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Title: THE ASYMPTOTIC BOUNDARY LAYER ON A CIRCULAR CYLINDER IN AXISYMMETRIC MICROPOLAR FLUID FLOW AND CONSTITUTIVE THEORIES

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The asymptotic boundary layer problem involving an axial incompressible Newtonian fluid flow past a circular cylinder parallel to its axis was investigated by Stewartson (1955). He addressed himself primarily to the Mangler's (1948) derivation, namely, the character of the boundary layer on a circular cylinder is equivalent to that on a flat plate owing to the boundary layer thickness being small compared with the radius of the body. Stewartson found from his investigation that if the velocity of the main stream is constant, the skin friction on the cylinder is increased at the corresponding points of a flat plate due to the effect of the transverse curvature of the cylinder, and, for the same reason, that the boundary layer thickness is slightly reduced in comparison with that of the flat plate. Thus, there certainly exist differences between the behavior of boundary layer on a circular cylinder and that on a flat plate. In this thesis we

investigate the behavior of the asymptotic boundary layer of an axial incompressible micropolar fluid flow on a circular cylinder, and obtain the boundary layer solutions and their characteristics arising out of the orientable nature of the fluid medium. The present investigation is found to lead to the confirmation of the longstanding famous conjecture of Eringen (1966) that the theory of micropolar fluids may have a mechanism capable of explaining drag reduction near a solid boundary. Expressions for the velocity and microrotation fields in the boundary layer as well as those of skin-friction and boundary layer thickness are obtained. This thesis also presents a review of several existing continuum and microcontinuum constitutive theories of great interest to orientate the recent trend in the field of continuum mechanics and to provide for ready reference.

# The Asymptotic Boundary Layer on a Circular Cylinder in Axisymmetric Micropolar Fluid Flow and Constitutive Theories

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## INTRODUCTION

It is well-known that the mass distribution function for a material is not as smooth as assumed in the classical continuum mechanics. In fact, the sequence consisting of the ratios formed by $\frac{\Delta M}{\Delta V}$, where $\Delta M$ is the total mass of a small volume element $\Delta V$ of a homogeneous material at any given time, is assumed to possess a limit known as mass density in classical continuum mechanics. But in reality, this sequence does not have a unique limit, when the size of $\Delta V$ is below a certain critical volume $\Delta V^*$. Contrary to this actual state of affairs the classical theories of continua are based on the rather inaccurate assumption that all material bodies possess continuous mass densities. Moreover, the classical laws of motion and the axioms of constitution are inaccurately assumed to be valid for every part of the medium regardless of its size. Thus, the continua of the classical theory are just dense assemblages of point masses, devoid of internal structures. Inherent in the classical viewpoint are drastic limitations on the extent to which continuum descriptions of macroscopic behavior can successfully bring out the fine structure of matter. This inadequacy of the classical continuum approach to describe the macroscopic manifestations of microscopic events like the production of internal angular momentum due to the microspin motion, coupled with a strong motivation for extending the range of applicability of continuum mechanics in the study of real material behavior, has led to the fast development of theories of the so-called microcontinua. Microcontinuum mechanics is intended to describe the fine structures of materials which the classical continua are devoid of. Duhem (1893) was the first to introduce the idea of an oriented-medium by viewing a continuum as an assemblage of points, each of which is associated with a set of three mutually perpendicular vectors known as directors capable of deforming independently of the deformations of the points. Subsequently, E. and F. Cosserat (1909) built a unified theory for deformable bars, surfaces and bodies on the concept of a three-space director-oriented continuum. The oriented-medium so defined has since become known as the microcontinuum. In theories of microcontinua, a material is regarded not only as a set of structured particles with mass and velocity, but also as consisting of substructures which can support microinertia and spin-inertia. Because of the interactions among microelements, theories of micro-continua admit couple stresses besides Cauchy stresses. These theories also involve mechanics of orientation of the elements constituting the microcontinuum, and of the production of internal angular momentum on account of their intrinsic rotation and deformations. The common feature of all the theories of microcontinua is that they all take into account the couple stress and body couples so that the continuum under consideration is polar in nature, exhibiting an asymmetric stress tensor. Among all the existing theories of microcontinua, the theory of micropolar fluids, due to Eringen (1966), seems to be one of the most promising ones in its mathematical simplicity and elegance as well as its physical applicability. In fact, such important fluids as animal blood, liquid crystals, polymeric fluids, fluids containing certain additives fall into this category. Therefore, it should be worthwhile to employ the theory of micropolar fluids to study boundary layer phenomena.

The study of the asymptotic boundary layer of an incompressible Newtonian fluid of constant properties on the exterior of a right circular cylinder with flow parallel to the cylinder axis was made by Stewartson (1955). Around 1950, the investigation of the boundary layer on a slender body of revolution had grown significant with an increased frequency of prototype flight and test models under low pressure conditions. Seban and Bond (1951) examined the skin friction in the boundary layer upon the leading edge of a right circular cylinder and found that the initial effect of the curvature of the cylinder is to increase the skin friction in comparison with the flat plate solution of Blasius. This very same effect was shown by Stewartson to prevail even in the asymptotic region of the boundary layer under the same geometry. Especially when the boundary layer thickness becomes comparable to the size of the body, the effect under discussion would be enhanced due to the greater momentum and energy exchanges inside the layer.

It is the main purpose of this thesis to bring out the asymptotic boundary layer characteristics on a circular cylinder in axial incompressible micropolar fluid flow which has not been investigated before.

In Chapter 1 we discuss several important classical constitutive theories for ready reference. Chapter 2 presents a review of five major microcontinuum constitutive theories advanced so far by Eringen and Suhubi (1964), Eringen (1964, 1966), Allen, Desilva, and Kline (1967), and Kirwan (1968), together with Mindlin's (1964) microstructure theory based on Hamilton's variational principle. In the present thesis, a review of Mindlin's theory is also made for purposes of an interesting comparison with other microcontinuum approaches, especially the linear theory of simple micro-elastic solids by Eringen and Suhubi (1964). In Chapter 3 a set of boundary-layer equations with appropriate boundary and matching conditions is derived and solved for the present flow problem, obtaining expressions for the skin friction, the velocity profile, and the boundary layer thickness. In Chapter 4 we conclude this thesis with a discussion of the results and scope of further work.

# 1. A REVIEW OF CONSTITUTIVE THEORIES IN CLASSICAL CONTINUUM MECHANICS

## 1.1 Preliminary Remarks

Nonlinearity of deformation and flow fields in materials is a well-established fact. Therefore, the physical cause-effect relationships should be explained accordingly.

Both the Newtonian theory of hydrodynamics and the Hooke's law of elasticity do not admit nonlinearity in their response functions [for example, the Merrington swelling effect (1943), the Weissenberg climbing effect (1947), the lengthening and shortening of twisted bars]. The failure of these classical theories of continua to manifest the effects of nonlinearity has prompted a rapid development of nonlinear continuum mechanics. Such classical nonlinear theories as Reiner-Rivlin theory (1945, 1948), Rivlin-Ericksen theory (1955), Green-Rivlin theory (1957), Noll's theory (1958), Oldroyd theory (1958) have been very successful in explaining nonlinear behaviors of materials. However, their general constitutive equations are too complicated and too unwieldly to solve many of the physical problems. Seth (1962, 1964, 1966) then recognized that the ever-increasing complexity of the nonlinear constitutive equations of continuous media and their ad-hoc generalizations resulted as a consequence of using ordinary measures of deformation and its rate such as the Cauchy strain measure in the constitutive equations of rheologically nonlinear materials. In other words, the constitutive equations have to be complicated as long as we use the classical measures of deformation and rate-of-deformation in their formulations. To avoid unnecessary complications in stress-strain relationships and at the same time to predict results fairly compatible with the experimental investigations, Seth generalized the classical measures of deformation as well as the rate of deformation. This concept has yielded suitable constitutive equations applicable to physical problems, as demonstrated by Narsimhan and Sra (1969). Seth's approach to the constitutive equations of continua has proven to be a major departure from the conventional methodologies in non-linear continuum mechanics.

In the following section, we discuss several important classical constitutive theories for ready reference.

## 1.2 Newtonian Theory of Hydrodynamics

The constitutive equation of classical hydrodynamics of viscous fluids is

$$\underset{\sim}{t} = (-p+\lambda \operatorname{tr} \underset{\sim}{d})\underset{\sim}{I} + 2\mu \underset{\sim}{d}, \qquad (1.2.1)$$

where

$\underset{\sim}{t}$ = stress tensor,

$\underset{\sim}{d}$ = deformation-rate tensor,

$\underset{\sim}{I}$ = unit tensor,

$\lambda$ = dilatational viscosity,

$\mu$ = shear viscosity, and

$p$ = hydrostatic pressure.

It is clear that (1.2.1) is a constitutively linear relation in the deformation-rate tensor $\underset{\sim}{d}$. Some fluids of ordinary experience, e.g., water, alcohol, air, mercury, fall into the domain of Newtonian fluids to a fair degree of accuracy. (1.2.1) has no mechanism of explaining such phenomena as the Weissenberg effect (1947), the Merrington effect (1943), varying flow rates and torques in the Poiseuille and Couette flows, respectively, and variable viscosity. Fluids characterized by these anomalous behaviors are termed non-Newtonian fluids. Typical examples are: condensed milk, liquid lubricants, pastes, plastics, colloids, high polymers, blood, asphalts, protein solutions, and so on. The ever-increasing use of a number of non-Newtonian fluids in recent industrial and biological investigations has stimulated several researchers to propose adequate mathematical models for such fluids.

## 1.3 Theories of Non-Newtonian Fluids

The class of non-Newtonian fluids was divided by Bhatnagar (1962) into the following three subclasses:

subclass I: visco-inelastic fluids (or Stokesian fluids),

subclass II: time-dependent fluids, and

subclass III: visco-elastic fluids.

Subclass I (Visco-Inelastic Fluids)

The constitutive equation for visco-inelastic fluids does not involve the time derivatives of the stress and deformation-rate components. This subclass of fluids exhibits diverse behavior in response to applied stress. Thus, it is customary to further divide visco-inelastic fluids into the following three categories:

Bingham plastics,

Pseudoplastic and dilatant fluids, and

Reiner-Rivlin fluids.

Bingham Plastics: This fluid obeys the Newtonian constitutive equation (1.2.1) but differs from the Newtonian fluid in sustaining a certain finite stress called the yield-stress before the flow begins.

Pseudoplastic and Dilatant Fluids: Tomita (1959) obtained a constitutive equation for pseudoplastic and dilatant fluids in the following form:

$$\underset{\sim}{t} = -p\underset{\sim}{I} + \mu_1 \underset{\sim}{d}[II_{\underset{\sim}{d}}]^{n-1/2} , \qquad (1.3.1)$$

where

$\mu_1$ = coefficient of viscosity,

$n$ = rheological constant, and

$II_{\underset{\sim}{d}}$ = the second invariant of $\underset{\sim}{d}$.

The main difference between the pseudoplastic $(n < 1)$ and dilatant fluids $(n > 1)$ is that in the former category of fluids the apparent viscosity (i.e., the viscosity measured by a viscometer) decreases with an increase in the rate of shear, while the opposite effect holds true for the latter. Note that when $n = 1$, (1.3.1) reduces to the constitutive equation of an incompressible Newtonian fluid. It is noteworthy that (1.3.1) contains only one viscosity coefficient.

Reiner-Rivlin Fluids (1945, 1948): This theory is based on the assumption that the stress tensor $\underset{\sim}{t}$ which is isotropic can be expressed in a power series of the first deformation-rate tensor $\underset{\sim}{d}$. With this assumption and the Cayley-Hamilton theorem (i.e., a matrix satisfies its characteristic equation), Reiner (1945) and Rivlin (1948) deduced for incompressible, isotropic viscous fluids the following constitutive equation:

$$\underset{\sim}{t} = -p\underset{\sim}{I} + a_1\underset{\sim}{d} + a_2\underset{\sim}{d}^2 , \qquad (1.3.2)$$

where

$\underset{\sim}{t}$ = stress tensor,

$\underset{\sim}{d}$ = first deformation-rate tensor,

$a_1$ = coefficient of viscosity,

$a_2$ = coefficient of cross-viscosity, and

$p$ = hydrostatic pressure.

$a_1$ and $a_2$ are functions of material properties (e.g., temperature and specific volume) and also of the three invariants of $\underset{\sim}{d}$. The rheological coefficients $a_1$ and $a_2$ are unknown and the theory, by itself, has no way of specifying them explicitly. There has been so far no experimental evidence that supports the existence of this class of fluids in nature or in industry. Experiments have also contradicted the theoretical prediction of the existence of two normal stresses in certain viscometric flows, when the rate of shear becomes appreciably larger.

<u>Subclass II (Time-Dependent Fluids)</u>

<u>Rheopectic Fluids</u>: This fluid exhibits an increase in viscosity with increasing time, while subjected to a steady rate of shear under isothermal conditions.

<u>Thixotropic Fluids</u>: This fluid possesses the opposite property, namely, its viscosity decreases as time increases under the same conditions as in rheopectic fluids.

These behaviors of time-dependent fluids are attributed to the whole chain of molecular deformations that occur subsequent to an impressed disturbance. Due to our lack of knowledge for representing the mechanism of the breaking and of reformation of molecular chains, no definite constitutive equation has been established thus far for this subclass of non-Newtonian fluids.

Subclass III (Visco-Elastic Fluids)

In elastic fluids one can no longer ignore the strain measure used however small it may be, as it is responsible for the recovery of the original state and for the reverse flow that follows the removal of stress. These material strains are determined by the stress history of the fluid and cannot be specified kinematically in terms of the large overall movements of the fluids. During the flow the natural state of the fluid changes constantly and tries to attain the instantaneous state of the deformed state, yet it does not succeed completely. This lag measures the elasticity of the fluid.

There exist two types of approaches to describing such a class of fluids:

i) Relaxation Theory, and

ii) Rivlin-Ericksen Theory, Green-Rivlin Theory, and Noll's Theory.

i) Relaxation Theory. In this approach to a mathematical theory of visco-elastic fluids, the elasticity is integrated into the constitutive equations by introducing stress-relaxation times and strain-retardation times.

Oldroyd Constitutive Theory: The constitutive equation proposed by Oldroyd (1958) is of the following form:

$$t_{ik}^{(e)} + \lambda_1 \frac{\mathcal{D} t_{ik}^{(e)}}{\mathcal{D} t} + \mu_0 t_{jj}^{(e)} d_{ik} - \mu_1 (t_{ij}^{(e)} d_{jk} + t_{jk}^{(e)} d_{ij}) + \nu_1 t_{j\ell}^{(e)} d_{j\ell} \delta_{ik}$$

$$= 2\eta_0 (d_{ik} + \lambda_2 \frac{\mathcal{D} d_{ik}}{\mathcal{D} t} - 2\mu_2 d_{ij} d_{jk} + \nu_2 d_{j\ell} d_{j\ell} \delta_{ik}) \qquad (1.3.3)$$

for incompressible, isotropic elastico-viscous fluids, where

$$t_{ik}^{(e)} = t_{ik} + p\delta_{ik} \quad \text{(deviatoric part of the stress tensor),} \qquad (1.3.4)$$

$$2d_{ik} = v_{i,k} + v_{k,i} \quad \text{(deformation-rate tensor), and} \qquad (1.3.5)$$

$$\frac{\mathcal{D}}{\mathcal{D} t} = \text{Jaumann derivative operator.}$$

By definition, the Jaumann derivative of a tensor $b_{\cdot i \cdot \cdot}^{\cdot \cdot k \cdot \cdot}$ is given by

$$\frac{\mathcal{D}}{\mathcal{D} t} b_{\cdot i \cdot \cdot}^{\cdot \cdot k \cdot \cdot} = \frac{\partial}{\partial t} b_{\cdot i \cdot \cdot}^{\cdot \cdot k \cdot \cdot} + v^m b_{\cdot i \cdot \cdot , m}^{\cdot \cdot k \cdot \cdot}$$

$$+ \Sigma w_i^m b_{\cdot m \cdot \cdot}^{\cdot \cdot k \cdot \cdot} + \Sigma' w_m^k b_{\cdot i \cdot \cdot}^{\cdot \cdot m \cdot \cdot}, \qquad (1.3.6)$$

where $\Sigma(\Sigma')$ stands for summation of similar terms, one for each covariant (contravariant) index and $w_{ij}$ is the spin tensor defined by $2w_{ij} = v_{i,j} - v_{j,i}$.

Here

$v_i$ = velocity vector,

$p$ = fluid pressure,

$\delta_{ik}$ = Kronecker delta,

$d_{ii} = 0$ for all $p$,

$\eta_0$ = coefficient of viscosity,

$\lambda_1$ = relaxation time constant,

$\lambda_2$ $(< \lambda_1)$ = retardation time constant, and

$\mu_0, \mu_1, \mu_2, \nu_1$, and $\nu_2$ are arbitrary scalar physical constants, each with the dimensions of time.

Oldroyd (1950) discussed two particular types of liquids of this class, namely Liquid A and Liquid B. They are derivable from (1.3.3) as follows:

Liquid A:

$$\eta_0 > 0, \quad \lambda_1 = -\mu_1 > \lambda_2 = -\mu_2 \geq 0, \quad \mu_0 = \nu_1 = \nu_2 = 0 \tag{1.3.7}$$

Liquid B:

$$\eta_0 > 0, \quad \lambda_1 = \mu_1 > \lambda_2 = \mu_2 \geq 0, \quad \mu_0 \, \eta \, \nu_1 = \nu_2 = 0 \tag{1.3.8}$$

The liquids A and B would exhibit very different bulk properties.

The elasticity of the fluid has been accounted by relaxation and retardation times $\lambda_1$ and $\lambda_2$, and the linearity of the Newtonian constitutive equation has been broken by introducing quadratic terms in the deformation-rate, and the products of stress and the deformation-rate. There is no physical theory that determines the characteristic times $\lambda_1$ and $\lambda_2$ in the constitutive equation. We finally mention that the nonlinearity of the equation has been introduced in a very arbitrary manner, rather than on a concrete basis.

Maxwell Fluids: Walter (1962) showed that the general equations of state of an isotropic incompressible elastico-viscous liquid have the forms:

Liquid A':

$$t_{ij} = -pg_{ij} + t_{ij}^{(e)} \tag{1.3.9}$$

$$t_{ij}^{(e)}(x,t) = 2\int_{-\infty}^{t} \psi(t-t') \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^n}{\partial x^j} d_{mn}(x',t')dt' \tag{1.3.10}$$

Liquid B':

$$t^{ij} = -pg^{ij} + t^{(e)ij} \tag{1.3.11}$$

$$t^{(e)ij}(x,t) = 2\int_{-\infty}^{t} \psi(t-t') \frac{\partial x^i}{\partial x'^m} \frac{\partial x^j}{\partial x'^n} d^{mn}(x',t')dt' \tag{1.3.12}$$

Here $x'^i = x'^i(x,t,t')$ is the position at time $t'$ of the element which is instantaneously at the point $x^i$ at time $t$. $\psi(t-t')$ is defined as

$$\psi(t-t') = \int_0^{\infty} \frac{N(\tau)}{\tau} e^{-(t-t')/\tau} d\tau , \qquad (1.3.13)$$

where $N(\tau)$ is called the relaxation spectrum [ $N(\tau)$ is defined such that $N(\tau)d\tau$ represents the total viscosity of the Maxwell elements with relaxation times between $\tau$ and $\tau + d\tau$. ]

Oldroyd's liquids A and B and the Newtonian liquid are special cases of A' and B'.

<u>Reiner-Philippoff Theory:</u>

$$t_{ij}^{(e)} = \mu_0 + \frac{\mu_\infty - \mu_0}{1+\left(\frac{1}{2\tau_0}\right)^2 \left(\sum_{n=1}^{3} \sum_{m=1}^{3} t_{nm}^{(e)} t_{mn}^{(e)}\right)} d_{ij} , \qquad (1.3.14)$$

where

$t_{ij}^{(e)}$ = deviatoric stress tensor,

$d_{ij}$ = first deformation-rate tensor, and

$\mu_0, \mu_\infty, \tau_0$ are adjustable positive parameters.

It is apparent from (1.3.14) that for very small or large values of $\tau_0$ the fluids behave like Newtonian, and for the intermediate values of $\tau_0$, they are markedly non-Newtonian.

<u>ii) Rivlin-Ericksen Theory, Green-Rivlin Theory and Noll's Theory</u>

<u>Rivlin-Ericksen Theory</u> (1955): Starting with the assumption that the stress at a point $x$ at time $t$ is a function of the

gradients, in the spatial system of velocity, acceleration, second and higher accelerations at the point $\underset{\sim}{x}$ at atime $t$, Rivlin and Ericksen formulated the constitutive equation

$$\underset{\sim}{t} = a_0 \underset{\sim}{I} + \sum_{i=1}^{N} a_i (\underset{\sim}{M}_i + \underset{\sim}{M}_i^T) \qquad (1.3.15)$$

for incompressible, isotropic visco-elastic fluids. $a_i$'s are unknown functions of the second and third invariants of the kinematic tensors $\underset{\sim}{d}^{(1)}, \underset{\sim}{d}^{(2)}, \ldots, \underset{\sim}{d}^{(n)}$ to preserve their forms under rigid motions. $\underset{\sim}{d}^{(j)}$ is the j-th material derivative of the square of the line-element. $\underset{\sim}{M}_i$ $(i = 1, 2, \ldots, m)$ are certain tensor products formed for the kinematic tensors $\underset{\sim}{d}^{(1)}, \underset{\sim}{d}^{(2)}, \ldots, \underset{\sim}{d}^{(n)}$, and $\underset{\sim}{M}_i^T$ is the transpose of $\underset{\sim}{M}_i$.

One disadvantage of this theory is that (1.3.15) is very complicated by the presence of several higher order kinematic tensors $\underset{\sim}{d}^{(j)}$ $(j = 1, 2, \ldots, n)$ and unknown functions of their invariants. However, Rivlin-Ericksen theory covers both inelastic and elastic fluids, and it has been very successful in obtaining normal stresses which are not necessarily equal. A large number of fluids like aqueous solution of polyacrylamid, polyisobutylene fall into this class of fluids.

Green-Rivlin Theory (1957): Assuming that the stress $\underset{\sim}{T}$ depends on the complete deformation history of a fluid, expressed by

$g_{pq} = x^m_{,p}(\tau)x^n_{,q}(\tau)\delta_{mn}$ over the range $-\infty < \tau \leq t$, and that the stress is a continuous function of the gradients of velocity and accelerations, Green and Rivlin finally obtained the following constitutive equation for visco-elastic fluids:

$$\underset{\sim}{T} = \beta_0(t)\underset{\sim}{I} + \sum_{j=1}^{5} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \sum_{i=0}^{R} \beta_i(t, \tau_1, \tau_2, \ldots, \tau_j)(\underset{\sim}{M}_i^{(j)} + \underset{\sim}{M}_i^{(j)T})$$

$$\times \; d\tau_1 d\tau_2 \ldots d\tau_j \, , \qquad (1.3.16)$$

where $\underset{\sim}{M}_i^{(j)}$ $(i = 1, 2, \ldots, R)$ are certain tensor products formed from the tensors $g_{pq}(\tau_j)$ and the kinematic tensors $\underset{\sim}{d}^{(1)}, \underset{\sim}{d}^{(2)}, \ldots, \underset{\sim}{d}^{(n)}$ already defined in the Rivlin-Ericksen theory, and $\underset{\sim}{M}_i^{(j)}$ is multilinear in the tensors $g_{pq}(\tau_j)$; $\underset{\sim}{M}_i^{(j)T}$ is the transpose of $\underset{\sim}{M}_i^{(j)}$; $\beta$'s are continuous functions of $t, \tau_1, \tau_2, \ldots, \tau_j$.

Green-Rivlin theory is a further generalization of Rivlin-Ericksen theory. Due to the similar structure of constitutive equation between the two theories, the remarks made in the Rivlin-Ericksen theory apply even more strongly to the present theory. For this very reason, these two theories have not been employed in their most general forms to solve any physical problems.

Noll's Theory (1958): Assuming that the stress in an incompressible fluid at time $\tau$ depends, to within a hydrostatic pressure, on the past history of the so-called relative deformation gradient up to time $\tau$, Noll derived the following constitutive equation:

$$\underset{\sim}{t} = -p\underset{\sim}{I} + \overset{\infty}{\underset{s=0}{\mathcal{F}}} [G(s)], \tag{1.3.17}$$

where $\mathcal{F}$ is the constitutive functional and $G(s)$ represents the history of the relative deformation gradient. This theory is somewhat similar in concept to the theory of Green and Rivlin. The solution of any problem with this theory requires the experimental determination of the three material constants, that is, the viscosity function and the two normal stress functions.

## Seth's Approach to Non-Linear Constitutive Equations

Narasimhan and Sra (1969) extended the concept of generalized measures of deformation-rates, pioneered by Seth (1962, 1964, 1966), to derive a physically applicable constitutive equation for visco-elastic fluids whose flows depend not only on velocity gradients but also on acceleration gradients.

They obtained the following generalized measures of deformation-rates:

$$\underset{\sim}{D}^* = \frac{k}{m^q n^q} [\underset{\sim}{I} - (\underset{\sim}{I} - 2m\underset{\sim}{D})^{n/2}]^q , \tag{1.3.18}$$

and

$$\underset{\sim}{B}^* = \frac{k'}{m'^{q'} n'^{q'}} [\underset{\sim}{I} - (\underset{\sim}{I} - 2m'\underset{\sim}{B})^{n'/2}]^{q'} , \tag{1.3.19}$$

where

$\underset{\sim}{D}^*$ = generalized first deformation-rate tensor,

$\underset{\sim}{B}^*$ = generalized second deformation-rate tensor,

$\underset{\sim}{D}$ = classical first deformation-rate tensor,

$\underset{\sim}{B}$ = classical second deformation-rate tensor,

$n, n', q$, and $q'$ are called measure indices, and

$m$ and $m'$ are parameters used to maintain the dimensions of the various terms on both sides of (1.3.18) and (1.3.19).

With such generalized deformation-rate measures, Seth established that a linear stress strain-velocity relation (1.2.1) would suffice for constructing constitutive equations for rheological materials. Hence, the following linear constitutive relation was chosen by Narasimhan and Sra (1969, 1971, 1972):

$$\underset{\sim}{t} = -p\underset{\sim}{I} + 2\mu\underset{\sim}{D}^* + 4\eta\underset{\sim}{B}^* , \tag{1.3.20}$$

where $\mu$ and $\eta$ are the classical viscosity and viscoelasticity coefficients.

Now, (1.3.20) becomes

$$\underset{\sim}{t} = -(p+F_0+G_0)\underset{\sim}{I} + F_1\underset{\sim}{D} + F_2\underset{\sim}{D}^2 + G_1\underset{\sim}{B} + G_2\underset{\sim}{B}^2 \tag{1.3.21}$$

through the use of the Cayley-Hamilton theorem in the expansions involved. For specific values of $n, q, n',$ and $q',$ which are to be determined through experiments analogously to that of viscosity coefficients, the $F'$s are known functions of the invariants of $\underset{\sim}{D}$ and $G'$s are known functions of the invariants of $\underset{\sim}{B}$ with a finite number of terms in each case. The non-linearity in (1.3.21) has condensed itself into three terms in each of the tensors $\underset{\sim}{D}$ and $\underset{\sim}{B}$. Since the use of $\underset{\sim}{B}^*$ is found to accomplish the goal of predicting the well-known visco-elastic phenomena, any higher order kinematic tensors beyond $\underset{\sim}{B}$ are not necessary. Furthermore, it must be noted that (1.3.21) does not involve any unknown response coefficients. Moreover, these constitutive theories have been applied by various workers for solving shearing flows, helical flows and secondary flows reaching good agreement with experiments.

We finally list for the sake of completeness several customarily used constitutive equations based on empirical or semi-empirical laws characterizing the shear stress $\tau$.

<u>Prandtl Fluids:</u>

$$\tau = A \sin^{-1}(cu_x) \tag{1.3.22}$$

<u>Eyring Fluids:</u>

$$\tau = B^{-1}u_x + C \sin(\tau/A) \tag{1.3.23}$$

Powell-Eyring Fluids:

$$\tau = Au_x + B \sinh^{-1}(Cu_x) \tag{1.3.24}$$

Williamson Fluids:

$$\tau = u_x[A/(B+u_x)+\mu_\infty], \tag{1.3.25}$$

where

$u_x = \frac{du}{dx}$ = velocity gradient,

$u_\infty$ = limiting viscosity at infinite rate of shear, and

A, B, C are constants which are typical of the particular fluid.

All of these equations are considerably more difficult to put into use than the power law and usually do not offer any compensating advantages.

Rabinowitsch Fluids:

$$u_x = \frac{\tau}{\mu_0}(1+C\tau^2), \tag{1.3.26}$$

where C and $\mu_0$ are constants typical of the fluid. This is an approximate rheological equation which can describe, for example, polyethylene and polystyrene malts.

# 2. A REVIEW OF MAJOR CONSTITUTIVE THEORIES IN MICROCONTINUUM MECHANICS

## 2.1 Preliminary Remarks

In this chapter the theory of micropolar fluids, due to Eringen, is presented together with the fundamental equations of microcontinua for later use. In addition, for the purpose of comparison and ready reference, a brief review of the following constitutive theories in microcontinuum mechanics is given:

linear theory of simple micro-elastic solids,

linear theory of simple microfluids,

linear theory of simple deformable directed fluids,

linear theory of fluids containing nonrigid structures, and

Mindlin's theory of microstructure in linear elasticity.

## 2.2 Microcontinuum Constitutive Theories

Linear Theory of Simple[1] Micro-Elastic Solids: Eringen and Suhubi (1964) presented a properly invariant nonlinear continuum theory of simple micro-elastic solids which are capable of supporting local stress moments and body moments, and are also influenced by the local inertial spin. The underlying idea of the theory is that each

[1]The term "simple" refers to the first order gradient of the various arguments retained in the constitutive equations.

material volume element contains micro-volume elements which can translate, rotate, and deform independently of the motion of the macro-volume elements, and the material points are assumed to undergo a certain affine transformation in the process of deformation.

The constitutive equations are expressed in terms of the derivatives of a stress potential with respect to $\underset{\sim}{e}$ (the classical strain tensor), $\underset{\sim}{\epsilon}$ (the micro-displacement gradient tensor), and $\underset{\sim}{\gamma}$ (a micro-deformation tensor). The general polynomial form of the constitutive equations are extremely lengthy and complicated. Here, we give only the linearized constitutive equations for a simple micro-elastic solid

$$t_{k\ell} = [-\pi+(\lambda+\tau)e_{mm}+\eta\epsilon_{mm}]\,\delta_{k\ell} + 2(\mu+\sigma)e_{k\ell} + \eta_1\epsilon_{\ell k} + \eta_2\epsilon_{k\ell}, \qquad (2.2.1)$$

$$\bar{t}_{k\ell} = [-\pi+(\lambda+2\tau)e_{mm}+(2\eta-\tau)\epsilon_{mm}]\delta_{k\ell} + 2(\mu+2\sigma)e_{k\ell} + (\eta_1+\eta_2-\sigma)(\epsilon_{k\ell}+\epsilon_{\ell k}), \qquad (2.2.2)$$

$$\begin{aligned}\lambda_{k\ell m} = {} & (a_1\gamma_{mrr}+a_2\gamma_{rmr}+a_3\gamma_{rrm})\delta_{k\ell} \\ & +(a_4\gamma_{\ell rr}+a_1\gamma_{r\ell r}+a_5\gamma_{rr\ell})\delta_{km} \\ & + (a_5\gamma_{krr}+a_3\gamma_{rkr}+a_6\gamma_{rrk})\delta_{\ell m} \qquad (2.2.3) \\ & + a_7\gamma_{\ell mk} + a_8(\gamma_{mk\ell}+\gamma_{k\ell m}) + a_9\gamma_{\ell km} + a_{10}\gamma_{m\ell k} \\ & + a_{11}\gamma_{k\ell m},\end{aligned}$$

and where

$t_{k\ell}$ = stress tensor,

$\bar{t}_{k\ell}$ = microstress average tensor,

$\lambda_{k\ell m}$ = first stress moment tensor,

$2e_{k\ell} = u_{k,\ell} + u_{\ell,k}$,

$\epsilon_{k\ell} = \phi_{k\ell} + u_{\ell,k}$,

$\gamma_{k\ell m} = -\phi_{k\ell,m}$,

$\delta_{k\ell}$ = Kronecker delta, and

$\pi, \lambda, \mu, \tau, \sigma, \eta, \eta_1, \eta_2, a_i$ $(i = 1, 2, \ldots, 11)$ are elastic coefficients.

Here $u_k$ and $\phi_{k\ell}$ are respectively the displacement vector and the micro-displacement tensor which are the basic unknowns of the theory. The micro-elastic properties of an isotropic linear elastic solid are completely described by 18 elastic coefficients, $\pi$ being zero for a natural stress free state.

Linear Theory of Simple Microfluids: Eringen (1964) developed the fluid counterpart of the theory of simple micro-elastic solids and obtained the linearized constitutive equations

$$\underset{\sim}{t} = [-\pi+\lambda \operatorname{tr} \underset{\sim}{d}+\lambda_0 \operatorname{tr}(\underset{\sim}{b}-\underset{\sim}{d})]\underset{\sim}{I} + 2\mu\underset{\sim}{d} + 2\mu_0(\underset{\sim}{b}-\underset{\sim}{d}) + 2\mu_1(\underset{\sim}{b}^T-\underset{\sim}{d}), \quad (2.2.4)$$

$$\underset{\sim}{\bar{t}} = [-\pi+\lambda \operatorname{tr} \underset{\sim}{d}+\eta_0 \operatorname{tr}(\underset{\sim}{b}-\underset{\sim}{d})]\underset{\sim}{I} + 2\mu\underset{\sim}{d} + \zeta_1(\underset{\sim}{b}-\underset{\sim}{b}^T-2\underset{\sim}{d}), \quad (2.2.5)$$

$$\lambda_{k\ell m} = (\gamma_1 a_{mrr} + \gamma_2 a_{rmr} + \gamma_3 a_{rrm})\delta_{k\ell} + (\gamma_4 a_{\ell rr} + \gamma_5 a_{r\ell r} + \gamma_6 a_{rr\ell})\delta_{km}$$

$$+ (\gamma_7 a_{krr} + \gamma_8 a_{rkr} + \gamma_9 a_{rrk})\delta_{\ell m} + \gamma_{14} a_{\ell mk} + \gamma_{15} a_{m\ell k}, \quad (2.2.6)$$

where

$\underset{\sim}{d}$ = deformation-rate tensor ($2d_{k\ell} = v_{k,\ell} + v_{\ell,k}$),

$\underset{\sim}{b}$ = micro-deformation rate tensor ($b_{k\ell} = \nu_{k\ell}$[2/] $+ v_{k,\ell}$),

$\underset{\sim}{a}$ = gyration gradient ($a_{k\ell m} = \nu_{k\ell,m}$),

$\underset{\sim}{I}$ = unit tensor,

$\delta_{ij}$ = Kronecker delta,

tr = trace,

T = transpose, and

$\lambda, \lambda_0, \mu, \mu_0, \mu_1, \eta_0, \zeta_1$, and $\gamma_i$ ($i = 1, 2, \ldots, 15$) are viscosity coefficients.

Fluids whose mechanical behavior is governed by (2.2.4), (2.2.5), and (2.2.6) are termed simple microfluids. Anisotropic fluids, vortex fluids, and fluids in which other intrinsic gyrational effects are important, are conjectured to fall into the domain of simple microfluids. Immediate application of the present theory is not possible due to the mathematical difficulties encountered upon application. Further, no information is available on the signs of the numerous viscosity coefficients which appear in the theory.

---

[2/] See (2.2.8).

Owing to the analytical complexity of the theory of simple microfluids, Eringen (1966) formulated the theory of micropolar fluids that possesses certain simplicity and elegance in its mathematical formulation, by imposition of no-microstretch and micro-isotropy on simple microfluids.

Theory of Micropolar Fluids: Many fluids found in industry and biological laboratories such as animal blood, polymeric fluids, liquid crystals, fluids containing certain additives are known to be constituted by bar-like elements which undergo only rigid motions (i.e., translation and rotation). These fluids fall into the category known as micropolar fluids. In the theory of micropolar fluids, the micro-deformation tensor $\underset{\sim}{\chi}_K$ defined by the affine transformation

$$\underset{\sim}{\xi} = \underset{\sim}{\chi}_K \Xi_K \tag{2.2.7}$$

is orthogonal, that is, $\underset{\sim}{\chi}\underset{\sim}{\chi}^T = \underset{\sim}{I} = \underset{\sim}{\chi}^T\underset{\sim}{\chi}$, in order to support only microrotations. In other words, no-microstretch is allowed in the motion of a microelement. Consequently, the gyration tensor $\nu_{k\ell}$ and the first stress moment tensor $\lambda_{k\ell m}$ become skew-symmetric, That is,

$$\nu_{k\ell} = -\nu_{\ell k}$$

and (2.2.8)

$$\lambda_{k\ell m} = -\lambda_{km\ell} ,$$

where $\nu = \dot{\underset{\sim}{\chi}}\underset{\sim}{\chi}^{-1}$. The constitutive equations for micropolar fluids are derivable from those of simple microfluids with (2. 2. 8).

$$t_{k\ell} = (-\pi+\lambda v_{r,r})\delta_{k\ell} + \mu(v_{k,\ell}+v_{\ell,k}) + \kappa(v_{\ell,k}-\epsilon_{k\ell r}\nu_r), \tag{2.2.9}$$

$$m_{k\ell} = \alpha\nu_{r,r}\delta_{k\ell} + \beta\nu_{k,\ell} + \gamma\nu_{\ell,k}, \tag{2.2.10}$$

where

$t_{k\ell}$ = stress tensor,

$m_{k\ell}$ = couple stress tensor,

$\delta_{k\ell}$ = Kronecker delta,

$v_k$ = velocity vector,

$\nu_k$ = microrotation vector, and

$\lambda, \mu, \kappa$ are known as the viscosity coefficients, while $\alpha$, $\beta$, and $\gamma$ are called gyroviscosity coefficients.

The micropolar fluid flow has two prominent departures from the case of Navier-Stokes theory, that is, the presence of the couple stress and the asymmetry of the stress tensor. The constitutive equations of the linear micropolar fluid involve six material coefficients, which conform to inequalities forced by the Clausius-Duhem thermodynamical inequality:

$$
\begin{aligned}
&3\lambda + 2\mu + \kappa \geqq 0, \\
&2\mu + \kappa \geqq 0, \\
&\kappa \geqq 0, \quad \gamma \geqq 0, \\
&|\beta| \leqq \gamma, \; 3\alpha + \beta + \gamma \geqq 0, \quad \text{for } \theta \text{ (= temperature)} > 0.
\end{aligned}
\tag{2.2.11}
$$

This class of fluids reduces to the Navier-Stokes fluids with the suppressions of the micropolar material constants $\kappa$, $\alpha$, $\beta$, $\gamma$ and with the vanishing of the body moment.

<u>Linear Theory of Simple Deformable Directed Fluids</u>: To develop the constitutive equations for their linear theory of simple fluids with deformable microelements, Allen, Desilva and Kline (1967) derived a canonical form of energy equation which, when coupled with the entropy production inequality, served as a guide to define a simple deformable directed fluid. The microstructure was described by a frame of director vectors $\underset{\sim}{d}_\alpha$ $(\alpha = 1, 2, 3)$ at each point $\underset{\sim}{x}$ of the fluid. This director frame associates with it not only the conventional translation velocity but also a rotational velocity and an ability to deform, thereby assigning structure and orientation to each micro-element. In addition to such kinematical variables as velocity $v_k$, velocity gradient $v_{k,m}$, a measure of relative changes in the lengths of, and the angles between, the directors $W_{(mn)}$, the spin tensor of the directors $W_{[mn]}$, Allen et al. included the vectors $\underset{\sim}{d}_\alpha$ in the arguments of the constitutive equations, since they were

interested in fluid suspensions and in the effect of particle orientation on the flow. The explicit constitutive equations are given by

$$t^{ij} = A^{ijkm}[v_{(k,m)} - W_{(km)}] + F^{ijkm}W_{(km)}$$

$$+ B^{ijkm}[v_{[k,m]} - W_{[mk]}] , \qquad (2.2.12)$$

$$\bar{t}^{ij} = C^{ijkm}[v_{(k,m)} - W_{(km)}] + G^{ijkm}W_{(km)} , \qquad (2.2.13)$$

$$\lambda^{ijk} = E^{ijkpqr}W_{(rq),p} + D^{ijkpqr}W_{[rq],p} ,$$

where

$t^{ij}$ = stress tensor,

$\bar{t}^{ij}$ = microstress average tensor,

$\lambda^{ijk}$ = first stress moment tensor, and

parentheses enclosing subscripts denote the symmetric part and the square brackets enclosing subscripts denote the skew-symmetric part of the various field variables. For example, $v_{(k,m)} = \frac{1}{2}(v_{k,m}+v_{m,k})$ and $w_{[mn]} = \frac{1}{2}[w_{mn}-w_{nm}]$.

Further, in the above equations, $A^{ijkm}$, $B^{ijkm}$, $C^{ijkm}$, $D^{ijkpqr}$, $E^{ijkpqr}$, $F^{ijkm}$, $G^{ijkm}$ are material coefficient tensors and are functions of $\underset{\sim}{d}_{\alpha}$, specific volume, and temperature.

As pointed out by Toupin (1964), a homogeneous deformation is uniquely determined by the motion of any three linearly independent vectors like a director frame. Since the deformation of the microelements in the theory of Eringen and Suhubi obeys a certain affine relative transformation, we may regard the model of a material with a triad of deformable directors assigned to each point of the continuum as identical to the Eringen-Suhubi formulation.

Kirwan's Theory: Kirwan (1968) formulated a constitutive theory for a fluid containing nonrigid microelements by employing a measure of deformation-rate of the microelements as an additional constitutive variable to the ordinary deformation-rate tensor, the kinematics being set up by the introduction of director frame within the fluid. Thus, this theory allows for nonrigid responses of the fluid microelements. Two new physical principles were utilized in developing constitutive equations. One principle required that the constitutive equations uniquely determine the stresses and couple stresses from the objective variables and vice versa. This requirement could be thought of as an invertible mapping between stress and deformation. In this way, any assumptions regarding the microelement kinematics are automatically reflected in the stresses. As the second principle, Kirwan assumed that only the phenomenological coefficients which appear in the Clausius-Duhem inequality have physical significance. Requiring the phenomenological coefficient tensors to be isotropic and using the above two newly introduced concepts, Kirwan gave the following linear constitutive equations:

$$t_{ji} = [A_1 D_{mm} + E_1 w_{mm}]\delta_{ji} + 2A_2 D_{ji} + 3E_2 w_{(ji)}$$
$$+ 2E_3[w_{[ij]} + W_{ij}], \quad (2.2.15)$$

$$\bar{t}_{ji} = [A_1 D_{mm} + B_1 w_{mm}]\delta_{ji} + 2A_2 D_{ji} + 2B_2 w_{(ji)}, \quad (2.2.16)$$

$$\begin{aligned}\lambda_{kij} = {} & \lambda_1[\delta_{ki} w_{mm,j} + \delta_{jk} w_{mm,i} + \delta_{ij}(w_{kn,n} + w_{nk,n})] \\ & + \lambda_2[\delta_{ki} w_{nj,n} + \delta_{jk} w_{in,n}] + \lambda_3[\delta_{ki} w_{jn,n} + \delta_{jk} w_{ni,n}] \\ & + \lambda_4[\delta_{ji} w_{mm,k} + \lambda_5 w_{ji,k} + \lambda_6[w_{kj,i} + w_{ik,j}] \\ & + \lambda_7 w_{ij,k} + \lambda_8[w_{jk,i} + w_{ki,j}] ,\end{aligned} \quad (2.2.17)$$

where

$t_{ji}$ = stress tensor,

$\bar{t}_{ji}$ = microstress average tensor,

$\lambda_{kij}$ = first stress moment tensor,

$D_{mn}$ = deformation-rate tensor,

$W_{mn}$ = vorticity tensor of the fluid,

$w_{mn}$ = microdeformation-rate tensor, and

A's, B's, E's, $\lambda$'s are viscosity coefficients.

This theory has been shown to reduce to the established equations of Eringen (1966) and Allen and Kline (1968), when the microelements are assumed to be rigid.

Mindlin's Theory: Mindlin (1964) derived from Hamilton's variational principle a linear theory of a three-dimensional elastic continuum which has some of the properties of a crystal lattice as a result of inclusion, in the theory, of the idea of the unit cell. The unit cell can be interpreted as a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material.

The constitutive equations of Mindlin's elastic materials were derived by differentiating the potential energy-density function $W$ with respect to the deformation-rate tensor $d_{ij}$, relative deformation tensor $\gamma_{ij}$ (i.e., the difference between the macro-displacement gradient tensor and the micro-deformation tensor), and micro-deformation gradient tensor $\kappa_{ijk}$ (i.e., the macrogradient of the micro-deformation tensor). That is,

$$t_{ij} = \frac{\partial W}{\partial d_{ij}} + \frac{\partial W}{\partial \gamma_{ij}} = \text{stress tensor},$$

$$\bar{t}_{ij} = \frac{\partial W}{\partial d_{ij}} = \text{microstress average tensor, and}$$

$$\lambda_{ijk} = \frac{\partial W}{\partial \kappa_{ijk}} = \text{first stress moment tensor.}$$

For a centrosymmetric, isotropic linear elastic material, the potential energy-density function is given in the following form:

$$W = \frac{1}{2}\lambda d_{ii}d_{jj} + \mu d_{ij}d_{ij} + \frac{1}{2}b_1\gamma_{ii}\gamma_{jj} + \frac{1}{2}b_2\gamma_{ij}\gamma_{ij} + \frac{1}{2}b_3\gamma_{ij}\gamma_{ji} + g_1\gamma_{ii}d_{ii}$$

$$+ g_2(\gamma_{ij}+\gamma_{ji})d_{ij} + a_1\kappa_{iik}\kappa_{kjj} + a_2\kappa_{iik}\kappa_{jkj} + \frac{1}{2}a_3\kappa_{iik}\kappa_{jjk}$$

$$+ \frac{1}{2}a_4\kappa_{ijj}\kappa_{ikk} + a_5\kappa_{ijj}\kappa_{kik} + \frac{1}{2}a_8\kappa_{iji}\kappa_{kjk} + \frac{1}{2}a_{10}\kappa_{ijk}\kappa_{ijk}$$

$$+ a_{11}\kappa_{ijk}\kappa_{jki} + \frac{1}{2}a_{13}\kappa_{ijk}\kappa_{ikj} + \frac{1}{2}a_{14}\kappa_{ijk}\kappa_{jik} + \frac{1}{2}a_{15}\kappa_{ijk}\kappa_{kji} \, . \qquad (2.2.18)$$

Hence,

$$t_{ij} = (\lambda+g_1)\delta_{ij}d_{kk} + 2(\mu+g_2)d_{ij} + (g_1+b_1)\delta_{ij}\gamma_{kk} + (g_2+b_2)\gamma_{ij}$$

$$+ (g_2+b_3)\gamma_{ii} \, , \qquad (2.2.19)$$

$$\overline{t}_{ij} = \lambda\delta_{ij} + 2\mu d_{ij} + g_1\delta_{ij}\gamma_{kk} + g_2(\gamma_{ij}+\gamma_{ji}), \qquad (2.2.20)$$

$$\lambda_{ijk} = a_1(\kappa_{mmi}\delta_{jk}+\kappa_{kmm}\delta_{ij}) + a_2(\kappa_{mmj}\delta_{ij}+\kappa_{mkm}\delta_{ij})$$

$$+ a_3\kappa_{mmk}\delta_{ij} + a_4\kappa_{imm}\delta_{jk} + a_5(\kappa_{jmm}\delta_{ik}+\kappa_{mim}\delta_{jk})$$

$$+ a_8\kappa_{mjm}\delta_{jk} + a_{10}\kappa_{ijk} + a_{11}(\kappa_{kij}+\kappa_{jki}) + a_{13}\kappa_{ikj} + a_{16}\kappa_{ijk}$$

$$+ a_{15}\kappa_{kji} \, , \qquad (2.2.21)$$

where

$\delta_{ij}$ = Kronecker delta, and

$\lambda$, $\mu$, $b_1$, $b_2$, $b_3$, $g_1$, $g_2$, a's are elastic coefficients.

This theory is analogous to the general theory of simple microelastic materials of Eringen and Suhubi (1964).

## 2.3 Fundamental Equations of Microcontinua

Basic principles of microcontinuum mechanics, namely conservation of mass, conservation of microinertia moments, balance of momenta, balance of first stress moments, conservation of energy and entropy inequality lead to the fundamental equations (Eringen, 1966).

Equation of continuity:

$$\frac{\partial \rho}{\partial t} + (\rho v_m)_{;m} = 0 \tag{2.3.1}$$

Conservation of microinertia moments:

$$\frac{\partial i_{k\ell}}{\partial t} + i_{k\ell;m} v_m - i_{\ell m} \nu_{mk} - i_{km} \nu_{m\ell} = 0 \tag{2.3.2}$$

Conservation of linear momentum:

$$t_{k\ell;k} + \rho(f_\ell - \dot{v}_\ell) = 0 \tag{2.3.3}$$

Conservation of moment of momentum:

$$t_{\ell k} - \bar{t}_{k\ell} + \lambda_{rk\ell;r} + \rho(\ell_{k\ell} - \dot{\sigma}_{k\ell}) = 0; \bar{t}_{[k\ell]} = 0 \tag{2.3.4}$$

Conservation of energy:

$$\rho\dot{\epsilon} - t_{k\ell} v_{\ell;k} - (\bar{t}_{k\ell} - t_{k\ell})\nu_{k\ell} - \lambda_{mk\ell}\nu_{\ell k;m} + q_{m;m} - \rho h = 0 \tag{2.3.5}$$

<u>Entropy inequality</u>:

$$\rho\dot{\eta} - \left(\frac{q_k}{\theta}\right)_{;k} - \rho\frac{h}{\theta} \geq 0\ , \qquad (2.3.6)$$

where

$\rho$ = mass density,

$f_\ell$ = body force,

$i_{k\ell}$ = microinertia moment,

$\bar{t}_{k\ell}$ = micro-stress average,

$\ell_{k\ell}$ = the first body moment per unit mass,

$\dot{\sigma}_{k\ell}$ = inertial spin,

$\epsilon$ = internal energy density per unit mass,

$q_m$ = heat flux vector,

$h$ = heat source per unit mass,

$t_{k\ell}$ = stress tensor,

$v_\ell$ = velocity vector,

$\nu_{k\ell}$ = gyration tensor,

$\lambda_{mk\ell}$ = the first stress moments,

$\eta$ = entropy density, and

$\theta$ = absolute temperature.

A semicolon followed by an index indicates covariant differentiation with respect to general curvilinear coordinates.

The basic field equations of a micropolar fluid motion which result by substituting the constitutive equations (2. 2. 9) and (2. 2. 10) into the balance laws (2 3. 3) and (2. 3. 4) become in the linear case [Eringen (1966)]:

Equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{v}) = 0, \tag{2. 3. 7}$$

Linear momentum equation:

$$\rho \frac{D\underset{\sim}{v}}{Dt} = \rho \underset{\sim}{f} - \nabla p + \kappa \nabla \times \underset{\sim}{\nu} - (\mu+\kappa)\nabla \times \nabla \times \underset{\sim}{v} + (\lambda+2\mu+\kappa)\nabla\nabla \cdot \underset{\sim}{v}, \tag{2. 3. 8}$$

Angular momentum equation:

$$\rho j \frac{D\underset{\sim}{\nu}}{Dt} = \rho \underset{\sim}{\ell} + \kappa(\nabla \times \underset{\sim}{v} - 2\underset{\sim}{\nu}) - \gamma \nabla \times \nabla \times \underset{\sim}{\nu} + (\alpha+\beta+\gamma)\nabla(\nabla \cdot \underset{\sim}{\nu}), \tag{2. 3. 9}$$

where

$D/Dt$ = material derivative operator,

$\rho$ = mass density,

$p$ = hydrostatic pressure,

$j$ = constant microinertia moment,

$\underset{\sim}{v}$ = linear velocity vector,

$\underset{\sim}{\nu}$ = microrotation vector,

$\underset{\sim}{\ell}$ = first body moments per unit mass,

$\underset{\sim}{f}$ = body force per unit mass,

$\lambda$, $\mu$, $\kappa$ are the viscosity coefficients and $\alpha$, $\beta$, $\gamma$, the gyroviscosity coefficients.

We refer the motion of the fluid to a system of cylindrical coordinates in anticipation of the problem to be treated in the next chapter.

Equations of Motion in Cylindrical Coordinate System $(r, \theta, z)$ with Axial Symmetry and Incompressibility Condition:

Equation of continuity:

$$\frac{\partial(ru_r)}{\partial r} + \frac{\partial(ru_z)}{\partial z} = 0, \tag{2.3.10}$$

Radial component of linear momentum equation:

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}$$

$$= f_r - \frac{1}{\rho}\frac{\partial p}{\partial r} + (\nu + \frac{\kappa}{\rho})(\nabla^2 u_r - \frac{u_r}{r^2}) - \frac{\kappa}{\rho}\frac{\partial n_\theta}{\partial z}, \tag{2.3.11}$$

Azimuthal component of linear momentum equation:

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}$$

$$= f_\theta + (\nu + \frac{\kappa}{\rho})(\nabla^2 u_\theta - \frac{u_\theta}{r^2}) + \frac{\kappa}{\rho}(\frac{\partial n_r}{\partial z} - \frac{\partial n_z}{\partial r}), \tag{2.3.12}$$

Axial component of linear momentum equation:

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z}$$

$$= f_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + (\nu + \frac{\kappa}{\rho})\nabla^2 u_z + \frac{\kappa}{\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}(rn_\theta)\right] , \qquad (2.3.13)$$

Radial component of angular momentum equation:

$$\frac{\partial n_r}{\partial t} + u_r \frac{\partial n_r}{\partial r} + u_z \frac{\partial n_r}{\partial z} - \frac{u_\theta n_\theta}{r}$$

$$= \frac{r}{\rho j}(\nabla^2 n_r - \frac{n_r}{r^2}) - \frac{\kappa}{\rho j}(\frac{\partial u_\theta}{\partial z} + 2n_r) , \qquad (2.3.14)$$

Azimuthal component of angular momentum equation:

$$\frac{\partial n_\theta}{\partial t} + u_r \frac{\partial n_\theta}{\partial r} + \frac{u_\theta n_r}{r} + u_z \frac{\partial n_\theta}{\partial z}$$

$$= \frac{\gamma}{\rho j}(\nabla^2 n_\theta - \frac{n_\theta}{r^2}) + \frac{\kappa}{\rho j}(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} - 2n_\theta), \qquad (2.3.15)$$

Axial component of angular momentum equation:

$$\frac{\partial n_z}{\partial t} + u_r \frac{\partial n_z}{\partial r} + u_z \frac{\partial n_z}{\partial z} = \frac{\gamma}{\rho j}\nabla^2 n_z + \frac{\kappa}{\rho j}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - 2n_z\right], \qquad (2.3.16)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} ,$$

and $u_r$, $u_\theta$, $u_z$; $n_r$, $n_\theta$, $n_z$; and $f_r$, $f_\theta$, $f_z$ are the velocity components, microrotation components and body forces in the cylindrical coordinates along the r-, $\theta$- and z-directions respectively.

# 3. THE ASYMPTOTIC BOUNDARY LAYER ON A CIRCULAR CYLINDER IN AXISYMMETRIC INCOMPRESSIBLE MICROPOLAR FLUID FLOW

## 3.1 Preliminary Remarks

Stewartson (1955) obtained an asymptotic series solution for the stream function $\psi = \psi(\eta, \xi)$, where $\eta$ and $\xi$ are dimensionless independent variables taken along the radial and axial directions, respectively, in the boundary layer on a semi-infinite circular cylinder in an axial incompressible Newtonian fluid flow. Application of the method of successive approximations to the linear momentum equation led to the following form of solution:

$$\frac{\partial \psi}{\partial \eta} = 1 + \sum_{s,t=1}^{\infty} \frac{P'_{s,t}(\eta)}{\xi^{s-1}[\ln \frac{2\xi}{C}]^t} . \qquad (3.1.1)$$

In particular, for $s = 1$ in the series on the righthand-side of (3.1.1),

$$\frac{\partial \psi}{\partial \eta} = 1 + \sum_{t=1}^{\infty} \frac{F'_t(\eta)}{[\ln \frac{2\xi}{C}]^t} + O(\frac{1}{\xi \ln \xi}) ,$$

where the prime indicates partial differentiation with respect to $\eta$, and $\ln C$ = Euler's constant. His analysis showed that if the velocity of the main stream is constant, the skin friction on the cylinder is increased at the corresponding points of a flat plate due to the effect of transverse curvature of the cylinder, and, for the same reason, that the boundary layer thickness is slightly reduced in comparison with that of the flat plate.

We believe that it would be of practical interest to examine the corresponding boundary layer flow problem using micropolar theory of fluids, since it is a more realistic model of fluids than the classical theory of viscous fluids

On account of the introduction of the gyration tensor and micro-inertia moment tensor in Eringen's theory of micropolar fluids, the field equations will consist of linear and angular momentum equations. They will be nonlinear in character and coupled in the two unknown field vectors, that is, linear velocity and microrotation vectors. We will follow here the Peddieson and Mcnitt's (1970) approach for investigating the boundary layer flow for a micropolar fluid.

In Section 3.2 we formulate our problem in a concrete manner, deriving a set of boundary-layer equations with appropriate boundary and matching conditions. Those equations will be solved with the aid of the method of successive approximations in Section 3.3. Finally, in the fourth and last section, results for the local effective wall shear stress, the velocity profile, and the boundary layer thickness are

## 3.2 The Statement of the Problem

We consider the asymptotic boundary layer induced by a steady, axial, incompressible micropolar fluid flow on a semi-infinite circular cylinder of radius $a$, whose axis is taken along the positive z-direction in cylindrical polar coordinate system $(r, \theta, z)$. The coordinate $r$ represents the transverse distance from the axis and $v$, $u$, the components of the fluid velocity in the r- and z-directions, respectively. The body force and body moment are neglected in our problem. The velocity field and the microrotation field for our problem are of the form:

$$\underset{\sim}{v} = (u_r, u_\theta, u_z): \quad u_r = v(r, z), \quad u_\theta = 0, \quad u_z = u(r, z). \tag{3.2.1}$$

$$\underset{\sim}{\nu} = (n_r, n_\theta, n_z): \quad n_r = 0, \quad n_\theta = n(r, z), \quad n_z = 0. \tag{3.2.2}$$

We will treat only the case of uniform mainstream velocity $U$ at zero incidence for simplicity.

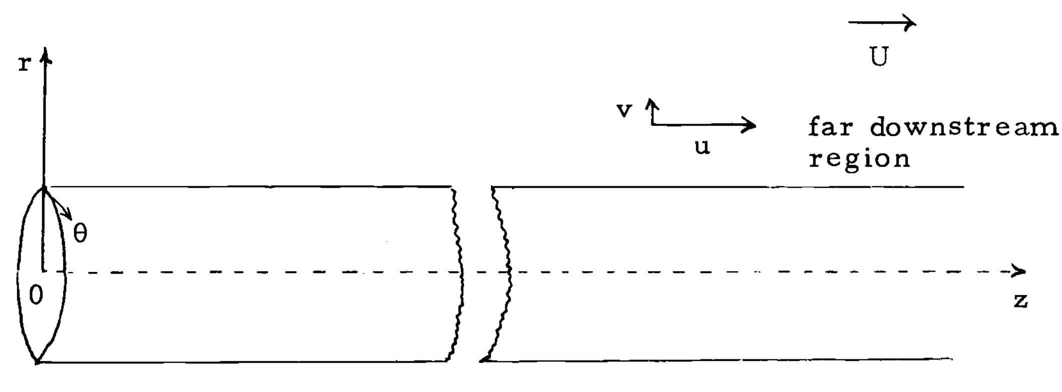


Figure 3.2.1. The geometry of the problem in the cylindrical coordinates.

The Equations of Motion

With the velocity and microrotation fields give by (3.2.1) and (3.2.2), respectively, the equations (2.3.10) through (2.3.16) of continuity and motion reduce to the following simpler forms:

Equation of continuity:

$$\frac{\partial(rv)}{\partial r} + \frac{\partial(ru)}{\partial z} = 0, \tag{3.2.3}$$

r-component linear momentum equation:

$$v\frac{\partial v}{\partial r} + u\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + (\nu + \frac{\kappa}{\rho})(\nabla^2 v - \frac{v}{r^2}) - \frac{\kappa}{\rho}\frac{\partial n}{\partial z}, \tag{3.2.4}$$

z-component linear momentum equation:

$$v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + (\nu + \frac{\kappa}{\rho})\nabla^2 u + \frac{\kappa}{\rho}(\frac{n}{r} + \frac{\partial n}{\partial r}), \tag{3.2.5}$$

θ-component angular momentum equation:

$$v\frac{\partial n}{\partial r} + u\frac{\partial n}{\partial z} = \frac{\gamma}{\rho j}(\nabla^2 n - \frac{n}{r^2}) + \frac{\kappa}{\rho j}(\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r} - 2n), \tag{3.2.6}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

Boundary Layer Equations

The boundary-layer magnitude estimates for micropolar fluid flow developed in our problem are based on the analysis of Peddieson and Mcnitt (1970). To study the inner (i.e., boundary-layer) flow the following new variables are introduced:

$$\bar{v} = R^{-1/2}v, \quad \bar{u} = u, \quad \bar{n} = R^{1/2}n, \quad \bar{p} = p, \quad \bar{r} = R^{-1/2}r, \quad \bar{z} = z, \tag{3.2.7}$$

where the barred and unbarred quantities are unrestricted field variables and inner variables, respectively, and $R$ is the Reynolds number, $R^{-1/2}$ being small compared with unity, i.e., $R^{-1/2} \ll 1$.

Inserting the new variables (i.e., the unbarred variables) into (3.2.3) through (3.2.6), we finally obtain the following boundary layer equations:

Equation of continuity:

$$\frac{\partial(rv)}{\partial r} + \frac{\partial(ru)}{\partial z} = 0 , \tag{3.2.8}$$

r-component linear momentum equation:

$$\frac{\partial p}{\partial r} = 0 \quad (\text{i.e.,} \quad p = p(z)) , \tag{3.2.9}$$

z-component linear momentum equation:

$$v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} = -\frac{1}{r}\frac{\partial p}{\partial z} + (\nu + \frac{\kappa}{\rho})[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})] + \frac{\kappa}{\rho}(\frac{n}{r} + \frac{\partial n}{\partial r}), \qquad (3.2.10)$$

$\theta$-component angular momentum equation:

$$v\frac{\partial n}{\partial r} + u\frac{\partial n}{\partial z} = -\frac{\kappa}{\rho j}(\frac{\partial u}{\partial r} + 2n) + \frac{\gamma}{\rho j}[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial n}{\partial r}) - \frac{n}{r^2}]. \qquad (3.2.11)$$

It is to be noted here that the above boundary layer equations contain the terms characterizing the asymmetry of the stress tensor, the couple stress, and the microspin-inertia tensor.

Boundary and Matching Conditions

$$v(a,z) = u(a,z) = 0 \quad \text{(hyper-stick condition)}, \qquad (3.2.12)$$

$$\lim_{r \to \infty} u(r,z) = U(z) \quad (z > 0), \qquad (3.2.13)$$

where $U(z)$ is the velocity of the fluid in the main stream and is taken to be proportional to $z^m$.

$$\lim_{r \to \infty} n(r,z) = 0, \qquad (3.2.14)$$

$$n(a,z) = n_0(z). \qquad (3.2.15)$$

The matter of proper microrotation boundary condition is as yet unresolved. At present, there seems to be no physical theory that enables us to choose the appropriate boundary conditions on the microrotation $n$. Hence, under such circumstances, one has to resort to an existence problem to settle the matter. Accordingly, we pose the problem, namely what sort of microrotation boundary conditions would permit the desired type of flow under consideration and also be consistent with the velocity boundary conditions? Here, by investigating this consistency requirement, we will establish the actual determination of the microrotation boundary condition permitting the asymptotic boundary layer flow in question. Thus, we choose the boundary condition (3. 2. 15) and will discuss its nature later in the next section.

## 3. 3 The Solution of the Boundary Layer Equations

The hydrostatic pressure $p$ is obtained by putting $u = U = Cz^m$ in (3. 2.10), yielding:

$$-\frac{1}{\rho}\frac{dp}{dz} = \frac{mU^2}{z} \tag{3. 3. 1}$$

Note that if $m = 0$, there is no pressure gradient in the flow. Accordingly, there is no problem of boundary-layer separation.

We first define a stream function using the equation of continuity as follows:

$$(\nu + \frac{\kappa}{\rho}) z\psi(\eta, \xi), \tag{3.3.2}$$

where

$$\eta = \frac{r^2 U}{2z(\nu + \frac{\kappa}{\rho})}, \tag*{$(3.3.3)_1$}$$

$$\xi = \frac{2z(\nu + \frac{\kappa}{\rho})}{Ua^2}. \tag*{$(3.3.3)_2$}$$

Then, the velocity components $v$ and $u$ are expressed in terms of the stream function $\psi$.

$$u = \frac{1}{r}\frac{\partial}{\partial r}[(\nu + \frac{\kappa}{\rho})z\psi] = U\psi_\eta, \tag{3.3.4}$$

$$v = -\frac{1}{r}\frac{\partial}{\partial z}[(\nu + \frac{\kappa}{\rho})z\psi]$$

$$= \frac{-(\nu + \frac{\kappa}{\rho})}{r}[\psi + (1-m)(\xi\psi_\xi - \eta\psi_\eta)], \tag{3.3.5}$$

where the subscripts $\eta$ and $\xi$ indicate partial differentiation as specified.

Next, the microrotation can be expressed in the form:

$$n = \frac{U}{r}\Phi(\eta, \xi), \tag{3.3.6}$$

where $\Phi(\eta, \xi)$ is assumed to be bounded for large $\eta$ and $\xi$ so that the flow may approach a potential flow as $r \to \infty$ for fixed $z$.

After substituting $\psi$, $\Phi$, and their derivatives into both (3.2.10) and (3.2.11), we find that for uniform mainstream velocity, namely $m = 0$,

z-component linear momentum equation:

$$2\eta\psi_{\eta\eta\eta} + (2+\psi)\psi_{\eta\eta} = \xi(\psi_\eta\psi_{\eta\xi} - \psi_\xi\psi_{\eta\eta}) - A\Phi_\eta, \tag{3.3.7}$$

$\theta$-component angular momentum equation:

$$D\eta\Phi_{\eta\eta} + \xi(\psi_\xi\Phi_\eta - \psi_\eta\Phi_\xi) + \psi\Phi_\eta - \left(\frac{\psi}{2\eta} - \frac{1}{2}\psi_\eta + \frac{\xi}{2\eta}\psi_\xi + AB\xi\right)\Phi$$

$$= AB\xi\eta\psi_{\eta\eta}, \tag{3.3.8}$$

where

$$A = \frac{\kappa}{\mu+\kappa}, \quad B = \frac{a^2}{j}, \quad \text{and} \quad D = \frac{2\gamma}{(\mu+\kappa)j}.$$

The boundary and matching conditions are now expressed in terms of $\psi$ and $\Phi$ as follows:

$$\psi_\eta = \psi + \xi\psi_\xi = 0 \quad \text{at} \quad \eta\xi = 1, \tag{3.3.9}$$

$$\psi_\eta \to 1 \quad \text{as} \quad \eta \to \infty \quad \text{for all} \quad \xi, \tag{3.3.10}$$

and as $\xi \to 0+$, provided $\eta\xi > 1$,

$$\Phi \text{ is bounded for large } \eta \text{ and } \xi, \text{ and} \tag{3.3.11}$$

$$\Phi = \Phi_0 \quad \text{at} \quad \eta\xi = 1. \tag{3.3.12}$$

(3.3.7) and (3.3.8) are highly nonlinear in character and form a coupled system of two partial differential equations involving the stream function $\psi$ and the microrotation represented by the function $\Phi$. These nonlinear coupled partial differential equations are, in general, extremely difficult to solve. Here, the method of successive approximations can be appropriately used in order to solve for $\psi$ and $\Phi$.

Before seeking to determine the solution of the system of equations (3.3.7) and (3.3.8), we note that in the boundary layer, $\xi\eta = O(1)$, since $\xi\eta = (\frac{r}{a})^2 = (\frac{a+\delta}{a})^2 \approx 1 + \frac{2\delta}{a} = O(1)$, where $\delta$ denotes the boundary layer thickness. In the asymptotic region ($\xi \gg 1$), the terms $AB\xi\Phi$ and $AB\xi\eta\psi_{\eta\eta}$ in (3.3.8) are of the same order[3/] when $\xi\Phi$ is of the same order as $\psi_{\eta\eta}$ for fixed $\eta$. Following Stewartson (1955), the relation that $AB\xi\Phi \sim AB\xi\eta\psi_{\eta\eta}$ is possible by selecting the asymptotic series expansions for $\psi_{\eta\eta}$ and $\Phi$ as follows:

$$\psi_{\eta\eta} = \sum_{s=1}^{\infty} \frac{G_s''(\eta)}{[\ln(\frac{2\xi}{C})]^s} + O\left[\frac{1}{\xi \ln(\frac{2\xi}{C})}\right], \tag{3.3.13}$$

and

---

[3/] We use the notation $\sim$ to express the same order of magnitude, whenever such a need arises in our subsequent discussion.

$$\xi\Phi = \sum_{s=1}^{\infty} \frac{H_s(\eta)}{[\ln(\frac{2\xi}{C})]^s} + O\left[\frac{1}{\xi \ln(\frac{2\xi}{C})}\right] , \tag{3. 3. 14}$$

where $G_s''(\eta)$ and $H_s(\eta)$ are unknown coefficients of $[\ln(\frac{2\xi}{C})]^{-s}$ and the double prime stands for the second derivative with respect to $\eta$.

With (3. 3. 13) and (3. 3. 14), the order of magnitude of the other terms in (3. 3. 8) can be shown to be small compared to those of $AB\xi\Phi$ and $AB\xi\eta\psi_{\eta\eta}$ within powers of $[\ln(\frac{2\xi}{C})]^{-s}$ for $s \geq 1$ and fixed $\eta$. It suffices to check only the first dominant term of each expansion for purposes of estimating terms of the desired order of magnitude:

$$\begin{aligned}
\eta\,\Phi_{\eta\eta} &\sim \frac{\eta H_1''(\eta)}{\xi[\ln(\frac{2\xi}{C})]} , \\
\xi\psi_\xi \Phi_\eta &\sim \frac{G_1(\eta)H_1'(\eta)}{\xi[\ln(\frac{2\xi}{C})]^3} , \\
\xi\psi_\eta \Phi_\xi &\sim \frac{H_1(\eta)}{\xi[\ln(\frac{2\xi}{C})]} , \\
\psi\Phi_\eta &\sim \frac{\eta H_1'(\eta)}{\xi[\ln(\frac{2\xi}{C})]} ,
\end{aligned} \tag{3. 3. 15}$$

$$\frac{\psi\Phi}{\eta} \sim \frac{H_1(\eta)}{\xi[\ln(\frac{2\xi}{C})]} ,$$

$$\psi_\eta \Phi \sim \frac{H_1(\eta)}{\xi[\ln(\frac{2\xi}{C})]} ,$$

$$\frac{\xi\psi_\xi \Phi}{\eta} \sim \frac{\eta^{-1} G_1(\eta) H_1(\eta)}{\xi[\ln(\frac{2\xi}{C})]^3} , \qquad (3.3.15 \text{ cont.})$$

$$\xi\Phi \sim \frac{H_1(\eta)}{[\ln(\frac{2\xi}{C})]} ,$$

$$\xi\eta\psi_{\eta\eta} \sim \frac{G_1(\eta)}{[\ln(\frac{2\xi}{C})]} ,$$

Therefore, (3.3.8) may now be approximated by

$$-AB\xi\Phi = AB\xi\eta\psi_{\eta\eta} \qquad (3.3.16)$$

or

$$\Phi = -\eta\psi_{\eta\eta} . \qquad (3.3.17)$$

We regard (3.3.16) or (3.3.17) as the first approximation of (3.3.8) and use it to construct the approximate series solution of $\psi$.

Putting (3.3.17) into (3.3.7), we obtain

$$2\eta\psi_{\eta\eta\eta} + (2+\psi)\psi_{\eta\eta} = \xi(\psi_\eta\psi_{\eta\xi} - \psi_\xi\psi_{\eta\eta}) + A(\eta\psi_{\eta\eta\eta} + \psi_{\eta\eta}) , \qquad (3.3.18)$$

which simplifies into the following form:

$$(2-A)\eta\psi_{\eta\eta\eta} + (2-A+\psi)\psi_{\eta\eta} = \xi\,(\psi_\eta \psi_{\eta\xi} - \psi_\xi \psi_{\eta\eta}). \qquad (3.3.19)$$

Following Stewartson (1955), in order to obtain the first and crucial term of the asymptotic series for $\psi(\eta, \xi)$, we can take $\xi\psi_\xi \to 0$, $(\xi\psi_\xi)_\eta \to 0$ as $\xi \to \infty$, for each value of $\eta$. The right-hand side of (3.3.19) becomes zero in the limit $\xi \to \infty$ and the boundary conditions change to

$$\psi = \psi_\eta = 0 \quad \text{at} \quad \eta = 0 \qquad (3.3.20)_1$$

and

$$\psi_\eta \to 1 \quad \text{as} \quad \eta \to \infty . \qquad (3.3.20)_2$$

Hence, we have

$$(2-A)\eta\psi_{\eta\eta\eta} + (2-A+\psi)\psi_{\eta\eta} = 0 . \qquad (3.3.21)$$

Integrating (3.3.21) once gives

$$\eta\psi_{\eta\eta} = K(\xi)\exp\left\{-\int_0^1 \frac{\psi(\eta,\xi)}{(2-A)\eta}\,d\eta\right\}, \qquad (3.3.22)$$

where $K(\xi)$ is an arbitrary function of integration. Near $\eta = 0$, (3.3.22) becomes either

$$\psi_{\eta\eta} \sim \eta^{-1} \qquad (3.3.23)$$

or

$$\psi_{\eta\eta} = 0 , \qquad (3.3.24)$$

when $K(\xi) = 0$, except possibly at $\eta = 0$.

Since the choice (3.3.23) does not lead to a solution satisfying the boundary condition on $\psi_\eta$ at $\eta = 0$, we choose (3.3.24) as our first approximation of $\psi$. Thereby, we obtain the first approximation

$$\psi = \eta. \tag{3.3.25}$$

Note that this is simply the stream function of the undisturbed stream and satisfies (3.3.19). The boundary conditions are, however, violated since

$$\psi_\eta = 1 \neq 0 \quad \text{at} \quad \eta\xi = 1$$

and

$$\psi = \eta \neq 0 \quad \text{at} \quad \eta\xi = 1.$$

We can always improve our approximation by substituting $\psi = \eta$ back into (3.3.22) and applying the boundary condition on $\psi_\eta$ at $\eta\xi = 1$ instead of at $\eta = 0$. Then

$$\eta\psi_{\eta\eta} = K(\xi)e^{-\eta/(2-A)}. \tag{3.3.26}$$

Integrating (3.3.26) once with the boundary condition on $\psi_\eta$ as $\eta \to \infty$, we have

$$\psi_\eta = 1 - K(\xi)\int_\eta^\infty \frac{e^{-z/(2-A)}dz}{z}. \tag{3.3.27}$$

When $\eta$ is small (i.e., $\eta \ll 1$),

$$\int_{\eta}^{\infty} \frac{e^{-z/(2-A)}dz}{z} = -E_i\left(-\frac{\eta}{2-A}\right)$$

$$= -\ln\left(\frac{\eta C}{2-A}\right) + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+1)!}\left(\frac{\eta}{2-A}\right)^{n+1}, \tag{3.3.28}$$

where $\ln C = 0.577\ldots$ and is Euler's constant. Using the boundary condition $\psi_\eta$ at $\eta\xi = 1$, we can determine the unknown function $K(\xi)$

$$K(\xi) = \frac{1}{\ln\frac{(2-A)\xi}{C}} + O\left(\frac{1}{\xi \ln \xi}\right). \tag{3.3.29}$$

Hence, we have for our second approximation to $\psi_\eta$

$$\psi_\eta = 1 - \frac{1}{\ln\frac{(2-A)\xi}{C}} \int_{\eta}^{\infty} \frac{e^{-z/(2-A)}dz}{z}. \tag{3.3.30}$$

Since the correction term is very small when $\xi$ is large except near $\eta\xi = 1$, we may write $\psi$ in the form of an asymptotic expansion:

$$\psi = \eta + \sum_{s=1}^{\infty} \frac{G_s(\eta)}{\left[\ln\frac{(2-A)\xi}{C}\right]^s} + O\left(\frac{1}{\xi \ln \xi}\right). \tag{3.3.31}$$

The investigation of the properties of $G_s(\eta)$ is in order. If our expansion is asymptotic, then it must be possible to make the error in

$\psi$ as small as we please by taking $\xi$ large enough with a sufficient number of terms. This requirement can be shown to be met, because of the analogous asymptotic series expansion of (3. 3. 31) to the corresponding expansion for a Newtonian fluid.

Substituting

$$\psi(\eta, \xi) = \eta + \sum_{s=1}^{\infty} \frac{G_s(\eta)}{[\ln \frac{(2-A)\xi}{C}]^s}$$

in (3. 3. 19) and comparing coefficients of $[\ln \frac{(2-A)\xi}{C}]^{-s}$, we obtain

$$\begin{aligned}(2-A)\eta G_s''' + (2-A+\eta)G_s'' = &-(s-1)G'_{s-1} - G_{s-1}G_1'' \\ &- \sum_{t=1}^{s-2} [tG_t'G'_{s-t-1} + G_t(tG''_{s-t-1} - G''_{s-t}]\end{aligned} \tag{3. 3. 32}$$

In the determination of the $G$'s, we will take as boundary conditions

$$\begin{aligned}\psi = 0 \quad &\text{at} \quad \eta = 0\,, \\ \psi_\eta = 0 \quad &\text{at} \quad \eta\xi = 1\,, \quad \text{and} \\ \psi_\eta \to 1 \quad &\text{as} \quad \eta \to \infty \quad \text{for all} \quad \xi\,.\end{aligned} \tag{3. 3. 33}$$

The boundary condition $(3.\ 3.33)_1$ is inaccurate. In this connection, Stewartson (1955) showed that the error incurred by taking the boundary condition on $\psi$ at $\eta = 0$ is sufficiently small for large

z. This result is also valid for the micropolar case owing to the similar form of expansion of the stream function. We now prove that if

$$G_t'(\eta) = D_t \ln \frac{\eta C}{2-A} + E_t + A_t \eta (\ln \frac{\eta C}{2-A})^2 + B_t \eta \ln \frac{\eta C}{2-A} + O(\eta) \tag{3.3.34}$$

near $\eta = 0$, for $1 \leq t \leq s-1$, where A's, B's, D's, and E's are constants, then $G_s'$ has a similar form near $\eta = 0$ for all $s \geq 1$. From (3.3.34) it follows that near $\eta = 0$

$$G_t = D_t \eta \ln \frac{\eta C}{2-A} + (E_t - D_t)\eta + \frac{1}{2} A_t \eta^2 (\ln \frac{\eta C}{2-A})^2$$
$$+ \frac{1}{2}(B_t - A_t)\eta^2 \ln \frac{\eta C}{2-A} + O(\eta^2) , \tag{3.3.35}$$

since $G_t(\eta) = 0$ at $\eta = 0$. Furthermore, differentiating (3.3.34), we obtain

$$G_t'' = D_t \eta^{-1} + A_t (\ln \frac{\eta C}{2-A})^2 + (2A_t + B_t) \ln \frac{\eta C}{2-A} + O(1) . \tag{3.3.36}$$

Substituting (3.3.34), (3.3.35) and (3.3.36) into (3.3.32) yields:

$$(2-A)\eta G_s''' + (2-A+\eta) G_s'' = (2-A) A_s (\ln \frac{\eta C}{2-A})^2$$
$$+ (2-A)(B_s + 4A_s) \ln \frac{\eta C}{2-A} + O(1), \tag{3.3.37}$$

where

$$(2-A)A_s = -\sum_{t=1}^{s-2} \{tD_t D_{s-t-1}\},$$

$$(2-A)(B_s+4A_s) = -(s-1)D_{s-1} - D_1 D_{s-1}$$
$$-\sum_{t=1}^{s-2} [t(D_t E_{s-t-1} + D_{s-t-1} E_t) + (tD_t D_{s-t-1} - D_t D_{s-t})]. \tag{3.3.38}$$

Now (3.3.37) can be rearranged as

$$(\eta G_s'')' + (\frac{1}{2-A})\eta G_s'' = A_s(\ln \frac{\eta C}{2-A})^2 + (B_s+4A_s)\ln \frac{\eta C}{2} + O(1). \tag{3.3.39}$$

Integrating (3.3.39) once gives

$$\eta G_s'' = A_s \eta(\ln \frac{\eta C}{2-A})^2 + (B_s+2A_s)\eta(\ln \frac{\eta C}{2-A}) + D_s + O(\eta). \tag{3.3.40}$$

Therefore, with the help of the method of mathematical induction,

$$G_s' = D_s \ln \frac{\eta C}{2-A} + E_s + A_s \eta(\ln \frac{\eta C}{2-A})^2 + B_s \eta \ln \frac{\eta C}{2-A} + O(\eta) \tag{3.3.41}$$

near $\eta = 0$ for all $s \geq 1$.

Thus, near $\eta = 0$,

$$\psi_\eta = 1 + \sum_{s=1}^{\infty} \frac{D_s \ln \frac{\eta C}{2-A} + E_s}{[\ln \frac{(2-A)\xi}{C}]^s} + O(\frac{1}{\xi \ln \xi}). \tag{3.3.42}$$

Now, we can determine the unknown coefficients A's, B's, D's, and E's from the boundary conditions. When $\eta\xi = 1$, $\psi_\eta = 0$, so that from (3.3 42) we have

$$1 = \sum_{s=1}^{\infty} \frac{D_s}{[\ln \frac{(2-A)\xi}{C}]^{s-1}} - \sum_{s=1}^{\infty} \frac{E_s}{[\ln \frac{(2-A)\xi}{C}]^s}. \tag{3.3.43}$$

This result leads to

$$D_1 = 1 \quad \text{and} \quad D_s = E_{s-1} \quad \text{for} \quad s > 1. \tag{3.3.44}$$

Expressing A's and B's in terms of $D_t$, we obtain from (3.3.38) and (3.3.44)

$$(2-A)A_s = -\sum_{t=1}^{s-2} \{tD_t D_{s-t-1}\},$$

$$B_s + 4A_s = -\frac{2(s-1)}{2-A} D_{s-1} \tag{3.3.45}$$

$$-\frac{1}{2-A} \sum_{t=1}^{s-2} \{2(t-1)D_{s-t}D_t + tD_t D_{s-t-1}\}.$$

Since the $E$'s are determined from the condition that $G_s'(\eta)$ approaches to zero as $\eta$ goes to infinity, and $A$'s, $B$'s, $D$'s are dependent only on the $E$'s, we may determine as many of them as we please by successive substitution.

The equation governing $G_1$ is:

$$(2-A)\eta G_1''' + (2-A+\eta)G_1'' = 0. \tag{3.3.46}$$

Integrating once gives

$$G_1'' = D_1 \eta^{-1} e^{-\eta/(2-A)}. \tag{3.3.47}$$

Further integrations result in

$$G_1' = -D_1 \int_\eta^\infty \frac{e^{-z/(2-A)}dz}{z} \tag{3.3.48}$$

and

$$G_1 = -D_1 \eta \int_\eta^\infty \frac{e^{-z/(2-A)}dz}{z} - (2-A)D_1(1-e^{-\eta/(2-A)}). \tag{3.3.49}$$

From (3.3.28) and (3.3.30) it now follows that

$$D_1 = 1 \quad \text{and} \quad E_1 = 0. \tag{3.3.50}$$

Taking into account only the first dominant term of the asymptotic expansion of $\psi$, we obtain $\psi_{\eta\eta}$ in the following form:

$$\psi_{\eta\eta} = \frac{\eta^{-1}}{\ln \frac{(2-A)\xi}{C}} + O\left[\frac{1}{(\ln \frac{(2-A)\xi}{C})^2}\right]. \tag{3.3.51}$$

From (3.3.17) the microrotation function $\Phi$ is finally expressed by

$$\Phi = -\eta\psi_{\eta\eta} = \frac{-1}{\ln \frac{(2-A)\xi}{C}} + O\left(\frac{1}{(\ln \xi)^2}\right). \tag{3.3.52}$$

Now that the desired solution for the asymptotic boundary layer flow has been established, we may discuss the boundary and matching conditions on the microrotation $n$. The matching condition (3.2.14) is satisfied in the limit $r \to \infty$, as shown by

$$n = \frac{U}{r}\,\Phi = \frac{U}{r}(-\eta\psi_{\eta\eta}) = \frac{-U}{r \ln \frac{2(2\mu+\kappa)z}{\rho U a^2 C}} \to 0$$

for fixed large $z$.

The boundary condition on the microrotation at $r = a$, is:

$$n_0(z) = \left(\frac{U}{a}\right) \frac{1}{\ln \frac{2(2\mu+\kappa)z}{\rho U a^2 C}} \tag{3.3.53}$$

for large $z$.

## 3.4 Boundary Layer Thickness and Skin-Friction

The shear stress $t_{rz}$ in physical components in cylindrical coordinate system may be obtained from (2.2.9) with covariant differentiation replacing ordinary differentiation. Thus,

$$t_{rz} = (\mu+\kappa)\frac{\partial u}{\partial r} + u\frac{\partial v}{\partial z} + \kappa n . \qquad (3.4.1)$$

Using (3.2.7), we get for the shear stress in terms of boundary layer variables,

$$R^{-1/2} t_{rz} = (\mu+\kappa)\frac{\partial u}{\partial r} + \kappa n . \qquad (3.4.2)$$

At the wall of the cylinder, (3.4.2) becomes

$$R^{-1/2} t_{w} = (\mu+\kappa)\frac{\partial u}{\partial r}\Big|_{w} + \kappa n_0 , \qquad (3.4.3)$$

where the subscript $w$ denotes that the expression is evaluated at the boundary. In accordance with the expression for the classical wall shear stress, we regard $R^{-1/2} t_w$ as our micropolar wall shear stress, which we denote by $t_{w.m.}$. Hence,

$$t_{w.m.} = (\mu+\kappa)\frac{\partial u}{\partial r}\Big|_{w} + \kappa n_0 = (\mu+\kappa)\frac{\partial \eta}{\partial r}\frac{\partial^2 \psi}{\partial \eta^2}\Big|_{w} + \kappa n_0 . \qquad (3.4.4)$$

Now when $\eta$ is small,

$$t_{w.m.} = (1+\frac{1}{2}\Delta)\frac{2\mu U}{a}\left[\frac{1}{\ln\frac{(1+\frac{1}{2}\Delta)4\mu z}{\rho U a^2 C}}\right] + O\left[\frac{1}{\left(\ln\frac{(1+\frac{1}{2}\Delta)4\mu z}{\rho U a^2 C}\right)^2}\right], \tag{3.4.5}$$

where $\Delta = \frac{\kappa}{\mu} \neq 0$ and if $\Delta = 0$, (3.4.5) reduces to the case of a Newtonian fluid. The expression for the couple stress $m_{r\theta}$ may be obtained through (2.2.10), in terms of physical components,

$$m_{r\theta} = -\beta(\frac{n}{r}) + \gamma(\frac{\partial n}{\partial r}) . \tag{3.4.6}$$

In terms of boundary layer variables,

$$R^{-1} m_{r\theta} = -\beta(\frac{n}{r}) + \gamma(\frac{\partial n}{\partial r}) . \tag{3.4.7}$$

On the wall of the cylinder, (3.4.7) becomes

$$m_w = -\beta(\frac{n}{r})_w + \gamma(\frac{\partial n}{\partial r})_w . \tag{3.4.8}$$

Since $n = -\frac{U}{r}\eta\psi_{\eta\eta}$, the couple stress at the wall is expressed by

$$m_w = +[\frac{\beta U}{r^2}\eta\psi_{\eta\eta}]_w + [\frac{\gamma U}{r^2}\eta\psi_{\eta\eta}]_w$$

$$= (\beta+\gamma)\frac{U}{a^2}\left[\frac{1}{\ln\frac{(1+\frac{1}{2}\Delta)4\mu z}{\rho U a^2 C}}\right] + O\left[\frac{1}{\left(\ln\frac{(1+\frac{1}{2}\Delta)4\mu z}{\rho U a^2 C}\right)^2}\right]. \tag{3.4.9}$$

Now, we may calculate the effective wall shear stress $t_{eff.}$ defined by

$$t_{eff.} = [-t_{rz} + \frac{m_{r\theta}}{r}]_{r=a} . \tag{3.4.10}$$

That is,

$$t_{eff.} = \frac{2\mu U}{a}(-1 - \frac{1}{2}\Delta+\beta^*+\gamma^*)\left[\frac{1}{\ln \frac{(1+\frac{1}{2}\Delta)4\mu z}{\rho U a^2 C}}\right] + O\left[\frac{1}{\left(\ln \frac{(1+\frac{1}{2}\Delta)4\mu z}{\rho U a^2 C}\right)^2}\right], \tag{3.4.11}$$

where

$$\beta^* = \frac{\beta}{2\mu a^2} \geq 0 \quad \text{and} \quad \gamma^* = \frac{\gamma}{2\mu a^2} \geq 0 .$$

The effective wall shear given by (3.4.11) is found to be reduced in the micropolar case compared to the Newtonian case if the material coefficients satisfy the following condition:

$$\frac{1}{2}\Delta - \beta^* - \gamma^* \leq 0. \tag{3.4.12}$$

The micropolar skin-friction coefficient denoted by $C_{f.m.}$ is

$$C_{f.m.} = \frac{t_{eff.}}{\frac{1}{2}\rho U^2} = \frac{4\nu}{Ua} \frac{(-1-\frac{1}{2}\Delta+\beta^*+\gamma^*)}{\ln \frac{(1+\frac{1}{2}\Delta)4\nu z}{Ua^2 C}} . \tag{3.4.13}$$

We may now form the ratio of the micropolar skin-friction coefficient to the Newtonian skin-friction coefficient, $C_{f.n.}$, which is obtained by setting $\Delta = \beta = \gamma = 0$.

$$\frac{C_{f.m.}}{C_{f.n.}} = (1+\frac{1}{2}\Delta-\beta^*-\gamma^*) \frac{\ln(\frac{4\nu z}{Ua^2 C})}{\ln \frac{(-1+\frac{1}{2}\Delta)4\nu z}{Ua^2 C}} . \tag{3.4.14}$$

In view of (3.4.12),

$$1 + \frac{1}{2}\Delta - \beta^* - \gamma^* \leq 1 . \tag{3.4.15}$$

Hence, the ratio

$$C_f = \frac{C_{f.m.}}{C_{f.n.}} \leq 1. \tag{3.4.16}$$

This confirms Eringen's conjecture.

Next, we obtain the expression for the micropolar velocity profile at large distances from the cylinder when $z$ is large. Since the axial component of the velocity, $u$, is the first derivative of the stream function $\psi$ with respect to $\eta$ multiplied by the uniform mainstream velocity U, we find that

$$\frac{u}{U} \simeq 1 - \frac{(1+\frac{1}{2}\Delta)}{\bar{r}^2} \frac{e^{-(\frac{2\bar{r}^2}{2+\Delta})}}{\ln(1+\frac{1}{2}\Delta)\bar{z}}, \tag{3.4.17}$$

where

$$\bar{r} = \frac{r}{2}\left(\frac{U}{\nu z}\right)^{1/2},$$

and

$$\bar{z} = \frac{4\nu z}{Ua^2 C}.$$

Finally, we estimate the boundary layer thickness on the cylinder. In the boundary layer, the radial coordinate $r$ can be written as $r = a + r'$, where $a$ is the radius of the cylinder and $r'$ is less than or equal to the boundary layer thickness $\delta$. The partial derivative operator $\frac{\partial}{\partial r}$ can now be replaced by $\frac{\partial}{\partial r'}$, since $\frac{\partial}{\partial r} = \frac{\partial}{\partial r'}\frac{\partial r'}{\partial r} = \frac{\partial}{\partial r'}$. Then, (3.2.10) can be rewritten in the form

$$v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + (\nu + \frac{\kappa}{\rho})(\frac{\partial^2 u}{\partial r^2} + \frac{1}{a+r'}\frac{\partial u}{\partial r'}) + \frac{\kappa}{\rho}(\frac{n}{a+r'} + \frac{\partial n}{\partial r'}). \tag{3.4.18}$$

From (3.4.17), we get the following relationship:

$$u \ln(1+\frac{1}{2}\Delta)\bar{z} \approx U\ln(1+\frac{1}{2}\Delta)\bar{z} - \frac{(1+\frac{1}{2}\Delta)}{\bar{r}^2} e^{-2\bar{r}^2/2+\Delta}. \tag{3.4.19}$$

Note that the last term of (3.4.19) depends entirely on $r$ and it will approach zero in the asymptotic sense as $r \to \infty$ (i.e., after passing through the outer edge of the boundary layer). This suggests that the main contribution to the longitudinal component of the velocity comes from the first term, namely $U \ln(1+\frac{1}{2}\Delta)\bar{z}$ in (3.4.19) in the asymptotic sense as $r \to \infty$. Thus, the potential flow $U$ is modified by a factor $\ln(1+\frac{1}{2}\Delta)\bar{z}$, due to the presence of the cylinder. This concept fixes the outer edge of the boundary layer, allowing for estimating the boundary layer thickness $\delta$. We now recall the Prandtl's assumption that the viscous forces are of the same order of magnitude as the inertial forces in the boundary layer. Making use of this assumption, (3.4.18) yields the following estimate for $\delta$:

$$u \frac{\partial u}{\partial z} \sim (\nu + \frac{\kappa}{\rho}) \frac{u}{r'^2}$$

or (3.4.20)

$$\frac{1}{2} \frac{\partial}{\partial z} (u^2) \sim (\nu + \frac{\kappa}{\rho}) \frac{u}{r'^2} .$$

Hence,

$$\frac{[U \ln(1+\frac{1}{2}\Delta)\bar{z}]^2}{z} \sim \frac{(\nu+\frac{\kappa}{\rho})[U \ln(1+\frac{1}{2}\Delta)\bar{z}]}{\delta^2} . \tag{3.4.21}$$

Solving for $\delta$ gives

$$\delta \sim \left[\frac{(1+\Delta)\nu z}{U \ln(1+\frac{1}{2}\Delta)\bar{z}}\right]^{1/2} = \left[\frac{(1+\Delta)\nu z}{U \ln \dfrac{(1+\frac{1}{2}\Delta)4\nu z}{Ua^2 C}}\right]^{1/2} . \tag{3.4.22}$$

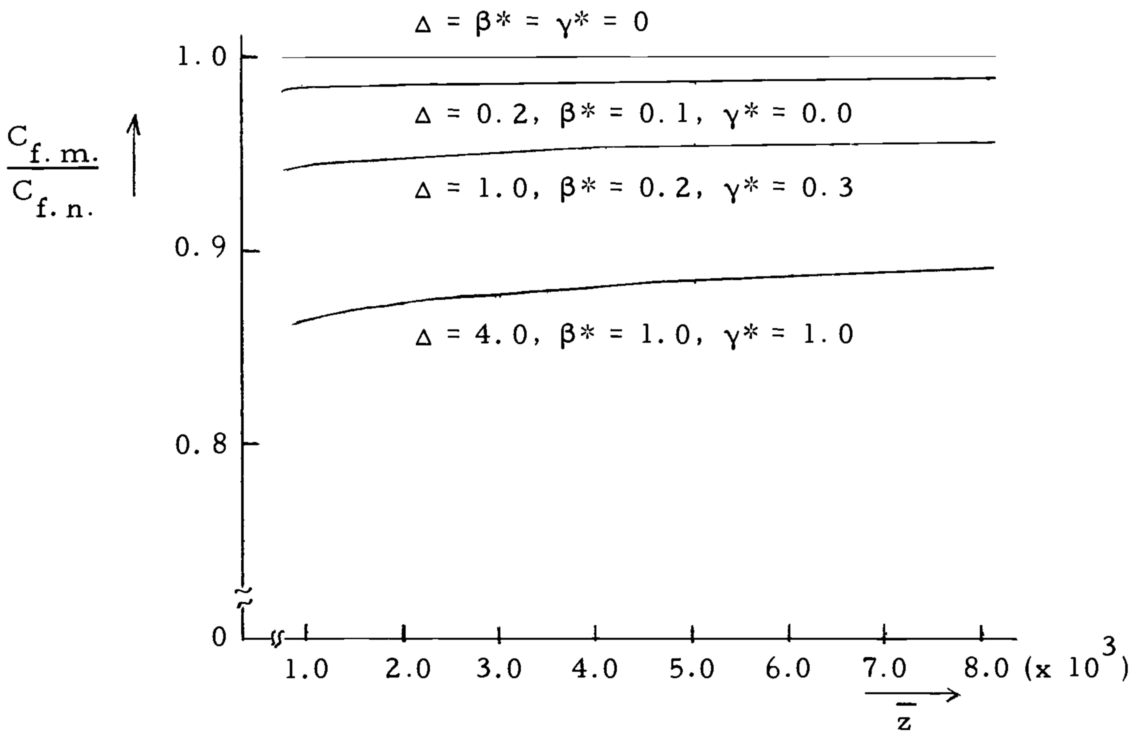


Figure 3.4.1. The ratio of the micropolar to Newtonian effective wall shear stresses at various local points $\bar{z}$.

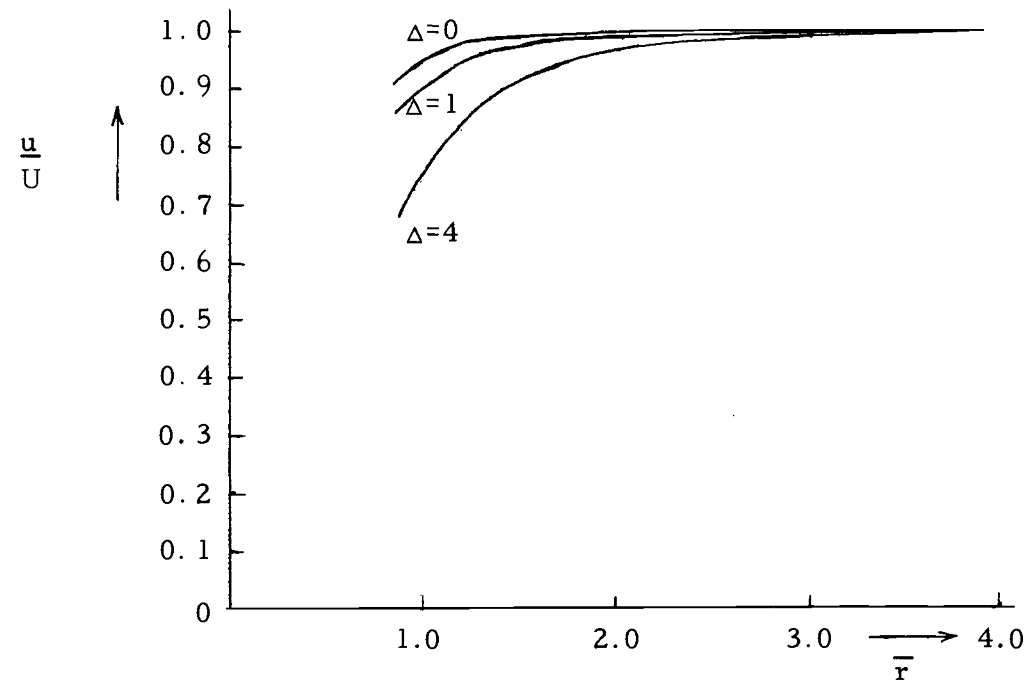


Figure 3.4.2. The velocity profile.

# 4. DISCUSSION OF THE RESULTS AND SCOPE OF FURTHER WORK

## 4.1 Discussion of the Results

Our present analysis has shown that if the micropolar material coefficients $\Delta$, $\beta$, $\gamma$ are restricted by the upper bound relation (3.4.12), then there occurs drag reduction at the wall of the cylinder in an axial incompressible micropolar fluid flow in relation to the case of a Newtonian fluid. This result is clearly due to the presence of the couple stress which counteracts the action of the shear stress, thereby reducing the drag near the wall of the cylinder (see Figure 3.4.1). Therefore, with the restriction (3.4.12) on $\kappa$, $\beta$, $\gamma$, the longstanding famous conjecture of Eringen (1966) is confirmed that the theory of micropolar fluids may have a mechanism capable of explaining drag reduction near a rigid body. The analysis of the results also reveals that the micropolar velocity profile is flattened in comparison to the Newtonian counterpart, as shown in Figure 3.4.2. The micropolar boundary layer thickness is shown to be larger than the Newtonian one. The increase in the boundary layer thickness is attributed to the presence of the micropolar viscosity coefficient $\kappa$.

## 4.2 Scope of Further Work

The present study of the asymptotic boundary layer on a circular cylinder in axial incompressible micropolar fluid flow has been limited only to the case dealing with a uniform mainstream velocity, that is, $m = 0$, due to the complex nature of the equations of the boundary layer involving the stream function $\psi$ and the microrotation function $\Phi$. When $m = 0$, there is no pressure gradient and therefore the phenomenon of boundary-layer separation does not arise. However, it should be definitely more interesting to examine the case where the mainstream velocity is proportional to $z^m$ with $-1/2 < m < 1$, from the practical consideration that an exact knowledge of the trailing edge phenomena on a body of revolution in general is yet to be gained. Therefore, a further work of the present investigation should be carried out in a more general setting with $m$ not necessarily equal to zero so that the more important problem of boundary-layer separation, whose phenomenon is always associated with the formation of vortices and with large energy losses in the wake of the body, may be treated for a better understanding of the trailing edge phenomena under consideration.

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