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Decay rates to equilibrium for nonlinear plate equations with degenerate, geometrically-constrained damping

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Abstract

We analyze the convergence to equilibrium of solutions to the nonlinear Berger plate evolution equation in the presence of localized interior damping (also referred to as *geometrically constrained damping*). Utilizing the results in [24], we have that any trajectory converges to the set of stationary points $\mathcal{N}$. Employing standard assumptions from the theory of nonlinear unstable dynamics on the set $\mathcal{N}$, we obtain the rate of convergence to an equilibrium. The critical issue in the proof of convergence to equilibria is a unique continuation property (which we prove for the Berger equation) that provides a gradient structure for the dynamics. We also consider the more involved von Karman evolution, and show that the same results hold *assuming a unique continuation property* for solutions, which is presently a challenging open problem.

Keywords: nonlinear plate equations, attractors, geometrically constrained damping, unique continuation, convergence to equilibrium

AMS Mathematics Subject Classification 2010: 37L30, 74K20, 35Q74, 35B40

1 Introduction

1.1 PDE Model

The Berger and von Karman plate equations are well-known in nonlinear elasticity, and each constitutes a basic model describing the nonlinear oscillations of a thin plate with large transverse displacements [37] (and references therein). In this treatment we study models of a dynamic nonlinear plate with negligible thickness (as is common in the modeling of thin structures [37]). We consider the presence of nonlinear, geometrically constrained interior damping (localized damping), whose support lies in a boundary layer or collar. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega = \Gamma$ that is sufficiently smooth. The problem is described below:

$$u_{tt} + \Delta^2 u + d(\mathbf{x})g(u_t) = f(u) + p \quad \text{in} \quad \Omega \times (0,\infty) \equiv Q \tag{1.1}$$
$$u\big|_{t=0} = u_0, \quad u_t\big|_{t=0} = u_1.$$

taken with clamped boundary conditions :

$$u = \partial_\nu u = 0 \quad \text{in} \quad \Gamma \times (0, \infty) \equiv \Sigma. \tag{1.2}$$

Remark 1.1. In the body of this treatment we consider clamped boundary conditions for the sake of exposition. The analysis in [24] for the von Karman plate accommodates all standard plate boundary conditions [37]—clamped, hinged, and free. The analysis below can be carried out mutatis mutandis for hinged boundary conditions, however, we want to clarify that in the analysis of the Berger evolution, free boundary conditions present a novel issue which is not present for von Karman plates. We discuss this further in Section 7.

The external pressure $p \in L_2(\Omega)$ plays an essential role in shaping nontrivial stationary solutions to the above equation. The localizing function $d(\mathbf{x}) \in L_\infty(\Omega)$ is nonnegative almost everywhere in Ω, and $d(\mathbf{x}) \geq c_0 > 0$ for $\omega \subset \text{supp } d$, where ω is an open collar of the boundary Γ.

Remark 1.2. We pause here to mention that other studies have investigated the use of a *partial collar* for localized interior damping. For instance, in the case of the wave equation, see the recent paper [20] which presents results on global attractors for the critical semilinear wave equation with such damping. Such a result would represent an improvement of the results presented herein, however, we will critically make use of a unique continuation result from [34] which will require our damping region ω to contain the full boundary.

The damping function $g(\cdot) \in C^1(\mathbb{R})$ is a monotone increasing function with $g(0) = 0$.

Remark 1.3. Note that for the general analysis (local attractors, their dimensionality, and regularity) *we need only take* $g \in C(\mathbb{R})$. The additional assumption that $g \in C^1(\mathbb{R})$ is needed in analyzing unique continuation properties of the nonlinear evolution (described below).

We consider the principal "physical" nonlinearities associated with the theory of large deflections for plates: Berger [5, 16] and von Karman [18] (and references therein). To minimize assumptions on $f(u)$, we restrict a bulk of our attention to the Berger nonlinearity. In Section 6 we will discuss the more involved von Karman equations, and extend our results for the Berger evolution to the von Karman evolution under an additional assumption.

The Berger nonlinearity is given by

$$f_B(u) = \left[\kappa ||\nabla u||^2 - \Upsilon\right] \Delta u, \tag{1.3}$$

where $\kappa > 0$ is a physical parameter and $\Upsilon \geq 0$ represents the strength of in-plane loading.This nonlinearity was suggested in [5] as a simplification of the von Karman equations. In the context of long-time behavior for plate evolution equations, the Berger nonlinearity was considered in [12, 16]. It also falls under the considerations of abstract second order (plate-like) equations, as presented in [18, Chapters 2 and 8]; in this reference, results on well-posedness and aspects of long-time behavior are unified for many nonlinear evolutions (including the von Karman and Berger equations).

1.2 Previous Considerations

The investigation of the long-time dynamics of partial differential equations has attracted attention for many years. In the case of unstable dynamics, the existence of a compact, and possibly smooth, finite dimensional attracting set for the dynamics generated by (1.1) is of great recent interest [18] (and references thererin). Studies of hyperbolic-like (wave or plate) dynamics with dissipation have been a topic of substantial activity for more than 30 years [6, 16, 17, 15, 19, 20, 32, 33]. While the existence of attractors is commonplace for dynamical systems that exhibit inherent smoothing effects (e.g. parabolic-like) [45, 8, 21, 41, 2, 39, 30], it is a much less evident phenomena for hyperbolic-like dynamics.

The existence and properties of compact global attractors for energy-type (weak) solutions to (1.1) with all standard boundary conditions, taken with the von Karman nonlinearity have been studied closely in the past 20 years. In particular, the existence and properties of compact global attractors was fleshed out in the case of nonlinear, monotone, fully supported interior damping without any restriction on the "size" of the damping. For a concise discussion of these results (with pertinent references), please consult [24]; for a complete exposition, see the recent monograph [18], which contains an historical account of the results for von Karman evolutions.

In contrast to fully supported damping, localized interior dissipation also arises in the control and long-time behavior of PDEs [33, 20, 7, 31, 47]. The implementation of such damping, for a general localization, constitutes a physically motivated attempt to obtain controllability and stability results for damping active "small" subsets of the domain. To a certain extent, localized interior dissipation represents an intermediate point between boundary dissipation and fully supported interior dissipation; localized dissipation is robust with respect to boundary conditions, while also *not requiring active damping on the full interior* of the domain (which can be physically impractical or mathematically unnecessary). However, obtaining results for localized dissipation can be more demanding than for full interior damping, i.e. $d(\mathbf{x}) \ge c_0 > 0$ for all x in Ω, since the associated energy methods require geometric techniques to control the energy of the plate.

With reference to equation (1.1), taken with the von Karman nonlinearity, the authors have recently provided results [24] on the existence of compact attractors of finite dimension, with regularity greater than that of the energy space. Specifically, these results hold for (1.1) with *all types of standard plate boundary conditions*, with *geometrically constrained dissipation* in an open collar of the boundary which may be nonlinear of *any polynomial growth* at infinity, and with *no restriction on the size* of the damping.

Additionally, the methods presented in [24] extend in straightforward way utilizing the Berger nonlinearity in place of the von Karman when clamped or hinged boundary conditions are taken for the plate. This follows from the fact that, abstractly, the Berger nonlinearity satisfies the same key estimates as the von Karman nonlinearity (i.e. local Lipschitz bound on the state space, with an appropriate bound on the unsigned portion of the potential energy). In the case of free boundary conditions, the results of [24] do not carry over from the von Karman evolution to the Berger (we elaborate on this in Section 7).

In [24], the ability to show existence and finite dimensionality of the compact attractor hinges upon a class of "observability" type energy estimates, which in turn depend on a novel method of localization of multipliers that allow for smooth propagation of the damping from a boundary collar into the interior. (For von Karman evolutions, this propagation was shown even in the presence of *free boundary conditions*, which do not comply with the Lopatinski conditions [42].) Two specific estimates are needed in the analysis in [24] (distinguished by the quadratic or linear nature of the lower order terms on the RHS), and in our analysis below (Lemma 3.3 and Lemma 5.2), we critically utilize estimates which are analogous to those in [24].

In many cases it is of interest to have *additional* knowledge of decay rates to equilibrium (in the context of hyperbolic-like dynamics). Showing convergence to equilibria under appropriate assumptions yields that the dynamics ultimately stabilize to a stationary state; this stands as a refinement to showing only the existence of an attractor for the dynamics. Indeed, for many dynamical systems trajectories stabilize to the attractor, but exhibit nontrivial behavior in the limit (e.g. periodic or chaotic behavior). For this reason, when possible, showing convergence to equilibria is a stronger result which demonstrates that the end behavior is *static*. In the case of plates, this indicates convergence to a *static deformation*.

Many authors have studied decay rates for the wave equation with dissipation. The exponential attraction property for the nonlinear wave equation with linear, interior damping was obtained in [3] (see [36] for more recent discussion). For wave equations with nonlinear boundary damping, an analysis was performed in [14]. The rates of stabilization to an equilibrium for von Karman equations with purely internal, nonlinear damping, *requiring sufficiently* large damping were derived in [17]. Later, this was done in [19], where the effects of boundary dissipation were studied by removing the condition imposed in [17]—namely, a necessarily large damping parameter. Again, appealing to the abstract setup in [18], these results also hold for Berger evolutions.

We conclude this section by mentioning that many of the techniques (e.g. the multiplier analysis) utilized here are adapted from boundary control techniques for the nonlinear wave equation and von Karman equation. For the nonlinear wave equation with boundary damping, there is vast literature; see [38, Chapters 2.5 and 3.3] and references therein for a review. The recent monograph [18] and references therein provide a rather complete account of dissipation in the von Karman plate model acting through hinged or free boundary conditions. However, for the Berger plate model under consideration in this treatment, the utilization of boundary damping represents, to the knowledge of the authors, a novel avenue of study. A recent paper [48] addresses the well-posedness and exponential decay of solutions for a generalized Berger beam model with static and dynamic damping present on the free end of the beam. At present, it is unclear whether these methods are adaptable to the Berger plate. The primary issue occurs for any plate model with inhomogeneous boundary terms which manifest themselves in energy balance equation and must be accommodated. See Section 7 for more discussion. Thus, at present, it appears that the results and techniques presented herein are not directly adaptable to the case of boundary damping.

1.3 Goals and New Challenges Encountered

In this paper, we provide novel results on nonlinear plate models with localized interior damping. The main goal is to produce decay rates to equilibrium for solutions to equations as described in Theorem 4.1 below. This is, to the knowledge of the authors, the first such result obtained with localized damping, although the techniques are motivated by studies of nonlinear boundary dissipation. Showing uniform convergence to an equilibrium in the case of *multiple equilibria* in PDE systems is rather subtle. This is true even in the case of *linear, fully-supported* damping, in the presence of a nonlinearity which is merely *subcritical*. The problem at hand is the inherent instability near the equilibria, where a trajectory entering an arbitrarily small neighborhood of an equilibrium may have a "change of heart" and begin converging to another point in the attractor. Thus, it is essential to control the behavior in such a neighborhood via *hard estimates*.

The primary challenge in setting up the problem is demonstrating that the dynamics of the Berger evolution possess a gradient structure; this involves showing a so called *unique continuation result* for Berger's plate equation. (In the case of von Karman plate, this remains an open problem.)

We emphasize that the approach and methods herein are (requisitely) non-standard, as we consider geometrically constrained damping. First, we work only with energy level—*weak*—solutions and *recover additional regularity* of these solutions when necessary. In order to obtain the estimates needed for the above results, we must appeal to energy methods which require the localization technique developed in [24]. Additionally, we make no assumptions on the *degeneracy of the nonlinear damping function g at the origin* and present the decay rate itself as a function of the damping.

Lastly, we note that much of the challenge in analyzing the convergence to equilibria for these dynamics lies in the technical difficulties associated with: (a) nonlinear and geometrically constrained damping, and (b) long-time behavior of hyperbolic-like dynamics (with infinite dimensional instability—spectra on the imaginary axis).

Hence, the main novelties of this treatment are:

(1) Unique continuation for Berger's plate model. We prove that a critical unique continuation property is satisfied in the case of $f = f_B$, which to our knowledge is a result which should be useful for a wide class of problems in nonlinear elasticity. For this, we utilize a result in [34] which shows that a related unique continuation property is satisfied for regular solutions when we assume that ω is a neighborhood of the entire boundary Γ. In order to apply this result, our starting point is to show that the solutions to (1.1)-(1.2) are more regular than the state space. This will critically depend on the recent "observability" type inequality mentioned above.

(2) Decay rates to equilibrium. We show that the trajectories corresponding to (1.1) converge to equilibrium, and we moreover derive the rate of convergence of solutions to equilibria points *as a function of the degeneracy of the damping at the origin.*

Remark 1.4. We note that this second result is interesting from the control-theoretic point of view. Indeed, if uniform decay rates to equilibria are established for individual solutions, then 'local' controllability theory can be employed in order to construct boundary controls which steer individual trajectories to desired states.

1.4 Notation and Conventions

In this paper, we denote $H^s(D)$ as the Sobolev space of order $s \in \mathbb{R}$ on domain D, the norm (and corresponding inner product) in $H^s(D)$ as $||\cdot||_s$ and $||\cdot||_0 \equiv ||\cdot||_{L_2(D)}$, and for simplicity norms and inner products written without subscript are taken to be $L_2(D)$ of the appropriate domain D. Additionally, we employ the notation that $H^s_0(D)$ gives the closure of $C_0^\infty(D)$ in the $||\cdot||_s$ norm. For the body of this treatment, we take

$$f(u) = f_B(u) = [||\nabla u||^2 - \Upsilon]\Delta u,$$

where $\Upsilon \geq 0$ and we have taken $\kappa = 1$ without loss of generality. In Section 6, we provide a discussion of the von Karman evolution equation.

For the remainder of the treatment, we take the localizing function to be $d(\mathbf{x}) \in L_\infty(\Omega)$ is non-negative almost everywhere in Ω, and $d(\mathbf{x}) \geq c_0 > 0$ for $\omega \subset \text{supp } d$, where ω is an open collar of the boundary Γ.

2 Preliminaries

We begin by recalling the energies associated to equation (1.1), which are given by

$$E(t) = \frac{1}{2}\big(||\Delta u||^2 + ||u_t||^2\big), \quad \mathcal{E}(t) = E(t) + \Pi(u),$$

with

$$\Pi(u) = \frac{1}{4}\Big(||\nabla u||^4 - 2\Upsilon||\nabla u||^2 - 4\int_\Omega pu\Big). \tag{2.1}$$

The above (linear) energy $E(t)$ dictates the physical *finite energy* space or so called *state space* for our analysis $\mathcal{H} \equiv H_0^2(\Omega) \times L_2(\Omega)$.

The following general estimates hold for the Berger nonlinearity (the corresponding estimates for the von Karman equation will be discussed in Section 6): For $f : H_0^2(\Omega) \to L_2(\Omega)$ given by $f(u) = (||\nabla u||^2 - \Upsilon)\Delta u$, we have

$$||f(u) - f(w)||_{-\delta} \le C(1 + ||u||_1^2 + ||w||_2^2)||u - w||_{2-\delta} \tag{2.2}$$

$$\Big|\frac{\Upsilon}{2}||\nabla u||^2 + (p, u)\Big| \le \epsilon\big(||\Delta u||^2 + \frac{1}{2}||\nabla u||^4\big) + C_\epsilon, \tag{2.3}$$

for all $\delta \in [0, 1/2)$ and for $\epsilon > 0$. The inequality in (2.2) follows from (taking $U = ||\nabla u||$ and $W = ||\nabla w||$):

$$\begin{aligned} ||f(u) - f(w)|| &\le ||U^2\Delta u - W^2\Delta w|| + \Upsilon||\Delta(u - w)|| \\ &= ||U^2\Delta u - U^2\Delta w + U^2\Delta w - W^2\Delta w|| + C||u - w||_2 \\ &\le C\big\{(1 + U^2)||u - w||_2 + (U^2 - W^2)||w||_2\big\} \\ &\le C\big\{1 + U^2)||u - w||_2 + (U + W)||w||_2\Big|||u||_1 - ||w||_1\Big|\big\} \\ &\le C(1 + ||u||_1^2 + ||w||_2^2)||u - w||_2, \end{aligned} \tag{2.4}$$

where in the final lines we utilize the continuous embedding $H^2 \hookrightarrow H^1$, the equivalence $||\nabla \cdot|| \sim ||\cdot||_1$, and Young's inequality. The proof of (2.3) follows directly from the superlinearity of $\Pi(u)$ [18, Section 1.5.1].

The issue of well-posedness for weak (finite-energy) solutions to nonlinear plate equations is recent [18, 23]. Difficulties arise concerning uniqueness of weak solutions, owing to the criticality and nonlocality of both the Berger and von Karman plate equations. In this paper, we focus on clamped boundary conditions (though we discuss free boundary conditions in Section 7), and we will be considering *generalized* nonlinear semigroup solutions [4, 43] (which are also *weak*—variational—solutions in this setup). For a detailed and complete discussion regarding the well-posedness and regularity of von Karman and Berger solutions the reader is referred to [18, 35]. Now, let us recall the main well-posedness result [16, 12, 18]:

Theorem 2.1. *With reference to problem (1.1)-(1.2) with initial data* $(u_0, u_1) \in \mathcal{H}$*, there exists a* *unique global solution of finite-energy (i.e. $(u, u_t) \in C([0,T]; \mathcal{H})$ for any $T > 0$), where (u, u_t) depends continuously on $(u_0, u_1) \in \mathcal{H}$.*

Thus, for any initial data in the finite energy space $(u_0, u_1) \in \mathcal{H}$, there exists a well-defined semiflow $S_t(u_0, u_1) \equiv \big(u(t), u_t(t)\big) \in \mathcal{H}$ which varies continuously with respect to the initial data in $\mathcal{H}$. At this point, it will be convenient to introduce an elliptic operator A: $\mathcal{D}(A) \subset L_2(\Omega) \to L_2(\Omega)$ given by $Au = \Delta^2 u$, where $\mathcal{D}(A)$ incorporates the clamped boundary conditions. By elliptic regularity $\mathcal{D}(A^{1/2}) \equiv H_0^2(\Omega)$. The corresponding generator of S_t is given by $\mathcal{A}(u,v) \equiv \Big(v, -Au - d(\mathbf{x})g(v) + f(u) + p\Big)$ with $\mathcal{D}(\mathcal{A}) = \{(u,v) \in \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}); Au + d(\mathbf{x})g(v) \in L_2(\Omega)\}$. For initial data taken in $\mathcal{D}(\mathcal{A})$, the corresponding solutions are regular and remain invariant in $\mathcal{D}(\mathcal{A})$ [4, 40, 43]. With an additional assumption that $g(s)$ is bounded polynomially at infinity, one has $\mathcal{D}(\mathcal{A}) \subset H^4(\Omega) \times H^2(\Omega)$ [18]. Equipped with the regularity of the domain $\mathcal{D}(\mathcal{A})$, one derives the energy identity for all regular solutions. Due to the density of the embedding $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}$ and the monotonicity of the damping the same energy equality remains valid for all generalized solutions. Thus the energy identity is satisfied for all strong and generalized (semigroup) solutions (complete details of this argument are given in [18]):

$$\mathcal{E}(t) + \int_s^t \int_\Omega d(\mathbf{x})g(u_t)u_t = \mathcal{E}(s) \tag{2.5}$$

for all $0 < s < t$.

Remark 2.1. Note that the above energy identity and the bound in (2.3) imply that generalized solutions are bounded in time in the topology of $\mathcal{H}$, i.e.

$$\sup_{[0,+\infty)} E(t) \le C < +\infty. \tag{2.6}$$

For the reader's convenience we will summarize the main results of [24], which pertain to the long-time behavior of the solutions to (1.1) taken with $f = f_V$. We reiterate that in what follows, localizing function $d(\mathbf{x}) \in L_\infty(\Omega)$ is nonnegative almost everywhere in Ω, and $d(\mathbf{x}) \ge c_0 > 0$ for $\omega \subset \text{supp } d$, where ω is any open collar of the boundary Γ. These results are also valid for Berger's nonlinearity $f = f_B$ in a straightforward way, owing to the development of the method in [24] and the estimate (2.2).

The damping is determined by the functions $d(\mathbf{x})$ and $g(s)$. The function g is $C^1(\mathbb{R})$ and, we now recall the *asymptotic* growth condition *from below* which will be imposed on $g(s)$. Such a condition is typical [37], and needed to obtain uniform decay rates of solutions in hyperbolic-like dynamics; specifically, it allows control of the kinetic energy for large frequencies:

Assumption 1. *There exist positive constants $0 < m \le M < \infty$ and a constant $p \ge 1$ such that*

$$m \le g'(s) \le M|s|^p, \quad |s| \ge 1$$

Theorem 2.2. *([24]) Suppose Assumption 1 is in force. Let $d(\mathbf{x}) \ge c_0 > 0$ in ω, where $\omega \subset$ supp d is any full collar of the boundary Γ. Then for all generalized solutions corresponding to initial data $||(u_0, u_1)||_{\mathcal{H}} \le R$, there exist compact attractor $\mathbf{A}_R \in \mathcal{H}$ for the dynamics. This is to say that for* *any R sufficiently large, the dynamical system* (W_R, S_t) *(with* $W_R = \{y \in \mathcal{H} : \mathcal{E}(y) \le R\}$*) admits a compact attractor* $\mathbf{A}_R$.

Remark 2.2. The above theorem can be rephrased by asserting that the dynamical system (W, S_t) possesses *local attractors* (with respect to the function $\mathcal{E}(\cdot)$).

The next theorem hinges upon a standard assumption in the theory of attractors which produces finite dimensionality and additional regularity for the attractor.

Theorem 2.3. *([24]) In addition to Assumption 1, assume that there exists* $m, M > 0$*, and* $\gamma < 1$ *such that* $0 < m \le g'(s) \le M[1 + sg(s)]^\gamma$*, for all* $s \in \mathbb{R}$*. Then,*

(a) the attractor $\mathbf{A}_R$ *is regular, which is to say* $\mathbf{A}_R \subset H^4(\Omega) \times H^2(\Omega)$ *is a bounded set in that topology, and*

(b) the fractal dimension of $\mathbf{A}_R$ *is finite.*

Remark 2.3. The additional assumption in the above theorem is satisfied for polynomial-type and certain exponential behavior at infinity (see [18, Remark 9.4.5, pp. 490–491]).

The existence of a *global* attractor $\mathbf{A}$—or more precisely, the independence of $\mathbf{A}_R$ on R, for R sufficiently large—follows from the *strictness* of the Lyapunov functional associated to the dynamics. In the present case, the Lyapunov functional is the nonlinear energy $\mathcal{E}(u, u_t) = \mathcal{E}(t)$, and its strictness reduces to the following unique continuation condition, denoted UC:

Definition 1 (UC Property)**.** *We say that the system satisfies the UC property iff the following implication is valid for any weak solution* (u, u_t) *to (1.1): there exists* $T > 0$ *such that*

$$u_t = 0 \ a.e \ \ in \ supp \ d \times (0, T) \Rightarrow \ u_t = 0 \ a.e \ in \ \Omega \times (0, T).$$

It is clear that the UC property holds if $d(\mathbf{x}) > 0$ a.e. in Ω. The validity of the UC property allows us to show that the dynamical system under consideration has a *strict* Lyapunov function, which yields a *gradient structure* for the dynamics. (See [18, Chapter 7] for a general discussion, and [24] for a more detailed discussion in the present case.) For dynamics with gradient structure all local results become global.

Theorem 2.4. *([24]) Assume that the UC property holds for weak solutions to* (1.1). *Then the attractor is global, i.e* $\mathbf{A}_R = \mathbf{A}$ *for some* $R > 0$*. Moreover, all the results of Theorem 2.2 and Theorem 2.3 apply to the global attractor* $\mathbf{A}$.

The need for the *UC assumption* above is due to current difficulties surrounding unique continuation results for the von Karman plate ($f = f_V$). While there is a rich theory of unique continuation results pertaining to various PDE dynamics [29, 44, 46] these results (proved by Carleman's estimates) require intrinsically that the operators involved are *local* in nature. This is not the case with von Karman evolutions, which involve an elliptic solver (the *Airy stress function*), and are thus nonlocal.

However, this property is satisfied for generalized (and hence weak) solutions to (1.1)-(1.2) for $f = f_B$ (see Section 3). Showing the UC property constitutes the first major step in the proof of the main results, which we now state:

Theorem 2.5 (Main Result 1)**.** *Consider* (1.1) *with* $p \in L_2(\Omega)$ *and clamped boundary conditions, and take* $f = f_B$*. Also, let* $d \in L_\infty(\Omega)$ *with* $d(\mathbf{x})$ *nonnegative a.e.* Ω*, and* $d(\mathbf{x}) \ge c_0 > 0$ *on* ω*. Take*

$g \in C^1(\mathbb{R})$ and let Assumption 1 be in force. Then the attractor $\mathbf{A}_R$ in Theorem 2.2 and Theorem 2.3 is global (in the sense of Theorem 2.4 above).

Theorem 2.6 (Main Result 2)**.** *Assume that, in addition to the hypotheses above, the following hold: i) There exist positive constants m, M, s_0 and $\gamma < 1$ such that*

$$g'(s) > m, \quad |s| > s_0 \quad \text{and} \quad g'(s) \leq M[1 + sg(s)]^\gamma, \quad for \;\; all \quad s \in \mathbb{R}.$$

ii) The set of equilibria is finite and hyperbolic.
Then, for any $U_0 = (u_0, u_1) \in \mathcal{H}$ there exists $e \in \mathcal{N}$ such that the following decay rate holds for the trajectory $S_t U_0$:

$$||S_t U_0 - e||_{\mathcal{H}} \leq C\sigma(t), \quad t > T_0,$$

where σ satisfies the equation:

$$\sigma_t + Q(\sigma) = 0, \;\; \sigma(0) = \sigma_0 = C(e, U_0).$$

The function $Q(s) \sim h^{-1}(s)$, where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, concave and monotone increasing function such that:
(a) $h(0) = 0$ and
(b) $s^2 + [g(s)]^2 \leq h(g(s)s)$ for $|s| \leq 1$.

Remark 2.4. Note that we have not specified the degree of degeneracy of the damping at the origin; rather, in line with [18] and other similar investigations, we provide decay rates as a function of the degeneracy. Clearly, when $g(s)$ is linear near the origin, the corresponding decay rates to equilibria are exponential.

Remark 2.5. We also note that our assumptions on the properties of the equilibria may not be sharp. Recent work utilizing the Lojasiewicz-Simon inequality has shown that assumptions on the finiteness of equilibria may not be necessary. See Remark 4.1 and references contained in the section for further discussion.

3 Preliminary Step: Unique Continuation for Berger's Plate

In this section we prove that the *UC* condition (as defined above) holds for generalized solutions to the Berger plate equation. For this, we shall invoke the deep result obtained in [34] (whose proof relies on Carleman estimates) which we now cite:

Theorem 3.1 ([34])**.** *Consider*

$$v_{tt} + \Delta^2 v + \sum_{|\alpha| \leq 2} B_\alpha(x,t)\partial_x^\alpha v + D(x,t)v_t = 0, \quad in \;\; \Omega \times (0,T),$$

where

$$B_\alpha(x,t) \in L_\infty(\Omega \times (0,T)) \quad for \; |\alpha| = 2,$$

$$B_\alpha(x,t) \in L_p(\Omega \times (0,T)) \quad for \; p > 2dim(\Omega) \; and \; |\alpha| \leq 1,$$

and

$$D(x,t) \in L_\infty(\Omega \times (0,T)).$$

Suppose v is a solution to the above equation in $\Omega \times (0,T)$, with $v \in L_2(0,T;H^3(\Omega))$ and $v_t \in L_2(0,T;H^1(\Omega))$. Let E be an open subset of Ω which contains $\partial\Omega$. If $v = 0$ in $E \times (0,T)$ then $v = 0$ in $\Omega \times (0,T)$.

Now, we give the main result of this section:

Theorem 3.2. *Suppose $u \in C(0,T;H^2(\Omega)) \cap C^1(0,T;L_2(\Omega))$ is a weak solution to*

$$u_{tt} + \Delta^2 u + d(\mathbf{x})g(u_t) - f_B(u) = p(\mathbf{x}), \tag{3.1}$$

with $g \in C^1(\mathbb{R})$, $d(\mathbf{x}) \in L_\infty(\Omega)$ nonnegative a.e. in Ω, and $d(\mathbf{x}) \geq c_0 > 0$ for $\omega \subset supp\ d$, where ω is an open collar of the boundary Γ. Then, $d(\mathbf{x})u_t = 0$ almost everywhere $\Omega \times (0,T)$ implies that $u_t = 0$ almost everywhere $\Omega \times (0,T)$.

Proof. Assume that u is a weak solution to (3.1) from the class $u \in C(0,T;H^2(\Omega)) \cap C^1(0,T;L_2(\Omega))$ and $d(\mathbf{x})u_t = 0$ almost everywhere in $\Omega \times (0,T)$. The assumptions on $d(\mathbf{x})$ imply that $u_t = 0$ on $\omega \subset supp\ d$. In order to make use of Theorem 3.1, we first need to improve the regularity of our solutions (when the damping term $d(x)u_t = 0$).

Remark 3.1. Although it is often that case that additional regularity can be shown for the dynamics when the damping term is taken to be zero, we emphasize that is a wholly nontrivial problem in the present case. Our difficulty arises in producing a "regularization" estimate in the case of geometrically constrained damping. The proof of additional regularity of trajectories (the necessary step in utilizing Theorem 3.1) depends in a critical way on a recent "observability" type estimate produced in [24], which requires highly specialized cutoff functions. A similar technique will be utilized in the discussion and proof of Lemma 5.2 below.

The inequality below will be instrumental in showing the additional smoothness of the solutions. We present it now as a preliminary lemma:

Lemma 3.3. *Let $u(\tau)$ and $w(\tau)$ be two solutions to* (1.1) *from the class*

$$C\big(0,\infty;H_0^2(\Omega)\big) \cap C^1\big(0,\infty;L_2(\Omega)\big)$$

such that

$$||u(\tau)||_2^2 + ||u_t(\tau)||^2 \leq R^2, \qquad ||w(\tau)||_2^2 + ||w_t(\tau)||^2 \leq R^2, \quad \tau \in [0,\infty).$$

and $z(\tau) = u(\tau) - w(\tau)$. Then for $\eta > 0$ and a given fixed constant T, the following estimate holds for all $s \in [0,\infty)$:

$$E_z(s+T) + \int_s^{s+T} E_z(\tau) \leq C(R,T)D_s^{s+T}(z) + C(R,T)\sup_{\tau\in[s,s+T]}||z(\tau)||_{2-\eta}^2 \tag{3.2}$$

where $E_z(t) = \frac{1}{2}\big(||z_t||^2 + ||\Delta z||^2\big)$ and $D_s^{s+T}(z) = \displaystyle\int_s^{s+T} (d(\mathbf{x})[g(u_t) - g(w_t)], z_t)$.

Proof. The proof is based on the method of localization of the multipliers developed in [24]. We note that, by straightforward calculation, the function f_B satisfies

$$(f_B(u) - f_B(w), z_t) = \frac{1}{2}\frac{d}{dt}\big(||\nabla z||^2\big)\big[\Upsilon - ||\nabla u||^2\big] + \big(||\nabla u|| - ||\nabla w||\big)\big(||\nabla u|| + ||\nabla w||\big)(\Delta w, z_t). \quad (3.3)$$

Also, the uniform boundedness of solutions in time (Remark 2.1) allows us to consider the constant R corresponding to any interval for the dynamics. So integrating the above equality implies (along with the compactness of the Sobolev embeddings) that

$$\left|\int_s^{s+T} (f_B(u) - f_B(w), z_t)\right| \le C(R,T,\epsilon) \sup_{\tau\in[s,s+T]} ||z(\tau)||^2_{2-\eta} + \epsilon \int_s^{s+T} E_z(\tau) \quad (3.4)$$

Utilizing an approach similar to that of [[24], Lemma 4.1], and the last inequality above, we obtain the desired estimate in (3.2). □

With the observability inequality above, we may improve the regularity of trajectories.

Lemma 3.4. *Let assumptions of Theorem 2.3 be satisfied and take u be a solution to (1.1)*-(1.2). *Then this solution has additional regularity; that is, $u_t \in L_\infty\big(0,T;H^4(\Omega)\cap H^1_0(\Omega)\big) \cap W^{1,\infty}(0,T;H^2(\Omega))$.*

Proof. The proof of Lemma 3.4 critically employs observability inequality (3.2) and we also enforce our hypothesis that $d(\mathbf{x})u_t = d(\mathbf{x})w_t = 0$ (and hence $d(\mathbf{x})z_t = 0$) a.e. on $\Omega\times(0,T)$. Then (3.2) gives that

$$E_z(s+T) + \int_s^{s+T} E_z(\tau) \le C(R,T) \sup_{\tau\in[s,s+T]} ||z(\tau)||^2_{2-\eta} \quad (3.5)$$

We then analyze the energy identity for z (following from (2.5)) on $[s, s+T]$:

$$E_z(s+T) + D_s^{s+T}(z) = E_z(s) + \int_s^{s+T} (f_B(u) - f_B(w), z_t). \quad (3.6)$$

Since $D_s^{s+T}(z) = 0$, taking into account (3.5) and (3.6) we have

$$E_z(s) + \int_s^{s+T} E_z(\tau) \le C(R,T) \sup_{[s,s+T]} ||z(\tau)||^2_{2-\eta} + \Big|\int_s^{s+T} (f_B(u) - f_B(w), z_t)\Big|,$$

for $0 < \eta < 1/2$, which together with (3.4) yields

$$E_z(s) + \int_s^{s+T} E_z(\tau) \le C(R,T) \sup_{[s,s+T]} ||z(\tau)||^2_{2-\eta}$$

We now consider the functions $u(t) = w(t+h)$ and $w(t)$ scaled by the factor h^{-1}, and take $z^h(t) = \frac{w(t+h)-w(t)}{h}$. After applying the above inequality for $z^h(t)$ we have

$$E_{z^h}(s) \le C(R,T) \sup_{[s,s+T]} ||z^h(\tau)||^2_{2-\eta}. \quad (3.7)$$

The compactness of the RHS of (3.7) is crucial. Since the embedding $H^2(\Omega) \hookrightarrow H^{2-\eta}(\Omega)$ is compact it allows us to bound:

$$||z^h(\tau)||^2_{2-\eta} \leq \epsilon E_{z^h}(\tau) + C(\epsilon)||z^h(\tau)||^2.$$

Noting

$$||z^h(t)|| \leq \frac{1}{h}\int_0^h ||z_t(\tau + t)||d\tau \leq C(R)$$

uniformly in h, we have:

$$E_{z^h}(s) \leq \epsilon \sup_{[s,s+T]} E_{z^h}(\tau) + C(\epsilon, R, T).$$

Taking the supremum over $s \in [0, \infty)$ we obtain

$$\sup_{[0,\infty)} E_{z^h}(\tau) \leq \epsilon \sup_{[0,\infty)} E_{z^h}(\tau) + C(\epsilon, R, T).$$

Taking the limit as $h \searrow 0$, we see that

$$||u_t(\tau)||_2 + ||u_{tt}(\tau)|| \leq constant,$$

for all $0 \leq \tau < \infty$ and hence we have that $u_t \in C(0, \infty; (H^2 \cap H^1_0)(\Omega)) \cap C^1(0, \infty; L_2(\Omega))$. Moreover, since $u_t \in H^2(\Omega) \subset C(\Omega)$, we have $g(u_t) \in C(\Omega) \subset L_2(\Omega)$ by the continuity of g. Elliptic regularity for

$$\Delta^2 u = -u_{tt} - d(\mathbf{x})g(u_t) - f_B(u) + p(\mathbf{x})$$

with the clamped boundary conditions yields

$$||u(\tau)||^2_4 \leq C, \text{ for all } \tau \in [0, \infty].$$

This gives the boundedness of the solutions in the space $H^4(\Omega) \times H^2(\Omega)$. But utilizing a bootstrapping procedure to repeat this argument we achieve additional regularity for u and its time derivatives *by assuming only an additional derivative on* g (namely, $g \in C^1(\mathbb{R})$).
Consequently, we have that $u_t \in L_\infty(0, T; H^4(\Omega) \cap H^1_0(\Omega)) \cap W^{1,\infty}(0, T; H^2(\Omega))$. □

Having established the additional regularity of the trajectories we now aim at completing the proof of Theorem 3.2. Let us consider the equation which u_t solves by differentiating (3.1). The equation which (ζ, ζ_t) satisfies is (taking $\zeta = u_t$):

$$\zeta_{tt} + \Delta^2\zeta + d(\mathbf{x})g'(u_t)\zeta_t - (||\nabla u||^2 - \Upsilon)\Delta\zeta = 2(\nabla u, \nabla u_t)\Delta u. \tag{3.8}$$

We are now in a position to use Theorem 3.1. For this, we consider two cases:

<u>*Case 1*</u> $(\nabla u, \nabla u_t) = 0$ for almost every $t \in (0, T)$:

In this case, the hypotheses of Theorem 3.1 are satisfied for the function ζ (noting that the set E in this case is defined by $\omega \subset$ *supp* d) in (3.8), and we conclude that $\zeta = u_t = 0$ a.e. in $\Omega \times (0, T)$ as desired.

Case 2 $(\nabla u(t), \nabla u_t(t)) \neq 0$ *on some set* $t \in \Theta$:

We consider the restriction of (3.8) to the set $\omega \subset$ *supp* d, recalling that ω is an open collar of the boundary. Then, since $\zeta = u_t = 0$ for all $\mathbf{x} \in \omega$, the equation (3.8) reads: $0 = (\nabla u, \nabla u_t)\Delta u$ on Θ. Since we have assumed that $(\nabla u, \nabla u_t) \neq 0$, this implies that $\Delta u = 0$ in the open, connected set ω which contains a portion of (in fact, the entire) boundary. Additionally, u has clamped boundary conditions: $u|_\Gamma = \partial_\nu u|_\Gamma = 0$. One of the fundamental results of unique continuation is given by the well known Holmgren Theorem [28] for noncharacteristic surfaces (for which the analyticity of coefficients is required). A standard application of this result yields that $u(t) = 0$ for any $t \in \Theta$, and thus $\zeta = u_t = 0$ for $x \in \Omega$ and all $t \in \Theta$. $\square$

Hence we conclude that the UC property holds for generalized and weak solutions to the Berger plate equation. We may now apply Theorem 2.4 to the Berger evolution with geometrically constrained damping to arrive at a global attractor with the aforementioned properties, and the proof of Theorem 2.5 is complete.

Remark 3.2. We pause to point out that in the unique continuation proof above (critical to our main results) we have made indispensable use of the fact that the damping region ω contain the *entire boundary* Γ. The question of whether or not such a unique continuation is possible for a partial collar of the boundary is open in the present case. We point to [20] for an example of a study of long-time behavior of the wave equation with dissipation localized to a partial collar. This analysis uses Carleman's estimates and associated techniques developed by V. Isakov [29]; utilization in the present case will require microlocal techniques.

4 Rate of Stabilization to Equilibria

We now address the primary question of interest: uniform stabilization rates to equilibria points. For gradient systems, this rate is related to the rates of convergence of individual trajectories to equilibria; indeed, when the dynamical system is gradient, we have the characterization of the global attractor as the unstable manifold, namely, $\mathbf{A} = \mathcal{M}^u(\mathcal{N})$ where $\mathcal{N}$ is the set of stationary points

$$\mathcal{N} = \{V \in \mathcal{H} : S_t V = V \text{ for all } t \geq 0\}$$

and we have that

$$\lim_{t\to\infty} d_{\mathcal{H}}(S_t W | \mathcal{N}\ ) = 0 \quad \text{for any } W \in \mathcal{H} \tag{4.1}$$

(the distance here is the Hausdorff semidistance). This implies eventual proximity to the unstable manifold of the set of equilibria for any trajectory.

Of course, an interesting question is whether individual trajectories stabilize to a specific equilibrium, or simply converge to the set $\mathcal{N}$. It is possible for a trajectory in a given dynamical system to exhibit convergence to an attractor with no ultimate convergence to a specific equilibria point. In fact, convergence to a single equilibrium can be shown, provided that the set of equilibria is *finite* (see Corollary 2.32 [16]). Thus, under the general assumption that the set of equilibria $\mathcal{N}$ is *finite*, one has that any $x \in \mathbf{A}$ belongs to some full trajectory $\gamma = \{(u(t), u_t(t)), t \in \mathbb{R}\}$, and for any $\gamma \in \mathbf{A}$, there exists a pair $\{e, e*\} \in \mathcal{N}$ such that

$$(u(t), u_t(t)) \to (e, 0), \quad in\ \mathcal{H} \quad as\ t \to \infty; (u(t), u_t(t)) \to (e*, 0), \quad in\ \mathcal{H} \quad as\ t \to -\infty; \tag{4.2}$$

Regarding decay rates, it is known that the degree of degeneracy at the origin ($g'(0) = 0$), is the determining quantity. For fully supported interior damping, with the additional assumption of *hyperbolicity of all equilibria*, trajectories converge to an equilibrium point at a specified rate (depending on degeneracy of the damping at the origin). Recall the *hyperbolicity condition* (which is a standard assumption in dynamical systems theory): an equilibrium e is hyperbolic if the spectrum of the linearization of the semiflow S_t near e does not intersect the unit circle in the complex plane. For our purposes, we define a hyperbolic equilibrium e to be one such that the equation

$$Au = \langle f'(e), u \rangle \tag{4.3}$$

(where $'$ denotes the Frèchet derivative) has only the trivial solution. If the linearization of the semiflow S_t exists near e, then the two definitions are equivalent [18, Remark 8.4.5].

At this point, we reiterate our main result in this work:

Theorem 4.1. *Assume that the following hold:*
i) There exist positive constants m, M, s_0 and $\gamma < 1$ such that

$$g'(s) > m, \quad |s| > s_0 \quad and \quad g'(s) \leq M[1 + sg(s)]^\gamma, \quad for\ \ all \quad s \in \mathbb{R}.$$

ii) The set of equilibria is finite and hyperbolic.
Then, for any $U_0 = (u_0, u_1) \in \mathcal{H}$ there exists $e \in \mathcal{N}$ such that the following decay rate holds for the trajectory $S_t U_0$:

$$||S_t U_0 - e||_{\mathcal{H}} \leq C\sigma(t), \quad t > T_0$$

where σ satisfies the equation:

$$\sigma_t + Q(\sigma) = 0, \ \sigma(0) = \sigma_0 = C(e, U_0).$$

The function $Q(s) \sim h^{-1}(s)$ where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, concave and monotone increasing function such that:
(a) $h(0) = 0$ and
(b) $s^2 + [g(s)]^2 \leq h(g(s)s)$ for $|s| \leq 1$.

Remark 4.1. While the property of finiteness of equilibria points is generic with respect to a given load p (Sard's Theorem), it is another endeavor to show convergence to equilibria *without* assuming finiteness of the set $\mathcal{N}$, i.e., given p, show convergence of any trajectory some $e \in \mathcal{N}$. In fact, there is a new tool addressing this issue that has been developed in series of papers [25, 26, 27, 9, 10, 49, 1] (and references therein) which is based on the validity of the so-called Lojasiewicz inequality. The Lojasiewicz gradient inequality refers to an analytic function defined on a real Hilbert space V, $F : V \to \mathbb{R}$, and states that for any point $a \in V$ there is a neighborhood $U(a) \in V$ and two constants $\theta \in (0, 1/2], C > 0$ such that

$$|F(u) - F(a)|^{1-\theta} \leq C||DF(u)||_{V'}, \quad \forall u \in U(a) \subset V \tag{4.4}$$

In the case of evolution equations, the functional F is related to potential energy of the system (e.g., $\Pi(\cdot)$). Having the Lojasiewicz inequality provides a tool for proving stabilization of trajectories to specific equilibria which are stationary points of the dynamics [25, 26, 9, 10, 49] without making the finiteness or hyperbolicity assumption on $\mathcal{N}$. It turns out that Lojasiewicz inequality is satisfied in the case of the von Karman problem with fully supported interior damping [13]. It was also shown that by assuming hyperbolicity of stationary solutions, the Lojasiewicz exponent θ is optimal and equal to $1/2$. By using the Lojasiewicz inequality, [13] proves that the individual trajectories of the von Karman evolution with nonlinear, fully supported interior damping (which is mildly degenerate at the origin) indeed stabilize asymptotically to a particular equilibrium. In the case of non-degenerate damping, the rate of convergence to equilibria are also established.

Although this method provides a means of circumventing assumptions on $\mathcal{N}$, the arguments referenced in [13] (emanating from [25, 26, 9, 10, 49]) depend strongly on the fact that (i) the damping has full geometric support and (ii) the degeneracy at the origin is mild. At the present time, it is not known whether similar result should be expected for geometrically constrained damping.

5 Proof of Theorem 4.1

Proof. The proof relies on geometric considerations of trajectories in a vicinity of an equilibrium point. The critical assumption here is that there are finite equilibria, each of which is hyperbolic; this reduces the problem of convergence in the vicinity of an equilibrium to a linear problem. We utilize the approach taken in the proof of Theorem 2.7 in [19].

By (4.1), we have that for any $W = (w_0, w_1) \in \mathcal{H}$, there exists an equilibrium point $E = (e, 0) \in \mathcal{N}$ such that the weak solution $W(t) = (w(t), w_t(t))$ satisfies

$$W(t) = S(t)W \underset{\mathcal{H}}{\to} E, \qquad t \to \infty \tag{5.1}$$

If we consider

$$Y(t) = (y(t), y_t(t)) \equiv W(t) - E = (w(t) - e, w_t(t)),$$

then from (5.1) we have that for any $\epsilon > 0$, there exists $T_0 > 0$ such that for all $T > T_0$

$$\int_{T-1}^{T} E_y(w(t) - e)dt = \int_{T-1}^{T} E_y(t)dt \le \epsilon, \tag{5.2}$$

where $E_y(t) = \frac{1}{2}\big(||y_t||^2 + ||\Delta y||^2\big)$. We now take ϵ sufficiently small, such that the only equilibrium in $B_{\mathcal{H}}(e, \epsilon)$ is e. By definition of equilibrium, $Y(t)$ satisfies the following equation:

$$y_{tt} + d(\mathbf{x})g(y_t) + \Delta^2 y + F(y + e) - F(e) = 0, \tag{5.3}$$

where $F(w) = -(||\nabla w||^2 - \Upsilon)\Delta w$. We now introduce some additional notation:

$$\xi(y(t)) \equiv E_y(t) + \Phi(y(t)),$$

where

$$\Phi(w) \equiv \int_0^1 \big(F(w\cdot s+e)-F(e),w\big)_{L_2(\Omega)}ds.$$

Here

$$\frac{d}{dt}\Phi(y(t)) = \big(F(y(t)+e)-F(e),y_t(t)\big)_{L_2(\Omega)}.$$

Multiplying (5.3) by y_t and integrating over $[s,t]$, we have

$$\xi(y(t)) + \widetilde{D}_s^t(y) \le \xi(y(s)) \tag{5.4}$$

where

$$\widetilde{D}_s^t(y) \equiv \int_s^t \int_\Omega d(\mathbf{x})g(y_t(\tau))y_t(\tau)\ d\Omega d\tau$$

for any $0\le s\le t$. Here the computations are first performed for strong solutions then passing to the limit on generalized solutions.

Regarding the functional $\xi(y(t))$, we provide the proposition below whose proof is now standard [18] (and references therein) and follows from the Sobolev embeddings and the inequality (2.2):

Proposition 5.1. *Let $y(t)$ be the solution of (5.3). Then the functional $\xi(y(t))$ satisfies the following properties :*

1) $\xi(y(t))\ge 0$ for all $t\ge 0$.

2) If $\|Y(t)\|_{\mathcal{H}}^2 = E_y(t)\le R^2$ for $t\in[0,T]$, then for all $t\in[0,T]$ we have:

$$|E_y(t)-\xi(y(t))| \le \epsilon\,\|y(t)\|_2^2 + C(\epsilon,R)\,\|y(t)\|^2 \tag{5.5}$$

$$E_y(t) \le C\big(\xi(y(t))+\|y(t)\|^2\big) \le \overline{C}\big(E_y(t)+\|y(t)\|^2\big) \tag{5.6}$$

where C and $\overline{C}$ are positive constants depending on R.

5.1 Refined Observability Estimate

Now, we give the following lemma which is the main novel ingredient of the proof of Theorem 4.1. This estimate is a refinement of the observability estimate presented above in Lemma 3.3.

Lemma 5.2. *Let u and w be the solutions to equation (1.1) satisfying*

$$||u(\tau)||_2^2 + ||u_t(\tau)||^2 \le R^2, \qquad ||w(\tau)||_2^2 + ||w_t(\tau)||^2 \le R^2, \quad \tau\in[0,\infty).$$

and take $\psi\equiv\mu z$, where $\mu\in C^\infty(\Omega)$, with supp $(\nabla\mu)$ contained in the damping region ω, and $z=u-w$ solves the following problem:

$$z_{tt}+\Delta^2 z + d(\mathbf{x})[g(u_t)-g(w_t)] = f(u)-f(w) \quad in \quad Q_T\equiv\Omega\times(0,T), \tag{5.7}$$
$$z(0)=u_0-w_0;\ z_t(0)=u_1-w_1,$$

with clamped boundary conditions. Then we have

$$\int_0^T E_z(t) \leq \; C\Big\{E_z(T) + E_z(0) + \int_{Q_T} \{G(z) + F(z)\}(X\nabla\psi) + \int_{Q_T} \{G(z) + F(z)\}z\Big\}$$

$$+C\int_0^T \int_\omega |z_t|^2 + C(T)l.o.t.^z \tag{5.8}$$

where

$$E_z(t) = \frac{1}{2}\left(||\Delta z||^2 + ||z_t||^2\right), \qquad l.o.t.^z \equiv \sup_{[0,T]} ||z(t)||^2_{2-\eta} \quad for \quad \eta \in (0, 1/2)$$

and

$$F(z) \equiv -(f(u) - f(w)), \qquad G(z) \equiv d(\mathbf{x})\big(g(u_t) - g(w_t)\big)$$

Proof. In order to show the above energy estimate we apply the localization method given in [24]. We begin with defining two cut off functions λ and μ. Let $\lambda(\cdot)$, $\mu(\cdot) \in C^\infty(\Omega)$ be such that $0 \leq \lambda(\mathbf{x}) \leq 1$ and $0 \leq \mu(\mathbf{x}) \leq 1$ and

$$\lambda(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \tilde{\omega} \\ 0, & \mathbf{x} \in \Omega \setminus \omega \end{cases} \tag{5.9}$$

,

$$\mu(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in \widetilde{\omega_1} \\ 1, & \mathbf{x} \in \Omega \setminus \omega_1 \end{cases} \tag{5.10}$$

where ω_1, $\widetilde{\omega_1}$, and $\widetilde{\omega}$ are the sets satisfying $\widetilde{\omega_1} \subset \omega_1 \subset \widetilde{\omega} \subset \omega \subset\subset \Omega$.

Remark 5.1. We emphasize that these cut-off functions satisfy the following facts :
(a) $\omega_1 \subset$ supp (λ)
(b) supp (λ) and supp (μ) overlap inside the damping region ω
(c) supp $(\lambda) \cup$ supp$(\mu) = \Omega$.

Now, we define the variables $\phi = \lambda z$ and $\psi = \mu z$. The use of the cut-off functions will produce commutators active in the regions of ω where the cut-off functions are non-constant. Now, multiplying (5.7) by λ and making use of the commutator symbol given by

$$[D_1, D_2]f = D_1(D_2 f) - D_2(D_1 f),$$

we arrive at

$$\phi_{tt} + \Delta^2\phi + \lambda G(z) - \lambda F(z) = [\Delta^2, \lambda]z$$

If we multiply the above equality by ϕ, integrate by parts in time, and use Green's Theorem, we have

$$\int_Q \left\{|\Delta\phi|^2 - |\phi_t|^2\right\} = \; \int_Q [\Delta^2, \lambda]z\phi$$

$$-\int_Q \lambda\left(G(z) - F(z)\right)\phi - (\phi_t, \phi)\big|_0^T.$$

Making use of standard splitting and Sobolev embeddings, we arrive at

$$\int_0^T \left\{||\Delta\phi||^2 - ||\phi_t||^2\right\} \leq \; C_1(E_z(T) + E_z(0)) + \int_Q [\Delta^2, \lambda] z\phi$$

$$- \int_Q \lambda \left(G(z) - F(z)\right) \phi \tag{5.11}$$

Now, we must explicitly estimate the commutator $\int_Q [\Delta^2, \lambda] z\phi$, whose estimation depends only on the use of Green's theorem:

$$\int_Q [\Delta^2, \lambda] z\phi = \int_Q \Big\{\Delta^2(\lambda z)(\lambda z) - \Delta^2 z(\lambda^2 z)\Big\}$$

$$\leq \epsilon \int_Q |\Delta z|^2 + C_2(\lambda, T) l.o.t.^z \tag{5.12}$$

Considering (5.11) and (5.12) we obtain

$$\int_0^T \left\{||\Delta\phi||^2 - ||\phi_t||^2\right\} \leq \; C_\lambda \Big\{(E_z(T) + E_z(0))\Big\}$$

$$+\epsilon \int_Q |\Delta z|^2 - \int_Q \lambda \left(G(z) - F(z)\right) \phi$$

$$+C(\lambda, T) l.o.t.^z \tag{5.13}$$

Now, we use a second multiplier $X \cdot \nabla\psi$, where $\psi \equiv \mu z$ and $X = \mathbf{x} - \mathbf{x}_0 \in \mathbb{R}^2$. If we firstly multiply (5.7) by μ we obtain

$$\psi_{tt} + \Delta^2\psi + \mu G(z) - \mu F(z) = [\Delta^2, \mu] z \tag{5.14}$$

Multiplying the above equality by $X\nabla\psi$, using Green's formula, and taking into account that $\text{supp}(\mu) \cap \Gamma = \emptyset$ we have

$$\int_Q \Big(|\Delta\psi|^2 + |\psi_t|^2\Big) \leq c_0(E_z(T) + E_z(0)) + \int_Q [\Delta^2, \mu] z(X\nabla\psi)$$

$$+ \int_Q \mu \left(F(z) - G(z)\right)(X\nabla\psi) \tag{5.15}$$

We note that $\text{supp}(\nabla\mu) \subset \{x \in \Omega : \lambda(x) \equiv 1\} = \widetilde{\omega}$. Therefore using (5.11) we have the following inequality for the commutator:

$$\int_Q [\Delta^2, \mu] z(X\nabla\psi) = \; \int_0^T \int_{\text{supp}(\nabla\mu)} [\Delta^2, \mu] z(X\nabla\psi) \leq C(\mu) \int_0^T \int_{\text{supp}(\nabla\mu)} |\Delta\psi|^2 + C(T, \mu) l.o.t.^z$$

$$\leq C(\mu) \int_0^T \int_{\{\lambda = 1\}} |\Delta z|^2 + C(T, \mu) l.o.t.^z$$

So from the above estimates we obtain

$$\int_0^T \{||\Delta\psi||^2 + ||\psi_t||^2\} \le c(\lambda,\mu)\Big\{(E_z(T) + E_z(0)) + \int_0^T \int_\omega |z_t|^2$$
$$+ \left|\int_Q G(z)z\right| + \int_Q F(z)z + \int_Q G(z)(X\nabla\psi)$$
$$+ \int_Q F(z)(X\nabla\psi) + c(T)l.o.t.^z\Big\} + \epsilon\int_Q |\Delta z|^2 \tag{5.16}$$

If we put in together the estimates (5.13) and (5.16) to obtain an estimate on the total energy we have

$$\int_0^T \{||\Delta\phi||^2 + ||\phi_t||^2 + ||\Delta\psi||^2 + ||\psi_t||^2\} \le \ c(\lambda,\mu)\Big\{(E_z(T) + E_z(0))$$
$$+ \int_0^T \int_\omega |z_t|^2 + \left|\int_Q G(z)z\right| + \int_Q F(z)z$$
$$+ \int_Q G(z)(X\nabla\psi) + \int_Q F(z)(X\nabla\psi) + C(T)l.o.t.^z\Big\} + \epsilon\int_Q |\Delta z|^2$$

Since the left hand side of the above inequality *overestimates* the total energy $E_z(t)$ we get

$$\int_0^T E_z(t) \le \widetilde{C}\{(E_z(T) + E_z(0)) + \int_0^T \int_\omega |z_t|^2$$
$$+ \left|\int_Q G(z)z\right| + \int_Q F(z)z + \int_Q G(z)(X\nabla\psi)$$
$$+ \int_Q F(z)(X\nabla\psi) + C(T)l.o.t.^z\} \tag{5.17}$$

where $\widetilde{C}$ depends on the cut-off functions. This yields the desired estimate. □

5.2 Decay Rates: Completion of Proof

Now, we assume that y is a weak solution to (5.3) such that $\sup_{t\in[0,T]} E_y(t) \le R^2$. We aim to apply the estimate (5.8). For this, we need to introduce a function h and an auxiliary function H_0 which capture the behavior of the damping function g, and will ultimately yield the decay rates. Consider a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ which is a concave, strictly increasing, continuous function with the following properties:

$$(a)\ h(0) = 0, \quad \text{and} \quad (b)\ s^2 + [g(s)]^2 \le h\big(sg(s)\big) \qquad \text{for} \quad |s| \le s_0 \tag{5.18}$$

Such a function can always be constructed due to the monotonicity of g [18, Appendix B]. From h, we construct $H_0 \equiv h(\tilde{c}\cdot s)$, for some constant $\tilde{c} > 0$.

After repeating the calculations of the proof of Lemma 5.2 for $y(t)$, if we split the region of integration (with respect to $|y(t)| \le 1$ and $|y(t)| \ge 1$), and take into account the asymptotic growth condition from below imposed on g (assumption (i) in Theorem 4.1), we arrive at

$$\int_0^T \int_\omega |y_t|^2 \leq C_1(T)(I + H_0)(\widetilde{D}_0^T(y)).$$

Taking into account (2.2) we have

$$\int_0^T \int_\Omega F(y)y \leq \epsilon \int_0^T ||y(t)||_2^2 dt + C(R,T,\epsilon)l.o.t.^y$$

and similarly

$$\int_0^T \int_\Omega F(y)X\nabla\psi \leq C(R)\int_0^T ||y(t)||_2||\psi(t)||_1 \leq \epsilon \int_0^T ||y(t)||_2^2 + C(R,T,\epsilon)l.o.t.^y,$$

Now, considering the last three inequalities in (5.8) we take

$$\int_0^T E_y(t)dt \leq C_0\Big\{E_y(T) + E_y(0) + \Big|\int_Q G(y)y\Big|$$

$$+\Big|\int_Q G(y)X\nabla y\Big|\Big\} + C_1(T)(I + H_0)(\widetilde{D}_0^T(y)) + C(R,T)l.o.t.^y$$

In standard fashion, using the polynomial growth condition in assumption (i) of Theorem 4.1, the dissipation terms may be estimated by $\widetilde{D}_0^T(y)$ and lower order terms [24]. Then, utilizing the energy relation (5.4) (first for strong solutions, then for weak via the limit transition) we have:

$$\xi(T) + \int_0^T E_y(t)dt \leq C_1(T)(I + H_0)(\xi(0) - \xi(T)) + C_2(T,R)l.o.t.^y \tag{5.19}$$

In dealing with rate of decay problems, the fundamental issue is to dispense with the lower order terms in the above estimate. To this end, we now prove the following lemma:

Lemma 5.3. *Let y be a weak solution to (5.3) such that $\sup_{t\in[0,T]} E_y(t) \leq R^2$. Moreover assume that (5.2) holds for a preassigned small ϵ. Then, there exists a positive constant ϵ_0 such that*

$$l.o.t.^y \leq C(R,T,\epsilon)(I + H_0)(\xi(0) - \xi(T))$$

provided $\epsilon \leq \epsilon_0$ and $T > T_0$, where T_0 is the same as in Theorem 4.1.

Proof. We utilize a compactness-uniqueness argument, applied first to strong solutions, and then extended to weak solutions via approximation. The compactness component of the argument follow from the fact that $H^2(\Omega)$ compactly embeds into $H^{2-\eta}(\Omega)$, and hence energy solutions are compact with respect to lower order terms. The uniqueness property will result from the fact that stationary solutions are locally unique (finiteness assumption), and that the corresponding linearized equation about any equilibrium has only the trivial solution (hyperbolicity assumption). The existence of a small neighborhood about e is needed in order to assert uniqueness of an equilibrium 'selected' by a trajectory.

Let $\epsilon_0 > 0$ be such that for the stationary solution e, there is no other stationary solutions w with

$$E_y(w-e) \leq \epsilon_0, \qquad \epsilon \leq \epsilon_0; \tag{5.20}$$

then, assume that the desired estimate in Lemma 5.3 does not hold. In this case, there exists a sequence $\{y_n(t)\} = \{w_n(t) - e\}$ of generalized solutions to equation (5.3) such that (a) $E_{y_n}(t) \leq R^2$ for all $t \in [0,T]$, (b) the bound in (5.2) holds, and (c)

$$\frac{l.o.t.^{y_n}}{(I+H_0)((\xi(0)-\xi(T)))} \to \infty, \text{ when } n \to \infty.$$

By (5.4), since we have that

$$\frac{l.o.t.^{y_n}}{(I+H_0)(\widetilde{D}_0^T(y_n))} \to \infty, \text{ when } n \to \infty, \tag{5.21}$$

the monotonicity of the function H_0 implies that

$$\int_0^T \int_\Omega d(\mathbf{x}) g(y_{nt}(t)) y_{nt}(t) d\Omega dt \to 0, \qquad H_0(\widetilde{D}_0^T(y_n)) \to 0, \tag{5.22}$$

Since $E_{y_n}(t) \leq R^2$, there exists a subsequence (also labeled y_n) such that

$$(y_n, y_{nt}) \to (y, y_t) \text{ *-weakly in } L_\infty(0,T; H_0^2(\Omega) \times L_2(\Omega)). \tag{5.23}$$

Now, we claim that

$$d(\mathbf{x}) y_{nt}(t) \to 0 \quad \text{ a.e. in } \Omega \times (0,T)$$

for a subsequence (again, also labeled y_n). Indeed, by (5.22), splitting the region of integration, we have

$$\int_0^T \int_\Omega d(\mathbf{x}) \left| y_{nt}(t) \right| d\Omega dt \leq C\lambda^{-1} + \int_{\mathscr{U}} d(\mathbf{x}) \left| y_{nt} \right| d\Omega dt,$$

where $\mathscr{U} = \{(t,x) \in (0,T) \times \Omega : |y_{nt}| \leq \lambda\}$. Using (5.18) for $|y_{nt}| \leq \lambda$ we take

$$\limsup_{n\to\infty} \int_0^T \int_\Omega d(\mathbf{x}) \left| y_{nt}(t) \right| d\Omega dt \leq \delta_\lambda + C_\delta H_0(\widetilde{D}_0^T(y_n))$$

which, together with (5.22), implies

$$d(\mathbf{x}) y_{nt}(t) \to 0 \quad \text{ a.e. in } \Omega \times (0,T).$$

Taking into account that the system satisfies UC property (see Theorem 3.2) (in the case $f = f_B$) we have that

$$y_t(t) = 0 \quad \text{ a.e. in } \Omega \times (0,T).$$

Hence, we can assume that

$$(y_n, y_{nt}) \to (y, 0) \text{ *-weakly in } L_\infty(0,T; H_0^2(\Omega) \times L_2(\Omega)) \tag{5.24}$$

and utilizing Aubin's compactness argument, this implies

$$y_n \to y \quad \text{strongly in } C(0,T;H^{2-\eta}(\Omega)) \text{ for any } \eta > 0. \tag{5.25}$$

Now, we prove that $y \equiv 0$. We note that when discussing the issue of stabilization of weak solutions to an equilibrium, the fact that weak/generalized solutions satisfy the variational form of the problem turns out to be critical. In this treatment, we consider generalized nonlinear semigroup solutions; these can be shown to be weak variational solutions in a straightforward fashion [18]. Thus the validity of energy relation (2.5) is of paramount importance here. Now, using (5.24)-(5.25) and passing to the limit (in the distributional sense) in (5.3), we have

$$\Delta^2 y + F(y+e) - F(e) = 0, \qquad \text{in } \Omega \times (0,T) \tag{5.26}$$

(where $y(t)$ is a variational solution of (5.26)). Since e is a stationary solution, the above implies that $w \equiv y + e$ is also a stationary solution. On the other hand, from the weak lower semicontinuity of the energy, weak convergence in (5.24), and the fact in (5.2), we have

$$\int_{T-1}^{T} E_y(t)dt = E_y(t) = E_y(w(t)-e)dt \leq \epsilon \leq \epsilon_0,$$

which together with assumption (5.20), yields that $w = e$. So we conclude that $y \equiv 0$ in (5.24) and (5.25), which yields,

$$y_n \to 0 \quad \text{strongly in } C(0,T;H^{2-\eta}(\Omega)) \text{ for any } \eta > 0. \tag{5.27}$$

Now, to reach the contradiction, let us rescale the sequence y_n as follows:

$$z_n = \frac{y_n}{\theta_n}$$

where, $\theta_n = \sqrt{l.o.t.^{y_n}}$. We note that by (5.27), the fact that $\theta_n \to 0$ and by (5.21) we have

$$\frac{1}{\theta_n^2} \int_0^T \int_\Omega d(\mathbf{x}) g(y_{nt}) y_{nt} \, d\Omega dt \to 0$$

From the energy estimate in (5.19), we also have that there exists some $K > 0$ such that $E_{z_n}(t) \leq K$ for all $t \in [0,T]$. So, again we may assume that there exists an element $(z, z_t) \in L_\infty(0,T;H_0^2(\Omega) \times L_2(\Omega))$ such that

$$(z_n, z_{nt}) \to (z, z_t) \text{ *-weakly in } L_\infty(0,T;H_0^2(\Omega) \times L_2(\Omega)) \tag{5.28}$$

and

$$z_n \to z \quad \text{strongly in } C(0,T;H^{2-\eta}(\Omega)) \text{ for any } \eta > 0. \tag{5.29}$$

In particular, arguing as before, we have that for some subsequence z_n

$$z_{nt} \to 0, \quad \text{a.e. in } \Omega \times (0,T).$$

Now, to obtain a differential equation for z, we consider the difference $F(e+z_n)-F(e)$ as $n\to\infty$: we note that

$$\frac{F(e+z_n)-F(e)}{\theta_n}\to F'(e)z \qquad \text{in } L_\infty(0,T;H^{-2}(\Omega)), \tag{5.30}$$

where F' is the Frechet derivative of F. The following conservative type estimates from [17]

$$||F'(e)z||_{H^{-2}(\Omega)}\le C(R)\,||z||_2\,,$$

$$\left|(F'(u)-F'(v),z)_{L_2(\Omega)}\right|\le C_R\,||u-v||_{2-\epsilon}\,||z||_2\,,\qquad z\in H_0^2(\Omega)$$

yield (5.30). Now, if we divide both sides of equation (5.3) by θ_n, after writing y_n for y, passing to the limit, and taking into account (5.30), we conclude that the function z solves

$$\Delta^2 z+F'(e)z=0 \qquad \text{in } \Omega\times(0,T), \tag{5.31}$$

in the variational sense. Since the equilibrium e is assumed to be hyperbolic, we have that the only solution to (5.31) is the zero solution. So, $z\equiv 0$ in (5.28) and (5.29), which is impossible since

$$1=l.o.t.^{z_n}\to l.o.t.^{z}=0.$$

Hence we finally conclude that the desired estimate in Lemma 5.3 holds. □

Now, in order to complete the proof of Theorem 4.1, we take into account the result of Lemma 5.3 in (5.19) and we obtain

$$\xi(T)\le C(I+H_0)(\xi(0)-\xi(T))$$

for $C>0$. Here we choose $T>0$ (depending on the solution) such that (5.2) holds with $\epsilon\le\epsilon_0$, where ϵ_0 is given in Lemma 5.3. Using the strictly monotonicity of $(I+H_0)$ we have (from the above estimate) that

$$\xi((m+1)T)+(I+H_0)^{-1}(C^{-1}\xi((m+1)T))\le\xi(mT), \qquad m=1,2,...$$

Now, applying Proposition B.3.3 [18, p. 746] we have that

$$\xi(mT)\le\sigma(m), \quad m=0,1,2,..., \tag{5.32}$$

where $\sigma(t)$ solves the equation

$$\sigma_t+Q(\sigma)=0,$$
$$\sigma(0)=\sigma_0=\xi(0)$$

and $Q(s)\sim h^{-1}(s)$, where $h:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous, concave and monotone increasing function such that $s^2\le h(g(s)s)$ for $|s|\le 1$. Now, using (5.6) and Lemma 5.3 we have

$$E_y(mT)\le C\Big(\xi(mT)+\sup_{[mT,(m+1)T]}\|y(t)\|^2\Big)\le C\xi(mT), \qquad m=0,1,2,...$$

which together with (5.32) yields

$$E_y(mT) \leq \overline{C}\sigma(m), \quad m = 0, 1, 2, ...,$$

The final conclusion follows from the continuity of the flow and the semigroup property. $\square$

6 Von Karman Plate

In this section we give a concise discussion on the plate equation (1.1) with the von Karman nonlinearity

$$f(u) = f_V(u) = [v(u) + F_0, u], \tag{6.1}$$

where the Airy's stress function $v(u)$ satisfies the following elliptic problem:

$$\begin{aligned} \Delta^2 v(u) &= -[u, u] \text{ in } \Omega \\ \partial_\nu v(u) &= v(u) = 0 \text{ on } \Gamma, \end{aligned} \tag{6.2}$$

with the von Karman bracket given by

$$[u, w] = u_{xx}w_{yy} + u_{yy}w_{xx} - 2u_{xy}w_{xy}. \tag{6.3}$$

Here the internal force $F_0 \in H^\theta(\Omega) \cap H_0^1(\Omega)$, $\theta > 3$. The energies associated to this model are given by

$$E(t) = \frac{1}{2}\big(||\Delta u||^2 + ||u_t||^2\big), \quad \widehat{E}(t) = E(t) + \frac{1}{4}||\Delta v(u)||^2$$
$$\mathscr{E}(t) = E(t) + \Pi(u),$$

where

$$\Pi(u) = \frac{1}{4}\int_\Omega \big(|\Delta v(u)|^2 - 2[F_0, u]u - 4pu\big). \tag{6.4}$$

and the energy $E(t)$ denotes the norm in the state space $\mathcal{H} \equiv H_0^2(\Omega) \times L_2(\Omega)$.

The presence of the von Karman bracket in the model, along with appropriate regularity properties imposed on F_0, assures that the total energy is bounded from below. This can be seen from the following inequality [18, 15]:

$$||u||^2 \leq \epsilon\big(||A^{1/2}u||^2 + ||\Delta v(u)||^2\big) + M_{\epsilon,p,F_0}$$

for every $\epsilon > 0$, $M > 0$. (This ensures that solutions to the equation are global in time bounded—see Remark 2.1).

On the other hand, the function $f = f_V : (H^2 \cap H_0^1)(\Omega) \to L_2(\Omega)$ satisfies the following local Lipschitz property [[18], Corollary 1.4.5]:

$$||f(u) - f(w)||_{-\delta} \leq C(||u||_2^2 + ||w||_2^2)||u - w||_{2-\delta} \tag{6.5}$$

for all $\delta \in [0, 1]$, which plays a critical role in obtaining the same result of Theorem (4.1).

As mentioned above, the main results of the previous work [24] were obtained for *local attractors*. This is due to the fact that for the von Karman plate evolution, the validity of the unique continuation property is a difficult open question. The main obstacle in obtaining this property for the von Karman plate with localized damping is the entirely non-local character of the Airy function (which prevents the applicability of Carleman's estimates).

At present, it is unknown whether the propagation of dissipation phenomenon across the entire domain holds for *weak* damping localized to a "small" set. Therefore, the results in [24] which apply "locally" become global by *assuming the UC property*. Hence, if for a particular type of damping function $d(\mathbf{x})$ (or equivalently, a specific geometric damping region) the UC property can be shown for the von Karman evolution, then the local results in [24] are upgraded, and our analysis for the Berger plate in this work carries over to the von Karman plate. We now state this as a corollary:

Corollary 6.1. *Assume the UC property is valid for the von Karman plate ((1.1) with $f = f_V$, as above). Then, under the same assumptions, the results of Theorem 4.1 hold for solutions to the von Karman evolution.*

We point out that the result of Theorem (4.1) confirms a conjecture made in [24] concerning the ability to replace strict positivity of the damping coefficient $d(\cdot)$ with the UC property.

7 Free Boundary Conditions

In this section we describe the free boundary conditions associated to the plate dynamics. We also discuss how the Berger nonlinearity is, in some sense, more sensitive to free boundary conditions than the von Karman nonlinearity.

We begin with a quick description of the standard free-clamped boundary conditions for the plate [37, 18]:

$$\begin{aligned} \mathcal{B}_1 u \equiv \Delta u + (1-\mu)B_1 u = 0 \quad &\text{on} \quad \Gamma_1, \\ \mathcal{B}_2 u \equiv \partial_\nu \Delta u + (1-\mu)B_2 u - \mu_1 u - (\beta(\mathbf{x})u^3 + c(\mathbf{x})u_t) = 0 \quad &\text{on} \quad \Gamma_1, \\ u = \partial_\nu u = 0 \text{ (clamped)} \quad &\text{on} \quad \Gamma_0, \end{aligned} \tag{7.1}$$

where we have partitioned the boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ (with Γ_0 not empty). We assume that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ in order to avoid technical boundary issues associated with elliptic regularity. The boundary operators B_1 and B_2 are given by [37]:

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx} = -\partial_\tau^2 u - \nabla \cdot \nu(\mathbf{x})\partial_\nu u \ ,$$

$$B_2 u = \partial_\tau \left[\left(\nu_1^2 - \nu_2^2\right) u_{xy} + \nu_1\nu_2 \left(u_{yy} - u_{xx}\right) \right] = \partial_\tau \partial_\nu \partial_\tau u,$$

where $\nu = (\nu_1, \nu_2)$ is the outer normal to Γ, $\tau = (-\nu_2, \nu_1)$ is the unit tangent vector along Γ. The parameter μ_1 and coefficient functions c and β are nonnegative; the constant $0 < \mu < 1$ has the meaning of the Poisson modulus.

The term $c(\mathbf{x})u_t$ corresponds to a (possibly degenerate) frictional damping coefficient on the free boundary of the plate. This term is typical in the analysis of long-time behavior of free plates taken with *interior damping*; see [18, p. 196 and pp. 488-489].

Remark 7.1. When working with the Berger nonlinearity, there is no *uniquness* problem like that associated to the von Karman nonlinearity ($[u,u]=0 \Rightarrow u=0$ with for u free boundary conditions). This indicates that the critical bounds associated to the Berger nonlinear term ((2.2) and (2.3)) can be obtained *without static damping given by the term* $\beta(\mathbf{x})u^3$. This is contrast to the case of the von Karman nonlinearity, where this term is needed to obtain the associated estimates [18, pp. 208, 489].

We now point out that the structure of the Berger nonlinearity produces an additional boundary term not present in the von Karman model when considering free boundary conditions:

$$-([v(u),u],u_t) = \frac{1}{4}\frac{d}{dt}||\Delta v(u)||^2 \tag{7.2}$$

$$-(||\nabla u||^2 \Delta u, u_t) = \frac{1}{4}\frac{d}{dt}||\nabla u||^4 - \boxed{\int_{\Gamma_1} (\partial_\nu u) u_t d\Gamma}. \tag{7.3}$$

For the analysis of long time behavior, we have the following calculation (taking $z = u - w$):

$$\begin{aligned}(f_B(u) - f_B(w), z_t) = &\frac{\left(||\nabla u||^2 - \Upsilon\right)}{2}\frac{d}{dt}||\nabla z||^2 \\ &- \left(||\nabla u|| - ||\nabla w||\right)\left(||\nabla u|| + ||\nabla w||\right)(\Delta w, z_t) \\ &+ \left(\Upsilon - ||\nabla u||^2\right)\int_{\Gamma_1}(\partial_\nu z) z_t d\Gamma\end{aligned}$$

The critical estimate for long time behavior analysis in this case is then:

$$\left|\left(\Upsilon - ||\nabla u||^2\right)\int_{\Gamma_1}(\partial_\nu z) z_t d\Gamma\right| \le C(\Upsilon, R, \epsilon, \Omega)||z||^2_{3/2,\Omega} + \epsilon||z_t||^2_{0,\Gamma}.$$

We note that the term $\epsilon||z_t||^2_\Gamma$ appearing on the boundary can be accommodated by the damping term associated to $c(\mathbf{x})u_t$ present in the boundary conditions (which account for frictional damping). This indicates that the function $c(\mathbf{x})$ must be *bounded from below* on the free portion of the boundary Γ_1 (and hence cannot be degenerate in this configuration). However, owing to the analysis in [18], this type of boundary damping alone *is sufficient to stabilize* the nonlinear plate model (von Karman or Berger) to the attractor, rendering localized dissipation (as in this treatment) inert. Thus, for Berger evolutions in the presence of localized interior damping, it is only meaningful to consider convergence to the attractor for standard hinged or clamped boundary conditions.

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