AN ABSTRACT OF THE THESIS OF

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Title: Division Algebra Representations of SO(4,2)

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Representations of $\mathrm{SO}(4, 2; \mathbb{R})$ are constructed using 4×4 and 2×2 matrices with elements in $\mathbb{H}' \otimes \mathbb{C}$. Using 2×2 matrix representations of $\mathbb{C}$ and $\mathbb{H}'$, the 4×4 representation is interpreted in terms of 16×16 real matrices. Finally, the known isomorphism between the conformal group and $\mathrm{SO}(4, 2; \mathbb{R})$ is written explicitly in terms of the 4×4 representation. The 4×4 construction should generalize to matrices with elements in $\mathbb{K}' \otimes \mathbb{K}$ for $\mathbb{K}$ any normed division algebra over the reals and $\mathbb{K}'$ any split algebra over the reals.

Division Algebra Representations of SO(4,2)

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Joshua James Kincaid, Author

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Personal

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TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF FIGURES

LIST OF TABLES

DIVISION ALGEBRA REPRESENTATIONS OF SO(4,2)

1 INTRODUCTION

1.1 Motivation

The Freudenthal-Tits magic square of Lie algebras provides an interesting relationship between the four normed division algebras over the reals—$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$—and certain Lie algebras [10, 20, 2]. Most notably, it relates four of the five exceptional Lie algebras—$\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$—to groups of 3×3 matrices with elements coming from the octonions (the unique nonassociative, noncommutative normed division algebra over the reals). The fifth exceptional Lie algebra, g_2, generates the automorphism group of the octonions, so in a sense the magic square reveals that *all* of the exceptional Lie algebras can be understood through properties of the octonions.

While the algebras appearing in the magic square are well understood, the corresponding *Lie groups* are not. In particular, the geometric significance of E_7 and E_8 is not clear [1]. While understanding these groups is an interesting topic in its own right, the suspected relevance of the groups E_6 and E_8 to theoretical physics [11, 16, 5, 13, 19, 6, 17] provides further encouragement for their study. Unfortunately, it is not yet clear how best to proceed in a description of E_7, let alone E_8; it is hoped that the present work represents an initial step toward such a description.

Specifically, we consider a simplified magic square involving groups of 2×2 matrices rather than 3×3 matrices. Previous work [3, 18, 8, 22] has established a close analogy between the first two rows of the 2×2 and 3×3 magic square, interpreting the first rows respectively as $\mathrm{SU}(2;\mathbb{K})$ and $\mathrm{SU}(3;\mathbb{K})$ and the second rows respectively as $\mathrm{SL}(2;\mathbb{K})$ and $\mathrm{SL}(3;\mathbb{K})$. In focusing on the $\mathbb{H}' \otimes \mathbb{C}$ slot of the 2×2 magic square, this work represents a first step toward providing a unified description of the third row of the 2×2 magic square at the group level. It is hoped that such a description will then extend naturally to a description of the third row of the 3×3 magic square, in turn providing an identification between E_7 and an as-of-yet unidentified octonionic matrix group.

1.2 Summary

In Chapter 2 we give a brief overview of necessary background material. Section 2.1 provides a standard construction of the normed division algebras $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ and their split counterparts. In Section 2.2 we introduce the groups of orthogonal, unitary, and symplectic matrices, identify these as Lie groups, and introduce the notation for Lie groups and algebras. Finally, in Section 2.3 we introduce the Freudenthal-Tits magic square and, following [22], present the interpretation of the first two rows of the magic square incorporating the split division algebras as $\mathrm{SU}(3;\mathbb{K})$ and $\mathrm{SL}(3;\mathbb{K})$, respectively. Section 2.3 concludes with a discussion of the 2×2 magic square, one cell of which is the focus of this work.

Chapter 3 contains the primary results of this thesis, which are representations of the Lie group $\mathrm{SO}(4,2;\mathbb{R})$ in terms of 4×4 and 2×2 matrices with elements in $\mathbb{H}' \otimes \mathbb{C}$. Ideally, the 2×2 representation will admit an interpretation as "$\mathrm{SU}(2;\mathbb{H}' \otimes \mathbb{C})$". Chapter 3 concludes with an explicit construction of the well-known isomorphism between $\mathrm{SO}(4,2;\mathbb{R})$ and the conformal group on $3+1$ dimensional spacetime in terms of the new 4×4 representations constructed here.

1.3 Notation

Before proceeding, some comments must be made on the notation we employ. We adhere throughout to the following not-necessarily-standard conventions:

- All sums are *explicitly* written using capital-sigma notation. In particular, symbols of the form $p^\mu \mathbf{\Gamma}_\mu$ do *not*, in themselves, indicate sums despite the superficial resemblance to Einstein's summation convention.

- Matrices are indicated by bold-faced symbols.
- In order to avoid ambiguity, the underlying field of all matrix groups is explicitly stated.

2 BACKGROUND

We present here a review of the mathematical structures that we will be using.

2.1 Normed Division Algebras

We start with a brief overview of normed division algebras

2.1.1 Cayley-Dickson Construction

The Cayley-Dickson construction [4, 1] provides a convenient way to construct a new algebra from an old one. In particular, beginning with the one-dimensional algebra $\mathbb{R}$, the Cayley-Dickson construction can be used to generate algebras of dimension 2^n for any integer n with the property that each new algebra contains the previous one in an obvious way. These algebras all have a natural sense of conjugation, and the first four algebras generated in this way are precisely the four normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. In this section we demonstrate this procedure by explicitly constructing the normed division algebras.

Begin with an algebra $\mathbb{K}$ equipped with an involutive antiautomorphism, $a \to \overline{a}$, called conjugation. On the space of ordered pairs, $\mathbb{K} \times \mathbb{K}$, define addition element-wise, multiplication by[1]

$$(a, b)(c, d) = (ac - \overline{d}b, da + b\overline{c}) \tag{2.1}$$

and conjugation by

$$\overline{(a, b)} = (\overline{a}, -b). \tag{2.2}$$

[1]The convention here matches that in [4], which differs from [1]. The difference between conventions is best understood in terms of the construction in 2.1.2: the convention from [4] corresponds to the statement $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}l$, while the convention in [1] corresponds to $\mathbb{O} = \mathbb{H} \oplus l\mathbb{H}$.

With these definitions, $\mathbb{K}\times\mathbb{K}$ becomes an algebra with conjugation, and there is a natural inclusion from $\mathbb{K}$ into $\mathbb{K}\times\mathbb{K}$ given by $a \hookrightarrow (a, 0)$.

A few properties follow directly from these definitions by straightforward calculation. First, multiplication in $\mathbb{K}\times\mathbb{K}$ will be commutative if and only if $\overline{a} = a$ for all $a \in \mathbb{K}$. Second, multiplication in $\mathbb{K}\times\mathbb{K}$ will be associative if and only if multiplication in $\mathbb{K}$ is both commutative and associative.

Now we can construct the normed division algebras. We start with $\mathbb{R}$ and take conjugation to be the identity map, so that $\overline{a} = a$ for $a \in \mathbb{R}$. Then we construct the complex numbers by defining $\mathbb{C} = \mathbb{R}\times\mathbb{R}$, with multiplication and conjugation defined according to (2.1) and (2.2). Next, define the quaternions as $\mathbb{H} = \mathbb{C}\times\mathbb{C}$ and the octonions as $\mathbb{O} = \mathbb{H}\times\mathbb{H}$, with multiplication and conjugation likewise defined in both cases. From the properties mentioned above, our choice of conjugation on $\mathbb{R}$, together with the fact that real multiplication is associative and commutative, guarantees that multiplication in $\mathbb{C}$ will be associative and commutative as well. However, conjugation in $\mathbb{C}$ is not the identity map, so multiplication in $\mathbb{H}$ will not be commutative, which leads in turn to the fact that multiplication in $\mathbb{O}$ is not associative.

If one attempts to extend this process to $\mathbb{O}\times\mathbb{O}$, one finds that the resulting algebra contains zero divisors. As such, while $\mathbb{O}\times\mathbb{O}$ can be made into an algebra in this way, the result is not a division algebra. In fact, the Hurwitz theorem [14] tells us that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the *only* normed division algebras over the reals.

Having defined the normed division algebras, we define the *real part* of an element s in any one of them by

$$\Re e[s] = \frac{1}{2}\left(s + \overline{s}\right). \tag{2.3}$$

Writing $s = (a, b) \in \mathbb{K}\times\mathbb{K}$ it follows from this definition that

$$\Re e[s] = \left(\frac{1}{2}(a + \overline{a}), 0\right) = (\Re e[a], 0), \tag{2.4}$$

from which we see that the real part of any element in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ is a real number (thereby justifying the name). We define the imaginary part of an element $s \in \mathbb{K} \times \mathbb{K}$ by

$$\Im m[s] = s - \Re e[s]. \tag{2.5}$$

Note that this definition of $\Im m[s]$ differs by a factor of i from the traditional definition on $\mathbb{C}$ and, in any case, does not yield a real number. Finally, we define the *squared norm*, $|s|^2$, of an element of $s = (a, b) \in \mathbb{K} \times \mathbb{K}$ via

$$|s|^2 \equiv \overline{s}s. \tag{2.6}$$

As with $\Re e[s]$, this associates a real number to s, which can be seen by computing

$$\overline{s}s = (\overline{a}, -b)(a, b) = (\overline{a}a + \overline{b}b, b\overline{a} - b\overline{a}) = (\overline{a}a + b\overline{b}, 0) = |a|^2 + |b|^2. \tag{2.7}$$

Finally, we note the following important consequences of these definitions. First, for $s \in \mathbb{R}$ we have $s = \overline{s}$ so that $|s|^2 = s^2 \geq 0$, from which it follows that $|s|^2 \geq 0$ for $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, with equality holding if and only if $s = 0$. Second, if $\Re e[s] = 0$, then we have $s = -\overline{s}$ so that

$$s^2 = -s\overline{s} = -|s|^2 \leq 0. \tag{2.8}$$

2.1.2 Alternate Construction

An alternate construction can be developed by the introduction of successive "square roots of -1" (algebra elements defined by the fact that they satisfy $x^2 = -1$) which we call *imaginary units*. Once again we begin with $\mathbb{R}$. Now, define the imaginary unit i (i.e., set $i^2 = -1$) and define the set $\mathbb{C}$ as

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i. \tag{2.9}$$

Elements of this space have the form $x + yi$ for real numbers x and y, and if we define multiplication by formal distribution we see that this yields the same algebra as (2.1) with $\mathbb{K} = \mathbb{R}$ under the identification $i \equiv (0, 1)$. If we now define a second, independent imaginary unit j, we can construct the quaternions as

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j. \tag{2.10}$$

This is a four-dimensional space, and the construction yields the natural basis, $\{1, i, j, ij\}$, where ij can be verified to be a third imaginary unit. Following convention, we define

$$k \equiv ij,$$

so that an arbitrary quaternion can be written as $x + yi + zj + wk$ for real x, y, z, w. The multiplication table follows if we assume associativity and is given in Table 2.1. Figure 2.1 provides an intuitive summary of Table 2.1, in the sense that multiplication of an element by the next element in the counterclockwise direction gives the third, while multiplication by the next element in the clockwise direction gives *minus* the third.

$\times$	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

TABLE 2.1: Multiplication table for basis imaginary units in $\mathbb{H}$

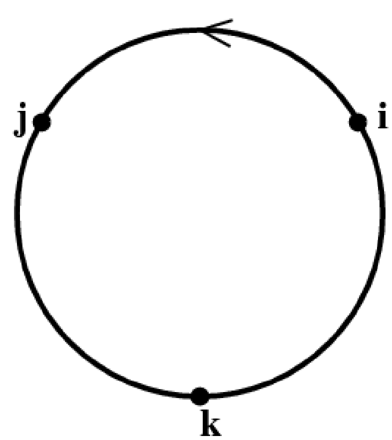

FIGURE 2.1: The quaternionic multiplication table

Proceeding once more in this manner we construct the octonions as $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}l$, where l is yet another imaginary unit. This gives the octonionic basis $\{1, i, j, k, kl, jl, il, l\}$, on which we impose multiplication as defined by Table 2.2 and summarized by Figure 2.2.

$\times$	i	j	k	kl	jl	il	l
i	-1	k	$-j$	jl	$-kl$	l	il
j	$-k$	-1	i	$-il$	l	kl	jl
k	j	$-i$	-1	$-l$	il	$-jl$	kl
kl	$-jl$	il	l	-1	i	$-j$	$-k$
jl	kl	l	$-il$	$-i$	-1	k	$-j$
il	l	$-kl$	jl	j	$-k$	-1	$-i$
l	$-il$	$-jl$	$-kl$	k	j	i	-1

TABLE 2.2: Multiplication table for basis imaginary units in $\mathbb{O}$

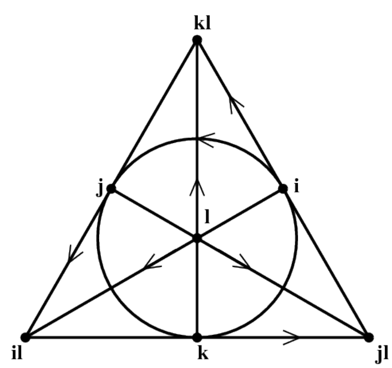

FIGURE 2.2: The octonionic multiplication table

In Figure 2.2, the straight lines are to be thought of as circles corresponding to the imaginary units in a quaternionic subalgebras. An examination of Table 2.2 shows that octonionic multiplication is not, in general, associative. It is, however, alternative, which is to say that the subalgebra generated by any two elements in $\mathbb{O}$ is associative.

In each case, we take conjugation to be the map $a \to \overline{a}$ defined by replacing each imaginary unit with its negative, $i \to -i$, $j \to -j$, and so on. We can then define the real and imaginary parts and the norm as in (2.3), (2.5), and (2.6).

Although we have only explicitly *defined* three imaginary units, and our bases contain only 1, 3, or 7 imaginary units, the spaces we have constructed actually contain an infinite number of them. Specifically, as noted in (2.8), any element with real part zero will square to a negative number. Thus, if s is nonzero and satisfies $\Re e[s] = 0$, then the element $s/|s|$ satisfies $s^2 = -1$. It follows that the imaginary units in $\mathbb{O}$ constitute a 7-sphere, S^7, while those in $\mathbb{H}$ constitute a 3-sphere, $S^3 \subset S^7$.

2.1.3 Split Algebras

For each of the normed division algebras constructed above we can construct a corresponding *split* algebra. These split algebras, $\mathbb{R}'$, $\mathbb{C}'$, $\mathbb{H}'$, and $\mathbb{O}'$, are constructed in a manner similar to that of the normed division algebras, with the caveat that instead of appending roots of -1 we sometimes append roots of unity. Specifically, we begin by defining $\mathbb{R}' = \mathbb{R}$. Then $\mathbb{C}'$ is defined as

$$\mathbb{C}' = \mathbb{R}' \oplus \mathbb{R}'L \tag{2.11}$$

where

$$L^2 = 1 \tag{2.12}$$

by definition. Holding with convention, we continue to call L an imaginary unit. Proceeding, we define

$$\mathbb{H}' = \mathbb{C}' \oplus \mathbb{C}'K, \tag{2.13}$$

and

$$\mathbb{O}' = \mathbb{H}' \oplus \mathbb{H}'J, \tag{2.14}$$

with

$$K^2 = J^2 = -1. \tag{2.15}$$

We also define the imaginary unit I via

$$I \equiv JK, \tag{2.16}$$

and note that

$$I^2 = -1 \tag{2.17}$$

while

$$(KL)^2 = (JL)^2 = (IL)^2 = 1. \tag{2.18}$$

The full multiplication in $\mathbb{O}'$ is given in Table 2.3.

$\times$	I	J	K	KL	JL	IL	L
I	-1	K	$-J$	JL	$-KL$	$-L$	IL
J	$-K$	-1	I	$-IL$	$-L$	KL	JL
K	J	$-I$	-1	$-L$	IL	$-JL$	KL
KL	$-JL$	IL	L	1	$-I$	J	K
JL	KL	L	$-IL$	I	1	$-K$	J
IL	L	$-KL$	JL	$-J$	K	1	I
L	$-IL$	$-JL$	$-KL$	$-K$	$-J$	$-I$	1

TABLE 2.3: Multiplication table for basis imaginary units in $\mathbb{O}'$

As in the case of $\mathbb{O}$, $\mathbb{O}'$ is alternative but not associative. However, the split algebras contain zero-divisors and are therefore *not* division algebras; for example, in $\mathbb{C}'$ (and therefore in the others) we have

$$(1 - L)(1 + L) = 1 - L + L - L^2 = 1 - L + L - 1 = 0. \tag{2.19}$$

As with the division algebras, we take conjugation to be the map $a \to a^*$ defined by replacing each imaginary unit with its negative, $L \to -L$, $J \to -J$, and so on. In the interest of clarity, we denote conjugation within a split algebra with an asterisk rather than a bar as was done for the division algebras. We retain the definition of real and imaginary parts and of the squared norm, and note that as a result of these definitions we have $|L|^2 = |KL|^2 = |JL|^2 = |IL|^2 = -1$; i.e., imaginary units that square to 1 have *negative* "squared norm". Our convention is to label imaginary units of a normed *division* algebra with lower-case letters from the middle of the alphabet, while labelling imaginary units of *split* algebra with upper-case letters.

It should be mentioned here that each of these algebras could be constructed through several alternate "paths". For example, we could have defined $\mathbb{H}'$ as

$$\mathbb{H}' = \mathbb{C} \oplus \mathbb{C}L, \tag{2.20}$$

and $\mathbb{O}'$ as

$$\mathbb{O}' = \mathbb{H} \oplus \mathbb{H}L. \tag{2.21}$$

In any case the constructions lead to the same algebra, with only the ordering of the bases and some signs being different.

As a final note, we introduce a secondary notation that will facilitate summing over imaginary units. Explicitly, we define the labels

$$\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\} = \{1, i, j, k, kl, jl, il, l\} \subset \mathbb{O}, \tag{2.22}$$

and

$$\{e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}\} = \{1, I, J, K, KL, JL, IL, L\} \subset \mathbb{O}', \tag{2.23}$$

where equality is understood to preserve the order.

2.2 Matrix Groups and Algebras

We now give a brief overview of the matrix groups and algebras used here.

2.2.1 Matrix Groups

Matrix groups are constructed from $r \times r$ matrices satisfying various conditions using matrix multiplication as the group operation. The most commonly discussed matrix groups are built from matrices whose components lie in the real or complex fields, in which case they can be classified to some extent in terms of their determinants; we give a basic taxonomy of such matrices here. In the sequel, however, we will be interested in matrices over more general algebras. For example, the elements of our matrices will, in some cases, lie in the algebra $\mathbb{H}' \otimes \mathbb{C}$. Due to the nonabelian nature of $\mathbb{H}'$, the determinant of such a matrix is not well-defined in the general case. As such, slightly different definitions from those presented here will be required.

To begin, we fix notation by writing $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ whenever our discussion can apply equally well to either case and define $\mathbb{M}_r(\mathbb{F})$ to be the collection of all $r \times r$ matrices with elements in $\mathbb{F}$.

For $\mathbf{A}, \mathbf{B} \in \mathbb{M}_r(\mathbb{F})$, the determinant map satisfies

$$\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B} \tag{2.24}$$

$$\det \mathbf{A}^T = \det \mathbf{A} \tag{2.25}$$

$$\det \mathbf{A}^\dagger = \overline{\det \mathbf{A}} \tag{2.26}$$

$$\det a\mathbf{A} = a^n \det \mathbf{A}, \tag{2.27}$$

and, if $\mathbf{A}$ is invertible,

$$\det \mathbf{A}^{\mathbf{-1}} = \frac{1}{\det \mathbf{A}}. \tag{2.28}$$

The first matrix group of interest is $\mathrm{GL}(r;\mathbb{F})$, defined to be the set of invertible $r \times r$ real matrices:

$$\mathrm{GL}(r;\mathbb{F}) \equiv \{\mathbf{A} \in \mathbb{M}_r(\mathbb{F}) : \det \mathbf{A} \neq 0\}. \tag{2.29}$$

Within $\mathrm{GL}(r;\mathbb{F})$ we have the subgroup $\mathrm{SL}(r;\mathbb{F})$, defined by

$$\mathrm{SL}(r;\mathbb{F}) \equiv \{\mathbf{A} \in \mathrm{GL}(r;\mathbb{F}) : \det \mathbf{A} = 1\}. \tag{2.30}$$

Other subgroups of $\mathrm{GL}(r;\mathbb{F})$ in which we will be interested are the orthogonal and unitary groups. Specifically, the orthogonal group of order (m,n) is defined as

$$\mathrm{O}(m,n;\mathbb{F}) \equiv \{\mathbf{O} \in \mathrm{GL}(m+n;\mathbb{F}) : \mathbf{O}^T\mathbf{G}\mathbf{O} = \mathbf{G}\}, \tag{2.31}$$

where $\mathbf{G}$ is a diagonal matrix with elements

$$\mathbf{G}_{\mu\nu} = \begin{cases} 0, & \mu \neq \nu \\ 1, & \mu = \nu \leq m \\ -1, & m < \mu = \nu \end{cases}. \tag{2.32}$$

If we consider the vector space V consisting of column vectors with $m+n$ elements from $\mathbb{F}$, then $\mathbf{G}$ determines a bilinear, indefinite, symmetric quadratic form $\widehat{G}$ on $V \times V$ via the map

$$\widehat{G}(v,w) = v^T\mathbf{G}w, \tag{2.33}$$

for any v and w in V. Then the elements of $\mathrm{O}(m,n;\mathbb{F})$ are seen to be precisely those linear maps acting on V through left multiplication that preserve (2.33).[2]

The unitary group $\mathrm{U}(m,n;\mathbb{F})$ is defined analogously, replacing transpose with *conjugate* transpose:

$$\mathrm{U}(m,n;\mathbb{F}) \equiv \{\mathbf{U} \in \mathrm{GL}(m+n;\mathbb{F}) : \mathbf{U}^\dagger\mathbf{G}\mathbf{U} = \mathbf{G}\}, \tag{2.34}$$

where $\mathbf{G}$ is as in (2.32). Once again considering the vector space V, we can see that $\mathbf{G}$ can also be used to determine a *sesquilinear*, indefinite, symmetric quadratic form $\widehat{G}$ on $V \times V$ via

$$\widehat{G}(v,w) = v^\dagger\mathbf{G}w, \tag{2.35}$$

[2]Given a vector space V, a quadratic form ϕ on $V \times V$ that is linear in the first variable and either linear or sesquilinear in the second is sometimes called a (bilinear or sesquilinear) "metric". Alternatively, the term metric is sometimes reserved for cases where ϕ satisfies additional requirements, such as being positive-definite or symmetric. We avoid any ambiguity by forgoing the use of the term altogether.

Then, as with $\mathrm{O}(m,n;\mathbb{F})$, $\mathrm{U}(m,n;\mathbb{F})$ represents the set of linear transformations on V that preserve (2.35).

A third matrix group can be defined in terms of the matrix

$$\mathbf{J} = \begin{pmatrix} 0 & \mathbf{I}_r \\ -\mathbf{I}_r & 0 \end{pmatrix}, \tag{2.36}$$

where $\mathbf{I}_r$ is the $r \times r$ identity matrix. In that case we define the symplectic group as

$$\mathrm{Sp}(2r;\mathbb{F}) = \left\{ \mathbf{S} \in \mathrm{GL}(2r;\mathbb{F}) : \mathbf{S}^\dagger \mathbf{J} \mathbf{S} = \mathbf{J} \right\} \tag{2.37}$$

The structure of $\mathbf{J}$ allows it to determine a sesquilinear, *antisymmetric* quadratic form $\widehat{J}$ defined as

$$\widehat{J}(v,w) = v^\dagger \mathbf{J} w, \tag{2.38}$$

and we see as in the other cases that $\mathrm{Sp}(2r;\mathbb{F})$ is the set of linear transformations on V that preserve $\widehat{J}$.

If we now take the intersection of each of $\mathrm{O}(m,n;\mathbb{F})$ and $\mathrm{U}(m,n;\mathbb{F})$ with $\mathrm{SL}(m+n;\mathbb{F})$, we get the *special* orthogonal and unitary groups, $\mathrm{SO}(m,n;\mathbb{F})$ and $\mathrm{SU}(m,n;\mathbb{F})$. It turns out, though we do not show here, that $\mathrm{Sp}(2r;\mathbb{F}) \subset \mathrm{SL}(2r;\mathbb{F})$, so there is no special symplectic group. Finally, we note that if $\mathbb{F} = \mathbb{R}$, the unitary and orthogonal groups are identical.

The above groups are examples of *Lie groups*; that is, they carry the structure of a smooth differentiable manifold in such a way that matrix multiplication and inversion are smooth maps. While a full discussion of the topological and geometric properties of such groups is beyond the scope of this work, the following foray into differential geometry will prove useful.

Given a matrix Lie group, G, consisting of $r \times r$ matrices with elements in $\mathbb{F}$, we can consider the tangent space at the identity, denoted $T_{\mathbf{I}}G$. To define this space, we consider the set $\mathcal{C}$ of curves in G with certain special properties:

$$\mathcal{C} \equiv \{c : \mathbb{R} \to G : c(0) = \mathbf{I}, c(\alpha)c(\beta) = c(\alpha+\beta) \text{ for } \alpha,\beta \in \mathbb{R}\}. \tag{2.39}$$

Each such curve is a *one-parameter family* of group elements. For $c \in \mathcal{C}$ we define the tangent vector to c at the identity to be

$$\dot{c}(0) = \left.\frac{dc}{d\alpha}\right|_0 , \tag{2.40}$$

where the derivative is applied component wise. For example, if we take $G = \mathrm{SL}(2;\mathbb{R})$ and c to be the map

$$c(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \tag{2.41}$$

then $c(0) = \mathbf{I}$ and $\dot{c}(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We then define $T_{\mathbf{I}}G$ to be

$$T_{\mathbf{I}}G \equiv \{\dot{c}(0) : c \in \mathcal{C}\} . \tag{2.42}$$

Next, we define the exponential map $\exp : T_{\mathbf{I}}G \to G$ by the equation

$$\exp[\mathbf{A}] = \sum_{\mu=0}^{\infty} \frac{1}{\mu!}\mathbf{A}^{\mu} \tag{2.43}$$

for $\mathbf{A} \in T_{\mathbf{I}}G$. Specifically,

$$c(\alpha) = \exp[\alpha\mathbf{A}] \in \mathcal{C} \tag{2.44}$$

is a curve in G with tangent vector $\mathbf{A}$ at $\mathbf{I}$. For any element $\mathbf{g}$ in the connected component of the identity of G, there is a curve $c \in \mathcal{C}$ with $c(1) = \mathbf{g}$. There is thus a corresponding tangent vector $\mathbf{A} \in T_{\mathbf{I}}G$ such that $\exp[\mathbf{A}] = \mathbf{g}$. We therefore say that $T_{\mathbf{I}}G$ "generates" the connected component of the identity and call a basis of $T_{\mathbf{I}}G$ "generators" of G. Considering the real vector space structure of $T_{\mathbf{I}}G$, we can turn it into an algebra by defining an appropriate product,

$$[\cdot,\cdot] : T_{\mathbf{I}}G \times T_{\mathbf{I}}G \to T_{\mathbf{I}}G. \tag{2.45}$$

Specifically, $T_{\mathbf{I}}G$ becomes a *Lie algebra* if that product is bilinear, alternating, and satisfies the Jacobi identity

$$[\mathbf{A},[\mathbf{B},\mathbf{C}]] + [\mathbf{B},[\mathbf{C},\mathbf{A}]] + [\mathbf{C},[\mathbf{B},\mathbf{A}]] = \mathbf{0} \tag{2.46}$$

for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in G$. The standard product satisfying these conditions is the commutator, defined by

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}. \tag{2.47}$$

In that case, the algebra is generally denoted with the same letters as for G, only made lower-case and written in Fraktur. For example, the Lie algebra associated with $\mathrm{SL}(2; \mathbb{R})$ is denoted $\mathfrak{sl}(2; \mathbb{R})$. We also define the anticommutator here, which is given by

$$\{\mathbf{A}, \mathbf{B}\} = \mathbf{AB} + \mathbf{BA}. \tag{2.48}$$

Other Lie algebras that will be needed in Section 2.3 are

$$\mathfrak{sq}(r) = \left\{ \mathbf{A} \in \mathbb{M}_r(\mathbb{H}) : \mathbf{A}^\dagger = -\mathbf{A} \right\}, \tag{2.49}$$

$$\mathfrak{sa}(r; \mathbb{K}) = \left\{ \mathbf{A} \in \mathbb{M}_r(\mathbb{K}) : \mathbf{A}^\dagger = -\mathbf{A}, \mathrm{tr}(\mathbf{A}) = 0 \right\}, \tag{2.50}$$

$$\mathfrak{sl}(r; \mathbb{H}) = \{ \mathbf{A} \in \mathbb{M}_r(\mathbb{H}) : \Re e(\mathrm{tr}\mathbf{A}) = 0 \}, \tag{2.51}$$

$$\mathfrak{sp}(2r; \mathbb{H}) = \left\{ \mathbf{A} \in \mathbb{M}_r(\mathbb{H}) : \mathbf{A}^\dagger \mathbf{J} = -\mathbf{JA}, \Re e(\mathrm{tr}\mathbf{A}) = 0 \right\}, \tag{2.52}$$

$$\mathrm{der}(\mathbb{K}) = \{ F \in \mathrm{end}(\mathbb{K}) : F(ab) = F(a)b + aF(b), \forall a, b \in \mathbb{K} \}, \tag{2.53}$$

where $\mathbf{J}$ is as in (2.36) and we $\mathbb{K}$ can be be any of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. In the case of $\mathrm{der}(\mathbb{K})$, the product is given by

$$[F_1, F_2](a) = F_1(F_2(a)) - F_2(F_1(a)), \tag{2.54}$$

for all $F_1, F_2 \in \mathrm{end}(\mathbb{K})$ and $a \in \mathbb{K}$; for the matrix algebras the product is the usual commutator. From these definitions we see that

$$\mathfrak{sa}(r; \mathbb{R}) = \mathfrak{so}(r; \mathbb{R}) = \mathfrak{su}(r; \mathbb{R}), \tag{2.55}$$

$$\mathfrak{sa}(r; \mathbb{C}) = \mathfrak{su}(r; \mathbb{C}), \tag{2.56}$$

and that $\mathfrak{sa}(r; \mathbb{H})$ as simply the traceless subalgebra of $\mathfrak{sq}(r)$.

Before moving on, we record the following standard, but important, properties of the exponential map: For matrices $\mathbf{A}, \mathbf{B} \in G$ and constant $a \in \mathbb{K}$,

$$\begin{aligned}
\exp[0] &= \mathbf{I}, \\
\exp[\mathbf{A}\mathbf{B}\mathbf{A}^{-1}] &= \mathbf{A}\exp[\mathbf{B}]\mathbf{A}^{-1} \\
\exp[-\mathbf{A}] &= \exp[\mathbf{A}]^{-1} \\
\exp[a\mathbf{A}] &= \exp[\mathbf{A}]^{a} \\
\exp[\mathbf{A}]\mathbf{B} &= \begin{cases} \mathbf{B}\exp[\mathbf{A}], & [\mathbf{A}, \mathbf{B}] = 0 \\ \mathbf{B}\exp[-\mathbf{A}], & \{\mathbf{A}, \mathbf{B}\} = 0 \end{cases}, \\
\exp[\mathbf{A}^{T}] &= \exp[\mathbf{A}]^{T} \\
\exp[\mathbf{A}^{\dagger}] &= \exp[\mathbf{A}]^{\dagger} \\
\det\left(\exp[\mathbf{A}]\right) &= e^{\operatorname{tr}(\mathbf{A})}
\end{aligned} \tag{2.57}$$

Moreover, if $\mathbf{A}^2 = \pm\mathbf{I}$, we have that

$$\exp[\mathbf{A}\alpha] = \mathbf{I}f(\alpha) + \mathbf{A}h(\alpha) = \begin{cases} \mathbf{I}\cosh(\alpha) + \mathbf{A}\sinh(\alpha), & \mathbf{A}^2 = \mathbf{I} \\ \mathbf{I}\cos(\alpha) + \mathbf{A}\sin(\alpha), & \mathbf{A}^2 = -\mathbf{I} \end{cases}, \tag{2.58}$$

where the second equality serves to define the functions f and h.

We now consider certain special matrix groups that will be of particular interest to us. First, we consider the group $\mathrm{SO}(3, 1; \mathbb{R})$, which is the group of 4×4 matrices with real elements that have determinant 1 and satisfy

$$\mathbf{O}^{T}\mathbf{G}\mathbf{O} = \mathbf{G}, \tag{2.59}$$

where $\mathbf{G}$ is a diagonal matrix, $\mathbf{G} = \operatorname{diag}(1, 1, 1, -1)$. It follows from the symmetry of (2.59) that it imposes 10 constraints on the 16 possible components of $\mathbf{O}$. Taking the determinant of (2.59), we see that $\det\mathbf{O} = \pm 1$, so the requirement that $\det\mathbf{O} = 1$ does not impose any new constraints on the number of degrees of freedom. In fact, in general, $\mathrm{SO}(m, n; \mathbb{R})$ is the connected component of the identity in $\mathrm{O}(m, n; \mathbb{R})$, so they have the same dimension. Therefore, $\mathrm{SO}(3, 1; \mathbb{R})$ is six-dimensional. One way to identify this group is to find a set of generators for its Lie algebra, $\mathfrak{so}(3, 1; \mathbb{R})$, by finding six linearly independent matrices that exponentiate to elements of $\mathrm{SO}(3, 1; \mathbb{R})$. To find *these* elements we note that the corresponding quadratic form in the sense of (2.33) is the Lorentz metric, so we expect $\mathrm{SO}(3, 1; \mathbb{R})$ to be the Lorentz group. We therefore consider the standard rotation matrices, such as

$$\mathbf{R}_{xy} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.60}$$

and boosts such as

$$\mathbf{R}_{tz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\alpha & \sinh\alpha \\ 0 & 0 & \sinh\alpha & \cosh\alpha \end{pmatrix}. \tag{2.61}$$

The other basic rotations and boosts have similar forms and are named similarly: $\mathbf{R}_{yz}$ and $\mathbf{R}_{zx}$ are the rotations in the yz- and zx-planes, respectively, while $\mathbf{R}_{ty}$ and $\mathbf{R}_{tz}$ are the boosts in the y and z directions. At $\alpha = 0$, each of these matrices is equal to the identity, so they represent curves in the sense of (2.39). Taking their derivatives and evaluating at the origin, we find, for example,

$$\mathbf{r}_{xy} = \dot{\mathbf{R}}_{xy}\Big|_{\alpha=0} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.62}$$

and

$$\mathbf{r}_{tz} = \dot{\mathbf{R}}_{tz}\Big|_{\alpha=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.63}$$

where we have introduced the notation $\mathbf{r}_{\mu\nu}$ for the element of $\mathfrak{so}(3,1;\mathbb{R})$ that generates of $\mathbf{R}_{\mu\nu}$. The six matrices constructed in this way are readily found to be linearly independent, so we do indeed have a basis for $\mathfrak{so}(3,1;\mathbb{R})$. We therefore conclude that $\mathrm{SO}(3,1;\mathbb{R})$ is precisely the Lorentz group, with general elements formed from products of matrices of the form (2.60) and (2.61).

We next consider the complex matrix group $\mathrm{SL}(2;\mathbb{C})$. The general 2×2 matrix has four complex components, so eight real degrees of freedom. The requirement that the determinant be 1 removes one complex degree of freedom, leaving six real degrees of freedom. We therefore expect $\mathfrak{sl}(2;\mathbb{C})$ to be six dimensional. In this case we have no additional constraints, and we note from (2.57) that $\det(\exp[\mathbf{A}]) = 1$ if and only if $\mathrm{tr}(\mathbf{A}) = 0$. We therefore seek six linearly independent (over $\mathbb{R}$) traceless 2×2 matrices. Perhaps the most common choice consists of the Pauli matrices,

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.64}$$

together with

$$i\sigma_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, i\sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.65}$$

Noting that the matrices in (2.64) all square to $\mathbf{I}$ while those in (2.65) all square to $-\mathbf{I}$, we can exponentiate them (applying (2.58)) to get representative elements of $\mathrm{SL}(2;\mathbb{C})$.

Explicitly, the corresponding group elements are

$$\exp[\sigma_z\alpha] = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}, \exp[\sigma_y\alpha] = \begin{pmatrix} \cosh\alpha & -i\sinh\alpha \\ i\sinh\alpha & \cosh\alpha \end{pmatrix},$$

$$\exp[\sigma_x\alpha] = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}, \exp[i\sigma_z\alpha] = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix},$$

$$\exp[i\sigma_y\alpha] = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \exp[i\sigma_x\alpha] = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}.$$

We next consider $\mathrm{SO}(4,2;\mathbb{R})$, elements of which are 6×6 matrices satisfying

$$\mathbf{O}^T\mathbf{G}\mathbf{O} = \mathbf{G}. \tag{2.66}$$

Equation (2.66) has the same form as (2.59), except that here $\mathbf{G}$ is a 6×6 diagonal matrix, $\mathbf{G} = \mathrm{diag}(1,1,1,1,-1,-1)$. Due to symmetry, (2.66) imposes 21 restrictions on the 36 potential degrees of freedom, leaving 15. In particular, we expect this group to act on six-component column vectors in the form of rotations between the first four dimensions, corresponding boosts between the first four dimensions and each of the fifth and sixth, and a rotation in the fifth and sixth dimension. These give six, eight, and one basic transformation each, covering the expected 15. Explicitly, in analogy with (2.60) and (2.63) we have $\mathrm{SO}(3,1;\mathbb{R})$ rotations such as,

$$\mathbf{r}_{xy} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.67}$$

and SO$(3,1;\mathbb{R})$ boosts such as

$$\mathbf{r}_{tz} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.68}$$

along with new rotations such as

$$\mathbf{r}_{qy} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.69}$$

and new boosts such as

$$\mathbf{r}_{pq} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{2.70}$$

Finally, we consider SU$(2,2;\mathbb{C})$. For a generator $\mathbf{A}$ to generate an element of SU$(2,2;\mathbb{C})$ it must satisfy the condition

$$\exp[t\mathbf{A}]^{\dagger}\mathbf{G}\exp[t\mathbf{A}] = \mathbf{G}, \tag{2.71}$$

where $\mathbf{G} = \mathrm{diag}(1,1,-1-1)$. We can rewrite this as

$$\exp[t\mathbf{A}]^{\dagger}\mathbf{G}\exp[t\mathbf{A}]\mathbf{G}^{-1} = 1, \tag{2.72}$$

and then note from (2.57) and the fact that we require $\det(\exp[\mathbf{A}]) = 1$ that $\mathbf{A}$ can be a generator of $\mathrm{SU}(2,2;\mathbb{C})$ if and only if $\mathbf{A}$ is a traceless matrix satisfying

$$\mathbf{A}^\dagger = -\mathbf{GAG}. \tag{2.73}$$

In other words, $\mathfrak{su}(2,2;\mathbb{C})$ is the subalgebra of $\mathfrak{sa}(2,2;\mathbb{C})$ consisting of traceless matrices. We have sixteen complex degrees of freedom to begin, and (2.71) imposes eight complex conditions, reducing that to eight complex degrees of freedom. However, the requirement that the determinant be one, rather than just have norm 1, reduces one of those to a real degree of freedom, so we have fifteen real degrees of freedom with which to work. We therefore require fifteen 4×4 traceless matrices that are linearly independent over $\mathbb{R}$ and satisfy (2.73). We forgo listing generators for this representation because $\mathrm{SU}(2,2;\mathbb{C})$ is isomorphic to $\mathrm{SO}(4,2;\mathbb{R})$[12], so we have already constructed a representation of $\mathrm{SU}(2,2;\mathbb{C})$.

2.2.2 Clifford Algebras

A matrix Clifford algebra is a matrix algebra generated by a set of $n \times n$ matrices, $\{\mathbf{\Gamma}_\mu\}$, satisfying the anticommutation relations

$$\{\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_\nu\} = 2g_{\mu\nu}\mathbf{I} \tag{2.74}$$

for some constants $g_{\mu\nu} = g_{\nu\mu}$. For our purposes, we will assume

$$g_{\mu\nu} = \begin{cases} 0, & \mu \neq \nu \\ 1, & 1 \leq \mu \leq m \\ -1, & m < \mu \leq r \end{cases}, \tag{2.75}$$

for some integers m and r.

Given a matrix algebra, one can define the Jordan product

$$\mathbf{A} \circ \mathbf{B} = \frac{1}{2}(\mathbf{AB} + \mathbf{BA}) = \frac{1}{2}\{\mathbf{A}, \mathbf{B}\}. \tag{2.76}$$

Using (2.76), we define an (indefinite) inner product on the vector space $V = \mathrm{span}\,\{\mathbf{\Gamma}_\mu\}$ by

$$\langle \mathbf{A}, \mathbf{B} \rangle = \frac{1}{n} \mathrm{tr}\,(\mathbf{A} \circ \mathbf{B})\,. \tag{2.77}$$

The factor of $1/n$ is introduced so that the generators $\{\mathbf{\Gamma}_\mu\}$ comprise an orthonormal basis with respect to this inner product:

$$\langle \mathbf{\Gamma}_\mu, \mathbf{\Gamma}_\nu \rangle = \frac{1}{n} \mathrm{tr}\left(\frac{1}{2} \{\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_\nu\} \right) = \frac{1}{n} g_{\mu\nu} \mathrm{tr}\,(\mathbf{I}) = g_{\mu\nu}. \tag{2.78}$$

2.2.3 Special Orthogonal Groups

We now consider how to use a representation of the Clifford algebra $C\ell_{m,n}(\mathbb{R})$ to construct a representation of $\mathrm{SO}(m, n; \mathbb{R})$. Specifically, we will show that the homogenous quadratic elements of $C\ell_{m,n}(\mathbb{R})$ act as generators of $\mathrm{SO}(m, n; \mathbb{R})$ via the map

$$\mathbf{P} \to \mathbf{M}_{\mu,\nu} \mathbf{P} \mathbf{M}_{\mu,\nu}^{-1}, \tag{2.79}$$

where $\mathbf{M}_{\mu,\nu} = \exp[\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\nu \frac{\theta}{2}]$, $\mathbf{P} = \sum_\mu^{m+n} x^\mu \mathbf{\Gamma}_\mu$, and the matrices $\{\mathbf{\Gamma}_\mu\}$ are the generators of $C\ell_{m,n}(\mathbb{R})$,

In the following, we assume μ, ν, τ are indices no two of which are equal but which are otherwise arbitrary. The following properties follow from the Clifford algebra anti-commutation relation, $\{\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_\nu\} = 2 g_{\mu,\nu} \mathbf{I}$:

$$\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\mu \equiv \mathbf{\Gamma}_\mu^2 = \pm \mathbf{I}, \tag{2.80}$$

$$(\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\nu) \mathbf{\Gamma}_\tau = \mathbf{\Gamma}_\tau (\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\nu), \tag{2.81}$$

$$(\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\nu) \mathbf{\Gamma}_\nu = (\mathbf{\Gamma}_\nu)^2 \mathbf{\Gamma}_\mu = g_{\nu,\nu} \mathbf{\Gamma}_\mu, \tag{2.82}$$

$$(\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\nu) \mathbf{\Gamma}_\mu = -(\mathbf{\Gamma}_\mu)^2 \mathbf{\Gamma}_\nu = -g_{\mu,\mu} \mathbf{\Gamma}_\nu, \text{ and} \tag{2.83}$$

$$(\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\nu)^2 = -\mathbf{\Gamma}_\mu^2 \mathbf{\Gamma}_\nu^2 = \pm \mathbf{I}. \tag{2.84}$$

With these observations, we are prepared to see how $\mathrm{SO}(m, n; \mathbb{R})$ is generated by the matrices $\{\mathbf{\Gamma}_\mu\}$. We compute

$$\mathbf{M}_{\mu,\nu}\mathbf{P}\mathbf{M}_{\mu,\nu}^{-1} = \exp[\mathbf{\Gamma}_\mu\mathbf{\Gamma}_\nu\theta/2]\left(\sum_{\tau=1}^{\mu+\nu} x^\tau\mathbf{\Gamma}_\tau\right)\exp[-\mathbf{\Gamma}_\mu\mathbf{\Gamma}_\nu\theta/2]. \tag{2.85}$$

From property (2.80) and (2.57), we see that if $\mu = \nu$, then (2.85) becomes

$$\mathbf{M}_{\mu,\nu}\left(\sum_{\tau=1}^{m+n} x^\tau\mathbf{\Gamma}_\tau\right)\mathbf{M}_{\mu,\nu}^{-1} = \exp[(\mathbf{\Gamma}_\mu\mathbf{\Gamma}_\mu - \mathbf{\Gamma}_\mu\mathbf{\Gamma}_\mu)\,\theta]\left(\sum_{\tau=1}^{m+n} x^\tau\mathbf{\Gamma}_\tau\right) = \sum_{\tau=1}^{m+n} x^\tau\mathbf{\Gamma}_\tau, \tag{2.86}$$

so that $\mathbf{P}$ is left unchanged. On the other hand, for $\mu \neq \nu$, we see from properties (2.81) and (2.82) that $\mathbf{M}_{\mu,\nu}$ (and therefore $\mathbf{M}_{\mu,\nu}^{-1}$) commutes with all but two of the matrices $\mathbf{\Gamma}_\mu$; specifically,

$$\mathbf{\Gamma}_\tau\mathbf{M}_{\mu,\nu}^{-1} = \begin{cases}\mathbf{M}_{\mu,\nu}\mathbf{\Gamma}_\tau, & \tau = \mu \text{ or } \nu \\ \\ \mathbf{M}_{\mu,\nu}^{-1}\mathbf{\Gamma}_\tau, & \mu \neq \tau \neq \nu\end{cases} \tag{2.87}$$

Therefore, the action of $\mathbf{M}_{\mu,\nu}$ on $\mathbf{P}$ affects only the $\mu\nu$ subspace. To see what that action is we use (2.58) and write out

$$\begin{aligned}\mathbf{M}_{\mu,\nu}(p^\mu\mathbf{\Gamma}_\mu + p^\nu\mathbf{\Gamma}_\nu)\mathbf{M}_{\mu,\nu}^{-1} &= \mathbf{M}_{\mu,\nu}^2(p^\mu\mathbf{\Gamma}_\mu + p^\nu\mathbf{\Gamma}_\nu) = \exp[\mathbf{\Gamma}_\mu\mathbf{\Gamma}_\nu\theta](p^\mu\mathbf{\Gamma}_\mu + p^\nu\mathbf{\Gamma}_\nu) \\ &= [\mathbf{I}f(\theta) + \mathbf{\Gamma}_\mu\mathbf{\Gamma}_\nu h(\theta)]\,(p^\mu\mathbf{\Gamma}_\mu + p^\nu\mathbf{\Gamma}_\nu) \\ &= [p^\mu f(\theta) + p^\nu h(\theta)g_{\nu,\nu}]\,\mathbf{\Gamma}_\mu + [p^\nu f(\theta) - p^\mu h(\theta)g_{\mu,\mu}]\,\mathbf{\Gamma}_\nu.\end{aligned} \tag{2.88}$$

From (2.74), we have

$$(\mathbf{\Gamma}_\mu\mathbf{\Gamma}_\nu)^2 = -\mathbf{I} \tag{2.89}$$

for $\mu, \nu \leq m$ or $\mu, \nu > m$, so in those cases the action of $\mathbf{M}_{\mu,\nu}$ is precisely rotation from μ to ν in the $\mu\nu$-plane. On the other hand we have

$$(\mathbf{\Gamma}_\mu\mathbf{\Gamma}_\nu)^2 = \mathbf{I} \tag{2.90}$$

for $\mu \leq m < \nu$ or vice-versa, so that $\mathbf{M}_{\mu,\nu}$ yields a boost in the positive ν direction. Thus, we have constructed $\mathrm{SO}(n, m; \mathbb{R})$.

2.3 Magic Squares

We now turn to a brief discussion of the relationship between the division algebras and certain Lie groups, which can be summarized in terms of "magic squares" of Lie algebras. A complete construction of these (and related) magic squares can be found in [2]; we give here a summary of the information relevant to this work.

2.3.1 The 3 × 3 Magic Square

The Freudenthal-Tits magic square is given in Table 2.4, where, according to the Cartan-Killing classification [15],

$$a_r = \mathfrak{su}(r+1;\mathbb{R}) \tag{2.91}$$

$$b_r = \mathfrak{so}(2r+1;\mathbb{R}) \tag{2.92}$$

$$c_r = \mathfrak{sp}(2r;\mathbb{R}) \tag{2.93}$$

$$d_r = \mathfrak{so}(2r;\mathbb{R}), \tag{2.94}$$

and f_4, e_6, e_7, and e_8 are four of the five exceptional Lie algebras.[3]

$\mathbb{K}_1\backslash\mathbb{K}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	a_1	a_2	c_3	f_4
$\mathbb{C}$	a_2	$a_2 \oplus a_2$	a_5	e_6
$\mathbb{H}$	c_3	a_5	d_6	e_7
$\mathbb{O}$	f_4	e_6	e_7	e_8

TABLE 2.4: The Freudenthal magic square

The interpretation here, following the Tits construction, is that the algebra in the

[3]We work throughout with real Lie algebras; i.e., *real* linear combinations of the elements in $T_{\mathbf{I}}G$.

$(\mathbb{K}_1, \mathbb{K}_2)$ slot is isomorphic to the algebra

$$\mathrm{der}(\mathbb{K}_1) \oplus \mathrm{der}(\mathbb{J}) \oplus \mathbb{K}_1' \otimes \mathbb{J}', \tag{2.95}$$

where $\mathbb{J}$ is the algebra of Hermitian 3×3 matrices with elements in $\mathbb{K}_2$, $\mathbb{K}_1'$ is the subspace of $\mathbb{K}$ orthogonal to the identity under the inner product

$$\langle a, b \rangle = \Re e[a\bar{b}], \tag{2.96}$$

and $\mathbb{J}'$ is the subspace of $\mathbb{J}$ orthogonal to the identity under the inner product

$$\langle \mathbf{A}, \mathbf{B} \rangle = \frac{1}{2} \mathrm{tr}\,(\mathbf{AB})\,; \tag{2.97}$$

i.e., $\mathbb{J}'$ is the algebra of *traceless* Hermitian matrices with elements in $\mathbb{K}_2$. The Tits construction (2.95) is not obviously symmetric, so the symmetric nature of Table 2.4 is somewhat surprising. This symmetry is explained by the Vinberg construction [21], wherein Table 2.4 remains unchanged, but the interpretation is different. In the Vinberg construction, the algebra in the $(\mathbb{K}_1, \mathbb{K}_2)$ slot is seen to be isomorphic to the algebra

$$\mathrm{der}(\mathbb{K}_1) \oplus \mathrm{der}(\mathbb{K}_2) \oplus \mathfrak{sa}(3; \mathbb{K}_1 \otimes \mathbb{K}_2), \tag{2.98}$$

in which case the symmetry is manifest.

Replacing the division algebras $\mathbb{K}_1$ in Table 2.4 with their split counterparts, one obtains a new magic square, shown in Table 2.5, where the extra labels on the exceptional Lie algebras indicate particular real forms.

Tables 2.4 and 2.5 describe isomorphisms between certain interesting Lie algebras and Lie algebras constructed in part from 3×3 matrices with elements in the division algebras and their split counterparts. Of particular interest is the presence of the exceptional Lie algebras. As discussed in [8, 17, 11], the *Lie group* E_6 has potential application in particle physics, and one might suspect that understanding the structure of the other exceptional Lie groups could be applicable there as well[13, 16, 6]. The presence of these groups in the magic square suggests an approach to determining their group structure, which is the long term goal of which this work is a small part.

$\mathbb{K}_1\backslash\mathbb{K}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(3;\mathbb{R})$	$\mathfrak{su}(3;\mathbb{C})$	$\mathfrak{sq}(3)$	$f_{4(52)}$
$\mathbb{C}'$	$\mathfrak{sl}(3;\mathbb{R})$	$\mathfrak{sl}(3;\mathbb{C})$	$\mathfrak{sl}(3;\mathbb{H})$	$e_{6(26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6;\mathbb{R})$	$\mathfrak{su}(3,3;\mathbb{C})$	$\mathfrak{sp}(6;\mathbb{H})$	$e_{7(25)}$
$\mathbb{O}'$	$f_{4(-4)}$	$e_{6(-2)}$	$e_{7(5)}$	$e_{8(24)}$

TABLE 2.5: The split algebra version of the Freudenthal magic square

2.3.2 The 2×2 Magic Square

In [2], a construction of a related magic square for 2×2 matrices is given. This magic square, given in Table 2.6, is constructed by analogy with (2.98),

$$\mathrm{der}(\mathbb{K}_1) \oplus \mathrm{der}(\mathbb{K}_2) \oplus \mathfrak{sa}(2;\mathbb{K}_1 \otimes \mathbb{K}_2), \tag{2.99}$$

with an important exception. As discussed in [2], one must replace $\mathrm{der}(\mathbb{K})$ with $\mathfrak{so}\,(\Im m\mathbb{K})$. In the case where $\mathbb{K}$ is associative, these two algebras are the same, but they differ when $\mathbb{K}$ is not associative, as in the case of $\mathbb{O}$ and $\mathbb{O}$'. It is therefore necessary to replace $\mathrm{der}(\mathbb{O})$ by $\mathfrak{so}(7;\mathbb{R})$ wherever it occurs in this construction, with a similar replacement being made for $\mathrm{der}(\mathbb{O}')$.

$\mathbb{K}_1\backslash\mathbb{K}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2;\mathbb{R})$	$\mathfrak{so}(3;\mathbb{R})$	$\mathfrak{so}(5;\mathbb{R})$	$\mathfrak{so}(9;\mathbb{R})$
$\mathbb{C}'$	$\mathfrak{so}(2,1;\mathbb{R})$	$\mathfrak{so}(3,1;\mathbb{R})$	$\mathfrak{so}(5,1;\mathbb{R})$	$\mathfrak{so}(9,1;\mathbb{R})$
$\mathbb{H}'$	$\mathfrak{so}(3,2;\mathbb{R})$	$\mathfrak{so}(4,2;\mathbb{R})$	$\mathfrak{so}(6,2;\mathbb{R})$	$\mathfrak{so}(10,2;\mathbb{R})$
$\mathbb{O}'$	$\mathfrak{so}(5,4;\mathbb{R})$	$\mathfrak{so}(6,4;\mathbb{R})$	$\mathfrak{so}(8,4;\mathbb{R})$	$\mathfrak{so}(12,4;\mathbb{R})$

TABLE 2.6: The split algebra version of the 2×2 magic square

Observe that the first row of Table 2.6 consists of the rotation groups in dimensions 2, 3, 5, and 9 respectively, the elements of the second row are the Lie algebras associated to the Lorentz groups in dimension 3, 4, 6, and 10 respectively, and the third row is composed of the Lie algebras for the corresponding conformal groups. This raises the question of whether an analogous relationship holds in the 3×3 case in Table 2.5. In fact, Freudenthal was able to show that $E_{7(-25)}$ can be interpreted as the conformal group corresponding to $E_{6(-26)}$[9], suggesting that the interpretation of the third row as the conformal group corresponding to the second row may be correct.

So far, we have considered magic squares of Lie algebras. Each of these can be trivially expressed as a magic square of Lie groups as well, but the geometric interpretation of the group level magic squares is not so well understood. In [7], Manogue and Dray conjecture that the elements of the group version of Table 2.6 take the form $\mathrm{SU}(2;\mathbb{K}_1 \otimes \mathbb{K}_2)$, defined to be the set of 2×2 matrices $\mathbf{A}$ satisfying

$$\det(\mathbf{A}\mathbf{X}\mathbf{A}^\dagger) = \det(\mathbf{X}), \tag{2.100}$$

whenever $\mathbf{X}$ has the form

$$\mathbf{X} = \begin{pmatrix} A & \overline{a} \\ a & A^* \end{pmatrix} \tag{2.101}$$

for some $A \in \mathbb{K}_1$ and $a \in \mathbb{K}_2$ and where, for now, the conjugation implied by $\dagger$ is with respect to $\mathbb{K}_1$ only. Again, this raises the question of whether a similar description can be given the for 3×3 magic square. One of the aims of the present work is to test whether this is, in fact, an appropriate definition of $\mathrm{SU}(2;\mathbb{K}_1 \otimes \mathbb{K}_2)$ by explicitly constructing a 2×2 matrix group representing the $\mathbb{H}' \otimes \mathbb{C}$ spot in the 2×2 magic square.

3 RESULTS

We introduce here a new description of the groups in the 2×2 magic square in terms of division algebras.

3.1 A New Look at the 2×2 Magic Square

Being a brief overview of the matrix groups and algebras used here.

3.1.1 Matrix Groups in the 2×2 Magic Square

As discussed above, matrix groups are constructed from $n \times n$ matrices satisfying various conditions using matrix multiplication as the group operation. The construction given above were all expressed in terms of determinants of matrices over $\mathbb{R}$ or $\mathbb{C}$. In the present case, however, we will be interested in matrices over algebras of the form $\mathbb{K}' \otimes \mathbb{K}$, where $\mathbb{K}$ is a normed division algebra and $\mathbb{K}'$ is a split algebra. We will shortly restrict further to matrices over $\mathbb{H}' \otimes \mathbb{C}$, but for the time being we remain general. The common characterisation of real and complex matrix groups in terms of their determinants does not immediately generalize to matrices over these algebras, due to the potentially nonabelian nature of multiplication in $\mathbb{K}$ and/or $\mathbb{K}'$, which prevents the determinant of such a matrix from being well-defined in the general case. As such, we will proceed with an explicit construction of the relevant matrix groups, and then use our construction to *define* an appropriate notion of determinant.

3.1.2 Clifford Algebras

We begin by constructing representations of certain Clifford algebras over $\mathbb{R}$ in terms of matrices over $\mathbb{K}' \otimes \mathbb{K}$ where $\mathbb{K}$ is a division algebra and $\mathbb{K}'$ is a split division algebra. In principle, the octonionic case would present additional problems due to the non-associativity of $\mathbb{O}$. However, the construction that we will develop can in all cases be naturally expressed in terms of *real* matrices of suitable dimension, provided one has an appropriate representation of $\mathbb{K}$ and $\mathbb{K}'$ in terms of real matrices. For the octonions, such a representation will necessarily include a multiplication ordering rule, since matrix multiplication within the representation will be associative. Moreover, all multiplication involved in the construction involves at most two octonionic units, in which case associativity holds as a consequence of alternativity. We may therefore include the octonionic case without problem.

It is sufficient to consider the case $\mathbb{O}\otimes\mathbb{O}'$, as the other combinations are all obtained simply by projecting onto the appropriate subalgebra.

First, we consider a matrix of the form

$$\mathbf{X} = \begin{pmatrix} A & \overline{a} \\ a & -A^* \end{pmatrix}, \tag{3.1}$$

with $A \in \mathbb{O}'$ and $a \in \mathbb{O}$. For definiteness, we set

$$a = x^1 + x^2 i + x^3 j + x^4 k + x^5 kl + x^6 jl + x^7 il + x^8 l = \sum_{\mu=1}^{8} x^\mu e_\mu \tag{3.2}$$

and

$$A = x^9 + x^{10} I + x^{11} J + x^{12} K + x^{13} KL + x^{14} JL + x^{15} IL + x^{16} L = \sum_{\mu=9}^{1} 6x^\mu e_\mu. \tag{3.3}$$

Then $\mathbf{X}$ can be written as

$$X = \sum_{\mu=1}^{16} x^\mu \sigma_\mu, \tag{3.4}$$

where

$$\sigma_\mu = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \mu = 1 \\ e_\mu \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & 2 \le \mu \le 8 \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \mu = 9 \\ e_\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & 10 \le \mu \le 16 \end{cases} . \tag{3.5}$$

The notation is drawn from the fact that σ_1, σ_2, and σ_9 are just the Pauli spin matrices, commonly denoted σ_x, σ_y, and σ_z respectively.

We note before moving on the anticommutation and commutation relations

$$\{\sigma_\mu, \sigma_\nu\} = 0, \qquad 1 \le \mu < \nu \le 9 \tag{3.6}$$

$$[\sigma_\mu, \sigma_\nu] = 0, \qquad \mu \le 9 < \nu \tag{3.7}$$

$$\{\sigma_\mu, \sigma_\nu\} = 0, \qquad 10 \le \mu < \nu \le 16. \tag{3.8}$$

We see also that

$$\sigma_{10}^2 = \sigma_{11}^2 = \sigma_{12}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.9}$$

while the other $\{\sigma_\mu\}$ square to the identity. Finally, we observe that the matrices $\{\sigma_\mu\}$ constitute a basis for a sixteen-dimensional vector space, of which $\mathbf{X}$ represents an arbitrary element.

We now consider the matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \mathbf{X} \\ \widetilde{\mathbf{X}} & 0 \end{pmatrix} \equiv \sum_{\mu=1}^{16} x^\mu \mathbf{\Gamma}_\mu, \tag{3.10}$$

where tilde represents trace reversal,

$$\widetilde{\mathbf{X}} = \mathbf{X} - \mathrm{tr}(\mathbf{X})\mathbf{I}, \tag{3.11}$$

and we have introduced the sixteen matrices $\boldsymbol{\Gamma}_\mu$. Explicitly, we see that

$$\boldsymbol{\Gamma}_\mu = \begin{cases} \sigma_1 \otimes \sigma_\mu & 1 \leq \mu \leq 10 \\ i\sigma_2 \otimes \sigma_\mu & 11 \leq \mu \leq 16 \end{cases} \tag{3.12}$$

From this characterization and the above commutation relations, we see that

$$\{\boldsymbol{\Gamma}_\mu, \boldsymbol{\Gamma}_\nu\} = 2g_{\mu,\nu}\mathbf{I}, \tag{3.13}$$

where $\mathbf{I}$ is the identity matrix and

$$g_{\mu,\nu} = \begin{cases} 0 & \mu \neq \nu \\ 1 & 1 \leq \mu = \nu < 13 \\ -1 & 14 \leq \mu = \nu \leq 16 \end{cases} . \tag{3.14}$$

But these are precisely the anticommutation relations necessary to generate a representation of the Clifford algebra $\mathrm{C}\ell_{12,4}(\mathbb{R})$, so $\mathbf{P}$ represents an arbitrary element of the vector space underlying $\mathrm{C}\ell_{12,4}(\mathbb{R})$.

Finally, we observe that this construction admits an obvious generalization to the case where $\mathbb{K}$ is a split algebra and/or $\mathbb{K}'$ is a division algebra, but we are not concerned with such cases here and so say no more about them.

3.2 $\mathbf{SO(4,2;\mathbb{R})}$ and the Conformal Group

We seek now to identify a real representation of $\mathrm{SO}(4,2;\mathbb{R})$ that satisfies certain "nice" conditions. Specifically, we want our representation to contain a representation of $\mathrm{SO}(3,1;\mathbb{R})$ in an obvious way and we want it to be linked explicitly to $\mathrm{SU}(2;\mathbb{H}'\otimes\mathbb{C})$.

Ideally, the construction developed here will extend naturally to the other groups in the 2×2 magic square and admit an analogue that can be applied to the 3×3 magic square. Finally, we will show how $\mathrm{SO}(4, 2; \mathbb{R})$ can be interpreted as the conformal group acting on a normed vector space of Lorentzian signature by transforming the representation constructed here into one in which the conformal operations are explicit.

3.2.1 $\mathbf{SO}(\mathbf{4}, \mathbf{2}; \mathbb{R})$

We apply the process outlined in Section 2.2.3 to the representation of $\mathrm{SO}(4, 2; \mathbb{R})$ constructed implicitly in Section 3.1.2. However, we change notation here in order to better match standard conventions, so we will repeat the construction explicitly. In doing so, some details will be referred to earlier discussion. We begin by considering the matrix

$$\mathbf{X} = \begin{pmatrix} A & \overline{a} \\ a & -A^* \end{pmatrix} \tag{3.15}$$

where $a = x + iy \in \mathbb{C}$ and $A = z + tL + qK + pKL \in \mathbb{H}'$. We may write (3.15) as

$$\mathbf{X} = z\sigma_z + x\sigma_x + y\sigma_y + t\sigma_L + q\sigma_K + p\sigma_{KL}, \tag{3.16}$$

where σ_x, σ_y, and σ_z are the Pauli spin matrices and σ_L, σ_K, and σ_{KL} are simply the identity multiplied by the respective split-quaternionic imaginary basis units.

We now consider the matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \mathbf{X} \\ \widetilde{\mathbf{X}} & 0 \end{pmatrix} \tag{3.17}$$

where as before tilde denotes trace reversal. We can write $\mathbf{P}$ as

$$\mathbf{P} = \sum_{\mu} \mu \mathbf{\Gamma}_\mu, \tag{3.18}$$

with μ running over the coefficients $\{t, x, y, z, p, q\}$, and the $\{\mathbf{\Gamma}_\mu\}$ are 4×4 matrices over $\mathbb{H}' \otimes \mathbb{C}$. As discussed in Section 3.1.2, the matrices $\{\mathbf{\Gamma}_\mu\}$ thus constructed generate

$C\ell_{4,2}(\mathbb{R})$. Explicitly, in the notation of (3.16) we have

$$\begin{aligned}
&\mathbf{\Gamma}_z = \sigma_x \otimes \sigma_z,\ \mathbf{\Gamma}_x = \sigma_x \otimes \sigma_x,\ \mathbf{\Gamma}_y = \sigma_x \otimes \sigma_y, \\
&\mathbf{\Gamma}_t = i\sigma_y \otimes \sigma_L,\ \mathbf{\Gamma}_q = i\sigma_y \otimes \sigma_K,\ \mathbf{\Gamma}_p = i\sigma_y \otimes \sigma_{KL},
\end{aligned} \tag{3.19}$$

or, in more traditional notation,

$$\begin{aligned}
&\mathbf{\Gamma}_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\ \mathbf{\Gamma}_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
&\mathbf{\Gamma}_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},\ \mathbf{\Gamma}_t = \begin{pmatrix} 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \\ -L & 0 & 0 & 0 \\ 0 & -L & 0 & 0 \end{pmatrix}, \\
&\mathbf{\Gamma}_q = \begin{pmatrix} 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \\ -K & 0 & 0 & 0 \\ 0 & -K & 0 & 0 \end{pmatrix},\ \mathbf{\Gamma}_p = \begin{pmatrix} 0 & 0 & KL & 0 \\ 0 & 0 & 0 & KL \\ -KL & 0 & 0 & 0 \\ 0 & -KL & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{3.20}$$

The homogeneous quadratic elements of the algebra generated by these matrices then generates a representation of $\mathrm{SO}(4,2;\mathbb{R})$ precisely as discussed in Section 2.2.3.

We are seeking a real representation, so we require a way to express these as real matrices while retaining the essential anticommutation relations. The solution is provided by finding suitable representations for $\mathbb{C}$ and $\mathbb{H}'$ in the form of 2×2 real matrices. We can do this by making use of the Pauli spin matrices, using the facts that

$$(i\sigma_y)^2 = -\mathbf{I}, \tag{3.21}$$

$$\sigma_x^2 = \sigma_z^2 = \mathbf{I}, \tag{3.22}$$

and all three of these matrices are purely real. If we identify

$$\begin{aligned} 1 &\to \mathbf{I} \\ i &\to i\sigma_y \end{aligned} \tag{3.23}$$

for $\{1, i\} \subset \mathbb{C}$ and

$$\begin{aligned} 1 &\to \mathbf{I} \\ L &\to \sigma_z \\ K &\to -i\sigma_y \\ KL &\to \sigma_x, \end{aligned} \tag{3.24}$$

for $\{1, L, K, KL\} \subset \mathbb{H}'$, then we see that the appropriate multiplication tables are preserved. Thus we can write elements of $\mathbb{H}' \otimes \mathbb{C}$ by taking tensor products of these representations. Specifically, we can now write the matrices $\{\mathbf{\Gamma}_\mu\}$ as

$$\mathbf{\Gamma}_x = \sigma_x \otimes \sigma_x \otimes \mathbf{I} \otimes \mathbf{I},\ \mathbf{\Gamma}_y = \sigma_x \otimes -i\sigma_y \otimes \mathbf{I} \otimes i\sigma_y,\ \mathbf{\Gamma}_z = \sigma_x \otimes \sigma_z \otimes \mathbf{I} \otimes \mathbf{I},$$

$$\mathbf{\Gamma}_t = i\sigma_y \otimes \mathbf{I} \otimes \sigma_z \otimes \mathbf{I},\ \mathbf{\Gamma}_q = i\sigma_y \otimes \mathbf{I} \otimes -i\sigma_y \otimes \mathbf{I},\ \mathbf{\Gamma}_p = i\sigma_y \otimes \mathbf{I} \otimes \sigma_x \otimes \mathbf{I}.$$

In this form, the fifteen generators of $\mathrm{SO}(4, 2; \mathbb{R})$ are

$$\begin{aligned}
&\mathbf{\Gamma}_t\mathbf{\Gamma}_x = \sigma_z \otimes \sigma_x \otimes \sigma_z \otimes \mathbf{I},\ \mathbf{\Gamma}_t\mathbf{\Gamma}_y = -\sigma_z \otimes i\sigma_y \otimes \sigma_z \otimes i\sigma_y,\ \mathbf{\Gamma}_t\mathbf{\Gamma}_z = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \mathbf{I}, \\
&\mathbf{\Gamma}_x\mathbf{\Gamma}_y = \mathbf{I} \otimes \sigma_z \otimes \mathbf{I} \otimes i\sigma_y,\ \mathbf{\Gamma}_y\mathbf{\Gamma}_z = \mathbf{I} \otimes \sigma_x \otimes \mathbf{I} \otimes i\sigma_y,\ \mathbf{\Gamma}_z\mathbf{\Gamma}_x = \mathbf{I} \otimes i\sigma_y \otimes \mathbf{I} \otimes \mathbf{I}, \\
&\mathbf{\Gamma}_q\mathbf{\Gamma}_x = -\sigma_z \otimes \sigma_x \otimes i\sigma_y \otimes \mathbf{I},\ \mathbf{\Gamma}_q\mathbf{\Gamma}_y = \sigma_z \otimes i\sigma_y \otimes i\sigma_y \otimes i\sigma_y,\ \mathbf{\Gamma}_q\mathbf{\Gamma}_z = -\sigma_z \otimes \sigma_z \otimes i\sigma_y \otimes \mathbf{I}, \\
&\mathbf{\Gamma}_p\mathbf{\Gamma}_x = \sigma_z \otimes \sigma_x \otimes \sigma_x \otimes \mathbf{I},\ \mathbf{\Gamma}_p\mathbf{\Gamma}_y = -\sigma_z \otimes i\sigma_y \otimes \sigma_x \otimes i\sigma_y,\ \mathbf{\Gamma}_p\mathbf{\Gamma}_z = \sigma_z \otimes \sigma_z \otimes \sigma_x \otimes \mathbf{I}, \\
&\mathbf{\Gamma}_t\mathbf{\Gamma}_p = -\mathbf{I} \otimes \mathbf{I} \otimes i\sigma_y \otimes \mathbf{I},\ \mathbf{\Gamma}_t\mathbf{\Gamma}_q = \mathbf{I} \otimes \mathbf{I} \otimes \sigma_x \otimes \mathbf{I},\ \mathbf{\Gamma}_p\mathbf{\Gamma}_q = -\mathbf{I} \otimes \mathbf{I} \otimes \sigma_z \otimes \mathbf{I}.
\end{aligned} \tag{3.25}$$

We also set

$$\mathbf{r}_{\mu\nu} = \mathbf{\Gamma}_\mu\mathbf{\Gamma}_\nu, \tag{3.26}$$

and denote the corresponding group element generated by $\mathbf{r}_{\mu\nu}$ by

$$\mathbf{R}_{\mu\nu} \equiv \exp[\mathbf{r}_{\mu\nu}\phi/2]. \tag{3.27}$$

There are two steps in this construction at which we can project onto a representation of $\mathrm{SO}(3,1;\mathbb{R})$ by projecting from $\mathbb{H}'$ to $\mathbb{C}'$. First, in our definition of $\mathbf{X}$ we can set $p = q = 0$, which is equivalent to restricting A to be in $\mathbb{C}' \subset \mathbb{H}'$. Calling this projection Π, we then have

$$\Pi(\mathbf{\Gamma}_p) = \Pi(\mathbf{\Gamma}_q) = 0$$

and we are left with only the t, x, y, and z elements. The anticommutation relations are obviously still satisfied, and the remaining matrices $\{\mathbf{\Gamma}_\mu\}$ still generate $\mathrm{SO}(3,1;\mathbb{R})$ in the obvious way.

On the other hand, we can make the projection from $\mathbb{H}'$ to $\mathbb{C}'$ in the final step by restricting to elements where the third factor is $1 = \mathbf{I}$ or $L = \sigma_x$. However, in this case we get an extra generator:

$$\mathbf{r}_{pq} = \mathbf{\Gamma}_p\mathbf{\Gamma}_q = -\mathbf{I} \otimes \mathbf{I} \otimes \sigma_z \otimes \mathbf{I}.$$

Exponentiating $\mathbf{r}_{pq}$ (in the $\mathbb{H}' \otimes \mathbb{C}$ representation) to find the corresponding group element, we get

$$\mathbf{R}_{pq} = e^{-L\theta/2}\mathbf{I}, \tag{3.28}$$

$\mathbf{R}_{pq}$ clearly commutes with $\mathbf{R}_{\mu\nu} \in \mathrm{SO}(3,1;\mathbb{R})$, so in fact this is a projection onto

$$\mathrm{SO}(3,1;\mathbb{R}) \times \mathbb{R} \subset \mathrm{SO}(4,2;\mathbb{R}) \tag{3.29}$$

So far we have considered transformations of the form (2.79) acting on $\mathrm{span}(\{\mathbf{\Gamma}_\mu\})$; i.e., transformations of $\mathbf{P}$. But, in light of the off-diagonal structure of the matrices $\{\mathbf{\Gamma}_\mu\}$, we can also consider the effect (2.79) has on $\mathbf{X}$. First, we observe that for matrices of the form (3.15), and therefore for the $\{\sigma_\mu\}$ defined by (3.16), trace-reversal corresponds to conjugation in $\mathbb{H}'$:

$$\widetilde{\sigma_\mu} = \sigma_\mu^*. \tag{3.30}$$

Then, considering (3.17), matrices $\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\nu$ take the form

$$\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\nu = \begin{pmatrix} \sigma_\mu \sigma_\nu^* & 0 \\ 0 & \sigma_\mu^* \sigma_\nu \end{pmatrix}. \tag{3.31}$$

In particular,

$$\exp[\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\nu \theta/2] = \begin{pmatrix} \exp[\sigma_\mu \sigma_\nu^* \theta/2] & 0 \\ 0 & \exp[\sigma_\mu^* \sigma_\nu \theta/2] \end{pmatrix}, \tag{3.32}$$

so we can write

$$\exp[\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\nu \theta/2] \mathbf{P} \exp[-\mathbf{\Gamma}_\mu \mathbf{\Gamma}_\nu \theta/2] = \tag{3.33}$$

$$\begin{pmatrix} 0 & \exp[\sigma_\mu \sigma_\nu^* \theta/2] \mathbf{X} \exp[-\sigma_\mu^* \sigma_\nu \theta/2] \\ \exp[\sigma_\mu^* \sigma_\nu \theta/2] \widetilde{\mathbf{X}} \exp[-\sigma_\mu \sigma_\nu^* \theta/2] & 0 \end{pmatrix}. \tag{3.34}$$

Thus, we have a 2×2 representation generated by the $\{\sigma_\mu\}$ matrices acting on $\mathbf{X}$ via

$$; \mathbf{X} \longmapsto \exp[\sigma_\mu \sigma_\nu^* \theta/2] \mathbf{X} \exp[-\sigma_\mu^* \sigma_\nu \theta/2]. \tag{3.35}$$

These transformations do not appear to have the general form

$$\mathbf{X} \longmapsto \mathbf{N} \mathbf{X} \mathbf{N}^\dagger, \tag{3.36}$$

even if we restrict $\dagger$ to indicate conjugation in only $\mathbb{H}'$ or $\mathbb{C}$, which calls into question whether the group constructed here can be identified as $\mathrm{SU}(2; \mathbb{H}' \otimes \mathbb{C})$ as hoped.

We conclude this section by listing explicitly the fifteen 4×4 matrices defined by (3.27) with the $\{\mathbf{\Gamma}_\mu\}$ matrices as in (3.20):

$$\mathbf{R}_{xy} = \begin{pmatrix} e^{i\phi/2} & 0 & 0 & 0 \\ 0 & e^{-i\phi/2} & 0 & 0 \\ 0 & 0 & e^{i\phi/2} & 0 \\ 0 & 0 & 0 & e^{-i\phi/2} \end{pmatrix},$$

$$\mathbf{R}_{yz} = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} & 0 & 0 \\ i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ 0 & 0 & i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix},$$

$$\mathbf{R}_{zx} = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} & 0 & 0 \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ 0 & 0 & -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix},$$

$$\mathbf{R}_{qx} = \begin{pmatrix} \cos\frac{\phi}{2} & K\sin\frac{\phi}{2} & 0 & 0 \\ K\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\phi}{2} & -K\sin\frac{\phi}{2} \\ 0 & 0 & -K\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix},$$

$$\mathbf{R}_{qy} = \begin{pmatrix} \cos\frac{\phi}{2} & -K\otimes i\sin\frac{\phi}{2} & 0 & 0 \\ K\otimes i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\phi}{2} & K\otimes i\sin\frac{\phi}{2} \\ 0 & 0 & -K\otimes i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix},$$

$$\mathbf{R}_{qz} = \begin{pmatrix} e^{K\phi/2} & 0 & 0 & 0 \\ 0 & e^{-K\phi/2} & 0 & 0 \\ 0 & 0 & e^{-K\phi/2} & 0 \\ 0 & 0 & 0 & e^{K\phi/2} \end{pmatrix},$$

$$\mathbf{R}_{tp} = \begin{pmatrix} e^{K\phi/2} & 0 & 0 & 0 \\ 0 & e^{K\phi/2} & 0 & 0 \\ 0 & 0 & e^{K\phi/2} & 0 \\ 0 & 0 & 0 & e^{K\phi/2} \end{pmatrix},$$

$$\mathbf{R}_{tx} = \begin{pmatrix} \cosh\frac{\phi}{2} & L\sinh\frac{\phi}{2} & 0 & 0 \\ L\sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cosh\frac{\phi}{2} & -L\sinh\frac{\phi}{2} \\ 0 & 0 & -L\sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} \end{pmatrix},$$

$$\mathbf{R}_{ty} = \begin{pmatrix} \cosh\frac{\phi}{2} & -L\otimes i\sinh\frac{\phi}{2} & 0 & 0 \\ L\otimes i\sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cosh\frac{\phi}{2} & -L\otimes i\sinh\frac{\phi}{2} \\ 0 & 0 & L\otimes i\sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} \end{pmatrix},$$

$$\mathbf{R}_{tz} = \begin{pmatrix} e^{L\phi/2} & 0 & 0 & 0 \\ 0 & e^{-L\phi/2} & 0 & 0 \\ 0 & 0 & e^{-L\phi/2} & 0 \\ 0 & 0 & 0 & e^{L\phi/2} \end{pmatrix},$$

$$\mathbf{R}_{tq} = \begin{pmatrix} e^{KL\phi/2} & 0 & 0 & 0 \\ 0 & e^{KL\phi/2} & 0 & 0 \\ 0 & 0 & e^{KL\phi/2} & 0 \\ 0 & 0 & 0 & e^{KL\phi/2} \end{pmatrix},$$

$$\mathbf{R}_{px} = \begin{pmatrix} \cosh\frac{\phi}{2} & KL\sinh\frac{\phi}{2} & 0 & 0 \\ KL\sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cosh\frac{\phi}{2} & -KL\sinh\frac{\phi}{2} \\ 0 & 0 & -KL\sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} \end{pmatrix},$$

$$\mathbf{R}_{py} = \begin{pmatrix} \cosh\frac{\phi}{2} & -KL\otimes i\sinh\frac{\phi}{2} & 0 & 0 \\ KL\otimes i\sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cosh\frac{\phi}{2} & KL\otimes i\sinh\frac{\phi}{2} \\ 0 & 0 & -KL\otimes i\sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} \end{pmatrix},$$

$$\mathbf{R}_{pz} = \begin{pmatrix} e^{KL\phi/2} & 0 & 0 & 0 \\ 0 & e^{-KL\phi/2} & 0 & 0 \\ 0 & 0 & e^{-KL\phi/2} & 0 \\ 0 & 0 & 0 & e^{KL\phi/2} \end{pmatrix},$$

$$\mathbf{R}_{pq} = \begin{pmatrix} e^{-L\phi/2} & 0 & 0 & 0 \\ 0 & e^{-L\phi/2} & 0 & 0 \\ 0 & 0 & e^{-L\phi/2} & 0 \\ 0 & 0 & 0 & e^{-L\phi/2} \end{pmatrix}. \tag{3.37}$$

3.2.2 The Conformal Group

We now represent the known isomorphism between the conformal group and $\mathrm{SO}(4,2;\mathbb{R})$ by establishing that the transformations generated by (3.25), acting on a suitable space, are precisely the conformal transformations: three rotations, three boosts, four translations, four conformal translations, and a dilation. To do this, we let $V = \mathrm{span}(\{\mathbf{\Gamma}_\mu\})$ and consider $\mathbf{P} \in V$ as defined in (3.18). We also impose the constraints $p + q \neq 0$ and, using 2.78,

$$|\mathbf{P}|^2 = \langle \mathbf{P}, \mathbf{P} \rangle = -t^2 + x^2 + z^2 + y^2 - p^2 + q^2 = 0. \tag{3.38}$$

We then define

$$\mathbf{Q} = T\mathbf{\Gamma}_t + X\mathbf{\Gamma}_x + Y\mathbf{\Gamma}_y + Z\mathbf{\Gamma}_z, \tag{3.39}$$

with

$$\begin{aligned} T &= \frac{t}{p+q}, \\ X &= \frac{x}{p+q}, \\ Y &= \frac{y}{p+q}, \\ Z &= \frac{z}{p+q},. \end{aligned} \tag{3.40}$$

In other words,

$$\mathbf{P} = \mathbf{Q}(p+q) + p\mathbf{\Gamma}_p + q\mathbf{\Gamma}_q. \tag{3.41}$$

Formally, in terms of the inner product on V the coefficients (3.40) are seen to be

$$\begin{aligned} T &= -\langle \mathbf{\Gamma}_t, \mathbf{P} \rangle / \langle \mathbf{\Gamma}_p + \mathbf{\Gamma}_q, \mathbf{P} \rangle, \\ X &= \langle \mathbf{\Gamma}_x, \mathbf{P} \rangle / \langle \mathbf{\Gamma}_p + \mathbf{\Gamma}_q, \mathbf{P} \rangle, \\ Y &= \langle \mathbf{\Gamma}_y, \mathbf{P} \rangle / \langle \mathbf{\Gamma}_p + \mathbf{\Gamma}_q, \mathbf{P} \rangle, \\ Z &= \langle \mathbf{\Gamma}_z, \mathbf{P} \rangle \langle \mathbf{\Gamma}_p + \mathbf{\Gamma}_q, \mathbf{P} \rangle. \end{aligned} \tag{3.42}$$

We now consider how $\mathbf{Q}$ changes when elements of $\mathrm{SO}(4,2;\mathbb{R})$ act on $\mathbf{P}$. As a first observation, when the rotations—$\mathbf{R}_{xy}$, $\mathbf{R}_{yz}$, and $\mathbf{R}_{zx}$—and the boosts—$\mathbf{R}_{tx}$, $\mathbf{R}_{ty}$, and $\mathbf{R}_{tz}$—act on $\mathbf{P}$, the effect on $\mathbf{Q}$ is the same, since $p+q$ is unaffected.

The effect of $\mathbf{R}_{pq}$ on $p+q$ is seen to be

$$p+q \longmapsto p\cosh\theta + q\sinh\theta + q\cosh\theta + p\sinh\theta = (p+q)(\cosh\theta+\sinh\theta), \tag{3.43}$$

in which case we see that

$$\mathbf{Q} \longmapsto \mathbf{Q}/(\cosh\theta+\sinh\theta) = \mathbf{Q}e^{-\theta}. \tag{3.44}$$

For example, if we set

$$\mathbf{P}' = \mathbf{R}_{pq}\mathbf{P}\mathbf{R}_{pq}^{-1} \tag{3.45}$$

we find from (3.42) that

$$X' \equiv \langle \mathbf{\Gamma}_x, \mathbf{P}'\rangle / \langle \mathbf{\Gamma}_p + \mathbf{\Gamma}_q, \mathbf{P}'\rangle = \frac{x}{(p+q)(\cosh\theta+\sinh\theta)} = \frac{X}{\cosh\theta+\sinh\theta} = Xe^{-\theta}. \tag{3.46}$$

The same calculation shows that T, Y, and Z are similarly dilated, from which we conclude that $\mathbf{R}_{pq}$ represents the dilation.

The translations and conformal translations are best understood by considering null rotations of the form

$$\mathbf{a}_\mu = \mathbf{r}_{p\mu} - \mathbf{r}_{q\mu} \tag{3.47}$$

and

$$\mathbf{b}_\mu = \mathbf{r}_{p\mu} + \mathbf{r}_{q\mu}. \tag{3.48}$$

First, observe that

$$\begin{aligned}
(\mathbf{r}_{p\mu} \pm \mathbf{r}_{q\mu})^2 &= (\mathbf{\Gamma}_p\mathbf{\Gamma}_\mu \pm \mathbf{\Gamma}_q\mathbf{\Gamma}_\mu)^2 && (3.49)\\
&= (\mathbf{\Gamma}_p\mathbf{\Gamma}_\mu)^2 + (\mathbf{\Gamma}_q\mathbf{\Gamma}_\mu)^2 \pm \mathbf{\Gamma}_p\mathbf{\Gamma}_\mu\mathbf{\Gamma}_p\mathbf{\Gamma}_\mu\mathbf{\Gamma}_q\mathbf{\Gamma}_\mu \pm \mathbf{\Gamma}_q\mathbf{\Gamma}_\mu\mathbf{\Gamma}_p\mathbf{\Gamma}_\mu && (3.50)\\
&= 0, && (3.51)
\end{aligned}$$

where in the last equality we have employed the anticommutation relations

$$\{\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_\nu\} = g_{\mu\nu}\mathbf{I}.$$

As a result, we see that

$$\exp[\pm \mathbf{a}_\mu \theta/2] = \mathbf{I} \pm \mathbf{a}_\mu \theta/2, \tag{3.52}$$

and

$$\exp[\pm \mathbf{b}_\mu \theta/2] = \mathbf{I} \pm \mathbf{b}_\mu \theta/2. \tag{3.53}$$

Next, we compute, as an example, the action of $\mathbf{a}_x$ on $\mathbf{P}$. To begin, observe that $\mathbf{a}_x$ involves only $\mathbf{\Gamma}_p$, $\mathbf{\Gamma}_x$, and $\mathbf{\Gamma}_q$, so that t, y, and z will be unaffected. We then compute

$$\begin{aligned}
\exp[\mathbf{a}_x\theta/2]\mathbf{\Gamma}_x \exp[-\mathbf{a}_x\theta/2] &= \left(\mathbf{I} + \frac{\theta}{2}\mathbf{\Gamma}_p\mathbf{\Gamma}_x - \frac{\theta}{2}\mathbf{\Gamma}_q\mathbf{\Gamma}_x\right)\mathbf{\Gamma}_x\left(\mathbf{I} - \frac{\theta}{2}\mathbf{\Gamma}_p\mathbf{\Gamma}_x + \frac{\theta}{2}\mathbf{\Gamma}_q\mathbf{\Gamma}_x\right)\\
&= \left(\mathbf{\Gamma}_x + \frac{\theta}{2}\mathbf{\Gamma}_p - \frac{\theta}{2}\mathbf{\Gamma}_q\right)\left(\mathbf{I} - \frac{\theta}{2}\mathbf{\Gamma}_p\mathbf{\Gamma}_x + \frac{\theta}{2}\mathbf{\Gamma}_q\mathbf{\Gamma}_x\right)\\
&= \mathbf{\Gamma}_x + \frac{\theta}{2}\mathbf{\Gamma}_p - \frac{\theta}{2}\mathbf{\Gamma}_q + \frac{\theta}{2}\mathbf{\Gamma}_p + \frac{\theta^2}{4}\mathbf{\Gamma}_x\\
&\quad + \frac{\theta^2}{4}\mathbf{\Gamma}_q\mathbf{\Gamma}_p\mathbf{\Gamma}_x - \frac{\theta}{2}\mathbf{\Gamma}_q + \frac{\theta^2}{4}\mathbf{\Gamma}_p\mathbf{\Gamma}_q\mathbf{\Gamma}_x - \frac{\theta^2}{4}\mathbf{\Gamma}_x\\
&= \mathbf{\Gamma}_x + \theta\mathbf{\Gamma}_p - \theta\mathbf{\Gamma}_q.
\end{aligned} \tag{3.54}$$

A similar calculation shows that

$$\mathbf{\Gamma}_p \longmapsto \theta\mathbf{\Gamma}_x - \frac{\theta^2}{2}\mathbf{\Gamma}_q + \left(1 + \frac{\theta^2}{2}\right)\mathbf{\Gamma}_p, \tag{3.55}$$

and

$$\mathbf{\Gamma}_q \longmapsto \mathbf{\Gamma}_q + \theta\mathbf{\Gamma}_x + \frac{\theta^2}{2}\mathbf{\Gamma}_p. \tag{3.56}$$

Combining (3.54), (3.55), and (3.56), we find

$$x\mathbf{\Gamma}_x + p\mathbf{\Gamma}_p + q\mathbf{\Gamma}_q \longmapsto (x + (p+q)\theta)\,\mathbf{\Gamma}_x + \left(p + p\frac{\theta^2}{2} + q\frac{\theta^2}{2} + x\theta\right)\mathbf{\Gamma}_p + \left(q - x\theta - p\frac{\theta^2}{2}\right)\mathbf{\Gamma}_q. \tag{3.57}$$

Applying (3.42) to (3.57), we see that

$$X' = \frac{x}{p+q} + \theta = X + \theta. \tag{3.58}$$

In other words, $\mathbf{a}_x$ acting on $\mathbf{P}$ has the effect of translating $\mathbf{Q}$ by θ in the $\mathbf{\Gamma}_x$ direction. Similar calculations show that $\mathbf{a}_t$, $\mathbf{a}_y$, and $\mathbf{a}_z$ yield corresponding translations.

We now consider the effect of $\mathbf{b}_x$ acting on $\mathbf{P}$. Proceeding as for (3.54), one finds

$$\exp[\mathbf{b}_x\theta/2]\mathbf{\Gamma}_x \exp[-\mathbf{b}_x\theta/2] = \mathbf{\Gamma}_x + \frac{\theta}{2}\mathbf{\Gamma}_p + \frac{\theta}{2}\mathbf{\Gamma}_q, \tag{3.59}$$

$$\exp[\mathbf{b}_x\theta/2]\mathbf{\Gamma}_q \exp[-\mathbf{b}_x\theta/2] = -\theta\mathbf{\Gamma}_x - \frac{\theta^2}{2}\mathbf{\Gamma}_p + \left(1 - \frac{\theta^2}{2}\right)\mathbf{\Gamma}_q, \tag{3.60}$$

and,

$$\exp[\mathbf{b}_x\theta/2]\mathbf{\Gamma}_p \exp[-\mathbf{b}_x\theta/2] = \theta\mathbf{\Gamma}_x + \left(1 + \frac{\theta}{2}\right)\mathbf{\Gamma}_p + \frac{\theta^2}{2}\mathbf{\Gamma}_q. \tag{3.61}$$

Taken together, (3.59), (3.60), and (3.61) yield

$$x\mathbf{\Gamma}_x + p\mathbf{\Gamma}_p + q\mathbf{\Gamma}_q \longmapsto (x + (p-q)\theta)\,\mathbf{\Gamma}_x + \left(x\theta + p + p\frac{\theta^2}{2} - q\frac{\theta^2}{2}\right)\mathbf{\Gamma}_p + \left(x\theta + q - q\frac{\theta^2}{2} + p\frac{\theta^2}{2}\right)\mathbf{\Gamma}_q, \tag{3.62}$$

from which it follows that

$$\begin{aligned} X' &= \frac{x + (p-q)\theta}{x\theta + p + p\frac{\theta^2}{2} - q\frac{\theta^2}{2} + x\theta + q - q\frac{\theta^2}{2} + p\frac{\theta^2}{2}} \\ &= \frac{X + \frac{p-q}{p+q}\theta}{2X\theta + 1 + \frac{p-q}{p+q}\theta^2} \\ &= \frac{X + |\mathbf{Q}|^2\theta}{2X\theta + 1 + |\mathbf{Q}|^2\theta^2}, \end{aligned} \tag{3.63}$$

where in the last line we have used (3.38) to write

$$\begin{aligned} \frac{p-q}{p+q} = \frac{p^2-q^2}{(p+q)^2} &= \frac{-t^2 + x^2 + y^2 + z^2}{(p+q)^2} \\ &= -T^2 + X^2 + Y^2 + Z^2 \\ &= \langle \mathbf{Q}, \mathbf{Q} \rangle \equiv |\mathbf{Q}|^2. \end{aligned} \tag{3.64}$$

To see why this is a conformal translation, we note that an element v in an inner product space V satisfies

$$v^{-1} = \frac{v}{|v|^2}. \tag{3.65}$$

Then a conformal translation of v in the direction of the vector α is given by

$$v \longmapsto \left(v^{-1} + \alpha\right)^{-1} = \frac{v + \alpha|v|^2}{1 + 2\langle v, \alpha\rangle + |\alpha|^2|v^2}. \tag{3.66}$$

Taking $v = \mathbf{Q}$ and assuming $\alpha = \theta\mathbf{\Gamma}_x$, the $\mathbf{\Gamma}_x$ component of (3.66) becomes precisely (3.63). Again, a similar calculation shows that $\mathbf{b}_y$, $\mathbf{b}_z$, and $\mathbf{b}_t$ are the other conformal translations.

4 CONCLUSION

We have presented two new representations of $\mathrm{SO}(4,2;\mathbb{R})$ consisting of, respectively, 4×4 and 2×2 matrices with elements in $\mathbb{H}' \otimes \mathbb{C}$. The construction admits an obvious generalization to the other slots in the 2×2 magic square, and the potential identification of the 2×2 representation as $\mathrm{SU}(2;\mathbb{K}' \otimes \mathbb{K})$ supports the suggested identification of the slots in the 3×3 magic square as $\mathrm{SU}(3;\mathbb{K}' \otimes \mathbb{K})$, although the exact form, or even existence, of a generalization of this construction to the 3×3 case remains uncertain. There are two natural choices for a next step in our understanding of the Lie group magic square. First, there is the rigorous identification of our 2×2 representation with $\mathrm{SU}(2;\mathbb{K}' \otimes \mathbb{K})$, after which one may consider how to extend the construction to get an interpretatation of the 3×3 magic square as $\mathrm{SU}(3;\mathbb{K}' \otimes \mathbb{K})$. Second, one could attempt to use this construction to establish the third row of the 2×2 magic square as $\mathrm{SU}(2,2;\mathbb{K})$, in analogy with the known isomorphism $\mathrm{SO}(4,2;\mathbb{R}) \equiv \mathrm{SU}(2,2;\mathbb{C})$; generalizing this process to the 3×3 case, if possible, would then allow an identification of E_7 as $\mathrm{SU}(3,3;\mathbb{O})$. It seems likely that a combination of these two methods will be necessary if we are to complete our classification and achieve, in the end, a complete description of the Lie group E_8.

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