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This paper deals with finite abstract Witt rings in the case where -1 = 1 in the square class group. The results also apply to Witt rings of quadratic forms over fields where $\sqrt{-1}$ is in the field. The concept of relative rigidity is studied in finite abstract Witt rings via their associated linked quaternionic pairing. This topic is of interest in connection with the elementary type conjecture.

RELATIVE RIGIDITY IN ABSTRACT WITT RINGS

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dedicated to Bryan...

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RELATIVE RIGIDITY IN ABSTRACT WITT RINGS

§1. Introduction

This section will be devoted to a presentation of definitions and introductory material that will be used in the sequel. All notation is that of standard quadratic form theory, such as is found in the texts [L] or [M].

Let F be a field with characteristic F $\neq$ 2 and $\dot{F} = F - \{0\}$. A quadratic form of dimension n over F is a second-degree homogeneous polynomial in n variables over F. Thus, it has the form

$$f(X) = \sum_{i,j=1}^{n} a_{ij}X_iX_j \in F[X_1,..,X_n] = F[X].$$

Alternatively, $f(X) = X^tM_fX$ where $M_f = (a_{ij})$ and X^t is the row matrix $[X_1\ X_2\ \dots\ X_n]$. If f and g are quadratic forms over F of the same dimension n, f is isometric to g if there exists a nonsingular $n \times n$ matrix B such that $f(X) = g(BX)$. As is well known from elementary linear algebra (see [L] or [M]), every quadratic form is isometric to one of the type $f(X) = a_1X_1^2 + a_2X_2^2 + \dots + a_nX_n^2$ with $a_1,\dots,a_n \in \dot{F}$. In this case, f is said to be diagonalized and will be abbreviated as $\langle a_1,a_2,\dots,a_n\rangle$.

A quadratic form f is said to be isotropic if there exists an $x \in F^n$ with $x \neq (0,\dots,0)$ such that $f(x) = 0$; f is called anistropic otherwise. The Witt-Grothendieck ring of quadratic forms over F, denoted $\hat{W}(F)$, is the Grothendieck ring of differences of isometry classes of quadratic forms defined over F with addition given by the direct sum and multiplication induced by the tensor product. For example, we have $\langle a_1,\dots,a_n\rangle \oplus \langle b_1,\dots,b_m\rangle =$

$\langle a_1,\ldots,a_n, b_1,\ldots,b_m\rangle$, $\langle a_1,\ldots,a_n\rangle \otimes \langle b_1,\ldots,b_m\rangle$ = $\langle a_1b_1,\ldots,a_1b_m,\ldots,a_nb_1,\ldots,a_nb_1,\ldots,a_nb_m\rangle$. The Witt ring, W(F), is the quotient $\hat{W}(F)/H$ where $H \subseteq \hat{W}(F)$ is the ideal of $\hat{W}(F)$ generated by the hyperbolic plane $\langle 1,-1\rangle$. It is then a consequence of Witt's cancellation theorem (c.f.[L]) that each element of W(F) corresponds to a single isometry class of anisotropic quadratic forms.

An element $y \in F$ is represented by a quadratic form f if there exists an $x \in F^n$ such that $f(x) = y$. Denote by D(f) the set of elements in $\dot{F}$ represented by f. For an n-tuple of elements $a_1,\ldots,a_n \in \dot{F}$, $\langle\langle a_1,a_2,\ldots,a_n\rangle\rangle$ will denote the 2^n-dimensional quadratic form $f = \bigotimes_{i=1}^{n} \langle 1,a_i\rangle = \langle 1,a_1,a_2,\ldots,a_n,a_1a_2,\ldots,a_1a_2\ldots a_n\rangle$, called an n-fold Pfister form. In the special case of Pfister forms, the set D(f) is actually a subgroup of the multiplicative group $\dot{F}$ (c.f.[L]).

It is important to consider Witt rings from the abstract (or axiomatic) point of view. Here, we describe a class of rings called abstract Witt rings, which arise without reference to any field. It is not known whether every abstract Witt ring arises as the Witt ring of a field; however, every Witt ring of a field is an abstract Witt ring. The approach we take is via abstract quaternionic structures which was introduced in [MY].

An abstract quaternionic structure, or Q-structure, is defined to be a triple (G,Q,q) where G is an abelian group of exponent two (that is, $x^2 = 1$ for all $x \in G$) with a distinguished element denoted by -1, Q is an abelian group of exponent 2, and $q : G \times G \to Q$ is a map satisfying the following four properties:

Q1: (Symmetry) $q(a,b) = q(b,a)$ for all a, $b \in G$;

Q2: $q(a,-a) = 0$ for all $a \in G$;

Q3: (Bilinearity) $q(a,bc) = q(a,b) + q(a,c)$;

Q4: (Linkage) $q(a,b) = q(c,d)$ implies that there exists an $x \in G$ such that $q(a,b) = q(a,x)$ and $q(c,d) = q(c,x)$.

The following results, which will be used extensively without reference, follow immediately:

(i) $q(a,1) = 0$

(ii) $q(a,-ab) = q(a,b)$.

Let (G,Q,q) be a Q-structure. A quadratic form of dimension $n \geq 1$ over G is an n-tuple $f = \langle a_1,a_2,\ldots,a_n\rangle$ where $a_1,a_2,\ldots,a_n \in G$. The sum of f and a form $g = \langle b_1,\ldots,b_m\rangle$ is defined by $f \oplus g = \langle a_1,\ldots a_n, b_1,\ldots,b_m\rangle$ and their product is $f \otimes g = \langle a_1b_1,\ldots,a_1b_m,\ldots,a_nb_m\rangle$. Two forms are isometric (denoted $\cong$) under the following conditions

(1) $\langle a\rangle \cong \langle b\rangle$ if and only if $a = b$,

(2) $\langle a_1,a_2\rangle \cong \langle b_1,b_2\rangle$ if and only if $a_1a_2 = b_1b_2$ and $q(a_1,a_2) = q(b_1,b_2)$,

(3) for $n \geq 3$ isometry is defined inductively by:

$\langle a_1,\ldots,a_n\rangle \cong \langle b_1,\ldots,b_n\rangle$ if and only if there exist $a,b,c_3,\ldots,c_n \in G$ such that

$\langle a_2,a_3,\ldots,a_n\rangle \cong \langle a,c_3,\ldots,c_n\rangle$, $\langle a_1,a\rangle \cong \langle b_1,b\rangle$ and

$\langle b_2,b_3,\ldots,b_n\rangle \cong \langle b,c_3,\ldots,c_n\rangle$.

Exactly as in the field case, one now obtains an abstract Witt-Grothendieck ring $\hat{W}$ and an abstract Witt ring W associated to the quaternionic structure.

An element $x \in G$ is represented by a form f of dimension n if there exist $x_2,\ldots,x_n \in G$ such that $f \cong \langle x,x_2,\ldots,x_n\rangle$. Denote by D(f) the set of elements in G that are represented by f. Equivalently, in the case of a 1-fold Pfister form, $D\langle\langle a\rangle\rangle = \{x \in G \mid q(-a,x) = 0\}$. It is also true for the Witt ring of a quaternionic structure, that $D\langle\langle a_1,\ldots,a_n\rangle\rangle$ is a subgroup of G. If D(f) = G, then f is said to be universal.

It should be noted that there is a linked quaternionic pairing associated with the Witt ring of a field. Let F be a field with characteristic not 2, $G = \dot{F}/\dot{F}^2$, $-1 = [-1]$ in G and $Q = I^2F/I^3F$ where IF is the ideal of even dimensional forms in W(F). It follows from results in [L] that the pairing $q : G \times G \to Q$ given by $q(a,b) = \langle\langle -a,-b\rangle\rangle \bmod I^3F$ is a linked quaterionic pairing and the Witt ring associated to this pairing is precisely W(F). Throughout the remainder of the paper we shall work with abstract Witt rings R, keeping this association in mind. Further, we shall use G_R, Q_R and q_R to denote the associated linked quaterionic pairing $q_R : G_R \times G_R \to Q_R$ as necessary, dropping the subscripts when no confusion may arise.

REMARK 1.1: Other versions of "abstract Witt rings" have appeared in the literature. However, all axiomatizations considered have been shown to describe the same class of rings as those arising as Witt rings of linked quaternionic pairings. For details, see [M].

As an example of a Witt ring over a field, suppose $F = \mathbb{R}$ (or any real-closed field) and consider a form $f \in W(\mathbb{R})$. Since f is anisotropic, it cannot have coefficients of mixed signs in its diagonalization. Thus, at every dimension n there are exactly two anisotropic forms: $n\langle 1\rangle$ and $n\langle -1\rangle$, which implies that $W(\mathbb{R})$ is isomorphic to $\mathbb{Z}$ as commutative rings.

Now consider an abstract quaternionic structure and its associated Witt ring. Suppose $|G| = 1$, so $G = \{1\}$, and $-1 = 1$. Then $q(1,1) = 0$ so $Q = \{0\}$. Forms over G are of the type $\langle 1,1,\dots,1\rangle$ but since $-1 = 1$, the only anisotropic forms are $\emptyset$ (the empty form) and $\langle 1\rangle$. Hence, $R \cong \mathbb{Z}/2\mathbb{Z}$. This Q-structure is realized as the Q-structure of any algebraically closed field F, in particular, the complex field.

REMARK 1.2: The definitions made for linked quaternionic pairings which don't involve the linkage property may also be applied to arbitrary bilinear pairings. They will be used in this manner whenever convenient.

§2: Witt Rings of Elementary Type

We can extend the number of examples of Witt rings by considering a direct product and the usual group ring in the category of Witt rings. For details see [M]. Let (G_S, Q, q) be a linked quaternionic pairing and S its associated abstract Witt ring. Let Δ be any group of exponent two and form the group ring $R = S[\Delta]$. Then R is the Witt ring of a quaternionic structure with $G_R = G_S \times \Delta$. Now suppose (G_R, Q_1, q_1) and (G_{R_2}, Q_2, q_2) are linked quaternionic pairings with Witt rings R_1 and R_2. Define $R_1 \times R_2$ to be the Witt ring associated to the pairing $q : (G_{R_1} \oplus G_{R_2}) \times (G_{R_1} \oplus G_{R_2}) \to Q_1 \oplus Q_2$ given by $q((a_1,b_1), (a_2,b_2)) = (q_1(a_1,a_2), q_2(b_1,b_2))$. That q is linked can be found in [M].

In the study of Witt rings one problem, which has received a great deal of attention, is to classify the finitely generated Witt rings (ie. $|G_R| < \infty$). A finitely generated Witt ring is said to be of elementary type if it is isomorphic to a Witt ring obtained from the Witt rings of a finite field, a local field, the reals or the complexes using iteratively the operations of direct product and group ring formation. One possible characterization of finitely generated Witt rings (often referred to as the elementary type conjecture) is that every finitely generated Witt ring is of elementary type. Carson and Marshall [CM] have shown, with the aid of a computer, that every Witt ring with $|G| \leq 32$ is of elementary type, but the general case seems to be beyond the reach of current techniques.

§3: Relatively Rigid Elements

Throughout the remainder of this paper, we treat Witt rings R that satisfy $-1 = 1$ in G_R. (This corresponds to $\sqrt{-1} \in F$ in the case of a Witt ring over a field F). Fix an integer $n \geq 2$. Whenever $t_1, t_2, \ldots, t_n \in G$ are linearly independent let $[t_1, t_2, \ldots, t_n]$ denote the 2^n products $\{1, t_1, t_2, \ldots, t_1 t_2, \ldots, t_1 t_2 \cdots t_n\}$ and $[t_1, t_2, \ldots t_n]_0 = [t_1, \ldots, t_n] - \{1\}$.

DEFINITION 3.1: An element $a \in G_R$ is rigid if $D\langle 1,a\rangle = \{1,a\}$.

FACT 3.2: If Δ is an elementary 2-group, S and $R = S[\Delta]$ are Witt rings, then every element in $G_R - G_S$ is rigid.

PROOF: c.f.[M]. □

DEFINITION 3.3: $t_1, t_2, \ldots, t_n \in G_R$ are relatively rigid in R if

(i) $\langle\langle -t_1, -t_2, \ldots, -t_n\rangle\rangle \neq 0$,

(ii) for all $g \in [t_1, t_2, \ldots, t_n]_0$,

$$D\langle\langle -g\rangle\rangle \cap \Big(\prod_{\substack{h \in [t_1, \ldots, t_n]_0 \\ h \neq g}} D\langle\langle -h\rangle\rangle \Big) = \{1, -g\}.$$

Definition 3.3 is introduced in [J2], where an axiomatization of Witt rings of 2-Henselian dyadic valued fields is described.

PROPOSITION 3.4: Let R_1, R_2 be any Witt rings with $|G_{R_1}| \geq 2$ and $|G_{R_2}| \geq 2$. Then the Witt ring $R = R_1 \times R_2$ contains no relatively rigid elements t_1, t_2.

PROOF: Assume that R has two relatively rigid elements, say $t_1 = (u_1,v_1)$, $t_2 = (u_2,v_2)$. If $(u_1,v_1) = (1,v_1)$, by applying the definitions of §2 we have that $(u_2,1) \in D\langle\langle t_1\rangle\rangle$ and $(u_2,1) \in D\langle\langle t_2\rangle\rangle$. This is a contradiction if $u_2 \neq 1$. If $u_2 = 1$, then $(w,1) \in D\langle\langle t_1\rangle\rangle \cap D\langle\langle t_2\rangle\rangle$ for any $w \neq 1$, $w \in G_R$, which is also a contradiction. Thus, by symmetry, $u_1,u_2,v_1,v_2 \neq 1$.

Then, $D\langle\langle t_1\rangle\rangle \supseteq \{(1,1),\ (u_1,1),\ (1,v_1),\ (u_1,v_1)\}$

$$D\langle\langle t_2\rangle\rangle \supseteq \{(1,1),\ (u_2,1),\ (1,v_2),\ (u_2,v_2)\}$$

$$D\langle\langle t_1t_2\rangle\rangle \supseteq \{(1,1),\ (u_1u_2,1),\ (1,v_1v_2),\ (u_1u_2,v_1v_2)\}.$$

Thus, $\{(1,1),\ (u_1,1),\ (1,v_1),\ (u_1,v_1)\} \subseteq D\langle\langle t_1\rangle\rangle \cap (D\langle\langle t_2\rangle\rangle \cdot D\langle\langle t_1t_2\rangle\rangle)$ which contradicts the definition of relative rigidity. $\square$

One notices that this argument applies to any product of Witt rings $R = R_1 \times \cdots \times R_m$. Fact 3.2 and Proposition 3.4 immediately give:

COROLLARY 3.5: If there exists a finite Witt ring with no rigid elements and at least two relatively rigid elements, then the elementary type conjecture is false.

The preceeding corollary is the motivation for the remainder of this paper.

We conclude this section with some observations which provide a lower bound on the order of G_R whenever $t_1,\ldots,t_n$ are relatively rigid in R and R has no rigid elements. Since no element of $[t_1,\ldots,t_n]_0$ is rigid, we assume in the proof that there exist $s_i, s_{ij}, \ldots \in G - \{1\}$ such that $s_i \notin \{1,t_i\}$, $s_{ij} \notin \{1,t_it_j\}$, etc. and

$$D\langle\langle t_i\rangle\rangle \supseteq \{1,\ t_i,\ s_i,\ t_is_i\}$$

$$D\langle\langle t_it_j\rangle\rangle \supseteq \{1,\ t_it_j,\ s_{ij},\ t_it_js_{ij}\} \quad \ldots$$

$$D\langle\langle t_1t_2\cdots t_r\rangle\rangle \supseteq \{1,\ t_1\cdots t_r,\ s_{1\cdots r},\ t_1\cdots t_rs_{1\cdots r}\}.$$

THEOREM 3.6: If $t_1,\ldots,t_r$ are relatively rigid and not rigid, then

$$\dim\left(\prod_{g\varepsilon[t_1,\ldots,t_r]_0} D\langle\langle g\rangle\rangle\right) \geq 2^r - 1 + r,\ r \geq 2.$$

PROOF: Proceed by induction. Suppose $r = 2$, that is, t_1, t_2 are relatively rigid but neither is rigid. Then $D\langle\langle t_1\rangle\rangle \supseteq \{1, t_1, s_1, t_1s_1\}$. Clearly, $s_1 \notin [t_1, t_2]$ which implies $[t_1, t_2, s_1]$ is 3-dimensional. Also, $s_2 \notin [t_1, t_2, s_1]$ which implies $[t_1, t_2, s_1, s_2]$ is 4-dimensional. By definition of relative rigidity, $D\langle\langle t_1t_2\rangle\rangle \cap (D\langle\langle t_1\rangle\rangle \cdot D\langle\langle t_2\rangle\rangle) = \{1, t_1t_2\}$ which implies $S_{12} \notin [t_1, t_2, s_1, s_2]$ since $D\langle\langle t_1\rangle\rangle \cdot D\langle\langle t_2\rangle\rangle \supseteq [t_1, t_2, s_1, s_2]$. Therefore, $[t_1, t_2, s_1, s_2, s_{12}]$ is 5-dimensional, that is, $\dim\left(\prod_{g\varepsilon[t_1,t_2]_0} D\langle\langle g\rangle\rangle\right) \geq 5 = 2^2 - 1 + 2$. Now assume this result is true for $t_1,\ldots,t_{r-1}$, that is, $\dim\left(\prod_{g\varepsilon[t_1,\ldots,t_{r-1}]_0} D\langle\langle g\rangle\rangle\right) \geq 2^{r-1} - 1 + (r - 1)$. Let $t_1,\ldots,t_{r-1}, t_r$ be relatively rigid and not rigid.

Claim: $\left(\prod_{g\varepsilon[t_1,\ldots,t_{r-1}]_0} D\langle\langle g\rangle\rangle\right) \cap D\langle\langle t_r\rangle\rangle = \{1\}$.

Since $\langle\langle t_1,\ldots,t_{r-1}, t_r\rangle\rangle \neq 0$ we have $t_r \notin \prod_{g\varepsilon[t_1,\ldots,t_{r-1}]_0} D\langle\langle g\rangle\rangle$.

Also, $t_1,\ldots,t_r$ are relatively rigid so that $s \notin \prod_{g\varepsilon[t_1,\ldots,t_{r-1}]_0} D\langle\langle g\rangle\rangle$, whenever $s\,\varepsilon\, D\langle\langle t_r\rangle\rangle$, $s \notin \{1,t_r\}$, thereby establishing the claim.

Since t_r and s_r are independent of each other, we have dim

$$\left[\left(\prod_{g\varepsilon[t_1,\ldots,t_{r-1}]_0} D\langle\langle g\rangle\rangle\right) \cdot D\langle\langle t_r\rangle\rangle\right] \geq 2^{r-1}-1 + (r-1) + 2 = 2^{r-1}-1 + r + 1.$$

Since $D\langle\langle t_1t_r\rangle\rangle \cap \left(\prod_{g\varepsilon[t_1,\ldots,t_{r-1}]_0} D\langle\langle g\rangle\rangle\right) \cdot D\langle\langle t_r\rangle\rangle \subseteq \{1, t_1t_r\}$

by relative rigidity, this implies $s_{1r} \notin \prod_{g\varepsilon[t_1,\ldots,t_{r-1}]_0} D\langle\langle g\rangle\rangle \cdot D\langle\langle t_r\rangle\rangle$. Thus, $\dim\,[(\prod_{g\varepsilon[t_1,\ldots,t_{r-1}]_0} D\langle\langle g\rangle\rangle) \cdot D\langle\langle t_r\rangle\rangle \cdot D\langle\langle t_1 t_r\rangle\rangle] \geq 2^{r-1} - 1 + r + 2$. Continue in this manner, adding one element at a time from the set $[t_1,\ldots,t_{r-1}] \cdot t_r$, which has 2^{r-1} elements. Thus, one obtains, $\dim\,(\prod_{g\varepsilon[t_1,\ldots,t_r]_0} D\langle\langle g\rangle\rangle) \geq 2^{r-1} -1 + r + 2^{r-1} = 2^r-1 + r.$ $\square$

As an immediate consequence, we obtain the following

<u>COROLLARY 3.7</u>: If $t_1,\ldots,t_r$ are relatively rigid and $|G_R| = 2^d$ then the Witt ring R has a rigid element if $d < 2^r - 1 + r$.

§4. 3-Box Structures

In this section, we introduce an object that arises naturally in the study of relative rigidity in abstract Witt rings. Let $q : G \times G \to Q$ be any linked quaternionic pairing. Let $t_1, t_2 \in G$ (not necessarily relatively rigid in this section). Set $G_1 = D\langle\langle t_1\rangle\rangle$, $G_2 = D\langle\langle t_2\rangle\rangle$ and $G_3 = D\langle\langle t_1t_2\rangle\rangle$.

We now introduce the 3-box. For $a \in G_1$, $b \in G_2$ and $c \in G_3$ the 3-box $[\underline{a}, b, c]$ will represent the relationship $q(a,b) + q(b,c) + q(a,c) = q(a,t_2)$. Similarly, $[a,\underline{b},c]$ represents $q(a,b) + q(b,c) + q(a,c) = q(b,t_1)$ and $[a,b,\underline{c}]$ represents $q(a,b) + q(b,c) + q(a,c) = q(c,t_1) = q(c,t_2)$.

<u>LEMMA 4.1:</u> Suppose $a \in G_1$, $b \in G_2$, $c \in G_3$. Then the following are equivalent:

(i) $[\underline{a},b,c]$

(ii) $[a,\underline{b},c\,t_1t_2]$

(iii) $[a\,t_1,\ b,\ \underline{c\,t_1t_2}]$

(iv) $[\underline{a\,t_1}, b\,t_2, c\,t_1\,t_2]$

(v) $[a\,t_1,\ \underline{b\,t_2}\ ,\ c]$

(vi) $[a,\ b\,t_2,\ \underline{c}]$

<u>PROOF:</u> $[\underline{a}, b, c]$ if and only if $q(a,b) + q(a,c) + q(b,c) = q(a,t_2)$ if and only if $q(a,b) + q(a,c) + q(b,c) + q(a,t_2) + q(b,t_1) = q(b,t_1)$ if and only if $q(a,b) + q(a,ct_1t_2) + q(b,ct_1t_2) = q(b,t_1)$ if and only if $[a,\underline{b},c\,t_1t_2]$.

The other implications follow in an analogous manner. $\square$

PROPOSITION 4.2: Suppose $t_1, t_2 \varepsilon G$, $q : G \times G \to Q$ is a linked quaternionic pairing and the notation is as above, then

(i) for all $a \varepsilon G_1$, $b \varepsilon G_2$ there exist $c, c', c'' \varepsilon G_3$ such that

$[\underline{a},b,c]$, $[a,\underline{b},c']$ and $[a,b,\underline{c''}]$ hold.

(ii) for all $a \varepsilon G_2$, $c \varepsilon G_3$ there exist $b, b', b'' \varepsilon G_3$ such that

$[\underline{a},b,c]$, $[a,\underline{b'},c]$ and $[a,b'',\underline{c}]$ hold.

(iii) for all $b \varepsilon G_2$, $c \varepsilon G_3$ there exist $a, a', a'' \varepsilon G_1$ such that

$[\underline{a},b,c]$, $[a',\underline{b},c]$ and $[a'',b,\underline{c}]$ hold.

PROOF: (i) Since $a \varepsilon G_1$ and $b \varepsilon G_2$, we have $q(t_1,a) = q(t_2,b) = 0$ and $q(t_1,b) = q(t_1,ab) = q(t_1t_2,b)$. Since q is a linked quaternionic pairing, there exists an $x \varepsilon G$ such that $q(t_1,b) = q(t_1t_2,b) = q(t_1t_2,x) = q(ab, x)$. Thus, $q(t_1t_2,xb) = 0$, which implies $xb \varepsilon G_3$ Set $c' = xb$. Then,

$$\begin{aligned} q(a,b) + q(a,c') + q(b,c') &= q(a,b) + q(a,xb) + q(b,xb) \\ &= q(a,x) + q(b,x) \\ &= q(ab,x) \\ &= q(t_1,b). \end{aligned}$$

Thus, $[a,\underline{b},xb]$ holds. Choose $c = c't_1t_2$ and apply Lemma 4.1 to see that $[\underline{a}, b, xbt_1t_2]$ holds.

Next apply what we have just shown to find $c'' \varepsilon G_3$ so that $[t_1a,\underline{b},c'']$ holds. By Lemma 4.1 $[a,b,\underline{c''}]$ holds. This proves (i); (ii) and (iii) follow from (i) by symmetry. $\square$

This leads to the following:

DEFINITION 4.3: If $q : G \times G \to Q$ is a bilinear pairing (not necessarily linked); $t_1, t_2 \in G$; $G_1 = D\langle\langle t_1 \rangle\rangle$, $G_2 = D\langle\langle t_2 \rangle\rangle$ and $G_3 = D\langle\langle t_1 t_2 \rangle\rangle$ (with the obvious definition); and $G = G_1 \cdot G_2 \cdot G_3$ then $q : G \times G \to Q$ is a *3-box structure over t_1, t_2* if (i), (ii), and (iii) from Proposition 4.2 hold.

It is unknown under what conditions the converse of this proposition is true, ie. what assumptions are necessary for a 3-box structure to be a linked quaternionic pairing. As will be seen, one of the major advantages in working with a 3-box structure is that it gives us a concrete way to study some of the relationships in a linked quaternionic pairing that arise from the linkage property. Moreover, we have:

THEOREM 4.4: If $q : G \times G \to Q$ is a 3-box structure over t_1, t_2, then for any homomorphism $h' : Q \to Q'$, the induced pairing $q' = h \circ q : G \times G \to Q'$ is a 3-box structure over t_1, t_2.

PROOF: Since q is a bilinear pairing and h is a homomorphism, $q' = h \circ q$ is bilinear. Now suppose $a \in G_1$, $b \in G_2$. Since $q : G \times G \to Q$ is a 3-box structure, there exists $c \in G_3$ such that $[\underline{a}, b, c]$ holds. Since h is a homomorphism, $h \circ q(a,b) + h \circ q(a,c) + h \circ q(b,c) = h \circ q(a, t_2)$. Thus, $[\underline{a}, b, c]$ holds in Q' as well. Similarly for all other 3-boxes; hence, $q' : G \times G \to Q'$ is a 3-box structure. $\square$

REMARK 4.5: In general, the analogue of Theorem 4.4 fails for linked quaternionic pairings. For this reason, I believe that 3-box structures may be a useful tool in the study of abstract Witt rings.

§5. Examples of 3-Box Structures

The remainder of this paper will be devoted to the study of abstract Witt rings R which contain two relatively rigid elements, t_1 and t_2, in $G_R = G_1 \cdot G_2 \cdot G_3$ where $G_1 = D\langle\langle t_1\rangle\rangle = [t_1, u_i]_{i \varepsilon I}$, $\bar{G}_1 = [u_i]_{i \varepsilon I}$ $(\cong G_1/\langle t_1\rangle)$, $G_2 = D\langle\langle t_2\rangle\rangle = [t_2, v_j]_{j \varepsilon J}$, $\bar{G}_2 = [v_j]_{j \varepsilon J}$ $(\cong G_2/\langle t_2\rangle)$, $G_3 = D\langle\langle t_1 t_2\rangle\rangle = [t_1 t_2, w_k]_{k \varepsilon K}$, $\bar{G}_3 = [w_k]_{k \varepsilon K}$ $(\cong G_3/\langle t_1 t_2\rangle)$, and I, J, K are some indexing sets. The order of I, J, and K will be represented by $|I|$, $|J|$ and $|K|$ and $\dim G_1 = 1 + |I|$, $\dim G_2 = 1 + |J|$, $\dim G_3 = 1 + |K|$. It will be assumed that R contains no rigid elements. Such rings exist (see below), but in all known examples G_R is infinite. As observed in §3, should some Witt ring exist with G_R finite, one would have a counterexample to the elementary type conjecture. To begin, we prove:

<u>PROPOSITION 5.1</u>: If $q : G \times G \to Q$ is a linked quaternionic pairing with t_1, t_2 relatively rigid, then the elements $\{q(t_1,v_j), q(t_1,w_k), q(t_2,u_i), q(t_1,t_2)\}_{i\varepsilon I,\ j\varepsilon J,\ k\varepsilon K}$ must be linearly independent in Q.

<u>PROOF</u>: Suppose $\sum_{j\varepsilon J}\varepsilon_j\, q(t_1,v_j) + \sum_{k\varepsilon K}\gamma_k\, q(t_1,w_k) + \delta q(t_1,t_2) = 0$ where $\varepsilon_j, \gamma_k, \delta \ \varepsilon\ \{0,1\}$. Then by bilinearity, $q(t_1, \prod_{j,k} v_j^{\varepsilon_j} w_k^{\gamma_k} t_2^{\delta}) = 0$ implies $\prod_{j,k} v_j^{\varepsilon_j} w_k^{\gamma_k} t_2^{\delta} \ \varepsilon\ D\langle\langle t_1\rangle\rangle$ which contradicts the relative rigidity of t_1 and t_2 if any of the ε_j, γ_k or δ are 1. Similarly, suppose $\prod_{i\varepsilon I}\delta_j\, q(t_2,u_i) = 0$, then $q(t_2, \prod_I u_i^{\delta_i}) = 0$, implying $\prod_I u_i^{\delta_i} \ \varepsilon\ D\langle\langle t_2\rangle\rangle$, a contradiction.

Finally, suppose $\sum_J \varepsilon_j q(t_1,v_j) + \sum_K \gamma_k q(t_1,w_k) + \delta q(t_1,t_2) + \sum_I \delta_i q(t_2,u_i) = 0$ and at least one of the ε_j, γ_k, or δ are nonzero and one of the δ_i is nonzero. Then, $q(t_1, \prod_{J,K} v_j^{\varepsilon_j} w_k^{\delta_k} t_2^{\delta}) = q(t_2, \prod_I u_i^{\delta_i})$. Since we have a linked pairing, there exists an $x \in G$ such that $q(t_1,x) = q(t_2,x) = q(t_2, \prod_I u_i^{\delta_i})$. But then $q(t_1t_2,x) = 0$ implies $x \in G_3$, $q(t_2, x \prod_I u_i^{\delta_i}) = 0$ implies $x \prod_I u_i^{\delta_i} \in G_2$ while $\prod_I u_i^{\delta_i} \in G_1$. Hence $\prod_I u_i^{\delta_i} = x \cdot x \prod_I u_i^{\delta_i} \in G_1 \cap (G_2 \cdot G_3)$ implies $\prod_I u_i^{\delta_i} = 1$ or t_1, a contradiction. $\square$

A 3-box structure is called unramified if there exist $\overline{G}_2 \subseteq G_2$ and $\overline{G}_3 \subseteq G_3$ with $t_2 \notin \overline{G}_2$, $t_1t_2 \notin \overline{G}_3$ and $q(\overline{G}_2, \overline{G}_3) = 0$. The terminology "unramified" was chosen to be in accordance with that of Witt rings of dyadic valued fields, (c.f. Example 5.2). The main problem of §6 is to study relative rigidity in Witt rings R where G_R is finite by considering the 3-box structure in the unramified case.

We now give some examples.

EXAMPLE 5.2: Let $v : F \to Z$ be a 2-Henselian, discretely valued field with $\sqrt{-1} \in F$ and nonperfect residue class field $\overline{F}$ satisfying char $\overline{F} = 2$, $\overline{F} = \overline{F}^2(\overline{t})$ for $t \in F$ and $\wp(\overline{F}) = \overline{F}$. Here, $\overline{F}$ is the residue class field of F and $\wp(x) = x^2 + x$ is the Artin-Schreier operator. Choose an element $\pi \in F$ such that $v(\pi) = 1$. Then it is a consequence of the results of [J1] that $W(F) = R$ satisfies the conditions described in the beginning of this section with $t_1 = t$ and $t_2 = \pi$.

EXAMPLE 5.3: This example comes from the unramified case of the above. Let $\mathcal{F}$ be a field with char $\mathcal{F} = 2$ and $\mathcal{F} = \mathcal{F}^2 + t\mathcal{F}^2$ where $\mathcal{F}^2$

is the subfield of squares of $\mathcal{F}$ and $t \in \dot{\mathcal{F}} - \dot{\mathcal{F}}^2$. Set $G_1 = (\dot{\mathcal{F}}/\dot{\mathcal{F}}^2, \cdot)$, $\overline{G}_2 = (\mathcal{F}^2, +)$, $\overline{G}_3 = (t\mathcal{F}^2, +)$, $t_1 = t \in G_1$ and let t_2 be a formal element denoted by π in the sequel. Set $G = \langle\pi\rangle \times G_1 \times \overline{G}_2 \times \overline{G}_3$ and $Q = G \times \mathcal{F} (\cong G_1 \times \mathcal{F}^2 \times t\mathcal{F}^2 \cong G_1 \times \overline{G}_2 \times \overline{G}_3)$. Specify the pairing $q : G \times G \to Q$ by defining it on basis elements as follows:

(i) $q(\overline{G}_2, \overline{G}_3) = 0$,

(ii) $q(\pi, g_1) = (g_1, 0)$ in Q for $g_1 \in G_1$

(iii) $q(t, g_2) = (0, g_2)$ and $q(t, g_3) = (0, g_3)$ for $g_2 \in \overline{G}_2$, $g_3 \in \overline{G}_3$

(iv) $q(g_1, g_2) + q(g_1, g_3) = q(g_1, g_2+g_3) = (0, (g_2+g_3)g_1^{-1})$ for $g_1 \in G_1$, $g_1 \neq [t]$, $g_2 \in \overline{G}_2$ and $g_3 \in \overline{G}_3$,

(v) $q(G_1, G_1) = q(G_2, G_2) = q(G_3, G_3) = 0$.

Then we have:

<u>FACT 5.4</u>: The following 3-boxes hold

(a) $[\underline{1}, g_2, g_3]$ for all $g_2 \in \overline{G}_2$, $g_3 \in \overline{G}_3$

(b) $[1 + tx^2, \underline{y}^2, tx^2y^2]$

(c) $[1 + tx^2, x^2y^2t^2, \underline{ty}^2]$

<u>PROOF</u>: (a) This follows immediately from (i).

(b) $q(1 + tx^2, y^2) + q(1 + tx^2, tx^2y^2) + q(y^2, tx^2y^2)$

$= q(1 + tx^2, y^2 + tx^2y^2)$

$= (0, (y^2 + tx^2y^2)(1 + tx^2)^{-1})$

$= (0, y2)$.

Thus, $[1 + tx^2, \underline{y}^2, tx^2y^2]$ holds.

(c) $q(1 + tx^2, x^2y^2t^2) + q(x^2y^2t^2, ty^2) + q(1 + tx^2, ty^2)$

$= q(1 + tx^2, x^2y^2t^2 + ty^2)$

$= (0, ty^2(1 + tx^2)(1 + tx^2)^{-1})$

$= (0, ty^2)$.

Thus, $[1 + tx^2, x^2y^2t^2, \underline{ty}^2]$ holds. $\square$

From Lemma 4.1, Fact 5.4 and some elementary considerations it follows that $q : G \times G \to Q$ is a 3-box structure. (In fact, it is "reduced", c.f. Def. 6.7).

§6. The Unramified Case

The goal of this section is to study in the unramified case (ie. $q(\overline{G}_2, \overline{G}_3) = 0$) the properties of any finite G with a 3-box structure. We first establish the notation.

Let $G = G_1 \cdot G_2 \cdot G_3$ be as described in §5 and let $q : G \times G \to Q$ be a 3-box structure over t_1, t_2. By symmetry, we can assume without loss of generality, that $\dim G_2 \geq \dim G_3$. Let V be a $(1 + |I| + |J| + |K| + |I||J| + |I||K|)$ - dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space with basis $\{(t_1,t_2), (t_1,v_j), (t_1 w_k), (t_2,u_i), (u_i,v_j), (u_i,w_k)\}_{I,J,K}$. Let $S \subseteq V$ be the $(1 + |I| + |J| + |K|)$ - dimensional subspace generated by $\{(t_1,t_2), (t_1,v_j), (t_1,w_k), (t_2,u_i)\}_{I,J,K}$. We can map V to Q specializing in the obvious manner (ie. map $(u_i,v_j) \mapsto q(u_i, v_j)$ and so on).

We use the 3-box notation to identify elements in V as follows: $[\underline{u},v,w]$ represents the element $(t_2 u) + (u,v) + (u,w)$, $[u,v,\underline{t_1 t_2 w}]$ represents the element $(u,v) + (u,w) + (t_2,u) + (t_1,v) + (t_1,t_2) + (t_1,w)$, and so on. Let T denote the subspace of V generated by all the 3-boxes that arise in $G \times G \to Q$. We now prove results which give a lower bound for dim T. In the following, let $E = (t_1,\overline{G}_2) \subseteq V$.

<u>LEMMA 6.1</u>: Let $u_1 \in \overline{G}_1$, $v_1, v_2 \in \overline{G}_2$, $\omega_1, \omega_2 \in G_3$.

(i) If $[u_1, \underline{v_1}, \omega_1]$, $[u_1, \underline{v_2}, \omega_2] \in T + E$.

then $[u_1, \underline{v_1 v_2}, \omega_1\omega_2] \in T + E$

(ii) If $[u_1, v_1, \underline{\omega_1}]$, $[u_1, v_2, \underline{\omega_2}] \in T + E$

then $[u_1, v_1 v_2, \underline{\omega_1\ \omega_2}] \in T + E$.

<u>PROOF</u>: Immediate, expand by the definitions. $\square$

Fix $u_1 \in \overline{G}_1$. Define $\Gamma_3 = \{\omega \in G_3 \mid$ there exists $v \in \overline{G}_2$ with $[u_1, \underline{v}, \omega] \in T + E\}$ and $\tilde{\Gamma}_3 = \{\tilde{\omega} \in G_3 \mid$ there exists $v \in \overline{G}_2$ with $[u_1, v, \underline{\tilde{\omega}}] \in T + E\}$. Notice that Γ_3 and $\tilde{\Gamma}_3$ are nonempty subgroups of G_3 by Lemma 6.1.

LEMMA 6.2: Notation as above. One of the following must hold

(i) $\Gamma_3 = G_3$

(ii) $\tilde{\Gamma}_3 = G_3$

(iii) $\tilde{\Gamma}_3 = \Gamma_3$ and $[G_3 : \Gamma_3] = 2$.

PROOF: Suppose $\Gamma_3 \neq G_3$, $\tilde{\Gamma}_3 \neq G_3$ and fix $\omega^* \in G_3 - \Gamma_3$, $\tilde{\omega}^* \in G_3 - \tilde{\Gamma}_3$

(a) Let $\omega \in \Gamma_3$. Then $\omega\omega^*, \omega^* \notin \Gamma_3$. By Definition 4.3, we can find $v, v' \in \overline{G}_2$ with $[u_1, \underline{t_2}v, \omega^*]$, $[u_1, \underline{t_2}v', \omega\omega^*] \in T$. Then one has $[u_1, \underline{t_2}v, \omega^*] + [u_1, \underline{t_2}v', \omega\omega^*] \equiv [u_1, \underline{vv'}, \omega] + (t_2, \omega) \bmod E$.
In particular, $[u_1, \underline{vv'}, \omega] + (t_2, \omega) \equiv [u_1, vv', \underline{\omega}] \in T + E$.
which implies $\omega \in \tilde{\Gamma}_3$, ie. $\Gamma_3 \subseteq \tilde{\Gamma}_3$.

(b) We obtain $\Gamma_3 \supseteq \tilde{\Gamma}_3$ in an analogous manner: Let $\tilde{\omega} \in \tilde{\Gamma}_3$. Then $\tilde{\omega}\tilde{\omega}^*, \tilde{\omega}^* \notin \tilde{\Gamma}_3$. Find $v, v' \in \overline{G}_2$ with $[u_1, t_2v, \underline{\omega}^*]$, $[u_1, t_2v', \underline{\omega\omega}^*] \in T$.
Adding the two relations yields $[u_1, vv', \underline{\tilde{\omega}}] + (t_2, \tilde{\omega}) \in T + E$.
Hence, $[u_1, \underline{vv'}, \tilde{\omega}] \in T + E$ and $\tilde{\omega} \in \Gamma_3$.

(c) Now suppose $[G_3 : \Gamma_3] \geq 4$. Choose $\omega_1, \omega_2 \in G_3 - \Gamma_3$ with $\omega_1\omega_2 \notin \Gamma_3$. Pick $v_1, v_2 \in \overline{G}_2$ so that $[u_1, \underline{t_2v_1}, \omega_1]$ and $[u_1, \underline{t_2v_2}, \omega_2] \in T$. Then, as above, $[u_1, v_1v_2, \underline{\omega_1\omega_2}] \in T + E$ which implies $\omega_1\omega_2 \in \tilde{\Gamma}_3$. This is a contradiction since $\Gamma_3 = \tilde{\Gamma}_3$. $\square$

PROPOSITION 6.3: Suppose $[G_3 : \Gamma_3] \leq 2$, $v_1, \ldots, v_m$ is a basis for $\overline{G}_2$ and $n = \dim \overline{G}_3$. (Recall $m \geq n$). Then, relabeling if necessary, there exists $\omega_{ij} \in G_3$, linearly independent with $[u_1, \underline{v_1}, \omega_{11}], \ldots, [u_1, \underline{v_n}, \omega_{1n}] \in T + E$.

PROOF: Let $K = \{\omega \in \Gamma_3 \mid [u_1, \underline{1}, \omega] \in T + E\}$. Let F be the relation in $\overline{G}_2 \times \Gamma_3$ of all pairs (v, ω) such that $[u_1, \underline{v}, \omega] \in T + E$. Lemma 6.1 implies that for fixed $v \in \overline{G}_2$ with $(v, \omega) \in F$ that $(v, \omega') \in F$ if and only if $\omega\omega' \in K$. Denote the class $\omega \pmod K$ by $f(v)$ whenever $(v, \omega) \in F$. Thus, we obtain a function $f : \overline{G}_2 \to \Gamma_3/K$ which is linear by Lemma 6.1 and is surjective by the definition of Γ_3. Relabeling if necessary, we have a basis $v_1, v_2, \ldots, v_m$ of $\overline{G}_2$ so that $f(v_1), \ldots, f(v_s)$ $(s \leq m)$ forms a basis of Γ_3/K. Choose any $\omega_{11}, \ldots, \omega_{1s} \in \Gamma_3$, $\omega_{1i} \in f(v_i)$. This means $[u_1, \underline{v_i}, \omega_{1i}] \in T + E$. For each of $v_{s+1}, \ldots, v_n$ express $f(v_j) = \prod_{i=1}^{s} f(v_i)^{\varepsilon_{ij}}$, $j = s+1, \ldots, n$ and $\varepsilon_{ij} \in \{0,1\}$. Since $\dim \Gamma_3 \geq n$, $\dim \Gamma_3/K = s$, we can choose $\omega_{s+1}, \ldots, \omega_n \in K$ to be independent in K. Lemma 6.1 guarantees that $[u_1, \underline{v_j}, (\prod_{i=1}^{s} \omega_{ij}^{\varepsilon_{ij}})\omega_j] \in T + E$ as both $[u_1, \underline{1}, \omega_j]$ and $[u_1, \underline{v_j}, \prod_{i=1}^{s} \omega_{1i}^{\varepsilon_{ij}}] \in T + E$. Set $\omega_{1j} = (\prod_{i=1}^{s} \omega_{1i}^{\varepsilon_{ij}})\omega_j$. Evidently, $\omega_{11}, \ldots, \omega_{1s}, \omega_{1,s+1}, \ldots, \omega_{1n}$ are linearly independent in Γ_3, proving the proposition. $\square$

PROPOSITION 6.4: Suppose $[G_3 : \tilde{\Gamma}_3] \leq 2$, $v_1, \ldots, v_m$ is a basis for $\overline{G}_2$ and $n = \dim \overline{G}_3$. Then, relabeling if necessary, there exists $\tilde{\omega}_{1j} \in G_3$, linearly independent, with $[u_1, v_1, \underline{\tilde{\omega}_{11}}], \ldots, [u_1, v_n, \underline{\tilde{\omega}_{1n}}] \in T + E$.

PROOF: The proof is formally identical to the proof of Proposition 6.3 and hence is omitted. $\square$

If $\{u_i\}_{i\varepsilon I}$ is a basis for $\overline{G}_1$, $\{v_j\}_{j\varepsilon J}$ is a basis for $\overline{G}_2$, $\dim\overline{G}_2 = m$ and $\dim\overline{G}_3 = n$ (recall $m \geq n$), consider the set of $|I||J| + |I||K|$ elements $\{[u_i,\underline{v}_j,\omega_{ij}],\ [u_i,v_j,\underline{\tilde{\omega}}_{ij}]\} \subseteq T + E$ constructed as follows: For each $i\varepsilon I$ define $\Gamma_{i,3}$, $\tilde{\Gamma}_{i,3}$ as in Γ_3, $\tilde{\Gamma}_3$ above ($i=1$). If $\dim \Gamma_{i,3} \geq n$, by Proposition 6.3, choose ω_{ij} to be linearly independent for some subset of n j's in J with $[u_i,\underline{v}_j,\omega_{ij}] \varepsilon T + E$. For all $j\varepsilon J$ choose $\tilde{\omega}_{ij}$ with $[u_i,v_j,\underline{\tilde{\omega}}_{ij}] \varepsilon T + E$. This gives $m + n$ elements. If $\dim \Gamma_{i,3} < n$, by Proposition 6.2 $\tilde{\Gamma}_{i,3} = G_3$, so do the same as above, reversing the roles of $\Gamma_{i,3}$ and $\tilde{\Gamma}_{i,3}$. Set $T_o = \text{span}\{[u_i,\underline{v}_j,\omega_{ij}],$ $[u_i,v_j,\underline{\tilde{\omega}}_{ij}]\} \subseteq T + E$.

<u>THEOREM 6.5</u>: $\dim((T_o + E)/E) = |I||J| + |I||K|$, in particular, $\dim T \geq |I||J| + |I||K|$.

<u>PROOF</u>: Step 1: It is shown that for each $i \varepsilon I$ and any $S_i \subseteq J$, $\prod_{j\varepsilon S_i} \omega_{ij}\tilde{\omega}_{ij} \neq 1$. Consider $\sum_{S_i} ([u_i,\underline{v}_j,\omega_{ij}] + [u_i,v_j,\underline{\tilde{\omega}}_{ij}]) \varepsilon T_o + E$. Expanding by the definition and collecting terms, this is congruent mod E to $(u_i, \prod_{S_i} \omega_{ij}\tilde{\omega}_{ij}) + (t_1, \prod_{S_i} \tilde{\omega}_{ij}) \varepsilon T_o + E$. Suppose $\prod_{S_i} \omega_{ij}\tilde{\omega}_{ij} = 1$. This means that $(t_1, \prod_{S_i} \tilde{\omega}_{ij}) \varepsilon T_o + E$, that is $q(t_1, \prod_{S_i} \tilde{\omega}_{ij}) \varepsilon$ $q(t_1,\overline{G}_2)$ in Q. This contradicts relative rigidity unless $\prod_{S_i} \tilde{\omega}_{ij} = 1$. But then $\prod_{S_i} \omega_{ij} = 1$ also, contradicting the independence assumptions.

Step 2: Suppose that $\sum_{i,j} (a_{ij}[u_i,\underline{v}_j,\omega_{ij}] + b_{ij}\ [u_i,v_j,\underline{\tilde{\omega}}_{ij}]) \varepsilon E$, Consideration of the factors (u_i,v_j) in the usual basis expansion in V shows that $a_{ij} = b_{ij}$ for all $i \varepsilon I$, $j \varepsilon J$. Set $S_i = \{j \varepsilon J \mid a_{ij} = b_{ij} = 1\}$. Then, we have $\sum_i (\sum_{j\varepsilon S_i} ([u_i,\underline{v}_j,\omega_{ij}] + [u_i,v_j,\underline{\tilde{\omega}}_{ij}])) \varepsilon E$.

Expanding, we have $\sum_i ((u_i, \sum_{j\in S_i} \omega_{ij}\tilde{\omega}_{ij}) + (t_1, \prod_{j\in S_i} \tilde{\omega}_{ij})) \in E$.
Then, by the description of V, we must have $\prod_{j\in S_i} \omega_{ij}\tilde{\omega}_{ij} = 1$ for all $i \in I$. By Step 1, $S_i = \emptyset$ for all $i \in I$. $\square$

COROLLARY 6.6: Hypotheses and notation as above. If $Q_o = \mathrm{im}\ (V \to Q)$ then $\dim Q_o \leq 1 + |I| + |J| + |K|$.

PROOF: The result follows from Theorem 6.5 (recalling $T \subseteq \ker\ (V \to Q)$).

DEFINITION 6.7: Let (G, Q, q) be a 3-box structure over t_1, t_2. If $\dim Q = 1 + |I| + |J| + |K|$ and t_1, t_2 are relatively rigid, then (G, Q, q) is called a reduced relatively rigid 3-box structure. If $Q \to Q/\Delta$ is an epimorphism and if the induced 3-box structure $(G, Q/\Delta, q')$ is a reduced relatively rigid 3-box structure, we call $(G, Q/\Delta, q')$ a reduction of (G, Q, q).

That a 3-box structure is reduced is equivalent to saying that in a linked quaternionic pairing, Q has the minimum possible dimension. (c.f. Proposition 5.1).

THEOREM 6.8: If there exists a 3-box structure (G, Q, q) with two relatively rigid elements and no rigid elements, then there exists a reduction of (G, Q, q).

PROOF: Let $Q_o = \mathrm{im}\ (V \to Q)$. Choose elements $q_1, \ldots, q_r \in \{q(G_1, G_1), q(G_2, G_2), q(G_3, G_3)\}$ such that $\{q_1, \ldots, q_r\}$ is a basis of Q/Q_o. Set $\Delta = \langle q_1, \ldots, q_r \rangle$, the subgroup spanned by the basis. Then the map $q' : G \times G \to Q \to Q/\Delta$ is a bilinear pairing. It is clear from the choice of Δ that $\dim (Q/\Delta) = 1 + |I| + |J| + |K|$, and the relative rigidity of t_1, t_2 follows since the composition $Q_o \to Q \to Q/\Delta$ is an isomorphism. Thus, $(G, Q/\Delta, q')$ is a reduced relatively rigid 3-box structure over t_1, t_2. $\square$

THEOREM 6.9: If there exists a 3-box structure (G, Q, q), inside a linked quaternionic pairing, with two relatively rigid elements and no rigid elements, then there is a reduction of 3-box structures $(G, Q/\Delta, q)$ of (G, Q, q) with $S \cong Q/\Delta$.

PROOF: Since images of the basis of S must be linearly independent in Q by Proposition 5.1, the map $S \to Q$ must be injective. The proof of Theorem 6.8 shows that $S \to Q/\Delta$ remains injective. Surjectivity follows as $\dim S = 1 + |I| + |J| + |K| = \dim Q/\Delta$. $\square$

CONCLUDING REMARK 6.10: It would have been nice to conclusively resolve the existence or nonexistence of a reduced, relatively rigid finite 3-box structure. In the future, if the question is resolved in the negative, then it would imply by Proposition 4.2 that a finite abstract Witt ring with the same properties does not exist. Should it be answered constructively in the affirmative where $S \cong Q_0$ then one would have a starting point to look for a counterexample to the elementary type conjecture. More importantly however, I believe that some of the techniques introduced here may prove useful in studying finite abstract Witt rings.

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