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The recursive and stochastic representation of solutions to the Fourier transformed Navier-Stokes equations, as introduced by [34], is extended in several ways. First, associated families of functions known as majorizing kernels are analyzed, in light of their apparently essential role in the representation. Second, the theory is put on a more comprehensive foundation by constructing the basic recursive object, the multiplicative functional or its successor, the random field, without invoking the strong Markov property. This allows the theory to embrace a wider class of evolutionary equations. Third, this methodology, that has delivered global existence and uniqueness theorems for the Navier-Stokes equations given suitably small initial datum, is extended to obtain local in time existence and uniqueness results when the initial datum is arbitrarily large. Fourth, the theory is applied to the semi-linear KPP equation linking it with, and extending previously known results on the representation of solutions with branching Brownian motion.

Branching Processes and Partial Differential Equations

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John Christopher Orum, Author

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CHAPTER I

INTRODUCTION

A new and surprising model for the analysis of Navier-Stokes equations was discovered in 1997 by Le Jan and Sznitman [34]. This model is now part of a more coherent yet still nascent theory involving branching processes and other evolutionary partial differential equations. Some of this new theory is presented here.

§1.1 A brief introduction to the Navier Stokes equations

The Navier-Stokes equations are for an unknown velocity vector

$$u = u(x,t) = (u_k(x,t)_{1\leq k\leq n}) \in \mathbb{R}^n$$

and scalar pressure $p = p(x,t)$, and describe the evolution of the motion of an idealized incompressible viscous fluid of constant density filling all space. Here $n = 2$ or 3. We consider the Navier-Stokes equations in the following form, in which a rescaling of the variables has already been made, thereby reducing the number of free parameters that arise in their derivation:

$$\frac{\partial}{\partial t}u_k + u \cdot \nabla u_k = \nu\Delta u_k - \frac{\partial p}{\partial x_k} + g_k(x,t), \quad 1 \leq k \leq n, \tag{1}$$

$$\text{div } u = \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} = 0. \tag{2}$$

A more condensed expression is simply

$$u_t + u \cdot \nabla u = \nu\Delta u - \nabla p + g, \tag{3}$$

$$\nabla \cdot u = 0. \tag{4}$$

Here ν is the kinematic viscosity, and $g = (g_k(x,t)_{1\leq k\leq n})$ is the external forcing, a generalization of gravity, the body force typically found in applications. Equation (2) expresses the incompressibility condition. For simplicity the external forcing is assumed to be divergence free, and continuity of the solution at $t = 0$ is afforded by the assumption that the initial datum, $u(x, 0^+) = u_0(x)$, is divergence free. Once these are specified, the problem is to find the unknown solution $u(x,t)$ on $\mathbb{R}^n \times [0,T]$ for some positive T or on $\mathbb{R}^n \times [0,\infty)$. In the first case the solution is said to be local in time, and in the second case the solution is said to be global in time.

The Navier-Stokes equations never come like this in scientific or engineering applications: applied problems typically have boundaries and boundary conditions and other variables such as temperature; free surfaces, sloping beaches, sediment, walls, porous media, chemical kinetics, multi-component fluids, etc. So why study the Navier-Stokes equations in the form of (3) and (4)? One answer is that these model an idealization of flows far from boundaries. Another answer is that the same mathematical subtleties that occur in applications also appear in these equations, and understanding these subtleties becomes easier once extraneous concerns have been eliminated. One such subtlety — quite important mathematically — is that in three spatial dimensions these equations have never been shown to possess regular solutions for all time, given arbitrary initial datum, no matter how smooth. This problem has attracted a huge amount of research going back to Leray [36] whose introduction of weak solutions, specifically for the Navier-Stokes equations, was itself an important advance in the theory of partial differential equations.

§1.2 Summary of the main features of the branching process representation

The major discovery of Le Jan and Sznitman is that the equations obtained from applying the Fourier transform to the Navier-Stokes equations, the *FNS* equations, admit solutions that may be represented, pointwise, as the expected value of a multiplicative functional on a multi-type branching process. The purpose of this section is to explain this representation, with a bias toward making the sequel readable. This is not a proper summary of their paper. Several of their important constructions are omitted, in particular, the martingales. These reappear in the uniqueness results in Chapters 6 and 7 and in [7]. This summary is also interpreted in light of the understanding gained by the authors of [7], comprising the Focus Research Group (FRG) at Oregon State University. The concept of a majorizing kernel owes its existence to the FRG. Certain manipulations of equations are more detailed here than in either the FRG paper [7] or in [34].

Starting with equations (3) and (4) the computation of the Fourier transform is facilitated by expressing the vector valued $u(x,t)$ in terms of its components, and incorporating the incompressibility condition into the nonlinear term:

$$\frac{\partial u_k}{\partial t} + \frac{\partial}{\partial x_j}(u_k u_j) = \nu \frac{\partial^2 u_k}{\partial x_j \partial x_j} - \frac{\partial p}{\partial x_k} + g_k, \quad 1 \le k \le n, \tag{5}$$

$$\frac{\partial u_j}{\partial x_j} = 0. \tag{6}$$

Here the Einstein convention holds: repeated indices denote summation. The Fourier transform in the spatial variable is applied to these equations. The version of the Fourier transform used here is

$$\widehat{f} = \widehat{f}(\xi,t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x,t)dx. \tag{7}$$

This results in the system of of transformed equations

$$\frac{\partial \widehat{u}_k}{\partial t} + \frac{i}{(2\pi)^{n/2}} \xi_j (\widehat{u}_k * \widehat{u}_j) = -i\xi_k \widehat{p} - \nu|\xi|^2 \widehat{u}_k + \widehat{g}_k, \quad 1 \le k \le n, \tag{8}$$

$$\xi_k \widehat{u}_k = 0. \tag{9}$$

In vector form this is, using the notation $\mathbf{e}_\xi = \xi|\xi|^{-1}$ provided $\xi \neq 0$,

$$\frac{\partial \widehat{u}}{\partial t} + \frac{i|\xi|}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left[\widehat{u}(\eta,t)\cdot \mathbf{e}_\xi\right] \widehat{u}(\xi-\eta,t)d\eta = -i\xi\widehat{p} - \nu|\xi|^2\widehat{u} + \widehat{g}, \tag{10}$$

$$\widehat{u}(\xi,t)\cdot\xi = 0. \tag{11}$$

The standard technique of reducing the number of equations and the number of unknowns is the application of the orthogonal projection $\mathbf{P}_{\xi^\perp} : \mathbb{C}^n \to \langle\xi\rangle^\perp$, that projects onto the subspace that is the orthogonal complement of ξ, i.e. $\mathbf{P}_{\xi^\perp}(\xi) = 0$. Application of $\mathbf{P}_{\xi^\perp}$ will remove the pressure term, so that the resulting equation may be solved for velocity alone. Then the pressure may be recovered from the velocity solution with the following Poisson equation that is obtained by taking the divergence of (3) using (4) and the assumption that the forcing is divergence free [22, p. 35]:

$$\Delta p = -\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_j}. \tag{12}$$

An application of $\mathbf{P}_{\xi^\perp}$ to equations (10) and (11), using the hypothesis that the forcing term is divergence free, gives

$$\frac{\partial \widehat{u}}{\partial t} + i|\xi|(2\pi)^{-n/2}\int_{\mathbb{R}^n}\left[\widehat{u}(\eta,t)\cdot \mathbf{e}_\xi\right]\mathbf{P}_{\xi^\perp}\widehat{u}(\xi-\eta,t)d\eta = -\nu|\xi|^2\widehat{u}+\widehat{g}, \tag{13}$$

$$\mathbf{P}_{\xi^\perp}\widehat{u} = \widehat{u}. \tag{14}$$

These convert immediately to the integral equation

$$\begin{aligned}\widehat{u}(\xi,t) = {} & e^{-\nu|\xi|^2 t}\widehat{u}_0(\xi) + \\ & \int_0^t e^{-\nu|\xi|^2 s}\left\{\frac{-i|\xi|}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}[\widehat{u}(\eta,t-s)\cdot\mathbf{e}_\xi]\mathbf{P}_{\xi^\perp}\widehat{u}(\xi-\eta,t-s)d\eta + \widehat{g}(\xi,s)\right\}ds,\end{aligned} \tag{15}$$

that describes the evolution of the transformed velocity field $\widehat{u}(\xi,t)$ within the subspace of divergence free vector fields. Some notational simplification is achieved by introducing the following $\otimes_\xi$-product, a bilinear operation that depends on the parameter $\xi \in \mathbb{R}^n/\{0\}$:

$$z \otimes_\xi w = -i|\xi|^{-1}(z\cdot\xi)\mathbf{P}_{\xi^\perp}w = -i(z\cdot \mathbf{e}_\xi)\mathbf{P}_{\xi^\perp}w, \quad z,w\in\mathbb{C}^n. \tag{16}$$

This becomes the predominant algebraic operation at the nodes of the branching process. With this notation equation (15) becomes

$$\begin{aligned}\widehat{u}(\xi,t) = {} & e^{-\nu|\xi|^2 t}\widehat{u}_0(\xi) + \\ & \int_0^t e^{-\nu|\xi|^2 s}\left\{\frac{|\xi|}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\widehat{u}(\eta,t-s)\otimes_\xi \widehat{u}(\xi-\eta,t-s)d\eta + \widehat{g}(\xi,t)\right\}ds.\end{aligned} \tag{17}$$

A key idea introduced by the FRG subsequent to [34], is the notion of a *majorizing kernel.* This is a locally integrable function $h(\xi)$ with the property that for almost all $\xi\in\mathbb{R}^n$,

$$dK_\xi(\eta) = \frac{h(\eta)h(\xi-\eta)}{h{*}h(\xi)}d\eta \tag{18}$$

is the law of a fully supported random vector (assuming that nothing less than fully supported solutions to *FNS* are of interest). The probabilistic interpretation of (18) requires

$$0 < h(\xi) < \infty \quad \text{and} \quad 0 < h{*}h(\xi) < \infty \tag{19}$$

holding almost everywhere. A majorizing kernel must also satisfy the defining inequality

$$h{*}h(\xi) \leq B|\xi|h(\xi) \tag{20}$$

where B is a positive constant; the reason for this condition is explained below. The probability measure (18) is expressed either symmetrically as the law $K_\xi(d\xi_1, d\xi_2)$ (supported on the set $\xi_1 + \xi_2 = \xi$) of a pair of correlated random vectors summing to ξ, or as the law $dK_\xi(\eta)$ for just one of the two vectors. Examples of majorizing kernels appear below.

Equation (17) is adjusted for probabilistic interpretation using a majorizing kernel $h(\xi)$: Divide through by $h(\xi)$ and also multiply and divide by 2, $h(\eta)$, $h(\xi - \eta)$, $h{*}h(\xi)$, and $\nu|\xi|^2$. The result is the *FNS:h* equation:

$$\begin{aligned} \chi(\xi,t) &= e^{-\nu|\xi|^2 t}\chi_0(\xi) + \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2 s}\{\cdots\}\,ds \\ \{\cdots\} &= \frac{1}{2}m(\xi)\int_{\mathbb{R}^n} \chi(\eta, t-s) \otimes_\xi \chi(\xi-\eta, t-s)dK_\xi(\eta) + \frac{1}{2}\varphi(\xi, t-s) \end{aligned} \tag{21}$$

where

$$\begin{aligned} \chi(\xi,t) = \frac{\widehat{u}(\xi,t)}{h(\xi)}, \qquad \chi_0(\xi) = \frac{\widehat{u}_0(\xi)}{h(\xi)}, \qquad m(\xi) = \frac{2|\xi|h{*}h(\xi)}{\nu|\xi|^2(\sqrt{2\pi})^n h(\xi)}, \\ dK_\xi(\eta) = \frac{h(\eta)h(\xi-\eta)}{h{*}h(\xi)}d\eta, \qquad \varphi(\xi,t) = \frac{2\widehat{g}(\xi,t)}{\nu|\xi|^2 h(\xi)}. \end{aligned} \tag{22}$$

This is the form of the *FNS* equations that admits the interpretation in terms of a branching process: Let S_θ be an exponentially distributed random variable with parameter $\nu|\xi|^2$. Let K_θ be a Bernoulli random variable with parameter $\frac{1}{2}$. Let Ξ_1 and Ξ_2 be a pair of correlated random variables distributed as $K_\xi(d\xi_1, d\xi_2)$. Assume S_θ, K_θ, and Ξ_1 are independent. Then equation (21), interpreted probabilistically, may be written

$$\chi(\xi,t) = \mathbb{E}\begin{cases} \chi_0(\xi)\mathbf{1}_{[S_\theta \geq t]} + \\ \varphi(\xi, t - S_\theta)\mathbf{1}_{[S_\theta < t]}\mathbf{1}_{[K_\theta = 0]} + \\ m(\xi)\chi(\Xi_1, t - S_\theta) \otimes_\xi \chi(\Xi_2, t - S_\theta)\mathbf{1}_{[S_\theta < t]}\mathbf{1}_{[K_\theta = 1]}. \end{cases} \tag{23}$$

The recursive interpretation of (23) leads to the representation

$$\chi(\xi,t) = \mathsf{X}_\theta\big(\tau(\xi,t)\big) \tag{24}$$

for the solution of (21) where $\mathsf{X}_\theta(\tau(\xi,t))$ denotes a multiplicative functional on a branching process whose evolutionary history is recorded by the random tree $\tau(\xi,t)$. The branching process and multiplicative functional are described next. A proof that (24) holds appears in [7]. The proof involves conditioning on the first branching in $\tau(\xi,t)$ and appeals to the strong Markov property of the process, imitating the original Le Jan – Sznitman proof. The proof also requires the integrability of $\mathsf{X}_\theta(\tau(\xi,t))$, a point discussed shortly.

Here is how the branching process works: The elapsed time in the branching process runs contrary to the elapsed time in the partial differential equation. Starting at (ξ,t), a particle having frequency type $\Xi_\theta = \xi$ lives for an exponentially distributed length of time S_θ with parameter $\nu|\xi|^2$ (so its mean is $1/\nu|\xi|^2$) and then it dies. In discussing such particles it is useful to confuse the animate particle with its frequency. At the death of Ξ_θ a coin K_θ is tossed. If $K_\theta = 0$ no new particles are born. If $K_\theta = 1$, then two new particles, Ξ_1 and Ξ_2 are born, distributed as $K_\xi(d\xi_1,d\xi_2)$. The process is repeated independently for each of the new particles Ξ_1, Ξ_2, whose exponential lifetimes S_1, S_2, have parameters $\nu|\Xi_1|^2$, $\nu|\Xi_2|^2$, respectively. This process is iterated with Ξ_1 and Ξ_2 subject to the same holding laws and splitting rules, and branching continues long as there are living particles above the threshold $t = 0$. This results in a random tree, $\tau(\xi,t)$, as illustrated in Figure 1. Such a tree records the path of a branching random walk, comprised of particles $\{\Xi_v : v \in \mathcal{V}\}$ having exponential lifetimes $\{S_v : v \in \mathcal{V}\}$ which are either terminated or not according to the Bernoulli variables $\{K_v : v \in \mathcal{V}\}$, where the index set $\mathcal{V} = \{\theta, 1, 2, 11, 12, 111, 112, 121, \dots\}$ is the natural labeling scheme for binary reproduction. One can see that this branching process is the result of interpreting the left hand side of (23) as a random variable, and proceeding recursively and iteratively with this interpretation wherever possible, and then reducing the result of the iteration procedure to a skeleton of random frequencies, their exponential lifetimes, and the kismet Bernoulli variables.

Here is how the multiplicative functional works: Figure 1 shows how such a tree $\tau(\xi,t)$ may have two types of nodes: operational nodes, and input nodes. Operational nodes ($\bullet$) occur at branch points, when a particle dies and is replaced by two new particles. Input nodes ($\circ$) occur at the leaves, when a particle either dies and is not replaced, or else when it dies below the threshold $t=0$. If an input node, located at (ξ^*,t^*) say, occurs above $t=0$, the forcing is evaluated at this point: $\varphi(\xi^*,t^*)$. If this input node occurs below $t=0$, then the initial datum is evaluated: $\chi_0(\xi^*)$. The multiplicative functional combines these sampled values, working from the leaves toward the root, through the non-associative binary operations

$$(a,b)\mapsto m(\xi)a\otimes_\xi b \tag{25}$$

where the nesting of the operations corresponds to the branching structure of the tree. The parameter value of any particular $\otimes_\xi$-product is the frequency type Ξ_v of that particular operational node. The value of the multiplicative functional $\mathsf{X}_\theta\big(\tau(\xi,t)\big)$ is the value that is attained at the root of the tree located at (ξ,t). As an example, the multiplicative functional on the tree $\tau(\xi,t)$ shown in Figure 1, would assume the value

$$\mathsf{X}_\theta(\tau(\xi,t)) = m(\xi)\Big[m(\Xi_1)\chi_0(\Xi_{11})\otimes_{\Xi_1}\chi_0(\Xi_{12})\Big]\otimes_\xi\varphi(\Xi_2,t-S_\theta-S_2). \tag{26}$$

The representation $\chi(\xi,t)=\mathbb{E}\mathsf{X}_\theta(\tau(\xi,t))$ depends on the finiteness and integrability of the multiplicative functional. It is finite because by standardizing all the particle lifetimes to a single epoch in discrete time, the resulting critical binary Galton-Watson process terminates with probability one, and consequently, the branching random walk in continuous time also terminates with probability one. Integrability holds by requiring that for all $\xi\in\mathbb{R}^n$ and for all $t\geq 0$,

$$|\chi_0(\xi)|\leq 1,\quad |\varphi(\xi,t)|\leq 1,\quad\text{and}\quad m(\xi)\leq 1. \tag{27}$$

Then the multiplicative functional is bounded by 1 almost surely, because the inputs are all combined though ordinary multiplications and $\otimes_\xi$-products, and a simple geometric argument establishes that $|a\otimes_\xi b|\leq|a||b|$.

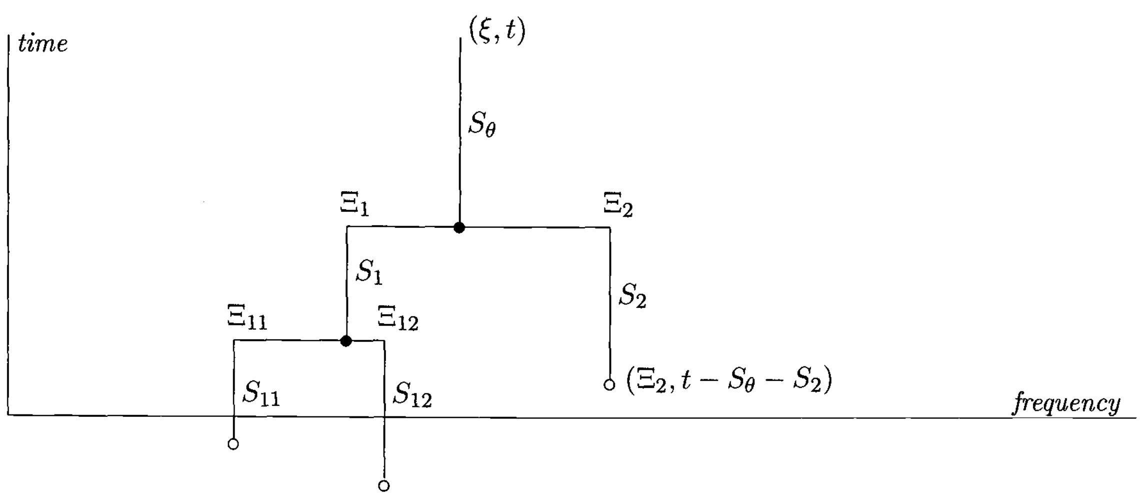

Figure 1: One realization of the random tree $\tau(\xi, t)$ representing the branching process. A particle of type $\xi \in \mathbb{R}^n$ lives for an exponentially distributed lifetime with parameter $\nu|\xi|^2$, then depending on the outcome of the Bernoulli random variable K_θ, it is replaced by either zero particles, or two particles distributed as $K_\xi(d\xi_1, d\xi_2)$, and the process begins anew. Here $K_\theta = 1$, $K_1 = 1$, and $K_2 = 0$. The multiplicative functional samples the initial datum and forcing at the input nodes ($\circ$) and then combines the results together at the internal nodes ($\bullet$) according to the operation $(a, b) \mapsto m(\xi)a \otimes_\xi b$.

The defining inequality (20) for majorizing kernels comes from the requirement $m(\xi) \leq 1$. In three dimensions the equality $h{*}h(\xi) = |\xi|h(\xi)$ has the solutions

$$h(\xi) = \frac{1}{\pi^3|\xi|^2} \qquad \text{and} \qquad h(\xi) = \frac{\beta}{2\pi}\frac{e^{-\beta|\xi|}}{|\xi|} \quad \beta \geq 0, \tag{28}$$

from which the constant B can be adjusted by scaling either the independent or dependent variable. Even more fully supported solutions to the inequality $h{*}h(\xi) \leq |\xi|h(\xi)$ in three dimensions are known [7]. The result is that the 3-dimensional Navier-Stokes equations have solutions admitting the Fourier side representation

$$\widehat{u}(\xi, t) = h(\xi)\mathbb{E}\mathrm{X}_\theta\big(\tau(\xi, t)\big) \tag{29}$$

for a variety of majorizing kernels. A more precise statement of this result is given in [7]; essentially whatever majorizing kernel $h(\xi)$ is chosen, there is a corresponding Banach space comprised of those functions whose Fourier transforms are commensurate with $h(\xi)$, and for suitably small initial datum and forcing as measured by the norm in this Banach space, the representation (29) holds, which in turn yields global existence and uniqueness results.

§1.3 Overview of new results

The stochastic model introduced by Le Jan and Sznitman depends on the three dimensional situation; in the context of their paper it boils down to properties of the Riesz kernel $\pi^{-3}|\xi|^{-2}$ on $\mathbb{R}^3$. This dimensional constraint is part of a more general phenomenon involving the properties of majorizing kernels. It is now known that there do not exist fully supported solutions to the convolution inequality

$$h * h(\xi) \leq B|\xi|h(\xi) \tag{30}$$

on $\mathbb{R}^2$ [45]. The problem of extending the Le Jan - Sznitman results to the 2-dimensional Navier-Stokes equations still remains, but it will require deeper ideas than simply trying to replace (28) by a 2-dimensional analogues. The closest thing to such an extension is that the local existence results described in Chapter 6 work equally well in dimensions 2 or 3.

Solutions to (30) are majorizing kernels of exponent $\theta = 1$; more general majorizing kernels of exponent $\theta \geq 0$ are defined (precisely in Chapter 2) as solutions to

$$h * h(\xi) \leq B|\xi|^{\theta} h(\xi). \tag{31}$$

Notwithstanding the non-existence result mentioned above, there still remains the problem of characterizing majorizing kernels of all exponents in all dimensions, and describing their properties.

Here is what is covered in the remaining chapters: Chapter 2 deals with prerequisite topics. Analysis begins in Chapters 3 where a modicum of progress is made on this apparently difficult problem of characterizing majorizing kernels. The integral representation described in Chapter 3 is extended in a utilitarian way in [7] but nothing close to a complete theory exists. Chapter 4 addresses analyticity questions about functions whose Fourier transforms are commensurate with majorizing kernels. Its utility is that certain partial differential equations within the scope of this methodology have solutions admitting holomorphic extensions; this chapter provides basic material for addressing this phenomenon. In Chapter 5 the model introduced by Le Jan and Sznitman is put on a more comprehensive foundation: the basic stochastic object, the multiplicative functional $\mathsf{X}_\theta\big(\tau(\xi, t)\big)$, is constructed without relying on any Markov process; it is replaced by a successor, the spacetime random field $\mathsf{X}_\theta(\xi, t)$. In Chapter 6, the theory is extended to cover local existence of solutions to the Navier-Stokes equations given arbitrarily large initial datum and forcing. In Chapter 7 the focus is on the semi-linear KPP equation. A Fourier side branching process representation of solutions is presented that complements known results on branching Brownian motion. Then these two branching processes, in physical and Fourier space, along with a second Fourier space branching process, and the Yule process for stochastic comparison, are utilized to explain the finite time blow-up of the KPP equation. As a corollary we see that the solution operator for the KPP equation preserves the class of non-negative definite functions that have integrable second derivatives.

CHAPTER 2
PREREQUISITES AND NOTATION

§2.1 Fourier transform calculus

The Fourier transform plays a central role in the theory. Since different versions of the Fourier transform exist with different factors appearing in formulae, the purpose of this section is to fix a version of the Fourier transform, record some of the consequences, and state some notation. Some local departures from this choice are made in the sequel, but these are explicitly noted. These departures occur when tying into the results of other authors, and when the Fourier-Stieltjes transform plays the role of the characteristic function of probability theory.

Given a complex valued function $f \in L^1(\mathbb{R})$, the Fourier transform

$$\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \cap C_0(\mathbb{R}^n) \tag{32}$$

is defined by

$$\widehat{f}(\xi) = (\mathcal{F}f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx. \tag{33}$$

Here $C(\mathbb{R}^n)$ denotes the space of continuous functions on $\mathbb{R}^n$ and $C_0(\mathbb{R}^n)$ denotes the subset of $C(\mathbb{R}^n)$ consisting of functions vanishing at infinity; $C^k(\mathbb{R}^n)$ denotes the space of functions possessing continuous partial derivatives of order $\leq k$.

The Gauss-Weierstrass inversion formula says that for each Lebesgue point x of f,

$$(2\pi)^{n/2} f(x) = \lim_{\epsilon\to 0} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi) e^{-\epsilon|\xi|^2} d\xi. \tag{34}$$

In view of (34) the inverse Fourier transform of $g(x) \in L^1(\mathbb{R}^n)$ is defined by

$$\widetilde{g}(x) = \check{\widehat{g}}(x) = (\mathcal{F}^{-1}g)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi) d\xi. \tag{35}$$

The notation $\check{g}$ denotes the composition of g with the linear map $x \mapsto -x$, that is, $\check{g}(x) = g(-x)$.

With the choice of factor $(2\pi)^{-n/2}$ in front of the integral, the transform becomes a L^2-isometry: if $f(x) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $\|f\|_2 = \|\widehat{f}\|_2$. This is a basis for the Plancherel

Theorems that extend the transform to a unitary operator $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. The same notation is used even though these are distinct transformations on different spaces. Although there is no need for this in the sequel, two representations of this Fourier–Plancherel transform are (for $n = 1$)

$$\widehat{f}(\xi) = L^2\text{–}\lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-ix\xi} f(x)dx \tag{36}$$

and

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{d}{d\xi} \int_{-\infty}^{\infty} \frac{e^{-ix\xi} - 1}{-ix} f(x)dx. \tag{37}$$

We have the following theorems for the Fourier transform and its inverse acting on L^1 and L^2 functions. These are standard theorems, proofs for dimension $n = 1$ may be found in [24], for example.

2.1 Theorem. *Given f and g in $L^1(\mathbb{R}^n)$, we have*

(*i*) $\widehat{f*g}(\xi) = (2\pi)^{n/2}\widehat{f}(\xi) \cdot \widehat{g}(\xi)$,

(*ii*) *if in addition $\widehat{f}$ and $\widehat{g}$ are in $L^1(\mathbb{R}^n)$, then*

$$\widehat{f \cdot g}(\xi) = (2\pi)^{-n/2}\widehat{f}(\xi)*\widehat{g}(\xi). \tag{38}$$

2.2 Theorem. *Given f and g in $L^2(\mathbb{R}^n)$, we have*

(*i*) $\mathcal{F}^{-1}(\widehat{f} \cdot \widehat{g})(x) = (2\pi)^{-n/2} f*g(x)$ *with the equality holding everywhere,*

(*ii*) $\widehat{f \cdot g}(\xi) = (2\pi)^{-n/2}\widehat{f}*\widehat{g}(\xi)$ *for all* $\xi \in \mathbb{R}^n$.

Another useful extension of the Fourier transform is from $\mathcal{S}$, the Schwartz class of C^∞–functions on $\mathbb{R}^n$ rapidly decreasing at infinity, to the dual space $\mathcal{S}'$ of temperate distributions on $\mathbb{R}^n$. Since $\mathcal{S} \subset L^1$, equation (33) is perfectly applicable for defining the transform on $\mathcal{S}$. In fact, the Fourier transform is an isomorphism of $\mathcal{S}$ onto itself, as well as a continuous self-adjoint linear operator on $\mathcal{S}$. Following the general procedure for extending operations from $\mathcal{S}$ to $\mathcal{S}'$ [23, p. 259], we obtain the extension $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ defined through the dual pairing

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle, \quad T \in \mathcal{S}', \varphi \in \mathcal{S}. \tag{39}$$

This extension is also a surjective homeomorphism. The formulae given in Theorems 2.1 and 2.2 do not necessarily apply to this extension of the Fourier transform, most notably, the convolution of two temperate distributions need not be defined. However, if f and g are in $L^2(\mathbb{R}^n)$, this extension of the Fourier transform can be used to extend Theorem 2.2:

2.3 Theorem. *Given f and g in $L^2(\mathbb{R}^n)$, then*

(*i*) $\widehat{f*g}(\xi) = (2\pi)^{n/2}\widehat{f}(\xi)\cdot\widehat{g}(\xi)$,

(*ii*) $\widehat{f\cdot g}(\xi) = (2\pi)^{-n/2}\widehat{f}*\widehat{g}(\xi)$.

The reason for bringing temperate distributions into the analysis, is that unlike ordinary differential equations whose solutions are curves in $\mathbb{R}^n$, and where all norms are equivalent, solutions to partial differential equations belong to infinite dimensional spaces, and finding the space where the solution exists is part of the problem. The technique of finding solutions informs the choice of function spaces. The branching process methodology induces its own method adapted function spaces — called majorization spaces — which vary according to the choice of majorizing kernel $h(\xi)$. For certain $h(\xi)$, the majorization spaces become subspaces of more familiar classical Banach spaces. The a priori use of temperate distributions covers this variability while retaining the Fourier transform. At the same time the convolution operation (which is not generally defined for temperate distributions) is also retained, thanks to the defining property of majorizing kernels.

One final extension of the Fourier transform is needed. This is the Fourier–Stieltjes transform defined on the space $M(\mathbb{R}^n)$ of complex Borel measures on $\mathbb{R}^n$. If $\mu \in M(\mathbb{R}^n)$ then its Fourier transform $\widehat{\mu}$ is defined either by recognizing μ as a temperate distribution, or by the formula

$$\widehat{\mu}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}d\mu(x). \tag{40}$$

The two definitions agree [23, p. 283]. If μ is a probability measure it is sometimes convenient to adjust the factor in front of the integral so that the transform becomes the complex conjugate of the characteristic function of probability theory.

§2.2 Convolution and delta–sequences

We make use of the following theorems on convolution. These are standard theorems; proofs may be found, for example, in [23] and in [24].

2.4 Theorem. *If f and g are in $L^1(\mathbb{R}^n)$, then $f * g$ is defined almost everywhere and $f * g \in L^1(\mathbb{R}^n)$ with*

$$\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}. \tag{41}$$

2.5 Theorem. *If $f \in L^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$, then $f * g$ is defined almost everywhere and $f * g \in L^2(\mathbb{R}^n)$ with*

$$\|f * g\|_{L^2} \le \|f\|_{L^1} \|g\|_{L^2}. \tag{42}$$

2.6 Theorem. *If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $1/p + 1/q = 1$ then $f * g$ is defined everywhere and is continuous and bounded with*

$$\|f * g\|_{L^\infty} \le \|f\|_{L^p} \|g\|_{L^q}. \tag{43}$$

The relationship between the norms stated in Theorems 2.4, 2.5, and 2.6 has the following generalization, that follows from Young's inequality [37, p. 91]:

2.7 Theorem. *Let $p, q, r \ge 1$ and $1/p + 1/q = 1 + 1/r$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ then $f * g \in L^r(\mathbb{R}^n)$.*

In the following theorem, the family of functions $\{\phi_\epsilon : \epsilon > 0\}$ is manufactured from any L^1-function $\phi(x)$ with $\int \phi(x)dx = 1$ by setting $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$. It is not necessary to assume that $\phi(x) \ge 0$. A proof may be found in [46, p. 72].

2.8 Theorem. *Let $1 \le p < \infty$. If $f \in L^p(\mathbb{R}^n)$ then $f * \phi_\epsilon \to f$ in $L^p(\mathbb{R}^n)$ as $\epsilon \to 0$.*

In applications in the sequel, $\phi(x)$ is a Gaussian — the density of a Gaussian random variable, and Theorem 2.8 is expressed as a corresponding statement holding for $k \to \infty$ under the correspondence $\epsilon = k^{-1}$ (and writing ϕ_k for $\phi_{k^{-1}}$). This is referred to as a delta–sequence, for the obvious reason that as $k \to \infty$, $\phi_k(x)$ is converging weakly to the delta function at the origin.

§2.3 The substitution lemma

A theorem used several times in the sequel is the following substitution lemma for conditional expectation and independence and is referred to simply as *the substitution lemma.* Proofs may be found in Durrett [19, p. 224] and in Bhattacharya and Waymire [8, p. 640].

2.9 Theorem. *Let* $X : (\Omega, \mathcal{F}) \to (S_1, \mathcal{S}_1)$ *and* $Y : (\Omega, \mathcal{F}) \to (S_2, \mathcal{S}_2)$ *be measurable maps and* $\phi : (S_1 \times S_2, \mathcal{S}_1 \times \mathcal{S}_2) \to \mathbb{C}$ *be measurable. Assume* $\phi(X, Y)$ *is integrable and that* $\sigma(X)$ *and* $\sigma(Y)$ *are independent. Then*

$$\mathbb{E}\Big\{\phi(X,Y)\,\Big|\,\sigma(Y)\Big\} = \Big[\mathbb{E}\phi(X,y)\Big]_{y=Y} \tag{44}$$

§2.3 The index set $\mathcal{V}$

The index set $\mathcal{V}$ is the collection of all finite words whose letters are taken from the two element set $\{1, 2\}$, with the empty word denoted by θ. A typical index element is $v = i_1 \dots i_n$ with $i_k \in \{1, 2\}$ for $k = 1, \dots, n$. That is,

$$\mathcal{V} = \{\theta\} \cup \bigcup_{n=1}^{\infty} \{1,2\}^n. \tag{45}$$

The set $\mathcal{V}$ is equipped with the following additional structure.

- The lexicographic order: $\theta < 1 < 2 < 11 < 12 < 21 < 22 < 111 \dots$.
- The length function $|\cdot| : \mathcal{V} \to \{0, 1, 2, \dots\}$ defined by $|\theta| = 0$, and if $v = i_1 i_2 \dots i_n$ then $|v| = n$.
- The truncation operation, that truncates a word at k letters: if $v = i_1 \dots i_k i_{k+1} \dots i_n$ then $v|k = i_1 \dots i_k$. If $k \geq |v|$, then $v|k = v$.
- The meet operation: $v \wedge w = v|n$ where $n = \max\big\{k \geq 0 : v|k = w|k\big\}$.
- The partial order: $v \preceq w$ if and only if $v \wedge w = v$.
- The concatenation operation: if $v = i_1 \dots i_k$, and $w = j_1 \dots j_m$, then

$$vw = i_1 \dots i_k j_1 \dots j_m. \tag{46}$$

- The boundary: $\partial \mathcal{V} = \{1,2\}^\infty$ consisting of all countably infinite words whose letters are taken from the set $\{1,2\}$.

It is useful to view objects indexed by $\mathcal{V}$ as belonging to an infinite triangular array with the rows ordered lexicographically, e.g.

$$(S_v : v \in \mathcal{V}) = \begin{bmatrix} S_\theta & & & \\ S_1 & S_2 & & \\ S_{11} & S_{12} & S_{21} & S_{22} \\ \dots & & & \end{bmatrix} = S_\theta^+. \tag{47}$$

The recursive definition will require an index shift that exploits the evident isomorphism between S_θ^+, S_1^+ and S_2^+ as partially ordered sets under the partial order $\preceq$. Here

$$(S_{1v} : v \in \mathcal{V}) = \begin{bmatrix} S_1 & & & \\ S_{11} & S_{12} & & \\ S_{111} & S_{112} & S_{121} & S_{122} \\ \dots & & & \end{bmatrix} = S_1^+ \tag{48}$$

and $S_2^+ = (S_{2v} : v \in \mathcal{V})$ is defined similarly. The superscript notation S_v^+ is used to denote the collection of all objects S_w such that $v \preceq w$.

§2.3 The sets $\mathcal{H}^{n,\theta}$ and other types of majorizing kernels

In the context of the analysis of Navier-Stokes and related equations (such as Burger's equation) a majorizing kernel of exponent θ is a non-negative locally integrable function $h(\xi)$ satisfying the inequality

$$h * h(\xi) \leq B|\xi|^\theta h(\xi) \tag{49}$$

for some positive constant B. The probabilistic interpretation of

$$dK_\xi(\eta) = \frac{h(\xi-\eta)h(\eta)}{h * h(\xi)} d\eta \tag{50}$$

requires that $h(\xi)$ be non-negative, and not be identically equal to zero. We also require the support of $h(\xi)$ to be a closed convex sub-semigroup $(W,+) \subset (\mathbb{R}^n,+)$ that contains the origin, and has the same topological dimension as $\mathbb{R}^n$, with

$$0 < h(\xi) < \infty \qquad \text{and} \qquad 0 < h * h(\xi) < \infty \tag{51}$$

holding almost everywhere on W. Typically W is a wedge or cone shaped region with the apex at the origin, unless W is all of $\mathbb{R}^n$. To be safe from pathologies we require that $h(\xi)$ be continuous wherever it is strictly positive.

With the exception of the majorizing kernels utilized in the analysis of Burger's equation, the majorizing kernels considered in the sequel are fully supported. The half-line support of the majorizing kernels for Burger's equation helps sanction the notion of defining more general majorizing kernels supported on W properly contained in $\mathbb{R}^n$. However, the additional generality afforded by taking $W \neq \mathbb{R}^n$ is presently not matched by any interesting application beyond Burger's equation. In fact, their use seems bizarre. Thus the notation $\mathcal{H}^{n,\theta}$ is used here to denote the collection of all *fully supported* majorizing kernels of exponent θ on $\mathbb{R}^n$.

Within $\mathcal{H}^{n,\theta}$ are the *standardized* or *normalized* majorizing kernels satisfying the bound

$$\sup_{\xi} \frac{h * h(\xi)}{|\xi|^{\theta} h(\xi)} = 1. \tag{52}$$

Pursuant to the analysis of other evolutionary partial differential equations, a majorizing kernel is any function $h(\xi)$ whose role in the theory is similar to that described in Chapter 1 for the Navier-Stokes equation. Examples of more general types of majorizing kernel appears in Chapter 7 in the analysis of the the KPP equation.

2.10 Remark. The following is perhaps relevant to any extension of the methodology involving less than fully supported majorizing kernels: if a majorizing kernel $h(\xi)$ is supported on the *cone about a of opening angle θ*,

$$\Gamma_{a,\theta} = \{\xi \in \mathbb{R}^n : \xi \cdot a > |\xi| \cos\theta\}, \tag{53}$$

where $a \in \mathbb{R}^n$, $|a| = 1$, and $0 < \theta < \pi/2$, then any function $\widehat{u}(\xi, t)$ that is dominated by such a majorizing kernel also has support in the cone $\Gamma_{a,\theta}$, and its inverse Fourier transform is the boundary value function of an analytic function defined on the open region $\mathbb{R}^n - i\Gamma^*_{a,\theta}$, where $\Gamma^*_{a,\theta} = \Gamma_{a,\pi/2-\theta}$ is the *dual cone.* Reed and Simon [47, pp. 19, 109] characterize the Fourier transforms of temperate distributions whose support lie in such symmetric cones, and remark on generalizations to arbitrary convex cones.

CHAPTER 3

THE STRUCTURE OF MAJORIZING KERNELS, PART I

§3.1 Background and introduction

The convolution inequality that defines a majorizing kernel for the Navier-Stokes equations comes from the bound on the multiplier $m(\xi)$ in the Fourier side branching process representation of solutions. In Chapter 7 a similar bound gives rise to a majorizing kernel for the Fourier transformed KPP equation. Presumably these are representative of the general situation for branching process representations of solutions to certain non-linear evolutionary partial differential equations. For *FNS* the bound $|m(\xi)| \leq 1$ is one of three bounds that together force the multiplicative functional to be integrable. The other bounds are on the initial datum and forcing: $|\chi_0(\xi)| \leq 1$ and $|\varphi(\xi,t)| \leq 1$. These bounds may appear brutal, but they are actually quite sharp in general, a fact readily verified by considering the non-linear ordinary differential equation $y' = y^2 - y$. The bound $|\chi_0(\xi)| \leq 1$ is sharp in the KPP equation; this is exploited to demonstrate its finite time blow-up, in Section 7.13.

For the Navier-Stokes equations in dimension 2 and 3 the inequality $|m(\xi)| \leq 1$ leads to the convolution inequality $h{*}h(\xi) \leq |\xi| h(\xi)$. Although devoid of the incompressibility condition, the 1-dimensional Burger's equation

$$\begin{aligned} &u_t + uu_x = \nu u_{xx}, \quad -\infty < x < \infty, \quad t \geq 0, \\ &u(x,0) = u_0(x), \end{aligned} \tag{54}$$

shares with the Navier-Stokes system the structure

$$u_t = \text{quadratic first order term} + \text{diffusion term}, \tag{55}$$

an analogy successfully pursued by Kreiss and Lorenz [31, p. 122]. This also leads to the convolution inequality $h{*}h(\xi) \leq |\xi| h(\xi)$, except that it is in dimension 1.

Here is a list of known solutions to the *equality* $h{*}h(\xi) = |\xi| h(\xi)$ in various dimensions:

(*i*) In dimension 1, for any $\alpha \in \mathbb{R}$,

$$h(\xi) = e^{-\alpha\xi}\mathbf{1}[\xi \geq 0], \tag{56}$$

(*ii*) In dimension 3, for any parameter $\beta > 0$,

$$h(\xi) = \frac{\beta}{2\pi}\frac{e^{-\beta|\xi|}}{|\xi|}, \tag{57}$$

(*iii*) In dimension $n \geq 3$,

$$h(\xi) = \frac{\Gamma^2(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})\pi^{(n+2)/2}}\frac{1}{|\xi|^{n-1}}. \tag{58}$$

This list is essentially complete (as a list of known solutions) up to the following modifications: in any dimension, multiplication by $e^{\alpha\cdot\xi}$, $\alpha \in \mathbb{R}^n$, maps the solution set onto itself; and, any of these solutions may be embedded into higher dimensions as singular measures supported on linear subspaces, as these measures will still satisfy the convolution equation. Such *thin set* solutions are not useful.

Solutions (*i*) in this list (with $\alpha > 0$) serve as majorizing kernels for the stochastic representation of solutions to Burger's equation constrained, for example, to lie in the Hardy space $H^2(\mathbb{R})$, i.e. the subspace of $L^2(\mathbb{R})$ consisting of those functions whose Fourier transforms vanish on the negative half line. As noted above in (55), Burger's equation is a natural choice for attempting to extend the probabilistic methodology introduced in [34] to different equations. The half line support of these majorizing kernels is an artifact of the non-existence of fully supported solutions to the convolution inequality $h * h(\xi) \leq |\xi|h(\xi)$ in dimension $n = 1$.

The family of solutions (*ii*) stem from Lemma 3.1 in the Le Jan – Sznitman paper [34] where it is asserted that

$$\psi_\alpha(\xi) = K_\xi(\psi_\alpha \otimes \psi_\alpha) \qquad \text{for} \qquad \psi_\alpha(\xi) = \frac{\pi^2}{2}\alpha|\xi|e^{-\alpha|\xi|}. \tag{59}$$

The kernel K_ξ is defined by equation (1.14) therein, and accordingly,

$$K_\xi(\psi_\alpha \otimes \psi_\alpha) = \frac{|\xi|}{\pi^3}\int_{\mathbb{R}^3}\frac{\psi_\alpha(\eta)\psi_\alpha(\xi-\eta)}{|\eta|^2|\xi-\eta|^2}d\eta = \frac{\pi^2}{2}\alpha|\xi|e^{-\alpha|\xi|} = \psi_\alpha(\xi). \tag{60}$$

This implies that

$$h(\xi) = \frac{1}{\pi^3}\frac{\psi_\alpha(\xi)}{|\xi|^2} = \frac{\alpha}{2\pi}\frac{e^{-\alpha|\xi|}}{|\xi|} \tag{61}$$

satisfies $h{*}h(\xi) = |\xi|h(\xi)$. The function

$$G(\xi) = G_2^{(3)}(\xi) = \frac{1}{4\pi}\frac{e^{-|\xi|}}{|\xi|} \tag{62}$$

is one of the infinite family of Bessel kernels introduced by Aronszajn and Smith [5]. This particular Bessel kernel, and the convolution with itself, are among several that may be expressed in terms of elementary functions, and for this reason they stand out as noteworthy examples. Beginning with this example, the analysis of the connection between majorizing kernels and Bessel kernels, as well as other related functions, is explored in Sections 3.4 and 6.5, and in [7].

Solutions *(iii)* are generalizations of the Le Jan – Sznitman solution

$$h(\xi) = \pi^{-3}\frac{1}{|\xi|^2} \tag{63}$$

to higher dimensions. These are the Riesz kernel solutions. They may be found by Fourier analysis, and this is done in Section 3.4. Attempting to extend this family of solutions to dimension $n = 2$, say by defining

$$h(\xi) = C_2\frac{1}{|\xi|}, \qquad \xi \in \mathbb{R}^2, \tag{64}$$

with some constant C_2, encounters the integrability problem that

$$h * h(\xi) = C_2^2 \int_{\mathbb{R}^2} \frac{1}{|\xi - \eta|}\frac{1}{|\eta|}d\eta \equiv \infty. \tag{65}$$

The results of this chapter are motivated by an attempt to enlarge this list and elucidate the structure of majorizing kernels, with emphasis on solutions to the equality $h * h(\xi) = |\xi|h(\xi)$. This appears to be a difficult problem with either the equality or inequality. A modicum of progress has been made, with the introduction of the integral representation given in Propositions 3.2 and 3.8, that is extended in a utilitarian way in [7] for the purpose of constructing majorizing kernels (with inequality) for the *FNS* equation. Still, nothing close to a complete theory of majorizing kernels exists. Moment relations are explored in Sections 3.5 and 3.6. The connection to Brownian motion and other stable processes is explored in Sections 3.7 and 3.8. Questions about the infinite divisibility of majorizing kernels are raised in Section 3.9. Finally, in the last section, a weaker property than infinite divisibility is considered, leading to results on the positive definiteness of majorizing kernels. It should be noted that the discussion of Bessel kernels in Chapter 6 could have been included here as part of the structure theory, but are included in Chapter 6 because of their immediate application to the *FNS* local existence results.

§3.2 An integral representation in one dimension

The motivation for seeking non-negative solutions to the convolution inequality

$$h{*}h(\xi) \leq |\xi| h(\xi) \tag{66}$$

in one dimension comes from the analysis of Burger's equation, and foreshadows the integral representation of solutions (*ii*) and (*iii*) above. We begin with a summary of one particular method of obtaining a stochastic and recursive representation of solutions to the Fourier transformed Burger's equation, distinguished by its close parallel with the Navier-Stokes theory presented in Chapter 1. Its rigorous construction requires simply a prudent translation to Burger's equation the theory presented in Chapter 5 for the Fourier Navier-Stokes equation.

After including a forcing term $g(x,t)$ on the right hand side of equation (54), the Fourier transform of the resulting equation is, in integrated form,

$$\widehat{u}(\xi,t) = e^{-\nu|\xi|^2 t}\widehat{u}_0(\xi) + \int_0^t e^{-\nu|\xi|^2 s}\Big\{\frac{-i\xi}{2\sqrt{2\pi}}\widehat{u} * \widehat{u}(\xi, t-s) + \widehat{g}(\xi, t-s)\Big\}ds. \tag{67}$$

Adjusted for probabilistic interpretation this becomes

$$\begin{aligned} \chi(\xi,t) &= e^{-\nu|\xi|^2 t}\chi_0(\xi) + \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2 s}\{\cdots\}\,ds \\ \{\cdots\} &= \frac{1}{2}m(\xi)\int_{-\infty}^{\infty}\chi(\eta,t-s)\chi(\xi-\eta,t-s)dK_\xi(\eta) + \frac{1}{2}\varphi(\xi,t-s), \end{aligned} \tag{68}$$

where

$$\chi(\xi,t) = \frac{\widehat{u}(\xi,t)}{h(\xi)}, \qquad \chi_0(\xi) = \frac{\widehat{u}_0(\xi)}{h(\xi)}, \qquad m(\xi) = \frac{-i\xi h{*}h(\xi)}{\nu|\xi|^2\sqrt{2\pi}h(\xi)},$$

$$dK_\xi(\eta) = \frac{h(\eta)h(\xi-\eta)}{h{*}h(\xi)}d\eta, \qquad \varphi(\xi,t) = \frac{2\widehat{g}(\xi,t)}{\nu|\xi|^2 h(\xi)}.$$

By using the majorizing kernel $h(\xi) = e^{-\alpha\xi}\mathbf{1}_{[\xi\geq 0]}$, $\alpha > 0$, the multiplier reduces to

$$m(\xi) = \frac{-i}{\nu\sqrt{2\pi}}, \tag{69}$$

and the splitting distribution $dK_\xi(\eta)$ is uniform on $[0, \xi]$. The adapted Banach spaces given by Definition 6.1 have analogues here, also denoted $\mathbb{B}_h$ and $\mathbb{B}_{h,T}$, except that $h(\xi)$, with its half-line support and exponential decay, puts $\mathbb{B}_h$ into a subspace of the Hardy space $H^p(\mathbb{R})$, for any p in the range $2 \le p \le \infty$ (at least).

Following the procedure discussed in Chapter 5, but applied to Burger's equation, produces the representation

$$\widehat{u}(\xi, t) = h(\xi)\mathbb{E}\mathsf{X}_\theta(\xi, t), \tag{70}$$

with the random scalar field $\mathsf{X}_\theta(\xi, t)$ admitting the recursive description

$$\mathsf{X}_\theta(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if } \lambda_\xi^{-1} S_\theta \ge t, \\ \varphi(\xi, t - \lambda_\xi^{-1} S_\theta) & \text{if } \lambda_\xi^{-1} S_\theta \ge t, K_\theta = 0, \\ m(\xi)\mathsf{X}_1(\Xi_1, t - \lambda_\xi^{-1} S_\theta)\mathsf{X}_2(\Xi_2, t - \lambda_\xi^{-1} S_\theta) & \text{if } \lambda_\xi^{-1} S_\theta < t, K_\theta = 1. \end{cases} \tag{71}$$

Interpreted pointwise, equation (70) relates the solution to the expected value of a multiplicative functional on a branching process whose particle types cascade toward the origin. Integrability of $X_\theta(\xi, t)$ is achieved by requiring that the initial datum and forcing satisfy the bounds

$$\widehat{u}_0(\xi) \le \sqrt{2\pi}\nu h(\xi), \qquad \widehat{g}(\xi, t) \le \frac{\sqrt{2\pi}}{2}\nu|\xi|^2 h(\xi). \tag{72}$$

Equipped with this summary, we return to the problem of solving the convolution inequality (66). Solutions $h(\xi)$ different from the one-sided exponential functions considered above will change the computation of the multiplicative functional slightly, due different multipliers $m(\xi)$. Define a generic $h(\xi)$ by

$$h(\xi) = h_X(\xi) = \mathbb{E}(e^{-\xi X})\mathbf{1}[\xi \ge 0] \tag{73}$$

where X is any random variable taking values in $\mathbb{R}$ satisfying the condition

$$\mathbb{E}(e^{-\xi X}) < \infty \quad \text{for all } \xi \ge 0. \tag{74}$$

Then $h(\xi)$ is essentially the bilateral Laplace-Stiltjies transform of $dF_X(t)$ where $F_X(t) = \mathbb{P}(X \le t)$, except that the transform is restricted to, and supported on, the right half line. Without this restriction, such transforms are known to be analytic in ξ, for ξ lying in some vertical strip in the complex plane [54, p. 238]. The condition (74) on the random variable X means that this vertical strip contains the entire right half plane. It turns out that $h(\xi)$ so defined satisfies $h * h(\xi) \leq |\xi| h(\xi)$. This is the content of Proposition 3.2.

3.1 Lemma. *For any $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, and $\xi \geq 0$,*

$$\frac{1}{\alpha - \beta}\left(e^{\alpha\xi} - e^{\beta\xi}\right) \leq \frac{\xi}{2}\left(e^{\alpha\xi} + e^{\beta\xi}\right). \tag{75}$$

Proof. Rearranging the inequality $1 + x \leq e^x$ gives

$$e^{-x} \leq \frac{1}{2}(1 + e^{-x}) - \frac{x}{2}e^{-x} = \frac{d}{dx}\left\{\frac{x}{2}(1 + e^{-x})\right\}. \tag{76}$$

Integrating this between 0 and y produces

$$\begin{cases} 1 - e^{-y} \leq \frac{y}{2}(1 + e^{-y}) & \text{if } y \geq 0 \\ 1 - e^{-y} \geq \frac{y}{2}(1 + e^{-y}) & \text{if } y \leq 0. \end{cases} \tag{77}$$

Equation (75) follows from the substitution $y = \xi\alpha - \xi\beta$, using $\xi \geq 0$. □

3.2 Proposition. *If $h(\xi) = \mathbb{E}(e^{-\xi X})\mathbf{1}[\xi \geq 0]$ and X satisfies $\mathbb{E}(e^{-\xi X}) < \infty$ for all $\xi \geq 0$, then $h*h(\xi) \leq \xi h(\xi)$, with equality when $\mathbb{P}(X = \alpha) = 1$, $\alpha \in \mathbb{R}$.*

Proof. For $\xi \geq 0$,

$$h*h(\xi) = \int_0^\xi h(\xi - \eta)h(\eta)d\eta = \int_0^\xi \mathbb{E}(e^{-(\xi-\eta)X_1})\mathbb{E}(e^{-\eta X_2})d\eta \tag{78}$$

where $X_1 \stackrel{d}{=} X$ and $X_2 \stackrel{d}{=} X$ are independent, and defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By independence of X_1 and X_2,

$$h*h(\xi) = \int_0^\xi \mathbb{E}(e^{-\xi X_1}e^{-\eta(X_2 - X_1)})d\eta, \tag{79}$$

and by Fubini's Theorem

$$h*h(\xi) = \mathbb{E}\left(\int_0^\xi e^{-\xi X_1}e^{-\eta(X_2 - X_1)}d\eta\right). \tag{80}$$

Let N and M denote events that X_1 and X_2 agree and disagree, respectively:

$$N = [X_1 = X_2] = \{\omega \in \Omega : X_1(\omega) = X_2(\omega)\}, \tag{81}$$

$$M = [X_1 \neq X_2] = \{\omega \in \Omega : X_1(\omega) \neq X_2(\omega)\}. \tag{82}$$

Using the notation $\mathbb{E}(Y;A) = \int_A Y(\omega)d\mathbb{P}(\omega)$,

$$\begin{aligned} h{*}h(\xi) &= \mathbb{E}\left(\int_0^\xi e^{-\xi X_1}e^{-\eta(X_2-X_1)}d\eta; N\right) + \mathbb{E}\left(\int_0^\xi e^{-\xi X_1}e^{-\eta(X_2-X_1)}d\eta; M\right) \\ &= \xi\,\mathbb{E}\left(e^{-\xi X_1}; N\right) + \mathbb{E}\left(\frac{e^{-\xi X_2}-e^{-\xi X_1}}{X_1-X_2}; M\right). \end{aligned} \tag{83}$$

An application of Lemma 3.1 with $\alpha = -X_2$ and $\beta = -X_1$ gives

$$\begin{aligned} h{*}h(\xi) &\leq \xi\,\mathbb{E}\left(e^{-\xi X_1}; N\right) + \mathbb{E}\left(\frac{\xi}{2}\left(e^{-\xi X_2}+e^{-\xi X_1}\right); M\right) \\ &= \frac{1}{2}\xi\,\mathbb{E}\left(e^{-\xi X_2}+e^{-\xi X_1}; N\right) + \frac{1}{2}\xi\,\mathbb{E}\left(e^{-\xi X_2}+e^{-\xi X_1}; M\right) \\ &= \frac{1}{2}\xi\,\mathbb{E}(e^{-\xi X_1}) + \frac{1}{2}\xi\,\mathbb{E}(e^{-\xi X_2}) = \xi h(\xi). \end{aligned} \tag{84}$$

□

§3.3 The idea of an integral representation and its supporting lemma.

The 1-dimensional integral representation in the previous section (Proposition 3.2) involves mixing exponential functions according to the law of the random variable X, adumbrating solutions to $h{*}h(\xi) = |\xi|h(\xi)$ in *any* dimension admitting integral representations. In this section functions of the form

$$G(\xi) = \int_0^\infty p_n(\xi,t)d\gamma(t), \quad \xi \in \mathbb{R}^n \tag{85}$$

are considered as candidates for solving $G{*}G(\xi) = |\xi|G(\xi)$. The integrand

$$p_n(\xi,t) = \frac{1}{(\sqrt{4\pi t})^n}e^{-|\xi|^2/4t}, \quad \xi \in \mathbb{R}^n \tag{86}$$

is the Gaussian kernel in n-dimensions, and γ is a measure supported on the positive half line. For certain choices of the measure γ, solutions (ii) and (iii) may be obtained.

In general γ is a positive measure; in the special case that γ is a probability measure, $G(\xi)$ becomes a weighted average of scaled Gaussian densities. Whether or not γ is a probability measure, it may happen that $\lim_{\xi\to 0} G(\xi) = \infty$. It is desirable that there be no other $\xi \in \mathbb{R}^n$ such that $G(\xi) = \infty$; this is the only requirement made on the representing measure γ. Once this requirement is made, it follows that $G(\xi)$ is a strictly decreasing function of $|\xi|$:

$$\begin{aligned} 0 \neq |\xi_1| < |\xi_2| \quad &\Rightarrow \quad p_n(\xi_2, t) < p_n(\xi_1, t) \quad \forall t > 0 \\ &\Rightarrow \quad G(\xi_2) < G(\xi_1) < \infty. \end{aligned}$$

One benefit of the representation (85) stems from the following lemma. The idea is that by pushing the convolution operation onto the representing measures, the higher dimensional problem may be reduced to a problem about measures supported on $[0, \infty)$. This lemma may be viewed as a generalization of the well-known Theorem 3.13 below about subordinating Brownian motions.

3.3 Lemma. *Suppose that*

$$G_1(\xi) = \int_0^\infty p_n(\xi, t)\gamma_1(dt) \quad \textit{and} \quad G_2(\xi) = \int_0^\infty p_n(\xi, t)\gamma_2(dt). \tag{87}$$

Then the convolution of G_1 and G_2 may be computed through the convolution of the measures γ_1 and γ_2:

$$G_1 * G_2(\xi) = \int_0^\infty p_n(\xi, t)\gamma_1 * \gamma_2(dt). \tag{88}$$

Proof. Fix $\xi \in \mathbb{R}^n$, apply the Tonelli Theorem repeatedly.

$$
\begin{aligned}
G_1 * G_2(\xi) &= \int_{\mathbb{R}^n} G_1(\xi-\eta)G_2(\eta)d\eta \\
&= \int_{\mathbb{R}^n} \int_0^\infty p_n(\xi-\eta,t_1)d\gamma_1(t_1) \int_0^\infty p_n(\eta,t_2)d\gamma_2(t_2)\ d\eta \\
&= \int_{\mathbb{R}^n} \int_0^\infty \int_0^\infty p_n(\xi-\eta,t_1)p_n(\eta,t_1)d\gamma_1(t_1)d\gamma_2(t_2)\ d\eta \\
&= \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} p_n(\xi-\eta,t_1)p_n(\eta,t_2)d\eta\ d\gamma_1(t_1)d\gamma_2(t_2) \\
&= \int_0^\infty \int_0^\infty p_n(\xi,t_1) * p_n(\xi,t_2)\ d\gamma_1(t_1)d\gamma_2(t_2) \\
&= \int_0^\infty \int_0^\infty p_n(\xi,t_1+t_2)\ d\gamma_1(t_1)d\gamma_2(t_2) = \int_0^\infty p_n(\xi,t)\ \gamma_1 * \gamma_2(dt).
\end{aligned} \tag{89}
$$

The last equality follows from the definition of the convolution of two measures:

$$\gamma_1 * \gamma_2(A) = \iint \mathbf{1}_A(t_1+t_2)d\gamma_1(t_1)d\gamma_2(t_2). \tag{90}$$

It is a theorem, e.g. [23, p. 281], that for any bounded Borel measurable function h,

$$\int h(t)\gamma_1 * \gamma_2(dt) = \iint h(t_1+t_2)d\gamma_1(t_1)d\gamma_2(t_2). \tag{91}$$

□

§3.4 The Riesz kernel and Bessel kernel solutions

In this section the integral representation of the previous section, and Lemma 3.3, is applied to the convolution equality $h * h(\xi) = |\xi|h(\xi)$, and the Riesz and Bessel kernel solutions.

Fourier analysis may be used to initially guess, and then prove, that in dimension $n \geq 3$, $G(\xi) = C_n|\xi|^{1-n}$ satisfies $G * G(\xi) = |\xi|G(\xi)$ on $\mathbb{R}^n$, where the C_n are certain constants. Indeed, for the version of the Fourier transform used here, that satisfies

$$\mathcal{F}^{-1}(f * g) = (2\pi)^{n/2}\mathcal{F}^{-1}(f)\mathcal{F}^{-1}(g) \tag{92}$$

we have

$$\mathcal{F}^{-1}(|\xi|^{\alpha-n}) = \mathrm{c}(n,\alpha)\frac{1}{|x|^{\alpha}} \tag{93}$$

where

$$\mathrm{c}(n,\alpha) = 2^{\alpha-\frac{n}{2}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}. \tag{94}$$

An indication of the computation behind equation (93) is given in Rudin [48, p. 205].

Assuming that the Fourier transform calculus holds as when the convolution of the two *distributions* is well-defined, then by equations (92) and (93)

$$\mathcal{F}^{-1}(|\xi|^{1-n} * |\xi|^{1-n}) = (2\pi)^{-n/2}[\mathrm{c}(n,1)]^2|x|^{-2} \tag{95}$$

and

$$\mathcal{F}^{-1}(|\xi|^{2-n}) = \mathrm{c}(n,2)|x|^{-2}. \tag{96}$$

Putting these together with the given factor suggests that

$$h(\xi) = \frac{\mathrm{c}(n,2)}{(2\pi)^{n/2}[c(n,1)]^2}\frac{1}{|\xi|^{n-1}} = \frac{\Gamma^2(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})\pi^{(n+2)/2}}\frac{1}{|\xi|^{n-1}} \tag{97}$$

solves the convolution equation, as both $h * h(\xi)$ and $|\xi|h(\xi)$ have the same transform.

Proposition 3.5 below verifies the extension of (92) to the class of distributions considered here, establishing that $h(\xi)$ given by equation (97) does indeed solve the convolution equation. However, this may also be proved directly — and more easily — using the integral representation (85).

3.4 Theorem. *The distribution*

$$G_R(\xi) = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{n/2}}\frac{1}{|\xi|^{n-1}}, \quad \xi \in \mathbb{R}^n, \quad n \geq 3, \tag{98}$$

has the integral representation

$$G_R(\xi) = \int_0^\infty \frac{1}{\sqrt{t}}p_n(\xi,t)dt, \tag{99}$$

and satisfies the convolution equation

$$G_R{*}G_R(\xi) = \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}\frac{\pi}{2}|\xi|G_R(\xi). \tag{100}$$

Before proving this we consider a heuristic argument for finding the measure $t^{-1/2}dt$. Suppose that $G(\xi) = C_n|\xi|^{1-n}$, which is expected to verify $G * G(\xi) = |\xi|G(\xi)$ for some constant C_n, has an integral representation of the form (85) for some measure γ. The goal is to find this measure, which is now assumed to have a density $\gamma(t)$. According to Lemma 3.3 then,

$$G{*}G(\xi) = \int_0^\infty p_n(\xi,t)\gamma{*}\gamma(t)dt. \tag{101}$$

Now exploit the fact that $G * G(\xi) = C_n|\xi|^{2-n}$ is harmonic on $\mathbb{R}^n/\{0\}$. (There appears to be no additional benefit in computing the distributional Laplacian $\Delta G * G(\xi)$ on all of $\mathbb{R}^n$.) Starting from $\Delta G * G(\xi) = 0$ on $\mathbb{R}^n/\{0\}$, take the Laplacian inside the integral, use the fact that $p_n(\xi,t)$ solves the heat equation, and then integrate by parts. The result is that for $\xi \neq 0$,

$$\Delta G{*}G(\xi) = \int_0^\infty \Delta p_n(\xi,t)\gamma{*}\gamma(t)dt$$

$$= -\int_0^\infty \frac{\partial}{\partial t}p_n(\xi,t)\gamma{*}\gamma(t)dt = \int_0^\infty p_n(\xi,t)\frac{\partial}{\partial t}\gamma{*}\gamma(t)dt = 0. \tag{102}$$

From this, and the assumption that $\gamma(t) \geq 0$, we may extract the identity

$$\gamma{*}\gamma(t) \equiv c\mathbf{1}_{[t\geq 0]} \tag{103}$$

where c is a positive constant.

The following calculation solves this for $\gamma(t)$. Provided that $a > -1$ and $b > -1$,

$$t^a\mathbf{1}_{[t\geq 0]}{*}t^b\mathbf{1}_{[t\geq 0]} = \int_0^t (t-s)^a s^b ds = \int_0^1 (t-tu)^a (tu)^b du \tag{104}$$

$$= t^{a+b+1}\int_0^1 (1-u)^a u^b du = B(a+1,b+1)t^{a+b+1}\mathbf{1}_{[t\geq 0]}, \tag{105}$$

where

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^1 x^{m-1}(1-x)^{n-1}dx \tag{106}$$

is the Euler beta function. In particular,

$$t^{-1/2}\mathbf{1}_{[t\geq 0]}*t^{-1/2}\mathbf{1}_{[t\geq 0]} = B(\tfrac{1}{2},\tfrac{1}{2})\mathbf{1}_{[t\geq 0]} = \pi\mathbf{1}_{[t\geq 0]}. \tag{107}$$

This suggests that the representing measure for $G(\xi) = C_n|\xi|^{1-n}$ on $\mathbb{R}^n$, $n \geq 3$, should be $d\gamma(t) = t^{-1/2}dt$, up to multiplication by a constant.

Although several steps in this heuristic argument are without rigorous justification, e.g. taking the Laplacian inside the integral in equation (102), the result is correct.

Proof of Theorem 3.4. By a straightforward calculation,

$$\begin{aligned} G_R(\xi) &= \int_0^\infty \frac{1}{\sqrt{t}}p_n(\xi,t)dt = \int_0^\infty t^{(-n-1)/2}\frac{1}{(4\pi)^{n/2}}e^{-|\xi|^2/4t}dt \\ &= \frac{1}{(4\pi)^{n/2}}\int_0^\infty t^{(-n+1)/2}e^{-|\xi|^2/4t}\frac{dt}{t} = \frac{1}{(4\pi)^{n/2}}\int_0^\infty u^{(n-1)/2}e^{-u|\xi|^2/4}\frac{du}{u} \\ &= \frac{1}{(4\pi)^{n/2}}\left(\frac{4}{|\xi|^2}\right)^{(n-1)/2}\int_0^\infty u^{(n-1)/2}e^{-u}\frac{du}{u} = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{n/2}}\frac{1}{|\xi|^{n-1}}. \end{aligned} \tag{108}$$

The convolution is computed using Lemma 3.3 and equation (107):

$$\begin{aligned} G_R*G_R(\xi) &= \int_0^\infty p_n(\xi,t)\pi dt = \frac{\pi}{(4\pi)^{n/2}}\int_0^\infty t^{-n/2}e^{-|\xi|^2/4t}dt \\ &= \frac{\pi}{(4\pi)^{n/2}}\int_0^\infty t^{(-n+2)/2}e^{-|\xi|^2/4t}\frac{dt}{t} = \frac{\pi}{(4\pi)^{n/2}}\int_0^\infty u^{(n-2)/2}e^{-u|\xi|^2/4}\frac{du}{u} \\ &= \frac{\pi}{(4\pi)^{n/2}}\left(\frac{4}{|\xi|^2}\right)^{(n-2)/2}\int_0^\infty u^{(n-2)/2}e^{-u}\frac{du}{u} = \frac{\Gamma(\frac{n-2}{2})}{4\pi^{(n-2)/2}}\frac{1}{|\xi|^{n-2}}. \end{aligned} \tag{109}$$

Combining equations (108) and (109) shows that $G_R(\xi)$ satisfies the equation

$$G_R*G_R(\xi) = \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}\frac{\pi}{2}|\xi|G_R(\xi) \tag{110}$$

and that

$$h(\xi) \stackrel{\text{def}}{=} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\frac{2}{\pi}G_R(\xi) = \frac{\Gamma^2(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})\pi^{(n+2)/2}}\frac{1}{|\xi|^{n-1}} \tag{111}$$

satisfies the convolution equation $h*h(\xi) = |\xi|h(\xi)$. □

We return to the problem of extending (92) to the class of distributions considered here. Its proof uses the integral representation for the Riesz kernels. Perhaps this is unavoidable. It does not appear to be possible, for example, to give a proof based on the fact that when $n/2 < \alpha < n$, the distribution $|\xi|^{-\alpha}$ is the sum of an L^1-function and an L^2-function. A general theorem that says that (92) holds for temperate distributions when either the product or the convolution is well-defined is elusive. Such a theorem is stated, but without proof, in Champeney [13, p. 142]. The original work of Schwartz may be the best source.

3.5 Proposition. *Suppose that* $0 < \alpha < n$, $0 < \beta < n$, *and* $0 < \alpha + \beta < n$. *Then the convolution on* $\mathbb{R}^n$ *of the distributions* $|\xi|^{\alpha-n}$ *and* $|\xi|^{\beta-n}$ *has inverse Fourier transform*

$$\mathcal{F}^{-1}\big(|\xi|^{\alpha-n} * |\xi|^{\beta-n}\big) = (2\pi)^{n/2}\,\mathcal{F}^{-1}\big(|\xi|^{\alpha-n}\big) \cdot \mathcal{F}^{-1}\big(|\xi|^{\beta-n}\big). \tag{112}$$

Proof. Let $R_\gamma(\xi) = |\xi|^{\gamma-n}$, for $0 < \gamma < n$. This has the integral representation

$$R_\gamma(\xi) = \frac{2^\gamma \pi^{n/2}}{\Gamma(\frac{n-\gamma}{2})} \int_0^\infty p_n(\xi, t) t^{(\gamma-2)/2} dt, \tag{113}$$

and by Lemma 3.3 and calculations similar to (108) and (109) we have

$$R_\alpha * R_\beta(\xi) = \pi^{n/2} \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})\Gamma(\frac{n-\alpha-\beta}{2})}{\Gamma(\frac{n-\alpha}{2})\Gamma(\frac{n-\beta}{2})\Gamma(\frac{\alpha+\beta}{2})} R_{\alpha+\beta}(\xi). \tag{114}$$

Define the approximation $R_\alpha^{(N)}(\xi)$ to $R_\alpha(\xi)$ by

$$R_\alpha^{(N)}(\xi) = \int_{N^{-1}}^N p_n(\xi, t) t^{(\alpha-2)/2} dt. \tag{115}$$

It is straightforward to check that $R_\alpha^{(N)}(\xi) \in \mathcal{S}$ and consequently, using this fact along with $R_\beta(\xi) \in \mathcal{S}'$, we have

$$\mathcal{F}^{-1}(R_\alpha^{(N)} * R_\beta(\xi)) = (2\pi)^{n/2}(\mathcal{F}^{-1} R_\alpha^{(N)})(\mathcal{F}^{-1} R_\beta). \tag{116}$$

Since $R_\alpha * R_\beta(\xi)$ is a temperate function [23, p. 258], the value of the functional acting on any $\psi \in \mathcal{S}$ may be computed as the integral

$$\langle R_\alpha * R_\beta, \psi\rangle = \int_{\mathbb{R}^n} R_\alpha * R_\beta(\xi)\psi(\xi) d\xi, \tag{117}$$

and moreover $R_{\alpha+\beta}(\xi)|\psi(\xi)| \in L^1(\mathbb{R}^n)$, making it a dominating function for the sequence $R_\alpha^{(N)} * R_\beta(\xi)\psi(\xi)$, and

$$\lim_{N\to\infty} \int_{\mathbb{R}^n} R_\alpha^{(N)} * R_\beta(\xi)\psi(\xi)d\xi = \int_{\mathbb{R}^n} R_\alpha * R_\beta(\xi)\psi(\xi)d\xi, \tag{118}$$

from which we conclude

$$\lim_{N\to\infty} R_\alpha^{(N)} * R_\beta = R_\alpha * R_\beta \quad \text{(convergence in } \mathcal{S}'\text{)}. \tag{119}$$

Similarly (and this requires $\alpha + \beta < n$),

$$\lim_{N\to\infty} (2\pi)^{n/2}(\mathcal{F}^{-1}R_\alpha^{(N)})(\mathcal{F}^{-1}R_\beta) = (2\pi)^{n/2}(\mathcal{F}^{-1}R_\alpha)(\mathcal{F}^{-1}R_\beta) \quad \text{(convergence in } \mathcal{S}'\text{)}. \tag{120}$$

Using the fact that the transform $\mathcal{F}^{-1}: \mathcal{S}' \to \mathcal{S}'$ is a homeomorphism, the two convergent sequences are in one-to-one correspondence and their limit points correspond:

$$\mathcal{F}^{-1}(R_\alpha * R_\beta) = (2\pi)^{n/2}(\mathcal{F}^{-1}R_\alpha)(\mathcal{F}^{-1}R_\beta). \tag{121}$$

This completes the proof. □

We return to the list in the beginning of Section 3.1. An integral representation for solution (ii) is obtained simply by recognizing this function as a Bessel kernel. The Bessel kernels were introduced by Calderón [11] and by Aronszajn and Smith [5]. Here it is sufficient to define them according to the integral representation (85), taking the measure γ to be the law of a gamma random variable:

$$\gamma(t)dt = \frac{1}{\Gamma(\frac{\alpha}{2})} t^{\alpha/2-1}e^{-t}dt. \tag{122}$$

3.6 Definition. *The Bessel kernel of order $\alpha > 0$ on $\mathbb{R}^n$ is the function $G_\alpha : \mathbb{R}^n \to [0, \infty]$ defined through the integral representation*

$$G_\alpha(\xi) = G_\alpha^{(n)}(\xi) = \frac{1}{\Gamma(\frac{\alpha}{2})(4\pi)^{n/2}} \int_0^\infty t^{(\alpha-n)/2-1}e^{-|\xi|^2/4t-t}dt. \tag{123}$$

An immediate benefit of this particular definition is that in light of Lemma 3.3, the Bessel kernels on $\mathbb{R}^n$ form a semigroup under convolution.

Actually, the gamma random variables belong to a two parameter family of distributions. Building both parameters into the integral representation achieves a slight generalization over Definition 3.6. Certain results of this generalization are recorded for computational reference.

3.7 Proposition. *Let*

$$G_{\alpha,\beta}^{(n)}(\xi) = \int_0^\infty p_n(\xi,t)\gamma_{\frac{\alpha}{2},\beta^2}(t)dt \tag{124}$$

where

$$\gamma_{\frac{\alpha}{2},\beta^2}(t) = \frac{1}{\Gamma(\frac{\alpha}{2})}\beta^\alpha t^{\alpha/2-1}e^{-\beta^2 t} \tag{125}$$

is the density of a gamma random variable with shape parameter $\alpha/2$ *and scale parameter* β^2. *Then*

1. $\int_{\mathbb{R}^n} G_{\alpha,\beta}^{(n)}(\xi)d\xi = 1$
2. $G_{\alpha_1,\beta}^{(n)} * G_{\alpha_2,\beta}^{(n)}(\xi) = G_{\alpha_1+\alpha_2,\beta}^{(n)}(\xi)$
3. $\left(\mathcal{F}^{-1}G_{\alpha,\beta}^{(n)}\right)(x) = (2\pi)^{-n/2}\left(\dfrac{\beta^2}{\beta^2+|x|^2}\right)^{\alpha/2}$
4. $G_{\alpha,\beta}^{(n)}(\xi) = \beta^n G_{\alpha,1}^{(n)}(\beta\xi)$

Proof. The first assertion is established by exchanging the order of integration, justified by Tonelli's Theorem, integrating the Gaussian kernel to 1, and then integrating the gamma density to 1. The second assertion follows from Lemma 3.3 and the fact that for the convolution of two gamma distributions that have the same scale parameter, it is the shape parameters that add. For the third statement, the order of integration is exchanged giving

$$\begin{aligned}\left(\mathcal{F}^{-1}G_{\alpha,\beta}^{(n)}\right)(x) &= (2\pi)^{-n/2}\frac{\beta^\alpha}{\Gamma(\frac{\alpha}{2})}\int_0^\infty e^{-t|x|^2}t^{\alpha/2}e^{-\beta^2 t}\frac{dt}{t} \\ &= (2\pi)^{-n/2}\frac{\beta^\alpha}{\Gamma(\frac{\alpha}{2})}\int_0^\infty \left(\frac{t}{\beta^2+|x|^2}\right)^{\alpha/2}e^{-t}\frac{dt}{t} \\ &= (2\pi)^{-n/2}\left(\frac{\beta^2}{\beta^2+|x|^2}\right)^{\alpha/2}. \end{aligned} \tag{126}$$

A straightforward change of variables established the fourth assertion. ∎

There are two identities that are useful in computing the Bessel kernels of order $n-1$ and $n+1$. These are

$$e^{-2ab} = \frac{a}{\sqrt{\pi}} \int_0^\infty e^{-b^2t-a^2/t} \frac{dt}{t^{3/2}}, \tag{127}$$

and

$$e^{-2ab} = \frac{a}{\sqrt{\pi}} \int_0^\infty e^{-a^2t-b^2/t} \frac{dt}{t^{1/2}}. \tag{128}$$

An elementary proof for these identities can be found in Jones [27, p. 315]. Equations (127) and (128) are also referred to as *subordination identities* by Taylor [51, pp. 218, 220] who gives a more sophisticated proof in the context of a discussion of the classical evolution equations. A straightforward application of these identities yields, respectively,

$$G_{n-1}^{(n)}(\xi) = \frac{1}{\Gamma(\frac{n-1}{2})(4\pi)^{n/2}} \int_0^\infty e^{-t-|\xi|^2/4t} \frac{dt}{t^{3/2}} = \frac{2\sqrt{\pi}}{\Gamma(\frac{n-1}{2})(4\pi)^{n/2}} \frac{e^{-|\xi|}}{|\xi|}, \tag{129}$$

and

$$G_{n+1}^{(n)}(\xi) = \frac{1}{\Gamma(\frac{n+1}{2})(4\pi)^{n/2}} \int_0^\infty e^{-t-|\xi|^2/4t} \frac{dt}{t^{1/2}} = \frac{\sqrt{\pi}}{\Gamma(\frac{n+1}{2})(4\pi)^{n/2}} e^{-|\xi|}. \tag{130}$$

From (129) and (130) we obtain

$$G_2^{(3)}(\xi) = \frac{1}{4\pi} \frac{e^{-|\xi|}}{|\xi|}, \qquad G_4^{(3)}(\xi) = \frac{1}{8\pi} e^{-|\xi|}. \tag{131}$$

Consequently $G(\xi) = 2G_2^{(3)}(\xi)$ is a solution to the convolution equation $h * h(\xi) = |\xi| h(\xi)$. Scaling both the independent and dependent variables (see Lemma 4.2) shows that $\beta^2 G(\beta\xi)$ solves the same equation. Combining this with the previously discussed results on the Riesz kernels proves the following proposition:

3.8 Proposition. *All of the known fully supported and rotationally invariant solutions to $G * G(\xi) = |\xi| G(\xi)$ on $\mathbb{R}^n$ (listed in the beginning of section 3.1) have the integral representation*

$$G(\xi) = \int_0^\infty p_n(\xi, t)\gamma(t)dt, \qquad \xi \in \mathbb{R}^n \tag{132}$$

for some representing measure $\gamma(t)dt$. These are, in dimension $n = 3$, the scaled Bessel kernel solutions

$$G(\xi) = \frac{\beta}{2\pi} \frac{e^{-\beta|\xi|}}{|\xi|}, \qquad \textit{with} \qquad \gamma(t) = 2\beta e^{-\beta^2 t}, \quad \beta > 0; \tag{133}$$

and in dimension $n \geq 3$, the Riesz kernel solutions

$$G(\xi) = \frac{\Gamma^2(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})} \frac{1}{\pi^{\frac{n}{2}+1}} \frac{1}{|\xi|^{n-1}} \qquad \textit{with} \qquad \gamma(t) = \frac{2}{\pi} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})} t^{-1/2}. \tag{134}$$

§3.5 Random variables, moments, and the convolution equation

The convolution equation $h{*}h(\xi) = |\xi|h(\xi)$ on $\mathbb{R}^n$, $h(\xi) \geq 0$, has the following probabilistic interpretation when $h(\xi)$ is integrable. The scaling of both the independent and dependent variables according to

$$\beta \cdot h(\xi) \stackrel{\text{def}}{=} \beta^{n-1} h(\beta\xi) \tag{135}$$

forms a 1-parameter group of transformations mapping the solution set onto itself. (The group is the multiplicative group of positive real numbers.) This transformation also scales the integral $\int \beta \cdot h(\xi) d\xi$, and consequently there is a member of the solution set that integrates to 1. Without loss of generality then, when considering $h(\xi) \in L^1(\mathbb{R}^n)$ it may be assumed that $h(\xi)$ is the density of a random variable X. Under this assumption, the convolution equation $h{*}h(\xi) = |\xi|h(\xi)$, is equivalent to the statement

$$X_1 + X_2 \stackrel{d}{=} \widehat{X} \tag{136}$$

where X_1 and X_2 are independent, both distributed as X, and $\widehat{X}$ has the *size-biased* distribution of X. The concept of size-biased is usually formulated for nonnegative random variables. Here it is extended to random variables taking values in $\mathbb{R}^n$: let Y be random variable in $\mathbb{R}^n$ with $\mathbb{E}(|Y|) < \infty$, then $\widehat{Y}$ has the corresponding size-biased distribution if

$$\mathbb{E}(g(\widehat{Y})) = \frac{\mathbb{E}(|Y|g(Y))}{\mathbb{E}|Y|} \tag{137}$$

for every positive Borel function g.

The formulation of the convolution equation as a size-biased equation gives a direct way of computing certain moment restrictions on $|X|$. The analysis of the convolution equation $h{*}h(\xi) = |\xi|h(\xi)$ via the sized-biased equation $X_1 + X_2 = \widehat{X}$ is first done in the case that the density of X has the integral representation

$$G_X(\xi) = \int_0^\infty p_n(\xi, t)\gamma(t) dt, \tag{138}$$

and then simply under the hypothesis that X is rotationally invariant. Unfortunately these moment restriction are of no help in actually solving the convolution equation, or even excluding the existence of solutions in certain dimensions.

The connection between (138) and the Gauss transform is now established. The n-dimensional Gauss transform of the law of a strictly positive random variable A is the law of the random variable $\sqrt{A}Z$ where Z is an n-dimensional multivariate standard normal random variable that is independent of A. (Some less obvious and more interesting properties of the Gauss transform are discussed in [14, p. 108]).

3.9 Proposition (well-known). *Let Z be a standard normal random variable on $\mathbb{R}^n$, and let T be independent of Z with density $\gamma(t)$ on $[0,\infty)$. Then $X = \sqrt{2T}Z$ has density*

$$G_X(\xi) = \int_0^\infty p_n(\xi,t)\gamma(t)dt. \tag{139}$$

Proof. The joint density of the pair (Z,T) on $\mathbb{R}^n \times [0,\infty)$ is

$$f_{Z,T}(z,t) = \frac{\gamma(t)}{(2\pi)^{n/2}} e^{-|z|^2/2} \tag{140}$$

where $z = (z_1,\ldots,z_n) \in \mathbb{R}^n$ and $t > 0$. Let (X,S) denote the pair of random variables defined through the transformation

$$\begin{cases} X = \sqrt{2T}Z, \\ S = T, \end{cases} \tag{141}$$

and for the corresponding change of variables, employ the notation

$$\begin{cases} \xi = \sqrt{2t}\, z, \\ s = t, \end{cases} \qquad \begin{cases} z = \xi/\sqrt{2s}, \\ t = s, \end{cases} \tag{142}$$

where $\xi = (\xi_1,\ldots\xi_n) \in \mathbb{R}^n$ is the argument of the density of X. The joint density of (X,S) is

$$f_{X,S}(\xi,s) = f_{Z,T}(z(\xi,s),t(\xi,s)) \left|\frac{\partial(z,t)}{\partial(\xi,s)}\right| \tag{143}$$

and the Jacobian is easily computed as

$$\left|\frac{\partial(z,t)}{\partial(\xi,s)}\right| = \left|\frac{\partial(z_1,\ldots,z_n,t)}{\partial(\xi_1,\ldots,\xi_n,s)}\right| = \frac{\partial t}{\partial s}\cdot\prod_{i=1}^n \frac{\partial z_i}{\partial \xi_i} = (2s)^{-n/2}. \tag{144}$$

Then

$$f_{X,S}(\xi, s) = \frac{\gamma(s)}{(4\pi s)^{n/2}} e^{-|\xi|^2/4s}, \tag{145}$$

and the marginal density of X in the pair (X, S) is

$$G_X(\xi) = \int_0^\infty f_{X,S}(\xi, s)ds = \int_0^\infty \frac{1}{(4\pi s)^{n/2}} e^{-|\xi|^2/4s}\gamma(s)ds. \tag{146}$$

□

Proposition 3.9 makes transparent the relationship between the moments of $G_X(\xi)$ and $\gamma(t)$. Let

$$\mu_s = \int_{\mathbb{R}^n} |\xi|^s G_X(\xi)d\xi \tag{147}$$

and

$$\gamma_s = \int_0^\infty t^s \gamma(t)dt. \tag{148}$$

Here s is any positive real number so perhaps μ_s and γ_s should be called *generalized absolute moments* and *generalized moments* respectively, but for brevity all are referred to as moments.

3.10 Lemma. *The moments μ_s and γ_s are related by*

$$\mu_s = 2^s \frac{\Gamma(\frac{n+s}{2})}{\Gamma(\frac{n}{2})} \gamma_{s/2} \tag{149}$$

Proof. Since $|Z|^2$ is a chi-squared random variable with n degrees of freedom, it follows that

$$\mathbb{E}|Z|^s = \frac{2^{-n/2}}{\Gamma(\frac{n}{2})} \int_0^\infty x^{s/2} x^{n/2} e^{-x/2} \frac{dx}{x} = 2^{s/2} \frac{\Gamma(\frac{n+s}{2})}{\Gamma(\frac{n}{2})}. \tag{150}$$

Now apply Proposition 3.9 and use the independence of Z and T:

$$\mu_s = \mathbb{E}|X|^s = \mathbb{E}(\sqrt{2T})^s \mathbb{E}|Z|^s = 2^{s/2}\gamma_{s/2} 2^{s/2} \frac{\Gamma(\frac{n+s}{2})}{\Gamma(\frac{n}{2})}. \tag{151}$$

□

3.11 Theorem. *Let X be a random variable on $\mathbb{R}^n$ satisfying the size-biased equation $X_1 + X_2 \stackrel{d}{=} \widehat{X}$ where X, X_1 and X_2 are independent and identically distributed. Suppose that the density of X enjoys the representation*

$$G_X(\xi) = \int_0^\infty p_n(\xi, t)\gamma(t)dt. \tag{152}$$

Then the following moment relation holds for the representing measure γ:

$$(\gamma*\gamma)_s\Gamma(s+\tfrac{n}{2}) = 2\gamma_{s+\frac{1}{2}}\Gamma(s+\tfrac{n}{2}+\tfrac{1}{2}). \tag{153}$$

Proof. According to Lemma 3.3 the density of $\widehat{X}$ has the form

$$G_X*G_X(\xi) = \int_0^\infty p_n(\xi,t)\gamma*\gamma(t)dt, \tag{154}$$

and by Lemma 3.10

$$\mathbb{E}|\widehat{X}|^{2s} = 2^{2s}(\gamma*\gamma)_s\frac{\Gamma(s+\frac{n}{2})}{\Gamma(\frac{n}{2})}. \tag{155}$$

At the same time, using the fact that $\widehat{X}$ is distributed as size-biased X, let $g(x) = |x|^{2s}$ in the definition of size–biased, equation (137), to conclude that

$$\mathbb{E}|\widehat{X}|^{2s} = \frac{1}{\mathbb{E}|X|}\mathbb{E}|X|^{2s+1} = \frac{1}{\mathbb{E}|X|}2^{2s+1}\gamma_{s+\frac{1}{2}}\frac{\Gamma(s+\frac{1}{2}+\frac{n}{2})}{\Gamma(\frac{n}{2})} \tag{156}$$

with

$$\mathbb{E}|X| = \int_{\mathbb{R}^n}|\xi|G_X(\xi)d\xi = \int_{\mathbb{R}^n}G_X*G_X(\xi)d\xi = 1. \tag{157}$$

Equating these to expressions for $\mathbb{E}|\widehat{X}|^{2s}$ gives equation (153). □

3.12 Theorem. *Let X be a rotationally invariant random variable on $\mathbb{R}^n$ satisfying the size-biased equation*

$$X_1+X_2 \stackrel{d}{=} \widehat{X} \tag{158}$$

where X, X_1 and X_2 are independent and identically distributed. Then the moments $\mu_k = \mathbb{E}|X|^k$ satisfy the infinite system of equations

$$\mu_1 = 1;\quad \mu_{2k+1} = \Gamma(k+\tfrac{n}{2})\Gamma(\tfrac{n}{2})\sum_{j=0}^{k}\binom{k}{j}\frac{\mu_{2k-2j}\,\mu_{2j}}{\Gamma(k-j+\frac{n}{2})\Gamma(j+\frac{n}{2})},\quad \textit{for } k\geq 1. \tag{159}$$

The first few moment relations given by (159) are

$$\mu_1 = 1,\quad \mu_3 = 2\mu_2,\quad \mu_5 = 2\mu_4+2\Big(\frac{n+2}{n}\Big)\mu_2^2,\quad \mu_7 = 2\mu_6+6\Big(\frac{n+4}{n+2}\Big)\mu_2\mu_4. \tag{160}$$

Remarkably, the relation between the second and third moment is independent of the dimension. Perhaps this could be construed as evidence that dimension as a parameter can not be varied, i.e. that a solution to (158) exists only in dimension 3. (See Open Problem 4.23).

In the following proof of this theorem ω_n denotes the uniform measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, normalized to a probability measure by dividing by $2\pi^{n/2}/\Gamma(\frac{n}{2})$. We let $\mathbf{e}_\xi = \xi/|\xi|$ and $\rho = |\xi|$, and use the notation $d\omega_n(\mathbf{e}_\xi)$ when the measure ω_n is integrated against a continuous function. The Fourier transform of ω_n is

$$\widehat{\omega_n}(x) = \int_{\mathbb{R}^n} e^{i\mathbf{e}_\xi \cdot x} d\omega_n(\mathbf{e}_\xi) = 2^\nu \Gamma(\tfrac{n}{2}) \frac{J_\nu(|x|)}{|x|^\nu}, \tag{161}$$

where $\nu = (n-2)/2$. It is convenient to temporarily adopt the particular version of the Fourier transform implicit in equation (161), and for this version, $\widehat{f * g} = \widehat{f}\,\widehat{g}$. Folland [23, p. 247] provides an indication of this computation of $\widehat{\omega_n}(x)$ in terms of Bessel functions by obtaining a differential equation satisfied by the radial component, starting with the fact that $(1 - |\xi|^2)\omega_n$ is the zero measure. This is also worked out in detail in Donoghue [18, p. 202].

Proof of Theorem 3.12. Let $\vec{\mu}$ denote the law of $|X|$ on $[0, \infty)$. Under the hypothesis of rotational invariance of X, knowing either μ or $\vec{\mu}$ is equivalent to knowing the other, and $\mu_k = \vec{\mu}_k$ for all $k \geq 0$. The Fourier transform of μ may be computed as the Hankel transform of $\vec{\mu}$:

$$\widehat{\mu}(x) = (\mathcal{H}_\nu \vec{\mu})(r) = 2^\nu \Gamma(\tfrac{n}{2}) \int_0^\infty \frac{J_\nu(r\rho)}{(r\rho)^\nu} d\vec{\mu}(\rho), \tag{162}$$

where $r = |x|$. Indeed,

$$\begin{aligned}
\widehat{\mu}(x) &= \int_{\mathbb{R}^n} e^{-i\xi \cdot x} d\mu(\xi) \\
&= \iint_{S^{n-1} \times [0,\infty)} e^{-i\rho \mathbf{e}_\xi \cdot x} d(\omega_n \otimes \vec{\mu})(\mathbf{e}_\xi, \rho) \\
&= \int_0^\infty \int_{S^{n-1}} e^{-i\mathbf{e}_\xi \cdot \rho x} d\omega_n(\mathbf{e}_\xi) d\vec{\mu}(\rho) \\
&= \int_0^\infty \widehat{\omega_n}(\rho|x|) d\vec{\mu}(\rho) = 2^\nu \Gamma(\tfrac{n}{2}) \int_0^\infty \frac{J_\nu(|x|\rho)}{(|x|\rho)^\nu} d\vec{\mu}(\rho).
\end{aligned} \tag{163}$$

The plan is to compare moments through the equation

$$(\mathcal{H}_\nu \vec{\mu})^2(r) = (\mathcal{H}_\nu \rho \vec{\mu})(r) \tag{164}$$

which is just the Hankel transform analogue of $\widehat{\mu}^2(x) = \widehat{|\xi|\mu}(x)$. The integrand in the right hand side of (163) has the well-known power series expansion

$$\frac{J_\nu(z)}{z^\nu} = \frac{1}{2^\nu}\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{4^k k!\Gamma(\nu+k+1)}. \tag{165}$$

Comparing this with the power series expansion of the exponential function establishes that for $z > 0$ and $\nu > -\frac{1}{2}$

$$\frac{J_\nu(z)}{z^\nu} \leq \frac{1}{2^\nu}\sum_{k=0}^{\infty}\frac{z^{2k}}{4^k k!\Gamma(k+\frac{1}{2})} = \frac{1}{2^\nu\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!} \leq \frac{1}{2^\nu\sqrt{\pi}}e^z. \tag{166}$$

This is used to justify the exchanging the order or integration and summation in computing (164). The abscissa of convergence of the Laplace transform of μ is $\sigma_0 = -R$ where R is the radius of convergence of the following power series:

$$\sum_{k=0}^{\infty}\frac{\vec{\mu}_k}{k!}r^k = \sum_{k=0}^{\infty}\int_0^\infty \frac{r^k\rho^k}{k!}d\vec{\mu}(\rho) = \int_0^\infty e^{r\rho}d\vec{\mu}(\rho). \tag{167}$$

Of course, this is certain to hold as an equality between finite magnitudes only for $|r| < R$. It turns out that $R \geq 1$ because the moments of X satisfy $\mu_k \leq k!$ (See Lemma 4.3). It follows that for $|r| < 1$,

$$\int_0^\infty e^{r\rho}d\vec{\mu}(\rho) < \infty \tag{168}$$

making $e^{r\rho}$ a dominating function for $(r\rho)^{-\nu}J_\nu(r\rho)$ with respect to the measure $\vec{\mu}$, as long as $-\frac{1}{2} < \nu$, and $|r| < 1$. Then by Fubini's Theorem, both sides of (164) may be evaluated by exchanging the order of summation and integration, and integrating term by term. The result is

$$\left\{\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\frac{n}{2})\mu_{2k}r^{2k}}{4^k k!\Gamma(k+\frac{n}{2})}\right\}^2 = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\frac{n}{2})\mu_{2k+1}r^{2k}}{4^k k!\Gamma(k+\frac{n}{2})}. \tag{169}$$

Collecting coefficients of r^{2k} gives the desired result, equation (159). $\square$

A converse of Theorem 3.12 is that if X is a rotationally invariant random variable on $\mathbb{R}^n$ whose absolute moments μ_k satisfy (159), and if

$$\mu_{2k} \leq M4^k\, k!\,\Gamma(k+\tfrac{n}{2}) \quad \text{for all} \quad k \geq 0 \tag{170}$$

for some constant M independent of k, or more generally, if

$$\limsup_{k\to\infty} \tfrac{1}{2k}\mu_{2k}^{1/2k} < \infty, \tag{171}$$

then X satisfies the size-biased equation (158). Here is a short explanation. Either of these conditions assure that the Hankel transform of $\vec{\mu}$, the law of $|X|$, may be computed by exchanging the order of summation and integration when the integrand $J_\nu(r\rho)/(r\rho)^\nu$ is expanded in the power series (165). The resulting series is

$$(\mathcal{H}_\nu\vec{\mu})(r) = \Gamma(\tfrac{n}{2})\sum_{k=0}^{\infty}\frac{(-1)^k\mu_{2k}r^{2k}}{4^k k!\Gamma(k+\frac{n}{2})} \tag{172}$$

that represents the transform locally at the origin. It has a positive radius of convergence, R_1 say, that is related to the limit in (171). The Hankel transform of $\rho\vec{\mu}$, which is the law of $|\widehat{X}|$, also has radius of convergence R_1, and has the power series expansion

$$(\mathcal{H}_\nu\rho\vec{\mu})(r) = \Gamma(\tfrac{n}{2})\sum_{k=0}^{\infty}\frac{(-1)^k\mu_{2k+1}r^{2k}}{4^k k!\Gamma(k+\frac{n}{2})}. \tag{173}$$

Condition (159) implies that for $r < R_1$,

$$(\mathcal{H}_\nu\vec{\mu})^2(r) = (\mathcal{H}_\nu\rho\vec{\mu})(r). \tag{174}$$

Thanks to just even powers of r appearing in the Hankel transforms, the corresponding Fourier transforms are necessarily analytic in the open ball $B(0, R_1) \in \mathbb{C}^n$ of radius R_1 centered at the origin. Inside $B(0, R_1)$ the same identity holds for the Fourier transforms. This identity persists on all of $\mathbb{R}^n$ by analytic continuation: according to the theory presented in Chapter 4, if the Fourier transform of a positive function on $\mathbb{R}^n$ is locally analytic at the origin, then it is necessarily holomorphic on a tube in $\mathbb{C}^n$ whose width is determined at the origin. The equality of the characteristic functions of $X_1 + X_2$ and $\widehat{X}$ establishes the advertised distributional equality between the random variables: $X_1 + X_2 \overset{d}{=} \widehat{X}$.

§3.6 The duplication formula for the gamma function

Suppose that the random variable X has the density of the Bessel kernel $G_2(\xi)$ on $\mathbb{R}^3$ whose representing measure is exponential, as established in Section 3.4:

$$G_2(\xi) = \frac{1}{4\pi}\frac{e^{-|\xi|}}{|\xi|} = \int_0^\infty p_n(\xi,t)e^{-t}dt. \tag{175}$$

A direct computation of the moments of X shows that

$$\mathbb{E}|X|^{2s} = \frac{1}{4\pi}\int_{\mathbb{R}^3} |\xi|^{2s-1}e^{-|\xi|}d\xi = \int_0^\infty r^{2s+1}e^{-r}dr = \Gamma(2s+2). \tag{176}$$

At the same time, by Lemma 3.10,

$$\mathbb{E}|X|^{2s} = 2^{2s}\Gamma(s+1)\frac{\Gamma(s+\frac{3}{2})}{\Gamma(\frac{3}{2})}. \tag{177}$$

Equating these two expressions and letting $z = s+1$ yields

$$\Gamma(2z)\sqrt{\pi} = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}), \tag{178}$$

which is the duplication formula for the gamma function. The explanation for this lies in equivalence between the subordination identity (127) and the duplication formula. This equivalence is discussed in Taylor [51, p. 265].

§3.7 Subordinated Brownian motion

Lemma 3.3 shows how the convolution of two functions — within a certain class — may be pushed onto the convolution of the two representing measures. This works for any pair of measures, finite or not. Now suppose that γ_1 and γ_2 are the laws of random variables, T_1 and T_2, say. A probability model that illuminates Lemma 3.3 interprets $G_1(\xi)$ and $G_2(\xi)$ as the densities of independent Brownian motions sampled at the times T_1 and T_2, respectively. Then the sum of the induced random variables is controlled by the the addition T_1+T_2. Here is how this works: Let $\mathfrak{X}_t$, $\mathfrak{X}_t^{(1)}$ and $\mathfrak{X}_t^{(2)}$ be *i.i.d.* Brownian motions on $\mathbb{R}^n$, for example, Brownian motions such that for any Borel set $A \in \mathbb{R}^n$ and for all $t \geq 0$,

$$\mathbb{P}(\mathfrak{X}_t \in A) = \int_A \frac{1}{(\sqrt{4\pi t})^n}e^{-|\xi|^2/4t}dt. \tag{179}$$

Randomize the sampling times for $\mathfrak{X}_t^{(1)}$ and $\mathfrak{X}_t^{(2)}$, as T_1 and T_2, respectively, where the pair (T_1, T_2) is independent of the three Brownian motions. It is not necessary to stipulate independence between T_1 and T_2. Then $\mathfrak{X}_{T_1}^{(1)} + \mathfrak{X}_{T_2}^{(2)}$ and $\mathfrak{X}_{T_1+T_2}$ have the same distribution.

3.13 Theorem. *With the hypotheses on the Brownian motions and sampling times spelled out in the previous paragraph,* $\mathfrak{X}_{T_1}^{(1)} + \mathfrak{X}_{T_2}^{(2)} \stackrel{d}{=} \mathfrak{X}_{T_1+T_2}$.

Proof. A fundamental property of Brownian motion is its independent increments. This may be used to establish that $\mathfrak{X}_{t_1}^{(1)} + \mathfrak{X}_{t_2}^{(2)} \stackrel{d}{=} \mathfrak{X}_{t_1+t_2}$ for any deterministic times t_1 and t_2. The extension to random times may be accomplished using the substitution lemma: for any Borel set $A \subseteq \mathbb{R}^n$,

$$\mathbb{P}(\mathfrak{X}_{T_1}^{(1)} + \mathfrak{X}_{T_2}^{(2)} \in A) = \tag{180}$$

$$\mathbb{E}\left\{\mathbb{E}\left\{\mathbf{1}[\mathfrak{X}_{T_1}^{(1)} + \mathfrak{X}_{T_2}^{(2)} \in A] \,\middle|\, \sigma(T_1, T_2)\right\}\right\} = \tag{181}$$

$$\mathbb{E}\left\{\left[\mathbb{E}\left(\mathbf{1}[\mathfrak{X}_{t_1}^{(1)} + \mathfrak{X}_{t_2}^{(2)} \in A]\right)\right]_{t_1=T_1, t_2=T_2}\right\} = \tag{182}$$

$$\mathbb{E}\left\{\left[\mathbb{E}\left(\mathbf{1}[\mathfrak{X}_{t_1+t_2} \in A]\right)\right]_{t_1=T_1, t_2=T_2}\right\} = \tag{183}$$

$$\mathbb{E}\left\{\mathbb{E}\left\{\mathbf{1}[\mathfrak{X}_{T_1+T_2} \in A] \,\middle|\, \sigma(T_1, T_2)\right\}\right\} = \mathbb{P}(\mathfrak{X}_{T_1+T_2} \in A). \tag{184}$$

In the application of this lemma, the Brownian motions $\mathfrak{X}_t$ $\mathfrak{X}_t^{(1)}$, $\mathfrak{X}_t^{(2)}$ assume values in the state space $\big(C[0,\infty), \mathcal{C}\big)$, where $\mathcal{C}$ is the Borel σ-field determined by the metric

$$\rho(\omega_1, \omega_2) = \sum_{n=1}^{\infty} \max_{0 \le t \le n} \big(|\omega_1(t) - \omega_2(t)| \wedge 1\big). \tag{185}$$

This σ-field coincides with the σ-field generated by the finite dimensional cylinder sets

$$C = \{\omega \in C[0,\infty) : \omega(t_1), \ldots, \omega(t_n) \in A\} \tag{186}$$

where A is Borel in $\mathbb{R}^n$ [29, p. 60]. The random variables T_1 and T_2 assumes values in $\big([0,\infty), \mathcal{B}\big)$ where $\mathcal{B}$ is the Borel σ-field on the half line. The integrable function φ is the composition of the indicator with the map that evaluates a given continuous function at a given argument, for example, $\varphi(\mathfrak{X}_t, T) = \mathbf{1}_{[\mathfrak{X}_T \in A]}$. $\square$

§3.8 Other semigroups of stable laws, and Schoenberg's Theorem

Lemma 3.3 is a generalization of Theorem 3.13 with a more general measure $d\gamma(t)$ playing the role of the random sampling time T. The probabilistic interpretation suggests ways that Lemma 3.3 may be generalized by replacing the Brownian process with another stable process. Let $q_n^{(\alpha)}(\xi, t)$ denote the marginal at t of a symmetric stable process of order α on $\mathbb{R}^n$ with $0 < \alpha \leq 2$. An easier description is

$$q_n^{(\alpha)}(\xi, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-t|x|^\alpha} e^{-ix\cdot\xi} dx, \tag{187}$$

with the Fourier transform adjusted here for interpretation as the inversion of a characteristic function. We could define $q_n^{(\alpha)}(\xi, t)$ for α outside $(0, 2]$, but it would not give probability measures, because $e^{-t|x|^\alpha}$ is positive definite only for α in this range. These stable densities satisfy

$$\underbrace{q_n^{(\alpha)} * \cdots * q_n^{(\alpha)}}_{\kappa \text{ factors}}(\xi, t) = q_n^{(\alpha)}(\xi, \kappa t) = \kappa^{-n/\alpha} q_n^{(\alpha)}(\kappa^{-1/\alpha}\xi, t), \tag{188}$$

a fact directly connected with the defining property of the α–exponent strictly stable multivariate random variables: such a random variable X, at least in the rotationally invariant case, enjoys the distributional equality

$$X_1 + \cdots + X_\kappa \stackrel{d}{=} \kappa^{-1/\alpha} X, \tag{189}$$

where X, and $X_1, \cdots, X_\kappa$ are independent and identically distributed.

The only two radially symmetric stable laws with densities that have expression in terms of elementary functions are the multivariate Cauchy and Gaussian distributions. For these we have

$$q_n^{(1)}(\xi, t) = \pi^{-(n+1)/2} \Gamma\Big(\frac{n+1}{2}\Big) \frac{t}{(t^2 + |\xi|^2)^{(n+1)/2}}, \tag{190}$$

and

$$q_n^{(2)}(\xi, t) = p_n(\xi, t) = (4\pi t)^{-n/2} e^{-|\xi|^2/4t}, \tag{191}$$

respectively. For the other values of α we may obtain a power series expansions of the radial part of the stable density by writing (187) as a Hankel transform, and then expanding either the exponential or Bessel function into a series depending on weather $\alpha < 1$ or $\alpha \geq 1$ [52, p. 212].

The essential property of the Gaussian kernels exploited by Lemma 3.3 is the isomorphism between the convolution semigroup and the additive semigroup of non-negative real numbers,

$$\left(\{p_n(\xi,t) : t \geq 0\}, *\right) \cong \left(\mathbb{R}^+, +\right), \tag{192}$$

a fact that follows directly from the relation

$$p_n(\xi,t_1) * p_n(\xi,t_2) = p_n(\xi,t_1+t_2). \tag{193}$$

Since this isomorphism property is also shared by the convolution semigroup of the stable laws, Lemma 3.3 may be generalized by replacing $p_n(\xi,t)$ by $q_n^{(\alpha)}(\xi,t)$. Even further generalizations are possible, using the transition densities of Levy processes on $\mathbb{R}^n$, but these are not considered here.

What is not obvious, and therefore expressed in the following theorem, is that this generalization using stable densities does not produce a larger class of functions. In particular, it does not enlarge the class of available candidates for solving the convolution inequality $h * h(\xi) \leq |\xi| h(\xi)$, although the alternative representation may purchase some computational efficiency. An example of the use of the stable laws in this capacity may be found in [7, pp. 5015-5021].

3.14 Theorem. *Let $G(\xi)$ have the representation*

$$G(\xi) = \int_0^\infty q_n^{(\alpha)}(\xi,t)\,d\nu(t), \qquad 0 < \alpha \leq 2, \tag{194}$$

for some measure ν, not necessarily finite. Then there exists a measure γ such $G(\xi)$ has the representation

$$G(\xi) = \int_0^\infty p_n(\xi,t)\,d\gamma(t). \tag{195}$$

It is interesting that a special case of this theorem may be proved analytically using Schoenberg's Theorem.

3.15 Theorem (Schoenberg). *A function $F(r)$ defined for $r \geq 0$ has the property that for every integer n, the function*

$$\Phi_n(\xi) = F(|\xi|) \tag{196}$$

is a function of positive type on $\mathbb{R}^n$ if and only if there exists a positive measure μ of finite total mass such that

$$F(r) = \int_0^\infty e^{-r^2\lambda} d\mu(\lambda). \tag{197}$$

This is proved in Donoghue [18, p. 205]. The finiteness of the measure occurs because of an appeal to Helly's selection theorem. This restriction carries over to the proof of the special case.

Proof of Theorem 3.14, special case. The special case is when $\alpha = 1$ and when the total mass of ν is finite. Then the hypothesis is that $G(\xi)$ has the representation

$$G(\xi) = \int_0^\infty q_n^{(1)}(\xi, t) d\nu(t). \tag{198}$$

For the purpose of applying Schoenberg's Theorem, let $\overline{q}_n(r,t) = q_n^{(1)}(|\xi|, t)$ where $r = |\xi|$. The function

$$F(r) = \int_0^\infty \overline{q}_n(r,t) d\nu(t) \tag{199}$$

has the property that in *any* dimension, m say,

$$\Phi_m(\xi) = F(|\xi|) = \int_0^\infty \overline{q}_n(r,t) d\nu(t) \tag{200}$$

is positive definite. To see this, observe that if $m = n$ then $q_n^{(1)}(|\xi|, t)$ is obviously positive definite on $\mathbb{R}^n$, its transform implicit in (187). The key point is that even when $m \neq n$, the functions

$$\xi \rightarrowtail \overline{q}_n(|\xi|, t) = C_n \frac{t}{(t^2 + |\xi|^2)^{(n+1)/2}}, \qquad \xi \in \mathbb{R}^m \tag{201}$$

are still positive definite, for all $t > 0$. Indeed, up to a scaling of the independent and dependent variables, these are Fourier transforms of Bessel kernels, (see Proposition 3.7). The function $\Phi_m(\xi)$ is essentially a mixture of the functions given by equation (201). Applying Schoenberg's Theorem obtains

$$F(r) = \int_0^\infty e^{-\lambda r^2} d\mu(\lambda) \tag{202}$$

for some finite measure μ, and therefore

$$G(\xi) = \int_0^\infty e^{-\lambda|\xi|^2} d\mu(\lambda). \tag{203}$$

Let μ^* denote the pushforward of μ under the involution $\lambda \to (4\lambda)^{-1}$, meaning that for all Borel sets $A \subseteq [0, \infty)$,

$$\mu^*(A) = \mu\big(\{x \in \mathbb{R} : (4x)^{-1} \in A\}\big). \tag{204}$$

Then $G(\xi)$ has the representation (195) where $d\gamma(t) = (4\pi t)^{n/2} d\mu^*(t)$. $\square$

The proof of general case of Theorem 3.14 depends on the following lemma. Feller discusses the one dimensional version of this in a Feller sort of way [20, p. 336]. We need the convolution semigroups of one-sided stable laws $\{u^{(\beta)}(\xi, t) : t \geq 0\}$ of characteristic exponent $0 < \beta \leq 1$ whose Laplace transforms are

$$\int_0^\infty e^{-\lambda\xi} u^{(\beta)}(\xi, t) dx = e^{-t\lambda^\beta}. \tag{205}$$

For these densities the analogue of (188) is

$$\underbrace{u^{(\beta)} * \cdots * u^{(\beta)}}_{\kappa \text{ factors}}(\xi, t) = u^{(\beta)}(\xi, \kappa t) = \kappa^{-1/\beta} u^{(\beta)}(\kappa^{-1/\beta}\xi, t). \tag{206}$$

3.16 Lemma. *For any $0 < \alpha \leq 2$ and $0 < \beta \leq 1$*

$$q_n^{(\alpha\beta)}(\xi, t) = \int_0^\infty q_n^{(\alpha)}(\xi, s) u^{(\beta)}(s, t) ds. \tag{207}$$

Proof. Fix $t > 0$ and let $q(\xi, t)$ be defined by the right hand side of (207). The function $q(\xi, t)$ is rotationally invariant and it can be recognized as the density of a strictly stable random variable of exponent $\alpha\beta$ on $\mathbb{R}^n$ by showing that the for any positive integer λ, the λ-fold convolution with itself satisfies

$$q^{*\lambda}(\xi, t) = \lambda^{-n/\alpha\beta} q(\lambda^{-1/\alpha\beta}\xi, t). \tag{208}$$

First,

$$q^{*\lambda}(\xi, t) = \int_0^\infty q_n^{(\alpha)}(\xi, s) u^{(\beta)}(s, \lambda t) ds \tag{209}$$

by the generalization of Lemma 3.3 and using the semigroup properties of $u^{(\beta)}(s,t)$. Next, using the stability properties of $u^{(\beta)}(s,t)$

$$q^{*\lambda}(\xi,t) = \int_0^\infty q_n^{(\alpha)}(\xi,s)u^{(\beta)}(\lambda^{-1/\beta}s,t)\lambda^{-1/\beta}ds, \tag{210}$$

and the variable substitution $s = \lambda^{1/\beta}s'$ achieves (208), using properties of $q_n^{(\alpha)}(\xi,s)$ and the definition of $q(\xi,t)$. This holds for any $t > 0$. Now letting t vary,

$$q(\xi,t_1) * q(\xi,t_2) = q(\xi,t_1+t_2) \tag{211}$$

again by the generalization of Lemma 3.3. This characterizes $\{q(\xi,t) : t \geq 0\}$ as a semigroup of rotationally invariant strictly stable laws, and identifies it up to semigroup isomorphism with $\{q_n^{(\alpha\beta)}(\xi,t) : t \geq 0\}$. In order to exclude the possibility that

$$q(\xi,t) = q_n^{(\alpha\beta)}(\xi,ct) \tag{212}$$

for some $c \neq 1$, take the inverse Fourier transform of both sides of (207) (with the transform adjusted to be considered a characteristic function) and note that the transform of $q_n^{(\alpha\beta)}(\xi,t)$ is essentially the Laplace transform of $u^{(\beta)}(s,t)$ evaluated at $|x|^\alpha$:

$$\begin{aligned}\int_{\mathbb{R}^n} q(\xi,t)e^{i\xi\cdot x}d\xi &= \int_0^\infty e^{-s|x|^\alpha}u^{(\beta)}(s,t)ds \\ &= e^{-t|x|^{\alpha\beta}} \\ &= \int_{\mathbb{R}^n} q_n^{(\alpha\beta)}(\xi,t)e^{i\xi\cdot x}d\xi. \end{aligned}\tag{213}$$

The first equality uses the definition of $q(\xi,t)$, with an exchange in the order of integration. Since the transforms of $q(\xi,t)$ and $q_n^{(\alpha\beta)}(\xi,t)$ agree for all $t > 0$, c = 1 in equation (212). □

Proof of Theorem 3.14, general case. Suppose

$$G(\xi) = \int_0^\infty q_n^{(\alpha)}(\xi,t)d\nu(t) \tag{214}$$

for some $0 < \alpha < 2$. By means of Lemma 3.16,

$$G(\xi) = \int_0^\infty \left(\int_0^\infty q_n^{(2)}(\xi,s)u^{(\alpha/2)}(s,t)ds\right)d\nu(t), \tag{215}$$

and by Tonelli's Theorem

$$G(\xi) = \int_0^\infty p_n(\xi, s)\gamma(s)ds \tag{216}$$

where

$$\gamma(s) = \int_0^\infty u^{(\alpha/2)}(s,t)d\nu(t). \tag{217}$$

$\square$

§3.9 Size-biased and infinitely divisible random variables.

A random variable X taking values in $\mathbb{R}^n$ is infinitely divisible if for each positive integer k there exist *i.i.d.* random variables $Y_1^{(k)}, \dots, Y_k^{(k)}$ such that

$$X \stackrel{d}{=} Y_1^{(k)} + \cdots + Y_k^{(k)}. \tag{218}$$

The notion may be extended to non-negative functions $f : \mathbb{R}^n \to \mathbb{R}^+ \cup \{\infty\}$ that are not necessarily integrable. Such a function is infinitely divisible if for each positive integer k there exists a non-negative function g_k such that

$$f = g_k^{*k} = \underbrace{g_k * \cdots * g_k}_{k \text{ factors}}. \tag{219}$$

With this extension the known solutions to $h{*}h(\xi) = |\xi|h(\xi)$ presented at the beginning of this chapter are all infinitely divisible. Indeed, the exponential distribution is infinitely divisible, and multiplication by any $e^{\alpha\cdot\xi}$ does not alter the infinite divisibility of this or any other solution to the convolution equation. The Bessel kernels are infinitely divisible, as they belong to a convolution semigroup as noted in Section 3.4. Finally the infinitely divisibility of the Riesz kernel solutions may be seen by the following convolution formula that appears in Proposition 3.5: With $0 < \alpha, \beta, \alpha + \beta < n$,

$$\left(|x|^{\alpha-n} * |x|^{\beta-n}\right)(y) = \int_{\mathbb{R}^n} |z|^{\alpha-n}|y-z|^{\beta-n}dz \tag{220}$$

$$= \frac{c_{n-\alpha-\beta}c_\alpha c_\beta}{c_{\alpha+\beta}c_{n-\alpha}c_{n-\beta}}|y|^{\alpha+\beta-n}; \qquad c_\alpha := \pi^{-\alpha/2}\Gamma(\frac{\alpha}{2}). \tag{221}$$

For yet another example, coming from a different type of convolution equation, in Chapter 7 the function

$$h(\xi) = \frac{6}{\sqrt{2\pi}}\frac{\pi\xi}{\sinh \pi\xi} \tag{222}$$

is shown to solve the convolution equation

$$h * h(\xi) = \sqrt{2\pi}(1 + |\xi|^2)h(\xi) \tag{223}$$

and suitably normalized, this is the density of an infinitely divisible random variable whose associated Levy process is the Meixner process [49, p. 24]. The infinite divisibility of all of these solutions suggests the following:

3.17 Open problem. *Prove or disprove, adding as needed hypotheses on the symmetry of $h(\xi)$, positive definiteness of $h(\xi)$, or support of $h(\xi)$: In any dimension non-negative solutions to $h{*}h(\xi) = |\xi|h(\xi)$, or more generally the equation $h{*}h(\xi) = |\xi|^\theta h(\xi)$, $\theta > 0$, or even $h{*}h(\xi) = g(\xi)h(\xi)$ for certain $g(\xi)$, are necessarily infinitely divisible.*

This appears to be a difficult problem that will probably not be solved until there is a complete characterization of the solutions to such convolution equations in all dimensions. Adhering to the methodology of approaching hard problems by first considering easier ones, observe that if $h(\xi)$ is infinitely divisible and solves $h{*}h(\xi) = |\xi|h(\xi)$, then $|\xi|h(\xi)$ is also infinitely divisible. This observation leads to the following ancillary question, that recognizes that perhaps the convolution equation itself is not necessary for the conclusion: *If X is an infinitely divisible random variable, is the size-biased random variable $\widehat{X}$ necessarily infinitely divisible?* The answer is no. Some examples of infinitely divisible random variables whose size-biased random variables are, and are not, infinitely divisible, are given next, and then an open problem is stated.

In each of the following cases both the original and size–biased random variable are infinitely divisible:

1. Any gamma random variable.

2. A geometric random variable X with probability mass function $\mathbb{P}\{X = k\} = qp^k$, $k \geq 0$, has $\widehat{X} \stackrel{d}{=} 1 + X_1 + X_2$ where X_1 and X_2 are independent and distributed as X.

3. For a Poisson random variable Y, $\widehat{Y} \stackrel{d}{=} 1 + Y$.

These examples are fairly easy. Poisson random variables are central to discussions of infinite divisibility, see for example [50].

The first of the following three counter-examples relies on the arithmetic structure of the support of the random variable, the second is analytic. The third is included for general mathematical interest, as it involves a property of the Riemann zeta function.

3.18 Counterexample (arithmetical). *There exists an infinitely divisible random variable Z such that $\widehat{Z}$ (having the size-biased distribution of Z) is not infinitely divisible.*

Proof. Let X be a (shifted) geometric random variable, that is, with probability mass function

$$\mathbb{P}(X = k) = qp^k \quad \text{for} \quad k \geq 0, \tag{224}$$

with $q = 1 - p \in (0,1)$. The probability generating function of X is

$$G_X(s) = \mathbb{E}(s^X) = q(1 - ps)^{-1} \tag{225}$$

with nth root

$$[G_X(s)]^{1/n} = \frac{q^{1/n}}{(1 - ps)^{1/n}} = q^{1/n} \sum_{k=0}^{\infty} \binom{-1/n}{k} (-1)^k p^k s^k \tag{226}$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} \qquad (\alpha \neq 0). \tag{227}$$

The parity of the number of negative terms in the numerator of $\binom{-1/n}{k}$ is matched by the factor $(-1)^k$, so that the coefficient of s^k is always positive. Since $[G(1)]^{1/n} = 1$, these positive coefficients sum to 1, and there exists a well-defined random variable $Y^{(n)}$ whose probability generating function is $G_{Y^{(n)}}(s) = [G_X(s)]^{1/n}$. In fact, let $Y^{(n)}$ be such that

$$\mathbb{P}(Y^{(n)} = k) = q^{1/n} \binom{-1/n}{k} (-1)^k p^k \quad \text{for} \quad k \geq 0. \tag{228}$$

This shows that X is infinitely divisible and gives an explicit construction of the *i.i.d.* summands such that

$$Y_1^{(n)} + \cdots + Y_n^{(n)} \stackrel{d}{=} X. \tag{229}$$

Let $Z = \alpha X_1 + \beta X_2$ where X_1 and X_2 are independent, both distributed as X, and $\alpha < \beta < 2\alpha$, $\beta \neq \frac{3}{2}\alpha$. Then Z is infinitely divisible, and assumes values in the set

$$L = \{m\alpha + n\beta : m \geq 0, n \geq 0\} = \{0, \alpha, \beta, \alpha+\beta, 2\alpha, \cdots\}. \tag{230}$$

Observe that $\beta - \alpha \notin L$ and $(\alpha+\beta)/2 \notin L$.

The size-biased random variable $\widehat{Z}$ is not infinitely divisible. Indeed, $\widehat{Z}$ assumes values in $L/\{0\}$, since $\mathbb{P}(\widehat{Z} = 0) = 0$, but otherwise $\widehat{Z}$ assumes the same set of values as Z, but with different (size-biased) probabilities. Suppose $\widehat{Z} \stackrel{d}{=} T_1 + T_2$ where independent random variables T_1 and T_2 have the same distribution. This will produce a contradiction. The smallest value assumed by $\widehat{Z}$ with positive probability is α, hence $\mathbb{P}(T_1 = \frac{1}{2}\alpha) > 0$. The second smallest value assumed by $\widehat{Z}$ with positive probability is β, hence one of the following holds:

$$\mathbb{P}\big(T_1 = \tfrac{1}{2}\beta\big) > 0 \quad \text{or} \quad \mathbb{P}\big(T_1 = \beta - \tfrac{1}{2}\alpha\big) > 0. \tag{231}$$

The first case implies that

$$\mathbb{P}\big(\widehat{Z} = \tfrac{1}{2}(\alpha+\beta)\big) = \mathbb{P}\big(T_1 = \tfrac{1}{2}\alpha\big)\mathbb{P}\big(T_2 = \tfrac{1}{2}\beta\big) > 0; \tag{232}$$

the second case implies that

$$\mathbb{P}\big(\widehat{Z} = 2\beta - \alpha\big) = \mathbb{P}^2\big(T_1 = \beta - \tfrac{1}{2}\alpha\big) > 0. \tag{233}$$

Both of these values lie outside the support of $\widehat{Z}$. □

3.19 Counterexample (analytic). *Let Z be a rotationally invariant $\mathbb{R}^n$-valued random variable. Assume that Z has a strictly positive density, $f_Z(x)$ say, and in particular, that $f_Z(0) < \infty$. Then the size-biased random variable $\widehat{Z}$ is not infinitely divisible.*

Proof. Suppose $\widehat{Z}$ were infinitely divisible. Then there is a random variable Y such that two independent copies of Y sum to $\widehat{Z}$ in distribution. Moreover Y itself is infinitely divisible, and so the characteristic function of Y, say $\phi_Y(\xi)$, is never zero. Whereas $\widehat{Z}$ is rotationally invariant, its characteristic function $\phi_{\widehat{Z}}(\xi)$ is both real and rotationally invariant. The relationship $\phi_Y^2(\xi) = \phi_{\widehat{Z}}(\xi)$ holds; and by continuity, and the fact that

$\phi_Y(\xi)$ is never zero, $\phi_Y(\xi)$ is everywhere the positive square root of $\phi_{\widehat{Z}}(\xi)$. Thus $\phi_Y(\xi)$ is rotationally invariant and the same holds Y. Let $g(x) \geq 0$ be the Radon–Nikodym derivative of the law of Y with respect to Lebesgue measure. (The singular part of the law of Y in the Lebesgue decomposition with respect to Lebesgue measure must be zero, because if Y has atoms so would $\widehat{Z}$.) Using the rotational invariance of $g(x)$, and the hypothesis that $f_Z(0) < \infty$,

$$f_{\widehat{Z}}(x)\Big|_{x=0} = |x| f_Z(x) \frac{1}{\mathbb{E}|Z|}\Big|_{x=0} = 0 = g * g(0) = \int_{\mathbb{R}^n} g^2(x)dx, \tag{234}$$

arriving at the conclusion $g(x) \equiv 0$, contradicting the fact that $g(x)$ is a probability density. □

3.20 Counterexample (number theoretic). *Fix $\sigma > 1$ and let Z_σ denote the random variable whose characteristic function is related to the Riemann zeta function via*

$$\phi_{Z_\sigma}(t) = \zeta^{-1}(\sigma)\zeta(\sigma - it). \tag{235}$$

Then Z_σ is infinitely divisible but $\widehat{Z_\sigma}$ (having the size-biased distribution of Z_σ) is not infinitely divisible.

Proof. The infinite divisibility of Z_s is proved in Chung [16, pp. 256–259]. The random variable Z_σ assumes values in the additive semigroup

$$L = \{\log n : n = 1, 2, 3, \dots\} \tag{236}$$

while $\widehat{Z_\sigma}$ assumes values in $L' = L/\{0\}$, a fact associated with the equation

$$\zeta'(\sigma)\phi_{\widehat{Z_\sigma}}(t) = \zeta(\sigma)\phi'_{Z_\sigma}(t). \tag{237}$$

Applying the same argument as given in the proof of Counterexample 3.18 obtains the fact that $\widehat{Z_\sigma}$ is not infinitely divisible. □

In all these counterexamples we see that where an obstruction to the infinite divisibility of the size-biased random variable occurs, it occurs at the origin. This has to do with the absolute value function going to zero, at zero.

3.21 Open problem. *Characterize the infinitely divisible random variables whose size-biased distributions are also infinitely divisible.*

Even though it is not known if solutions to the convolution equation $h*h(\xi) = |\xi|^\theta h(\xi)$ are necessarily infinitely divisible, the next section established that certain solutions to convolution equations of this type share an important property with the symmetric infinitely divisible laws.

§3.10 The positive definiteness of majorizing kernels

Known rotationally invariant solutions to convolution equations of the form $h*h(\xi) = g(\xi)h(\xi)$ are either integrable and positive definite, or else they are distributions of positive type. No counter-examples are known. A partial reason for this is given in this section, although it might not be the key reason: the arguments presented here depend on there being an *even* number of convolution factors, and there could exist alternative arguments that generalize to equations having triple convolutions, for example.

3.22 Theorem. *Let $h(\xi)$, not necessarily positive, solve the convolution equation*

$$h{*}h(\xi) = g(\xi)h(\xi), \tag{238}$$

and suppose that $h(\xi)$ is the sum of an L^1-function and an L^2-function, and satisfies the condition $h(\xi) = h(-\xi)$ for all $\xi \in \mathbb{R}^n$. Suppose that $f(\xi) = [g(\xi)]^{-1}$ is a bounded integrable function with $\widetilde{f}(x) = (\mathcal{F}^{-1}f)(x)$ strictly positive and integrable. Then $\widetilde{h}(x) = (\mathcal{F}^{-1}h)(x)$ is strictly positive.

3.23 Theorem. *Let $h(\xi) > 0$ solve the convolution equation*

$$h{*}h(\xi) = g(\xi)h(\xi), \tag{239}$$

and suppose that $h(\xi)$ is integrable and satisfies the condition $h(\xi) = h(-\xi)$ for all $\xi \in \mathbb{R}^n$. Suppose that the multiplicative inverse of $g(\xi)$ has the integral representation

$$f(\xi) = \frac{1}{g(\xi)} = \int_0^\infty p_n(\xi,t)\,d\gamma(t) = \int_0^\infty \frac{1}{(\sqrt{4\pi t})^n} e^{-|\xi|^2/4t}\,d\gamma(t) \tag{240}$$

for some positive measure γ. Then $(\mathcal{F}^{-1}h)(x)$ is strictly positive.

To prove these theorems the key idea is that $(\mathcal{F}^{-1}h)(x)$ is the convolution of a strictly positive function and the square of a real valued function. Knowing the specific function $g(\xi)$ can facilitate implementing this idea. Since trying to prove the most general case encounters technical difficulties with the Fourier transform operational calculus, the hypotheses on $g(\xi)$ are taken to be sufficiently general to cover many interesting convolution equations, but at the same time, permit fairly straightforward proofs of these theorems. These conditions include the case $g(\xi) = |\xi|^\theta$ for various $\theta > 0$, that arise in the analysis of the Navier-Stokes equations, and the case $g(\xi) = 1 + |\xi|^2$ which occurs in the analysis of the FKPP equation. Generalizations of these theorems are certainly possible.

Proof of Theorem 3.22. We may assume that $h(\xi)$ is integrable. To see this, write the equation as

$$f(\xi)h{*}h(\xi) = h(\xi), \qquad f(\xi) = [g(\xi)]^{-1} \in L^1 \cap L^\infty. \tag{241}$$

Let $h(\xi) = h_1(\xi) + h_2(\xi)$ where $h_1 \in L^1$ and $h_2 \in L^2$. Then

$$\begin{aligned} \int h(\xi)d\xi &= \sum_{i,j=1}^{2} \int f(\xi)h_i * h_j(\xi)d\xi \\ &\le \|f\|_{L^\infty}\|h_1 * h_1\|_{L^1} + 2\|f\|_{L^\infty}\|h_1 * h_2\|_{L^1} + \|f\|_{L^1}\|h_2 * h_2\|_{L^\infty} \\ &\le \|f\|_{L^\infty}\|h_1\|_{L^1}^2 + 2\|f\|_{L^\infty}\|h_1\|_{L^1}\|h_2\|_{L^2} + \|f\|_{L^1}\|h_2\|_{L^2}^2 < \infty. \end{aligned} \tag{242}$$

This uses Theorems 2.4, 2.5, and 2.6.

Taking the inverse Fourier transform of equation (241) (justified below), implies that $\widetilde{h}(x)$ solves

$$\widetilde{h}(x) = (\mathcal{F}^{-1}h)(x) = \{\widetilde{f}(x)\}{*}\{(\mathcal{F}^{-1}h)^2(x)\} = \widetilde{f} * \widetilde{h}^2(x). \tag{243}$$

The hypothesis on $h(\xi)$ implies that $\widetilde{h}(x)$ is real, and consequently $\widetilde{h}(x)$ is strictly positive, being the convolution of a positive function and a non-negative function.

Obtaining (243) is now justified. Let $\{\phi_k(\xi)\}_{k=0}^\infty$ be a delta sequence of Gaussians, each element of which integrates to 1. Then as $k \to \infty$,

$$h{*}h{*}\phi_k(\xi) \xrightarrow{L^1} h * h(\xi) \tag{244}$$

and since $f(\xi) \in L^\infty$, we also have the L^1–convergence

$$f(\xi)h{*}h{*}\phi_k(\xi) \xrightarrow{L^1} f(\xi)h{*}h(\xi). \tag{245}$$

Let $\widetilde{\phi}_k(x) = (\mathfrak{F}^{-1}\phi_k)(x)$ and note that $\lim_{k\to\infty}\widetilde{\phi}_k(x) \equiv (2\pi)^{-n/2}$. Applying $\mathfrak{F}^{-1}$ to the sequence in the left hand side of (245) obtains, for each k,

$$\begin{aligned} \mathfrak{F}^{-1}\Big\{f(\xi)h{*}h{*}\phi_k(\xi)\Big\} &= (2\pi)^{-n/2}\Big\{\widetilde{f}(x)\Big\}{*}\Big\{\left(\mathfrak{F}^{-1}h{*}h{*}\phi_k\right)(x)\Big\} \\ &= (2\pi)^{n/2}\Big\{\widetilde{f}(x)\Big\}{*}\Big\{\widetilde{h}^2(x)\cdot\widetilde{\phi}_k(x)\Big\}. \end{aligned} \tag{246}$$

The first equality holds by Theorem 2.1 as

$$f(\xi), \quad h{*}h{*}\phi_k(\xi), \quad \widetilde{f}(x), \quad \left(\mathfrak{F}^{-1}h{*}h{*}\phi_k\right)(x) = \widetilde{h}^2(x)\cdot\widetilde{\phi}_k(x) \tag{247}$$

are all L^1-functions. The second equality holds by Theorem 2.1. We have the pointwise convergence

$$(2\pi)^{n/2}\Big\{\widetilde{f}(x)\Big\}{*}\Big\{\widetilde{h}^2(x)\cdot\widetilde{\phi}_k(x)\Big\} \longrightarrow \widetilde{f}(x){*}\widetilde{h}^2(x), \tag{248}$$

and the convergence is also in $L^\infty \cap C_0$, because the entire sequence is the image of a Cauchy sequence under the continuous transformation between Banach spaces:

$$\mathfrak{F}^{-1} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \cap C_0(\mathbb{R}^n). \tag{249}$$

Thus the pointwise limit agrees with the L^∞-limit, and the limits of the two Cauchy sequences are Fourier transform pairs: $f(\xi)h{*}h(\xi) \longleftrightarrow \widetilde{f} * \widetilde{h}^2(x)$. $\square$

Proof of Thereom 3.23. First, write the convolution equation as

$$f(\xi)h * h(\xi) = \int_0^\infty h * h(\xi)p_n(\xi,t)d\gamma(t) = h(\xi). \tag{250}$$

Then for any $x \in \mathbb{R}^n$, using the hypothesis that $h(\xi) > 0$,

$$\int_{\mathbb{R}^n}\left[\int_0^\infty \left| e^{ix\cdot\xi}\, h * h(\xi)\, p_n(\xi,t)\right| d\gamma(t)\right] d\xi = \|h(\xi)\|_{L^1} < \infty, \tag{251}$$

and it follows by the Fubini-Tonelli theorem that

$$\begin{aligned}
(\mathcal{F}^{-1}h)(x) &= (2\pi)^{-n/2}\int_0^\infty \left[\int_{\mathbb{R}^n} e^{ix\cdot\xi} h * h(\xi) p_n(\xi,t)d\xi\right] d\gamma(t) \\
&= (2\pi)^{-n/2}\int_0^\infty \left\{(\mathcal{F}^{-1}h)^2(x)\right\} * \left\{\widetilde{p}_n(x,t)\right\} d\gamma(t) \\
&= \left\{(\mathcal{F}^{-1}h)^2(x)\right\} * \left\{(\mathcal{F}^{-1}f)(x)\right\}. \qquad (252)
\end{aligned}$$

The symmetry of $h(\xi)$ implies that its transform is real, and therefore $(\mathcal{F}^{-1}h)(x)$ is the convolution of a non-negative function and a strictly positive function. The middle equality is justified by applying the same argument as the preceding theorem, using the fact that

$$\widetilde{p}_n(x,t) = (\mathcal{F}^{-1}p)(\xi,t) \in L^1 \cap L^\infty. \qquad (253)$$

Here $(\mathcal{F}^{-1}f)(x) = \int_0^\infty \widetilde{p}_n(x,t)\, d\gamma(t)$ by definition, since there is no a priori assignment of $f(\xi) = [g(\xi)]^{-1}$ to any particular space where the Fourier transform is defined. $\square$

CHAPTER 4

THE STRUCTURE OF MAJORIZING KERNELS, PART II

§4.1 Introduction: a question about log-convexity

This chapter continues with the structure theory of majorizing kernels, discussing those majorizing kernels whose inverse Fourier transforms are analytic. It is motivated by the following two facts and their juxtaposition, leading to Question 4.1 below:

(*i*) The sets $\mathcal{H}^{n,\theta}$ (consisting of majorizing kernels on $\mathbb{R}^n$ of exponent θ) are logarithmically convex. That is, if $h_1(\xi) \geq 0$ and $h_2(\xi) \geq 0$ satisfy

$$h_1 * h_1(\xi) \leq B_1 |\xi|^\theta h(\xi), \qquad \text{and} \qquad h_2 * h_2(\xi) \leq B_2 |\xi|^\theta h(\xi), \tag{254}$$

then $h(\xi) = [h_1(\xi)]^{\sigma_1}[h_2(\xi)]^{\sigma_2}$ verifies the inequality $h * h(\xi) \leq B|\xi|^\theta h(\xi)$ where $B = B_1^{\sigma_1} B_2^{\sigma_2}$, $\sigma_1 + \sigma_2 = 1$ and $0 \leq \sigma_1, \sigma_2 \leq 1$. Moreover, if $h_1(\xi) \in \mathcal{H}^{n,\theta_1}$ and $h_2(\xi) \in \mathcal{H}^{n,\theta_2}$, then $h(\xi) = [h_1(\xi)]^{\sigma_1}[h_2(\xi)]^{\sigma_2} \in \mathcal{H}^{n,\theta_3}$, where $\theta_3 = \sigma_1\theta_1 + \sigma_2\theta_2$.

(*ii*) The inverse Fourier transform of

$$h(\xi) = \frac{\beta}{2\pi} \frac{e^{-\beta|\xi|}}{|\xi|}, \qquad \xi \in \mathbb{R}^3 \tag{255}$$

(which solves $h * h(\xi) = |\xi| h(\xi)$ for any $\beta > 0$) is analytic on the tube $\mathbb{R}^3 + iU_\beta = \{x + iy \in \mathbb{C}^3 : |y| < \beta\}$. The fact that $(\mathcal{F}^{-1}h)(x)$ is analytic on some tube $\mathbb{R}^3 + iU_\epsilon$ may be determined directly from the equation $h * h(\xi) = |\xi| h(\xi)$, along with the condition $h(\xi) \in L^1(\mathbb{R}^3)$, without actually knowing this family of solutions.

The first fact is useful in constructing exponent θ majorizing kernels across a range of values for θ, applicable in the construction of Banach spaces $\mathbb{B}_h$ and $\mathbb{B}_{h,T^*}$ used in the local existence and uniqueness proofs for Navier-Stokes equations. The utility of the second fact is this: Suppose $u(x,t) \in \mathbb{R}^3$ is a solution to Navier-Stokes equations such that $|\widehat{u}(\xi,t)| \leq K h(\xi)$ for some constant K with $h(\xi)$ given by equation (255). Then the components $u_j(x,t)$ of $u(x,t)$, which are given by

$$u_j(x,t) = (\mathcal{F}^{-1}\widehat{u}_j)(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^3} e^{i\xi\cdot x} \widehat{u}_j(\xi,t) d\xi, \quad 1 \leq j \leq 3, \tag{256}$$

each extend in the spatial variable to a holomorphic function of several complex variables. This particular $h(\xi)$ is an example of a type of majorizing kernel that may be called *analytic-majorizing* (even though it is the inverse Fourier transform that is analytic, not $h(\xi)$ that is analytic). It is natural to ask: Is the sub-collection of such majorizing kernels $\mathcal{H}_A^{n,\theta} \subset \mathcal{H}^{n,\theta}$ say, also log-convex? The confluence of log-convexity and analyticity of the inverse Fourier transform (within a certain class of majorizing kernels) leads to the following question, that is slightly more general than asking if $\mathcal{H}_A^{n,\theta}$ is log-convex:

4.1 Question. *Let $h_1(\xi) \geq 0$ and $h_2(\xi) \geq 0$ solve*

$$h_1{*}h_1(\xi) \leq |\xi|^{\theta_1} h_1(\xi) \quad \textit{and} \quad h_2{*}h_2(\xi) \leq |\xi|^{\theta_2} h_2(\xi) \tag{257}$$

respectively, and suppose both of these have analytic inverse Fourier transforms. Then does $h(\xi) = [h_1(\xi)]^{\sigma_1}[h_2(\xi)]^{\sigma_2}$, which necessarily solves $h{*}h(\xi) \leq |\xi|^{\theta_3} h(\xi)$, $\theta_3 = \sigma_1\theta_1 + \sigma_2\theta_2$, also have analytic inverse Fourier transform?

It may not be obvious that the answer is yes. The analyticity discussed in Theorem 4.7 below traces back to the *equality* in $h{*}h(\xi) = |\xi|^{\theta} h(\xi)$, and when this is replaced by an *inequality*, as occurs when taking log-convex combinations, trying to prove an analogous result encounters an inequality that "goes the wrong way".

Using this question as a starting point, this chapter presents some results about majorizing kernels whose inverse Fourier transforms are analytic, the moments of such majorizing kernels, and the connections between these things. It is organized as follows: First assertions (i) and (ii) above are justified. Question 4.1 is made precise and well-posed, and then answered by Theorem 4.17. A comparison with a related theorem in harmonic analysis is made. Finally open problems are stated.

§4.2 Motivating statements justified; theorems about moments, analyticity

The first assertion, about the logarithmic convexity of $\mathcal{H}^{n,\theta}$, and its generalization, may be proved by applying Hölders inequality to the integral expressing the convolution. This is done in detail in Corollary 2.1 of [7]. The rest of this section deals with the second assertion.

4.2 Lemma. *Suppose that $h(\xi) \geq 0$ solves $h*h(\xi) = B\,|\xi|^\theta\, h(\xi)$ and $h(\xi) \in L^1(\mathbb{R}^n)$. Let $h_1(\xi) = B^{-1}h(\xi)$ and let $h_2(\xi) = \beta^{n-\theta}h_1(\beta\xi) = \beta^{n-\theta}B^{-1}h(\beta\xi)$. Then $h_1(\xi)$ solves $h_1*h_1(\xi) = |\xi|^\theta h_1(x)$ and $h_2(\xi)$ also solves $h_2*h_2(\xi) = |\xi|^\theta h_2(x)$ for any $\beta > 0$. If the particular value $\beta = \bar{\beta} \stackrel{\text{def}}{=} (M/B)^{1/\theta}$ is used, where $M = \int h(\xi)d\xi$, then $\int h_2(\xi)d\xi = 1$. For this choice $\beta = \bar{\beta}$, the moments $\nu_k = \int |\xi|^k h(\xi)d\xi$ and $\mu_k = \int |\xi|^k h_2(\xi)d\xi$ are related by*

$$\nu_k = B\bar{\beta}^{k+\theta}\mu_k = M\Big(\frac{M}{B}\Big)^{k/\theta}\mu_k. \tag{258}$$

Proof. Just compute: $h_1*h_1(\xi) = B^{-2}h*h(\xi) = B^{-2}B|\xi|^\theta h(\xi) = |\xi|^\theta h_1(\xi)$, and for any $\beta > 0$

$$\begin{aligned} h_2*h_2(\xi) &= \beta^{2n-2\theta}h_1*h_1(\beta\xi) = \beta^{2n-2\theta}\int h_1(\beta\xi-\beta\eta)h_1(\beta\eta)d\eta \\ &= \beta^{n-2\theta}\int h_1(\beta\xi-\eta')h_1(\eta')d\eta' \\ &= \beta^{n-2\theta}h_1*h_1(\beta\xi) = \beta^{n-\theta}|\xi|^\theta h_1(\beta\xi) = |\xi|^\theta h_2(\xi). \end{aligned} \tag{259}$$

Since

$$\begin{aligned} \int h_2(\xi)d\xi &= \beta^{-\theta}\int h_1(\beta\xi)\beta^n d\xi = \beta^{-\theta}\int h_1(\xi)d\xi = \beta^{-\theta}B^{-1}\int h(\xi)d\xi \\ &= \beta^{-\theta}\frac{M}{B}, \end{aligned} \tag{260}$$

the choice of $\beta = \bar{\beta} = (M/B)^{1/\theta}$ yields $\int h_2(\xi)d\xi = 1$. Finally

$$\begin{aligned} \nu_k = \int |\xi|^k h(\xi)d\xi &= \int |\bar{\beta}\xi|^k h(\bar{\beta}\xi)\bar{\beta}^n d\xi = \bar{\beta}^{k+\theta}B\int |\xi|^k\bar{\beta}^{n-\theta}B^{-1}h(\bar{\beta}\xi)d\xi \\ &= \bar{\beta}^{k+\theta}B\int |\xi|^k h_2(\xi)d\xi = B\bar{\beta}^{k+\theta}\mu_k. \end{aligned} \tag{261}$$

□

4.3 Lemma. *If $h(\xi) \geq 0$ solves $h*h(\xi) = B\,|\xi|^\theta\, h(\xi)$, where $\theta \geq 1$, and $\|h(\xi)\|_{L^1} = M < \infty$, then the absolute moments $\nu_k = \int |\xi|^k h(\xi)d\xi$ are bounded by $\nu_k \leq k!\,\big(\frac{M}{B}\big)^{k/\theta}M$.*

Proof. As in Lemma 4.2, let $h_2(\xi) = \bar{\beta}^{n-\theta}B^{-1}h(\bar{\beta}\xi)$, with $\bar{\beta} = \big(M/B\big)^{1/\theta}$. The moments $\mu_k = \int |\xi|^k h_2(\xi)d\xi$ will be computed by letting X, X_1, and X_2 be *i.i.d.* random variables on $\mathbb{R}^n$ with law $h_2(\xi)$. First observe that $\mu_0 = 1$ and either $\mu_s \leq 1$ for all $s \geq 0$, or else

$\mu_s > 1$ for some $s_0 > 0$. The bound $\mu_k \leq k!$ for all $k \geq 1$ is established: In the first case this is obvious since $1 \leq k!$, so assume that $\mu_{s_0} > 1$ for some $s_0 > 0$. Since

$$\frac{d^2}{ds^2}\mu_s = \int (\log|\xi|)^2 |\xi|^s h(\xi) d\xi \geq 0, \tag{262}$$

it follows that μ_s is a convex function of s and therefore increasing on the interval (s_0, ∞) using $\mu_0 = 1$, and $\mu_{s_0} > 1$. (The proof will show that μ_s exists for all $s \geq 0$.) The argument is by induction. Let k_0 denote the smallest integer such that $\mu_{k_0+1} > 1$ while $\mu_0, \ldots, \mu_{k_0} \leq 1$. Then certainly $\mu_j \leq j!$ for $1 \leq j \leq k_0$. Consider μ_{k+1} for $k \geq k_0 + 1$ with the induction hypothesis being $\mu_j \leq j!$ for $1 \leq j \leq k$. Since μ_s is increasing on $(k_0 + 1, \infty) \subset (s_0, \infty)$, it follows that

$$\begin{aligned} \mu_{k+1} \leq \mu_{k+\theta} = \mathbb{E}|X|^{k+\theta} &= \int |\xi|^k |\xi|^\theta h_2(\xi) d\xi = \int |\xi|^k h_2 {*} h_2(\xi) d\xi \\ &= \mathbb{E}|X_1 + X_2|^k \leq \sum_{j=0}^{k} \binom{k}{j} \mu_j \mu_{k-j} \leq \sum_{j=0}^{k} \binom{k}{j} j!(k-j)! = (k+1)k!, \end{aligned} \tag{263}$$

which completes the inductive step, and so $\mu_k \leq k!$ for all $k \geq k_0$, and hence for all k. Combining $\mu_k \leq k!$ with equation (258) completes the proof. $\square$

4.4 Lemma. *If $h(\xi) \geq 0$ solves $h{*}h(\xi) = B\,|\xi|^\theta\, h(\xi)$, where $\theta \geq 1$, and $\|h(\xi)\|_{L^1} = M < \infty$, then $e^{t|\xi|}h(\xi) \in L^1(\mathbb{R}^n)$ for any $t < B^{1/\theta}M^{-1/\theta}$.*

Proof. Fix $t < B^{1/\theta}M^{-1/\theta}$. The series expansion of $e^{t|\xi|}$ combined with Tonelli's Theorem gives

$$\begin{aligned} \int_{\mathbb{R}^n} e^{t\cdot|\xi|} h(\xi) d\xi &= \int_{\mathbb{R}^n} \sum_{k=0}^{\infty} \frac{t^k |\xi|^k}{k!} h(\xi) d\xi = \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \frac{t^k |\xi|^k}{k!} h(\xi) d\xi \\ &\leq M \sum_{k=0}^{\infty} t^k \left(\frac{M}{B}\right)^{k/\theta} = M \sum_{k=0}^{\infty} r^k < \infty, \end{aligned} \tag{264}$$

where $r < 1$ is defined by $t = rB^{1/\theta}M^{-1/\theta}$. $\square$

The following theorem is fundamental, so a proof is included. No specific reference is known. Note that the hypothesis does not require $h(\xi)$ to be positive.

4.5 Theorem. *Let* $f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} h(\xi) d\xi$ *where* $h(\xi) \in L^1(\mathbb{R}^n)$, *and suppose that* $e^{t|\xi|}h(\xi) \in L^1(\mathbb{R}^n)$ *for some* $t > 0$. *Then* $f(x)$ *has a holomorphic extension to the tube* $\mathbb{R}^n + iU_t \subset \mathbb{C}^n$ *where* $\mathbb{R}^n + iU_t = \{x + iy \in \mathbb{C}^n : |y| < t\}$ *given by*

$$f(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iz\cdot\xi} h(\xi) d\xi. \tag{265}$$

Proof. For fixed ξ, the function $z \longrightarrow e^{iz\cdot\xi}h(\xi)$ is analytic on $\mathbb{C}^n$, except perhaps on a set of measure zero where $h(\xi) = \infty$ or else is not defined. Redefining $h(\xi)$ on a set of measure zero so that it is everywhere finite, if necessary, the following formula holds for all $\xi \in \mathbb{R}^n$:

$$e^{iz\cdot\xi}h(\xi) = \left(\frac{1}{2\pi i}\right)^n \int_\Gamma \frac{e^{iw\cdot\xi}h(\xi)dw_1 \cdots dw_n}{(w_1 - z_1)\cdots(w_n - z_n)}. \tag{266}$$

The domain of integration is the set $\Gamma = \{w \in \mathbb{C}^n : |w_j - \zeta_j| = r_j, 1 \le j \le n\}$ where $\zeta = (\zeta_1, \ldots, \zeta_n)$ is fixed, and equation (266) holds, at least, for any z that is an element of the polydisc $\Delta(\zeta; r) = \Delta(\zeta_1, \ldots, \zeta_n; r_1, \ldots, r_n) = \{z \in \mathbb{C}^n : |z_j - \zeta_j| < r_j, 1 \le j \le n\}$. See for example the proof of Osgood's Lemma in [26, p. 2]. It is worth observing that with the exception of dimension $n = 1$, $\Gamma \subsetneq \partial\Delta(\zeta; r)$, and if $n = 1$ then $\Gamma = \partial\Delta(\zeta; r)$. This fact is not exploited here in this proof.

Choose any $z \in \mathbb{R}^n + iU_t$ and polydisc $\Delta(\zeta; r)$ centered at ζ such that $z \in \Delta(\zeta; r)$ and $\overline{\Delta}(\zeta; r) \subset \mathbb{R}^n + iU_t$. Consider the function

$$G(w, \xi) = \frac{e^{iw\cdot\xi}h(\xi)}{(w_1 - z_1)\cdots(w_n - z_n)} \tag{267}$$

defined on $\Gamma \times \mathbb{R}^n$ with $\Gamma \subset \overline{\Delta}(\zeta; r)$ described above. Since Γ is compact, there is a minimum distance R_0 from z to Γ and

$$\frac{1}{(w_1 - z_1)\cdots(w_n - z_n)} \le \frac{1}{R_0^n} \tag{268}$$

for all $w \in \Gamma$. Moreover $G(w, \xi) \in L^1(\Gamma \times \mathbb{R}^n, |dw| \times d\xi)$, where $|dw|$ is the product of the arc-length measures: $|dw| = |dw_1| \cdots |dw_n|$. This uses the hypothesis that $e^{t|\xi|}h(\xi)$ is integrable for some $t > 0$:

$$\begin{aligned}\int_{\mathbb{R}^n}\int_\Gamma |G(w,\xi)|\,|dw|d\xi &\le \int_{\mathbb{R}^n}\int_\Gamma \frac{|e^{iw\cdot\xi}h(\xi)|}{R_0^n}|dw|d\xi \le \int_{\mathbb{R}^n}\int_\Gamma \frac{e^{t|\xi|}|h(\xi)|}{R_0^n}|dw|d\xi \\ &= \frac{(2\pi)^n r_1\cdots r_n}{R_0^n}\int_{\mathbb{R}^n} e^{t|\xi|}|h(\xi)|d\xi < \infty.\end{aligned} \tag{269}$$

The inequality $|e^{iw\cdot\xi}| \leq e^{t|\xi|}$ holds because if $w = x + iy \in \Gamma \subset \mathbb{R}^n + iU_t$, then $|y \cdot \mathbf{e}_\xi| \leq |y| < t$ with $\mathbf{e}_\xi = \xi/|\xi|$. For any $z \in \Delta(\zeta; r)$, using the integrability of $G(w, \xi)$ and Fubini's Theorem,

$$\begin{aligned} f(z) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi\cdot z} h(\xi) d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\frac{1}{2\pi i}\right)^n \int_\Gamma \frac{e^{iw\cdot\xi} h(\xi)}{(w_1 - z_1)\cdots(w_n - z_n)} dw d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\frac{1}{2\pi i}\right)^n \int_\Gamma G(w, \xi) dw d\xi \\ &= \left(\frac{1}{2\pi i}\right)^n \int_\Gamma \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} G(w, \xi) d\xi dw \\ &= \left(\frac{1}{2\pi i}\right)^n \int_\Gamma \frac{\varphi(w)}{(w_1 - z_1)\cdots(w_n - z_n)} dw \end{aligned} \tag{270}$$

where

$$\varphi(w) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi\cdot w} h(\xi) d\xi \tag{271}$$

is defined for $w \in \Gamma$; and $\varphi(w)$ is continuous on Γ by the dominated convergence theorem, using $\Gamma \subset \mathbb{R}^n + iU_t$. Still keeping $z \in \Delta(\zeta; r)$ fixed, the series expansion

$$\frac{1}{(w_1 - z_1)\cdots(w_n - z_n)} = \sum_{\nu_1,\ldots,\nu_n=0}^{\infty} \frac{(z_1 - \zeta_1)^{\nu_1}\cdots(z_n - \zeta_n)^{\nu_n}}{(w_1 - \zeta_1)^{\nu_1+1}\cdots(w_n - \zeta_n)^{\nu_n+1}} \tag{272}$$

is absolutely and uniformly convergent for $w \in \Gamma$. Substitute the expansion into (270) and interchange the order of summation and integration, using the boundedness of the continuous function on the compact set Γ. It follows that $f(z)$ has the power series expansion

$$f(z) = \sum_{\nu_1,\ldots,\nu_n=0}^{\infty} a_{\nu_1\cdots\nu_n} (z_1 - \zeta_1)^{\nu_1}\cdots(z_n - \zeta_n)^{\nu_n} \tag{273}$$

where

$$a_{\nu_1\cdots\nu_n} = \left(\frac{1}{2\pi i}\right)^n \int_\Gamma \frac{\varphi(w)dw}{(w_1 - z_1)^{\nu_1+1}\cdots(w_n - z_n)^{\nu_n+1}} \tag{274}$$

and $\varphi(w)$ is given by (271) □

The only topological fact about U_t that gets used in this proof is that U_t is a nonempty open set. The same proof works *mutatis mutandis* for the following theorem with U_t

replaced by V°. It is recorded here because of the similarity with Theorem 4.5, and it is used in the sequel in the proof of the log-convexity of $\mathcal{H}_A^{n,\theta}$.

4.6 Theorem. *Let* $(\mathcal{F}^{-1}h)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} h(\xi) d\xi$ *where* $h(\xi) \in L^1(\mathbb{R}^n)$, *and suppose that* $V = \{t \in \mathbb{R}^n : e^{t\cdot\xi} h(\xi) \in L^1(\mathbb{R}^n)\}$ *has nonempty interior* V°. *Then* $(\mathcal{F}^{-1}h)(x)$ *extends to a holomorphic function* $f(z)$ *on* $\mathbb{R}^n + iV^\circ \subset \mathbb{C}^n$, *given by*

$$f(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iz\cdot\xi} h(\xi) d\xi. \tag{275}$$

The sequential application of Lemma 4.4 and Theorem 4.6 gives the following theorem, and this justifies the second part of statement *(ii)* given at the beginning of this chapter.

4.7 Theorem. *If* $h(\xi) \geq 0$ *solves* $h{*}h(\xi) = B\,|\xi|^\theta\, h(\xi)$, *where* $\theta \geq 1$, *and* $\|h(\xi)\|_{L^1} = M < \infty$, *then* $f(x) = (\mathcal{F}^{-1}h)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} h(\xi) d\xi$ *extends to a holomorphic function* $f(z)$ *on the tube* $\mathbb{R}^n + iU_t = \{x + iy \in \mathbb{C}^n : |y| < t\}$ *where* $t = B^{1/\theta} M^{-1/\theta}$.

This bound is sharp in one sense and not sharp in another. It is sharp because in dimension $n = 1$ the function $h(\xi) = Be^{-\alpha\xi}\mathbf{1}_{[\xi\geq 0]}$ satisfies $h{*}h(\xi) = B|\xi|h(\xi)$ with $M = B\alpha^{-1}$. According to the theorem $(\mathcal{F}^{-1}h)(x)$ extends to an analytic function on the strip $\{x + iy : |y| < \alpha\}$. In fact, the inverse Fourier transform is

$$(\mathcal{F}^{-1}h)(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha - ix} \tag{276}$$

which extends to a holomorphic function on the set $\{x+iy : y > -\alpha\}$. In higher dimensions or in the case that $h(\xi)$ is radially symmetric, this bound is not sharp. For example, the Bessel kernel solution given by equation (133) satisfy the convolution equality with $B = 1$, $\theta = 1$, and

$$M = \frac{\beta}{2\pi} \int_{\mathbb{R}^3} \frac{e^{-\beta|\xi|}}{|\xi|} d\xi = \frac{2}{\beta}, \tag{277}$$

and the theorem gives the region of analyticity being a tube of diameter $\beta/2$ while the actual bound is twice this: The inverse Fourier transform of these Bessel kernels are (see Section 3.6)

$$\Big(\mathcal{F}^{-1} \frac{2}{\beta} G^{(3)}_{2,\beta}\Big)(x) = \frac{2}{\beta} (2\pi)^{-n/2} \Big(\frac{\beta^2}{\beta^2 + |x|^2}\Big)^{\alpha/2}. \tag{278}$$

§4.3 Analytic-majorizing kernels and their integrability

Question 4.1 is made well-posed by specifying the domain of the inverse Fourier transform, and then made precise by saying what it means for an inverse Fourier transform of a majorizing kernel to be analytic.

Recall that a majorizing kernel $h(\xi) \in \mathcal{H}^{n,\theta}$ is a positive locally integrable function $h(\xi)$ (belonging to the space $\mathcal{S}'$ of temperate distributions on $\mathbb{R}^n$), that solves the nonlinear convolution inequality $h*h(\xi) \leq B|\xi|^\theta h(\xi)$. The Fourier transform on $\mathcal{S}'$ is the standard extension of

$$(\mathcal{F}\varphi)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\varphi(x)dx \tag{279}$$

as a continuous self-adjoint linear operator on the Schwartz class $\mathcal{S}$ of C^∞-functions rapidly decreasing at infinity [23, p. 259].

Analyticity of $(\mathcal{F}^{-1}h)(x)$ is taken to mean the weakest thing possible, requiring only that it be analytic in some neighborhood of the origin. This idea is analogous to Lukacs treatment of analytic characteristic functions [38, p. 191]. According to Lukacs, an analytic characteristic function is
"a characteristic function which coincides with a regular analytic function in some neighborhood of the origin in the complex z-plane".

4.8 Definition. *A majorizing kernel $h(\xi)$ is* analytic-majorizing *if there exists a function $f(z)$ admitting power series expansion*

$$f(z) = \sum_{\nu_1,\ldots,\nu_n=0}^{\infty} a_{\nu_1\cdots\nu_n} z_1^{\nu_1}\cdots z_n^{\nu_n} \tag{280}$$

that converges for all $z = (z_1, \ldots z_n) \in \Delta(0;\rho)$, where $\Delta(0;\rho)$ is the open polydisc $\Delta(0;\rho) = \{z \in \mathbb{C}^n : |z_j| < \rho_j,\ 1 \leq j \leq n\}$, and the distribution $(\mathcal{F}^{-1}h)(x) \in \mathcal{S}'$ coincides with $f(x)$ in the neighborhood $B(0;\rho) = \Delta(0;\rho) \cap \mathbb{R}^n$. The subset of $\mathcal{H}^{n,\theta}$ consisting of analytic-majorizing kernels is denoted $\mathcal{H}_A^{n,\theta}$.

This is a well-defined property. A distribution $T \in \mathcal{D}'$ is equal to zero in the neighborhood of a given point if for every $\varphi \in \mathcal{D}$ with support within this neighborhood, we have $\langle T, \varphi\rangle = 0$. Two distributions T_1 and T_2 agree locally if their difference $T_1 - T_2$ vanishes locally [25, p. 144]. The same applies to the temperate distributions $\mathcal{S}' \subset \mathcal{D}'$. Note that the term *analytic-majorizing* refers to this property of the inverse Fourier transform $(\mathcal{F}^{-1}h)(x)$, not that $h(\xi)$ is analytic.

It turns out that the inverse Fourier transform of a majorizing kernel is analytic everywhere on $\mathbb{R}^n$ if it is locally analytic at the origin. Even more, the width of its holomorphic extension into a tubular region in $\mathbb{C}^n$ is determined locally at the origin. As a first step in the proof of this, the following theorem shows that given an analytic-majorizing kernel $h(\xi) \in \mathcal{S}'$, $h(\xi)$ is actually in $L^1(\mathbb{R}^n)$. The proof requires some topology, which is first reviewed (basic reference: Folland, [23]).

The vector space $\mathcal{S}$ becomes a topological vector space with the family of seminorms

$$\|\varphi\|_{(m,\alpha)} = \sup_{x\in\mathbb{R}^n} (1+|x|)^m |\partial^\alpha \varphi(x)|. \tag{281}$$

Endowed with the weakest topology making all of the seminorm maps continuous, $\mathcal{S}$ becomes a Frechet space, that is, a complete, metrizable, locally convex topological vector space. A sequence φ_n converges to zero in this topology if all of its seminorms converge to zero, but the convergence of the seminorms need not be uniform. In order to verify that $\varphi_k \to 0$ it suffices to check that $\|\varphi_k\|_{(m,\alpha)} \to 0$ for an arbitrary m and α. The space $\mathcal{S}'$ of temperate distributions (continuous linear functionals on $\mathcal{S}$) is endowed with the weak-star topology: a sequence T_k in $\mathcal{S}'$ converges to T_0 if and only if $\langle T_k, \varphi \rangle$ converges to $\langle T_0, \varphi \rangle$ for every φ in $\mathcal{S}$. The topology on $\mathcal{S}'$ is not metrizable.

In the proof of the following theorem, we use the version of the Fourier transform whose inverse corresponds to the characteristic function of probability theory.

4.9 Theorem. $\mathcal{H}_A^{n,\theta} \subset L^1(\mathbb{R}^n) \cap \mathcal{H}^{n,\theta}$.

Proof. Let $h \in \mathcal{H}_A^{n,\theta}$. Of course $h \in \mathcal{H}^{n,\theta}$ already; the important thing is to show that $h \in L^1(\mathbb{R}^n)$. The hypothesis is that h is a locally integrable function and the temperate distribution $\widetilde{h} = \mathcal{F}^{-1}h$ agrees with an analytic function $f(x)$ on an open ball $B(0,\epsilon)$. Let $\widetilde{h}(0)$ denote $f(0)$.

Let $\phi_k(x) = (2\pi)^{-n/2}k^{n/2}e^{-\frac{1}{2}k|x|^2}$ and $\widetilde{\phi}_k(\xi) = e^{-\frac{1}{2k}|\xi|^2}$ be sequences of Gaussians (which for any fixed k are Fourier transform pairs). Then ϕ_k is a delta–sequence while

$\widetilde{\phi}_k$ converges to 1 pointwise as $k \to \infty$. According to the extension of the inverse Fourier transform from $\mathcal{S}$ to $\mathcal{S}'$

$$\langle \widetilde{h}, \phi_k \rangle = \langle h, \widetilde{\phi}_k \rangle \quad \text{for all} \quad k \geq 1. \tag{282}$$

The right hand side has limit $\int h(x)dx$ by the monotone convergence theorem. It remains to show that $\langle \widetilde{h}, \phi_k \rangle \to \widetilde{h}(0) < \infty$. The immediate difficulty in asserting this is that $\phi_k \to \delta_0$ in $\mathcal{S}'$ and δ_0 acts on continuous functions, whereas $\widetilde{h}$ itself is a temperate distribution. The action of the delta-sequence is isolated by means of a cut-off function ψ with the following properties:

(*i*) $\psi \in \mathcal{S}$,

(*ii*) $\psi \equiv 1$ on $B(0, \epsilon/2)$,

(*iii*) $\psi \equiv 0$ on $B(0, \epsilon)^c$.

Then for all $k \geq 1$

$$\langle \widetilde{h}, \phi_k \rangle = \langle \widetilde{h}, \psi\phi_k \rangle + \langle \widetilde{h}, (1-\psi)\phi_k \rangle \tag{283}$$

and the right hand side has the following limits:

$$\lim_{k\to\infty} \langle \widetilde{h}, \psi\phi_k \rangle = \lim_{k\to\infty} \int_{B(0,\epsilon)} f(x)\psi(x)\phi_k(x)dx = f(0)\psi(0) = \widetilde{h}(0), \tag{284}$$

$$\lim_{k\to\infty} \langle \widetilde{h}, (1-\psi)\phi_k \rangle = 0. \tag{285}$$

The first holds because ϕ_k is a delta-sequence and the product $f\psi$ is continuous, while the second holds because $\|(1-\psi)\phi_k\|_{(m,\alpha)} \to 0$ for arbitrary (m, α). To verify this, fix (m, α). By the product rule and triangle inequality

$$\left|\partial^\alpha\big((1-\psi)\phi_k\big)\right| \leq \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \left|\partial^\beta\big(1-\psi(x)\big)\right| \cdot \left|\partial^\gamma \phi_k(x)\right|. \tag{286}$$

Since all of the proper derivatives $\partial^\beta(1-\psi(x))$ vanish outside the region $\epsilon/2 \leq |x| \leq \epsilon$, let

$$C = \sup_{\beta\leq\alpha} \sup_{\frac{\epsilon}{2}\leq|x|\leq\epsilon} \left|\partial^\beta(1-\psi(x))\right|. \tag{287}$$

This is a finite constant, being the maximum over a finite collection of supremums of continuous functions over a compact set. Accordingly, it is sufficient to show that

$$\lim_{k\to\infty}\sum_{\gamma\leq\alpha}\sup_{|x|\geq\frac{\epsilon}{2}}(1+|x|)^m(1+C)\frac{\alpha!}{\beta!\gamma!}|\partial^\gamma\phi_k(x)|=0, \tag{288}$$

and since this is a finite sum, it suffices to show that for arbitrary γ,

$$\lim_{k\to\infty}\left\{\sup_{|x|\geq\frac{\epsilon}{2}}(1+|x|)^m|\partial^\gamma\phi_k(x)|\right\}=0. \tag{289}$$

The following facts about $\partial^\gamma e^{-\frac{1}{2}k|x|^2}$ may be established by induction:

(i) There exists a polynomial $p_\gamma=p_\gamma(k,x_1,\ldots,x_n)$ of total degree $|\gamma|$ such that $\partial^\gamma e^{-\frac{1}{2}k|x|^2}=p_\gamma(k,x_1,\ldots,x_n)e^{-\frac{1}{2}k|x|^2}$

(ii) The degree of p_γ as a polynomial in k is $|\gamma|$.

(iii) The degree of p_γ as a polynomial in $x_1,\ldots,x_n$ is $|\gamma|$.

Consider the following substitutions made to the polynomial $p_\gamma(k,x_1,\ldots,x_n)$: replace each occurrence of k^m ($m\leq|\gamma|$) by $k^{|\gamma|}$, replace each occurrence of x_i by $r=|x|$, and finally replace each numerical coefficient with its absolute value. Making these substitutions with an implicit use of the triangle inequality gives

$$|\partial^\gamma e^{-\frac{1}{2}k|x|^2}|\leq k^{|\gamma|}q_\gamma(r)e^{-\frac{1}{2}kr^2} \tag{290}$$

where $q_\gamma(r)$ is a polynomial of degree $|\gamma|$ whose coefficients are all positive. Then returning to equation (289)

$$\lim_{k\to\infty}\left\{\sup_{|x|\geq\frac{\epsilon}{2}}(1+|x|)^m|\partial^\gamma\phi_k(x)|\right\}\leq\lim_{k\to\infty}\left\{\sup_{r\geq\frac{\epsilon}{2}}(1+r)^m k^{|\gamma|+\frac{n}{2}}q_\gamma(r)e^{-\frac{1}{2}kr^2}\right\}. \tag{291}$$

Each term in the expansion of the right side of (291) has the form

$$Ck^{|\gamma|+\frac{n}{2}}r^j e^{-\frac{1}{2}kr^2}\quad\text{where}\quad j\leq|\gamma|+m, \tag{292}$$

(and the constant C is not the same for each term). Each term reaches a maximum at $r=\sqrt{j/k}$. By taking k sufficiently large,

$$\sup_{r\geq\epsilon/2}Ck^{|\gamma|+n/2}r^j e^{-\frac{1}{2}kr^2}=Ck^{|\gamma|+n/2}\epsilon^j 2^{-j}e^{-\frac{k}{2}(\frac{\epsilon}{2})^2}, \tag{293}$$

and the limit of each of these terms is 0 as $k \to \infty$. This completes the proof, since $\int h(x)dx = \lim_{k\to\infty}\langle h, \widetilde{\phi}_k \rangle = \lim_{k\to\infty}\langle \widetilde{h}, \phi_k \rangle = \widetilde{h}(0) < \infty$. □

§4.4 The log-convexity of $\mathcal{H}_A^{n,\theta}$

Before addressing the log-convexity of $\mathcal{H}_A^{n,\theta}$, we consider a counter-example: the inverse Fourier transform of a function $g(\xi) \in L^1(\mathbb{R})$ need not be holomorphic on any strip, even if it is analytic in the unit disc centered at the origin. This sheds some light on Theorem 4.16 below, which asserts that for analytic-majorizing kernels, the inverse Fourier transform is always holomorphic on some ϵ-tube $\mathbb{R}^n + iU_\epsilon$.

4.10 Example. Let $\widehat{c}(\xi) = \sqrt{\frac{\pi}{2}}e^{-|\xi|}$ for $\xi \in \mathbb{R}^1$, and define $g(\xi)$ by

$$g(\xi) = \widehat{c}(\xi) + \sum_{n=1}^{\infty} \frac{1}{2^n} e^{-in\xi} \frac{1}{n} \widehat{c}(\frac{1}{n}\xi). \tag{294}$$

The factors of 2^{-n} assure integrability, and Fubini's Theorem gives

$$\begin{aligned}
(\mathcal{F}^{-1}g)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} g(\xi) d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \widehat{c}(\xi) d\xi + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{1}{2^n} \int_{-\infty}^{\infty} e^{i\xi x} e^{i\xi n} \frac{1}{n} \widehat{c}(\frac{\xi}{n}) d\xi \\
&= \frac{1}{1+x^2} + \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{1}{1+n^2(x-n)^2} \right).
\end{aligned} \tag{295}$$

This has a meromorphic extension to $\mathbb{C}$ with poles at $z = \pm i$ and at $z = n \pm in^{-1}$ for $n \geq 1$; it is analytic in the unit disc but not analytic on any strip containing the real axis.

Let $f(x) = (\mathcal{F}^{-1}h)(x)$ be the inverse Fourier transform of $h(\xi) \in \mathcal{H}_A^{n,\theta}$. Define functions $f_j(\cdot x_j \cdot)$ for $1 \leq j \leq n$ by setting all the coordinates except x_j equal to zero:

$$f_j(\cdot x_j \cdot) = f(0, \ldots, 0, x_j, 0, \ldots, 0). \tag{296}$$

Let $\alpha_k^{(j)}$ and $\beta_k^{(j)}$ denote the (single coordinate) moments and the absolute (single coordinate) moments of $h(\xi)$ respectively:

$$\alpha_k^{(j)} = \int_{\mathbb{R}^n} \xi_j^k h(\xi) d\xi, \quad \beta_k^{(j)} = \int_{\mathbb{R}^n} |\xi_j|^k h(\xi) d\xi, \quad \xi = (\xi_1, \ldots, \xi_n). \tag{297}$$

The key idea in the proof of the following theorem comes from Lucaks [38, p. 20].

4.11 Theorem. *Let $h(\xi) \in \mathcal{H}_A^{n,\theta}$. Then all of the moments of $h(\xi)$ that involve just a single coordinate ξ_j exist: $\alpha_k^{(j)} \le \beta_k^{(j)} < \infty$, for $1 \le j \le n$.*

Proof. Let $g(x)$ be any function. For all $k \ge 1$, define the difference operators Δ_k^t by

$$\Delta_1^t g(x) = g(x+t) - g(x-t), \tag{298}$$

and the higher order difference operators by induction:

$$\Delta_{k+1}^t g(x) = \Delta^t \Delta_k^t g(x). \tag{299}$$

For the function $g(x) = e^{ix\xi}$, induction shows that $\Delta_k^t e^{ix\xi} = (2i \sin t\xi)^k e^{ix\xi}$.

By hypothesis, $f(x) = (\mathcal{F}^{-1}h)(x)$ is the inverse Fourier transform of an analytic-majorizing kernel. Then each of the functions $f_j(\cdot x_j \cdot)$, $1 \le j \le n$, has a power series expansion valid for $|x_j| < \rho_j$:

$$f_j(\cdot x_j \cdot) = \sum_{\nu_j=0}^{\infty} a_{\nu_j} x_j^{\nu_j}. \tag{300}$$

Since all of the derivatives of $f_j(\cdot x_j \cdot)$ exist at $x_j = 0$, they may be expressed using the inferior limits and difference operators:

$$f_j^{(k)}(0) = \lim_{t \to 0} \frac{\Delta_k^t f_j(0)}{(2t)^k} = \liminf_{t \to 0} \frac{\Delta_k^t f_j(0)}{(2t)^k}. \tag{301}$$

Using the continuity of the absolute value function,

$$\liminf_{t \to 0} \left| \frac{\Delta_{2k}^t f_j(0)}{(2t)^{2k}} \right| = \left| \liminf_{t \to 0} \frac{\Delta_{2k}^t f_j(0)}{(2t)^{2k}} \right| = \left| f_j^{(2k)}(0) \right| < M < \infty. \tag{302}$$

At the same time,

$$\begin{aligned} \left| \frac{\Delta_{2k}^t f_j(0)}{(2t)^{2k}} \right| &= \left| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \Delta_{2k}^t (e^{i\xi_j x_j}) \Big|_{x_j=0} \frac{1}{(2t)^{2k}} h(\xi) d\xi \right| \\ &= \left| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\frac{2i \sin \xi_j t}{2t} \right)^{2k} h(\xi) d\xi \right| \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\frac{\sin \xi_j t}{t} \right)^{2k} h(\xi) d\xi. \end{aligned} \tag{303}$$

Then by Fatou's Lemma (and of course, using $h(\xi) \geq 0$),

$$M > \liminf_{t\to 0} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\frac{\sin \xi_j t}{t}\right)^{2k} h(\xi) d\xi$$
$$\geq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \liminf_{t\to 0} \left(\frac{\sin \xi_j t}{t}\right)^{2k} h(\xi) d\xi = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \xi_j^{2k} h(\xi) d\xi. \tag{304}$$

This establishes that the moments $\alpha_{2k}^{(j)}$ exist for $k \geq 1$. Lemma 4.2 shows that for a probability measure, the absolute moment function μ_s is a convex function of s. Exactly the same argument shows that the continuous extension $\beta_s^{(j)}$ of $\beta_k^{(j)}$ is also a convex function of s. Since $\beta_{2k}^{(j)} = \alpha_{2k}^{(j)} < \infty$ for all k, using this convexity, $\alpha_k^{(j)} \leq \beta_k^{(j)} < \infty$ for all k. □

4.12 Lemma. *Each $f_j(\cdot x_j \cdot)$ has the power series expansion*

$$f_j(\cdot x_j \cdot) = \frac{1}{(2\pi)^{n/2}} \sum_{k=0}^{\infty} \frac{i^k \alpha_k^{(j)} x_j^k}{k!} \tag{305}$$

that converges for $|x_j| < \rho_j$, where the ρ_j are the polydisc radii given in Definition 4.8.

Proof. The coefficients of $f_j(\cdot x_j \cdot)$ in equation (305) are identified using the standard theory of Taylor series [9, p. 543] and the existence of the moments $\beta_k^{(j)}$. Since

$$\int_{\mathbb{R}^n} \left| \frac{\partial^k}{\partial x_j^k} \left\{ e^{ix_j\xi_j} h(\xi) \right\} \right| d\xi \leq \int_{\mathbb{R}^n} |\xi_j|^k h(\xi) d\xi = \beta_k^{(j)} < \infty, \tag{306}$$

the derivatives $f_j^{(k)}(\cdot x_j \cdot)$ may be computed [23, p. 54] as

$$\frac{\partial^k}{\partial x_j^k} \left\{ \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix_j\xi_j} h(\xi) d\xi \right\} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\partial^k}{\partial x_j^k} \left\{ e^{ix_j\xi_j} h(\xi) \right\} d\xi. \tag{307}$$

In particular,

$$f_j^{(k)}(\cdot x_j \cdot)\Big|_{x_j=0} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} i^k \xi_j^k h(\xi) d\xi = \frac{1}{(2\pi)^{n/2}} i^k \alpha_k^{(j)}, \tag{308}$$

and the coefficients a_{ν_j} in the power series (280) are $a_{\nu_j} = \frac{1}{k!} f_j^{(k)}(0)$. □

4.13 Theorem. *Let $h(\xi) \in L^1(\mathbb{R}^n)$, $h(\xi) \geq 0$, $\alpha_k = \int \xi_j^k h(\xi) d\xi$, and $\beta_k = \int |\xi_j|^k h(\xi) d\xi$ for some particular $1 \leq j \leq n$. Then the power series*

$$F_e(z_j) = \sum_{k=0}^{\infty} \frac{\alpha_{2k} z_j^{2k}}{(2k)!}, \quad F_m(z_j) = \sum_{k=0}^{\infty} \frac{\alpha_k z_j^k}{k!}, \quad \textit{and} \quad F_a(z_j) = \sum_{k=0}^{\infty} \frac{\beta_k z_j^k}{k!}, \tag{309}$$

share the same radius of convergence, $\rho\,(=\rho_j)$ say. If $\rho \neq 0$, then $e^{t|\xi_j|}h(\xi) \in L^1(\mathbb{R}^n)$ for all $t < \rho$.

Proof. Let ρ_e, ρ, and ρ_a denote the convergence radii of $F_e(z_j)$, $F_m(z_j)$, and $F_a(z_j)$ respectively. Observe that

$$\limsup \left|\frac{\alpha_{2k}}{(2k)!}\right|^{\frac{1}{2k}} \leq \limsup \left|\frac{\alpha_k}{k!}\right|^{\frac{1}{k}} \leq \limsup \left|\frac{\beta_k}{k!}\right|^{\frac{1}{k}} \tag{310}$$

and therefore $\rho_a \leq \rho \leq \rho_e$. The inequality on the left holds because the largest cluster point of the subsequence does not exceed the largest cluster point of the original sequence; the inequality on the right holds because the for each k, $\alpha_k \leq \beta_k$. If $\rho_e = 0$, the conclusion of the theorem is trivial, so assume $0 < t < \rho_e$. Expanding $e^{t|\xi_j|}$ in a power series, and using Tonelli's Theorem gives

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\beta_k}{k!} t^k = \int_{\mathbb{R}^n} e^{t|\xi_j|} h(\xi) d\xi &< \int_{\mathbb{R}^n} \left(e^{t|\xi_j|} + e^{-t|\xi_j|}\right) h(\xi) d\xi \\ &= 2\sum_{k=0}^{\infty} \frac{\beta_{2k}}{(2k)!} t^{2k} = 2\sum_{k=0}^{\infty} \frac{\alpha_{2k}}{(2k)!} t^{2k} < \infty. \end{aligned} \tag{311}$$

This shows that if $t < \rho_e$ then $t < \rho_a$, and hence $\rho_e \leq \rho_a$, closing the chain of inequalities: $\rho_e = \rho = \rho_a$. This also establishes the second conclusion of the theorem, that $e^{t|\xi_j|}h(\xi) \in L^1(\mathbb{R}^n)$ provided $t < \rho$. □

4.14 Theorem. *If $h(\xi) \in \mathcal{H}_A^{n,\theta}$, and ρ_j $(1 \leq j \leq n)$ are the polydisc radii given in Definition 4.8, then $e^{t_j|\xi_j|}h(\xi) \in L^1(\mathbb{R}^n)$ whenever $t_j < \rho_j$.*

Proof. By definition, each of the functions $f_j(\cdot x_j \cdot)$ has a power series expansion at the origin with radius of convergence ρ_j. Using Lemma 4.12 these are precisely the power series

$$f_j(\cdot x_j \cdot) = \frac{1}{(2\pi)^{n/2}} \sum_{k=0}^{\infty} \frac{i^k \alpha_k^{(j)} x_j^k}{k!} = \frac{1}{(2\pi)^{n/2}} F_m(ix_j) \tag{312}$$

and by the last sentence of Theorem 4.13 $e^{t_j|\xi_j|}h(\xi) \in L^1(\mathbb{R}^n)$ for $t_j < \rho_j$. □

The next theorem extends the conclusion of Theorem 4.14 to include the general form $e^{t\cdot\xi}h(\xi)$, where t is first assumed to lie in a convex polyhedron constrained by the polydisc $\Delta(0;\rho)$, and then is allowed to lie in a maximal convex open set.

4.15 Theorem. *If $h(\xi) \in \mathcal{H}_A^{n,\theta}$, then there exists an open set $V^\circ \subseteq \mathbb{R}^n$, such that $0 \in V^\circ$ and for all $t \in V^\circ$, $e^{t\cdot\xi}h(\xi) \in L^1(\mathbb{R}^n)$. The largest such open set is convex.*

Proof. For any fixed $v_1 \dots v_m \in \mathbb{R}^n$ define $a(w,\xi)$ by

$$a(w,\xi) = \frac{e^{w\cdot\xi}}{\sum_{j=1}^m e^{v_j\cdot\xi}} \tag{313}$$

It turns out that $0 < a(w,\xi) \le 1$ whenever w lies inside the convex hull of $v_1 \dots v_m$. Indeed, suppose $w = \sum \sigma_j v_j$ with $\sigma_j \ge 0$ and $\sum \sigma_j = 1$. Then

$$a(w,\xi) = \frac{\prod_{j=1}^m (e^{v_j\cdot\xi})^{\sigma_j}}{\sum_{j=1}^m e^{v_j\cdot\xi}} \tag{314}$$

and the numerator is between the maximum and the minimum of the summands in the denominator.

Suppose $0 < t_j < \rho_j$ for $1 \le j \le n$. Define v_j^+ and v_j^- by $v_j^\pm = (0,\dots,0,\pm t_j,0,\dots,0)$. These are the vertices of a polyhedron in $\mathbb{R}^n$; let V_n denote the interior of the convex hull of $v_1^\pm,\dots,v_n^\pm$. Define $a(t,\xi)$ as in equation (314) using this set of vertices. By Theorem 4.14, $e^{t_j|\xi|}h(\xi) \in L^1(\mathbb{R}^n)$. For any $t \in V_n$ in

$$e^{t\cdot\xi} = a(t,\xi)\sum_{j=1}^n (e^{v_j^+\cdot\xi} + e^{v_j^-\cdot\xi}) \le \sum_{j=1}^n (e^{t_j|\xi|} + e^{-t_j|\xi|}) \le 2\sum_{j=1}^n e^{t_j|\xi|}, \tag{315}$$

so that $e^{t\cdot\xi}h(\xi) \in L^1(\mathbb{R}^n)$. This exhibits an open set $V_n \subseteq \mathbb{R}^n$ such that $0 \in V_n$ and $e^{t\cdot\xi}h(\xi) \in L^1(\mathbb{R}^n)$ for all $t \in V_n$. Repeating this argument shows that any such open set may be assumed to be convex, and the largest such open set is convex. $\square$

4.16 Theorem. *If $h(\xi) \in \mathcal{H}_A^{n,\theta}$, then $(\mathcal{F}^{-1}h)(x)$ has a holomorphic extension*

$$f(z) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{iz\cdot\xi}h(\xi)d\xi \tag{316}$$

defined on $\mathbb{R}^n + iV^\circ$ where V° is the nonempty interior of the convex set $\{t \in \mathbb{R}^n : e^{t\cdot\xi}h(\xi) \in L^1(\mathbb{R}^n)\}$. In particular, $\mathbb{R}^n + iV^\circ$ contains an ϵ-tube $\mathbb{R}^n + iU_\epsilon = \{x + iy \in \mathbb{C}^n : |y| < \epsilon\}$.

Proof. By Theorem 4.15 the open set V° is nonempty and convex, and contains the origin, since $h(\xi)$ is integrable. By Theorem 4.6, $f(z)$ is holomorphic on $\mathbb{R}^n + iV^\circ$. $\square$

Note that the width of the tube $\mathbb{R}^n + iU_\epsilon$ is determined by the size of the sphere U_ϵ, which is determined by the polydisc radii ρ_j given by the local analyticity of $(\mathcal{F}^{-1}h)(x)$ at the origin.

4.17 Theorem. *Question 4.1 has a positive answer, and in particular the set $\mathcal{H}_A^{n,\theta}$ is log-convex.*

Proof. Suppose that $h_i(\xi) \in \mathcal{H}_A^{n,\theta_i}$ for $i = 1, 2$. Let $V_i = \{t \in \mathbb{R}^n : e^{t\cdot\xi}h_i(\xi) \in L^1(\mathbb{R}^n)\}$ for $i = 1, 2$. By Theorem 4.15 the interiors of these sets are nonempty. Consider any log-convex combination $h(\xi) = [h_1(\xi)]^{\sigma_1}[h_2(\xi)]^{\sigma_2}$. Let V° denote the corresponding convex combination of sets: $V^\circ = \sigma_1 V_1^\circ + \sigma_2 V_2^\circ$. If $t \in V^\circ$, then Hölder's inequality with conjugate exponents σ_1^{-1} and σ_2^{-1} establishes the integrability of $e^{t\cdot\xi}h(\xi) = [e^{t_1\cdot\xi}h_1(\xi)]^{\sigma_1}[e^{t_2\cdot\xi}h_2(\xi)]^{\sigma_2}$; and by Theorem 4.6 $(\mathcal{F}^{-1}h)(x)$ has a holomorphic extension to the region $\mathbb{R}^n + iV^\circ$. $\square$

§4.5 Summary and open problems

A positive integrable function $h(\xi)$ that satisfies the convolution equality $h*h(\xi) = |\xi|^\theta h(\xi)$ has the property that $(\mathcal{F}^{-1}h)(x)$ is analytic, as long as $\theta \geq 1$. The method of proof uses the absolute moments of $h(\xi)$. This method does not appear to be extendible to the case that $\theta < 1$ or to the case that $h(\xi)$ satisfies the convolution inequality, although there are probably analogues to Theorem 4.7 covering these cases.

A fairly easy proof of the log-convexity of $\mathcal{H}_A^{n,\theta}$ may be given once Theorem 4.16 is established. The hard part is getting to Theorem 4.16 from our definition of of $\mathcal{H}_A^{n,\theta}$. In this way, the log-convexity question turns out to be less interesting than the structure theory that it motivates: The membership requirement for $h(\xi) \in \mathcal{H}_A^{n,\theta}$ is weakened to a local analyticity condition on $(\mathcal{F}^{-1}h)(x)$ at the origin, and when this condition is met, it follows that $h(\xi) \in L^1(\mathbb{R}^n)$. It also follows that $(\mathcal{F}^{-1}h)(x)$ is analytic on all of $\mathbb{R}^n$ and that the width of its holomorphic extension into a tubular region in $\mathbb{C}^n$ is determined locally at the origin. This is deduced using moment and growth conditions on $h(\xi)$ that follow from the local analyticity of $(\mathcal{F}^{-1}h)(x)$. Both of these results depend essentially on the fact that majorizing kernels are positive functions.

This global conclusion about $(\mathcal{F}^{-1}h)(x)$ given the local hypothesis is similar to the following theorem in harmonic analysis:

Theorem: Let $f(x)$ be of positive type and belong to the class C^{2k} in some neighborhood of the origin; then $f(x)$ is everywhere C^{2k}.

This is stated and proved in Donoghue [18, p. 186]. The method of proof used therein is quite different from the methods used in this chapter.

The results of this chapter are introductory; here are some problems indicating directions for further research.

4.18 Open problem. *Extend Theorem 4.7 to include majorizing kernels of exponent* $\theta < 1$.

An extension of Theorem 4.7 could not include $\theta = 0$ because there are no majorizing kernel of exponent 0 that satisfy the equality. The Cauchy densities satisfy the inequality, but their inverse Fourier transforms are not analytic.

4.19 Open problem. *Prove or disprove: The extremal elements of* $\mathcal{H}_A^{n,\theta}$ *satisfy the convolution equality. Extremal means in the sense of log-convexity, but only up to multiplication by* $e^{t\cdot\xi}$.

An extremal element is one which can not be expressed as a proper log-convex combination of other elements: According to this definition, $\mathcal{H}_A^{n,\theta}$ does not contain extremal elements because for any $h(\xi) \in \mathcal{H}_A^{n,\theta}$, the set of $t \in \mathbb{R}^n$ such that in $e^{t\cdot\xi}h(\xi) \in \mathcal{H}_A^{n,\theta}$ is not compact. For this reason we have to consider extremal elements modulo the multiplication by an exponential factor $e^{t\cdot\xi}$.

4.20 Open problem. *Prove or disprove:* $\mathcal{H}_A^{n,\theta} = \mathcal{H}^{n,\theta} \cap L^1(\mathbb{R}^n)$.

We already have $\mathcal{H}_A^{n,\theta} \subset \mathcal{H}^{n,\theta} \cap L^1(\mathbb{R}^n)$ by Theorem 4.9. Since integrable majorizing kernels satisfying the equality are in $\mathcal{H}_A^{n,\theta}$, it might be possible to prove this by using extremal elements, if Open problem 4.19 has a positive solution.

4.21 Open problem. *Sharpen the bound given in Theorem 4.7 under the additional hypothesis that the majorizing kernels are rotationally invariant.*

4.22 Open problem. *Describe the topological structure of* $\mathcal{H}^{n,\theta}$ *and* $\mathcal{H}_A^{n,\theta}$.

The topological structure depends on the what topology is imposed on these sets. Determining the correct topology is part of the problem. The closure of $\mathcal{H}^{n,\theta}$ and $\mathcal{H}_A^{n,\theta}$ under log-convex combinations show that under any reasonable topology these sets are connected. Yet the majorizing kernels that satisfy the *equality* may or may not form a connected subset. There is some evidence that in dimension $n = 6$ a continuum of rotationally invariant solutions to $h*h(\xi) = |\xi|^2 h(\xi)$ exists between the Riesz and Bessel kernel solutions. This evidence comes from the solution to the second order differential equation arising as the inverse Fourier transform of this convolution equation. After a change of variables putting the ordinary differential equation into Hamiltonian form, the Riesz kernel corresponds to a center, and the Bessel kernel corresponds to a homoclinic orbit. In between, there is a continuum of solutions to the differential equation. It is not known if these correspond to *positive* solutions to the convolution equation.

4.23 Open problem. *Prove or disprove: the only rotationally invariant integrable solutions to the convolution equality,* $h*h(\xi) = |\xi|^\theta h(\xi)$, $h(\xi) > 0$ *are Bessel kernels, up to scaling the independent and dependent variables.*

CHAPTER 5

STRUCTURE THEORY FOR THE BRANCHING PROCESS REPRESENTATION OF SOLUTIONS TO THE FOURIER NAVIER-STOKES EQUATION

§5.1 Introduction

The probabilistic representation of solutions to *FNS:h* may be obtained without invoking the strong Markov property, as was done in [34] and [7]. This is an improvement on the methodology used therein, and it should permit the probabilistic representations of solutions to partial differential equations using processes with non-exponential holding times; this would be quite natural for those partial or ordinary differential equations whose conversion to an integral equation requires something other than an exponential integrating factor. The theory presented here could be presented in greater generality, but this is unnecessary, as it is expected that the key ideas are easily transported.

The basic construction is the spacetime random field $\mathsf{X}_\theta(\xi, t)$. Pointwise, this is a multiplicative functional on a discrete time branching random walk $\{\Xi_v : v \in \mathcal{V}\}$ started at $\Xi_\theta = \xi$, the point of interest, and augmented by the collection of random variables $\{(S_v, K_v) : v \in \mathcal{V}\}$. The random variables in the array $\{(S_v, K_v, \Xi_v) : v \in \mathcal{V}\}$ are intrinsic to the branching process and are certainly essential to computation of $\mathsf{X}_\theta(\xi, t)$. However, for some more theoretical purposes it is better to replace $\{\Xi_v : v \in \mathcal{V}\}$ by one of two other possible arrays, the first being an array of $\mathfrak{F}$-valued random variables $\{F_v : v \in \mathcal{V}\}$, where $\mathfrak{F}$ denotes the space of all Borel maps from $\mathbb{R}^n$ into itself. This exploits the fact that for any $v \in \partial\mathcal{V}$ the genealogical line $\{\Xi_{v|n} : n \geq 0\}$ is a Markov chain, any Markov chain may be viewed as iterated random mappings. But to avoid the dependence on the theory of iterated random mappings, instead of this replacement, we replace $\{\Xi_v : v \in \mathcal{V}\}$ with the second possibility, an array $\{U_v : v \in \mathcal{V}\}$ of *i.i.d.* random variables uniformly distributed $I = [0, 1]$.

With this replacement, the branching random walk $\{\Xi_v : v \in \mathcal{V}\}$, with all its genealogical dependence, becomes ancillary to the simpler array $\{U_v : v \in \mathcal{V}\}$ that has no dependence structure, and it turns out that by treating *this* as the primary source of randomness, it becomes easier to prove things. Theorem 5.5 for instance. Another benefit is that it brings coherence to the stochastic process across all values of ξ, allowing for example, the samplewise computation of the inverse Fourier transform $\mathcal{F}^{-1}\mathrm{X}_\theta(\xi,t)(\omega)$.

§5.2 The construction of the branching random walk

The family $p(\xi,\cdot)$ defined next is a family of Borel probability measures on $\mathbb{R}^n$ with the property that for any $G \in \mathcal{B}(\mathbb{R}^n)$, the map $\xi \to p(\xi, G)$ is Borel. Let

$$p(\xi,G) = \begin{cases} \displaystyle\int_G \frac{h(\xi-\eta)h(\eta)}{h*h(\xi)}d\eta & \text{if } h*h(\xi) < \infty, \\ \delta_0(G) & \text{if } h*h(\xi) = \infty, \end{cases} \qquad \xi \in \mathbb{R}^n, G \in \mathcal{B}(\mathbb{R}^n). \tag{317}$$

The second line here is a technical completion that is useful for further constructions based on $p(\xi, G)$, but is otherwise inconsequential.

5.1 Theorem. *There exists a probability measure* $\mathfrak{m}$ *on the space* $\mathfrak{F}$ *of Borel maps from* $\mathbb{R}^n$ *into itself such that*

$$\mathfrak{m}\{\mathfrak{f} \in \mathfrak{F} : \mathfrak{f}(\xi) \in G\} = p(\xi, G) \tag{318}$$

for all ξ *in* $\mathbb{R}^n$ *and and all* $G \in \mathcal{B}(\mathbb{R}^n)$.

The most general form of this theorem is stated and proved in Kifer [30, p. 8]. At this point we could construct an *i.i.d.* array of $\mathfrak{F}$-valued random variables $\{F_v : v \in \mathcal{V}\}$ and use this to construct the branching random walk $\{\Xi_v : v \in \mathcal{V}\}$. Instead, a more self-contained alternative is given next by Theorem 5.2 in which the random maps are parameterized by the unit interval, and selected uniformly therein according to the array of random variable $\{U_v : v \in \mathcal{V}\}$. This approach is followed here.

5.2 Theorem. *Let* Ξ *and* U *be independent random variables defined on the same probability space, where* Ξ *assumes values in* $\mathbb{R}^n$ *but is otherwise arbitrary, and* U *is uniform on* $I = [0,1]$. *Then given any non-negative measurable function* $h(\xi)$ *on* $\mathbb{R}^n$ *such that* $h*h(\Xi)$ *is finite almost surely, there exists a* $\sigma(\Xi, U)$*-measurable random variable* X *such that for any Borel set* $G \subset \mathbb{R}^n$, $\mathbb{P}\big(X \in G \big| \sigma(\Xi)\big) = p(\Xi, G)$.

Proof. There exists a measurable function $f : \mathbb{R}^n \times I \to \mathbb{R}^n$ such that for any $\xi \in \mathbb{R}^n$, and any Borel set $G \subset \mathbb{R}^n$, $\mathbb{P}\big(f(\xi,U) \in A\big) = p(\xi, A)$. The construction of such a measurable function is outlined below. Let $X = f(\Xi, U)$; this is clearly $\sigma(\Xi, U)$-measurable. Then we have

$$\begin{aligned}
\mathbb{P}(X \in A \,|\, \sigma(\Xi)) &= \mathbb{P}(f(\Xi, U) \in A \,|\, \sigma(\Xi)) \\
&= \Big[\mathbb{P}(f(\xi, U) \in A)\Big]_{\xi\,=\,\Xi} \\
&= \Big[p(\xi, A)\Big]_{\xi\,=\,\Xi} \\
&= p(\Xi, A). \qquad (319)
\end{aligned}$$

The second equality uses the independence of U and Ξ and the substitution lemma. It remains to construct f. Since I and $I^\infty = \prod_{k=1}^\infty I_k$ are measurably isomorphic, it suffices to construct it on $\mathbb{R}^n \times I^\infty$. Let $\mathcal{Q}_k$, $k \geq 1$ denote the set of semi-open intervals $[a_1, b_1) \times \cdots \times [a_n, b_n)$ whose corners belong to $2^{-k+1}\mathbb{Z}^n$. Call $(a_1, \ldots a_n)$ the distinguished corner. Use the first coordinate of I^∞ to select a member $R_1 \in \mathcal{Q}_1$ according to the transition probabilities $p(\xi, Q_1)$, $Q_1 \in \mathcal{Q}_1$. This means that the pre-image of any Q_1 under the selection function has Lebesgue measure equaling $p(\xi, Q_1)$. Thereafter, use the first k coordinates of I^∞ to select $R_k \in \mathcal{Q}_k \cap R_{k-1}$ according to the relative probabilities $p(\xi, Q_k)/p(\xi, R_{k-1})$, $Q_k \in \mathcal{Q}_k \cap R_{k-1}$. The result is a decreasing sequence of rectangles $R_1 \supset R_2 \supset \cdots$. By passing to distinguished corners amongst these rectangles, a sequence of measurable functions $f_k : I^k \times \mathbb{R}^n \subset I^\infty \times \mathbb{R}^n \to \mathbb{R}^n$ may be obtained, whose pointwise limit f is both well defined and measurable. By construction, for any $\xi \in \mathbb{R}^n$ and with $U = (U_1, U_2, \ldots)$ uniform on I^∞,

$$\mathbb{P}(f(\xi, U) \in Q) = p(\xi, Q) \quad \text{for all} \quad Q \in \cup_{k=1}^\infty \mathcal{Q}_k, \qquad (320)$$

which extends immediately to any Borel set G in place of Q. □

Either Theorem 5.1 or Theorem 5.2 may be used to construct a discrete time branching random walk with starting frequency $\Xi_\theta = \xi$. Here Theorem 5.2 is used; statements in the sequel have analogues with I replaced by $\mathfrak{F}$ and U_v replaced by F_v. For all $v \in \mathcal{V}$ let $\Xi_{v1} = f(\Xi_v, U_v)$ and $\Xi_{v2} = \Xi_v - \Xi_{v1}$. This results in a coherent family of branching random walks $\{\Xi_v : v \in \mathcal{V}\}$, parameterized by $\Xi_\theta = \xi$, such that for any $v \in \partial\mathcal{V}$ the sequence is $\{\Xi_{v|n}\}_{n=1}^{\infty}$ is a Markov chain with

$$\mathbb{P}\big(\Xi_{v|(n+1)} \in G \,\big|\, \sigma(\Xi_{v|n})\big) = \int_G \frac{h(\Xi_{v|n} - \eta)h(\eta)}{h * h(\Xi_{v|n})} d\eta. \tag{321}$$

From now on the more symmetric notation $\mathbf{d}_1$ and $\mathbf{d}_2$ will be used: We write

$$\Xi_{v1} = \mathbf{d}_1(\Xi_v, U_v) \qquad \text{and} \qquad \Xi_{v2} = \mathbf{d}_2(\Xi_v, U_v), \tag{322}$$

instead of

$$\Xi_{v1} = f(\Xi_v, U_v) \qquad \text{and} \qquad \Xi_{v2} = \Xi_v - \Xi_{v1}. \tag{323}$$

§5.3 The random field $\mathsf{X}_\theta(\xi, t)$ and solutions to *FNS:h*

The random field $\mathsf{X}_\theta(\xi, t)$ is constructed by separating the recursive and random aspects. First a recursive deterministic function is defined, and then the randomness is introduced be replacing some of the arguments with random variables.

Define the deterministic function

$$\mathbf{x} : \mathbb{R}^3 \times [0, T^*) \times \prod_{v \in \mathcal{V}} (\mathbb{R}^+ \times \{0,1\} \times I)_v \to \mathbb{C}^3 \cup \{\infty\} \tag{324}$$

according to the following recursive scheme that may or may not terminate in a finite number of steps: For the first step

$$\mathbf{x}(\xi, t\,; s_\theta^+, k_\theta^+, u_\theta^+) = \begin{cases} \chi_0(\xi) & \text{if } \lambda_\xi^{-1} s_\theta \geq t, \\ \varphi(\xi, t - \lambda_\xi^{-1} s_\theta) & \text{if } \lambda_\xi^{-1} s_\theta < t,\ k_\theta = 0, \\ m(\xi)\mathbf{x}(\mathbf{d}_1(\xi, u_\theta), t - \lambda_\xi^{-1} s_\theta\,; s_1^+, k_1^+, u_1^+) & \\ \quad \otimes_\xi \mathbf{x}(\mathbf{d}_2(\xi, u_\theta),\, t - \lambda_\xi^{-1} s_\theta\,; s_2^+, k_2^+, u_2^+) & \text{if } \lambda_\xi^{-1} s_\theta < t,\ k_\theta = 1. \end{cases} \tag{325}$$

This exploits the fact that while the indices shift between the arrays (s_v^+, k_v^+, u_v^+) and $(s_{v1}^+, k_{v1}^+, u_{v1}^+)$ say, the poset structure of the two index sets remain the same. On the subsequent steps (and this includes the first step $v = \theta$ as well)

$$\mathbf{x}(\xi, t\,; s_v^+, k_v^+, u_v^+) = \begin{cases} \chi_0(\xi) & \text{if } \lambda_\xi^{-1} s_\theta \geq t, \\ \varphi(\xi, t - \lambda_\xi^{-1} s_v) & \text{if } \lambda_\xi^{-1} s_v < t, k_v = 0, \\ m(\xi)\mathbf{x}(\mathrm{d}_1(\xi, u_v), t - \lambda_\xi^{-1} s_v\,; s_{v1}^+, k_{v1}^+, u_{v1}^+) & \\ \quad \otimes_\xi \mathbf{x}(\,\mathrm{d}_2(\xi, u_v),\, t - \lambda_\xi^{-1} s_v\,; s_{v2}^+, k_{v2}^+, u_{v2}^+) & \text{if } \lambda_\xi^{-1} s_v < t, k_v = 1. \end{cases} \tag{326}$$

If this scheme terminates in a finite number of steps then $\mathbf{x}(\xi, t\,; s_\theta^+, k_\theta^+, u_\theta^+)$ evaluates to a finite $\otimes_\xi$-product. On the subset where finite halting does not occur, the assignment

$$\mathbf{x}(\xi, t; s_v^+, k_v^+, u_v^+) = \infty$$

is made. Note that at this point, the domain of the function is not equipped with a measure, and so any question about the size of the non-halting set is not well-posed. This does become an issue next, when a measure is imposed by replacing the arrayed arguments of $\mathbf{x}(\xi, t\,; s_\theta^+, k_\theta^+, u_\theta^+)$ with random variables.

For each $v \in \mathcal{V}$ define the spacetime random field $\mathsf{X}_v(\xi, t)$ on $\mathbb{R}^3 \times [0, T^*)$ by substituting into $\mathbf{x}(\xi, t\,; s_\theta^+, k_\theta^+, u_\theta^+)$ the random variables (S_v^+, K_v^+, U_v^+) for the arguments $(s_\theta^+, k_\theta^+, u_\theta^+)$, with the index set adjusted accordingly. (This means that S_v replaces s_θ, S_{v1} replaces s_1, etc.) In particular,

$$\mathsf{X}_\theta(\xi, t) = \mathbf{x}(\xi, t\,; S_\theta^+, K_\theta^+, U_\theta^+), \tag{327}$$

and

$$\mathsf{X}_1(\xi, t) = \mathbf{x}(\xi, t\,; S_1^+, K_1^+, U_1^+), \qquad \mathsf{X}_2(\xi, t) = \mathbf{x}(\xi, t\,; S_2^+, K_2^+, U_2^+). \tag{328}$$

Then evidently for any $v \in \mathcal{V}$,

$$\mathsf{X}_v(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if } \lambda_\xi^{-1} S_v \geq t, \\ \varphi(\xi, t - \lambda_\xi^{-1} S_v) & \text{if } \lambda_\xi^{-1} S_v < t, K_v = 0, \\ m(\xi)\mathsf{X}_{v1}(\Xi_1, t - \lambda_\xi^{-1} S_v) \otimes_\xi \mathsf{X}_{v2}(\Xi_2, t - \lambda_\xi^{-1} S_v) & \text{if } \lambda_\xi^{-1} S_v < t, K_v = 1. \end{cases} \tag{329}$$

On the right hand side $\mathsf{X}_{v1}(\Xi_1, t - \lambda_\xi^{-1} S_v)$ denotes the random field — whose randomness derives from the functional dependence on $(S_{v1}^+, K_{v1}^+, U_{v1}^+)$ — evaluated at the random point $(\Xi_1, t - \lambda_\xi^{-1} S_v)$, and similarly for $\mathsf{X}_{v2}(\Xi_2, t - \lambda_\xi^{-1} S_v)$.

After replacing the arrayed arguments in $\mathbf{x}(\xi, t\,; s_v^+, k_v^+, u_v^+)$ with random variables, the possible non-halting of the recursive scheme (329) occurs with zero probability. This is best proved by considering the underlying stochastic model.

5.3 Proposition. *For any fixed (ξ, t), the random variable $\mathsf{X}_v(\xi, t)$ is a finite concatenation of ordinary products and $\otimes_\xi$-products almost surely (weather or not $\mathsf{X}_v(\xi, t)$ is integrable).*

Proof. The underlying branching process $\tau(\xi, t)$ is a continuous time multi-type branching process. Suppress the information on particle types, and standardize all particle lifetimes to a single epoch in discrete time, and the result is a binary Galton-Watson branching process in discrete time, in which the objects in each generation produce zero descendents with probability p_0 and two descendents with probability p_2. The probabilities p_0 and p_2 are just the two probabilities of the Bernoulli random variables K_v, which we assume to have mean 1. Thus the Galton-Watson process is critical binary (the mean number of descendents is 1 exactly) and it is well known that if this parameter is ≤ 1 then the extinction probability of the process is 1, e.g. [55, p. 4], [6, p. 7]. Thus the same holds for the tree $\tau(\xi, t)$ in continuous time, and the number of input nodes is finite almost surely. $\square$

5.4 Proposition. *If $\mathsf{X}_v(\xi, t)$ is integrable at a point (ξ, t) for some $v \in \mathcal{V}$ then $\mathsf{X}_v(\xi, t)$ is integrable for all $v \in \mathcal{V}$ and $\mathbb{E}\mathsf{X}_v(\xi, t)$ does not depend on v.*

Proof. By definition $\mathsf{X}_v(\xi, t) = \mathbf{x}(\xi, t; S_v^+, K_v^+, U_v^+)$, and the arrays (S_v^+, K_v^+, U_v^+) all have the same distribution independent of $v \in \mathcal{V}$, by construction. Thus for any fixed (ξ, t) the random variables $\mathsf{X}_v(\xi, t)$ all have the same distribution; and if any one is integrable then all are integrable with the same expectation. $\square$

Precisely what is meant by a solution to *FNS:h* is stated next. The data are measurable functions:

(initial datum) $\quad \chi_0 : \mathbb{R}^n \to \mathbb{C}^n \quad \text{with} \quad \xi \cdot \chi_0(\xi) = 0,$

(forcing) $\quad \varphi : \mathbb{R}^n \times [0, T^*) \to \mathbb{C}^n \quad \text{with} \quad \xi \cdot \varphi(\xi, t) = 0 \;\text{ for all }\; t \in [0, T^*).$

A *solution* to *FNS:h* for is a measurable function such that

$$\chi(\xi,t) = e^{-\nu|\xi|^2 t}\chi_0(\xi) + \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2 s}\{\cdots\}\,ds \tag{330}$$
$$\{\cdots\} = \frac{1}{2}m(\xi)\int_{\mathbb{R}^n} \chi(\eta,t-s)\otimes_\xi \chi(\xi-\eta,t-s)dK_\xi(\eta) + \frac{1}{2}\varphi(\xi,t-s)$$

holds *a.e.* for $(\xi,t)\in\mathbb{R}^n\times[0,T^*)$. This definition of a solution is technically different than the one given in [34] and [7].

Note that if we replace such a solution $\chi(\xi,t)$ by the right hand side of (330), the function has been altered on at most a set of $(n+1)$-dimensional measure zero (by definition). We may then replace $\chi(\xi,t)$ by this modification in the right hand side as well, without changing the value of the double integral, as both $dK_\xi(\eta)$ and the exponential are absolutely continuous with respect to Lebesgue measure. Thus if two functions are considered to be equivalent if they agree *a.e.* then there is a version of $\chi(\xi,t)$ satisfying the integral equation *everywhere*. It can then be shown that for this version, the following properties hold:

1. $\chi(\xi,t)$ is continuous in t for any fixed ξ,
2. $\xi\cdot\chi(\xi,t)=0$ for all $t\in[0,T^*)$.

5.5 Theorem. *If the random field* $\mathrm{X}_\theta(\xi,t)$ *is everywhere integrable on* $\mathbb{R}^3\times[0,T^*)$*, then* $\chi(\xi,t)\overset{def}{=}\mathbb{E}\mathrm{X}_\theta(\xi,t)$ *solves FNS:h therein.*

Proof. Compute $\mathbb{E}\mathrm{X}_\theta(\xi,t)$ by conditioning on $\mathcal{F}_1=\sigma(S_\theta,K_\theta,U_\theta)$:

$$\mathbb{E}\{\mathrm{X}_\theta(\xi,t)\,|\,\mathcal{F}_1\} = \chi_0(\xi)\mathbf{1}\left[\lambda_\xi^{-1}S_\theta\ge t\right] + \varphi(\xi,t-\lambda_\xi^{-1}S_\theta)\mathbf{1}\left[\lambda_\xi^{-1}S_\theta<t,\;K_\theta=0\right]$$
$$+\mathbb{E}\left\{\begin{array}{c} m(\xi)\mathbf{x}(\Xi_1,t-\lambda_\xi^{-1}S_\theta\,;S_1^+,K_1^+,U_1^+) \\ \otimes_\xi\,\mathbf{x}(\,\Xi_2,\,t-\lambda_\xi^{-1}S_\theta\,;S_2^+,K_2^+,U_2^+)\end{array}\middle|\,\mathcal{F}_1\right\}\mathbf{1}\left[\lambda_\xi^{-1}S_\theta<t,\;K_\theta=1\right] \tag{331}$$

where $\Xi_1=\mathbf{d}_1(\xi,U_\theta)$ and $\Xi_2=\mathbf{d}_2(\xi,U_\theta)$. The conditional expectation in the third term is now computed using the independence structure of the random variables involved, the substitution lemma, the fact that the $\otimes_\xi$-product is bilinear, and Theorem 5.5 coupled with Proposition 5.4:

$$
\begin{aligned}
&\mathbb{E}\left\{\begin{array}{l} m(\xi)\mathbf{x}(\Xi_1, t-\lambda_\xi^{-1}S_\theta\,; S_1^+, K_1^+, U_1^+) \\ \qquad \otimes_\xi \mathbf{x}(\,\Xi_2,\, t-\lambda_\xi^{-1}S_\theta\,; S_2^+, K_2^+, U_2^+) \end{array}\middle|\, \mathcal{F}_1\right\} \\
&= \left[\mathbb{E}\left(\begin{array}{l} m(\xi)\mathbf{x}(\xi_1, t-\lambda_\xi^{-1}s_\theta\,; S_1^+, K_1^+, U_1^+) \\ \qquad \otimes_\xi \mathbf{x}(\xi_2,\, t-\lambda_\xi^{-1}s_\theta\,; S_2^+, K_2^+, U_2^+) \end{array}\right)\right]\left\{\begin{array}{l} \xi_1 = \Xi_1, \\ \xi_2 = \Xi_2, s_\theta = S_\theta \end{array}\right. \\
&= \left[\begin{array}{l} m(\xi)\mathbb{E}\mathbf{x}(\xi_1, t-\lambda_\xi^{-1}s_\theta\,; S_1^+, K_1^+, U_1^+) \\ \qquad \otimes_\xi \mathbb{E}\mathbf{x}(\xi_2,\, t-\lambda_\xi^{-1}s_\theta\,; S_2^+, K_2^+, U_2^+) \end{array}\right]\left\{\begin{array}{l} \xi_1 = \Xi_1, \\ \xi_2 = \Xi_2, s_\theta = S_\theta \end{array}\right. \\
&= \left[\begin{array}{l} m(\xi)\mathbb{E}\mathsf{X}_1(\xi_1, t-\lambda_\xi^{-1}s_\theta) \\ \qquad \otimes_\xi \mathbb{E}\mathsf{X}_2(\xi_2,\, t-\lambda_\xi^{-1}s_\theta) \end{array}\right]\left\{\begin{array}{l} \xi_1 = \Xi_1, \\ \xi_2 = \Xi_2, s_\theta = S_\theta \end{array}\right. \\
&= m(\xi)\chi(\Xi_1, t-\lambda_\xi^{-1}S_\theta)\otimes_\xi \chi(\Xi_2,\, t-\lambda_\xi^{-1}S_\theta). \qquad (332)
\end{aligned}
$$

Upon replacing this in equation (331) and taking the expectation of both sides, we obtain

$$
\chi(\xi,t) = \mathbb{E}\left\{\begin{array}{l} \chi_0(\xi)\mathbf{1}\left[\lambda_\xi^{-1}S_\theta \geq t\right] + \\ \varphi(\xi, t-\lambda_\xi^{-1}S_\theta)\mathbf{1}\left[\lambda_\xi^{-1}S_\theta < t,\ K_\theta = 0\right] + \\ m(\xi)\chi(\Xi_1, t-\lambda_\xi^{-1}S_\theta)\otimes_\xi \chi(\Xi_2,\, t-\lambda_\xi^{-1}S_\theta)\mathbf{1}\left[\lambda_\xi^{-1}S_\theta < t,\ K_\theta = 1\right]. \end{array}\right. \qquad (333)
$$

This is *FNS:h* expressed probabilistically. □

§5.4 A martingale associated with the pointwise evaluation of $\mathsf{X}_\theta(\xi, t)$

The following martingale and attendant constructions are useful in uniqueness proofs. To start with, it is based on an existing solution to *FNS:h* on $\mathbb{R}^n \times [0, T^*]$, denoted by

$\gamma(\xi, t)$. Define the sequence of deterministic functions

$$\mathbf{x}^{(m)} : \mathbb{R}^3 \times [0, T^*) \times \prod_{v \in \mathcal{V}} (\mathbb{R}^+ \times \{0, 1\} \times I)_v \to \mathbb{C}^3 \tag{334}$$

starting with

$$\mathbf{x}^{(0)}(\xi, t\,; s_\theta^+, k_\theta^+, u_\theta^+) = \gamma(\xi, t), \tag{335}$$

and thereafter according to the recursive formula

$$\mathbf{x}^{(m+1)}(\xi, t\,; s_\theta^+, k_\theta^+, u_\theta^+) = \begin{cases} \chi_0(\xi) & \text{if } \lambda_\xi^{-1} s_\theta \geq t, \\ \varphi(\xi, t - \lambda_\xi^{-1} s_\theta) & \text{if } \lambda_\xi^{-1} s_\theta < t, k_\theta = 0, \\ m(\xi)\mathbf{x}^{(m)}(f_1(\xi, u_\theta), t - \lambda_\xi^{-1} s_\theta\,; s_1^+, k_1^+, u_1^+) & \\ \quad \otimes_\xi \mathbf{x}^{(m)}(f_2(\xi, u_\theta),\, t - \lambda_\xi^{-1} s_\theta\,; s_2^+, k_2^+, u_2^+) & \\ & \text{if } \lambda_\xi^{-1} s_\theta < t, k_\theta = 1. \end{cases} \tag{336}$$

For each $v \in \mathcal{V}$ and $n \geq 0$ define the spacetime random field $\mathsf{X}_v^{(n)}(\xi, t)$ as before, by substituting the array of random variables (S_v^+, K_v^+, U_v^+) for the arguments (s_v^+, k_v^+, u_v^+) with the indices adjusted accordingly. Then for all $v \in \mathcal{V}$

$$\mathsf{X}_v^{(0)}(\xi, t) = \gamma(\xi, t) \tag{337}$$

and for all $n \geq 0$,

$$\mathsf{X}_v^{(n+1)}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if } \lambda_\xi^{-1} S_v \geq t, \\ \varphi(\xi, t - \lambda_\xi^{-1} S_v) & \text{if } \lambda_\xi^{-1} S_v < t, K_v = 0, \\ m(\xi)\mathsf{X}_{v1}^{(n)}(\Xi_{v1}, t - \lambda_\xi^{-1} S_v) \otimes_\xi \mathsf{X}_{v2}^{(n)}(\Xi_{v2}, t - \lambda_\xi^{-1} S_v) & \text{if } \lambda_\xi^{-1} S_v < t, K_v = 1. \end{cases} \tag{338}$$

Of course $\mathsf{X}_\theta^{(n)}(\xi, t)$ does not really depend on the entire array of random variables, just those up to the *nth* level. For example, the randomness in $\mathsf{X}_\theta^{(1)}(\xi, t)$ derives entirely from $(S_\theta, K_\theta, U_\theta)$. In general, $\mathsf{X}_\theta^{(n)}(\xi, t)$ is $\mathcal{F}_n$-measurable where $\mathcal{F}_n$ is defined for $n \geq 1$ by

$$\mathcal{F}_n = \sigma(S_v, K_v, U_v : |v| \leq n - 1), \tag{339}$$

and $\mathcal{F}_0 = \{\varnothing, \Omega\}$. This may be seen by drawing a few diagrams.

For fixed (ξ, t) the random variable $\mathsf{X}_\theta^{(n)}(\xi, t; \gamma)$ is the multiplicative functional obtained by truncating the branching process at the ends of the *nth* generation lineages, and, wherever the branching process would have been started anew, the known solution is input instead. For this reason the random field $\mathsf{X}_\theta^{(n)}(\xi, t; \gamma)$ will be referred to as the *nth* level truncation of $\mathsf{X}_\theta(\xi, t)$. If we want to emphasize that an element of the sequence — martingale actually — is based on a particular solution, $\gamma = \gamma(\xi, t)$ say, we write $\mathsf{X}_\theta^{(n)}(\xi, t; \gamma)$. In most applications, this martingale is based on an essentially bounded solution, but this is not necessary for its almost sure convergence. Theorem 5.8 addresses this. The boundedness of $\gamma(\xi, t)$ is one way of assuring each element of the sequence is integrable.

5.6 Theorem. *If the sequence $\left(\mathsf{X}_v^{(n)}(\xi, t) : n \geq 0\right)$ is based on a solution $\gamma(\xi, t) \in L^\infty(\mathbb{R}^n \times [0, T^*))$, then for any fixed $\xi \in \mathbb{R}^n$ and $0 \leq t < T^*$, $\mathsf{X}_v^{(n)}(\xi, t)$ is integrable for all $v \in \mathcal{V}$ and truncation levels $n \geq 0$, and $\mathbb{E}\mathsf{X}_v^{(n)}(\xi, t) = \gamma(\xi, t)$.*

Proof. The sequence $\left\{\mathsf{X}_\theta^{(n)}(\xi, t) : n \geq 0\right\}$ is based on a solution $\gamma(\xi, t)$ that is assumed to be in $L^\infty(\mathbb{R}^n \times [0, \infty))$; we may take $\gamma(\xi, t)$ to be a version that is everywhere bounded by R, say. Whereas each $\mathsf{X}_v^{(n)}(\xi, t)(\omega)$ is a $\otimes_\xi$-product of at most 2^n vectors, we have

$$\left|\mathsf{X}_v^{(n)}(\xi, t)(\omega)\right| \leq \max\left\{1, R^{2^n}\right\}. \tag{340}$$

Thus each $\mathsf{X}_v^{(n)}(\xi, t)$ is integrable. The rest of the proof is by induction starting with $\mathbb{E}\mathsf{X}_\theta^{(0)}(\xi, t) = \mathbb{E}\gamma(\xi, t) = \gamma(\xi, t)$. Compute $\mathbb{E}\mathsf{X}_\theta^{(n+1)}(\xi, t)$ by conditioning on $\mathcal{F}_1$:

$$\begin{aligned} \mathbb{E}\{\mathsf{X}_\theta^{(n+1)}(\xi, t) \,|\, \mathcal{F}_1\} &= \chi_0(\xi)\mathbf{1}\left[\lambda_\xi^{-1}S_\theta \geq t\right] + \varphi(\xi, t - \lambda_\xi^{-1}S_\theta)\mathbf{1}\left[\lambda_\xi^{-1}S_\theta < t,\, K_\theta = 0\right] \\ &\quad + \mathbb{E}\left(\begin{array}{c} m(\xi)\mathsf{X}_1(\Xi_1, t - \lambda_\xi^{-1}S_\theta) \\ \otimes_\xi \mathsf{X}_2^{(n)}(\Xi_2, t - \lambda_\xi^{-1}S_\theta) \end{array} \middle| \mathcal{F}_1\right)\mathbf{1}\left[\lambda_\xi^{-1}S_\theta < t,\, K_\theta = 1\right]. \end{aligned} \tag{341}$$

Computing the third term in the manner that was done in Theorem 5.5, and applying the inductive hypothesis, obtains

$$\mathbb{E}\{\mathsf{X}_\theta^{(n+1)}(\xi,t)\,|\,\mathcal{F}_1\} = \begin{cases} \chi_0(\xi)\mathbf{1}\Big[\lambda_\xi^{-1}S_\theta \geq t\Big]+ \\ \varphi(\xi,t-\lambda_\xi^{-1}S_\theta)\mathbf{1}\Big[\lambda_\xi^{-1}S_\theta < t,\ K_\theta = 0\Big]+ \\ \begin{pmatrix} m(\xi)\gamma(\Xi_1,t-\lambda_\xi^{-1}S_\theta) \\ \otimes_\xi\, \gamma(\Xi_2,\, t-\lambda_\xi^{-1}S_\theta) \end{pmatrix} \mathbf{1}\Big[\lambda_\xi^{-1}S_\theta < t,\ K_\theta = 1\Big]. \end{cases} \tag{342}$$

Taking the expected value of both sides and noting that the new right hand side is just the right hand side of *FNS:h* gives $\mathbb{E}\mathsf{X}_v^{(n+1)}(\xi,t) = \gamma(\xi,t)$, completing the induction step. □

The sequence $\{\mathsf{X}_\theta^{(n)}(\xi,t) : n \geq 0\}$ is actually a martingale. To show this it is useful to introduce a family of σ-fields indexed by $v \in \mathcal{V}$ and $n \geq 0$. These are defined recursively as follows:

$$\mathcal{F}_v^{(0)} = \{\varnothing, \Omega\} \tag{343}$$

and for all $n \geq 0$

$$\mathcal{F}_v^{(n+1)} = \sigma(S_v, K_v, U_v) \vee \mathcal{F}_{v1}^{(n)} \vee \mathcal{F}_{v2}^{(n)}. \tag{344}$$

5.7 Theorem. *For fixed* (ξ,t), $\{\mathsf{X}_\theta^{(n)}(\xi,t) : n \geq 0\}$ *is a martingale adapted to the filtration* $\mathcal{F}_n = \mathcal{F}_\theta^{(n)}$.

Proof. The proof is by induction on n starting with

$$\mathbb{E}\{\mathsf{X}_v^{(1)}(\xi,t)\,|\,\mathcal{F}_v^{(0)}\} = \mathbb{E}\mathsf{X}_v^{(1)}(\xi,t) = \mathsf{X}_v^{(0)}(\xi,t) \tag{345}$$

that holds for any $v \in \mathcal{V}$. The inductive hypothesis is that for all $v \in \mathcal{V}$,

$$\mathbb{E}\{\mathsf{X}_v^{(n)}(\xi,t)\,|\,\mathcal{F}_v^{(n-1)}\} = \mathsf{X}_v^{(n-1)}(\xi,t). \tag{346}$$

Let us compute:

$$\begin{aligned} \mathbb{E}\{\mathsf{X}_v^{(n+1)}(\xi,t)\,|\,\mathcal{F}_v^{(n)}\} &= \chi_0(\xi)\mathbf{1}[\lambda_\xi^{-1}S_v \geq t] \\ &+\varphi(\xi,t-\lambda_\xi^{-1}S_v)\mathbf{1}[\lambda_\xi^{-1}S_v < t,\ K_v = 0] \\ &+\mathbb{E}\left\{ \begin{matrix} m(\xi)\mathsf{X}_{v1}^{(n)}(\Xi_1,t-\lambda_\xi^{-1}S_v) \\ \otimes_\xi\, \mathsf{X}_{v2}^{(n)}(\Xi_2,t-\lambda_\xi^{-1}S_v) \end{matrix} \,\middle|\, \mathcal{F}_v^{(n)} \right\} \mathbf{1}[\lambda_\xi^{-1}S_v < t,\ K_v = 1]. \end{aligned} \tag{347}$$

The conditional expectation in the third term may be written as

$$m(\xi)\mathbb{E}\left\{\mathsf{X}_{v1}^{(n)}(\Xi_1, t-\lambda_\xi^{-1}S_v)\middle|\,\mathcal{F}_v^{(n)}\right\}\otimes_\xi \mathbb{E}\left\{\mathsf{X}_{v2}^{(n)}(\Xi_2, t-\lambda_\xi^{-1}S_v)\middle|\,\mathcal{F}_v^{(n)}\right\} \tag{348}$$

using conditional independence of the factors. Using

$$\mathcal{F}_v^{(n)} = \mathcal{F}_{v1}^{(n-1)} \vee \mathcal{F}_{v2}^{(n-1)} \vee \sigma(S_v, K_v, U_v), \tag{349}$$

and the role of independence in the conditional expectation, the factor on the left in (348) may be written as

$$\mathbb{E}\left\{\mathsf{X}_{v1}^{(n)}(\Xi_1, t-\lambda_\xi^{-1}S_v)\middle|\,\mathcal{F}_v^{(n)}\right\} = \mathbb{E}\left\{\mathsf{X}_{v1}^{(n)}(\Xi_1, t-\lambda_\xi^{-1}S_v)\middle|\,\mathcal{F}_{v1}^{(n-1)} \vee \sigma(S_v, K_v, U_v)\right\}. \tag{350}$$

The theorem that is being used here, e.g. [55, p. 88], is that if $\mathcal{H}$ and $\sigma(X) \vee \mathcal{G}$ are independent, then

$$\mathbb{E}\{X|\,\mathcal{G}\vee\mathcal{H}\} = \mathbb{E}\{X|\mathcal{G}\}, \quad \text{a.s.} \tag{351}$$

Here $X = \mathsf{X}_{v1}^{(n)}(\Xi_1, t-\lambda_\xi^{-1}S_\theta)$, $\mathcal{H} = \mathcal{F}_{v2}^{(n-1)}$, and $\mathcal{G} = \mathcal{F}_{v1}^{(n-1)} \vee \sigma(S_v, K_v, U_v)$. A similar expression holds for the factor on the right in (348). Now using the independence of $\sigma(S_v, K_v, U_v)$ and $\mathcal{F}_{v1}^{(n-1)}$, the substitution lemma, and the induction hypothesis, we my express this as

$$\mathbb{E}\left\{\mathsf{X}_{v1}^{(n)}(\Xi_1, t-\lambda_\xi^{-1}S_v)\middle|\,\mathcal{F}_{v1}^{(n-1)} \vee \sigma(S_v, K_v, U_v)\right\} = \mathsf{X}_{v1}^{(n-1)}(\Xi_1, t-\lambda_\xi^{-1}S_v). \tag{352}$$

Similarly,

$$\mathbb{E}\left\{\mathsf{X}_{v2}^{(n)}(\Xi_2, t-\lambda_\xi^{-1}S_v)\middle|\,\mathcal{F}_{v2}^{(n-1)} \vee \sigma(S_v, K_v, U_v)\right\} = \mathsf{X}_{v2}^{(n-1)}(\Xi_2, t-\lambda_\xi^{-1}S_v). \tag{353}$$

Putting (352) and (353) back into (348) and (347) using (350) obtains $\mathsf{X}_v^{(n)}(\xi, t)$, as expressed for example in the right hand side of (338), (but with n replaced by $n-1$). Explicitly,

$$\mathbb{E}\{\mathsf{X}_v^{(n+1)}(\xi, t)\,|\,\mathcal{F}_v^{(n)}\} = \mathsf{X}_v^{(n)}(\xi, t), \tag{354}$$

completing the induction step and the proof. □

5.8 Theorem. *For any fixed* (ξ, t)*, the martingale* $\big(\mathsf{X}_\theta^{(n)}(\xi, t; \gamma) : n \geq 0\big)$ *as described by equation (338), converges almost surely to the completed multiplicative functional* $\mathsf{X}_\theta(\xi, t)$.

Proof. Consider $\mathsf{X}_\theta(\xi, t)$ as a multiplicative functional on the branching process that is started with a single ancestral particle of type $\Xi_0 = \xi$. The branching process does not explode almost surely, and as a corollary to this fact, the random variable

$$N(t) = \inf\{n \geq 0 : \mathsf{X}_\theta^{(n)}(\xi, t; \gamma) = \mathsf{X}_\theta(\xi, t)\} \tag{355}$$

is finite almost surely. In terms of the branching process, $N(t)$ is one more than the highest number of splits up until elapsed time t along any genealogical line. The events

$$A_k = [N(t) \leq k] = \big[\mathsf{X}_\theta^{(k)}(\xi, t; \gamma) = \mathsf{X}_\theta^{(k+1)}(\xi, t; \gamma) = \cdots = \mathsf{X}_\theta(x, t)\big] \tag{356}$$

form an increasing and exhaustive sequence, implying the assertion to be proved. □

5.9 Remark. Theorem 5.8 is really about the comportment of the *nth* level truncations in relation to the completed multiplicative functional. There is no martingale convergence theorem at work here; in fact these martingales are not necessarily L^1-bounded.

CHAPTER 6
LOCAL EXISTENCE AND UNIQUENESS OF SOLUTIONS TO FOURIER NAVIER-STOKES EQUATIONS

In this chapter the stochastic methodology is applied to obtain local existence and uniqueness results for the Navier-Stokes equations given arbitrarily large initial datum, as measured by the norm on certain method adapted Banach spaces. Local means local in time. The Banach spaces are $\mathbb{B}_h$ and $\mathbb{B}_{h,T^*}$, whose construction require a majorizing kernel $h(\xi)$ of exponent $0 \le \theta < 1$. Examples of such majorizing kernels may be found in among the Bessel kernels, whose characteristic exponential decay at infinity put them in the class of analytic-majorizing kernels discussed in Chapter 4. By using such a majorizing kernel, $\mathbb{B}_h$ becomes a Banach space of analytic functions. Thus the existence and uniqueness results presented here, in the context of solutions to the *FNS:h* integral equation, may be translated to statements about the spatial analyticity of solutions to the Navier-Stokes equations.

§6.1 The method adapted Banach spaces

Detailed motivation for employing the Banach spaces $\mathbb{B}_h$ appears in the next section. Basically the methodology requires the initial datum to belong to this space, and the forcing term $g(x,t)$ to be such that for any fixed t

$$G_2\big(\sqrt{\nu\delta^{-1}}x\big) * g(x,t) \in \mathbb{B}_h, \tag{357}$$

where $G_2(x)$ is the Bessel kernel of order 2. One approach to the definition of the $\mathbb{B}_h$ would be the following:

$$\mathbb{B}_h = \Big\{ f \in [\mathcal{S}'(\mathbb{R}^n)]^n : \widehat{f} \in L^0(\mathbb{R}^n;\mathbb{C}^n),\ \widehat{f}(\xi)\big[h(\xi)\big]^{-1} \in L^\infty(\mathbb{R}^n;\mathbb{C}^n) \Big\} \tag{358}$$

Here $\mathcal{S}'(\mathbb{R}^n)$ is the space of temperate distributions on $\mathbb{R}^n$ and $[\mathcal{S}'(\mathbb{R}^n)]^n$ is the space of vector fields whose components are in $\mathcal{S}'(\mathbb{R}^n)$. The L^0–notation denotes measurability (measurable functions as opposed to temperate distributions); $L^0(\mathbb{R}^n;\mathbb{C}^n)$ is the space of measurable complex n-vectors on $\mathbb{R}^n$. This is essentially the approach taken in [7], but the exact approach used therein is not quite satisfactory, particularly for the purpose of defining Banach spaces of time-dependent functions, e.g. $\mathbb{B}_{h,T^*}$. Definition 6.1, below is better. This chapter is concerned exclusively with analytic-majorizing kernels, meaning that the class of function spaces $\mathbb{B}_h$ under consideration is smaller than that induced by more general majorizing kernels that are not necessarily integrable. The restriction to this smaller class of function spaces means that inverse Fourier transform acts on L^1–functions. Here $L^\infty(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n \times [0, T^*])$ are just more compact expressions for $L^\infty(\mathbb{R}^n, \mathbb{C}^n)$ and $L^\infty(\mathbb{R}^n \times [0, T^*], \mathbb{C}^n)$ respectively.

6.1 Definition. *The Banach spaces $\mathbb{B}_h$, $\widehat{\mathbb{B}}_h$ $\mathbb{B}_{h,T^*}$, $\widehat{\mathbb{B}}_{h,T^*}$ are defined as follows:*

$$\widehat{\mathbb{B}}_h = \{\chi(\xi)h(\xi) : \chi(\xi) \in L^\infty(\mathbb{R}^n)\},$$
$$\mathbb{B}_h = \mathcal{F}^{-1}\widehat{\mathbb{B}}_h = \{\mathcal{F}^{-1}\widehat{v}(\xi) : \widehat{v}(\xi) \in \widehat{\mathbb{B}}_h\},$$
$$\widehat{\mathbb{B}}_{h,T^*} = \{\chi(\xi,t)h(\xi) : \chi(\xi,t) \in L^\infty(\mathbb{R}^n \times [0,T^*]),\ \chi(\xi,t_0) \in L^\infty(\mathbb{R}^n)\ \forall t_0 \in [0,T^*]\},$$
$$\mathbb{B}_{h,T^*} = \mathcal{F}^{-1}\widehat{\mathbb{B}}_{h,T^*}.$$

The norms are

$$\|f(x)\|_{\mathbb{B}_h} = \|\widehat{f}(\xi)\|_{\widehat{\mathbb{B}}_h} = \left\|\frac{\widehat{f}(\xi)}{h(\xi)}\right\|_{L^\infty(\mathbb{R}^n)},$$
$$\|f(x,t)\|_{\mathbb{B}_{h,T^*}} = \|\widehat{f}(\xi,t)\|_{\widehat{\mathbb{B}}_{h,T^*}} = \sup_{0\le t\le T*}\left\|\frac{\widehat{f}(\xi,t)}{h(\xi)}\right\|_{L^\infty(\mathbb{R}^n)}$$

where $f(x) = (\mathcal{F}^{-1}\widehat{f}\,)(x)$, and $f(x,t) = (\mathcal{F}^{-1}\widehat{f}\,)(x,t)$.

For each of these Banach spaces the onus of membership lies on the Fourier side; this just reflects the fact that the analysis occurs on the Fourier side. The norms on these spaces are just a translations of the L^∞-norms naturally arising from the methodology. Note that equation (357) is just a more complicated way of expressing that for each t,

$$\frac{1}{\delta + \nu|\xi|^2} \cdot \widehat{g}(\xi,t) \in \widehat{\mathbb{B}}_h. \tag{359}$$

Presumably the results discussed here can be translated to statements about local existence of classical solutions to the Navier-Stokes equations provided the forcing term $g(x,t)$

has sufficient regularity, and that these classical solutions admit holomorphic extensions to tubes $\mathbb{R}^n + iU \subset \mathbb{C}^n$.

§6.2 The delta method: an exponential time-dependent transformation

Recall the Navier-Stokes equations for the velocity field of a viscous incompressible fluid:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u - \nabla p + g(x,t), \tag{360}$$

$$\nabla \cdot u = 0. \tag{361}$$

The time-dependent transformation $v(x,t) = u(x,t)e^{-\delta t}$ appears in the analysis of partial differential equations of parabolic type. In fact, one demonstration of a maximum principle for Burgers equation begins with this transformation [31, p. 132]. Applying this transformation to the Navier-Stokes equations gives the modified system of equations

$$\frac{\partial v}{\partial t} + \delta v + e^{\delta t} v \cdot \nabla v = \nu \Delta v - e^{-\delta t} \nabla p + e^{-\delta t} g(x,t), \tag{362}$$

$$\nabla \cdot v = 0. \tag{363}$$

However, we actually want to apply this transformation directly to the the *FNS:h* integral equation, anticipating that the newly induced δ terms will become useful. In fact they are, and we are lead to the exponent θ majorizing kernels defined by the inequalities

$$h * h(\xi) \leq B|\xi|^{\theta} h(\xi), \quad B > 0, \quad 0 \leq \theta < 1. \tag{364}$$

Many solutions of these are known.

The *FNS:h* equations are

$$\chi(\xi,t) = e^{-\nu|\xi|^2 t}\chi_0(\xi) + \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2 s}\{\cdots\}ds \tag{365}$$

$$\{\cdots\} = \frac{1}{2}m(\xi)\int_{R^n} \chi(\eta, t-s) \otimes_{\xi} \chi(\xi - \eta, t-s)dK_{\xi}(\eta) + \frac{1}{2}\varphi(\xi, t-s).$$

In terms of the function $\chi(\xi,t)$, the transformation becomes $\chi^{\star}(\xi,t) = \chi(\xi,t)e^{-\delta t}$, and the initial datum remains unchanged: $\chi_0^{\star}(\xi) = \chi_0(\xi)$. Applying this transformation gives the new integral equation, denoted *δ-FNS:h*:

$$\chi^\star(\xi,t) = e^{-\lambda_\xi t}\chi_0^\star(\xi) + \int_0^t \lambda_\xi e^{-\lambda_\xi s}\{\cdots\}ds$$

$$\{\cdots\} = \frac{1}{2}e^{\delta(t-s)}m^\star(\xi)\int_{R^n}\chi^\star(\eta,t-s)\otimes_\xi \chi^\star(\xi-\eta,t-s)dK_\xi(\eta)$$

$$+\frac{1}{2}e^{-\delta(t-s)}\varphi^\star(\xi,t-s) \tag{366}$$

where

$$\chi^\star(\xi,t) = \frac{\hat{v}(\xi,t)}{h(\xi)} = \frac{\widehat{u}(\xi,t)}{h(\xi)}e^{-\delta t} = \chi(\xi,t)e^{-\delta t}, \qquad \lambda_\xi = \nu|\xi|^2+\delta$$

$$m^\star(\xi) = \frac{2h*h(\xi)|\xi|}{(2\pi)^{n/2}(\nu|\xi|^2+\delta)h(\xi)}, \qquad \varphi^\star(\xi,t) = \frac{\nu|\xi|^2\varphi(\xi,t)}{\nu|\xi|^2+\delta} = \frac{2\widehat{g}(\xi,t)}{(\nu|\xi|^2+\delta)h(\xi)}. \tag{367}$$

Applying the same methodology as described in Section 5.3 leads to the stochastic representation of solutions to *δ-FNS:h*,

$$\chi^\star(\xi,t) = \mathbb{E}\mathsf{X}_\theta^\star(\xi,t). \tag{368}$$

The multiplicative functional (or random field) $\mathsf{X}_\theta^\star(\xi,t)$ has the usual recursive expression:

$$\mathsf{X}_\theta^\star(\xi,t) = \begin{cases} \chi_0(\xi) & \text{if } S_\theta \geq t, \\ e^{-\delta(t-S_\theta)}\varphi^\star(\Xi_1, t-\lambda_\xi^{-1}S_\theta) & \text{if } S_\theta < t, K_\theta = 0, \\ e^{+\delta(t-S_\theta)}m^\star(\xi)\mathsf{X}_1^\star(\Xi_1,t-\lambda_\xi^{-1}S_\theta)\otimes_\xi \mathsf{X}_2^\star(\Xi_2,t-\lambda_\xi^{-1}S_\theta) & \text{if } S_\theta < t, K_\theta = 1. \end{cases} \tag{369}$$

Of course, the representation holds just in case $\mathsf{X}_\theta^\star(\xi,t)$ is integrable. The exponential term $e^{\delta(t-S_\theta)}$ precludes global control of integrability, but this is balanced by the particular functional form of $m^\star(\xi)$, with the new δ term, allowing control of the size of $\mathsf{X}_\theta^\star(\xi,t)$ locally, over a finite time interval $0<t<T^*$, but with arbitrarily large initial datum and forcing. Conforming to this goal, the bound

$$e^{\delta T^*}m^\star(\xi) \leq \frac{1}{R} \tag{370}$$

is obtained next, with the result that each binary operation in the computation of $\mathsf{X}_\theta^\star(\xi,t)$ is controlled for increasing norm. That is, for

$$z,w\in\mathbb{C}^n, \quad 0\leq s,t\leq T^*, \quad |z|\leq R, \quad |w|\leq R, \tag{371}$$

and $\xi \neq 0$, the output of the binary operation has norm

$$\left| e^{\delta(t-s)} m^\star(\xi) z \otimes_\xi w \right| \leq e^{\delta T^*} m^\star(\xi) \left| z \otimes_\xi w \right| \leq \frac{1}{R} R^2 = R. \tag{372}$$

Accordingly, we have

$$\begin{aligned} m^\star(\xi) &= \frac{2h * h(\xi)|\xi|}{(2\pi)^{n/2}(\nu|\xi|^2 + \delta)h(\xi)} \\ &\leq \frac{2}{(2\pi)^{n/2}} \cdot \sup_{|\xi|} \left\{ \frac{h * h}{|\xi|^\theta h(\xi)} \right\} \cdot \sup_{|\xi|} \left\{ \frac{|\xi|^{1+\theta}}{\nu|\xi|^2 + \delta} \right\} \end{aligned} \tag{373}$$

where the multiplication and division by $|\xi|^\theta$ exploits the existence of majorizing kernels of exponent $0 \leq \theta < 1$. Taking $h(\xi)$ to be such a majorizing kernel, the middle factor is

$$B = \sup_{|\xi|} \frac{h * h(\xi)}{|\xi|^\theta h(\xi)}, \tag{374}$$

and the supremum on the right is

$$\sup_{|\xi|} \left\{ \frac{|\xi|^{1+\theta}}{\nu|\xi|^2 + \delta} \right\} = \frac{1}{2} \sqrt{\frac{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}{\nu^{1+\theta}\,\delta^{1-\theta}}}, \tag{375}$$

which is attained at

$$|\xi| = \sqrt{\frac{\delta}{\nu}\left(\frac{1+\theta}{1-\theta}\right)}. \tag{376}$$

Next, let

$$R = \frac{\sqrt{\nu^{1+\theta}\delta^{1-\theta}}}{Bk}, \tag{377}$$

which is the inverse of the left hand side of (375) upto a multiplicative factor. The auxiliary variable k is left unspecified for now, but will be determined later according to the goal of maximizing T^*. The reason for introducing k is explained in Remark 6.4 below. From the definition of R and equations (373) through (375), it follows that

$$m^\star(\xi) \leq \frac{c}{Rk}, \tag{378}$$

where

$$c = \frac{1}{(2\pi)^{n/2}} \sqrt{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}. \tag{379}$$

Solving equation (377) for δ obtains

$$\delta = (Rk)^{2/(1-\theta)} \left[\frac{B^2}{\nu^{1+\theta}} \right]^{1/(1-\theta)}, \tag{380}$$

and taking T^* to be

$$T^* = \frac{1}{\delta}\left(\log k - \log c\right) \tag{381}$$

achieves

$$e^{\delta T^*} m^\star(\xi) \;\le\; kc^{-1} \cdot \frac{c}{Rk} = \frac{1}{R}. \tag{382}$$

The value of k that maximizes T^* is now determined. The function

$$f(k) = k^{-2/(1-\theta)}(\log k - \log c) \tag{383}$$

captures the essential dependence of T^* on k, attaining its maximum at $k = k_m$, where

$$\log k_m = \frac{1-\theta}{2} + \log c. \tag{384}$$

The maximum value of $f(k)$ is

$$\sup_{k>0} f(k) = f(k_m) = \frac{(2\pi)^{n/(1-\theta)}}{2e\,(1+\theta)^{(1+\theta)/(1-\theta)}}. \tag{385}$$

Using this value, $k = k_m$, in equations (380) and (381) gives

$$\delta = e \cdot (1-\theta)\left(\frac{(1+\theta)^{(1+\theta)}R^2}{(2\pi)^n}\right)^{1/(1-\theta)}\left[\frac{B^2}{\nu^{1+\theta}}\right]^{1/(1-\theta)} \tag{386}$$

and

$$T^* = \frac{1}{2e}\left(\frac{(2\pi)^n}{(1+\theta)^{(1+\theta)}R^2}\right)^{1/(1-\theta)}\left[\frac{\nu^{1+\theta}}{B^2}\right]^{1/(1-\theta)}. \tag{387}$$

The main result of the preceding calculation is that with δ and T^* and determined by (386) and (387), then for any $0 \le s,t \le T^*$, the binary operation that appears in the recursive expression (369), including the attached multiplicative factor, satisfies

$$z, w \in \mathbb{C}^n, \quad |z|, |w| \le R \quad \Rightarrow \quad \left|e^{\delta(t-s)} m^\star(\xi) z \otimes_\xi w\right| \le R. \tag{388}$$

This forms the basis of the local existence results presented in Section 6.4.

§6.3 Remarks on the variables, units, and majorizing kernels

Here is a summary of the dependence relationship between n, θ, ν, B, R, δ and T^*. The dimension n is fixed, and θ is considered as given, constrained only by the existence of a majorizing kernel of exponent $0 \leq \theta < 1$ but is otherwise arbitrary. The space $\mathbb{B}_h$ is fixed, determined by some majorizing kernel $h(\xi)$ satisfying

$$h * h(\xi) \leq B|\xi|^{\theta} h(\xi) \tag{389}$$

but is otherwise arbitrary. Of course, the standardization of $h(\xi)$ has no effect on the computation of T^*, as the change $B \to 1$ brought about by standardization is balanced by the change $R \to BR$. Having fixed $\mathbb{B}_h$, with $h(\xi)$ either standardized or not, δ and T^*, which are determined from R, B, and ν, are inversely related: $2\delta T^* = 1 - \theta$.

6.2 Remark (on units). Let $[L]$ denotes units of length and $[T]$ denote units of time. In order to compare $\widehat{u}(\xi, t)$ and $h(\xi)$, both should have the same units, namely $[L]^4[T]^{-1}$. Then R is dimensionless, and *FNS:h* is a dimensionless equation, as is equation (366) that is obtained via the transformation

$$\chi^\star(\xi, t) = \chi(\xi, t)e^{-\delta t} \tag{390}$$

applied to *FNS:h*. In these equations

$$m(\xi) = \frac{2|\xi| h * h(\xi)}{(2\pi)^{n/2}\nu|\xi|^2 h(\xi)} \quad \text{and} \quad m^\star(\xi) = \frac{2h * h(\xi)|\xi|}{(2\pi)^{n/2}(\nu|\xi|^2 + \delta)h(\xi)} \tag{391}$$

are both dimensionless. When $0 \leq \theta < 1$, $m^\star(\xi)$ is bounded by the product of the dimensional quantity B, having units of $[L]^{1+\theta}[T]^{-1}$, and another factor having units inverse to those of B. The ratio $\nu^{1+\theta}/B^2$ has units of $[T]^{1-\theta}$, from which a time can be obtained. When $\theta = 1$, B and ν both have the same units, and between them no time or length can be obtained. Here are the units on all the variables (with $0 \leq \theta < 1$):

x	$[L]$	ν	$[L]^2[T]^{-1}$	k	dimensionless
ξ	$[L]^{-1}$	h	$[L]^4[T]^{-1}$	$\chi, \chi^\star$	dimensionless
t	$[T]$	B	$[L]^{1+\theta}[T]^{-1}$	$m, m^\star$	dimensionless
δ	$[T]^{-1}$	c	dimensionless	R	dimensionless

In equations (386) and (387) the quantity in the square bracket carries units; the other factor is dimensionless. If we standardize $h(\xi)$ and take $B = 1$, then the 1 so obtained carries units.

6.3 Remark (on standardizing the majorizing kernel). Recall that $\mathcal{H}^{n,\theta}$ admits a 1-parameter group of transformations

$$\beta{\cdot}h(\xi) \stackrel{def}{=} \beta^{n-\theta}h(\beta\xi), \quad \beta > 0, \tag{392}$$

by scaling both the independent and dependent variables. Thus if $h(\xi)$ is a majorizing kernel of exponent θ and constant B, as in equation (364), a standardized majorizing kernel $h_{std}(\xi)$ may be constructed using any transformation of the form (with B made dimensionless),

$$h_{std}(\xi) = B^{-1}\beta^{n-\theta}h(\beta\xi), \quad \beta > 0. \tag{393}$$

This includes the scaling of ξ alone by taking $\beta^{n-\theta} = B$.

6.4 Remark (on the auxiliary variable). The variable k is introduced because without it, the computation of T^* is not invariant under scaling the Fourier transform. To illustrate this, consider the computation of T^* with $k = 1$:

$$T^*(k=1) \;=\; \left[\frac{\nu^{1+\theta}}{B^2}\right]^{1/(1-\theta)} \left(\frac{1}{R^2}\right)^{1/(1-\theta)} \log\frac{(2\pi)^{n/2}}{\sqrt{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}}. \tag{394}$$

Were another version of the Fourier transform used, $\mathcal{F}_\lambda \stackrel{\text{def}}{=} \lambda\mathcal{F}$ say, then T^* would be computed as:

$$T^*(k=1) \;=\; \left[\frac{\nu^{1+\theta}}{B^2}\right]^{1/(1-\theta)} \left(\frac{1}{\lambda^2R^2}\right)^{1/(1-\theta)} \log\frac{\lambda(2\pi)^{n/2}}{\sqrt{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}}. \tag{395}$$

On the other hand, equation (387) is invariant under scaling the Fourier transform. The effect of optimization over k increases as $\theta \to 1$. For example, with $n = 3$ and $\theta = 0$, the ratio between T^* computed with $k = k_m$ and T^* computed with $k = 1$ is

$$\frac{T^*(k=k_m)}{T^*(k=1)} = \frac{(2\pi)^3}{2e}\left(\log\,(2\pi)^{3/2}\right)^{-1} \approx 16.6, \tag{396}$$

and with $n = 3$ and $\theta = 1/2$, this ratio is $\approx$1280. As $\theta \to 1$ this ratio increases to infinity.

§6.4 Local existence and uniqueness of solutions to *FNS:h*

With δ given by equation (386) in the construction of the branching process and recursively defined multiplicative functional $\mathsf{X}^\star_\theta(\xi,t)$, the control of norm increase at the binary nodes, e.g. equation (388), extends to the entire multiplicative functional, as long as $0 \le t \le T^*$.

6.5 Lemma. *If the initial datum and forcing satisfy the bounds*

$$\|\chi_0(\xi)\|_{L^\infty(\mathbb{R}^n)} \le R, \quad \|\varphi^\star(\xi,t)\|_{L^\infty(\mathbb{R}^n\times[0,T^*])} \le R, \tag{397}$$

then the multiplicative functional $\mathsf{X}^\star_\theta(\xi,t)$ given recursively by (369) satisfies the same bound. That is, almost surely,

$$|\mathsf{X}^\star_\theta(\xi,t)| \le R \tag{398}$$

for $0 \le t \le T^*$, where T^* is given by equation (387).

Proof. Almost all realizations of the branching process produce trees with at most finitely many binary nodes. Given such a realization, the computation of the multiplicative functional is done through a finite sequence of binary operations, where the nesting of the operations corresponds to the branching structure of the tree. At each stage in the computation, the two operational inputs are bounded by R, and as equation (388) shows, the output is bounded by R. This output is either the final value, or the input of another binary operation. After all binary operations are complete, the final value must be bounded by R. □

6.6 Theorem (Existence and Uniqueness). *Let $h(\xi)$ be a majorizing kernel of exponent $0 \le \theta < 1$, normalized to $\sup_{|\xi|} h * h(\xi)/|\xi|^\theta h(\xi) = 1$. Suppose that the initial datum and forcing satisfy the bounds*

$$\|\chi_0(\xi)\|_{L^\infty(\mathbb{R}^n)} \le R \quad \text{and} \quad \|\varphi^\star(\xi,t)\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \le R, \tag{399}$$

Then there exists a function $\chi(\xi,t)$ measurable in ξ, and continuous in t, that solves FNS:h, locally in time, at least on the interval $[0,T^*)$ where T^* is given by equation (387). The solution $\chi(\xi,t)$ satisfies the bound

$$\sup_\xi |\chi(\xi,t)| \le Re^{\delta t} \tag{400}$$

where δ is given by equation (386). This is the unique solution to FNS:h in the class $L^\infty(\mathbb{R}^n \times [0,T^*])$.

Proof. For existence, let $\chi^\star(\xi,t) = \mathbb{E}\mathsf{X}_\theta^\star(\xi,t)$ where $\mathsf{X}_\theta^\star(\xi,t)$ is defined recursively by (369) on $\mathbb{R}^3 \times [0,T^*]$. Lemma 6.5 explains the integrability of the random field therein. Arguing exactly as in Theorem 5.5 we conclude that $\chi^\star(\xi,t)$ solves δ-*FNS:h.* Of course $|\chi^\star(\xi,t)| \le R$ as well. This bound coupled with the definition $\chi^\star(\xi,t) = \chi(\xi,t)e^{-\delta t}$ gives the bound (400).

To establish uniqueness, observe that the following two statements are equivalent:

1. $\chi(\xi,t)$ uniquely solves *FNS:h* in the class $L^\infty(\mathbb{R}^n \times [0,T))$ for any $0 < T \le T^*$,
2. $\chi^\star(\xi,t)$ uniquely solves δ-*FNS:h* in the class $L^\infty(\mathbb{R}^n \times [0,T])$ for any $0 < T \le T^*$.

We verify the second statement. Suppose that $\gamma(\xi,t) \in L^\infty(\mathbb{R}^n \times [0,T_a^*))$ is some alternative solution to *FNS:h* up to an alternative time $T_a^* \le T^*$. Let $\gamma^\star(\xi,t) = \gamma(\xi,t)e^{-\delta t}$ be the corresponding solution to δ-*FNS:h.* Let R' denote the larger of the two solutions to δ-*FNS:h*:

$$R' = \max\left\{\|\chi^\star(\xi,t)\|_{L^\infty},\, \|\gamma^\star(\xi,t)\|_{L^\infty},\, R\right\}. \tag{401}$$

Here the R takes into account the the initial datum $\chi_0^\star(\xi)$ and forcing $\varphi^\star(\xi,t)$, and L^∞-norm refers to the Banach space $L^\infty(\mathbb{R}^n \times [0,T_a^*))$. Define T_1 by equation (386) with R replaced by R'; this gives a time interval $[0,T_1]$ on which the output of the binary operation is bounded by R' when the two inputs are bounded by R'. Equipped with this fact, fix ξ and $t \le T_1$ and consider the two martingales

$$\left(\mathsf{X}_\theta^{\star(n)}(\xi,t;\chi^\star) : n \ge 0\right), \qquad \left(\mathsf{X}_\theta^{\star(n)}(\xi,t;\gamma^\star) : n \ge 0\right), \tag{402}$$

that are based on the solutions $\chi^\star(\xi,t)$ and $\gamma^\star(\xi,t)$ respectively. Both of these martingales are bounded by R' almost surely, and converge in L^1 to their respective almost sure limits. But these limits are the same, being the completed multiplicative functional $\mathsf{X}_\theta^\star(\xi,t)$ whose inputs are consist entirely of forcing terms and the initial datum. It follows that the two solutions agree up to time T_1. Repeating this argument with the equation restarted at time T_1 and taking as initial datum the function $\chi^\star(\xi,T_1)$ establishes uniqueness up to time $T_2 = 2T_1$, and iterating gives uniqueness up to $T_a^\star$, which verifies the last statement of the theorem. □

§6.5 Bessel kernels as majorizing kernels

The utility of majorizing kernels of exponent $0 \le \theta < 1$ in Theorem 6.6 motivates the search for particular majorizing kernels of such exponents, and an exploration of their properties. In this section the Bessel kernels $G_\alpha^{(n)}(\xi)$ are considered as candidates for solving the convolution inequality $h * h(\xi) \le |\xi|^\theta h(\xi)$, with $0 < \theta < 1$. One benefit of exhibiting majorizing kernels in this class of functions, is that the Bessel kernels decay exponentially at infinity, and the corresponding Fourier majorization spaces are actually Banach spaces of analytic functions.

Some necessary facts about $G_\alpha^{(n)}(\xi)$ and $K_\nu(z)$ are first presented. The references are Jones [27, pp. 314–316] or Adams and Hedberg [2, pp. 10–12] for the firstly presented representation of the Bessel kernels; Aronszajn and Smith [5, pp. 413–417] for the representation of the Bessel kernels in terms of the Bessel functions $K_\nu(z)$, and the asymptotics of the Bessel kernels; Lebedev [35, pp. 108–120] for the integral representation for $K_\nu(z)$, and the recurrence relations satisfied by $K_\nu(z)$; and Watson [53, p. 440] for the integral representation of the product of two Bessel functions.

The Bessel kernel of order α on $\mathbb{R}^n$ may be defined (as in Definition 3.6) through the integral representation

$$\begin{aligned} G_\alpha^{(n)}(\xi) &= \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty p_n(t,\xi) t^{\alpha/2-1} e^{-t} dt \\ &= \frac{1}{\Gamma(\frac{\alpha}{2})(4\pi)^{n/2}} \int_0^\infty t^{(\alpha-n)/2-1} e^{-|\xi|^2/4t-t} dt, \end{aligned} \tag{403}$$

where $\xi \in \mathbb{R}^n$. It also has the representation

$$G_\alpha^{(n)}(\xi) = \frac{1}{2^{(n+\alpha-2)/2}\pi^{n/2}\Gamma(\frac{\alpha}{2})} K_{\frac{n-\alpha}{2}}(|\xi|)|\xi|^{\frac{\alpha-n}{2}}. \tag{404}$$

The Bessel function $K_\nu(z)$ is an analytic function of z except at $z = 0$, and for $z \ne 0$ it extends to an entire analytic function of ν. It has the integral form

$$\begin{aligned} K_\nu(z) &= \frac{1}{2} \int_{-\infty}^\infty e^{-z\cosh u - \nu u} du \\ &= \int_0^\infty e^{-z\cosh u} \cosh(\nu u) du \qquad |\arg z| < \frac{\pi}{2}, \quad \nu \text{ arbitrary}. \end{aligned} \tag{405}$$

Making the substitution $t = e^u$ gives the

$$K_\nu(z) = \frac{1}{2}\int_0^\infty e^{-zt/2-z/2t}\, t^{-\nu-1} e^{-t} dt, \tag{406}$$

and replacing t by $2t/z$ gives the integral representation

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-z^2/4t} t^{-\nu-1} e^{-t} dt, \qquad |\arg z| < \frac{\pi}{4}. \tag{407}$$

This establishes agreement between representations (403) and (404).

The integral form (405) shows that $K_\nu(z) = K_{-\nu}(z)$, and by differentiating with respect to ν under the integral, we see that $K_\nu(z)$ is an increasing function of ν, for fixed $z > 0$, when ν is real and positive. In addition, the following recurrence relations are satisfied by $K_\nu(z)$:

$$\frac{d}{dz}[z^\nu K_\nu(z)] = -z^\nu K_{\nu-1}(z), \tag{408}$$

$$\frac{d}{dz}[z^{-\nu} K_\nu(z)] = -z^{-\nu} K_{\nu+1}(z), \tag{409}$$

$$K_{\nu-1}(z) + K_{\nu+1}(z) = -2K'_\nu(z), \tag{410}$$

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z). \tag{411}$$

The product of $K_\mu(z)$ and $K_\nu(z)$ has the integral representation

$$K_\mu(z)K_\nu(z) = 2\int_0^\infty K_{\mu+\nu}(2z\cosh t)\cosh(\mu-\nu)t\, dt \tag{412}$$

$$= 2\int_0^\infty K_{\mu-\nu}(2z\cosh t)\cosh(\mu+\nu)t\, dt, \tag{413}$$

when $|\arg z| < \pi/4$ and μ and ν are unrestricted.

The Bessel kernel $G_\alpha(\xi) = G_\alpha^{(n)}(\xi)$ has the following asymptotics:

$$\text{As } \xi \to 0, \qquad \begin{cases} G_\alpha(\xi) \sim \dfrac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{n/2} \Gamma(\frac{\alpha}{2})} |\xi|^{\alpha-n} & \text{if } \alpha < n, \\ G_n(\xi) \sim \dfrac{1}{2^{n-1}\pi^{n/2}\Gamma(\frac{n}{2})} \log \dfrac{1}{|\xi|}, & \\ G_\alpha(\xi) \sim \dfrac{\Gamma(\frac{\alpha-n}{2})}{2^n \pi^{n/2}\Gamma(\frac{\alpha}{2})} & \text{if } \alpha > n. \end{cases} \tag{414}$$

$$\text{As } \xi \to \infty, \qquad G_\alpha(\xi) \sim \frac{1}{2^{\frac{n+\alpha-1}{2}} \pi^{\frac{n-1}{2}} \Gamma(\frac{\alpha}{2})} |\xi|^{\frac{\alpha-n-1}{2}} e^{-|\xi|}. \tag{415}$$

The following lemma shows how these asymptotics restrict the order of the Bessel kernel if it is to serve as a majorizing kernel of a particular exponent.

6.7 Lemma. *Let $G_\alpha(\xi) = G_\alpha^{(n)}(\xi)$ denote the Bessel kernel of order α on $\mathbb{R}^n$.*
If $\alpha < n/2$, then

$$\lim_{\xi \to 0} \frac{G_\alpha * G_\alpha(\xi)}{|\xi|^\theta G_\alpha(\xi)} = \begin{cases} \infty & \text{if } \theta > \alpha, \\ 2^{-\alpha} \dfrac{\Gamma(\frac{n-2\alpha}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})\Gamma(\alpha)} & \text{if } \theta = \alpha, \\ 0 & \text{if } \theta < \alpha. \end{cases} \tag{416}$$

If $\alpha = n/2$, then

$$\lim_{\xi \to 0} \frac{G_\alpha * G_\alpha(\xi)}{|\xi|^\theta G_\alpha(\xi)} = \begin{cases} \infty & \text{if } \theta \geq \alpha, \\ 0 & \text{if } \theta < \alpha. \end{cases} \tag{417}$$

If $n/2 < \alpha < n$, then

$$\lim_{\xi \to 0} \frac{G_\alpha * G_\alpha(\xi)}{|\xi|^\theta G_\alpha(\xi)} = \begin{cases} \infty & \text{if } \theta > n-\alpha, \\ 2^{\alpha-n} \dfrac{\Gamma(\frac{2\alpha-n}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})\Gamma(\alpha)} & \text{if } \theta = n-\alpha, \\ 0 & \text{if } \theta < n-\alpha. \end{cases} \tag{418}$$

If $\alpha = n$, then

$$\lim_{\xi \to 0} \frac{G_\alpha * G_\alpha(\xi)}{|\xi|^\theta G_\alpha(\xi)} = \begin{cases} \infty & \text{if } \theta > 0, \\ 0 & \text{if } \theta = 0. \end{cases} \tag{419}$$

If $n < \alpha$, then

$$\lim_{\xi \to 0} \frac{G_\alpha * G_\alpha(\xi)}{|\xi|^\theta G_\alpha(\xi)} = \begin{cases} \infty & \text{if} \quad \theta > 0, \\ \dfrac{\Gamma(\frac{2\alpha - n}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha - n}{2})\Gamma(\alpha)} & \text{if} \quad \theta = 0. \end{cases} \tag{420}$$

The asymptotics at infinity are

$$\lim_{|\xi| \to \infty} \frac{G_\alpha * G_\alpha(\xi)}{|\xi|^\theta G_\alpha(\xi)} = \begin{cases} \infty & \text{if} \quad \theta < \dfrac{\alpha}{2}, \\ 2^{-\alpha/2} \dfrac{\Gamma(\frac{\alpha}{2})}{\Gamma(\alpha)} & \text{if} \quad \theta = \dfrac{\alpha}{2}, \\ 0 & \text{if} \quad \theta > \dfrac{\alpha}{2}. \end{cases} \tag{421}$$

Proof. These statements follow directly from the asymptotics given by (414) and (415), and the convolution structure of the Bessel kernels, i.e. $G_\alpha * G_\alpha(\xi) = G_{2\alpha}(\xi)$. □

Lemma 6.7 says that when the desired exponent of the majorizing kernel is $0 < \theta < n/2$, it is necessary to take $\theta \leq \alpha \leq \min\{n - \theta, 2\theta\}$ as the order of the Bessel kernel. The following theorem establishes that this necessary condition is also sufficient. Lemma 6.7 also says that when the desired exponent of the majorizing kernel is $\theta > n/2$, there are no Bessel kernel solutions.

6.8 Theorem. *The Bessel kernel $G_\alpha(\xi)$ is a majorizing kernel of exponent θ if and only if $0 < \theta < n/2$ and $\theta \leq \alpha \leq \min\{n - \theta, 2\theta\}$.*

Proof. For $G_\alpha(\xi)$ to be a majorizing kernel of exponent θ it is necessary and sufficient that

$$\lim_{\xi \to 0} \frac{G_{2\alpha}(\xi)}{|\xi|^\theta G_\alpha(\xi)} < \infty, \qquad \lim_{|\xi| \to \infty} \frac{G_{2\alpha}(\xi)}{|\xi|^\theta G_\alpha(\xi)} < \infty. \tag{422}$$

The asymptotics at infinity imply $\alpha \leq 2\theta$, from (421). Since there are no Bessel kernels of order $\alpha = 0$, θ must be strictly positive. Take θ to be fixed and consider the implications of various α. It can not be the case that $\alpha \geq n$, lest $\theta = 0$ by (419) and (420). For the other possible values of α, the following implications hold from (416), (417), and (418), respectively:

$$\begin{aligned} \alpha < \tfrac{n}{2} \quad &\Rightarrow \quad \theta \leq \alpha < \tfrac{n}{2}, \\ \alpha = \tfrac{n}{2} \quad &\Rightarrow \quad \theta < \alpha = \tfrac{n}{2}, \\ \tfrac{n}{2} < \alpha < n \quad &\Rightarrow \quad \alpha \leq n - \theta. \end{aligned} \tag{423}$$

Each of these statements separately implies that $\theta < \frac{n}{2}$, and together, along with the previously noted constraint $\alpha < 2\theta$, they give the stated constraints on α. $\square$

When the desired exponent of the majorizing kernel is $0 < \theta < n/2$ and the order of the Bessel kernel is one of the two extremes $\alpha = \theta$ or $\alpha = \min\{n-\theta, 2\theta\}$, the normalization constant for the majorizing kernel may be computed exactly. In the following theorem $G_\theta(\xi) = G_\theta^{(n)}(\xi)$ denotes the Bessel kernel of order θ on $\mathbb{R}^n$.

6.9 Theorem. *For $\theta < n/2$, $G_\theta(\xi)$ is a majorizing kernel of exponent θ; the normalization*

$$h(\xi) = \frac{2^\theta \Gamma(\frac{n-\theta}{2})\Gamma(\theta)}{\Gamma(\frac{n-2\theta}{2})\Gamma(\frac{\theta}{2})} G_\theta(\xi) \tag{424}$$

satisfies $h{*}h(\xi) \le |\xi|^\theta h(\xi)$ with

$$\sup_{|\xi|} \frac{h * h(\xi)}{|\xi|^\theta h(\xi)} = \lim_{|\xi|\to 0} \frac{h * h(\xi)}{|\xi|^\theta h(\xi)} = 1. \tag{425}$$

For $n/3 \le \theta < n/2$, $G_{n-\theta}(\xi)$ is a majorizing kernel of exponent θ; the normalization

$$h(\xi) = \frac{2^\theta \Gamma(\frac{\theta}{2})\Gamma(n-\theta)}{\Gamma(\frac{n-2\theta}{2})\Gamma(\frac{n-\theta}{2})} G_{n-\theta}(\xi) \tag{426}$$

satisfies $h{*}h(\xi) \le |\xi|^\theta h(\xi)$ with

$$\sup_{|\xi|} \frac{h * h(\xi)}{|\xi|^\theta h(\xi)} = \lim_{|\xi|\to 0} \frac{h * h(\xi)}{|\xi|^\theta h(\xi)} = 1. \tag{427}$$

If $\theta = n/3$ then

$$h(\xi) = 2^{n/3} \frac{\Gamma(\frac{2n}{3})}{\Gamma(\frac{n}{3})} G_{\frac{2n}{3}}(\xi) \tag{428}$$

satisfies $h{*}h(\xi) = |\xi|^\theta h(\xi)$.

Proof. For the first assertion, define $F_1(z)$ by

$$F_1(z) = \frac{G_\theta * G_\theta(\xi)}{|\xi|^\theta G_\theta(\xi)}, \qquad z = |\xi|. \tag{429}$$

By Lemma 6.7

$$\lim_{z\to 0} F_1(z) = 2^{-\theta} \frac{\Gamma(\frac{n-2\theta}{2})\Gamma(\frac{\theta}{2})}{\Gamma(\frac{n-\theta}{2})\Gamma(\theta)} \ne 0,\ \infty. \tag{430}$$

The plan is to show that $F_1(z)$ is decreasing on $[0,\infty)$. Indeed, by the representation (404)

$$\begin{aligned} F_1(z) &= 2^{-\theta/2}\frac{\Gamma(\frac{\theta}{2})}{\Gamma(\theta)}\frac{K_{\frac{n-2\theta}{2}}(z)z^{\frac{n-\theta}{2}}}{K_{\frac{n-\theta}{2}}(z)z^{\frac{n-2\theta}{2}}}\frac{1}{z^\theta} \\ &= C_\theta\frac{z^{\nu_1}K_{\nu_1}(z)}{z^{\nu_2}K_{\nu_2}(z)}, \qquad \text{with} \quad \begin{cases} \nu_1 = \frac{n-2\theta}{2}, \\ \nu_2 = \frac{n-\theta}{2}, \end{cases} \end{aligned} \tag{431}$$

and where $C_\theta > 0$ is the indicated constant. Differentiating $F_1(z)$, and using the recurrence relation (413), gives

$$F_1'(z) = C_\theta\frac{z^{\nu_1}z^{\nu_2}\Big[K_{\nu_1}(z)K_{\nu_2-1}(z) - K_{\nu_1-1}(z)K_{\nu_2}(z)\Big]}{\Big[z^{\nu_2}K_{\nu_2}(z)\Big]^2}. \tag{432}$$

The quantity in the numerator is always negative, utilizing $\nu_1 < \nu_2$. To see this, apply the product representation (413) to obtain

$$\begin{aligned} &K_{\nu_1}(z)K_{\nu_2-1}(z) - K_{\nu_1-1}(z)K_{\nu_2}(z) = \\ &\quad 2\int_0^\infty \Big\{K_{\nu_2-\nu_1-1}(2z\cosh t) - K_{\nu_2-\nu_1+1}(2z\cosh t)\Big\}\cosh(\nu_1+\nu_2-1)t\,dt. \end{aligned} \tag{433}$$

Since $\nu_1 < \nu_2$, the inequality $|\nu_2-\nu_1-1| < |\nu_2-\nu_1+1|$ holds, and because $K_\nu(z)$ is an increasing function of ν, it follows that for any fixed $z > 0$,

$$K_{\nu_2-\nu_1-1}(2z\cosh t) - K_{\nu_2-\nu_1+1}(2z\cosh t) < 0. \tag{434}$$

This is integrated against the positive function $\cosh(\nu_1+\nu_2-1)t$ so the entire integral, and hence $F'(z)$, is always negative.

The maximum value attained by $F_1(z)$ at $z = 0$ indicates the correct normalization for $h(\xi)$ in equation (424).

For the second and third assertions, define $F_2(z)$ by

$$F_2(z) = \frac{G_{n-\theta} * G_{n-\theta}(\xi)}{|\xi|^\theta G_{n-\theta}(\xi)}, \qquad z = |\xi|. \tag{435}$$

Again by Lemma 6.7, but this time equation (418),

$$\lim_{z\to 0} F_2(z) = 2^{-\theta}\frac{\Gamma(\frac{n-2\theta}{2})\Gamma(\frac{n-\theta}{2})}{\Gamma(\frac{\theta}{2})\Gamma(n-\theta)}. \tag{436}$$

The representation (404) and the symmetry condition $K_{-\nu}(z) = K_\nu(z)$ give

$$F_2(z) = 2^{\frac{\theta-n}{2}}\frac{\Gamma(\frac{n-\theta}{2})}{\Gamma(n-\theta)}\frac{K_{\frac{2\theta-n}{2}}(z)z^{\frac{n-2\theta}{2}}}{K_{\frac{\theta}{2}}(z)z^{-\frac{\theta}{2}}}\frac{1}{z^\theta}$$

$$= C_{n,\theta}\frac{z^{\nu_1}K_{\nu_1}(z)}{z^{\nu_2}K_{\nu_2}(z)}, \qquad \text{with} \quad \begin{cases} \nu_1 = \frac{n-2\theta}{2}, \\ \nu_2 = \frac{\theta}{2}. \end{cases} \tag{437}$$

Note that $\nu_1 \le \nu_2$ with equality when $\theta = n/3$. (Recall the hypothesis: $n/3 \le \theta < n/2$.) Repeating the previous argument shows that for $\nu_1 < \nu_2$, $F_2(z)$ is strictly decreasing. The maximum of $F_2(z)$, given by equation (436), determines the normalization for $h(\xi)$ in equation (426). Finally, when $\theta = n/3$,

$$F_2(z) \equiv 2^{-n/3}\frac{\Gamma(\frac{n}{3})}{\Gamma(\frac{2n}{3})}, \tag{438}$$

and the inverse of the constant on the right hand side is the correct normalization for $h(\xi)$ in equation (428). □

CHAPTER 7

THE EQUATION OF KOLMOGOROV, PETROVSKII, AND PISKUNOV

§7.1 Background and introduction

The subject of this chapter is the Kolmogorov–Petrovskii–Piskunov (KPP) equation for $u = u(x,t)$ in the form

$$\begin{aligned} &u_t = u_{xx} + u^2 - u, \quad -\infty < x < \infty, \quad t \geq 0, \\ &u(x,0) = f(x). \end{aligned} \tag{439}$$

This equation (or the variant obtained from the substitution $u \to 1-u$) is also known as Fisher's equation because Fisher introduced

$$\frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} + mp(1-p) \tag{440}$$

as a model of the spread of an advantageous gene throughout a population [21]. For this reason it also called the Fisher–KPP or FKPP equation, but here the appellative "FKPP" pertains to the Fourier transformed KPP equation for $\widehat{u} = \widehat{u}(\xi, t)$,

$$\widehat{u}_t = -|\xi|^2\widehat{u} + \frac{1}{\sqrt{2\pi}}\widehat{u} * \widehat{u} - \widehat{u}. \tag{441}$$

Equation (439) is probably really too simple to be a compelling model of any physical situation. Its importance is historical and pedagogical; it is the classic simplest case of a reaction-diffusion system. Such systems typically have the form

$$\partial_t \mathbf{u}(x,t) - D\Delta\mathbf{u}(x,t) = F(\mathbf{u}(x,t)), \quad x, \mathbf{u} \in \mathbb{R}^n \tag{442}$$

for some nonlinear function $F(\mathbf{u}(x,t))$ and diffusion matrix D. There is an enormous amount of research on these equations, much of it centered around the existence of wave-front solutions. Many basic questions about global existence of solutions and stability are still open.

A few examples of reaction diffusion equations are given. Somewhat more complicated than the KPP equation is Nagumo's equation

$$u_t = u_{xx} - u(u-1)(u-a), \quad 0 < a < \tfrac{1}{2}, \tag{443}$$

considered by McKean [41] as capturing certain essential behavior of transmission of nerve impulses. A still more complicated example is the well-known FitzHugh-Nagumo model

$$\begin{aligned} u_t - \epsilon u_{xx} &= \sigma v - \gamma u, \\ v_t - v_{xx} &= -v(v-a)(v-b) - u. \end{aligned} \tag{444}$$

Here σ and γ are positive constants, $\epsilon \geq 0$ and $0 < a < b$. This system is a more tractable simplification of the Hodgkins-Huxley equations — also a reaction diffusion system — consisting of four coupled equations which was set up by 1959 Nobel prize winners Hodgkins and Huxley in their investigations on the transmission of impulses in nerves [4, p. 224], [42]. A traveling wave model arising from the Belousov-Zhabotinskii oscillating chemical reaction is the system

$$\begin{aligned} u_t &= u_{xx} + u(1 - u - rv), \\ v_t &= v_{xx} - buv, \end{aligned} \tag{445}$$

with $r, b > 0$, and $0 \leq u(x,t), v(x,t) \leq 1$ [43, p. 236].

Solutions to equations (443) and (444) admit solutions defined as the expected value of a functionals on branching Brownian motion, like the KPP equation, except more complicated. The general relationship between a given reaction-diffusion equation and the possible expression of solutions with such branching Brownian motion representations is a new and open problem.

The reason for our interest in the KPP equation is that it admits recursive stochastic representations of solutions in both physical and Fourier space, with the underlying processes being branching Brownian motion and a multi-type branching process respectively. Information flows both ways, with properties of the two branching processes being derived from the partial differential equation and visa versa.

It should be mentioned that the physical and Fourier space branching processes are not really dual to each other as they do not lend themselves to the analysis of phenomena in a symmetric or dual way. As an example, there exist solutions to the KPP equation whose qualitative behavior resembles the traveling wave solutions, an example of which is the following 1-parameter family of solutions in explicit form discovered by Ablowitz and

Zeppetella [1]:

$$u(x,t) = 1 - \frac{1}{\left[1 + r\exp\{(x-ct)/\sqrt{6}\}\right]^2}, \quad c = \frac{5}{\sqrt{6}}, \quad r > 0. \tag{446}$$

These solutions all satisfy $u(-\infty) = 0$ and $u(+\infty) = 1$ or visa versa. McKean [39], [40] proved results on such traveling waves by using the branching Brownian motion representation. But these traveling waves, having the typical property of being flat at both ends far away from the wave front, are not integrable; and in any distributional sense their spatial Fourier transforms are not functions. This rules out their analysis with any Fourier side branching process, at least without the introduction of new ideas.

This chapter starts with known results on the branching Brownian motion representation of solutions to (439), which are then extended. The basic stochastic functional is put into the recursive framework used in the analysis of other partial differential equations, and it is constructed without relying on the Markov property. This framework is used to prove the uniqueness of any steady state solutions, which then used to establish the L^1-convergence of the martingale associated with the steady state solution. This is part of an intertwined set of results involving the partial differential equation and the multiplicative functional. They stem from the simple observation that the previously known bound on the initial datum, $|f(x)| \leq 1$, is a bound by a steady state solution, $f(x) \equiv 1$, which serves as a separatrix distinguishing finite time blow-up from global existence. Analogous results hold when other steady state solutions serve as bounds on the initial datum, and for such solutions the multiplicative functional remains integrable. This is the first known example of the integrability of this sort of recursively defined stochastic functional holding globally in time under a condition other than the initial datum bounded in absolute value by 1.

Turning to the Fourier side, another recursively defined stochastic representation is presented. The underlying multi-type branching process is called the FKPP process. A large portion of this section is devoted to analyzing the convolution equation that defines for it a suitable majorizing kernel. After an excursion into the theory of elliptic functions, suitable majorizing kernels are found for both the periodic and unrestricted KPP equation. Finally, equipped with branching process representations in physical and Fourier space, and drawing on the obtained uniqueness results, a non-explosion result for the FKPP process is derived, and this in turn, is used to obtain finite time blow-up for the KPP equation.

§7.2 Review of McKean's branching Brownian motion representation

In 1937 Kolmogorov, Petrovskii, and Piskunov [44] proved results on the existence of traveling wave solutions to (439) complementing results of Fisher [21] also in 1937. In 1975 McKean extended these results by employing the branching Brownian motion representation of solutions [39]. The origin of such ideas is not clear. Expected values functionals on branching diffusion processes satisfying nonlinear partial differential equations of parabolic type appears in Watanabe, 1967 [15].

For the KPP equation, the underlying stochastic process may be informally described as follows: At time $t = 0$, a single particle executes a Brownian motion $\mathfrak{X}(t)$ starting from the origin. (McKean discusses the equation $u_t = \frac{1}{2}u_{xx} + u^2 - u$ and the Brownian motion is standard; here the Brownian motion corresponds to the heat equation $u_t = u_{xx}$.) After an exponential waiting time T independent of $\mathfrak{X}(t)$, with $\mathbb{P}(T > t) = e^{-t}$, the particle splits in two, and the new particles starting from $\mathfrak{X}(T)$ continue along independent Brownian paths and are subject to the same splitting rule. After an elapsed time $t > 0$ there are $Z(t)$ particles located at the points $\mathfrak{X}^{(i)}(t)$, $i = 1, \ldots, Z(t)$. The KPP equation has a solution admitting the representation

$$u(x,t) = \mathbb{E} \prod_{i=1}^{Z(t)} f\big(x - \mathfrak{X}^{(i)}(t)\big), \tag{447}$$

provided that the initial datum satisfies $|f(x)| \leq 1$ to insure the integrability of the random product. (McKean requires $0 \leq f \leq 1$ pursuant to the physical interpretation of the model. In fact, in Fisher's equation, p is a probability.) The representation (447) is proved as follows: Let $u(x,t)$ be defined by (447). Making use of the notation $\{H_t = e^{t\Delta} : t \geq 0\}$ to denote the semigroup of convolution operators that is convolution with the respective

Gauss-Weierstrass kernel, and conditioning on the time of the first split,

$$\begin{aligned} u(x,t) &= \mathbb{P}(T>t)\int_{-\infty}^{\infty} f(x-y)\mathbb{P}(\mathfrak{X}_t \in dy) \\ &\quad + \int_0^t \mathbb{P}(T\in ds)\int_{-\infty}^{\infty} u^2(x-y,t-s)\mathbb{P}(\mathfrak{X}_s \in dy) \\ &= e^{-t}H_t f(x) + \int_0^t e^{-s}H_s u^2(x,t-s)ds. \end{aligned} \tag{448}$$

Then after making the substitution $s \to t-s$ in the integral, differentiation produces (439). This is done rigorously in Theorem 7.2 below.

Inspection of this argument reveals an apparent use of the strong Markov property. The proof involves an implicit recursiveness; at the random branching point $(\mathfrak{X}(T), T)$, the quantity $u^2(x-\mathfrak{X}(T), t-T)$ is recognized as the expected value of the product of the two (factor) multiplicative functionals on the independent branches of the after-T process. At the same time, the simplicity of this argument suggests that the strong Markov property is not really needed. In fact, there *is* a way to do this without invoking the strong Markov property, and this carefully presented in the next section.

Also, no indication of how one would come up with such a representation is given by McKean. It is useful in this context of analysis of partial differential equations through recursively defined stochastic processes, to see how Duhamel's principle can guide its discovery. Without worrying about the technical details, Duhamel's principle says roughly that for an initial-boundary value problem on a domain Ω in $\mathbb{R}^n$, the solution to the inhomogeneous equation

$$\begin{aligned} &u_t - \Delta u = g, && \text{in } \Omega\times\mathbb{R}^+, \\ &u = 0, && \text{on } \partial\Omega\times\mathbb{R}^+, \\ &u(\cdot,0) = f, && \text{in } \Omega, \end{aligned} \tag{449}$$

may be expressed as

$$u(x,t) = H_t f(x) + \int_0^t H_{t-s} g(x,s)ds, \tag{450}$$

where H_t is the solution operator for the homogeneous equation. A more abstract variant has A as the infinitesimal generator of a semigroup of operators on a Banach space, $\mathcal{D}(A)$

the domain of A, and $g \in C([0,T), \mathcal{D}(A))$. The solution to

$$\frac{\partial u}{\partial t} = Au + g(t), \quad u(0) = f, \tag{451}$$

is given by

$$u(t) = e^{tA}f + \int_0^t e^{(t-s)A}g(s)ds. \tag{452}$$

It is this last form that makes the representation idea transparent. Simply treat the nonlinear term u^2 in the KPP equation as a forcing term, justified a posteriori, and let $A = \Delta - I$. Writing the solution as

$$u(t) = e^{tA}f + \int_0^t e^{(t-s)A}u^2(x,s)ds \tag{453}$$

and noting $e^{tA} = e^{-t}H_t$, gives (448) after the substitution $s \to t - s$.

The equivalence of the integral equation form and the differential equation form of the KPP equation is now established. The fundamental solution to the heat equation (the Gauss-Weierstrass kernel) is denoted by

$$K(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-|x|^2/4t}. \tag{454}$$

The action of H_t is convolution with $K(x,t)$ in the spatial variable. The proof of Theorem 7.2 follows somewhat the discussion of the volume potential for the inhomogeneous heat equation in [12, p. 329].

7.1 Lemma. *For any $\lambda > 1$ there is a constant M such that*

$$\left|\frac{\partial}{\partial x}K(x-y,t)\right| \le \frac{M}{\sqrt{t}}K(x-y,\lambda t), \tag{455}$$

$$\left|\frac{\partial^2}{\partial x^2}K(x-y,t)\right| \le \frac{M}{t}K(x-y,\lambda t), \tag{456}$$

$$\left|\frac{\partial}{\partial t}K(x-y,t)\right| \le \frac{M}{t}K(x-y,\lambda t). \tag{457}$$

Proof. For any $\sigma > 0$ and $0 < \alpha < 1$ there exists an M such that

$$z^\sigma e^{-z} \le Me^{-\alpha z}, \quad z \ge 0, \tag{458}$$

and the proof of these statements is an application of this fact. □

7.2 Theorem. *Let $u(x,t) \in L^\infty(\mathbb{R} \times [0,T])$ solve the KPP integral equation*

$$u(x,t) = e^{-t}H_t f(x) + \int_0^t e^{-s} H_s u^2(x, t-s) ds \tag{459}$$

with $f(x) \in L^\infty(\mathbb{R})$. Then $u(\cdot,t) \in L^\infty(\mathbb{R})$ for all $t \in [0,T]$, and $u(x,t)$ is a classical solution (differentiable in t, twice differentiable in x on $\mathbb{R} \times (0,T]$) to the initial value problem

$$\begin{aligned} &u_t = u_{xx} + u^2 - u, \quad -\infty < x < \infty, \quad 0 \le t \le T, \\ &u(x, 0^+) = f(x). \end{aligned} \tag{460}$$

Moreover, if $f(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then $u(\cdot,t) \in L^p(\mathbb{R})$ for all $t \in [0,T]$ and for all $1 \le p \le \infty$.

Proof. For any $t \in [0,T]$ we have by equation (459)

$$\left|u(x,t)\right| \le e^{-t} \left\| f(x) \right\|_{L^\infty} + (1 - e^{-t}) \tilde{R}^2 \tag{461}$$

where $\tilde{R} = \|u(x,t)\|_{L^\infty(\mathbb{R}\times[0,T])}$, hence $u(\cdot,t) \in L^\infty(\mathbb{R})$. We now let

$$R = \sup_{0 \le t \le T} \left\| u(\cdot, t) \right\|_{L^\infty(\mathbb{R})} \tag{462}$$

and employ R, not $\tilde{R}$, for the rest of the proof. If $f(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then again by equation (459),

$$\left\| u(\cdot,t) \right\|_{L^1} \le e^{-t} \left\| f(x) \right\|_{L^1} + e^{-t} R \int_0^t e^s \left\| u(\cdot, s) \right\|_{L^1} ds \tag{463}$$

and utilizing Gronwall's Lemma applied to the function $e^t \|u(\cdot,t)\|_{L^1}$ it follows that

$$\left\| u(\cdot,t) \right\|_{L^1} \le \left\| f(x) \right\|_{L^1} \exp\left\{(R-1)t\right\}. \tag{464}$$

Thus $u(\cdot,t) \in L^1(\mathbb{R})$ on $[0,T]$, and by convexity of the L^p–norms, $u(\cdot,t) \in L^p(\mathbb{R})$ for all $1 \le p \le \infty$ on $[0,T]$.

Next, provided that differentiation under the integral sign is guaranteed, a straightforward differentiation yields

$$\Big(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\Big)u(x,t)=-e^{-t}\int_{-\infty}^{\infty}K(x-y,t)f(y)dy$$

$$+e^{-t}\int_{-\infty}^{\infty}\Big(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\Big)K(x-y,t)f(y)dy$$

$$-e^{-t}\int_0^t\int_{-\infty}^{\infty}e^sK(x-y,t-s)u^2(y,s)dy$$

$$+e^{-t}\int_0^t\int_{-\infty}^{\infty}e^s\Big(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\Big)K(x-y,t-s)u^2(y,s)dy+u^2(x,t)$$

$$=-u(x,t)+u^2(x,t). \tag{465}$$

The rest of the proof is a justification of these steps. The constants arising in the estimates are denoted C_1, C_2, C_3, and C_4.

We first need the existence of the derivative $(\partial/\partial x)u(x,t)$ along with the estimate

$$\Big|\frac{\partial}{\partial x}u^2(x,t)\Big|\le\frac{C_3}{\sqrt{t}}. \tag{466}$$

The derivative of the first term in the right hand side of (459) may be computed as

$$\frac{\partial}{\partial x}e^{-t}H_tf(x)=e^{-t}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}K(x-y,t)f(y)dy. \tag{467}$$

The justification for taking the derivative under the integral is by Lebesgue dominated convergence: according to Lemma 7.1,

$$\Big|\frac{\partial}{\partial x}K(x-y,t)f(y)\Big|\le\frac{M}{\sqrt{t}}K(x-y,\lambda t)\big\|f(x)\big\|_{L^\infty}, \tag{468}$$

and for any fixed $(x,t)\in\mathbb{R}\times(0,T]$ and $\epsilon>0$, we have on an interval of length 2ϵ, the bound

$$K(x-y,\lambda t)\le G_{x,t,\epsilon}(y)\overset{\text{def}}{=}\sup\big\{K(x'-y,\lambda t):x-\epsilon<x'<x+\epsilon\big\}. \tag{469}$$

The dominating function $G_{x,t,\epsilon}(y)$ is clearly integrable since it is bounded and decays exponentially at infinity. Equation (467) along with the estimate (468), gives the bound

$$\Big|\frac{\partial}{\partial x}e^{-t}H_tf(x)\Big|\le\frac{C_1e^{-t}}{\sqrt{t}}. \tag{470}$$

Similar arguments establish that the derivatives $\partial/\partial t$ and $\partial^2/\partial x^2$ may be taken inside the integral in the first term on the right hand side of (459):

$$\frac{\partial^2}{\partial x^2}e^{-t}H_t f(x) = \int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2}K(x-y,t)f(y)dy \tag{471}$$

and

$$\frac{\partial}{\partial t}e^{-t}H_t f(x) = -e^{-t}H_t f(x) + \int_{-\infty}^{\infty} \frac{\partial}{\partial t}K(x-y,t)f(y)dy. \tag{472}$$

For the purpose of taking the derivatives of the second term in (459), we introduce the following family of functions defined for h sufficiently close to t:

$$v_h(x,t) = e^{-t}\int_0^{t-h}\int_{-\infty}^{\infty} e^s K(x-y,t-s)u^2(y,s)ds. \tag{473}$$

Since the singularity of the kernel $K(x-y,t-s)$ occurs at $x=y$, $t=s$, we can differentiate under the integral, justified by arguments similar to the one given in the previous paragraph. The derivatives exist and are continuous:

$$\frac{\partial v_h}{\partial x}(x,t) = e^{-t}\int_0^{t-h}\int_{-\infty}^{\infty} e^s \frac{\partial}{\partial x}K(x-y,t-s)u^2(y,s)dyds, \tag{474}$$

$$\frac{\partial^2 v_h}{\partial x^2}(x,t) = e^{-t}\int_0^{t-h}\int_{-\infty}^{\infty} e^s \frac{\partial^2}{\partial x^2}K(x-y,t-s)u^2(y,s)dyds, \tag{475}$$

and

$$\frac{\partial v_h}{\partial t}(x,t) = -v_h(x,t) + e^{-h}\int_{-\infty}^{\infty} K(x-y,h)u^2(y,t-h)dy$$
$$+ e^{-t}\int_0^{t-h}\int_{-\infty}^{\infty} e^s \frac{\partial}{\partial t}K(x-y,t-s)u^2(y,s)dyds. \tag{476}$$

Next, let

$$v(x,t) = e^{-t}\int_0^{t}\int_{-\infty}^{\infty} e^s K(x-y,t-s)u^2(y,s)dyds, \tag{477}$$

$$g(x,t) = e^{-t}\int_0^{t}\int_{-\infty}^{\infty} e^s \frac{\partial}{\partial x}K(x-y,t-s)u^2(y,s)dyds. \tag{478}$$

A comparison of $v(x,t)$ and $v_h(x,t)$ and a simple estimate established the uniform convergence of $v_h(x,t)$ to $v(x,t)$. Indeed,

$$\big|v(x,t) - v_h(x,t)\big| = e^{-t}\int_t^{t-h}\int_{-\infty}^{\infty} e^s K(x-y,t-s)u^2(y,s)dyds \le R^2 h. \tag{479}$$

We also have the the uniform convergence of $(\partial/\partial x)v_h(x,t)$ to $g(x,t)$:

$$\left|g(x,t) - \frac{\partial v_h}{\partial x}(x,t)\right| = e^{-t}\int_{t-h}^{t}\int_{-\infty}^{\infty} e^s \frac{\partial}{\partial x}K(x-y,t-s)u^2(y,s)dyds$$

$$\leq R^2\int_{t-h}^{t}(t-s)^{-1/2}\int_{-\infty}^{\infty}\frac{|x-y|}{2(t-s)^{1/2}}K(x-y,t-s)dyds \leq 2R^2\pi^{-1/2}\sqrt{h}. \tag{480}$$

From

$$v_h(x,t) = \int_{x_0}^{x}\frac{\partial v_h}{\partial x}(x',t)dx' + v_h(x_0,t), \tag{481}$$

and the uniform convergence of v_h to v and $\partial v_h/\partial x$ to g, we have

$$v(x,t) = \int_{x_0}^{x} g(x',t)dx' + v(x_0,t) \tag{482}$$

on any compact interval $[x_0, x]$, implying that

$$\frac{\partial}{\partial x}v(x,t) = g(x,t) = e^{-t}\int_0^t\int_{-\infty}^{\infty} e^s\frac{\partial}{\partial x}K(x-y,t-s)u^2(y,s)dyds. \tag{483}$$

From a calculation similar to (480) we also have the estimate

$$\left|\frac{\partial v}{\partial x}(x,t)\right| \leq 2R^2\pi^{-1/2}\sqrt{t}. \tag{484}$$

Combining this with the bound (470) gives

$$\left|\frac{\partial u}{\partial x}(x,t)\right| \leq \frac{C_1e^{-t}}{\sqrt{t}} + 2R^2\pi^{-1/2}\sqrt{t} \leq \frac{C_2}{\sqrt{t}} \tag{485}$$

for some positive constant C_2, and

$$\left|\frac{\partial}{\partial x}u^2(x,t)\right| \leq \frac{C_3}{\sqrt{t}} \tag{486}$$

where $C_3 = 2RC_2$.

We now consider the derivatives $(\partial^2/\partial x^2)$ and $(\partial/\partial t)$ applied to the second term on the right hand side of (459). An application of the identity

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial t}K(x-y,t-s)u^2(x,s)dy = \int_{-\infty}^{\infty}\frac{\partial^2}{\partial x^2}K(x-y,t-s)u^2(x,s)dy = 0 \tag{487}$$

gives

$$\frac{\partial^2 v_h}{\partial x^2} = e^{-t}\int_0^{t-h}\int_{-\infty}^{\infty} e^s\frac{\partial^2}{\partial x^2}K(x-y,t-s)\big[u^2(y,s) - u^2(x,s)\big]dyds. \tag{488}$$

The immediate goal is to show that

$$\frac{\partial^2 v}{\partial x^2} = g_2(x,t) \stackrel{\text{def}}{=} e^{-t}\int_0^t \int_{-\infty}^{\infty} e^s \frac{\partial^2}{\partial x^2} K(x-y,t-s)u^2(y,s)dyds. \tag{489}$$

The differentiability of $u^2(x,t)$, the mean value theorem, and the bound (486), obtains the following estimate, which establishes the uniform convergence of $\partial^2 v_h/\partial x^2$ to g_2:

$$\begin{aligned}\Big|g_2(x,t) - \frac{\partial^2 v_h}{\partial x^2}(x,t)\Big| &= e^{-t}\int_{t-h}^t \int_{-\infty}^{\infty} e^s \frac{\partial^2}{\partial x^2} K(x-y,t-s)\big[u^2(y,s)-u^2(x,s)\big]dyds \\ &\leq \frac{C_3}{\sqrt{4\pi}}\int_{t-h}^t \int_{-\infty}^{\infty} s^{-1/2}\Big\{\frac{|x-y|}{t-s} + \frac{|x-y|^3}{2(t-s)^2}\Big\} e^{-|x-y|^2/4(t-s)} \frac{dy}{2(t-s)^{1/2}} ds \\ &\leq \frac{BC_3}{\sqrt{4\pi}}\int_{t-h}^t (t-s)^{-1/2}s^{-1/2}ds \leq C_4\Big(\frac{h}{t-h}\Big)^{1/2}. \end{aligned} \tag{490}$$

Here B is a constant arising from the transformation $\rho = (x-y)/(2(t-s)^{1/2})$. From

$$\frac{\partial}{\partial x}v_h(x,t) = \int_{x_0}^x \frac{\partial^2}{\partial x^2}v_h(x',t)dx' + \frac{\partial}{\partial x}v_h(x_0,t) \tag{491}$$

and the uniform convergence of $\partial v_h/\partial x$ to $\partial v/\partial x$ and $\partial^2 v_h/\partial x^2$ to g_2, we have

$$\frac{\partial}{\partial x}v(x,t) = \int_{x_0}^x g_2(x',t)dx' + \frac{\partial}{\partial x}v_h(x,t)y \tag{492}$$

on compact intervals $[x_0,x]$, which implies that

$$\frac{\partial^2}{\partial x^2}v(x,t) = g_2(x,t). \tag{493}$$

Finally let

$$g_3(x,t) = -v(x,t) + u^2(x,t) + \frac{\partial^2}{\partial x^2}v(x,t). \tag{494}$$

From equations (475) and (476), we have

$$\frac{\partial v_h}{\partial t}(x,t) = -v_h(x,t) + e^{-h}\int_{-\infty}^{\infty} K(x-y,h)u^2(y,t-h)dy + \frac{\partial^2}{\partial x^2}v_h(x,t), \tag{495}$$

which converges uniformly to $g_3(x,t)$ as $h \to 0$. From

$$v_h(x,t) = \int_{t_0}^t \frac{\partial}{\partial t}v_h(x,t')dt' + \frac{\partial}{\partial t}v_h(x,t_0) \tag{496}$$

and the uniform convergence of v_h to v and $\partial v_h/\partial t$ to $g_3(x,t)$, we have on compact subintervals $[t_0, t] \subset [0, T]$

$$v(x,t) = \int_{t_0}^{t} g_3(x,t')dt' + \frac{\partial}{\partial t}v(x,t_0), \tag{497}$$

which implies that

$$\frac{\partial}{\partial t}v(x,t) = g_3(x,t). \tag{498}$$

Combining the appropriate equations establishes equation (465). □

§7.3 The stochastic recursion obtained without the Markov property

A basic object is the multiplicative functional

$$M_\theta(x,t) = \prod_{i=1}^{Z(t)} f(x - \mathfrak{X}^{(i)}(t)) \tag{499}$$

that is determined by the branching Brownian motion process and whose expected value solves the KPP equation. In this section this object is constructed without explicit appeal to the strong Markov property, or even the Markov property of Brownian motion. The fact that Brownian motion is a Markov process is incidental.

We would like to define the random product (499) recursively by writing the entire product in terms of self-similar functionals defined on the independent branches determined by the first split, say as

$$M_\theta(x,t) = \begin{cases} f(x - \mathfrak{X}_\theta(S_\theta)) & \text{if } S_\theta \geq t, \\ M_1\big(x - \mathfrak{X}_\theta(S_\theta), t - S_\theta\big) M_2\big(x - \mathfrak{X}_\theta(S_\theta), t - S_\theta\big) & \text{if } S_\theta < t. \end{cases} \tag{500}$$

The problem of making rigorous sense of this leads to a construction along the lines of the multiplicative functional for the *FNS:h* equation. The basic idea is the same, that of beginning with a deterministic function whose arguments are later replaced by arrays of random variables. While it is not necessary to repeat this construction for each partial differential equation under consideration, it is worth going into the details here as it involves Brownian motion instead of simple random variables.

There are two arrays of $\mathcal{V}$-indexed random objects from which the randomness of the multiplicative functional derives. These are $(S_v : v \in \mathcal{V})$ and $(\mathfrak{X}_v : v \in \mathcal{V})$. The S_v are *i.i.d.* exponential random variables with parameter 1. The $\mathfrak{X}_v = \mathfrak{X}_v(t)$ are *i.i.d.* Brownian motions with

$$\mathbb{P}(\mathfrak{X}_v(t) \in A) = \int_A \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} dx \tag{501}$$

for all $t > 0$ and Borel sets $A \subset \mathbb{R}$. These Brownian motions correspond to the heat equation $u_t = u_{xx}$.

Recall the notation of the superscript $^+$ used in arrays of $\mathcal{V}$-indexed objects to denote the collection of all successive objects in the partial order on $\mathcal{V}$. For example, for any $v \in \mathcal{V}$,

$$S_v^+ = \{S_{v1}, S_{v2}, S_{v11}, S_{v12}, S_{v21}, S_{v22}, \dots\}, \tag{502}$$

and $S_\theta^+ = (S_v : v \in \mathcal{V})$ denotes the entire $\mathcal{V}$-indexed array.

The deterministic function

$$\mathbf{m} : \mathbb{R} \times [0, T^*) \times \prod_{v \in \mathcal{V}} \left(C(\mathbb{R}^+, \mathbb{R}) \times \mathbb{R}^+ \right)_v \to \mathbb{R} \cup \{\infty\} \tag{503}$$

is defined according to the following recursive scheme. For the first step

$$\mathbf{m}(x, t\,; \mathfrak{x}_\theta^+, s_\theta^+) = \begin{cases} f(x - \mathfrak{x}_\theta(s_\theta)) & \text{if } s_\theta \geq t, \\ \mathbf{m}(x - \mathfrak{x}_\theta(s_\theta), t - s_\theta\,; \mathfrak{x}_1^+, s_1^+)\mathbf{m}(x - \mathfrak{x}_\theta(s_\theta), t - s_\theta\,; \mathfrak{x}_2^+, s_2^+) & \text{if } s_\theta < t. \end{cases} \tag{504}$$

This definition makes sense by virtue of the natural shift isomorphism between arrays $(s_v^+, \mathfrak{x}_v^+)$ and $(s_{v1}^+, \mathfrak{x}_{v1}^+)$ say, under the partial order induced by the index set $\mathcal{V}$. On the subsequent steps and including the first step $v = \theta$ as well,

$$\mathbf{m}(x, t\,; \mathfrak{x}_v^+, s_v^+) = \begin{cases} f\big(x - \mathfrak{x}_v(s_v)\big) & \text{if } s_v \geq t, \\ \mathbf{m}\big(x - \mathfrak{x}_v(s_v), t - s_v\,; \mathfrak{x}_{v1}^+, s_{v1}^+\big)\mathbf{m}\big(x - \mathfrak{x}_v(s_v), t - s_v\,; \mathfrak{x}_{v2}^+, s_{v2}^+\big) & \text{if } s_v < t. \end{cases} \tag{505}$$

If this scheme terminates in a finite number of steps then $\mathbf{m}(x, t\,; \mathfrak{x}_\theta^+, s_\theta^+)$ evaluates to a finite product. On the subset where finite halting does not occur, the assignment

$$\mathbf{m}(x, t; \mathfrak{x}_v^+ s_v^+) = \infty$$

is made. The issue of the size of the non-halting set becomes germane only after a measure is introduced by replacing the arrayed arguments with random variables. We will see that non-halting has probability zero.

For each $v \in \mathcal{V}$ define the spacetime random field $M_v(x,t)$ on $\mathbb{R}\times[0,T^*)$ by substituting the array of random variables $(S_v^+, \mathfrak{X}_v^+)$ for the arguments $(s_v^+, \mathfrak{x}_v^+)$ with the index set adjusted accordingly. (This means that S_v replaces s_θ, S_{v1} replaces s_1, etc.) In particular,

$$M_\theta(\xi,t) = \mathbf{m}(x,t\,;\mathfrak{X}_\theta^+, S_\theta^+), \tag{506}$$

and

$$M_1(\xi,t) = \mathbf{m}(x,t\,;\mathfrak{X}_1^+, S_1^+), \qquad M_2(\xi,t) = \mathbf{m}(x,t\,;\mathfrak{X}_2^+, S_2^+). \tag{507}$$

Then evidently for any $v \in \mathcal{V}$,

$$M_v(x,t) = \begin{cases} f(x - \mathfrak{X}_v(S_v)) & \text{if } S_v \geq t, \\ M_{v1}\big(x - \mathfrak{X}_v(S_v), t - S_v\big) M_{v2}\big(x - \mathfrak{X}_v(S_v), t - S_v\big) & \text{if } S_v < t. \end{cases} \tag{508}$$

On the right hand side $M_{v1}\big(x - \mathfrak{X}_v(S_v), t - S_v\big)$ denotes the random field — whose randomness derives from the functional dependence on $(\mathfrak{X}_{v1}^+, S_{v1}^+)$ — evaluated at the random point $(x - \mathfrak{X}_v(S_v), t - S_v)$.

7.3 Proposition. *For any fixed x and t the multiplicative functional $M_\theta(x,t)$ is finite (weather of not it is integrable) and the event of non-halting has probability zero.*

Proof. The number of factors in the random product $M_\theta(x,t)$ is completely determined by the number $Z(t)$ of Brownian particles at time t. We may forget the spatial diffusion; the collection of exponential holding times $(S_v : v \in \mathcal{V})$ is sufficient for computing $Z(t)$. By forgetting spatial diffusion, the process reduces to the Yule process, for which the number of particles at any given time is geometrically distributed. □

The event of non-halting before a fixed time t in the recursive definition of $M_\theta(x,t)$ is equivalent to the event of an explosion of the associated branching process before time t. Another proof of Proposition 7.3 is that explosions occur in branching Brownian motion with probability zero.

7.4 Proposition. *If $M_v(x,t)$ is integrable at a point (x,t) for some $v \in \mathcal{V}$ then, with x and t fixed, $M_v(x,t)$ is integrable for all $v \in \mathcal{V}$ and $\mathbb{E}M_v(x,t)$ does not depend on v.*

Proof. By definition $M_v(x,t) = \mathbf{m}(x,t;\mathfrak{X}_v^+, S_v^+)$ and the arrays $(\mathfrak{X}_v^+, S_v^+)$ all have the same distribution independent of $v \in \mathcal{V}$ by construction. Thus for any fixed (x,t) the random variables $M_v(x,t)$ all have the same distribution; and if any one is integrable then all are integrable with the same expectation. □

7.5 Theorem. *If the random field $M_\theta(x,t)$ is everywhere integrable on $\mathbb{R} \times [0, T^*)$ then $u(x,t) \stackrel{\text{def}}{=} \mathbb{E}M_\theta(x,t)$ solves the KPP equation therein.*

Proof. Compute $M_\theta(x,t)$ by conditioning on $\mathcal{F}_1 = \sigma(S_\theta, \mathfrak{X}_\theta)$:

$$\begin{aligned} \mathbb{E}\{M_\theta(x,t)\,|\,\mathcal{F}_1\} &= f\big(x - \mathfrak{X}_\theta(t)\big)\mathbf{1}[S_\theta \geq t] \\ &+ \mathbb{E}\left\{ \begin{array}{l} \mathbf{m}\big(x - \mathfrak{X}_\theta(S_\theta), t - S_\theta\,; S_1^+, \mathfrak{X}_1^+\big) \\ \times\mathbf{m}\big(x - \mathfrak{X}_\theta(S_\theta), t - S_\theta\,; S_2^+, \mathfrak{X}_2^+\big) \end{array} \middle| \mathcal{F}_1 \right\} \mathbf{1}[S_\theta < t]. \end{aligned} \tag{509}$$

The conditional expectation in the second term is now computed using the independence structure of the random variables involved, the substitution lemma, and Proposition 7.4:

$$\begin{aligned} &\mathbb{E}\left\{ \begin{array}{l} \mathbf{m}\big(x - \mathfrak{X}_\theta(S_\theta), t - S_\theta\,; S_1^+, \mathfrak{X}_1^+\big) \\ \times\mathbf{m}\big(x - \mathfrak{X}_\theta(S_\theta), t - S_\theta\,; S_2^+, \mathfrak{X}_2^+\big) \end{array} \middle| \mathcal{F}_1 \right\} \\ &= \left[\mathbb{E}\left(\begin{array}{l} \mathbf{m}\big(x - \mathfrak{x}_\theta(s_\theta), t - s_\theta\,; S_1^+, \mathfrak{X}_1^+\big) \\ \times\mathbf{m}\big(x - \mathfrak{x}_\theta(s_\theta), t - s_\theta\,; S_2^+, \mathfrak{X}_2^+\big) \end{array} \right) \right]_{s_\theta = S_\theta, \mathfrak{x}_\theta = \mathfrak{X}_\theta} \\ &= \left[\begin{array}{l} \mathbb{E}\mathbf{m}\big(x - \mathfrak{x}_\theta(s_\theta), t - s_\theta\,; S_1^+, \mathfrak{X}_1^+\big) \\ \times\mathbb{E}\mathbf{m}\big(x - \mathfrak{x}_\theta(s_\theta), t - s_\theta\,; S_2^+, \mathfrak{X}_2^+\big) \end{array} \right]_{s_\theta = S_\theta, \mathfrak{x}_\theta = \mathfrak{X}_\theta} \\ &= \left[\begin{array}{l} \mathbb{E}M_1\big(x - \mathfrak{x}_\theta(s_\theta), t - s_\theta\big) \\ \times\mathbb{E}M_2\big(x - \mathfrak{x}_\theta(s_\theta), t - s_\theta\big) \end{array} \right]_{s_\theta = S_\theta, \mathfrak{x}_\theta = \mathfrak{X}_\theta} \\ &= u\big(x - \mathfrak{X}_\theta(S_\theta), t - S_\theta\big)\, u\big(x - \mathfrak{X}_\theta(S_\theta), t - S_\theta\big). \end{aligned} \tag{510}$$

Replacing this in equation (509) and taking the expectation of both sides obtains

$$u(x,t) = \mathbb{E}f\big(x - \mathfrak{X}_\theta(t)\big)\mathbf{1}\left[S_\theta \geq t\right] + \mathbb{E}u^2(x - \mathfrak{X}_\theta(S_\theta), t - S_\theta)\mathbf{1}\left[S_\theta < t\right], \tag{511}$$

which the KPP equation expressed probabilistically (compare with (448)). $\square$

If we want to emphasize that the random field $M_\theta(x,t)$ is constructed from a particular initial datum, $f(x)$ say, the notation $M_\theta(x,t \mid f)$ is used. This should be carefully distinguished from the notation $M_\theta^{(n)}(x,t;u)$, introduced next, that denotes an element of a martingale sequence.

Just as in Section 5.4 where a martingale is associated with the multiplicative functional $\mathsf{X}_\theta(\xi,t)$ for the *FNS:h* equations, there is a natural martingale

$$\{M_\theta^{(n)}(x,t;u) : n \geq 0\} \tag{512}$$

that is associated with $M_\theta(x,t)$. Here $u = u(x,t) \in L^\infty(\mathbb{R} \times [0,T))$ is an existing solution to the KPP equation, and the martingale is said to be *based on* $u(x,t)$. It is possible to give a detailed construction of this martingale along the lines of the construction of the *FNS:h* martingale. This would involve a similar sequence of deterministic functions that in the end are replaced by random variables. However, it would be overly pedantic to repeat such a construction. Here it suffices to describe it recursively. Given any pair $(x,t) \in \mathbb{R} \times [0,T)$ we have

$$M_\theta^{(0)}(x,t;u) = u(x,t), \tag{513}$$

and for all $n \geq 0$

$$M_\theta^{(n+1)}(x,t;u) = \begin{cases} f(x - \mathfrak{X}_\theta(S_\theta)) & \text{if } S_\theta \geq t, \\ M_1^{(n)}\big(x - \mathfrak{X}_\theta(S_\theta), t - S_\theta\big) M_2^{(n)}\big(x - \mathfrak{X}_\theta(S_\theta), t - S_\theta\big) & \text{if } S_\theta < t. \end{cases} \tag{514}$$

Here is another way of describing the martingale: For fixed x and t the random variable $M_\theta^{(n)}(x,t;u)$ is the multiplicative functional on branching Brownian motion that is obtained by truncating at the ends of all of the *nth* generation lineages, and at the ramification point where the two new particles would have commenced new Brownian motions, the known solution is input instead — but squared because there are two new particles.

(The known solution $u(x,t)$ is evaluated at the random branch point if the elapsed time for the branching Brownian motion process is considered to run contrary to the evolution of the partial differential equation.) Otherwise, if an nth generation particle lives past elapsed time t, the initial datum is input as usual.

§7.4 Local and global existence: comparison with the Yule process

If the KPP equation (439) is given an initial condition that is constant, $f(x) = \alpha$ say, then all of the Brownian motion becomes extraneous to the computation of the solution. In this case the recursively defined multiplicative functional may be replaced by the simpler recursively defined functional that is obtained from (514) by suppressing all spatial dependence:

$$Y_v(t) = \begin{cases} \alpha & \text{if } S_v \geq t, \\ Y_{v1}(t - S_v)Y_{v2}(t - S_v) & \text{if } S_v < t. \end{cases} \tag{515}$$

The underlying stochastic process reduces to the Yule process, or exponential binary fission. The solution $u(x,t) = u(t) = \mathbb{E}Y_\theta(t)$ coincides with the generating function for the branching process evaluated at $s = \alpha$:

$$F(s,t) = \mathbb{E}s^{Z(t)}\Big|_{s=\alpha}. \tag{516}$$

Here $Z(t)$ counts the number of particles alive at time t; it is geometrically distributed. The generating function is usually defined for $|s| \leq 1$, because for $s > 1$, the random variable inside the right hand side of (516) is not necessarily integrable. Integrability holds only for $0 \leq t \leq T^*$, where

$$T^* = \begin{cases} \infty & \text{if } s \leq 1, \\ \log \dfrac{s}{s-1} & \text{if } s > 1, \end{cases} \tag{517}$$

and on this time interval we have

$$F(s,t) = \frac{se^{-t}}{1-(1-e^{-t})s}. \tag{518}$$

This is well-known; it may be obtained by solving either the forward or backward Kolmogorov equations for the Yule process [6]. Similar results hold for the KPP equation.

7.6 Proposition. *If the initial datum for the KPP equation satisfies* $|f(x)| \leq \alpha$ *then* $M_\theta(x,t)$ *is integrable on* $\mathbb{R} \times [0, T^*)$*, where*

$$T^* = \begin{cases} \infty & \text{if } \alpha \leq 1, \\ \log \dfrac{\alpha}{\alpha - 1} & \text{if } \alpha > 1. \end{cases} \tag{519}$$

In either case, $u(x,t) = \mathbb{E} M_\theta(x,t)$ *solves the KPP equation on the interval* $0 \leq t < T^*$ *and the following bound holds:*

$$|u(x,t)| \leq \frac{\alpha e^{-t}}{1 - (1 - e^{-t})\alpha}. \tag{520}$$

If $\alpha > 1$*, the initial datum* $f(x) \equiv \alpha$ *produces finite time blow-up, with blow-up time* T^* *given by (519).*

Proof. The total number of particles alive at time t is the same for the branching Brownian motion process and the exponential binary fission process. Let $Z(t)$ denote this common number. It is easy enough to couple these two processes just by forgetting the spatial variation in the diffusion process to produce the exponential binary fission process. Then we have

$$|M_\theta(x,t)| = \left| \prod_{i=1}^{Z(t)} f(x - \mathfrak{X}_t^{(i)}) \right| = \prod_{i=1}^{Z(t)} \left| f(x - \mathfrak{X}_t^{(i)}) \right| \leq \alpha^{Z(t)}. \tag{521}$$

Since $\alpha^{Z(t)}$ is integrable on $[0, T^*)$, the same holds for $|M_\theta(x,t)|$ and we have the bound

$$|u(x,t)| \leq \mathbb{E}\, |M_\theta(x,t)| \leq \mathbb{E} \alpha^{Z(t)} = \frac{\alpha e^{-t}}{1 - (1 - e^{-t})\alpha}. \tag{522}$$

If the initial datum is $f(x) \equiv \alpha$ then the inequalities in (522) are actually equalities and computing the blow-up time T^* is immediate. $\square$

Proposition 7.6 shows that the function that is identically 1 is a separatrix of sorts, distinguishing initial data that produces finite time blow-up from initial data for which global solutions exist. In itself this is unremarkable. However this should be seen in the following context: The constant functions 0 and 1 are the two trivial steady state solutions to the KPP equation. This means that they solve the ordinary differential equation

$$y'' + y^2 - y = 0. \tag{523}$$

It turns out that there are other steady state solutions which are not bounded by 1, for which an analogue of Proposition 7.6 holds. Moreover, the branching Brownian motion representation holds as well, for as long as the solution exists.

§7.5 The steady state solutions, and the L^1–convergence of the martingale

In Sections 7.10 and 7.11 a 1-parameter family of steady state solutions to the KPP equation is exhibited. The parameter c arises naturally as the first constant of integration for the corresponding ordinary differential equation (523), and for $-\frac{1}{6} \leq \mathrm{c} \leq 0$ the KPP equation has a strictly positive steady state solution $\psi_c(x)$ (called $y_c(x)$ in Sections 7.10 and 7.11). The extremes are

$$\psi_{-1/6}(x) \equiv 1 \quad \text{and} \quad \psi_0(x) = \frac{3}{2}\left[\cosh\left(\frac{x}{2}\right)\right]^{-2}. \tag{524}$$

For intermediate values of c, the $\psi_c(x)$ are periodic functions that are described in terms of Weierstrass elliptic functions. The salient feature of these solutions is their positivity; for all c in this range,

$$0 < \min_x \psi_c(x) \leq 1 \leq \max_x \psi_c(x) \leq \tfrac{3}{2}. \tag{525}$$

There are other steady state solutions but they are unbounded and not strictly positive.

The next few theorems make use of the almost sure convergence of the martingale $\{M^{(n)}(x,t;u) : n \geq 0\}$ introduced at the end of Section 7.3. One way to explain this convergence is by introducing the random variable $N(t)$ that is defined to be one more than the highest number of splits in any genealogical line up to elapsed time t in branching Brownian motion. Clearly $N(t)$ does not depend on x. Given a region $\mathbb{R} \times [0,T)$ on which a solution $u(x,t)$ to the KPP equation exists, $N(t)$ may also be described, for $t \leq T$, as

$$N(t) = \inf\{n \geq 0 : M_\theta^{(n)}(x,t;u) = M_\theta(x,t)\} \tag{526}$$

as long as the equality between $M_\theta^{(n)}(x,t;u)$ and $M_\theta(x,t)$ is understood as an identity between functional expressions, and that accidental numerical equalities do not count. By drawing a few diagrams one can see that $N(t)$ satisfies the bounds

$$1 + \log_2 Z(t) \leq N(t) \leq Z(t). \tag{527}$$

Let $A_k = A_k(t)$ denote the event $[N(t) \leq k]$. With t fixed the sequence $(A_k : k \geq 1)$ is clearly increasing, and the right hand bound in (527) implies

$$\mathbb{P}(A_k) = \mathbb{P}(N(t) \leq k) \geq \mathbb{P}(Z(t) \leq k) = 1 - (1 - e^{-t})^k \to 1. \tag{528}$$

Thus $(A_k : k \geq 1)$ is an increasing and exhaustive sequence. Noting that the event A_k is also the event

$$A_k = \left[M_\theta^{(k)}(x, t; u) = M_\theta^{(k+1)}(x, t; u) = \cdots = M_\theta(x, t)\right], \tag{529}$$

it follows that the martingale $\{M_\theta^{(n)}(x, t; u) : n \geq 0\}$ is eventually constant almost surely, and the constant value so attained coincides with the value of the completed functional $M_\theta(x, t)$.

Here is a more general way to describe the situation. Let the sequence of events A_k be defined by (529). Then the non-explosion of the branching process is equivalent to this sequence being exhaustive, and this gives an easy way to establish the almost sure convergence of the martingale to the completed multiplicative functional. This is essentially the content of Theorem 5.8.

7.7 Theorem. *If the initial datum for the KPP equation is any nonzero steady state solution $\psi(x)$ that is bounded and strictly positive, then $u(x, t) \equiv \psi(x)$ is the unique solution in the class $L^\infty(\mathbb{R} \times [0, \infty))$.*

Proof. Let $u(x, t) \equiv \psi(x)$, and suppose that for some $T > 0$, an alternative solution $v(x, t) \in L^\infty(\mathbb{R} \times [0, T_a))$ also solves the KPP equation with initial datum $\psi(x)$. We have to allow for the possibility that $v(x, t)$ exists on a shorter time interval than $u(x, t)$, hence the introduction of the time T_a. Define T_1 by

$$T_1 = \begin{cases} \infty & \text{if } R = 1, \\ \log \dfrac{R}{R-1} & \text{if } R > 1, \end{cases} \tag{530}$$

where

$$R = \max\left\{ \|v(x, t)\|_{L^\infty(\mathbb{R} \times [0, T_a))}, \sup_x \psi(x) \right\}. \tag{531}$$

We have $R \geq 1$ by the properties of $\psi(x)$. Fix $x \in \mathbb{R}$ and $t < T_1$, and let

$$\{M_\theta^{(n)}(x,t;u) : n \geq 0\} \qquad \text{and} \qquad \{M_\theta^{(n)}(x,t;v) : n \geq 0\}$$

denote the martingales based on solutions $u(x,t)$ and $v(x,t)$ respectively. Since $R \geq 1$ and the number of input factors is always bounded by $Z(t)$, we have the bounds

$$\left|M_\theta^{(n)}(x,t;u)\right| \leq R^{Z(t)}, \qquad \left|M_\theta^{(n)}(x,t;v)\right| \leq R^{Z(t)}, \tag{532}$$

independent of n, and $R^{Z(t)}$ is integrable since $t < T_1$. Then by standard martingale theory e.g. [55, p. 128], the two martingales are uniformly integrable, and they converge in L^1 to their respective almost sure limits, and the notation for these limits is introduced as well:

$$M_\theta^{(n)}(x,t;u) \xrightarrow{L^1} M_\theta^{(\infty)}(x,t;u) \quad \text{and} \quad M_\theta^{(n)}(x,t;v) \xrightarrow{L^1} M_\theta^{(\infty)}(x,t;v). \tag{533}$$

It is clear from the remarks preceding this theorem that

$$M_\theta^{(\infty)}(x,t;u) = M_\theta(x,t \mid \psi) \quad \text{and} \quad M_\theta^{(\infty)}(x,t;v) = M_\theta(x,t \mid \psi) \tag{534}$$

where $M_\theta(x,t \mid \psi)$ is the multiplicative functional associated with the initial datum $\psi(x)$. From the L^1-convergence (533), and equation (534) it follows that

$$\mathbb{E}M_\theta(x,t \mid \psi) = u(x,t) \quad \text{and} \quad \mathbb{E}M_\theta(x,t \mid \psi) = v(x,t), \tag{535}$$

so the two solutions agree up to time T_1. Repeating this argument with the equation restarted at time T_1 and and with initial datum $\psi(x) = u(x,T_1)$ gives uniqueness up to time $T_2 = 2T_1$. Iterating gives uniqueness of the steady state solution up to the time T_a that was associated with the possible alternative solution $v(x,t)$. This establishes the uniqueness of the steady state solution for all time $t > 0$. □

The following theorem shows that the uniqueness of the steady state solution implies the L^1-convergence of each of the martingales (parameterized by x) that are based on the steady state solution. The proof invokes Scheffe's Lemma, e.g. [55, p. 63], that states that if $(X_n : n \geq 1)$ is a sequence of random variables and $X_n \to X$ almost surely, then

$$\mathbb{E}|X_n| \to \mathbb{E}|X| \quad \text{implies} \quad \mathbb{E}|X_n - X| \to 0, \tag{536}$$

with the reverse implication holding if all the random variables are integrable.

7.8 Theorem. *The random field $M_\theta(x,t \mid \psi)$, that is constructed taking the initial datum to be any bounded and strictly positive steady state solution $\psi(x)$, is integrable for all x and t. For any fixed x and t the martingale that is based on this steady state solution $\psi(x)$ converges almost surely and in L^1 to the multiplicative functional $M_\theta(x,t \mid \psi)$:*

$$M_\theta^{(n)}(x,t;\psi) \xrightarrow{a.s.,\,L^1} M_\theta(x,t \mid \psi). \tag{537}$$

Proof. Fix x and t. Let $\{M_\theta^{(n)}(x,t;\psi) : n \geq 0\}$ denote the martingale based on the solution $\psi(x) > 0$ as described by (513) and (514). As the martingale is non-negative,

$$M_\theta^{(\infty)}(x,t;\psi) \stackrel{\text{def}}{=} \lim_{n\to\infty} M_\theta^{(n)}(x,t;\psi) \tag{538}$$

exists almost surely, by general martingale convergence theory e.g. [55]. By the remarks preceding Theorem 7.7, $M_\theta^{(\infty)}(x,t;\psi) = M_\theta(x,t \mid \psi)$. (Or this can be seen directly from these remarks without invoking the almost sure convergence of the non-negative martingale.) Then applying Fatou's lemma,

$$\mathbb{E}M_\theta(x,t \mid \psi) = \mathbb{E}(\liminf M_\theta^{(n)}(x,t;\psi)) \leq \liminf \mathbb{E}M_\theta^{(n)}(x,t;\psi) = \psi(x). \tag{539}$$

This establishes the integrability of $M_\theta(x,t \mid \psi)$. By Theorem 7.5, $u(x,t) := \mathbb{E}M_\theta(x,t \mid \psi)$ solves the KPP equation with initial datum $\psi(x)$. But by the uniqueness of the steady state solution, Theorem 7.7, $u(x,t)$ must actually equal $\psi(x)$. Then with the convergence being quite trivial

$$\mathbb{E}M_\theta^{(n)}(x,t;\psi) \to \mathbb{E}M_\theta(x,t \mid \psi) = \psi(x) \tag{540}$$

and as the random variables are positive anyway, we may applying Scheffe's lemma to conclude

$$\mathbb{E}\big|M_\theta^{(n)}(x,t;\psi) - M_\theta(x,t;\psi)\big| \to 0, \tag{541}$$

This establishes the L^1-convergence of the martingale. $\square$

§7.6 Global solutions for larger initial datum: existence and uniqueness

We now come to the main theorem, that addresses the branching Brownian motion representation for solutions to the KPP equation when the initial datum does not satisfy $|f(x)| \le 1$. The interest here is that this moves beyond the original bound stated by McKean. From a broader perspective, this is the first discovered example of a recursively defined expected value product solving a partial differential equation in which the integrability holds globally in time, under a less restrictive condition on the initial datum than it being bounded by 1 in absolute value. (Compare with the bound $|\chi_0(\xi)| \le 1$ stated in Section 1.2 that is required for the global integrability of the *FNS:h* multiplicative functional.)

7.9 Theorem. *Suppose that the initial datum for the KPP equation satisfies the bound $|f(x)| \le \psi(x)$ where $\psi(x)$ is a bounded and strictly positive steady state solution. Then $M_\theta(x,t \mid f)$ is integrable for all x and t, the solution $u(x,t)$ exists for all time enjoying the representation $u(x,t) = \mathbb{E}M_\theta(x,t \mid f)$, and $u(x,t)$ uniquely solves the KPP equation in the class $L^\infty(\mathbb{R} \times [0,T))$ for any $T > 0$.*

Proof. Let $M_\theta(x,t \mid f)$ and $M_\theta(x,t \mid \psi)$ denote the random fields constructed with initial data $f(x)$ and $\psi(x)$ respectively. We have integrability of $M_\theta(x,t \mid \psi)$ for x and t arbitrary. Of course integrability also holds for any smaller the initial datum, since almost surely, that is, on the event of non-explosion where the comparison can be made,

$$\big| M_\theta(x,t \mid f) \big| \le M_\theta(x,t \mid \psi). \tag{542}$$

Then by Theorem 7.5, $u(x,t) = \mathbb{E}M_\theta(x,t; f)$ solves the KPP equation with initial condition $|f(x)| \le \psi(x)$, and this representation holds for all time $t > 0$. Note that $|u(x,t)| \le \psi(x)$.

To show uniqueness, suppose that an alternative solution $v(x,t) \in L^\infty(\mathbb{R} \times [0,T_a))$ also solves the KPP equation with initial datum $f(x)$. Let

$$\big\{M_\theta(x,t;u) : n \ge 0\big\} \quad \text{and} \quad \big\{M_\theta(x,t;v) : n \ge 0\big\}$$

denote the martingales based on the solutions $u(x,t)$ and $v(x,t)$ respectively. Let $R \ge 1$

be defined by

$$R = \max\big\{\|v(x,t)\|_{L^\infty(\mathbb{R}\times[0,T_a))},\ \sup_x \psi(x)\big\}. \tag{543}$$

Define T_1 by

$$T_1 = \begin{cases} \infty & \text{if } R = 1, \\ \log \dfrac{R}{R-1} & \text{if } R > 1. \end{cases} \tag{544}$$

Then for any $t < T_1$ and $x \in \mathbb{R}$ the following bounds hold, independent of n:

$$\big|M_\theta^{(n)}(x,t;u)\big| \le R^{Z(t)}, \quad \big|M_\theta^{(n)}(x,t;v)\big| \le R^{Z(t)}. \tag{545}$$

Now the same argument given in the proof of Theorem 7.7 works here as well. Here it is again: As long as $t < T_1$, the two martingales are uniformly integrable and converge almost surely and in L^1 to their respective limits. But these limits are identical — it is the multiplicative functional $M_\theta(x,t \mid f)$ that is constructed from the initial datum $f(x)$. This establishes uniqueness of the solution up to time T_1. Repeating the argument with the equation restarted with the initial datum $u(x,T_1)$, and using the fact that $|u(x,T_1)| \le \psi(x)$, yields uniqueness up to time $T_2 = 2T_1$, and iterating gives uniqueness up to T_a. $\square$

§7.7 The representation with non-negative initial datum

The ideas of the previous two sections can be applied to the situation in which the initial datum for the KPP equation is non-negative. The resulting theorem is the best possible: as long as the solution exists, the multiplicative functional is integrable, and moreover, the solution may be represented as the expected value product on branching Brownian motion.

7.10 Theorem. *Suppose that $u(x,t) \in L^\infty\big(\mathbb{R}\times[0,T]\big)$ solves the KPP integral equation*

$$u(x,t) = e^{-t}H_t f(x) + \int_0^t e^{-s}H_s u^2(x,t-s)ds \tag{546}$$

with initial datum $f(x) \in L^\infty(\mathbb{R})$ with $f(x) \ge 0$. Then

(i) for any $T_a \le T$, $u(x,t)$ is the unique solution to (546) in the class $L^\infty\big(\mathbb{R}\times[0,T_a]\big)$,

(ii) $u(x,t) \ge 0$,

(iii) $u(x,t)$ is a classical solution to the initial value problem

$$\begin{aligned} &u_t = u_{xx} + u^2 - u, \quad -\infty < x < \infty, \quad 0 < t \leq T, \\ &u(x, 0^+) = f(x). \end{aligned} \tag{547}$$

(iv) for all $(x,t) \in \mathbb{R} \times [0,T]$ the branching Brownian motion representation is valid:

$$u(x,t) = \mathbb{E} M_\theta(x,t) = \mathbb{E} \prod_{i=1}^{Z(t)} f\big(x - \mathfrak{X}^{(i)}(t)\big). \tag{548}$$

Proof. Suppose that $v(x,t) \in L^\infty\big(\mathbb{R} \times [0, T_a]\big)$ also solves equation (546) with the same initial datum $f(x)$. Define R by recalling Theorem 7.2 and letting

$$R = \max\Big\{ \sup_{0 \leq t \leq T} \|v(\cdot,t)\|_{L^\infty},\ \sup_{0 \leq t \leq T} \|v(\cdot,t)\|_{L^\infty}, \|f(x)\|_{L^\infty} \Big\}, \tag{549}$$

Let T_1 denote the minimum of T, T_a and T_R, where

$$T_R = \begin{cases} \infty & \text{if } R \leq 1, \\ \log \dfrac{R}{R-1} & \text{if } R > 1. \end{cases} \tag{550}$$

Fix $x \in \mathbb{R}$ and $0 \leq t \leq T_1$. Let $M_\theta^{(n)}(x,t;u)$ and $M_\theta^{(n)}(x,t;v)$ denote the martingales based on $u = u(x,t)$ and $v = v(x,t)$ respectively. If $R > 1$ we have the bounds

$$\big|M_\theta^{(n)}(x,t;u)\big| \leq R^{Z(t)}, \quad \text{and} \quad \big|M_\theta^{(n)}(x,t;v)\big| \leq R^{Z(t)}; \tag{551}$$

if $R \leq 1$ we have the bounds

$$\big|M_\theta^{(n)}(x,t;u)\big| \leq 1, \quad \text{and} \quad \big|M_\theta^{(n)}(x,t;v)\big| \leq 1. \tag{552}$$

Since $t < T_1$, the bounding random variable $R^{Z(t)}$ is integrable, and the two martingales are uniformly integrable: they converge in L^1 to their respective almost sure limits, which in both cases is the completed multiplicative functional $M_\theta(x,t) = M_\theta(x,t \mid f)$. Consequently

$$u(x,t) = v(x,t) = \mathbb{E} M_\theta(x,t). \tag{553}$$

Thus, the two solutions agree up to time T_1, and on $\mathbb{R} \times [0, T_1]$, we have $u(x,t) \geq 0$ by the positivity of the factors in the integrable random variable $M_\theta(x,t)$. Repeating this argument with the equation restarted at time T_1 with initial datum $u(x,T_1) \geq 0$ gives uniqueness and positivity of the solution on $\mathbb{R} \times [0,T_2]$ where $T_2 = 2T_1$ (but adjusted accordingly so that T_2 does not exceed T or T_a). Note that by hypothesis the solution exists up to time $T \geq T_1$, so the equation can be restarted with initial datum $u(x,T_1)$ exactly, not $u(x,T_1-\epsilon)$. Repeating this argument as many times as necessary verifies (i) and (ii).

The assertion (iii) is just a restatement of Theorem 7.2, included here to emphasize that $u(x,t) < \infty$ everywhere on $\mathbb{R} \times [0,T]$.

For (iv), fix $x \in \mathbb{R}$ and $0 < t \leq T$ and consider the martingale $M_\theta^{(n)}(x,t;u)$. Each element of the martingale is a positive random variable, and integrable, as

$$M_\theta^{(n)}(x,t;u) \leq R^{2^n}. \tag{554}$$

Since $M_\theta^{(n)}(x,t;u)$ converges almost surely to $M_\theta(x,t)$ it follows from Fatou's lemma that

$$\mathbb{E}M_\theta(x,t) = \mathbb{E}\big(\liminf M_\theta^{(n)}(x,t;u)\big) \leq \liminf \big(\mathbb{E}M_\theta^{(n)}(x,t;u)\big) = u(x,t). \tag{555}$$

This established the integrability of $M_\theta(x,t)$. Let $w(x,t) = \mathbb{E}M_\theta(x,t)$ where now x and t are variables. By Theorem 7.7, $w(x,t)$ solves the KPP equation on $\mathbb{R} \times [0,T]$. Then by part (i) of this Theorem, $w(x,t) = u(x,t)$ and the inequality in (555) is actually an equality, so the representation is valid. □

§7.8 The KPP equation without the linear term

In the branching Brownian motion representation of solutions to the KPP equation, the exponential waiting times between the splitting of the Brownian particles may be traced back to the existence of the linear term $-u$. It is therefore of interest to see that a similar representation holds for the following equation obtained from KPP by removing this term:

$$u_t = u_{xx} + u^2, \quad u(x,0) = f(x). \tag{556}$$

The technique for dealing with this just as easily handles the generalization

$$u_t = u_{xx} + u^2 - \lambda u, \quad u(x,0) = f(x) \tag{557}$$

where $\lambda \geq 0$. We obtain for this generalization the representation

$$u(x,t) = (\delta+\lambda)\mathbb{E}\left\{ e^{\delta L(t)} \prod_{i=1}^{Z(t)} \frac{f(x - \mathfrak{X}^{(i)}(t))}{(\delta+\lambda)} \right\}, \tag{558}$$

as long as the random variable in the right hand side is integrable. The underlying stochastic model is branching Brownian motion with the exponential waiting times having parameter $\delta + \lambda$. Here $L(t)$ is the random variable that is the sum of the lifetimes of all particles in the branching Brownian motion process, up to elapsed time t. Of course $L(t)$ also depends on the parameter $\lambda + \delta$.

How to obtain this representation is outlined. Following the discussion in Section 7.2, equation (557) converts to the integral equation

$$u(x,t) = e^{-\lambda t} H_t f(x) + \int_0^t e^{-\lambda s} H_s u^2(x, t-s). \tag{559}$$

This is adjusted for the probabilistic interpretation with the introduction of the parameter $\delta > -\lambda$:

$$u(x,t) = e^{-(\delta+\lambda)t} e^{\delta t} H_t f(x) + \int_0^t (\delta+\lambda) e^{-(\delta+\lambda)s} e^{\delta s} (\delta+\lambda)^{-1} H_s u^2(x, t-s). \tag{560}$$

Following the standard procedure produces the representation $u(x,t) = \mathbb{E}M_\theta(x,t)$ with the following recursive definition of the $M_v(x,t)$, $v \in \mathcal{V}$:

$$M_v(x,t) = \begin{cases} e^{\delta t} f(x - \mathfrak{X}_v(S_v)) & \text{if } S_v \geq t, \\ \dfrac{e^{\delta S_v}}{(\delta+\lambda)} M_{v1}(x - \mathfrak{X}_v(S_v), t - S_v) M_{v2}(x - \mathfrak{X}_v(S_v), t - S_v) & \text{if } S_v < t. \end{cases} \tag{561}$$

If there are $Z(t)$ particles alive at elapsed time t, then the number of branch points is $Z(t) - 1$; and each binary operation contributes a factor of $(\delta+\lambda)^{-1}$. The time dependent multipliers $e^{\delta S_v}$ can be amalgamated as the single multiplicative factor $e^{\delta L(t)}$, giving the random product in equation (558).

Presumably one can argue that integrability, and hence this representation, holds for all $t > 0$, if the initial datum $f(x)$ bounded in absolute value by any positive bounded steady state solution. This program has not been carried out. Note that even with $\lambda = 0$, integrability holds locally, at least for sufficiently small δ and sufficiently small initial data. The sharpest criteria that delivers integrability with $\lambda = 0$ has not been determined. It appears likely that the restriction $\lambda \geq 0$ may be lifted also.

7.11 Open problem. *Explain or utilize the possible simultaneous scaling of the the initial datum and the solution by $(\delta + \lambda)^{-1}$ evident in equation (558).*

§7.9 Analysis on the Fourier side: the FKPP equation and process

Formally computing the Fourier transform of the integral equation (448) gives the following integral equation:

$$\widehat{u}(\xi, t) = e^{-t(1+|\xi|^2)}\widehat{u}_0(\xi) + \int_0^t e^{-s(1+|\xi|^2)} \frac{1}{\sqrt{2\pi}} \widehat{u} * \widehat{u}(\xi, t-s)ds. \tag{562}$$

Suitable adjustments leads to the Fourier side representation of the solution as the expected value of a multiplicative functional on a multi-type branching process. Equation (562) may also be obtained from the Fourier transformed KPP equation. The integral equation and the differential equation are equivalent.

In order to obtain the branching process representation, divide by a majorizing kernel $h(\xi)$ (to be determined) and make the other indicated adjustments:

$$\frac{\widehat{u}(\xi, t)}{h(\xi)} = e^{-t(1+|\xi|^2)} \frac{\widehat{u}_0(\xi)}{h(\xi)} + \int_0^t (1 + |\xi|^2) e^{-s(1+|\xi|^2)} \{\cdots\} \, ds \tag{563}$$

where

$$\{\cdots\} = \frac{h*h(\xi)}{\sqrt{2\pi}(1 + |\xi|^2)h(\xi)} \int_{-\infty}^{\infty} \frac{\widehat{u}(\xi - \eta, t-s)\widehat{u}(\eta, t-s)}{h(\xi - \eta)h(\eta)} \frac{h(\xi - \eta)h(\eta)}{h*h(\xi)} d\eta.$$

Assuming that a suitable majorizing kernel may be found, let

$$\begin{aligned} m(\xi) &= \frac{h*h(\xi)}{\sqrt{2\pi}(1 + |\xi|^2)h(\xi)}, \quad dK_\xi(\eta) = \frac{h(\xi - \eta)h(\eta)}{h*h(\xi)} d\eta, \\ \chi(\xi, t) &= \frac{\widehat{u}(\xi, t)}{h(\xi)}, \quad \chi_0(\xi) = \frac{\widehat{u}_0(\xi)}{h(\xi)}, \quad \text{and} \quad \lambda_\xi = 1 + |\xi|^2. \end{aligned} \tag{564}$$

Equation (563) is then more compactly expressed as

$$\begin{aligned} \chi(\xi, t) &= e^{-\lambda_\xi t}\chi_0(\xi) + \\ &\int_0^t \lambda_\xi e^{-\lambda_\xi s} \left\{ m(\xi) \int_{-\infty}^{\infty} \chi(\xi - \eta, t-s)\chi(\eta, t-s) dK_\xi(\eta) \right\} ds. \end{aligned} \tag{565}$$

This admits the usual probabilistic and recursive interpretation leading to the branching process representation of solutions, given recursively by

$$X_\theta(\xi,t) = \begin{cases} \chi_0(\xi) & \text{if } \lambda_\xi^{-1}S_\theta \geq t, \\ m(\xi)X_1(\Xi_1, t-\lambda_\xi^{-1}S_\theta)X_2(\Xi_2, t-\lambda_\xi^{-1}S_\theta) & \text{if } \lambda_\xi^{-1}S_\theta < t, \end{cases} \tag{566}$$

with the solution having the expected value representation

$$\widehat{u}(\xi,t) = h(\xi)\mathbb{E}X_\theta(\xi,t). \tag{567}$$

This representation is valid as long as $X_\theta(\xi,t)$ is integrable. Integrability is assured by requiring that the initial datum satisfies $|\widehat{u}_0(\xi)| \leq h(\xi)$, with $h(\xi)$ subject to the the constraint $m(\xi) \leq 1$.

Equation (566) is a compact expression that is intended to convey the recursively defined random field $X_\theta(\xi,t)$ whose full construction parallels $\mathsf{X}_\theta(\xi,t)$ for *FNS:h*, given in Section 5.3. Rather than repeating an analogue of that construction here, it suffices to describe it informally and comment on some important differences.

Just as with the *FNS* equations, the random field $X_\theta(\xi,t)$ is first constructed as function of the array

$$\{(S_v, U_v) : v \in \mathcal{V}\}. \tag{568}$$

Once this construction is made, we may fix ξ and t and see that the actual random variables involved in any realization of the process, the random variables

$$\{(S_v, \Xi_v) : v \in \mathcal{V}\}, \tag{569}$$

may be naturally organized into a branching process on which $X_\theta(\xi,t)$ is a multiplicative functional. This associated process is defined next. The random variables Ξ_v and U_v are related in the same way as the identically named random variables of Section 5.2

7.12 Definition (The FKPP process). *The FKPP process is a continuous-time multi-type branching process whose particle types are real numbers. It is parameterized by $\xi \in \mathbb{R}$ and elapsed time $t \geq 0$. Starting with index $v = \theta$, a single ancestral particle of type $\Xi_v = \xi$ lives for an exponentially distributed length of time with parameter $1 + |\Xi_v|^2$, and* *then splits into two new particles* Ξ_{v1} *and* Ξ_{v2}*, correlated according to* $\Xi_{v1} + \Xi_{v1} = \Xi_v$ *and distributed according to*

$$\mathbb{P}(\Xi_{v1} \in A \mid \Xi_v) = \int_A \frac{h(\Xi_v - \eta)h(\eta)}{h{*}h(\Xi_v)} d\eta. \tag{570}$$

These new particles, in turn, are subject to the same holding laws and splitting rules, and are independent of each other and the history of the process. After an elapsed time t *there are a random number* $\zeta(\xi, t)$ *of particles.*

Just as with the *FNS* branching process, the multiplicative functional $X_\theta(\xi, t)$ is computed by attaching initial datum $\chi_0(\cdot)$ to the input nodes of the associated tree $\tau(\xi, t)$, where the argument of the initial datum corresponds to the particle type. Then this random number of inputs is combined through a sequence of binary operations, where the order of the operations corresponds to the branching structure of the tree. At each binary node, the two arguments, a and b say, are combined through the ordinary multiplication

$$a, b \to m(\xi)ab. \tag{571}$$

The initial datum is multiplied together in this way — up the tree — until the root is reached, giving a particular value for $X_\theta(\xi, t)$ corresponding to the particular realization of the branching process.

The difference between the multiplicative functionals for the FKPP process and the *FNS* process are as follows:

1. For $X_\theta(\xi, t)$ the binary operation is commutative and associative, being ordinary multiplication. For $\mathsf{X}_\theta(\xi, t)$ the binary operation is neither commutative nor associative.

2. For the FKPP process a particle always splits at the end of its life; there is no termination without replacement. For the *FNS* process particles may terminate without replacement with probability one half.

It is the termination of particles without replacement that insures non-explosion in the *FNS* process. The non-explosion of the FKPP process is proved directly in Theorem 7.20.

So far a majorizing kernel for the FKPP equation has not been determined. This issue is addressed next.

§7.10 A majorizing kernel for the FKPP equation, with equality

The solution representation $\widehat{u}(\xi,t) = h(\xi)\mathbb{E}X_\theta(\xi,t)$ depends on the integrability of $X_\theta(\xi,t)$ that is achieved by making $m(\xi) \leq 1$ and taking $|\chi_0(\xi)| \leq 1$. The extremal case $m(\xi) = 1$ constrains the majorizing kernel to solve the nonlinear convolution equality

$$h{*}h(\xi) = \sqrt{2\pi}(1+|\xi|^2)h(\xi), \tag{572}$$

with $h(\xi) \geq 0$ naturally required for the probabilistic interpretation. The rest of this section is devoted to solving this equation by Fourier analysis. Before commencing, observe that appropriately scaled, both the bilateral exponential density and Cauchy density solve the corresponding convolution *inequality.* The use of these would introduce a multiplicative factor $m(\xi) \leq 1$ at each of the operational nodes in the branching process.

The analysis of (572) is simplified by looking for solutions that satisfy $h(\xi) = h(-\xi)$; in this case $y(x) = (\mathcal{F}^{-1}h)(x)$ is real valued, and the theory of real ordinary differential equations can be applied to the problem. This is not a severe restriction because the KPP equation itself is usually analyzed with real initial data and solutions, and part of the goal here is the dual analysis of the KPP equation with two stochastic processes: branching Brownian motion, and a multi-type branching process on the Fourier side. A bridge between the two process is the fact that if $h(\xi)$ is integrable and solves (572) then $y(x) = (\mathcal{F}^{-1}h)(x)$ solves the steady state KPP equation

$$\frac{d^2}{dx^2}y(x) = y(x) - y^2(x). \tag{573}$$

7.13 Question. *Are there solutions to the equation* $h*h(\xi) = \sqrt{2\pi}(1+|\xi|^2)h(\xi)$ *that are positive, integrable, and satisfy* $h(\xi) = h(-\xi)$*?*

It is easy to see that solutions to the differential equation (573) may be found, so the question really is: do these solutions have Fourier transforms that are positive functions? Before answering this question, some facts about solutions to (572) are recorded.

7.14 Proposition. *Let $h(\xi) \in L^2(\mathbb{R})$ solve $h*h(\xi) = \sqrt{2\pi}(1+|\xi|^2)h(\xi)$. Then $h(\xi)$ is actually bounded and continuous. It is not necessary to assume that $h(\xi) \geq 0$.*

Proof. This follows from the general result that given any $f, g \in L^2(\mathbb{R})$, $f*g$ is bounded and continuous on $\mathbb{R}$ (proved in detail in Gasquet [24, p. 181]). This is a special case of Theorem 2.6. □

7.15 Proposition. *If $h(\xi) \in L^1(\mathbb{R})$ solves $h*h(\xi) = \sqrt{2\pi}(1+|\xi|^2)h(\xi)$ and $h(\xi) \geq 0$, then $\|h(\xi)\|_{L^1} \geq \sqrt{2\pi}$.*

Proof. Denote the absolute moments of $h(\xi)$ by $\mu_k = \int |\xi|^k h(\xi) d\xi$. Integrating the equation gives $\mu_0^2 = \sqrt{2\pi}(\mu_0 + \mu_2) \geq \sqrt{2\pi}\mu_0$. □

This shows a rigid aspect of the convolution equation. This is different than say, the equation $h*h(\xi) = |\xi|^\theta h(\xi)$ on $\mathbb{R}^n$ whose 1-parameter group of transformations $h_\beta(\xi) = \beta^{n-\theta}h(\beta\xi)$, $\beta \in \mathbb{R}^+$, can transform any integrable solution into a solution of any given L^1-norm.

7.16 Proposition. *If $h(\xi) \in \mathcal{S}'$ is a non-negative function that solves $h*h(\xi) = \sqrt{2\pi}(1+|\xi|^2)h(\xi)$, and $y(x) = (\mathcal{F}^{-1}h)(x)$ is continuous in a neighborhood of the origin, then $h(\xi)$ is actually in $L^1(\mathbb{R})$ and $y(x) \in C^2(\mathbb{R})$.*

Proof. Theorem 4.9 states that if $h(\xi) \in \mathcal{S}'$ is a positive function whose Fourier transform is analytic in a neighborhood of the origin, then $h(\xi) \in L^1$. The same argument works, almost verbatim, with *analytic* replaced by *continuous.* Once it is established that $h(\xi)$ is in $L^1(\mathbb{R})$, the convolution equation shows that $|\xi|^2 h(\xi)$ is integrable, and its inverse Fourier transform is continuous. A convexity argument shows that $|\xi|h(\xi)$ is also integrable. Two applications of the Lebesgue dominated convergence theorem gives

$$\frac{\partial^2}{\partial x^2} y(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} |\xi|^2 h(\xi) d\xi \tag{574}$$

showing that $y(x) \in C^2(\mathbb{R})$. □

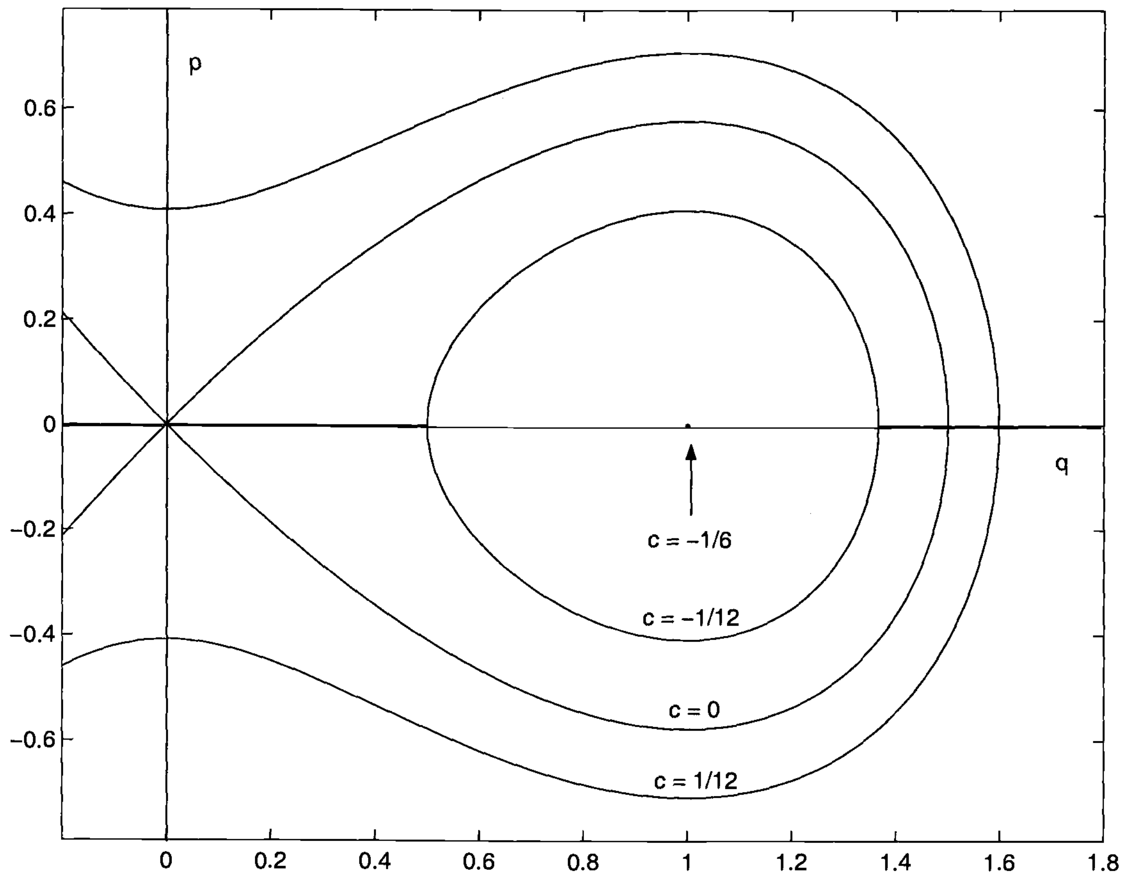

Figure 2: The level curves $\mathcal{H}(p,q) = c$.

Returning to Question 7.13, equation (573) is put into Hamiltonian form. Let $q = y$ and $p = y'$. The ordinary differential equation (573) becomes the Hamiltonian system with Hamiltonian $\mathcal{H}(p,q) = \frac{1}{3}q^3 - \frac{1}{2}q^2 + \frac{1}{2}p^2$:

$$\frac{dp}{dx} = -\frac{\partial \mathcal{H}}{\partial q} = q - q^2, \quad \frac{dq}{dx} = \frac{\partial \mathcal{H}}{\partial p} = p. \tag{575}$$

The family of curves $\mathcal{H}(p,q) = c$, where c is a constant are first integrals of the system. These are shown in Figure 2.

The system has two critical points: a hyperbolic fixed point at $(q,p) = (0,0)$ (with $c = 0$) and a center at $(q,p) = (1,0)$ (with $c = -\frac{1}{6}$). Also determined by the value $c = 0$ is the homoclinic orbit connecting the hyperbolic fixed point to itself, which is just part of the self-intersecting elliptic curve $\mathcal{H}(p,q)=0$. The closed curves inside the homoclinic orbit are determined by the values $-\frac{1}{6}<\mathrm{c}<0$. These are the bounded components of real elliptic curves having two components. The curves outside the homoclinic orbit are determined by the values $\mathrm{c}>0$; in this case the real elliptic curve is connected. The corresponding solutions to (573) are either bounded and periodic, bounded and integrable, or unbounded, as $-\frac{1}{6}<\mathrm{c}<0$, $\mathrm{c}=0$, or $\mathrm{c}>0$, respectively. Of course, this classification pertains to just those solutions with $q(0)>0$ (corresponding to $h(\xi)\geq 0$).

It turns out that only the homoclinic orbit produces a majorizing kernel, and even then only for a single choice of initial conditions. The solutions inside the homoclinic orbit fail to provide majorizing kernels because their Fourier transforms (as elements of $\mathcal{S}'$) are lattice supported measures. The solutions outside the homoclinic orbit fail to provide an integrable majorizing kernels because they are unbounded, or alternatively, by crossing the line $q=0$. These three cases are now considered in detail.

§7.10.1 Inside the homoclinic orbit

This is the case that $-\frac{1}{6}<\mathrm{c}<0$. The elliptic curve $\mathcal{H}(p,q)=\mathrm{c}$ corresponds to the differential equation

$$\left(\frac{dq}{dx}\right)^2=-\frac{2}{3}q^3+q^2+2\mathrm{c}, \tag{576}$$

which leads to elliptic integrals of the form

$$x=\int\frac{dq}{(-\frac{2}{3}q^3+q^2+2\mathrm{c})^{1/2}}. \tag{577}$$

Solutions are found by turning to the theory of elliptic functions. Basic references are [3], [33], and for a more algebraic and geometric treatment, Jones and Singerman [28]. Let $q=-6w+\frac{1}{2}$ putting the equation into Weierstrass normal form:

$$\left(\frac{dw}{dx}\right)^2=4w^3-g_2w-g_3,\quad\text{with}\quad g_2=\frac{1}{12},\; g_3=\frac{1+12\mathrm{c}}{6^3}. \tag{578}$$

The roots $e_1>e_2>e_3$ of the polynomial $P(w)=4w^3-g_2w-g_3$ are distinct, being the affine images of the three points where the original elliptic curve $\mathcal{H}(p,q)=\mathrm{c}$ crosses the

q-axis: see Figure 3. This may be verified algebraically by computing the discriminant

$$\Delta_P = 16(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 = \frac{c(6c-1)}{72} \tag{579}$$

which is strictly positive for $-\frac{1}{6} < c < 0$. This uses the fact that $\Delta_P = g_2^3 - 27g_3^2$ for polynomials in Weierstrass normal form [28, p. 274]. Since $P(w)$ has distinct roots, there exists a lattice $\Omega = \Omega(\omega_1, \omega_2)$ such that the Weierstrass elliptic function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Omega}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) \tag{580}$$

solves the differential equation

$$[\wp'(z)]^2 = 4[\wp(z)]^3 - g_2\wp(z) - g_3. \tag{581}$$

The direct relationship between g_2 and g_3 and the lattice $\Omega(\omega_1, \omega_2)$ is

$$g_2 = 60\sum_{\omega\in\Omega}{}'\omega^{-4} \quad \text{and} \quad g_3 = 140\sum_{\omega\in\Omega}{}'\omega^{-6}. \tag{582}$$

An inverse algorithm for computing ω_1 and ω_2 from g_2 and g_3 is given by Cohen [17, pp. 394–398]. First, recall the definition of the arithmetic-geometric mean (AGM) of two numbers.

7.17 Definition. *Let a and b be two positive real numbers. The arithmetic-geometric mean of a and b is the common limit of the two sequences defined by $a_0 = a$, $b_0 = b$, $a_{n+1} = (a_n + b_n)/2$, and $b_{n+1} = \sqrt{a_n b_n}$.*

Next, the following facts are relevant: $\wp(z)$ is a *real* function (meaning $\wp(\overline{z}) = \overline{\wp(z)}$) if and only if the lattice Ω is *real* (meaning $\overline{\Omega} = \Omega$). A real lattice Ω is either real rhombic ($\omega_1 = \overline{\omega}_2$) or real rectangular ($\omega_1$ is real, and ω_2 is purely imaginary). For a real lattice Ω the corresponding elliptic curve has either one or two components as Ω is real rhombic or real rectangular respectively. Since the analysis here is concerned with solutions inside the homoclinic orbit, the algorithm should give ω_1 real and ω_2 purely imaginary. In fact it does:

$$\omega_1 = \frac{\pi}{\mathrm{AGM}(\sqrt{e_1 - e_3}, \sqrt{e_1 - e_2})}, \qquad \omega_2 = \frac{\pi i}{\mathrm{AGM}(\sqrt{e_1 - e_3}, \sqrt{e_2 - e_3})}. \tag{583}$$

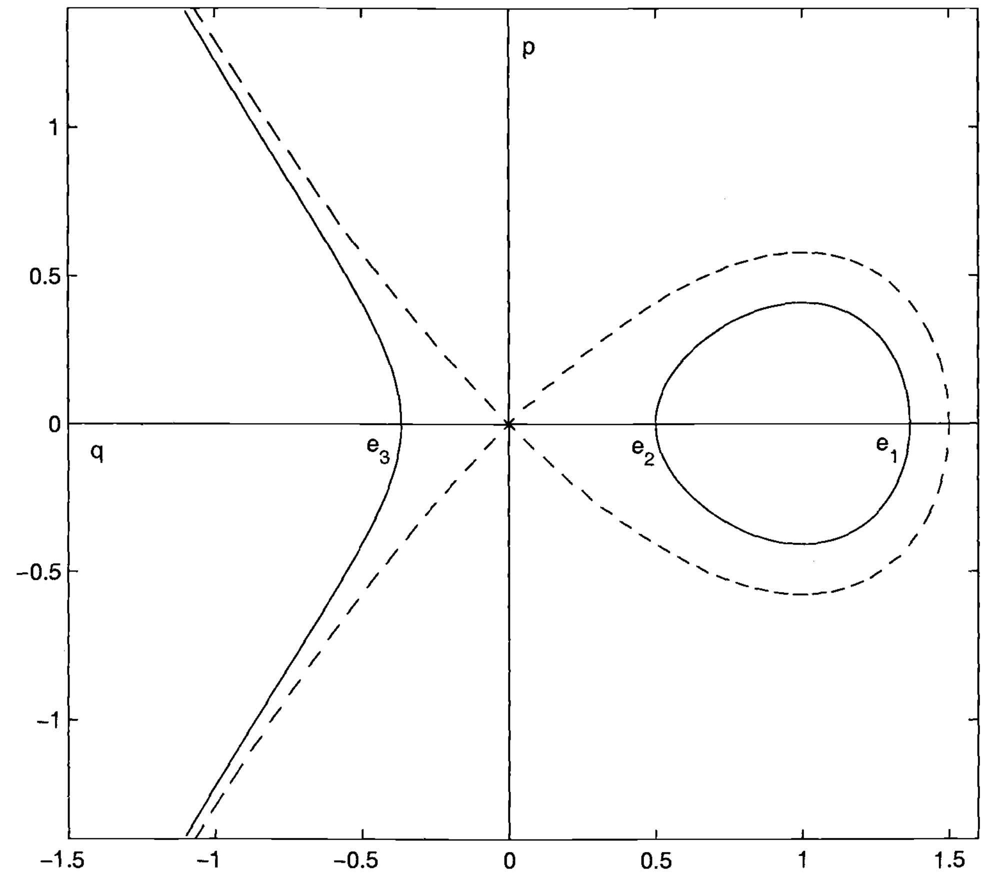

Figure 3: The roots $e_1 > e_2 > e_3$ of the polynomial $P(w) = 4w^3 - g_2 w - g_3$ are the three points where the elliptic curve $\mathcal{H}(p,q) = c$ crosses the q-axis. Here $c = -1/12$ with g_2 and g_3 determined by equation (578). The curve $\mathcal{H}(p,q) = 0$ is shown with the dashed line.

It may be verified that as c $\to 0$ from below, $\omega_1 \to \infty$ and $\omega_2 \to 2\pi i$. This should be compared with the solutions corresponding to the homoclinic orbit which are simply periodic with period $2\pi i$. Similarly as c $\to -1/6$ from above, $\omega_1 \to 2\pi$, and $|\omega_2| \to \infty$.

Equation (581) is solved by $\wp(z)$ for all complex z, but the goal here is find real solutions that traverse the bounded component of elliptic curve $\mathcal{H}(p,q) =$ c. This is not obtained by simply restricting $\wp(z)$ to the real numbers, because $\wp(z)$ has a pole of order 2 at each $\omega \in \Omega$. This problem is solved by recognizing that $\wp(\mathbb{R})$ is the unbounded component of the elliptic curve while the image of $\mathbb{R} + \frac{1}{2}\omega_2$ is the bounded component [28, p. 114]. Taking this into account, the periodic solutions to the equation $y'' = y - y^2$ that satisfy $y'(0) = 0$ are

$$\begin{cases} y_c(x) = -6\wp(x + \frac{1}{2}\omega_2) + \frac{1}{2}, \\ y_c(x + \frac{1}{2}\omega_1) = -6\wp(x + \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2) + \frac{1}{2}, \end{cases} \qquad -\infty < x < \infty, \tag{584}$$

with the parameter $-1/6 < c < 0$ determining $y(0)$. This uses the fact that the zeros of $\wp'(z)$ inside the fundamental parallelogram occur at $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, and $\frac{1}{2}(\omega_1 + \omega_2)$. Of course the particular $\wp$-function depends on the lattice $\Omega(\omega_1, \omega_2)$ which depends on c through equations (578) and (583). These two solutions may be distinguished by the fact that with $e_1 > e_2 > e_3$,

$$\wp(\tfrac{1}{2}\omega_1) = e_1, \quad \wp(\tfrac{1}{2}\omega_1 + \tfrac{1}{2}\omega_2) = e_2, \quad \wp(\tfrac{1}{2}\omega_2) = e_3. \tag{585}$$

This is stated without proof in Byrd [10, p. 310]. Another way to distinguish these solutions is given in Section 7.11 where $y_c(x)$ is shown to be positive definite by a direct computation of its Fourier series, and then of course the translate, $y_c(x + \frac{1}{2}\omega_2)$, is not positive definite. Since these solutions are periodic, their Fourier transforms can not serve as majorizing kernels for the unrestricted KPP equation. However, the theory carries over to the periodic KPP equation, and the family of functions $h_{per}(k) = (\mathcal{F}y_c)(k)$ are majorizing kernels for the periodic KPP equation. We will return to this in Section 7.11.

§7.10.2 Outside the homoclinic orbit

This is the case that $c > 0$. The inverse Fourier transform of an integrable majorizing kernel $h(\xi) \geq 0$ achieves its absolute maximum at the origin. Solutions corresponding to any curve outside the homoclinic orbit are unbounded; for this reason none of them could be the inverse Fourier transform of a non-negative integrable function. This proves the following proposition.

7.18 Proposition. *If $h(\xi) \in L^1(\mathbb{R})$ solves $h * h(\xi) = \sqrt{2\pi}(1 + |\xi|^2)h(\xi)$, with $h(-\xi) = h(\xi)$ and $h(\xi) \geq 0$, then $y(x) = (\mathcal{F}^{-1}h)(x)$ does not correspond to a curve outside the homoclinic orbit.*

Remarkably, it is possible to weaken the positivity and integrability conditions on $h(\xi)$ and still assert that $y(x) = (\mathcal{F}^{-1}h)(x)$ does not correspond to a curves outside the homoclinic orbit. This is because $y(x) > 0$ by the next proposition, and curves outside the homoclinic orbit cross the line $q = 0$.

7.19 Proposition. *Suppose $h(\xi)$ solves $h * h(\xi) = \sqrt{2\pi}(1 + |\xi|^2)h(\xi)$ with $h(-\xi) = h(\xi)$ and that $h(\xi)$ is the sum of an L^1-function and an L^2-function. Then $y(x) > 0$. It is not necessary to assume $h(\xi) \geq 0$.*

Proof. We may assume that $h(\xi)$ is integrable. To see this, write the equation as

$$f(\xi)h{*}h(\xi) = h(\xi), \tag{586}$$

where

$$f(\xi) = (2\pi)^{-1/2}(1 + |\xi|^2)^{-1} \in L^1 \cap L^\infty. \tag{587}$$

Let $h(\xi) = h_1(\xi) + h_2(\xi)$ where $h_1 \in L^1$ and $h_2 \in L^2$. Then

$$\begin{aligned} \int h(\xi)d\xi &= \sum_{i,j=1}^{2} \int f(\xi)h_i * h_j(\xi)d\xi \\ &\leq \|f\|_{L^\infty}\|h_1 * h_1\|_{L^1} + 2\|f\|_{L^\infty}\|h_1 * h_2\|_{L^1} + \|f\|_{L^1}\|h_2 * h_2\|_{L^\infty} \\ &\leq \|f\|_{L^\infty}\|h_1\|_{L^1}^2 + 2\|f\|_{L^\infty}\|h_1\|_{L^1}\|h_2\|_{L^2} + \|f\|_{L^1}\|h_2\|_{L^2}^2 < \infty. \end{aligned} \tag{588}$$

This uses Theorems 2.4, 2.5, and 2.6.

Taking the inverse Fourier transform of equation (586) (justified below), implies that $y(x)$ solves

$$y(x) = (\widetilde{f}{*}y^2)(x) = \sqrt{\tfrac{1}{2}\pi}\,\{e^{-|x|}\} * \{y^2(x)\} \tag{589}$$

where $\widetilde{f}(x) = (\mathcal{F}^{-1}f)(x)$. Since $h(\xi)$ is an even function, $y(x)$ is real, and consequently $y(x)$ is strictly positive, being the convolution of a positive function and a non-negative function.

Obtaining (589) is now justified. Let $\{\phi_k(\xi)\}_{k=0}^{\infty}$ be a delta sequence of Gaussians, each element of which integrates to 1. Then as $k \to \infty$,

$$h{*}h{*}\phi_k(\xi) \xrightarrow{L^1} h * h(\xi) \tag{590}$$

and since $f(\xi) \in L^\infty$,

$$f(\xi)h{*}h{*}\phi_k(\xi) \xrightarrow{L^1} f(\xi)h{*}h(\xi). \tag{591}$$

Let $\widetilde{\phi}_k(x) = (\mathfrak{F}^{-1}\phi_k)(x)$; note that we have $\lim_{k\to\infty} \widetilde{\phi}_k(x) \equiv (2\pi)^{-1/2}$. Applying $\mathfrak{F}^{-1}$ to the sequence in the left hand side of (591) obtains, for each k,

$$\begin{aligned} \mathfrak{F}^{-1}\Big\{f(\xi)h * h * \phi_k(\xi)\Big\} &= (2\pi)^{-1/2}\Big\{\widetilde{f}(x)\Big\}{*}\Big\{\left(\mathfrak{F}^{-1}h{*}h{*}\phi_k\right)(x)\Big\} \\ &= (2\pi)^{1/2}\Big\{\widetilde{f}(x)\Big\}{*}\Big\{y^2(x)\widetilde{\phi}_k(x)\Big\}. \end{aligned} \tag{592}$$

The first equality holds by Theorem 2.1 as

$$f(\xi), \quad h * h * \phi_k(\xi), \quad \widetilde{f}(x), \quad \left(\mathfrak{F}^{-1}h * h * \phi_k\right)(x) = y^2(x)\widetilde{\phi}_k(x) \tag{593}$$

are all L^1-functions. The second equality holds by Theorem 2.1. We have the pointwise convergence

$$(2\pi)^{1/2}\Big\{\widetilde{f}(x)\Big\}{*}\Big\{y^2(x)\cdot\widetilde{\phi}_k(x)\Big\} \longrightarrow \widetilde{f}(x){*}y^2(x), \tag{594}$$

and the convergence is also in $L^\infty \cap C_0$, because the entire sequence is the image of a Cauchy sequence under the continuous transformation between Banach spaces:

$$\mathfrak{F}^{-1} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \cap C_0(\mathbb{R}^n). \tag{595}$$

Thus the pointwise limit agrees with the L^∞-limit, and the limits of the two Cauchy sequences are Fourier transform pairs: $f(\xi)h{*}h(\xi) \longleftrightarrow \widetilde{f} * y^2(x)$. □

§7.10.3 The homoclinic orbit

This is the case $c = 0$. Solutions that traverse the homoclinic orbit may be found by a second integration. The curve $\mathcal{H}(p,q) = 0$ corresponds to the differential equation

$$\left(\frac{dq}{dx}\right)^2 = q^2(1 - \frac{2}{3}q) \tag{596}$$

which leads to

$$x = \int \frac{dq}{q(1-\frac{2}{3}q)^{1/2}}. \tag{597}$$

The integral may be computed with the substitution $q = \frac{3}{2}\cos^2\theta$. According to the location of the homoclinic orbit it may be assumed that θ lies in the interval $(-\pi/2, \pi/2)$. Then

$$\int \frac{d\theta}{\cos\theta} = \log|\sec\theta + \tan\theta| = -\frac{1}{2}x + C \tag{598}$$

so that

$$\sec\theta + \tan\theta = Be^{-\frac{1}{2}x} \tag{599}$$

and the constant B is determined by the initial conditions. In fact, according to the program of finding positive symmetric solutions in answer to Question 7.13, the correct choice of initial conditions is $q(0) = 3/2$ and $p(0) = 0$, which corresponds to $B = 1$. Letting $w = \sec^2\theta = 3/2y$ one arrives at

$$\sqrt{w} + \sqrt{w-1} = Be^{-\frac{1}{2}x}, \tag{600}$$

and solving for $\sqrt{w}$ gives

$$\sqrt{w} = \frac{1 + B^2e^{-x}}{2Be^{-x/2}}. \tag{601}$$

Finally, the general solution to equation (573) associated with the homoclinic orbit is

$$y(x) = \frac{3}{2}\left(\frac{2B}{e^{x/2} + B^2e^{-x/2}}\right)^2, \tag{602}$$

and the particular solution of interest is $y(x) = \frac{3}{2}[\cosh(x/2)]^{-2}$. It still remains to show that $h(\xi) = (\mathfrak{F}y)(\xi) \geq 0$ (when $B = 1$). This leads to questions about when a solution to a differential equation is positive definite and under what conditions that fact may be inferred from the differential equation itself. Theorems along these lines are apparently unknown.

The Fourier transform of $y(x) = \frac{3}{2}[\cosh(x/2)]^{-2}$ may be found by evaluating the Fourier integral using contour integration and the residue theorem. The singularities of $y(z)$ are simple poles that lie on the imaginary axis at $2\pi i(k + \frac{1}{2})$, $k \in \mathbb{Z}$. To evaluate the Fourier transform, fix ξ and let

$$I = \frac{1}{\sqrt{2\pi}}\frac{3}{2}\int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{\cosh^2(x/2)}dx = \frac{6}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{2 + e^x + e^{-x}}dx. \tag{603}$$

Consider the contour integral

$$L_R = \frac{6}{\sqrt{2\pi}} \int_{\Gamma_R} \frac{e^{-iz\xi}}{2 + e^z + e^{-z}} dz \tag{604}$$

where Γ_R is the rectangular anti-clockwise contour

$$\int_{\Gamma_R} = \int_{-R}^{R} + \int_{R}^{R+2\pi i} + \int_{R+2\pi i}^{-R+2\pi i} + \int_{-R+2\pi i}^{-R} \tag{605}$$

that surrounds the pole at $z = \pi i$. As $R \to \infty$ the contribution from the vertical segments vanish and the contribution from the horizontal segments combine to give

$$\begin{aligned} L = \lim_{R\to\infty} L_R &= \frac{6}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\xi z}}{2 + e^z + e^{-z}} dz + \frac{6}{\sqrt{2\pi}} \int_{\infty+2\pi i}^{-\infty+2\pi i} \frac{e^{-iz\xi}}{2 + e^z + e^{-z}} dz \\ &= \frac{6}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{2 + e^x + e^{-x}} dx - \frac{6}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{2\pi\xi} e^{-i\xi x}}{2 + e^x + e^{-x}} dx \\ &= (1 - e^{2\pi\xi}) I. \end{aligned} \tag{606}$$

At the same time, using the residue theorem and L'Hospital's rule, the finite contour integrals (for any R) evaluate to

$$\begin{aligned} L_R &= 2\pi i \operatorname{Res}\left(z = \pi i, \frac{6}{\sqrt{2\pi}} \frac{e^{-i\xi x}}{2 + e^z + e^{-z}}\right) \\ &= \frac{12\pi i}{\sqrt{2\pi}} \lim_{z\to\pi i} \left\{ (z - \pi i) \frac{e^{-i\xi z}}{2 + e^z + e^{-z}} \right\} = \frac{-12\pi\xi e^{\pi\xi}}{\sqrt{2\pi}}. \end{aligned} \tag{607}$$

Then

$$(1 - e^{2\pi\xi}) I = \frac{-12\pi\xi e^{\pi\xi}}{\sqrt{2\pi}}, \tag{608}$$

giving the Fourier transform

$$h(\xi) = (\mathfrak{F}y)(\xi) = \frac{6}{\sqrt{2\pi}} \frac{2\pi\xi}{e^{\pi\xi} - e^{-\pi\xi}} = \frac{6}{\sqrt{2\pi}} \frac{\pi\xi}{\sinh \pi\xi}. \tag{609}$$

§7.11 Majorizing kernels for the periodic KPP equations

The solutions inside the homoclinic orbit have Fourier transforms that are lattice supported measures. If these periodic functions are developed into Fourier series instead, the result is a class of majorizing kernels supported on $\mathbb{Z}$. These become the majorizing kernels for the periodic KPP equations

$$\begin{aligned} &u_t = u_{xx} + u^2 - u, \quad -\tfrac{1}{2}\omega_1 < x < \tfrac{1}{2}\omega_1, \quad t \geq 0, \\ &u(x,0) = f(x). \end{aligned} \tag{610}$$

Recall that the period ω_1 depends on the first constant of integration in solving (575). Recall also that the Fourier transform of a periodic integrable function $f(x)$ of period ω is

$$\widehat{f}(k) = (\mathfrak{F}f)(k) = \frac{\sqrt{2\pi}}{\omega} \int_{-\omega/2}^{\omega/2} f(x) e^{-ix\frac{2\pi k}{\omega}} dx, \tag{611}$$

and the inversion formula (which defines the inverse Fourier transform) is

$$f(x) = \left(\mathfrak{F}^{-1}\widehat{f}\,\right)(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ix\frac{2\pi k}{\omega}}, \tag{612}$$

with the sum converging in an appropriate function space depending on the nature of $f(x)$. If the function $f(x)$ is differentiable then the series (612) converges absolutely, a fact from the theory of Fourier series e.g. [51, p. 183]. When it makes sense, this version of the Fourier transform satisfies

$$\widehat{fg}(k) = (\mathfrak{F}fg)(k) = \frac{1}{\sqrt{2\pi}}(\mathfrak{F}f * \mathfrak{F}g)(k) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} \widehat{f}(k-j)\widehat{g}(j), \tag{613}$$

or equivalently,

$$\left(\mathfrak{F}^{-1}\widehat{f} * \widehat{g}\,\right)(x) = \sqrt{2\pi} f(x) g(x). \tag{614}$$

The development of the periodic function $y_c(x) = -6\wp(x + \frac{1}{2}\omega_2) + \frac{1}{2}$ into Fourier series is achieved using the calculus of residues. For $k \in \mathbb{Z}$, let

$$a_k = a_k(c) = \frac{\sqrt{2\pi}}{\omega_1} \int\limits_{-\omega_1/2}^{\omega_1/2} e^{-ix\frac{2\pi k}{\omega_1}} y_c(x) dx. \tag{615}$$

Fix $k \neq 0$ and consider the contour integral

$$L(k) = \int_\Gamma e^{-iz\frac{2\pi k}{\omega_1}} y(z) dz \tag{616}$$

where Γ is the rectangular anti-clockwise contour with corners at $-\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_1 + \omega_2$, and $-\frac{1}{2}\omega_1+\omega_2$. Along the path of integration the contributions from the vertical segments cancel and the contributions from the horizontal segments combine to give

$$L(k) = (1 - e^{2\pi k|\omega_2|/\omega_1}) \int_{-\omega_1/2}^{\omega_1/2} e^{-ix\frac{2\pi k}{\omega_1}} y(x)dx, \tag{617}$$

so that

$$a_k = \frac{\sqrt{2\pi}}{\omega_1} \frac{L(k)}{(1 - e^{2\pi k|\omega_2|/\omega_1})}. \tag{618}$$

By the residue theorem

$$L(k) = 2\pi i \ \operatorname{Res}\left(z = \tfrac{1}{2}\omega_2, \ -6e^{-iz\frac{2\pi k}{\omega_1}} \wp(z + \tfrac{1}{2}\omega_2) + \tfrac{1}{2}e^{-iz\frac{2\pi k}{\omega_1}}\right), \tag{619}$$

and thanks to the Weierstrass representation (580) this may be evaluated as

$$\begin{aligned} L(k) &= -12\pi i \lim_{z\to\frac{1}{2}\omega_2} \frac{d}{dz} \left\{(z - \tfrac{1}{2}\omega_2)^2 e^{-iz\frac{2\pi k}{\omega_1}} \wp(z + \tfrac{1}{2}\omega_2)\right\} \\ &= -\frac{24\pi^2 k}{\omega_1} e^{\pi k|\omega_2|/\omega_1}. \end{aligned} \tag{620}$$

Combining this with equation (618) gives for $k \neq 0$

$$a_k = \frac{6\sqrt{2\pi}}{\omega_1} \frac{\pi \frac{2\pi k}{\omega_1}}{\sinh\left(\frac{1}{2}|\omega_2|\frac{2\pi k}{\omega_1}\right)}. \tag{621}$$

Remarkably, these Fourier coefficients are, up to a multiplicative factor, the majorizing kernel $h(\xi)$ for the unrestricted KPP equation sampled at $\xi = |\omega_2|\omega_1^{-1}k$, for all $k \neq 0$:

$$a_k = \frac{4\pi^2}{\omega_1|\omega_2|} h\big(|\omega_2|\omega_1{}^{-1}k\big), \quad k \in \mathbb{Z}/\{0\}. \tag{622}$$

(The function $h(\xi)$ is given by (609).)

These Fourier coefficients are all positive. We have not yet computed a_0, but it also must be positive, as $y_c(x)$ is strictly positive. Setting aside the problem of a_0 for the moment, we may now define the family of majorizing kernels $h_{\text{per}}(k)$ on the integers by $h_{\text{per}}(k) = a_k$. Of course, there is an implicit dependence on the parameter c. These are the majorizing kernels for the periodic KPP equations (of period ω_1 also depending on c) and they satisfy

$$h_{\text{per}} * h_{\text{per}}(k) = \sqrt{2\pi}(1 + 4\pi^2 k^2 \omega_1^{-2}) h_{\text{per}}(k). \tag{623}$$

This may be verified directly by computing the transform of the steady state KPP equation (573), with periodic boundary conditions.

We now compute a_0 by evaluating equation (623) at $k = 0$. This gives

$$\sqrt{2\pi} a_0 = \sum_{j=-\infty}^{\infty} a_j a_{-j} = a_0^2 + 2 \sum_{j=1}^{\infty} a_j^2 \tag{624}$$

and consequently

$$a_0 = \sqrt{\tfrac{1}{2}\pi} \pm \sqrt{\tfrac{1}{2}\pi - 2\textstyle\sum_{j=1}^{\infty} a_j^2}. \tag{625}$$

The extreme values of the parameter c indicate the branch of the square root. As $c \to -\frac{1}{6}$, $y_c(x)$ approaches the constant function 1 having period 2π, and $a_0 \to \sqrt{2\pi}$. Consequently for c in a neighborhood of $-\frac{1}{6}$, the positive square root is taken. As $c \to 0$, the real period ω_1 of $y_c(x)$ approaches infinity, and the imaginary period ω_2 approaches $2\pi i$, and so from equation (615), $a_0 \to 0$. Consequently for c in a neighborhood of 0, the negative square root is taken. The number of sign changes in the radical must be odd, as c varies between $-\frac{1}{6}$ and 0; presumably there is a single sign change occurring at the particular value of c for which $a_0 = \sqrt{\pi/2}$.

It may be that the expression (625) can not be improved. One program that has not been carried out is the expression of the periodic solutions $y_c(x)$ in terms of the Jacobi elliptic functions, and this may help. But there is a reason for suspecting that the expression (625) admits no simplification: There are two constants associated with any $\wp$-function. These are

$$\eta_1 = \int_{z_0}^{z_0+\omega_1} \wp(z)dz, \quad \text{and} \quad \eta_2 = \int_{z_0}^{z_0+\omega_2} \wp(z)dz, \tag{626}$$

where $z_0 \notin \Omega$. These integrals are independent of z_0, as $\wp(z)$ has zero residues, and its anti-derivative is a single valued function. A fundamental relation between η_1, η_2, and the periods ω_1, ω_2 is Legendre's relation, e.g. [3, p. 274],

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i. \tag{627}$$

Because of this, and the equation

$$a_0 = \frac{\sqrt{2\pi}}{\omega_1}\Big(\frac{1}{2} - 6\eta_1\Big), \tag{628}$$

determining c_0, η_1, and η_2 are essentially equivalent. Lang [32, p. 249] gives the following formula for computing η_2 considered a function of the lattice $\Omega(\tau, 1)$ and referring to the second of the pair of fundamental periods:

$$\eta_2(\tau, 1) = \frac{(2\pi i)^2}{12}\left[-1 + 24\sum_{n=1}^{\infty}\frac{nq_\tau^n}{1-q_\tau^n}\right]. \tag{629}$$

Here $q_\tau = e^{2\pi i\tau}$ with τ lying in the upper half plane. Compared to the complexity of this expression, equation (625) for a_0 is better.

This ends the discussion of the majorizing kernels for the periodic KPP equation. The functions $y_c(x) = -6\wp(x + \frac{1}{2}\omega_2) + \frac{1}{2}$ with $-\frac{1}{6} < c < 0$, which are the periodic steady state solutions to the KPP equation, are elsewhere designated $\psi_c(x)$.

§7.12 The non-explosion of the FKPP process

The non-explosion of the FKPP process is derived from the uniqueness of the steady state solution. This refers to the unique integrable steady state solution $\psi(x) = \psi_0(x)$. Although not presented here, there are analogous results for the periodic KPP equations and the periodic steady state solutions $\psi_c(x)$ with $-\frac{1}{6} < c < 0$. For these there are corresponding periodic FKPP processes whose particle types are integers.

7.20 Theorem. *The FKPP process, as given by Definition 7.12, never explodes.*

Proof. Let $q(\xi, t)$ denote the probability that the FKPP branching process, started with a single ancestral particle of type ξ, does not explode by time t. Conditioning on the time of the first split, and using the independence of the two branches, we have

$$q(\xi, t) = e^{-\lambda_\xi t} + \int_0^t \lambda_\xi e^{-\lambda_\xi s}\int_{-\infty}^{\infty} q(\xi - \eta, t - s)q(\eta, t - s)dK_\xi(\eta)ds. \tag{630}$$

Let $Q(\xi, t) = h(\xi)q(\xi, t)$ and note that $Q(\xi, \cdot)$ inherits integrability from $h(\xi)$. Substituting this into (630) establishes that $Q(\xi,t)$ satisfies the integral equation

$$\begin{aligned} Q(\xi,t) &= h(\xi)e^{-\lambda_\xi t} + \int_0^t e^{-\lambda_\xi s}\frac{\lambda_\xi h(\xi)}{h*h(\xi)}\int_{-\infty}^{\infty} Q(\xi-\eta,t-s)Q(\eta,t-s)d\eta ds \\ &= h(\xi)e^{-\lambda_\xi t} + \int_0^t e^{-\lambda_\xi s}\frac{1}{\sqrt{2\pi}}Q*Q(\xi,t-s)ds \\ &= h(\xi)e^{-\lambda_\xi t} + e^{-\lambda_\xi t}\int_0^t e^{\lambda_\xi s}\frac{1}{\sqrt{2\pi}}Q*Q(\xi,s)ds. \end{aligned} \tag{631}$$

Differentiating yields the equivalent differential-convolution equation

$$\frac{\partial}{\partial t}Q(\xi,t) = -(1+|\xi|^2)Q(\xi,t) + \frac{1}{\sqrt{2\pi}}Q*Q(\xi,t), \tag{632}$$

with initial datum $Q(\xi,t) = h(\xi)$. Let $\widetilde{Q}(x,t) = (\mathcal{F}^{-1}Q)(x,t)$. Then $\widetilde{Q}(x,t)$ solves the KPP equation (439), with initial datum being a steady state solution

$$\widetilde{Q}(x,0) = \psi(x) = \mathcal{F}^{-1}h(\xi) = \frac{3}{2}\left[\cosh(\frac{x}{2})\right]^{-2}. \tag{633}$$

By its probabilistic construction, $0 \le Q(\xi,t) \le h(\xi)$ and consequently

$$\left|\widetilde{Q}(x,t)\right| \le \widetilde{Q}(0,t) \le \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} h(\xi)d\xi = \psi(0) = \frac{3}{2}, \tag{634}$$

so that $\widetilde{Q}(x,t) \in L^\infty(\mathbb{R}\times[0,\infty))$. By Theorem 7.7 the steady state solution is unique in this class. Hence $\widetilde{Q}(x,t) \equiv \psi(x)$, and using the injectivity of the inverse Fourier transform $\mathcal{F}^{-1} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$, we have $Q(\xi,\cdot) \equiv h(\xi)$ almost everywhere, and continuity implies the equality holds everywhere, and so $q(\xi,t) \equiv 1$, and the probability of explosion is zero. □

This utilizes the independence of the branches and the fact that for the function of interest, the appropriate binary operation at the branch points is multiplication. There is nothing special about the probability of non-explosion. The same technique works for the generating function

$$F(s;\xi,t) = \mathbb{E}s^{\zeta(\xi,t)} \tag{635}$$

where $\zeta(\xi,t)$ counts the number of particles (of any type) by elapsed time t in the FKPP process started with a single ancestral particle of type ξ.

7.21 Corollary. *Let $u_s(x,t)$ denote the solution to the KPP equation started with initial datum $u_s(x,0^+) = s\psi(x)$, with $|s| \leq 1$ and $\psi(x)$ given by equation (642). Then*

$$(\mathcal{F}u_s)(x,t) = h(\xi)F(s;\xi,t). \tag{636}$$

Proof. Fix s such that $|s| \leq 1$. Just as in the proof of Theorem 7.20, we have $F_s(\xi,t) = F(s;\xi,t)$ satisfying the integral equation

$$F_s(\xi,t) = se^{-\lambda_\xi t} + \int_0^t \lambda_\xi e^{-\lambda_\xi t'} \int_{\mathbb{R}} F_s(\xi-\eta, t-t')F_s(\eta, t-t')dK_\xi(\eta)dt', \tag{637}$$

and $G(\xi,t) = h(\xi)F_s(\xi,t)$ solves (632) but with different initial datum. Proceeding just as before, $u_s(x,t) = (\mathcal{F}^{-1}G)(x,t)$ solves the KPP equation with the initial datum $s\psi(x)$. □

§7.13 The finite time blow-up of the KPP equation

The finite time blow-up of the KPP equation is based on a stochastic comparison between the FKPP branching process and the Yule process. Once it is known that the FKPP process never explodes then the stochastic comparison between the random variables $\zeta(\xi,t)$ and $Z(t)$ becomes useful. Both $\zeta(\xi,t)$ and $Z(t)$ are expressed as functions of the common array $\{(\Xi_v, S_v) : v \in \mathcal{V}\}$ and in this way coupled. We express $\zeta(\xi,t)$ according to the following counting procedure:

$$\zeta(\xi,t) = 1 + \sum_{v\in\mathcal{V}} \mathbf{1}\left[\textstyle\sum_{k=0}^{|v|}(1+|\Xi_{v|k}|^2)^{-1}S_{v|k} < t\right]. \tag{638}$$

The interpretation of this formula is that the number of particles alive at time t is one more than the number of binary nodes appearing before the elapsed time t; and these are counted using the indicator functions. Similarly, for the Yule process,

$$Z(t) = 1 + \sum_{v\in\mathcal{V}} \mathbf{1}\left[\textstyle\sum_{k=0}^{|v|} S_{v|k} < t\right]. \tag{639}$$

For any $v \in \mathcal{V}$ it is obvious that

$$\left(1+|\Xi_v|^2\right)^{-1} S_v \leq S_v, \tag{640}$$

and consequently

$$\mathbf{1}\left[\textstyle\sum_{k=0}^{|v|} S_v < t\right] \leq \mathbf{1}\left[\textstyle\sum_{k=0}^{|v|}(1+|\Xi_v|^2)^{-1}S_v < t\right], \tag{641}$$

so that on the event of non-explosion, we have $Z(t) \leq \zeta(\xi,t)$. This holds for any ξ and t.

7.22 Theorem. *If the KPP equation has the initial datum $\alpha\psi(x)$, where $\alpha > 1$ and*

$$\psi(x) = \frac{3}{2}\cosh^{-2}\left(\frac{x}{2}\right), \tag{642}$$

then the solution ceases to exist sometime between $T_l^* < T_u^*$ where

$$T_l^* = \log\left(\frac{3\alpha}{3\alpha - 2}\right) \quad \textit{and} \quad T_u^* = \log\left(\frac{\alpha}{\alpha - 1}\right). \tag{643}$$

Proof. Let $u(x,t)$ solve the KPP equation on $\mathbb{R} \times [0, T^*)$ with initial datum $\alpha\psi(x)$, and suppose that $[0, T^*)$ is the largest time interval on which the solution exists. Then by Proposition 7.6, we have $T_l^* \leq T^*$ using the fact that the size of the initial datum is bounded by $\frac{3}{2}\alpha$. At the same time, on the Fourier side, assuming that the representation

$$\chi(\xi, t) = \mathbb{E}X_\theta(\xi, t) = \mathbb{E}\alpha^{\zeta(\xi,t)} \tag{644}$$

is valid, we have

$$\widehat{u}(\xi, t) = h(\xi)\mathbb{E}\alpha^{\zeta(\xi,t)} \geq h(\xi)\mathbb{E}\alpha^{Z(t)} = \frac{\alpha e^{-t}}{1 - (1 - e^{-t})\alpha} \tag{645}$$

which blows up for all ξ as $t \to T_u^*$. This means that $T^* \leq T_u^*$ because as $t \to T_u^*$,

$$u(0, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{u}(\xi, t) dx \to \infty. \tag{646}$$

It remains to validate (644). Since we are no longer in a situation where the initial datum is bounded by a majorizing kernel, it is not known a priori that the multiplicative functional is integrable, or if this represents yet another solution distinct from the solution of interest. Here is how this may be done: Let $T = T^* - \epsilon$. Working with the integrated form of the KPP equation, it can be shown in general that if $u(x,t) \in L^\infty\big(\mathbb{R} \times [0, T]\big)$ solves the KPP equation with initial datum satisfying $f(x) \in L^1 \cap L^\infty \cap C^2$ and $\Delta f(x) \in L^1(\mathbb{R})$, then $\Delta u(\cdot, t) \in L^1(\mathbb{R})$ on $[0, T]$. The specific initial datum $\alpha\psi(x)$ satisfies these conditions, so for the solution of interest, $|\xi|^2\widehat{u}(\xi, t) \in L^\infty(\mathbb{R})$ for all $0 \leq t \leq T$. Moreover, $u(\cdot, t)$ takes values in the Banach space $\mathbb{B}(\check{h}) = \mathbb{B}_{\check{h}}$ of Definition 6.1 built from the majorizing kernel

$$\check{h}(\xi) = \frac{\sqrt{2\pi}}{2} \cdot \frac{1}{\pi} \frac{\beta}{\beta^2 + |\xi|^2} \tag{647}$$

where $\beta > 0$ is arbitrary. This scaled Cauchy density has the property that

$$\breve{m}(\xi) \stackrel{\text{def}}{=} \frac{\breve{h} * \breve{h}(\xi)}{\sqrt{2\pi}\,\breve{h}(\xi)} = \frac{\beta^2 + |\xi|^2}{4\beta^2 + |\xi|^2} \leq 1. \tag{648}$$

Using this particular majorizing kernel the FKKP equation may be expressed as

$$\breve{\chi}(\xi, t) = e^{-t} e^{-t|\xi|^2} \breve{\chi}_0(\xi) + \int_0^t e^{-s} \breve{m}(\xi) e^{-s|\xi|^2} \int_{\mathbb{R}} \breve{\chi}(\xi - \eta) \breve{\chi}(\eta) d\breve{K}_\xi(\eta) ds \tag{649}$$

where

$$\breve{\chi}(\xi, t) = \frac{\widehat{u}(\xi, t)}{\breve{h}(\xi)}, \quad \breve{\chi}_0(\xi) = \frac{\widehat{u}_0(\xi)}{\breve{h}(\xi)}, \quad d\breve{K}_\xi(\eta) = \frac{\breve{h}(\eta)\breve{h}(\xi - \eta)}{\breve{h} * \breve{h}(\xi)} d\eta. \tag{650}$$

Assuming integrability of the multiplicative functional $\breve{X}_\theta(\xi, t)$, a fact established shortly, this produces the representation

$$\breve{\chi}(\xi, t) = \mathbb{E}\breve{X}_\theta(\xi, t), \tag{651}$$

with $\breve{X}_\theta(\xi, t)$ described recursively by

$$\breve{X}_\theta(\xi, t) = \begin{cases} e^{-t|\xi|^2} \breve{\chi}_0(\xi) & \text{if } S_\theta \geq t, \\ \breve{m}(\xi) e^{-S_\theta |\xi|^2} \breve{X}_1(\Xi_1, t - S_\theta) \breve{X}_2(\Xi_2, t - S_\theta) & \text{if } S_\theta < t. \end{cases} \tag{652}$$

As usual, the Ξ_1 particle is distributed according to $d\breve{K}_\xi(\eta)$, and the two new particles are completely correlated: $\Xi_1 + \Xi_2 = \xi$. The notable difference between this representation and (566) is the inclusion of the time dependent multipliers, and the fact that the exponential lifetimes $\{S_v : v \in \mathcal{V}\}$ are *i.i.d.* with mean 1.

We now consider the representation (651) and (652) with the specific initial datum in question:

$$\breve{\chi}_0(\xi) = \frac{\alpha h(\xi)}{\breve{h}(\xi)} = \frac{6\alpha\pi\xi(\beta^2 + |\xi|^2)}{\beta \sinh \pi\xi}. \tag{653}$$

Let $\breve{\chi}(\xi, t) \in L^\infty(\mathbb{R} \times [0, T])$ be defined by

$$\breve{\chi}(\xi, t) = \frac{\widehat{u}(\xi, t)}{\breve{h}(\xi)} = \sqrt{2\pi}\, \frac{\beta^2 + |\xi|^2}{\beta} \widehat{u}(\xi, t) \tag{654}$$

where $\widehat{u}(\xi, t)$ is the Fourier transform of the hypothesized solution $u(x, t) \in \mathbb{B}(\breve{h})$. Let

$$R = \max\left\{ \sup_{0 \leq t \leq T} \left\| \breve{\chi}(\cdot, t) \right\|_{L^\infty(\mathbb{R})}, \left\| \breve{\chi}_0(\xi) \right\|_{L^\infty(\mathbb{R})} \right\} \tag{655}$$

and let $T_1 = \min\{T, T_R\}$ where

$$T_R = \begin{cases} \infty & \text{if } R = 1, \\ \log \dfrac{R}{R-1} & \text{if } R > 1. \end{cases} \tag{656}$$

We would like to assert that the function $\check{\chi}(\xi, t)$ of equation (654) is a solution to (649) started with the initial datum (653), and this solution is unique in the class $L^\infty(\mathbb{R} \times [0, T])$. For this purpose we construct the solution $\check{\gamma}(\xi, t)$ to (649) using the recursively defined $\check{X}_\theta(\xi, t)$ given by (652). A comparison of the stochastic model underlying the recursive description (652) with the Yule process shows that the number of particles alive at time t is the geometrically distributed random variable $Z(t)$. Also, the multipliers for $\check{X}_\theta(\xi, t)$ which appear at the input and operational nodes are, respectively,

$$e^{-t|\xi|^2} \leq 1 \qquad \text{and} \qquad \check{m}(\xi) e^{-S_v|\xi|^2} \leq 1. \tag{657}$$

Consequently

$$\left|\check{X}_\theta(\xi, t)\right| \leq R^{Z(t)} \tag{658}$$

and $\check{X}_\theta(\xi, t)$ is integrable as long as $0 \leq t \leq T_1$. Then by an analogue of Theorem 5.5,

$$\check{\gamma}(\xi, t) \stackrel{\text{def}}{=} \mathbb{E}\check{X}_\theta(\xi, t) \tag{659}$$

solves (649). An application of a martingale argument similar to that given for (i) and (ii) of Theorem 7.10 shows that $\check{\chi}(\xi, t) = \check{\gamma}(\xi, t)$ and $\check{\chi}(\xi, t) \geq 0$ for $0 \leq t \leq T_1$. Repeating this argument by restarting the equation with initial datum $\check{\chi}(\xi, T_1)$ shows that $\check{\chi}(\xi, t)$ is the unique solution to (649) in the class $L^\infty(\mathbb{R} \times [0, T_2])$ where $T_2 = 2T_1$, and that $\check{\chi}(\xi, t)$ remains non-negative up to time T_2. Iterating this argument as many times as necessary yields the desired uniqueness of $\check{\chi}(\xi, t)$ up to time T, and that $\check{\chi}(\xi, t) \geq 0$ on $\mathbb{R} \times [0, T]$.

Armed with the knowledge that $\check{\chi}(\xi, t)$, and hence $\widehat{u}(\xi, t)$, is non-negative, we return to the FKPP equation in the form

$$\chi(\xi, t) = e^{-\lambda_\xi t} \chi_0(\xi) + \int_0^t \lambda_\xi e^{-\lambda_\xi s} \left\{ m(\xi) \int_{-\infty}^{\infty} \chi(\xi - \eta, t - s) \chi(\eta, t - s) dK_\xi(\eta) \right\} ds \tag{660}$$

with

$$\chi(\xi,t) = \frac{\widehat{u}(\xi,t)}{h(\xi)}, \qquad m(\xi) = 1, \qquad \chi_0(\xi) = \frac{\alpha h(\xi)}{h(\xi)} = \alpha, \tag{661}$$

(and the other functions given by (564)). The multiplicative functional $X_\theta(\xi,t)$ with this particular initial datum is described recursively by

$$X_\theta(\xi,t) = \begin{cases} \alpha & \text{if } \lambda_\xi^{-1} S_\theta \geq t, \\ X_1(\Xi_1, t - \lambda_\xi^{-1} S_\theta) X_2(\Xi_2, t - \lambda_\xi^{-1} S_\theta) & \text{if } \lambda_\xi^{-1} S_\theta < t. \end{cases} \tag{662}$$

We would like to complete the proof with an analogue of the martingale argument given in part (*iv*) of Theorem 7.10. The problem is that $\chi(\xi,t)$ is not known to be bounded a priori. (Recall the hypothesis of Theorem 7.10; see also the comments preceding Theorem 5.6.) We get around this by the following device. Define the sequence of random fields $\big(Z_\theta^{(k)}(\xi,t) : k \geq 0\big)$ just as in the martingale construction of Section 5.4, except that the sequence begins with $Z_\theta^{(0)}(\xi,t) \equiv 0$. Thereafter

$$Z_\theta^{(k+1)}(\xi,t) = \begin{cases} \alpha & \text{if } \lambda_\xi^{-1} S_\theta \geq t, \\ Z_1^{(k)}(\Xi_1, t - \lambda_\xi^{-1} S_\theta) Z_2^{(k)}(\Xi_2, t - \lambda_\xi^{-1} S_\theta) & \text{if } \lambda_\xi^{-1} S_\theta < t. \end{cases} \tag{663}$$

We know that $Z_\theta^{(k)}(\xi,t)$ is positive and integrable, enjoying the bounds

$$0 \leq Z_\theta^{(k)}(\xi,t) < \alpha^{2^k}. \tag{664}$$

An induction argument establishes that for all $k \geq 0$,

$$0 \leq \mathbb{E} Z^{(k)}(\xi,t) < \chi(\xi,t). \tag{665}$$

Yet the sequence $Z_\theta^{(k)}(\xi,t)$ still converges almost surely to the completed multiplicative functional $X_\theta(\xi,t)$. Then

$$\gamma(\xi,t) \stackrel{\text{def}}{=} \mathbb{E} X_\theta(\xi,t) = \mathbb{E}\big(\liminf Z_\theta^{(k)}(\xi,t)\big) \leq \liminf \big(\mathbb{E} Z_\theta^{(k)}(\xi,t)\big) \leq \chi(\xi,t). \tag{666}$$

Since $X_\theta(\xi,t)$ is integrable, we have by an analogue of Theorem 5.5, that $\gamma(\xi,t)$ solves the FKPP equation (660). It remains to show that $\gamma(\xi,t) = \chi(\xi,t)$ or equivalently, that $\widehat{w}(\xi,t) = \widehat{u}(\xi,t)$ where

$$\widehat{w}(\xi,t) = h(\xi)\gamma(\xi,t). \tag{667}$$

This follows from uniqueness: By Theorem 7.2 $u(\cdot,t)$, and hence $\widehat{u}(\cdot,t)$ are both square integrable on $[0,T]$, so the bound $0 \leq \widehat{w}(\xi,t) \leq \widehat{u}(\xi,t)$ implies that $\widehat{w}(\xi,t) \in L^2(\mathbb{R})$ for $0 \leq t \leq T$. Both $\widehat{w}(\xi,t)$ and $\widehat{u}(\xi,t)$ are integrable as well, as they solve the equation

$$\widehat{u}(\xi,t) = \alpha h(\xi) e^{-t(1+|\xi|^2)} + (2\pi)^{-1/2} \int_0^t e^{-s(1+|\xi|^2)} \widehat{u} * \widehat{u}(\xi, t-s) ds \tag{668}$$

implying that for fixed $0 \leq t \leq T$,

$$|\widehat{u}(\xi,t)| \leq \alpha h(\xi) + \frac{B}{(1+|\xi|^2)} \in L^1(\mathbb{R}) \tag{669}$$

where

$$B = (2\pi)^{-1/2} \sup_{0 \leq s \leq T} \|\widehat{u}(\cdot,s)\|^2_{L^2(\mathbb{R})}. \tag{670}$$

Let $w(x,t) = \left(\mathcal{F}^{-1}\widehat{w}\right)(x,t)$. Both $u(x,t)$ and $w(x,t)$ are bounded solutions to the KPP equation on $\mathbb{R} \times [0,T]$ started with the same initial datum $\alpha\psi(x)$, so by the uniqueness part of Theorem 7.10, $u(x,t)$ and $w(x,t)$, are really the same functions, and by injectivity of the inverse Fourier transform $\mathcal{F}^{-1} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ coupled with the continuity of the functions $\widehat{u}(\xi,t)$ and $\widehat{w}(x,t)$ we have $\widehat{u}(\xi,t) = \widehat{w}(x,t)$ holding everywhere. This equality extends immediately to $\check{\gamma}(\xi,t) = \check{\chi}(\xi,t)$, so the inequalities in (666) are actually equalities, and the representation (644) is valid. $\square$

Embedded in the previous proof is following remarkable fact, recorded here as a corollary to the theorem.

7.23 Corollary. *Suppose that $u(x,t)$ defined on $\mathbb{R}^n \times [0,T]$ is a classical solution to the KPP equation started with initial datum $f(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ that is non-negative definite (its Fourier transform satisfies $\widehat{f}(\xi) \geq 0$) and whose second derivative is integrable. Then at all later times $0 < t \leq T$, both $u(x,t)$ and $(\partial^2/\partial x^2)u(x,t)$ are integrable and $u(x,t)$ is non-negative definite.*

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