} nc_n F^{n-1} \dot F = \dot F \sum_{n=1}^{\infty} nc_n F^{n-1}$$

which converges uniformly and absolutely on $[0, T]$ by the "M test" since the series

$$\sum_{n=1}^{\infty} nc_n x^{n-1}$$

is the term wise derivative of a power series with infinite radius of convergence. Thus, $\ddot{\beta}(F)$ is given by

$$\ddot{\beta}(F) = \sum_{n=1}^{\infty} c_n \frac{d}{dt} F^n.$$

CHAPTER III

THE INTEGRAL EQUATION

Equation (7) can be transformed into an integral equation for $\ddot{y}$ as follows. Replacing β, $\dot{\beta}$ and $\ddot{\beta}$ by their series representations one obtains the equation

$$
\begin{aligned}
(12) \qquad a_0\ddot{y} + a_1\dot{y} + a_2 y = {} & k_0 \sum_{n=1}^{\infty} nc_n F^{n-1}\dot{F} \\
& + k_1 \sum_{n=1}^{\infty} c_n F^n + k_2\beta_0 \\
& + k_2 \sum_{n=1}^{\infty} c_n \int_0^t F^n ds \\
= {} & c_1\left\{k_0\dot{F} + k_1 F + k_2\int_0^t F ds\right\} + k_2\beta_0 \\
& + \sum_{n=2}^{\infty} c_n\left\{k_0 n F^{n-1}\dot{F} + k_1 F^n \right. \\
& \left. + k_2\int_0^t F^n ds\right\}
\end{aligned}
$$

where the terms which are linear in y, $\dot{y}$, $\ddot{y}$ and $\ddot{y}$ have been separated out. Replacing y, $\dot{y}$ and $\ddot{y}$ in equation (12) by their equivalent expressions

$$y = 1/2\int_o^t (t-t_1)^2 \dddot{y}(t_1)dt_1 + 1/2t^2 \ddot{y}_o + t\dot{y}_o + y_o$$

$$\dot{y} = \int_o^t (t-t_1)\dddot{y}(t_1)dt_1 + t\ddot{y}_o + \dot{y}_o$$

$$\ddot{y} = \int_o^t \dddot{y}(t_1)dt_1 + \ddot{y}_o$$

one obtains the equation

$$L\left(\dddot{y}\ \ddot{y}_o\ \overset{t}{\dot{y}_o}\ y_o\right) = \sum_{n=2}^{\infty} c_n\left\{k_o \frac{d}{dt} F^n + k_1F^n + k_2\int_o^t F^n dt\right\}$$

where L is given by

$$L = \int_o^t [a_o + a_1(t-t_1) + a_2\frac{(t-t_1)^2}{2}]\dddot{y}(t_1)dt_1$$

$$-c_1\int_o^t \Big\{k_o[b_1+b_2(t-t_1)]+k_1[b_o+b_1(t-t_1)+b_2\frac{(t-t_1)^2}{2}]$$

$$+k_2[b_o(t-t_1)+b_1\frac{(t-t_1)^2}{2} + b_2\frac{(t-t_1)^3}{6}]\Big\}\dddot{y}dt_1$$

$$+[a_o+a_1t+a_2\frac{t^2}{2}]\ddot{y}_o+[a_1+a_2t]\dot{y}_o+a_2y_o$$

(continued on next page)

$$-c_1\Big\{k_0[b_1+b_2t]\ddot{y}_0+k_0b_2\dot{y}_0$$

$$+k_1[b_0+b_1t+b_2\frac{t^2}{2}]\ddot{y}_0+k_1[b_1+b_2t]\dot{y}_0+k_1b_2y_0$$

$$+k_2[b_0t+b_1\frac{t^2}{2}+b_2\frac{t^3}{6}]\ddot{y}_0+k_2[b_1t+b_2\frac{t^2}{2}]\dot{y}_0+k_2b_2ty_0\Big\}$$

$$-c_1k_0b_0\dddot{y}(t) - k_2\beta_0.$$

Or, writing $G(t) = b_0t+b_1\frac{t^2}{2}+b_2\frac{t^3}{6}$ and $H(t) = a_0+a_1t+a_2\frac{t^2}{2}$,

$$L = -c_1k_0b_0\Big\{\dddot{y}(t)+\int_0^t[\frac{1}{b_0}\ddot{G}(t-t_1)+\frac{k_1}{k_0b_0}\dot{G}(t-t_1)$$

$$+\frac{k_2}{k_0b_0}G(t-t_1)-\frac{1}{c_1k_0b_0}H(t-t_1)]\dddot{y}dt_1$$

$$+[\frac{1}{b_0}\ddot{G}(t)+\frac{k_1}{k_0b_0}\dot{G}(t)+\frac{k_2}{k_0b_0}G(t)-\frac{1}{c_1k_0b_0}H(t)]\ddot{y}_0$$

$$+[\frac{1}{b_0}\dddot{G}(t)+\frac{k_1}{k_0b_0}\ddot{G}(t)+\frac{k_2}{k_0b_0}\dot{G}(t)-\frac{1}{c_1k_0b_0}\dot{H}(t)-\frac{k_2}{k_0}]\dot{y}_0$$

$$+[\frac{k_1}{k_0b_0}\dddot{G}(t)+\frac{k_2}{k_0b_0}\ddot{G}(t)-\frac{1}{c_1k_0b_0}\ddot{H}(t)-\frac{k_2b_1}{k_0b_0}]y_0+\frac{k_2}{c_1k_0b_0}\beta_0\Big\}$$

$$= -c_1k_0b_0\Big\{\dddot{y}(t) + \int_0^t K(t-t_1)\dddot{y}(t_1)dt_1$$

$$+K(t)\ddot{y}_0+[\dot{K}(t)-\frac{k_2}{k_0}]\dot{y}_0+[\ddot{K}(t)-\frac{k_2b_1}{k_0b_0}]y_0+\frac{k_2}{c_1k_0b_0}\beta_0\Big\}$$

where $K(t) \equiv \frac{1}{b_0}\ddot{G}(t)+\frac{k_1}{k_0b_0}\dot{G}(t)+\frac{k_2}{k_0b_0}G(t)-\frac{1}{c_1k_0b_0}H(t)$. Thus equation (12) is written as

$$(13) \quad \dddot{y}(t) + \int_0^t K(t-t_1)\dddot{y}(t_1)dt_1 =$$

$$-K(t)\ddot{y}_0 - [\dot{K}(t) - \frac{k_2}{k_0}]\dot{y}_0 - [\ddot{K}(t) - \frac{k_2 b_1}{k_0 b_0}]y_0 + A_1\beta_0$$

$$+A_2 \sum_{n=2}^{\infty} c_n \left\{k_0 \frac{d}{dt}F^n + k_1 F^n + k_2 \int_0^t F^n dt_1\right\}$$

where

$$F = \int_0^t \dot{G}(t-t_1)\dddot{y}(t_1)dt_1 + \dot{G}(t)\ddot{y}_0 + \ddot{G}(t)\dot{y}_0 + \dddot{G}(t)y_0,$$

$$\dot{F} = b_0\dddot{y}(t) + \int_0^t \ddot{G}(t-t_1)\dddot{y}(t_1)dt_1 + \ddot{G}(t)\ddot{y}_0 + \dddot{G}(t)\dot{y}_0,$$

$$K(t) = \frac{k_2}{k_0 b_0}G(t) + \frac{k_1}{k_0 b_0}\dot{G}(t) + \frac{1}{b_0}\ddot{G}(t) - \frac{1}{c_1 k_0 b_0}H(t),$$

$$G(t) = 1/6 b_2 t^3 + 1/2 b_1 t^2 + b_0 t,$$

$$H(t) = 1/2 a_2 t^2 + a_1 t + a_0,$$

$$A_1 = -\frac{k_2}{c_1 k_0 b_0},$$

$$A_2 = -\frac{1}{c_1 k_0 b_0}.$$

Equation (13) is a nonlinear integral equation of the sort treated by E. Schmidt (4, p.370) in his work on nonlinear integral equations except that equation (13) is of Volterra type. This, however, causes no particular difficulty in applying Schmidt's methods.

Consider now the equation

$$u(t) + \int_0^t K(t-t_1)u(t_1)dt_1 = v(t). \tag{14}$$

Since the kernel K is a polynomial it is relatively easy to obtain its "resolvent" kernel in closed form, i.e., the kernel in the equivalent equation

$$v(t) + \int_0^t \Gamma(t-t_1)v(t_1)dt_1 = u(t).$$

In fact, writing

$$K(t) = d_1 + d_2 t + \frac{d_3}{2}t^2 + \frac{d_4}{6}t^3$$

one obtains

$$\Gamma(t) = B_1 e^{r_1 t} + B_2 e^{r_2 t} + B_3 e^{r_3 t} + B_4 e^{r_4 t} \tag{15}$$

where the r_n (n=1,2,3,4) are distinct roots of the equation

$$x^4 + d_1 x^3 + d_2 x^2 + d_3 x + d_4 = 0$$

and the A_n (n=1,2,3,4) satisfy the equations

$$\sum_{n=1}^{4} B_n = -d_1,$$

$$\sum_{n=1}^{4} r_n B_n = -d_2 + d_1^2,$$

$$\sum_{n=1}^{4} r_n^2 B_n = -d_3 + 2d_1 d_2 - d_1^3,$$

$$\sum_{n=1}^{4} r_n^3 B_n = -d_4 + 2d_1 d_3 + d_2^2 - 3d_1^2 d_2 + d_1^4.$$

In the case of multiple roots a similar expression may be obtained for $\Gamma(t)$.

In deriving the expression for $\Gamma(t)$ it is sufficient to suppose $v(t)$ to be a continuous function. Noting that the right side of equation (13) is continuous one can thus employ the resolvent kernel Γ to rewrite equation (13) as

$$(16) \quad \ddot{y}(t) = \alpha_0(t)\ddot{y}_0 + \alpha_1(t)\dot{y}_0 + \alpha_2(t)y_0 + \alpha_3(t)\beta_0$$

$$+ A_2 \sum_{n=2}^{\infty} c_n\Big\{k_0\Big[\frac{d}{dt}F^n + \int_0^t \Gamma(t-t_1)\frac{d}{dt_1}F^n dt_1\Big]$$

$$+ k_1\Big[F^n + \int_0^t \Gamma(t-t_1)F^n dt_1\Big]$$

$$+ k_2\Big[\int_0^t F^n ds + \int_0^t\int_0^{t_1}\Gamma(t-t_1)F^n ds\, dt_1\Big]\Big\}$$

where

$$\alpha_0 = -K(t) - \int_0^t \Gamma(t-t_1)K(t_1)dt_1,$$

$$\alpha_1 = -[\dot{K}(t) - \frac{k_2}{k_0}] - \int_0^t \Gamma(t-t_1)[\dot{K}(t_1) - \frac{k_2}{k_0}]dt_1,$$

$$\alpha_2 = -[\ddot{K}(t) - \frac{k_2 b_1}{k_0 b_0}] - \int_0^t \Gamma(t-t_1)[\ddot{K}(t_1) - \frac{k_2 b_1}{k_0 b_0}]dt_1,$$

$$\alpha_3 = A_1[1 + \int_0^t \Gamma(t-t_1)dt_1].$$

The right-hand side of equation (16) is an "integral power series" in the arguments $\ddot{y}(t)$, y_0, $\dot{y}_0$, $\ddot{y}_0$ and β_0 which contains no term linear in $\ddot{y}(t)$ alone. (Note that if $\ddot{y}(t)$ be replaced by a constant x the general term in the series of (16) is of nth degree in the arguments x, y_0, $\dot{y}_0$, $\ddot{y}_0$. Thus the only linear terms on the right side of (16) are the first four.)

Making use of the expansion

$$(A + B + C + D)^n = \sum_{\sigma_m = n} \frac{n!}{m_0!m_1!m_2!m_3!} A^{m_0}B^{m_1}C^{m_2}D^{m_3}$$

$$(\sigma_m = m_0+m_1+m_2+m_3),$$

the series in equation (16) becomes

$$
\begin{aligned}
&\sum_{n=2}^{\infty} c_n \sum_{\sigma_m=n} \frac{n!}{m_0!\,m_1!\,m_2!\,m!}\, \ddot{y}_0^{\,m_0}\, \dot{y}_0^{\,m_1}\, y_0^{\,m_2} \times \\
&\quad \times \Big\{ k_0 \Big[\frac{d}{dt}\Big(\dot{G}^{m_0} \ddot{G}^{m_1} \dddot{G}^{m_2} \Big(\int_0^t \dot{G}\ddot{y}\,dt_1 \Big)^m \Big) \\
&\qquad\qquad + \int_0^t \Gamma \cdot \frac{d}{dt_1}\Big(\dot{G}^{m_0} \ddot{G}^{m_1} \dddot{G}^{m_2} \Big(\int_0^{t_1} \dot{G}\ddot{y}\,dt_2 \Big)^m \Big) dt_1 \Big] \\
&\quad + k_1 \Big[\dot{G}^{m_0} \ddot{G}^{m_1} \dddot{G}^{m_2} \Big(\int_0^t \dot{G}\ddot{y}\,dt_1 \Big)^m \\
&\qquad\qquad + \int_0^t \Gamma\, \dot{G}^{m_0} \ddot{G}^{m_1} \dddot{G}^{m_2} \Big(\int_0^{t_1} \dot{G}\ddot{y}\,dt_2 \Big)^m dt_1 \Big] \\
&\quad + k_2 \Big[\int_0^t \dot{G}^{m_0} \ddot{G}^{m_1} \dddot{G}^{m_2} \Big(\int_0^{t_1} \dot{G}\ddot{y}\,dt_2 \Big)^m dt_1 \\
&\qquad\qquad + \int_0^t \Gamma \int_0^{t_1} \dot{G}^{m_0} \ddot{G}^{m_1} \dddot{G}^{m_2} \Big(\int_0^{t_2} \dot{G}\ddot{y}\,dt_3 \Big)^m dt_2\, dt_1 \Big] \Big\}.
\end{aligned}
$$

Then noting that $\dot{G}(0) = b_0$, equation (16) is written

$$(17)\quad \ddot{y} = \alpha_0\ddot{y}_0 + \alpha_1\dot{y}_0 + \alpha_2 y_0 + \alpha_3\beta_0$$

$$+ A_2 \sum_{n=2}^{\infty} c_n \sum_{\sigma_m = n} \frac{n!}{m_0!m_1!m_2!m!} \times$$

$$\times\Big\{mk_0[g_M(b_0\ddot{y} + \int_0^t \dot{G}\ddot{y}dt_1)(\int_0^t \dot{G}\ddot{y}dt_1)^{m-1}$$

$$+ \int_0^t \Gamma g_M(b_0\ddot{y} + \int_0^{t_1}\dot{G}\ddot{y}dt_2)(\int_0^{t_1}\dot{G}\ddot{y}dt_2)^{m-1}dt_1]$$

$$+ (k_0\dot{g}_M + k_1 g_M)(\int_0^t \dot{G}\ddot{y}dt_1)^m$$

$$+ \int_0^t [(k_0\dot{g}_M + k_1 g_M)\Gamma + k_2 g_M](\int_0^{t_1}\dot{G}\ddot{y}dt_2)^m dt_1$$

$$+ k_2\int_0^t \Gamma \int_0^{t_1} g_M(\int_0^{t_2}\dot{G}\ddot{y}dt_3)^m dt_2 dt_1\Big\}\ddot{y}_0^{m_0}\dot{y}_0^{m_1}y_0^{m_2}$$

where, for brevity, $g_M(t) \equiv \dot{G}^{m_0}(t)\cdot\ddot{G}^{m_1}(t)\cdot\dddot{G}^{m_2}(t)$.

A dominant series will now be obtained for the series of (17).

On replacing the "argument function" $\ddot{y}(t)$ by the positive constant u and the arguments $\ddot{y}_0$, $\dot{y}_0$, y_0, β_0 by the positive constants v_0, v_1, v_2, v_3, respectively, in equation (17) one has

$$u = \alpha_o v_o + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$+ A_3 \sum_{n=2}^{\infty} c_n \sum_{\sigma_m = n} \frac{n!}{m_o!m_1!m_2!m!} \times$$

$$\times \Big\{ mk_o [g_M (b_o + \int_o^t \ddot{G} dt_1)(\int_o^t \dot{G} dt_1)^{m-1}$$

$$+ \int_o^t \Gamma g_M (b_o + \int_o^{t_1} \ddot{G} dt_2)(\int_o^{t_1} \dot{G} dt_2)^{m-1} dt_1]$$

$$+ (k_o \dot{g}_M + k_1 g_M)(\int_o^t \dot{G} dt_1)^m$$

$$+ \int_o^t [(k_o \dot{g}_M + k_1 g_M)\Gamma + k_2 g_M](\int_o^{t_1} \dot{G} dt_2)^m dt_1$$

$$+ k_2 \int_o^t \Gamma \int_o^{t_1} g_M (\int_o^{t_2} \dot{G} dt_3)^m dt_2 dt_1 \Big\} v_o^{m_o} v_1^{m_1} v_2^{m_2} u^m .$$

Denoting by $\tilde{f}$ the maximum of the absolute value of a function $f(t)$ on the interval $[0, T]$ one obtains the associated equation

$$u = \tilde{\alpha}_o v_o + \tilde{\alpha}_1 v_1 + \tilde{\alpha}_2 v_2 + \tilde{\alpha}_3 v_3$$

$$+ |A_3| \sum_{\sigma_m > 1} |c_{\sigma_m}| \frac{\sigma_m!}{m_o!m_1!m_2!m!} \times$$

$$\times \Big\{ m|k_o| [\tilde{g}_M (|b_o| + \int_o^T \tilde{\ddot{G}} dt_1)(\int_o^T \tilde{\dot{G}} dt_1)^{m-1}$$

(continued on next page)

$$+ \int_0^T \tilde{\Gamma}\tilde{g}_M(|b_0|+\int_0^T \tilde{\ddot{G}}dt_2)(\int_0^{t_1}\tilde{\ddot{G}}dt_2)^{m-1}dt_1]$$

$$+ (|k_0|\tilde{\dot{g}}_M + |k_1|\tilde{g}_M)(\int_0^T \tilde{\dot{G}}dt_1)^m$$

$$+ \int_0^T [(|k_0|\tilde{\dot{g}}_M + |k_1|\tilde{g}_M)\tilde{\Gamma} + |k_2|\tilde{g}_M](\int_0^{t_1}\tilde{\dot{G}}dt_2)^m dt_1$$

$$+ |k_2|\int_0^T \tilde{\Gamma}\int_0^{t_1}\tilde{g}_M(\int_0^T \tilde{\dot{G}}dt_2)^m dt_2 dt_1\Big\} v_0^{m_0} v_1^{m_1} v_2^{m_2} u^m$$

where it is understood $\tilde{g}_M \equiv \tilde{\dot{G}}^{m_0}\tilde{\ddot{G}}^{m_1}\tilde{\dddot{G}}^{m_2}$ and
$\tilde{\dot{g}}_M \equiv m_0\tilde{\dot{G}}^{m_0-1}\tilde{\ddot{G}}^{m_1+1}\tilde{\dddot{G}}^{m_2} + m_1\tilde{\dot{G}}^{m_0}\tilde{\ddot{G}}^{m_1-1}\tilde{\dddot{G}}^{m_2+1}$.

Or

$$(18) \quad u = \tilde{\alpha}_0 v_0 + \tilde{\alpha}_1 v_1 + \tilde{\alpha}_2 v_2 + \tilde{\alpha}_3 v_3$$

$$+ |A_2| \sum_{\sigma_m > 1} |c_{\sigma_m}| \frac{\sigma_m!}{m_0! m_1! m_2! m!} \times$$

$$\times \Big\{(1+\tilde{\Gamma}\; T)u|k_0|(|b_0|+\tilde{\ddot{G}}T)mX^{m-1}X_0^{m_0}X_1^{m_1}X_2^{m_2}$$

$$+(1+\tilde{\Gamma}\; T)\frac{v_0}{v_1}|k_0|m_0 X^m X_0^{m_0-1}X_1^{m_1+1}X_2^{m_2}$$

$$+(1+\tilde{\Gamma}\; T)\frac{v_1}{v_2}|k_0|m_1 X^m X_0^{m_0}X_1^{m_1-1}X_2^{m_2+1}$$

$$+[|k_2|(\tilde{\Gamma}\frac{T^2}{2} + T)+|k_1|(1+\tilde{\Gamma}\; T)]X^m X_0^{m_0}X_1^{m_1}X_2^{m_2}\Big\}$$

where

$$X \equiv \tilde{\dot{G}}Tu, \quad X_0 \equiv \tilde{\dot{G}}v_0, \quad X_1 \equiv \tilde{\ddot{G}}v_1, \quad X_2 \equiv \tilde{\ddot{G}}v_2.$$

Now, the series on the right side of (18) dominates that on the right side of (17) whenever $0 \leq t \leq T$ and

$$|\ddot{y}_0| < v_0, \quad |\dot{y}_0| < v_1, \quad |y_0| < v_2, \quad \tilde{y} < u$$

so that the series (17) converges uniformly and absolutely subject to these conditions provided the series (18) converges. But (18) can be rewritten

$$\begin{aligned}
u = {} & \tilde{\alpha}_0 v_0 + \tilde{\alpha}_1 v_1 + \tilde{\alpha}_2 v_2 + \tilde{\alpha}_3 v_3 \\
& + |A_2| \Big\{ \Big[|k_2| \Big(\tilde{\Gamma} \frac{T^2}{2} + T\Big) + |k_1| (1 + \tilde{\Gamma}\, T) \Big] \sum_{n=2}^{\infty} |c_n| (X + X_0 + X_1 + X_2)^n \\
& \quad + \big[(1 + \tilde{\Gamma}\, T) u |k_0| (|b_0| + \tilde{\ddot{G}} T) \big] \sum_{n=2}^{\infty} \frac{\partial}{\partial X} |c_n| (X + X_0 + X_1 + X_2)^n \\
& \quad + \frac{(1 + \tilde{\Gamma}\, T) v_0 |k_0| X_1}{v_1} \sum_{n=2}^{\infty} \frac{\partial}{\partial X_0} |c_n| (X + X_0 + X_1 + X_2)^n \\
& \quad + \frac{(1 + \tilde{\Gamma}\, T) v_1 |k_0| X_2}{v_2} \sum_{n=2}^{\infty} \frac{\partial}{\partial X_1} |c_n| (X + X_0 + X_1 + X_2)^n \Big\}
\end{aligned}$$

each of which converges for all values of v_0, v_1, v_2,

$\tilde{\dot{G}}$, $\tilde{\ddot{G}}$, $\tilde{\dddot{G}}$, T, u, by the definition of the c_n. Thus the series (17) converges for all $\ddot{y}_o$, $\dot{y}_o$, y_o, $\ddot{y}$ and $t \geqq 0$ and has the series (18) as a majorant.

Equation (17) is analogous to the equation

$$y = B_o x_o + B_1 x_1 + B_2 x_2 + B_3 x_3$$
$$+ \sum B_{m_o m_1 m_2 m_3 n} \, x_o^{m_o} x_1^{m_1} x_2^{m_2} x_3^{m_3} y^n$$

which, by the implicit function theorem, can be solved for y as a power series in the x_j provided certain conditions are fulfilled. Schmidt's result is analogous to this implicit function theorem.

CHAPTER IV

THE SOLUTION OF THE INTEGRAL EQUATION

Formal solution of equation (17). A formal solution of equation (17) is now obtained in the form of an infinite series. In succeeding sections conditions are obtained such that this series converges and represents the solution to (17). Substitute for $\ddot{y}$ in (17) the series

$$(19) \qquad \ddot{y} = \sum_{\sigma_p \geq 1} v_{p_0 p_1 p_2 p_3}(t) \ddot{y}_0^{p_0} \dot{y}_0^{p_1} y_0^{p_2} \beta_0^{p_3} \qquad (\sigma_p = p_0 + p_1 + p_2 + p_3),$$

where the coefficient functions $v_{m_0 m_1 m_2 m_3}$ are as yet undetermined. On equating coefficients of like powers in $\ddot{y}_0$, $\dot{y}_0$, y_0 and β_0 the $v_{m_0 m_1 m_2 m_3}$ can be determined.

Since $\sigma_m > 1$ in the series of equation (17) the only terms which are linear on the right side of (17) are the first four. Hence the linear terms are determined as

$$(20) \qquad v_{1000} = \alpha_0, \quad v_{0100} = \alpha_1, \quad v_{0010} = \alpha_2, \quad v_{0001} = \alpha_3.$$

On substituting (19) for $\ddot{y}$ on the right side of (17), the coefficient of $\ddot{y}_0^{n_0} \dot{y}_0^{n_1} y_0^{n_2} \beta_0^{n_3}$ is obtained and equated to $v_{n_0n_1n_2n_3}(t)$. It can be seen that the expression for $v_{n_0n_1n_2n_3}(t)$ will be in terms of only those $v_{p_0p_1p_2p_3}(t)$ for which $\sigma_p < \sigma_n$. This is immediately apparent in those terms of (17) for which $m = 0$, for in those terms $\ddot{y}$ does not even appear. For $m \geq 1$ consider the formal substitution of (19) for $\ddot{y}$ in those terms of (17) involving $(\int_0^t \dot{G}\ddot{y}dt_1)^m$:

$$\left(\int_0^t \dot{G} \sum_{\sigma_p \geq 1} v_{p_0p_1p_2p_3} \ddot{y}_0^{p_0} \dot{y}_0^{p_1} y_0^{p_2} \beta_0^{p_3} dt_1\right)^m$$
$$= \left(\sum_{\sigma_p \geq 1} \int_0^t \dot{G} v_{p_0p_1p_2p_3} dt_1 \ddot{y}_0^{p_0} \dot{y}_0^{p_1} y_0^{p_2} \beta_0^{p_3}\right)^m .$$

Letting $V_{p_0p_1p_2p_3} = \int_0^t \dot{G} v_{p_0p_1p_2p_3} dt_1$ this may be written as

$$\left(\sum_{\sigma_p \geq 1} V_{p_0p_1p_2p_3} \ddot{y}_0^{p_0} \dot{y}_0^{p_1} y_0^{p_2} \beta_0^{p_3}\right)^m$$

or, denoting $n_0 + n_1 + n_2 + n_3$ by σ_n, as

$$\sum_{j=0}^{m} \binom{m}{j} \left(\sum_{1 \leq \sigma_p < \sigma_n} V_{p_0p_1p_2p_3} \ddot{y}_0^{p_0} \dot{y}_0^{p_1} y_0^{p_2} \beta_0^{p_3}\right)^{m-j} \times$$
$$\left(\sum_{\sigma_p \geq \sigma_n} V_{p_0p_1p_2p_3} \ddot{y}_0^{p_0} \dot{y}_0^{p_1} y_0^{p_2} \beta_0^{p_3}\right)^{j} .$$

Now, for m = 1 one has for this term

$$\sum_{1 \leq \sigma_p < \sigma_n} V_{p_0 p_1 p_2 p_3} \ddot{y}_0^{p_0} \dot{y}_0^{p_1} y_0^{p_2} \beta_0^{p_3}$$

$$+ \sum_{\sigma_p \geq \sigma_n} V_{p_0 p_1 p_2 p_3} \ddot{y}_0^{p_0} \dot{y}_0^{p_1} y_0^{p_2} \beta_0^{p_3}.$$

But this is multiplied by $\ddot{y}_0^{m_0} \dot{y}_0^{m_1} y_0^{m_2}$ where $m_0 + m_1 + m_2 > 0$ so that the second term, i.e., the infinite sum for which $\sigma_p \geq \sigma_n$, cannot contribute to the evaluation of terms $V_{n_0 n_1 n_2 n_3}$ for which $n_0 + n_1 + n_2 + n_3 = \sigma_n$. For m > 1 the argument is very much the same. In the expression

$$\sum_{j=0}^{m} \binom{m}{j} \left(\sum_{1 \leq \sigma_p < \sigma_n} V_{p_0 p_1 p_2 p_3} \ddot{y}_0^{p_0} \dot{y}_0^{p_1} y_0^{p_2} \beta_0^{p_3} \right)^{m-j}$$

$$\left(\sum_{\sigma_p \geq \sigma_n} V_{p_0 p_1 p_2 p_3} \ddot{y}_0^{p_0} \dot{y}_0^{p_1} y_0^{p_2} \beta_0^{p_3} \right)^{j}$$

these terms for which j = 0 do not involve the V's for which $\sigma_p \geq \sigma_n$. For j = 1 these V's appear and are each multiplied by terms for which $\sigma_p \geq 1$ which results in terms for which $\sigma_p > \sigma_n$ and thus cannot contribute to the evaluation of the $v_{n_0 n_1 n_2 n_3}$ for which $n_0+n_1+n_2+n_3=\sigma_n$. For j > 1 this is even more true.

Hence for that part of the evaluation of $v_{n_0 n_1 n_2 n_3}$ from terms involving $(\int_0^t \dot{G}\ddot{y}dt_1)^m$ it is not necessary to substitute the whole infinite series (19) for $\ddot{y}$ but only that part of it for which $\sigma_p < \sigma_n$. It can be shown in a similar manner that the same statement is true for those terms involving $\ddot{y}(\int_0^t \dot{G}\ddot{y}dt_1)^{m-1}$ or $\int_0^t \ddot{G}\ddot{y}dt_1(\int_0^t \dot{G}\ddot{y}dt_1)^{m-1}$. This proves the statement made in the last paragraph.

That is, one first evaluates those $v_{n_0 n_1 n_2 n_3}$ for which $\sigma_n = 1$, then those for which $\sigma_n = 2$ in terms of the preceding ones, then those for which $\sigma_n = 3$, etc., so that expressions are obtained which define the $v_{n_0 n_1 n_2 n_3}$ explicitly instead of ones which define them by means of implicit relations.

<u>Majorant for the solution series (19).</u> Equation (17) can be rewritten in the form

$$\ddot{y}(t) = \alpha_0\ddot{y}_0 + \alpha_1\dot{y}_0 + \alpha_2 y_0 + \alpha_3\beta_0 + \sum_{\sigma_p>1} \alpha_{p_0 p_1 p_2 p}\begin{pmatrix} t \\ \ddot{y} \end{pmatrix} \ddot{y}_0^{p_0}\dot{y}_0^{p_1}y_0^{p_2} \tag{21}$$

where p is the "power" of $\ddot{y}$ in the general term and the general term is given by

$$(22)\qquad \alpha_{m_0 m_1 m_2 m} = A_2 c_{\sigma_m} \frac{\sigma_m!}{m_0!m_1!m_2!m!} \times$$

$$\times \Big\{ mk_0 [g_M(b_0\ddot{y}+\int_0^t \ddot{G}\ddot{y}dt_1)(\int_0^t \dot{G}\ddot{y}dt_1)^{m-1}$$

$$+ \int_0^t \Gamma g_M(b_0\ddot{y}+\int_0^{t_1}\ddot{G}\ddot{y}dt_2)(\int_0^{t_1}\dot{G}\ddot{y}dt_2)^{m-1}dt_1]$$

$$+ (k_0\dot{g}_M + k_1 g_M)(\int_0^t \dot{G}\ddot{y}dt_1)^m$$

$$+ \int_0^t [(k_0\dot{g}_M + k_1 g_M)\Gamma + k_2 g_M](\int_0^{t_1}\dot{G}\ddot{y}dt_2)^m dt_1$$

$$+ k_2\int_0^t \Gamma \int_0^{t_1} g_M(\int_0^{t_2}\dot{G}\ddot{y}dt_3)^m dt_2 dt_1\Big\}.$$

Absolute and uniform convergence of (21) is implied by that of (17).

Consider the equation

$$(23)\qquad x(t) = |\alpha_0|\ddot{y}_0 + |\alpha_1|\dot{y}_0 + |\alpha_2|y_0 + |\alpha_2|\beta_0$$

$$+ \sum_{\sigma_p > 1} |\alpha|_{p_0 p_1 p_2 p} \binom{t}{x} \ddot{y}_0^{p_0}\dot{y}_0^{p_1}y_0^{p_2}$$

where $|\alpha|_{m_0 m_1 m_2 m}$ is obtained from $\alpha_{m_0 m_1 m_2 m}$ by replacing coefficient functions by their absolute values, i.e., by replacing A_2, c_{σ_m}, b_0, k_j, $\dot{G}$, $\ddot{G}$, $\dddot{G}$, $\dot{G}_0$, Γ by their respective absolute values. Convergence of this series for all values of $\ddot{y}_o$, $\dot{y}_o$, y_o, x, and $t \geq 0$ follows from that of the series (18). The formal solution of (23),

$$(24) \qquad x = \sum_{\sigma_m \geq 1} w_{m_o m_1 m_2 m_3}(t) \ddot{y}_o^{m_o} \dot{y}_o^{m_1} y_o^{m_2} \beta_o^{m_3},$$

may be obtained in the same way as was (19). In fact, $w_{m_o m_1 m_2 m_3}$ is obtained from $v_{m_o m_1 m_2 m_3}$ on replacing A_2, c_{σ_m}, b_o, k_j, $\dot{G}$, $\ddot{G}$, $\dddot{G}$, $\dot{G}_e$, Γ by their absolute values, respectively. Hence it follows that

$$(25) \qquad |v|_{m_o m_1 m_2 m_3} \leq w_{m_o m_1 m_2 m_3} \qquad (0 \leq t \leq T)$$

so that if series (24) converges, so does the solution series (19).

Consider next the equation

$$(26) \qquad z = \tilde{\alpha}_o \ddot{y}_o + \tilde{\alpha}_1 \dot{y}_o + \tilde{\alpha}_2 y_o + \tilde{\alpha}_3 \beta_o + \sum_{\sigma_p > 1} \widetilde{|\alpha|}_{p_o p_1 p_2 p} \ddot{y}_o^{p_o} \dot{y}_o^{p_1} y_o^{p_2} z^p$$

where

$$\tilde{\alpha}_j = \max_{0 \leq t \leq T} |\alpha_j(t)|,$$

$$\widetilde{|\alpha|}_{p_o p_1 p_2 p} = \max_{0 \leq t \leq T} |\alpha|_{p_o p_1 p_2 p} \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

Again, convergence for all values of $\ddot{y}_o$, $\dot{y}_o$, y_o and z is ensured by the convergence of the series (18).

The solution series for (26),

$$z = \sum_{\sigma_m \geq 1} B_{m_0 m_1 m_2 m_3} \ddot{y}_0^{m_0} \dot{y}_0^{m_1} y_0^{m_2} \beta_0^{m_3}, \tag{27}$$

may also be obtained by equating coefficients, the $B_{m_0 m_1 m_2 m_3}$ being polynomials in a finite number of the $\widetilde{[\alpha]}_{p_0 p_1 p_2 p}$ with positive integral coefficients.

Before discussing the series (27), which exists as a solution to equation (26) by the implicit function theorem, it will be shown that

$$\tilde{w}_{m_0 m_1 m_2 m_3} \leqq B_{m_0 m_1 m_2 m_3} \tag{28}$$

so that the series (27) dominates the solution series (19). First of all

$$B_{1000} = \tilde{\alpha}_0 = \tilde{w}_{1000}, \qquad B_{0100} = \tilde{\alpha}_1 = \tilde{w}_{0100},$$

$$B_{0010} = \tilde{\alpha}_2 = \tilde{w}_{0010}, \qquad B_{0001} = \tilde{\alpha}_3 = \tilde{w}_{0001}.$$

For the proof of (28) it remains then to show that the inequalities

$$\tilde{w}_{m_0 m_1 m_2 m_3} \leqq B_{m_0 m_1 m_2 m_3} \qquad (\sigma_m = 1, 2, \ldots, n-1) \tag{29}$$

imply the validity of

$$\tilde{w}_{m_0 m_1 m_2 m_3} \leqq B_{m_0 m_1 m_2 m_3} \qquad (\sigma_m = n). \tag{30}$$

For the evaluation of the coefficients $w_{p_0p_1p_2p_3}$ where $\sigma_p = n$ one substitutes for x on the right side of (23) the sum

$$\sum_{1\leq\sigma_m<n} w_{m_0m_1m_2m_3}(t)\ddot{y}_0^{m_0}\dot{y}_0^{m_1}y_0^{m_2}\beta_0^{m_3}.$$

If, instead, the sum

$$(31)\qquad S_{n-1} = \sum_{1\leq\sigma_m<n} B_{m_0m_1m_2m_3}\ddot{y}_0^{m_0}\dot{y}_0^{m_1}y_0^{m_2}\beta_0^{m_3}$$

were substituted, the resulting coefficients $k_{m_0m_1m_2m_3}$ of terms in $\ddot{y}_0^{m_0}\dot{y}_0^{m_1}y_0^{m_2}\beta_0^{m_3}$ for which $\sigma_m = n$ would be greater than the corresponding coefficients $w_{m_0m_1m_2m_3}$, respectively, because of inequalities (29). Now

$$|\alpha|_{m_0m_1m_2m}\begin{pmatrix} t \\ S_{n-1}\end{pmatrix}\ddot{y}_0^{m_0}\dot{y}_0^{m_1}y_0^{m_2}$$

$$= |\alpha|_{m_0m_1m_2m}\begin{pmatrix} t \\ 1\end{pmatrix}\ddot{y}_0^{m_0}\dot{y}_0^{m_1}y_0^{m_2}S_{n-1}^{m}$$

and

$$|\alpha|_{m_0m_1m_2m}\begin{pmatrix} t \\ 1\end{pmatrix} \leq \widetilde{|\alpha|}_{m_0m_1m_2m}$$

so that the coefficients $k_{m_0m_1m_2m_3}$ $(\sigma_m = n)$ are each less than or equal to the corresponding coefficients obtained by substituting S_{n-1} for z on the right side of equation (26). But these coefficients are the $B_{m_0 m_1 m_2 m_3}$. Hence $w_{m_0 m_1 m_2 m_3}(t) \leqq B_{m_0 m_1 m_2 m_3}$ (with $\sigma_m = n$) for each value of t in $[0, T]$ and the result (30) follows. This completes the proof of (28).

It is now possible to give conditions which ensure the convergence of the solution series (19). For, from inequalities (25) and (28)

$$(32) \quad |v_{m_0 m_1 m_2 m_3}(t)| \leqq |v|_{m_0 m_1 m_2 m_3}(t)$$

$$\leqq w_{m_0 m_1 m_2 m_3}(t) \leqq \tilde{w}_{m_0 m_1 m_2 m_3} \leqq B_{m_0 m_1 m_2 m_3}$$

if $0 \leqq t \leqq T$ so that whenever the series (27) converges so does the solution series (19). In fact convergence of (27) implies absolute and uniform convergence of (19).

Convergence of the dominant series (27). Returning to the question of convergence of series (27) it should be noted that the coefficients $B_{m_0 m_1 m_2 m_3}$ are polynomials in a finite number of the $\widetilde{|\alpha|}_{p_0 p_1 p_2 p}$ with positive integral coefficients, since they are obtained by a process which consists only of substitution followed by additions and multiplications. Now, let the three positive constants M, r, R be chosen such that

$$(33)\quad \begin{cases} |\widetilde{\alpha}|_{p_0 p_1 p_2 p} \leqq \dfrac{(p_0+p_1+p_2+p_3)!}{p_0!p_1!p_2!p_3!} \dfrac{M}{r^{p_0+p_1+p_2+p_3} R^p} \, . \\ \tilde{\alpha}_j \leqq \dfrac{M}{r} \qquad (j = 0,\ 1,\ 2,\ 3). \end{cases}$$

Then the series

$$(34)\quad X = \frac{M}{r}(\ddot{y}_0+\dot{y}_0+y_0+\beta_0)$$

$$+ \sum_{\sigma'_p>1} \frac{(p_0+p_1+p_2+p_3)!}{p_0!p_1!p_2!p_3!} \frac{M\ddot{y}_0^{p_0}\dot{y}_0^{p_1}y_0^{p_2}\beta_0^{p_3}}{r^{p_0+p_1+p_2+p_3}R^p} X^p$$

$$(\sigma'_p = p_0+p_1+p_2+p_3+p)$$

dominates the series (26). That this can be done, resulting in a convergent series, follows since (26) is a convergent series. The formal solution series for (34) again has coefficients which are polynomials $P_{m_0 m_1 m_2 m_3}$ in a finite number of the coefficients which appear on the right side of (34). The only difference between these and the corresponding polynomials in the solution of (27) is that in the latter some of the terms are missing which do appear in the former. In light of this, the inequalities (31), and the fact that the polynomials possess positive integral coefficients it follows that

$$(35)\qquad B_{m_0 m_1 m_2 m_3} \leqq P_{m_0 m_1 m_2 m_3} .$$

Series (34), however, is the expansion of the right side of

$$(36) \qquad X = \frac{M}{\left(1 - \frac{\ddot{y}_0 + \dot{y}_0 + y_0 + \beta_0}{r}\right)\left(1 - \frac{X}{R}\right)} - M - M\,\frac{X}{R}.$$

Equation (36) has the equivalent form, for $|X| < R$ and $|\ddot{y}_0 + \dot{y}_0 + y_0 + \beta_0| < r$,

$$(37) \qquad \frac{M+R}{R^2}\cdot X^2 - X + \frac{M\frac{Y}{r}}{1-\frac{Y}{r}} = 0 \qquad (Y \equiv \ddot{y}_0+\dot{y}_0+y_0+\beta_0)$$

which has

$$(38) \qquad X = \frac{1 - \left(1 - 4\cdot\frac{M+R}{R^2}\cdot\frac{M\frac{Y}{r}}{1-\frac{Y}{r}}\right)^{\frac{1}{2}}}{2\cdot\frac{M+R}{R^2}}$$

as the only solution which vanishes with Y.

An equivalent form for (38) is

$$(39) \qquad X = \frac{1 - \left(1 - \frac{Y}{r}\right)^{-\frac{1}{2}}\cdot\left(1 - \frac{Y}{\alpha}\right)^{\frac{1}{2}}}{2\cdot\frac{M+R}{R^2}}, \qquad \alpha = r\left(\frac{R}{R+2M}\right)^2$$

which may be expanded into a convergent series in $\ddot{y}_0$, $\dot{y}_0$, y_0, β_0 provided

$$|\ddot{y}_0 + \dot{y}_0 + y_0 + \beta_0| < \alpha. \tag{40}$$

This series for the right side of (39) must then coincide with the formal series solution of (34). Hence (27) will converge whenever inequality (40) is satisfied. Thus the series solution (19) to the original equation (17) will also converge, absolutely and uniformly, under this condition.

Since uniform and absolute convergence of (19) and of (17) justifies the operations carried out in evaluating the solution series (19) it follows that (19) is a solution of equation (17) and hence of equation (13). Due to the uniform convergence of the series (19) for $\dddot{y}$ it follows that y can be obtained from (19) by integration. That is,

$$y = \sum_{\sigma_p \geq 1} \left\{ \int_0^t \int_0^{t_1} \int_0^{t_2} v_{p_0 p_1 p_2 p_3}(t_3)\,dt_3\,dt_2\,dt_1\, \ddot{y}_0^{p_0} \dot{y}_0^{p_1} y_0^{p_2} \beta_0^{p_3} \right\}$$

$$+ \frac{1}{2}\ddot{y}_0 t^2 + \dot{y}_0 t + y_0,$$

and this is the solution to the differential equation (7) with the specified initial conditions.

CHAPTER V

UNIQUENESS OF THE SOLUTION

It will now be shown that there exists a constant $h > 0$ such that if $|\ddot{y}_0| < h$, $|\dot{y}_0| < h$ and $|y_0| < h$ then a solution of (17) which is bounded by h must be unique.

For, suppose u and $u + v$ to be two bounded solutions to (17) with $v(t) \not\equiv 0$. Then from (17), subtracting u from $u + v$,

$$(41) \quad v = A_2 \sum_{\sigma_m > 1} c_{\sigma_m} \frac{\sigma_m!}{m_0!m_1!m_2!m!} \ddot{y}_0^{m_0} \dot{y}_0^{m_1} y_0^{m_2} \times$$

$$\times \Big\{ m k_0 g_M \Big[\Big(b_0[u+v] + \int_0^t \ddot{G}[u+v]dt_1\Big)\Big(\int_0^t \dot{G}[u+v]dt_1\Big)^{m-1}$$

$$- \Big(b_0 u + \int_0^t \ddot{G}u dt_1\Big)\Big(\int_0^t \dot{G}u dt_1\Big)^{m-1}\Big]$$

$$+ m k_0 \int_0^t \Gamma g_M \Big[\Big(b_0[u+v] + \int_0^{t_1} \ddot{G}[u+v]dt_2\Big)\Big(\int_0^{t_1} \dot{G}[u+v]dt_1\Big)^{m-1}$$

$$- \Big(b_0 u + \int_0^{t_1} \ddot{G}u dt_2\Big)\Big(\int_0^{t_1} \dot{G}u dt_2\Big)^{m-1}\Big]dt_1$$

$$+ (k_0 \dot{g}_M + k_1 g_M)\Big[\Big(\int_0^t \dot{G}[u+v]dt_1\Big)^m - \Big(\int_0^t \dot{G}u dt_1\Big)^m\Big]$$

(continued on next page)

$$+\int_0^t [(k_0\dot g_M + k_1 g_M)\Gamma + k_2 g_M][(\int_0^{t_1}\dot G[u+v]dt_2)^m$$

$$-(\int_0^{t_1}\dot G u dt_2)^m]dt_1$$

$$+k_2\int_0^t \Gamma \int_0^{t_1} g_M[(\int_0^{t_2}\dot G[u+v]dt_3)^m - (\int_0^{t_2}\dot G u dt_3)^m]dt_2 dt_1\}.$$

The right-hand side of (41) can be expressed in terms of "powers" of u. That is, for terms involving $(\int_0^s \dot G\cdot[u+v]ds_1)^m - (\int_0^s \dot G u ds_1)^m$ one has

$$(\int_0^s \dot G u ds_1 + \int_0^s \dot G v ds_1)^m - (\int_0^s \dot G u ds_1)^m$$

$$= \sum_{k=0}^{m-1}\binom{m}{k}(\int_0^s \dot G u ds_1)^k(\int_0^s \dot G v ds_1)^{m-k}$$

$$= \int_0^s \dot G v ds_1 \sum_{k=0}^{m-1}\binom{m}{k}(\int_0^s \dot G u ds_1)^k(\int_0^s \dot G v ds_1)^{m-k-1}$$

so that for $0 \leq s \leq T$

$$|(\int_0^s \dot G\cdot[u+v]ds_1)^m - (\int_0^s \dot G u ds_1)^m|$$

$$\leq \tilde v\int_0^s |\dot G|ds_1 \sum_{k=0}^{m-1}\binom{m}{k}(\int_0^s |\dot G|ds_1)^k(\int_0^s |\dot G|ds_1)^{m-k-1}\ \tilde u^k\ \tilde v^{m-k-1}$$

$$\leq \tilde v(\int_0^s |\dot G|ds_1)^m \sum_{k=0}^{m-1}\binom{m}{k}\tilde u^k\ \tilde v^{m-k-1}$$

$$\leq \tilde v\ (\tilde{\dot G}T)^m\ m(\tilde u + \tilde v)^{m-1}.$$

Similarly,

$$|(b_o[u+v] + \int_o^s \ddot{G}[u+v]ds_1)(\int_o^s \dot{G}[u+v]ds_1)^{m-1}$$

$$- (b_o u + \int_o^s \ddot{G}uds_1)(\int_o^s \dot{G}uds_1)^{m-1}|$$

$$= |(b_o[u+v]+\int_o^s \ddot{G}[u+v]ds_1) \sum_{k=0}^{m-1} \binom{m-1}{k} (\int_o^s \dot{G}uds_1)^k (\int_o^s \dot{G}vds_1)^{m-k-1}$$

$$- (b_o u + \int_o^s \ddot{G}uds_1)(\int_o^s \dot{G}uds_1)^{m-1}|$$

$$= |(b_o v + \int_o^s \ddot{G}vds_1)(\int_o^s \dot{G}uds_1)^{m-1}$$

$$+(b_o[u+v]+\int_o^s \ddot{G}[u+v]ds_1) \sum_{k=0}^{m-2} \binom{m-1}{k} (\int_o^s \dot{G}uds_1)^k (\int_o^s \dot{G}vds_1)^{m-k-1}|$$

$$\leq \tilde{v}\, \tilde{u}^{m-1} (|b_o| + \int_o^s |\ddot{G}|ds_1)(\int_o^s |\dot{G}|ds_1)^{m-1}$$

$$+ (\tilde{u}+\tilde{v})(|b_o| + \int_o^s |\ddot{G}|ds_1) \sum_{k=0}^{m-2} \binom{m-1}{k} \tilde{u}^k \tilde{v}^{m-k-1} (\int_o^s |\dot{G}|ds_1)^{m-1}$$

$$\leq \tilde{v}\, (|b_o| + \int_o^s |\ddot{G}|ds_1)(\int_o^s |\dot{G}|ds_1)^{m-1} [\tilde{u}^{m-1}$$

$$+ (\tilde{u}+\tilde{v}) \sum_{k=0}^{m-2} \binom{m-1}{k} \tilde{u}^k \tilde{v}^{m-k-2}]$$

$$\leq \tilde{v}(|b_o|+\int_o^s |\ddot{G}|ds_1)(\int_o^s |\dot{G}|ds_1)^{m-1} [\tilde{u}^{m-1} +(m-1)(\tilde{u}+\tilde{v})^{m-1}]$$

$$\leq \tilde{v}(|b_o| + \tilde{\ddot{G}}T)(\tilde{\dot{G}}T)^{m-1} m(\tilde{u}+\tilde{v})^{m-1}.$$

Thus, on $0 \leq t \leq T$, from equation (41)

$$|v| \leq |A_2| \sum_{\sigma_m > 1} |c_{\sigma_m}| \frac{\sigma_m!}{m_0! m_1! m_2! m!} m\tilde{v}(\tilde{u}+\tilde{v})^{m-1} \times$$

$$\times \Big\{ m|k_0|\tilde{g}_M(|b_0|+\tilde{\dot{G}}T)(\tilde{G}T)^{m-1}(1+\tilde{\Gamma}\,T)$$

$$+ (|k_0|\tilde{\dot{g}}_M + |k_1|\tilde{g}_M)(\tilde{\dot{G}}T)^m$$

$$+ [(|k_0|\tilde{\dot{g}}_M + |k_1|\tilde{g}_M)\tilde{\Gamma} + |k_2|\tilde{g}_M]T(\tilde{\dot{G}}T)^m$$

$$+ |k_2|\tilde{\Gamma}\tilde{g}_M \frac{T^2}{2} (\tilde{\dot{G}}T)^m \Big\} |\ddot{y}_0|^{m_0} |\dot{y}_0|^{m_1} |y_0|^{m_2}.$$

Recalling the series in (18) one observes that the above series is the term-wise derivative of a convergent power series and hence is convergent. Making the obvious substitution one rewrites the above as

$$(42) \qquad |v| \leq \sum_{\sigma_m > 1} \tilde{v}\, A_{m_0 m_1 m_2 m} |\ddot{y}_0|^{m_0} |\dot{y}_0|^{m_1} |y_0|^{m_2} (\tilde{u}+\tilde{v})^{m-1}.$$

Replacing $|v|$ by $\tilde{v}$ and dividing each member by this positive number it follows that

$$(43) \qquad 1 \leq \sum_{\sigma_m > 1} A_{m_0 m_1 m_2 m} |\ddot{y}_0|^{m_0} |\dot{y}_0|^{m_1} |y_0|^{m_2} (\tilde{u}+\tilde{v})^{m-1}.$$

But this series vanishes for $\ddot{y}_o = \dot{y}_o = y_o = \tilde{u}+\tilde{v} = 0$ and thus it follows that there is some positive number h such that if $|\ddot{y}_o| < h$, $|\dot{y}_o| < h$, $|y_o| < h$ and $\tilde{u}+\tilde{v} < 3h$, then inequality (43) is false. Hence if $|\ddot{y}_o| < h$, $|\dot{y}_o| < h$, $|y_o| < h$, $\tilde{u} < h$, $\widetilde{(u+v)} < h$ then inequality (43) is false contradicting the assumption that $v(t) \not\equiv 0$. This proves the assertion that the solution is unique if it along with the initial conditions are bounded by a small enough positive number.

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APPENDIX

APPENDIX

CALCULATION OF THE RESOLVENT KERNEL

In Chapter III the resolvent kernel Γ for the kernel K was obtained in closed form by solving a certain differential of fourth order. The methods employed there will now be extended to the more general case in which the kernel K is a polynomial of nth degree in s-t. That is, suppose

$$u(t) + \int_0^t K(t-s)u(s)ds = v(t)$$

is equivalent to

$$v(t) + \int_0^t \Gamma(t-s)v(s)ds = u(t)$$

and that

$$K(x) = \sum_{k=0}^{n} \frac{a_k}{k!} x^k .$$

Then from the reciprocal relation

$$\Gamma(x) + K(x) + \int_0^x K(x-y)\Gamma(y)dy = 0$$

one obtains by successive differentiation

$$\Gamma^{(p)} + K^{(p)} + \int_0^x K^{(p)}(x-y)\Gamma(y)dy + \sum_{j=0}^{p-1} a_j \Gamma^{(p-j-1)}$$

$$(p=0,1,\cdots,n+1).$$

(The differentiability of Γ can be verified by considering its Neumann series in terms of K.) Then, since K is a polynomial of nth degree, Γ satisfies the differential equation

$$\Gamma^{(n+1)} + a_0\Gamma^{(n)} + a_1\Gamma^{(n-1)} + \cdots + a_n\Gamma = 0$$

with the initial conditions

$$\Gamma^{(p)}(0) = -a_p - \sum_{j=0}^{p-1} a_j \Gamma^{(p-j-1)}(0), \qquad (p=0,1,\cdots,n).$$

Now, this differential equation can be solved if one solves the corresponding algebraic equation

$$z^{n+1} + a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

for its n+1 roots. (Note that $a_n \neq 0$, for it was supposed that K was a polynomial of degree n.) If the n+1 roots r_j

are distinct, then Γ takes the form

$$\Gamma = \sum_{j=0}^{n} A_j e^{r_j x}$$

where the $r_j \neq 0$ since $a_n \neq 0$. Now the A_j are constants which satisfy the system of linear equations

$$\sum_{j=0}^{n} r_j^p A_j = \Gamma^{(p)}(0), \qquad (p=0,1,\cdots,n)$$

where the $\Gamma^{(p)}(0)$ are as given above. The determinant of this system, being of Vandermonde type, is certainly different from zero because of the hypothesis that the roots r_j are all distinct. Thus the A_j can always be determined and the solution Γ can then be written.

In case the roots r_j are not all distinct, on the other hand, the form taken by Γ is somewhat different. Suppose, for example, that two of the roots, say r_0 and r_1, are equal and that the remaining n-1 roots are distinct from these and from each other. Then

$$\Gamma = (A_0 x + A_1)e^{r_0 x} + \sum_{j=2}^{n} A_j e^{r_j x}$$

where the A_j satisfy the linear system

$$pr_1^{p-1}A_0 + \sum_{j=1}^{n} r_j^p A_j = \Gamma^{(p)}(0), \qquad (p=0,1,\cdots,n).$$

Again the determinant does not vanish so that the A_j can be determined.

Similar results are obtained in those cases where three of the roots r_j are equal, or two pairs of equal roots occur, etc. The problem of obtaining Γ is thus reduced to that of finding the zeroes of a polynomial and then solving a system of linear algebraic equations.

The case of polynomial kernels would appear to be of particular interest also for the approximate solution of Volterra integral equations in which the kernel is analytic in s-t.

A more general discussion in which it is supposed only that K(s,t) satisfies a homogeneous differential equation of the type

$$\sum_{j=0}^{n} a_j(s) \frac{\partial^j}{\partial s^j} K(s,t) = 0$$

has been given by Volterra (5, p. 67).