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The objective of this investigation is the development of improved techniques for the estimation of robustness for dynamic systems with structured uncertainties, a problem which was approached by application of the Lyapunov direct method. This thesis considers the sign properties of the Lyapunov function derivative integrated along finite intervals of time, in place of the traditional method of the sign properties of the derivative itself.

This proposed approach relaxes the sufficient conditions of stability, and is used to generate techniques for the robust design of control systems with structured perturbations. The need for such techniques has been demonstrated by recent research interest in the area of robust control design.

The system considered is assumed to be nominally linear, with time-variant, nonlinear bounded perturbations. Application of the proposed technique warrants that estimates of robustness will either match or constitute an improvement upon those obtained by application of the traditional Lyapunov approach. The application of numerical procedures are used to demonstrate improvements in estimations of robustness for two-, three- and four-dimensional dynamic systems with one or more structured perturbations. The proposed numerical approaches obtain improved bounds, which are considered in the sense of their engineering aspects. To increase the accuracy of the numerical procedures, symbolic algebraic calculations are utilized.

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Redacted for Privacy

Head of Department of Mechanical Engineering

Redacted for Privacy

Dean of Graduate School

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CHAPTER 1

INTRODUCTION

1.1 Design Problem

The Lyapunov direct method has been utilized recently in a number of applications for control system analysis and design. This is the method most generally used for the determination of the stability of nonlinear and/or time-varying systems. Vannelli and Vidyasagar [47] have considered the application of the Lyapunov direct method to various theoretical and computational approaches for the estimation of the domain of attraction of autonomous nonlinear systems. Implementations of the general method of the Lyapunov direct method have thus resulted in such developments as the maximal Lyapunov function. As early as 1979, Brayton and Tong [8] introduced computer-generated Lyapunov functions which considered the Aizerman conjecture to obtain improved results. Subsequently, this approach has been widely used for the robust design of control systems.

Classes of linear, time-invariant dynamical systems, in which the system and input matrices, as well as the input itself, are uncertain, have been considered by Leitmann

[24]. The only available information concerning these uncertainties has been knowledge of the compact sets in which they may range. Based upon this knowledge, state feedback controls have been designed to assure uniform asymptotic stability of the zero state for all possible system responses. Barmish and Leitmann [2] completed research for uncertainty threshold estimation.

The control of dynamic systems which contain uncertain elements and are subject to uncertain inputs is often treated by the application of stochastic control theory. The construction of measured (or estimated) state feedback controls that provide a guarantee that system responses enter and remain within a particular neighborhood of the zero state after a finite interval of time was considered by Leitmann [23], as well as controller design for uncertain systems [25].

Thus, analysis of the stability robustness of linear, time-invariant systems subject to linear perturbations has been a matter of considerable research interest for a number of years. The two types of linear perturbations, time-variant and time-invariant, which clearly influence the analysis have been considered. The stability of linear time-invariant systems with time-invariant perturbations is directly addressed by testing for the negativity of the real parts of the eigenvalues, whereas time-variant cases are best accommodated by application of time domain Lyapunov stability analysis.

For the current investigation, asymptotically stable linear systems subject to time-variant, nonlinear perturbations are considered, and bounds are obtained for the perturbations to assure system stability. Considerable attention has already been focused upon the design of controllers for multi-variable, linear systems to the end that certain properties of the resulting system are preserved under various classes of perturbations occurring within the system. According to Patel and Toda [36], the bounds of perturbation can be computed numerically, providing a useful quantitative measure of robustness for asymptotically stable systems.

The Lyapunov approach to testing for the stability of state-space models was first applied by Yedavalli and Liang, generating bounds which tended to be conservative [52]. The principal theme of their approach was the reduction of the conservatism of the stability robustness bounds. In addition, application of the Lyapunov approach has also been used extensively for the estimation of controller robustness, thus renewing interest in the parameter space approach to control systems and providing alternatives to the classical methods of Routh-Hurwitz and Nyquist.

Thus, recent widespread interest in the robust design of control systems subject to structured perturbations has served to shift research activity toward parameter space methods, and enlargement of the scope of the approach to include the Lyapunov method as well as frequency domain concepts [17,8,50].

1.2 Objectives of the Investigation

The objective of the current investigation is the development of a new Lyapunov-based technique for the robust design of control systems subject to structured perturbations. Specifically, the proposed technique provides better results for two basic robust control design problems, the robust stability problem and the uncertain system stabilization problem. This approach involves application of the Lyapunov direct method to control design for time-variant, nonlinear systems with bounded perturbations.

In Chapter 2, general stability concepts and the basic theorems derivative from applications of the Lyapunov direct method are presented. In addition, generalizations of this methodology are discussed and the advantages of their mathematical apparatus are introduced. In Chapter 3, a new approach to the determination of robustness bounds is introduced, accompanied by consideration of improved stability criteria for relaxing Lyapunov stability conditions. The applications of the proposed technique are given practical demonstration in Chapter 4, with detailed analyses of the results from the perspective of computational programming. In the concluding chapter, results are summarized and proposed directions for further research are considered. Detailed considerations of programs and procedures used for the development of the proposed technique are included as appendices.

CHAPTER 2

THEORETICAL BACKGROUND

2.1 Introduction

For the successful design of given control systems for uncertain systems, stability is the most important subject to be determined. If the proposed system is linear and time-invariant, a number of stability criteria are available, including both the Nyquist and Routh stability criteria. However, if the proposed system is non-linear, and/or time-variant, then these criteria cannot be applied. The advantage of the application of the Lyapunov direct method is that the stability of a system can be determined without the need of solving the state equations, a process which can be either very difficult or nearly impossible when applied to nonlinear and/or time-variant state equations.

Examination of the stability of a given system is always the first and basic step in system analysis. If a system is disturbed in any manner at any given time, the issue is to determine the effect of the disturbance on subsequent output. If a system is initially in a state of equilibrium, then it will in theory remain in that state thereafter, and Lyapunov stability is concerned with the trajectories of a system when the initial state is near to equilibrium. From an engineering point of view, this is of the utmost importance since external disturbances (e.g., noise or component errors) are always present in any actual system. In systems which have disturbances from their surroundings, it is often procedurally difficult to obtain precise solutions. However, from the practical engineering point of view, it is of crucial importance to obtain measures that define allowable perturbation bounds so that the stability of the original system may be maintained. Considerable research attention has been devoted to this issue.

The proposed technique for the solution of this problem utilizes the parameter space approach to the robust design of control systems subject to structured perturbations. The two principal research directions established in the parameter space approach include the Lyapunov direct method and frequency domain concepts. This investigation is limited to an approach via the Lyapunov direct method. Application of the Lyapunov direct method allows for the analysis of nonlinear and time-variant systems, and is less time consuming than other options since it does not require calculation of characteristic polynomials of transfer functions.

Two principal directions have been established in research development in the area under consideration. The first, including the Bellman-Matrosov concept of vector Lyapunov function and the concept of maximal Lyapunov function, is engaged with the search for the best Lyapunov device to apply to the problems of robust stability. The second has been concentrated upon devising better robustness bounds by consideration of the properties of matrix equations, forming derivatives of Lyapunov functions along the system's solution. Both methods are considered in the material presented in this chapter in the context of systems with structured perturbations. The improvement of robustness bounds in parametric space has been the subject of several research approaches. For the current project, the parameter space approach to the robust design of control systems is subject to structured perturbations which are nonlinear and time-variant.

Progress in this area has lead to the design of more powerful, quicker, and lighter systems, which in turn require the development of new and more fully robust controllers. Aircraft, large space structures, manipulators, and robots constitute examples of technological areas in which the need for robust control systems is of particular utility. Design estimates currently in use, and based on Lyapunov-like theorems, have in simulations differed by as much as hundreds of percent from accurate values [41].

Thus, research directions in methods to improve the state of controller designs are discussed in this chapter. In the following sections, stability in the sense of Lyapunov, basic theorems and definitions of robustness are stated and defined, arriving at a generalization of Lyapunov direct methods for improved controller design. Finally, the Lyapunov derivative used for this project is considered in detail with the establishment of new criteria for a mathematical apparatus.

2.2 Stability and Robustness

In 1892, A. M. Lyapunov presented his first and second methods for determining the stability of dynamic systems as described by ordinary differential equations [30]. The first method consisted entirely of procedures in which the explicit form of the solutions of the differential equations were used for the analysis, a method which has proved to be fundamental to the stability analysis of equilibrium points. Unfortunately, this method is applicable only to local stability at the point of interest.

The second method, however, does not require the solution of differential equations. Thus, the Lyapunov's second method is suitable to the stability analysis of nonlinear systems for which exact solutions may not be obtained. The second, or direct, method provides improved stability evaluation and is therefore powerful, but at the same time it provides only sufficient conditions of stability.

Ogata [30] has explained the Lyapunov methods by the use of the energy function concept. From the classical theory of mechanics, it is accepted that a vibratory system is asymptotically stable if its total energy (a positive definite function) is continually decreasing (that is, the time derivative of the total energy must be negative definite) until reaching a state of equilibrium. The Lyapunov second method is based on a generalization of the fact that if the system has an asymptotically stable equilibrium state, then the stored energy of the system which is disturbed within the domain of attraction decays with increasing time. However, there is no simple way of defining an "energy function."

To circumvent this problem, Lyapunov introduced the so-called Lyapunov function, or a fictitious energy function which was more general than that of energy and more widely applicable. In point of fact, any scalar function satisfying the hypothesis of Lyapunov's stability theorems can serve as Lyapunov function. Lyapunov functions are dependent upon x_1, $x_2, \cdots$, x_n and t, and denoted by $V(x_1,x_2,\cdots,x_n,t)$ or simply by $V(x,t)$. If Lyapunov functions do not explicitly include t, then they are denoted by $V(x_1,x_2,\cdots,x_n)$ or $V(x)$. In Lyapunov's second method, the sign behavior of $V(x,t)$ and that of its time derivative $\dot{V}(x,t)$ provide information on the stability, asymptotic stability, or instability of an equilibrium state without requiring a direct solution. Thus, it can be shown that if a scalar function $V(x)$, where x is an n-dimensional vector, is positive definite, then the states x which satisfy

$$V(x) = C ,$$

where C is a positive constant, lie on a closed hypersurface in the n-dimensional state space, at least in the neighborhood of the origin. If $V(x) \to \infty$ as $\|x\| \to \infty$, then such a closed surface extends over the entire state space. The hypersurface $V(x) = C_1$ lies entirely inside the hypersurface $V(x) = C_2$ if $C_1 < C_2$.

Theorem 2.1. Suppose that a system is described by $\dot{x} = f(x,t)$, where $f(0,t) = 0$ for all t. The system is continuous when solutions exist. If there exists a scalar function $V(x,t)$, having continuous, first partial derivatives and satisfying the following conditions,

1. $V(x,t)$ is locally positive definite and
2. $\dot{V}(x,t) \leq 0$, $t \geq t_0$, $x \in B_r$,

then the equilibrium state at the origin is stable.

The equilibrium point at time t_0 is uniformly asymptotically stable over the interval $[t_0,\infty)$ if there exists a continuously differentiable decrescent locally positive definite function V such that $-\dot{V}$ is an l.p.d.f. (a locally positive definite function, as defined by Vidyasagar [49]). A good illustration is provided if we let $V = V(x)$ (i.e., V is not time-dependent), and $V(x)$ takes the constant values 0, C_1, $C_2, \cdots$ $(0 < C_1 < C_2 \cdots)$. Then, $V(x) = 0$ corresponds to the origin of the state plane, and $V(x) = C_1$, $V(x) = C_2, \cdots$ describes nonintersecting surfaces enclosing the origin of the state plane. Note that $V(x)$ is radially unbounded, or $V(x)\to\infty$ as $\|x\|\to\infty$, the surfaces extending over the entire state plane. Theorem 2.1 is a basic theorem of the direct method. For details, see the proofs in Chapter 5.2 from Vidyasagar [49].

Theorem 2.2. Suppose that a system is described by $\dot{x} = f(x,t)$ where $f(0,t) = 0$ for all $t \geq t_0$.

If there exists a scalar function $V(x,t)$, having continuous, first partial derivatives and satisfying the following conditions,

1. $V(x,t)$ is positive definite,
2. $\dot{V}(x,t)$ is negative definite, and
3. $\dot{V}(\phi(t;x_0,t_0),t)$ does not vanish identically in $t \geq t_0$ for any t_0 and any $x_0 \neq 0$,

where $\phi(t;x_0,t_0)$ denotes the trajectory or solution starting from x_0 at t_0, then the equilibrium state at the origin of the system is uniformly asymptotically stable in the large.

Note that if $\dot{V}$ is not negative definite, but only negative semi-definite, then the trajectory of a representative point can become tangent to some particular surface $V(x,t) = C$. However, since $\dot{V}(\phi(t;x_0,t_0),t)$ does not vanish identically in $t \geq t_0$ for any t_0 and any $x_0 \neq 0$, the representative point cannot remain at the tangent point and must therefore move toward the origin. However, if there exists a positive definite scalar function $V(x,t)$ such that $\dot{V}$ is identically zero, then the system can remain in a limit cycle. The equilibrium state at the origin, in this case, is said to be stable in the sense of Lyapunov.

Instability: If an equilibrium state $x = 0$ of a system is unstable, then there exists a scalar function $W(x,t)$ which determines the instability of the equilibrium state.

Theorem 2.3. Suppose that a system is described by $\dot{x} = f(x,t)$ where $f(0,t) = 0$ for all $t \geq t_0$. If there exists a scalar function $W(x,t)$, having continuous, first partial derivatives and satisfying the following conditions,

1. $W(x,t)$ is positive definite in some region about the origin and
2. $\dot{W}(x,t)$ is positive definite in the same region,

then the equilibrium state at the origin is unstable.

Consider the following linear system,

$$\dot{x} = Ax \ ,$$

where x is a state vector (n-dimensional vector) and A is an n x n matrix. When A is assumed to be nonsingular, a possible Lyapunov function can be chosen as

$$V(x) = x^T P x \ ,$$

where P is a positive definite matrix. The time derivative $V(x)$ along any trajectory is

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P (Ax) \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + PA) x \ .\end{aligned}$$

Since $V(x)$ was chosen to be positive definite, for asymptotic stability it is required that $\dot{V}(x)$ be negative definite. Therefore, it is further required that

$$\dot{V}(x) = -x^T Q x \ ,$$

where

$$Q = -(A^T P + PA): \quad \text{positive definite} \ .$$

It is convenient to first specify a positive definite matrix Q, then examine whether or not P determined from $A^T P + PA = -Q$ is positive definite. Note that it is a necessary and sufficient condition that P is positive definite.

Theorem 2.4. Consider the system described by

$$\dot{x} = Ax \ ,$$

where x is a state vector (n x n dimensional vector) and A is an n x n constant nonsingular matrix. It is a necessary and sufficient condition that the equilibrium state $x = 0$ be asymptotically stable in the large if, given any positive definite matrix Q, there exists a positive definite matrix P such that $A^T P + PA = -Q$. The scalar function $x^T P x$ is a Lyapunov function for this system.

In applying this theorem several important remarks are in order.

1. If $\dot{V}(x) = -x^TQx$ does not vanish identically along any trajectory, then Q may be chosen to be positive semi-definite.
2. If an arbitrary positive definite matrix as Q is chosen and solves the matrix equation $A^TP+PA = -Q$ to determine P, then the positive definiteness of P is a necessary and sufficient condition for the asymptotic stability of the equilibrium state $x = 0$.
3. The final result does not depend upon choice of a particular Q matrix so long as it is positive definite.
4. To determine the elements of the P matrix, the matrices $A^TP + PA$ and $-Q$ are equated element by element. This results in $n(n + 1)/2$ linear equations for the determination of the elements $P_{ij} = P_{ji}$ of P.
5. In determining whether or not there exists a positive definite matrix P, it is convenient to select $Q = I$, where I is the identity matrix. The elements of P are then determined from $A^TP+PA = -I$ and the matrix P is tested for positive definiteness.

From Lunze [27], the definition of the robustness of control systems is composed of two ingredients:

1. A system property (e.g., a stability margin in the frequency domain), and
2. A class of perturbations against which the system properties are robust (e.g., uncertain physical parameters, neglected actuator dynamics and non-linearity, modeling uncertainty, non-ideal controller implementation, sensor or actuator failure).

In the current study, feedback control systems are considered. Feedback control systems exhibit several important properties since the behavior of the overall system is produced by the properties as well as the interactions of its parts. Feedback makes it possible to stabilize inherently unstable systems, to improve the robustness against variations of the performance of some system part, or to attenuate unmeasurable external disturbances. In the control system, unknown disturbances may influence the performance of the process. In this case, the output is not only an answer to the control input, but also to disturbances, which are generally uncertain in the sense that they may be one of a set of possible disturbance signals. Uncertainties of these types occur to a lesser or greater extent in nearly all control systems since, for purposes of modeling and design, the system to be controlled must be taken out of its environment. Usually, it is not obvious which of the phenomena must be considered as a part of the actual plant, or as major connections between the plant and its environment, and which are not. Given these circumstances, it is important to be aware that the principal properties of systems are only weakly dependent upon such uncertainties. Robustness against unmodeled dynamic elements and perturbations may even be considered as a structural property of systems. The aim of robustness analysis is to cope with the difficulties of model uncertainties when designing feedback controllers.

In addition, for system analysis of the robust design of control systems, stability must be considered in view of the uncertainties of the system equations. This is particular true of the current investigation since stability is the principal area of research interest. Specific explanations and examples from Singh and Coelho [40] are considered in Chapter 4, in which VTOL (vertical take-off and landing) aircraft systems, with several parameters varying over time, result in substantial changes in dynamics. Moreover, the system equations then require an adequate controller to achieve satisfactory and stable performance when subjected to widely different flight conditions. For example, stability conditions vary with different flight conditions, different airspeeds or different pitch angles. However, parameters with certain conditions stated in the system equations can be determined if the VTOL continues to be stable over a large parametric space. The techniques proposed in the current investigation serve to improve the parametric range of the structured perturbations considered to be the robustness bounds.

2.3 Generalization of the Lyapunov Direct Method

Lyapunov's direct method is a logical alternative to algebraic methods of parameter analysis and the robust design of control systems subject to structured perturbations. An appropriate framework for the study of system (S) stability is the Bellman-Mastrov concept of vector Lyapunov functions, one of the most important recent generalizations of the Lyapunov direct method [48]. In its most simple terms, this concept is unique in that it assigns a Lyapunov function to each subsystem, thus establishing stability in a part of the state space. In this general setting, one means to consider structured uncertainties is to apply the notion and criteria of stability under structural perturbations.

As considered by Vidyasagar [48], the concept of vector Lyapunov functions calls for breaking a full system into subsystems, assigning the Lyapunov function to each component subsystems, and then proceeding with the investigation of the stability of the entire system. Siljak [39] considers a dynamic system as

$$S : \dot{x}_i = g_i(t,x_i) + h_i(t,x,E) , \quad i\epsilon N \qquad (2.3.1)$$

being composed of N subsystems, i.e.,

$$S_i : \dot{x}_i = g_i(t,x_i) , \quad i\epsilon N \qquad (2.3.2)$$

where $x_i(t)\epsilon R^{n_i}$ is the state of S_i and $x(t)\epsilon R^n$ is the state of S at $t\epsilon R$, $x = (x_1^T, x_2^T, \cdots, x_n^T)^T$ and $N = \{1,2,\cdots,N\}$. The N x N matrix $E = (e_{ij})$ represents uncertainty about the system S, while it is assumed that the elements $e_{ij}\epsilon[0,1]$ are constant but unknown numbers. The usual existence and unique properties of the solutions $x_E(t;t_0,x_0)$ of (2.2.1) are also assumed for all initial conditions $(t_0,x_0)\epsilon R \times R^n$ and all admissible E. The unique equilibrium $x = 0$ of the system S is invariant under structured perturbations, that is, it does not change in E.

Consider the interconnection functions $h_i(t,x,E)$, taking the form

$$h_i(t,x,E) = h_i(t,e_{i1}x_1,e_{i2}x_2,\cdots,e_{iN}x_N) \quad , \qquad i\epsilon N \qquad (2.3.3)$$

which indicate that an element e_{ij} of E represents the coupling from the subsystem S_j to the subsystem S_i. For this reason, the perturbations in E are termed structural perturbations, and stability under such perturbations is called connective stability. The notion of the N x N fundamental interconnection matrix, $\bar{E}=(\bar{e}_{ij})$, a binary matrix describing the basic structure of S is then introduced. The structural perturbations are then described by the element by element inequality, $E\le\bar{E}$, thereby stating the definition of connective stability:

Definition 2.1. A system S is connectively stable if the equilibrium $x = 0$ of S is stable in the sense of Lyapunov for all $E \le \bar{E}$.

A large number of variations upon this basic definition and related stability results within the framework of vector Lyapunov functions have been presented by Siljak [38].

It is not the intent of this presentation to focus upon these variations, or subsequent results obtained in the context of connective stability, rather the purpose of this study is to focus upon the relationship between stability analysis and developments in the area of robustness bounds. However, a brief review of the former is useful to an understanding of the presentation in this study. To denote the systems which most frequently occur, such that their unperturbed portion is linear and stationary, with perturbations expressed linearly with respect to x, Siljak [38] defines

$$S_E : \dot{x}_i = A_i x_i + \sum_{j=1}^{N} e_{ij} A_{ij} x_i , \qquad i \in N$$

which is composed of N subsystems:

$$S_i : \dot{x}_i = A_i x_i .$$

The subsystem S_E can then be rewritten in a compact form

$$S_E : \dot{x} = A_D x + A_C(E) x ,$$

where $A_D = \mathrm{diag}\{A_1, A_2, \cdots, A_N\}$ and $A_c = (e_{ij}A_{ij})$ are matrices of appropriate dimensions.

Assuming that each subsystem S_i is stable, that is, all eigenvalues of each A_i have negative real parts, for each S_i choose a norm-like Lyapunov function

$$u_i(x_i) = (x_i^T H_i x_i)^{1/2} ,$$

where H_i is the symmetric positive definite solution of the Lyapunov matrix equation

$$A_i^T H_i + H_i A_i = -G_i \ .$$

For the overall system, S_E, choose a Lyapunov function $V(x) = d^T V_i(x)$, where $V \epsilon R^N$ is a vector Lyapunov function with components as defined previously and $d \epsilon R_+^N$ is a positive vector. Calculating the Dini derivative $D^+ v(x)$ along the solutions of S_E and introducing some estimations, form the N x N matrix $W = (W_{ij})$ where

$$W_{ij} = \frac{1}{2} \frac{\lambda_m(G_i)}{\lambda_M(H_i)} - \overline{e}_{ij}\, \epsilon_i \qquad i=j$$

and

$$W_{ij} = -\overline{e}_{ij} \epsilon_{ij} \ . \qquad i \neq j$$

In conclusion, S_E is connectively stable if the matrix W is an M-matrix, which is equivalent to saying that W satisfies the following inequalities:

$$\begin{vmatrix} w_{11} & w_{12} & \cdots & \cdots & w_{1k} \\ w_{21} & w_{22} & \cdots & \cdots & w_{2k} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ w_{k1} & w_{k2} & \cdots & \cdots & w_{kk} \end{vmatrix} > 0 \quad for\ all\ k \in N \ .$$

The special case, when the system S_E is reduced to a single subsystem,

$$S_p \ : \ \dot{x} = A_N x + A_P x \ ,$$

is considered by Patel and Toda [35], where A_N is a matrix of a nominal system $S_N : \dot{x} = A_N x$ and $A_P x$ is an unstructured perturbation. In this case, in place of the matrices H_i, a single matrix H is to be found, obtained as the solution of the Lyapunov equation

$$A_N^T H + H A_N = -I \ ,$$

where I is an identity matrix. Furthermore, improvements of robustness bounds using transformations have been obtained recently by Yedavalli and Liang [52].

Robustness bounds obtained by the application of vector Lyapunov functions result in inequalities which differ from the bounds obtained by application of the scalar Lyapunov function approach. Siljak [39] used a simple example to show that the vector approach (that is, utilizing modal transformation) would contribute to better results. Yedavalli and Liang [52] obtained an improvement of robustness bounds by the consideration of state transformations. Subsequently, Becker and Grimm [5], for the case of unstructured perturbations, proved that robustness bounds obtained by application of the small gain theorem were always superior to those obtained by state transformation. Yedavalli and Liang [52] had stated that the question of finding the best transformation for either unstructured or structured, time-variant perturbations was still an open issue. For the current investigation, another approach toward obtaining better robustness bounds using scalar Lyapunov functions with time-variant, structured system perturbations has been adopted.

The mathematical basis of the proposed technique was established in two studies by Olas [31,32], who considered the sign properties of the integral of the Lyapunov derivative along a finite interval of time

$$\int_0^T \dot{V}(x(\tau,\ t_0,\ x_0))\,d\tau \ , \tag{2.4.1}$$

which is considered in place of the sign properties of the derivative $\dot{V}$ when investigating system stability. The interval of time is then considered as dependent upon t_0 and x_0 [31].

Corollary 2.1 is utilized for the estimation of robustness bounds:

Corollary 2.1. Consider a system

$$\dot{x} = f(x) \ , \qquad f(0) = 0 \tag{2.4.2}$$

where $f \epsilon C^{(1)}(R^n)$. Let all solutions of (2.4.2) be defined in the future. If there exist:

1) a continuously differentiable positive definite function $V(x)$,
2) a bounded function $T(x)$ defined for $x \ \epsilon\ R^n$ and having a positive lower bound, and
3) a continuous, positive-definite function $W(x)$,

such that the function

$$V^*(x) = \int_{-T(x)}^{0} \dot{V}(x(\tau,\ 0,\ x))\,d\tau \tag{2.4.3}$$

fulfills the condition

$$-V^*(x) \geq W(x)$$

and

$$V^*(0) = W(0) ,$$

then the trivial solution of (2.4.2) is globally asymptotically stable.

The procedure based on Corollary 2.1 is then a natural extension of the Lyapunov direct method procedure.

The first step following selection of the Lyapunov function candidate $V(x)$ is to check the sign of $\dot{V}$. It is denoted by $\Gamma_1 \subset R^n\backslash 0$ (i.e., a state space with the point zero excluded), where $\dot{V} \langle 0$. If $\Gamma_1 = R^n\backslash 0$, global asymptotic stability is ensured; if $\Gamma_1 \neq R^n\backslash 0$, the following steps are required. Observe that for each $x\epsilon\Gamma_1$, there exists an $\epsilon \rangle 0$ such that

$$\int_{-\epsilon}^{0} \dot{V}(x(\tau, 0, x))d\tau < 0$$

Therefore, further investigation of (2.4.3) is required only for $x\epsilon\Gamma_1$.

For the proposed technique, the perturbations are considered as structured, nonlinear time-variant, and systems are nominally linear. This class of systems is particularly suited to the utilization of Corollary 2.1 for the estimation of robustness bounds. Two approaches to the problem are considered. The first is based on the numerical generation of the worst solutions, starting from the points $x\epsilon\Gamma_1$. Given the class of systems under discussion, it is sufficient to consider only the points x belonging to the unit sphere, S_1, $x \epsilon S_1 \backslash \Gamma_1$. The numerical procedure derived in Chapter 3 is provided to serve as a check upon consideration of this step.

The second approach utilizes the fact that the class of systems under discussion allows for the derivation of the analytical expression of the difference

$$V(x(t,0,x_0)) - V(x_0) \quad . \tag{2.4.4}$$

For a case when V(x) is selected as the quadratic form, this leads to the analysis of the properties of matrices, which are used to describe how the difference in (2.4.4) behaves for different perturbations. Following this approach, it is possible to obtain the robustness bounds analytically. It is also possible that this approach, when applied to multi-dimensional systems, will result in a relatively simple technique. The details of these techniques and procedures, including application to aircraft dynamic systems, are discussed in Chapters 3 and 4.

CHAPTER 3

APPLICATION OF RELAXING LYAPUNOV STABILITY CONDITIONS TO ROBUSTNESS BOUNDS

3.1 Introduction

The Lyapunov direct method has been recently applied to a number of applications in the area of dynamic control system and other related areas. Leitman [25] applied the principle to controller design for uncertain systems, and Siljak [39] used the Lyapunov direct method to estimate the control of robustness bounds. In addition, Barmish and Leitmann [3] have used the technique for uncertainty threshold estimation.

The objective of the current investigation is to improve the robustness estimates of dynamic systems with structured uncertainties, using Lyapunov stability conditions to weaken the stability conditions formulated in classical Lyapunov theorems.

For purposes of analysis, the sign properties of the Lyapunov function derivative integrated along the finite interval of time are considered, rather than the sign properties of the derivative itself, which has been the traditional means of judging system stability. The system examined in the current investigation is assumed to be nominally linear, with time-variant, nonlinear bounded perturbations.

3.2 Relaxing Lyapunov Stability Conditions

Consider a system

$$\dot{x} = f(t,x), \quad f(t,0) = 0 , \tag{3.2.1}$$

where $f \epsilon C_{(t,x)}^{(0,1)}(Z)$, $Z = I_t^+ \times D$, $I_t^+ = \{t : t_- < t < \infty\}$, and $D = \{x \epsilon R^n : \|x\| < H\}$. Let $x(t)$ be a solution to (3.2.1) for $a < t < b$, where $(a,b) \subset I_t^+$. The following theory of integral continuity of solutions is valid for system (3.2.1).

Theorem 3.1. Select an arbitrary constant $\epsilon > 0$ and an interval $[\alpha,\beta] \epsilon (a,b)$. There exists a constant $\delta > 0$ such that

a) the solution $z(t)$, fulfilling $z(\gamma) = z_0$, where $\gamma \epsilon [\alpha,\beta]$ and $\|z(\gamma) - x(\gamma)\| < \delta$, is well-defined for $t \epsilon [\alpha,\beta]$, and

b) for $t \epsilon [\alpha,\beta]$, the relation is fulfilled by $\|z(t) - x(t)\| < \epsilon$.

It follows that the solution (3.2.1) comes from $x(t,t_0,x_0)$, $x(t_0,t_0,x_0) = x_0$. The following conclusion, based on Theorem 3.1, provides estimates for solutions neighboring the trivial solution $x = 0$.

Conclusion 3.1. For any $\epsilon_1 > 0$ and for any finite interval $[t_0, t_0+T]$, there exists $\delta_1 > 0$ such that for every x_0 fulfilling $\|x_0\| < \delta_1$, a solution

$x(t,t_0,x_0)$ exists, satisfying $\|x(t,t_0,x_0)\| < \epsilon_1$ for $t\epsilon[t_0,t_0+T]$.

3.3 Estimation of the Integral

For the sake of simplicity, the set $D = R^n$ is assumed. Let $V = V(t,x)$, defined on $I_t^+ \times R^n$, be a continuously differentiable function, such that it is locally positive-definite on some ball, B_r, centered at 0. Then, let $\dot{V}(t,x)$ denote a derivative of V along the system given in (3.2.1) and let $x(t,t_0,x_0)$ exist for $t\epsilon[t_0,t_0+T]$.

Then, consider the expression

$$\int_{t_0}^{t} \dot{V}(\tau,x(\tau,t_0,x_0))\ d\tau\ . \tag{3.3.1}$$

Conclusion 3.1 (section 3.2) can be used, taking into account the continuity of $\dot{V}(t,x)$ and the fact that $\dot{V}(t,0) = 0$. The following conclusion can then be obtained.

Conclusion 3.2. For any constant $\epsilon_3 > 0$ and for any interval $[t_0,\ t_0+T]$, there exists a constant $\delta_3 > 0$, such that if $\|x_0\|<\delta_3$, then

$$\left| \int_{t_0}^{t_0+u} \dot{V}(\tau,x(\tau,t_0,x_0))\ d\tau \right| < \epsilon_3$$

for $u\epsilon[t_0,\ t_0+T]$; when the above integral on the interval $[t_0,t]$ is estimated, a group property of the solution to (3.2.1) is used.

Assume that x_0 is so chosen that the solution exists and remains within the ball, B_r, on the interval $[t_0,t]$.

Let the sequence of time instants $\{\tau_i\}$, $i = 0,1,\cdots,m$, $\tau_0 = t$, $\tau_m = t_0$, $\tau_{i+1} < \tau_i$ divide the interval $[t_0,t]$ on m subintervals $[\tau_{i+1}, \tau_i]$, $i = 0,1,2,\cdots,m-1$. Then, write

$$\int_{t_0}^{t}\dot{V}(\tau,x(\tau,t_0,x_0))\,d\tau = \sum_{i=0}^{m-1}\int_{\tau_{i+1}}^{\tau_i}\dot{V}(\tau,x(\tau,t_0,x_0))\,d\tau \ . \tag{3.3.2}$$

The group property of the solution may then be expressed by the relation

$$x(\tau,t_0,x_0) = x(\tau,\bar{t};x(\bar{t},t_0,x_0)) \ , \tag{3.3.3}$$

valid for any τ, $\bar{t}\epsilon[t_0,t]$, letting $x_i = x(\tau_i,t_0,x_0)$ and

$$x(\tau,t_0,x_0) = x(\tau,\tau_i,x_i) \ . \tag{3.3.4}$$

Relation (3.3.4) allows the transformation of (3.3.2) to the final form

$$\int_{t_0}^{t}\dot{V}(\tau,x(\tau,t_0,x_0))\,d\tau = \sum_{i=0}^{m-1}\int_{\tau_{i+1}}^{\tau_i}\dot{V}(\tau,x(\tau,\tau_i,x_i))\,d\tau \ . \tag{3.3.5}$$

Thus, it is possible to formulate the following theorem:

Theorem 3.2. Consider a system given in (3.2.1) where $f\epsilon^{(0,1)}_{(t,x)}\ (I_t^+ \times R^n)$. If there exist

a) a continuously differentiable and locally positive-definite function V(t,x) and

b) a bounded function T(t,x) defined for $t\epsilon[0,\infty]$, $x\epsilon B_r$, and having a positive lower bound such that the function

$$V^*(t,x) = \int_{t-T(t,x)}^{t} \dot{V}(\tau, x(\tau,t,x))\,d\tau \qquad (3.3.6)$$

exists and fulfills the condition

$$V^*(t,x) \le 0 \ , \quad \text{for } (t,x) \in [0,\infty) \times B_r \qquad (3.3.7)$$

then the trivial solution of (3.2.1) is Lyapunov stable.

Proof: In principle, the proof differs little from the classical proof of the Lyapunov theorem. As is known, it is sufficient to prove stability for a selected initial instants t_0, and $t_0 = 0$ is selected. By virtue of the definition of $V(t,x)$, there exists a continuous and positive-definite function $W(x)$ such that

$$V(t,x) \ge W(x) > 0 \ , \quad \text{for } x \ne 0 \ , \ x \in B_r$$
$$V(t,0) = W(0) = 0 \ . \qquad (3.3.8)$$

Select the sphere $S_\epsilon = \{\|x\| = \epsilon\}$ such that $S \epsilon B_r$. By virtue of Weierstrauss' theorem, the lower bound of $W(x)$ on S_ϵ is attained at a certain point $\bar{x}$ of S_ϵ, that is,

$$\inf_{x \in S_\epsilon} W(x) = W(x^*) = \alpha > 0 \ . \qquad (3.3.9)$$

Utilizing the fact that the function $V(0,x)$ is continuous and $V(0,0) = 0$, it may be concluded that there exists a neighborhood $\|x\| < \delta_2 < \epsilon$, such that for every $\|x_0\| < \delta_2$, $0 \le V(0,x_0) < \alpha/2$.

Then, T denotes the upper bound of $T(t,x)$:

$$\sup_{[0,\infty)\times B_r} T(t,x) = T \ . \qquad (3.3.10)$$

Using Conclusion 3.2, δ_3 is chosen so that for every x_0 satisfying $\|x_0\|<\delta_3$,

$$\left|\int_0^u \dot{V}(\tau, x(\tau,0,x_0))\,d\tau\right| < \frac{\alpha}{2} \qquad (3.3.11)$$

for $u\epsilon[0,T]$. Then, denoting that $\delta_4 = \min(\delta_2, \delta_3)$, for $\|x_0\| < \delta_4$,

$$0 \le V(0,x_0) < \alpha/2 \qquad (3.3.12)$$

and

$$\left|\int_0^u \dot{V}(\tau, x(\tau,0,x_0))\,d\tau\right| < \frac{\alpha}{2} \qquad (3.3.13)$$

for $u\epsilon[0,T]$. Thus, consider an arbitrary non-trivial solution with initial condition x_0, such that $\|x_0\| < \delta_4$, given that trajectory of this solution remains entirely inside of the sphere S_ϵ, that is,

$$\| x(t,0,x_0) \| < \epsilon \ . \qquad \text{for} \quad t\epsilon[0,\infty) \qquad (3.3.14)$$

Then, assume the contrary, that is, at some instant $t = t^*$, the point of trajectory is for the first time located on S_ϵ:

$$\| x(t,0,x_0) \| < \epsilon \qquad \text{for} \quad t\epsilon[0,t^*)$$
$$\| x(t^*,0,x_0) \| = \epsilon \ . \qquad (3.3.15)$$

The corresponding value of $V(t,x)$ is obtained by writing

$$V(t^*, x(t^*,0,x_0)) = V(0,x_0) + \int_0^{t^*} \dot{V}(\tau, x(\tau,0,x_0))\,d\tau \ . \qquad (3.3.16)$$

Denote $x(t^*,0,x_0) = x^*$, $t^* = \tau_0$, then introduce inductively two finite sequences: the sequence of instants, $\{\tau_i\}$, $\tau_{i+1}<\tau_i$, and the sequence of points, $x_i \epsilon B_r$, by defining

$$\tau_1 = \tau_0 - T(\tau_0, x^*) \ , \ x_1 = x(\tau_1, 0, x_0) \ ,$$

$$\tau_{i+1} = \tau_i - T(\tau_i, x_i) \quad , \quad x_{i+1} = x(\tau_{i+1}, 0, x_0) \quad . \tag{3.3.17}$$

By virtue of the properties of the function T(t,x) for some $i = i^*$, $\tau_{i^*} \leq 0$, while for the previous instants $\tau_i > 0$. Then, 0 is accepted as the last i^*-th term of the sequence, with the observation that

$$\tau_{i^*-1} \leq T(\tau_{i^*-1}, x_{i^*-1}) \leq T \quad . \tag{3.3.18}$$

The integral on the right-hand side of (3.3.16) is then trans-formed, writing

$$\int_0^{t^*} \dot{V}(\tau, x(\tau, 0, x_0))\, d\tau = \sum_{i=0}^{m-1} \int_{\tau_{i+1}}^{\tau_i} \dot{V}(\tau, x(\tau, \tau_i, x_i))\, d\tau \quad . \tag{3.3.19}$$

From assumption (3.3.7), all of the above integration, with the exception of the last time step integration of the in-tegral (3.3.19), are less than or equal to zero. Using (3.3.16), it may be estimated

$$V(t^*, x(t^*, 0, x_0)) \leq V(0, x_0) + \left| \int_0^{\tau_{m-1}} \dot{V}(\tau, x(\tau, 0, x_0))\, dt \right| \,. \tag{3.3.20}$$

Therefore,

$$V(t^*, x(t^*, 0, x_0)) < \alpha \quad . \tag{3.3.21}$$

On the other hand, $x(t^*, 0, x_0) \epsilon S_\epsilon$. Thus, by virtue of (3.3.8) and (3.3.9)

$$V(t^*, x(t^*, 0, x_0)) \geq \alpha \quad , \tag{3.3.22}$$

contradicts (3.3.21) and proves the theorem. It should be noted that the integral (3.3.6) may be transformed to the form

$$\overline{V}(t,x) = V(t,x) - V(t-T(t,x), x(t-T(t,x), t, x)) \quad , \tag{3.3.23}$$

which is a representation of the difference between the values of the function V at the initial point (t,x) and at the point in the "past" defined by the pair

$$(t-T(t,x),x(t-T(t,x),t,x)) \ .$$

The right hand side of (3.3.23) should not be confused with the expression

$$V(t_T+T(t_T,x_T),x(t_T+T(t_T,x_T),t_T,x_T)) - V(t_T,x_T) \ , \quad (3.3.24)$$

in which the solution with the initial values t_T, x_T is shifted forward through the interval $T(t_T,x_T)$.

The following example shows that the fact that (3.3.24) is not positive does not imply stability. Consider a scalar equation

$$\dot{x} = \frac{1+sint+tcost}{1+t(1+sint)}\, x$$

and function $V = V(x) = x^2$. The solution to the equation is of the form

$$x(t,t_0,x_0) = \frac{1+t(1+sint)}{1+t_0(1+sint_0)}\, x_0$$

and, as may be seen, the trivial solution is unstable. To determine the function T, two instants t, t_1, $t_1 > t$, are selected, writing

$$V(x(t_1,t_0,x_0)) - V(x(t,t_0,x_0))$$

and

$$= \frac{x_0^2[(1+t_1(1+\sin t_1))^2-(1+t(1+\sin t))^2]}{[1+t_0(1+\sin t_0)]^2} \ .$$

For any t introduce

$$k = integer(\frac{t}{2\pi})$$

so that $2\pi \leq t < 2\pi(k+1)$, and choosing $T(t) = 2\pi(k+2) + 3\pi/2 - t$ so that $t_1 = t+T(t)$ is obtained as $t_1 = 2\pi(k+2) + 3\pi/2$.

It is easily seen that T(t) is bounded, for it lower bound, $t \epsilon [0,\infty)$, is positive. Then, for this selection of t_1,

$$V(x(t_1,t_0,x_0)) - V(x(t,t_0,x_0))$$

$$= \frac{x_0^2[1-\{1+t(1+\sin t)\}^2]}{[1+t_0(1+\sin t_0)]^2} ,$$

which demonstrates that the negative semi-definiteness of (3.3.24) does not ensure the stability of $x = 0$.

Finally, the integral (3.3.6) must be transformed to the form (3.3.23), which represents the difference between the values of the function V at the initial point (t,x) and the point previously defined as the pair

$$(t-T(t,x), x(t-T(t,x),t,x)) .$$

Then, proofs of the theorems for several conditions of the trivial solution to the system given in (3.2.1) are given:

Theorem 3.3. Consider a system (3.2.1), as defined in section 3.2. If there exist:

a) a continuously differentiable, locally positive-definite, and decrescent function V(t,x),

b) a bounded time function $T(t,x)$, defined for $t\epsilon[0,\infty]\subset I_t^{+}$, $x \epsilon B_r$, and having a positive lower bound, and

c) a continuous, locally positive-definite function $W(x)$, such that the function

$$-V^{*}(t,x) = -\int_{t-T(t,x)}^{t} \dot{V}(\tau,x(\tau,t,x))\,d\tau$$

fulfills the conditions

$$-V^{*}(t,x) \geq W(x) \ , \quad \text{for} \quad x \epsilon B_r$$

and

$$-V^{*}(t,0) = W(0) \ ,$$

then the trivial solution of (3.2.1) is asymptotically stable. In this theorem, it is not required that $V^{*}(t,x)$ is a continuous function.

As may be seen from Theorem 3.2, Theorem 3.3 is the stronger of the two. Since this is the case, it is only necessary to prove that

$$\lim_{t\to\infty} \|x(t,0,x_0)\| = 0 \ . \tag{3.3.26}$$

That is, given $\eta > 0$, there exists an instant t^{*} such that $\|x(t,0,x_0)\| \leq \epsilon$ for $t\epsilon[t^{*},\infty]$.

The functions $V(t,x)$ and $V^{*}(t,x)$ fulfill

$$\alpha(\|x\|) \leq V(t,x) \leq \beta(\|x\|) \ ,$$

$$V^{*}(t,x) \leq -\gamma(\|x\|) \ ,$$

where α, β, γ are functions belonging to Class k [49]. Therefore, it is enough to show that

$$V(t,x(t,0,x_0)) \leq \alpha(\epsilon) \quad \text{for } t\epsilon[t^{*},\infty) \ .$$

This proof consists of two parts. First, it must be shown that for an arbitrary $\delta > 0$, there exists an instant $\bar{t}$ such that the function V attains the value δ. In this process, V is a positive-definite decrescent function and the properties of V^* are used. Consequently, the fact that $V(t,x(t,0,x_0))$ is some finite time which remains below the assigned value allows establishment of the proof of (3.3.26).

Theorem 3.4. Consider a system (3.2.1), as defined in section 3.2, and let all solutions of (3.2.1) be defined in the future. If there exists:

a) a continuously differentiable, positive-definite and decrescent $V(t,x)$,

b) a bounded function $T(t,x)$ defined for $t\epsilon[0,\infty)$, $x \epsilon B_r$, and having a positive lower bound, and

c) a continuous, positive-definite function $W(x)$,

such that the function

$$V^* = \int_{t-T(t,x)}^{t} \dot{V}(\tau,x(\tau,t,x))\,d\tau$$

fulfills the conditions

$$-V^*(t,x) \geq W(x) \ , \qquad \text{for } t\epsilon[0,\infty),\ x\epsilon R^n$$

and

$$V^*(t,0) = W(0) \ ,$$

then the trivial solution of (3.2.1) is globally asymptotically stable.

The fact that the solutions of (3.2.1) are defined in the future enables estimation of the upper bound of the integral

$$\int_0^u \dot{V}(\tau, x(\tau, 0, x_0))\, d\tau \ . \qquad u \epsilon [0, T)$$

The following corollary provides global asymptotic stability conditions for the trivial solution of the autonomous system for the case when $V = V(x)$:

Corollary 3.1. Consider a system

$$\dot{x} = f(x) \ , \ f(0) = 0 \ , \tag{3.3.27}$$

where $f \epsilon C^{(1)}(R^n)$, letting all solutions of (3.3.26) be defined in the future. If there exist:

a) a continuously differentiable, positive-definite function $V(x)$,

b) a bounded function $T(x)$ defined for $x \epsilon R^n$, and having a positive lower bound, and

c) a continuous, positive-definite function $W(x)$,

such that

$$V^*(x) = \int_{-T(x)}^0 \dot{V}(x(\tau, 0, x))\, d\tau$$

fulfills the conditions

$$-V^*(x) \geq W(x)$$

and

$$V^*(0) = W(0) \ ,$$

then the trivial solution of (3.3.26) is globally asymptotically stable.

Thus, in this chapter applied theorems for robust controls and new approaches to dynamic systems have been introduced. In the following chapter, these approaches are applied to practical examples arising from actual dynamical systems, including aircraft dynamic modeling.

CHAPTER 4

RESULTS OF THE NUMERICAL APPROACH

4.1 Introduction

In this chapter, actual dynamical systems are approached by application of the Lyapunov direct method to demonstrate the improvement of robustness bounds. These examples are drawn from actual aircraft control systems, a methodology which has attracted considerable attention in testing robust controls since in their absence it is difficult to analyze these systems. The selected examples are used to demonstrate the practicality of the proposed techniques. Systems with structured perturbations are introduced for purposes of problem analysis with the proposed numerical and computational techniques. *FORTRAN* is the principal computer language used for programming, in conjunction with *MACSYMA*, a symbolic algebraic calculation application directed at the analytical solution of complex mathematical problems. *FORTRAN* has been applied to those problems easily expressed in terms of numerical calculations, while *MACSYMA* has been used for those problems which must be expressed in symbolic terms, including matrix calculations, linear equations and nonlinear polynomial equations. This application serves to minimize the large errors in the numerical approach, while providing methods for achieving exact solutions to problems. *MACSYMA* works with symbols, polynomial expressions, equations, and numbers, and can return results in either numeric or symbolic form.

In this chapter, Theorem 3.4 is applied to the investigation of the robustness of linear systems with structured uncertainties. The numerical procedure applied follow the flow chart given in Figure 4.1.

First, the quadratic form was selected as the Lyapunov function. The bounds of uncertainty were defined by calculating the Lyapunov derivatives. Then, new and higher bounds of uncertainty were selected, with the function $V^*(x)$ considered for the initial conditions belonging to a unit sphere. For these initial conditions, for which the Lyapunov derivative remains negative, there is no need to determine a solution. For the remaining solutions, integration is extended until the function $\dot{V}^*$ is negative.

The robust design of control systems subject to structural perturbations is a natural application of the results which are thus presented. Since the structural perturbations are bounded, solutions for these types of systems are defined in the future. A one-degree of freedom dynamic system with structured perturbations is investigated in section 4.2. Section 4.3 includes the example of a three-dimensional case, and more complicated dynamic systems are considered in the following section.

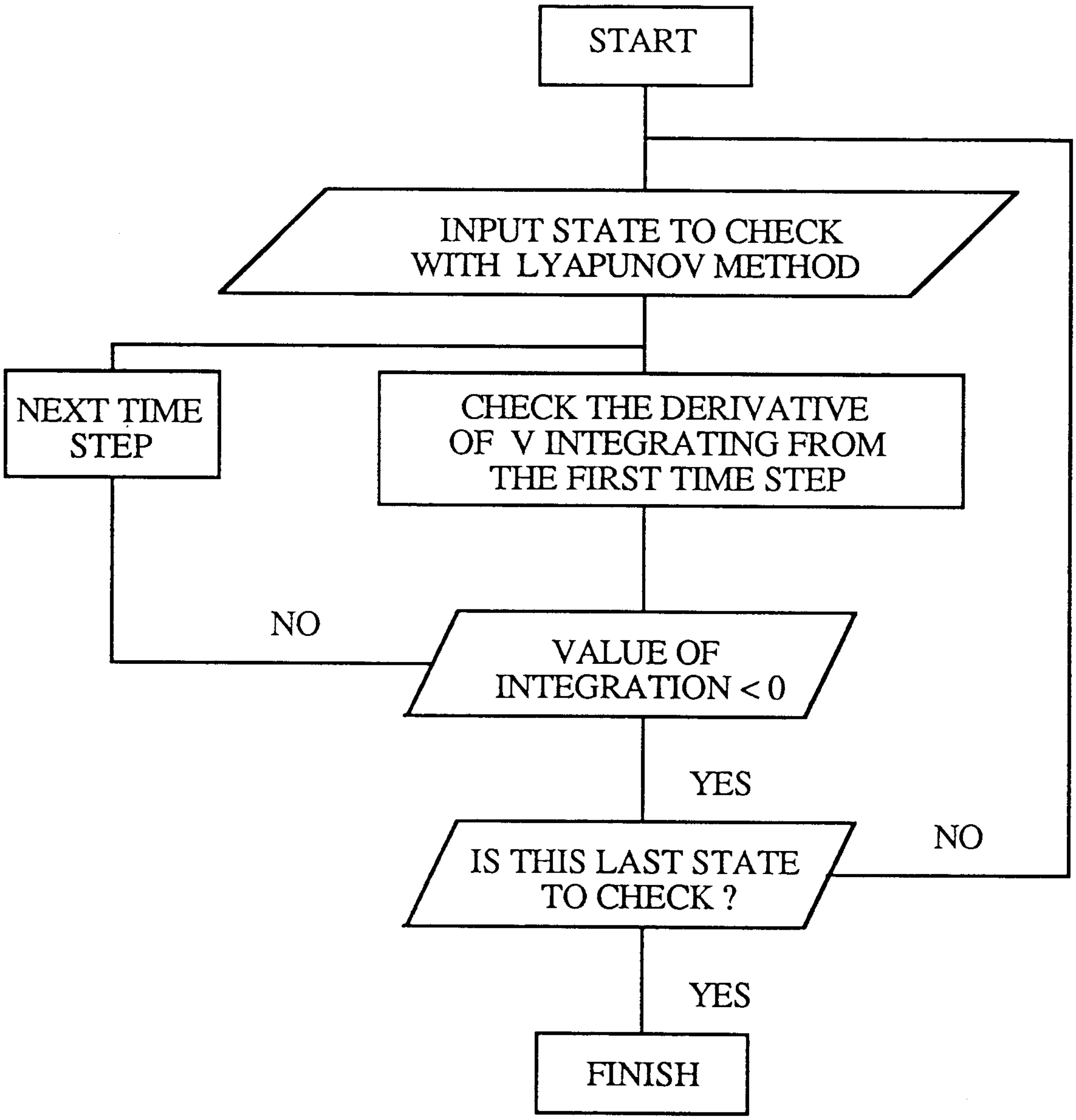


Figure 4.1 Flow Chart for Numerical Procedures.

4.2 Analysis of a Two-Dimensional System with a Single Structured Perturbation

Consider the system

$$\dot{x} = Ax + g(t,x)Gx , \qquad x \epsilon R^2$$

where g(t,x) is a scalar function,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} ,$$

and the bound k on g(t,x) is such that if $|g(t,x)|<k$, the trivial solution of (3.3.27) is asymptotically stable.

A quadratic form of the Lyapunov function V is chosen as Lyapunov candidate function $V = x^TPx$, $\dot{V} = x^T(A^TP+PA)x$. Solving for the matrix P to obtain the best Lyapunov function results in

$$P = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

and

$$V = x_1^2 + x_1x_2 + x_2^2 .$$

Then, V is differentiated with respect to time,

$$\frac{dV}{dt} = 2x_1\dot{x}_1 + \dot{x}_1x_2 + x_1\dot{x}_2 + 2x_2\dot{x}_2 ,$$

and $\dot{x}_1$ and $\dot{x}_2$ in the system equation are replaced:

$$\frac{dV}{dt} = 2x_1x_2+x_2^2+x_1[(-1+g)x_1-x_2]+2x_2[(-1+g)x_1-x_2]$$

$$= -\{(1-g)x_1^2+(1-2g)x_1x_2+x_2^2\} < 0 .$$

If $\dot{V}$ fulfills the condition $\dot{V} < 0$, then the above equation must fulfill the following conditions:

a) $1 - g > 0$, $g < 1$, and

b) $(1 - 2g)^2 - 4(1 - g) < 0$, $|g| < \frac{\sqrt{3}}{2} = 0.8660254$.

If the case is positive, then the system is asymptotic stable. The numerical procedure shown in Fig. 4.1 is based on the result $|g| = 0.96$, signifying a 10% improvement in the estimate of robustness.

To consider the unit sphere in two-dimensional space, first select k with the same extent of improvement, searching the areas that do not satisfy $\dot{V} < 0$. Corollary 3.1 is then used to check the values of

$$V^*(x) = \int_{-T(x)}^{0} \dot{V}(x(\tau, 0, x))\, d\tau$$

and

$$\bar{V}(t,x) = V(t,x) - V(t-T(t,x), x(t-T(t,x), t, x)) \quad . \qquad (4.1.1)$$

If the value of (4.1.1) is less than zero, this initial state is stable. In mathematical problem, differential equations are in ideal terms solved about the state variable. For the area which does not satisfy the condition $\dot{V} < 0$, integrate backward in time, selecting the time T,

$$x(-T) = e^{-AT} x(0) + \int_0^T e^{A(\tau - T)} g(t,x) Gx \, d\tau \ ,$$

the exact solution from the system equation, which may then be used to check the value of $\dot{V}$, the proposed technique. Further implementations of this procedure are discussed and solved in Appendix A.

After the described technique is applied to the one-degree of freedom case, the robustness bounds of $|g|$ can be improved by a factor of 10%.

4.3 Analysis of a Three-Dimensional System with Two Structured Perturbations

Consider the three-dimensional case posed by Siljak [39], originally introduced by Zhou and Khargonekar [54]. The following linear constant plant

$$S : \dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x$$

is driven by the output feedback

$$u = -\begin{bmatrix} 1-k_1 & 0 \\ 0 & 1-k_2 \end{bmatrix} y \, .$$

The final closed-loop system is

$$\hat{S} : \dot{x} = \begin{bmatrix} -2+k_1 & 0 & -1+k_1 \\ 0 & -3+k_2 & 0 \\ -1+k_1 & -1+k_2 & -4+k_1 \end{bmatrix} x \, .$$

This case is a two-degree of freedom dynamic system with two structured perturbations, k_1 and k_2. Applying the

Lyapunov direct method, $V = x^T P x$ and $\dot{V} = x^T(A^T P + PA)x$ and the equation $A^T P + PA = -I$ is solved, using the *MACSYMA* symbolic calculation program. It follows that

$$P = \begin{bmatrix} \frac{2}{7} & \frac{9}{476} & -\frac{1}{14} \\ \frac{9}{476} & \frac{83}{476} & -\frac{11}{476} \\ -\frac{1}{14} & -\frac{11}{476} & \frac{1}{7} \end{bmatrix},$$

where P is the exact solution and is positive-definite. Then, $\dot{V}$ is calculated by the matrix

$$A^T P + PA = \begin{bmatrix} \frac{3k_1-7}{7} & -\frac{25k_2+2k_1}{476} & \frac{2k_1}{7} \\ -\frac{25k_2+2k_1}{476} & \frac{36k_2-119}{119} & \frac{57k_2-2k_1}{476} \\ \frac{2k_1}{7} & \frac{57k_2-2k_1}{476} & \frac{k_1-7}{7} \end{bmatrix}. \qquad (4.3.1)$$

If the matrix $A^T P + PA$ had been negative-definite, the selected system would always be asymptotically stable. For $A^T P + PA$ to be negative-definite, the regions k_1 and k_2 should be $|k_1| < 1.60$ and $|k_2| < 2.74$. To find the regions for k_1 and k_2, the inequality equations from the given matrix (4.3.1) are solved. These inequality equations are derived in a manner such that the negativeness of the matrix $A^T P + PA$ observes the Routh-Hurwitz criterion.

For the current investigation, the regions considered above are considered to be rectangular spaces. Using the same proposed technique, but with the complications in this example that the case is three-dimensional with two structured perturbations, the k_1 and k_2 regions can be extended. For $|k_1| < 1.60$, the region of $|k_2|$ is extended to 2.97. Judged from the standpoint of computational requirements, the three-dimensional case with two structured perturbations takes much more time to achieve results than two-dimensional case with one structured perturbation. As may be seen from the results, the $|k_2|$ region is extended with an 8.3% improvement in robustness estimate.

4.4 Analysis of an Aircraft (VTOL) System

In this section, the method for the estimation of robustness described in Chapter 3 is applied to a specific VTOL aircraft (i.e., a helicopter). With respect to model dynamics, the linearized model of the VTOL aircraft in the vertical plane is described by:

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u \ . \tag{4.4.1}$$

The state vector $x \epsilon R^4$, and the components of x are:

x_1: horizontal velocity (knots),

x_2: vertical velocity (knots),

x_3: pitch rate (degree/sec), and

x_4: pitch angle (degrees).

The two-vector control is $u = [u_1 \ u_2]^T$, where

u_1 = "collective" pitch control and

u_2 = "longitudinal cyclic" pitch control.

Control is essentially achieved and maintained by varying the angle of attack with respect to air passing by the rotor blades. Collective control u_1 is principally used to control the vertical motion of the aircraft in an up and down direction, while the principal use of longitudinal cyclic control u_2 is to control the horizontal velocity of the aircraft.

For the model under consideration, nominal air speed is assumed to be 135 knots. Thus, for an airspeed of 135 knots, ΔA and ΔB are zero matrices in (4.4.1). For typical load and flight conditions for a VTOL aircraft at an airspeed of 135 knots, the matrices A and B are:

$$A = \begin{bmatrix} -0.0336 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1001 & 0.3681 & -0.707 & 1.42 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.52 & 4.49 \\ 0.0 & 0.0 \end{bmatrix}.$$

As the airspeed is changed, all of the elements of the first three rows of both matrices also change. The most significant changes take place in the elements a_{32}, a_{34} and b_{21}, and in the following example all the other elements are assumed to be constant. Thus, in the matrices $\Delta A(t)$ and

$\Delta B(t)$, the only non-zero parameters are $\Delta a_{32}(t)$, $\Delta a_{34}(t)$ and $\Delta b_{21}(t)$.

A control law of the form $u = kx$ is chosen, where the constant matrix k is obtained by solving the linear quadratic optimization problem for the nominal system from (4.4.1) with $\Delta A = 0$, $\Delta B = 0$. To obtain desirable handling characteristics at the nominal airspeed of 135 knots, the feedback gain, as provided by Sundararajan [44], is

$$k = \begin{bmatrix} -0.8143 & -1.2207 & 0.266 & 0.826 \\ -0.2582 & 1.178 & 0.0623 & -0.212 \end{bmatrix}.$$

It is then of interest to improve the robustness estimates for the linear controls on the bounds of variations with parameters. Thus, for the nominal part of the given system equation,

$$\dot{x} = Ax + Bu = (A + Bk)x = \bar{A}x$$

and

$$\bar{A} = A + Bk$$

$$= \begin{bmatrix} -0.442152 & -0.305248 & 0.147396 & -0.127576 \\ -0.877862 & -14.2805 & 0.47227 & 0.516586 \\ 3.435818 & 12.39558 & -1.895593 & -4.0914 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The system equation is therefore

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u ,$$

$$u = kx$$

and

$$\dot{x} = (A + Bk)x + (\Delta A + \Delta Bk)x$$
$$= \bar{A}x + \Delta\bar{A}x \ .$$

In the matrices ΔA and ΔB, $\Delta a_{32}(t)$, Δa_{34} and Δb_{21} are the only non-zero elements. With three parameters, the system is called a three-degree of freedom dynamic system with the structured perturbations Δa_{32}, Δa_{34} and Δb_{21}.

Applying the Lyapunov direct method, *MACSYMA* is used for $V = x^TPx$ and $\dot{V} = x^T(\bar{A}^TP+P\bar{A})x$ to solve the equation $\bar{A}^TP + P\bar{A} = -I$. Using this approach, there are 16 linear equations, but only six of these equations are dependent since the matrix P is symmetric. Finally, the matrix P is

$$P = \begin{bmatrix} 2.365108 & 0.190270 & 0.204453 & -1.215635 \\ 0.190270 & 0.348726 & 0.366103 & 0.079727 \\ 0.207453 & 0.366103 & 0.460888 & 0.170179 \\ -1.215635 & 0.079727 & 0.170179 & 2.187138 \end{bmatrix}$$

and the system equation is

$$\hat{S}:\ \dot{x} = \begin{bmatrix} -0.442152 & -0.305248 & 0.147396 & -0.127576 \\ \begin{matrix}-0.8143\Delta b_{21} \\ -0.877862\end{matrix} & \begin{matrix}-1.2207\Delta b_{21} \\ -14.2805\end{matrix} & \begin{matrix}0.266\Delta b_{21} \\ +0.47227\end{matrix} & \begin{matrix}0.826\Delta b_{21} \\ +0.516586\end{matrix} \\ 3.435818 & \Delta a_{32}+12.39558 & -1.895593 & \Delta a_{34}-4.0914 \\ 0 & 0 & 1 & 0 \end{bmatrix} x \ .$$

Thus, P is the exact solution and is positive-definite. Then, from the system equation, $\dot{V}$ is calculated for three structured perturbations,

$\bar{A}^T P + P\bar{A}$

$$= \begin{bmatrix} -0.309874\Delta b_{21} \\ -1.0 & \begin{matrix}-0.51623\Delta b_{21}\\+0.207453\Delta a_{32}\end{matrix} & -0.247506\Delta b_{21} & \begin{matrix}0.0922415\Delta b_{21}\\+0.207453\Delta a_{34}\end{matrix} \\ \begin{matrix}-0.51623\Delta b_{21}\\+0.207453\Delta a_{32}\end{matrix} & \begin{matrix}-0.85138\Delta b_{21}\\+0.732206\Delta a_{32}\\-1.0\end{matrix} & \begin{matrix}-0.354141\Delta b_{21}\\+0.460888\Delta a_{32}\end{matrix} & \begin{matrix}0.190725\Delta b_{21}\\+0.366103\Delta a_{34}\\+0.170179\Delta a_{32}\end{matrix} \\ -0.247506\Delta b_{21} & \begin{matrix}-0.354141\Delta b_{21}\\+0.460888\Delta a_{32}\end{matrix} & \begin{matrix}0.194767\Delta b_{21}\\-1.0\end{matrix} & \begin{matrix}0.3236085\Delta b_{21}\\+0.460888\Delta a_{34}\end{matrix} \\ \begin{matrix}0.0922415\Delta b_{21}\\+0.207453\Delta a_{34}\end{matrix} & \begin{matrix}0.190725\Delta b_{21}\\+0.366103\Delta a_{34}\\+0.170179\Delta a_{32}\end{matrix} & \begin{matrix}0.3236085\Delta b_{21}\\+0.460888\Delta a_{34}\end{matrix} & \begin{matrix}0.131709\Delta b_{21}\\+0.340358\Delta a_{34}\\-1.0\end{matrix} \end{bmatrix},$$

and $\dot{V} = x^T(\bar{A}^T P + P\bar{A})x$ is

$$\begin{aligned}\dot{V} = {} & (-0.309874\Delta b_{21} - 1.0)x_1^2 + (-1.03246\Delta b_{21} \\ & + 0.414906\Delta a_{32})x_1x_2 \\ & + (-0.85138\Delta b_{21} + 0.732206\Delta a_{32} - 1.0)x_2^2 \\ & + (-0.49501\Delta b_{21})x_1x_3 \\ & + (-0.708282\Delta b_{21} + 0.921776\Delta a_{32})x_2x_3 \\ & + (0.194767\Delta b_{21} - 1.0)x_3^2 \\ & + (0.184483\Delta b_{21} + 0.414906\Delta a_{34})x_1x_4 \\ & + (0.38145\Delta b_{21} + 0.732206\Delta a_{34} + 0.340358\Delta a_{32})x_2x_4 \\ & + (0.647217\Delta b_{21} + 0.921776\Delta a_{34})x_3x_4 \\ & + (0.131709\Delta b_{21} + 0.340358\Delta a_{34} - 1.0)x_4^2 .\end{aligned}$$

If $\dot{V}$ is always negative, that is, the matrix $\bar{A}^T P + P\bar{A}$ is always negative-definite, the system will be asymptotically stable. To fulfil this condition, the regions Δa_{32}, Δa_{34} and Δb_{21} have certain limits. In the case of $|\Delta a_{32}| < 0.43$, $|\Delta a_{34}| < 0.24$ and $|\Delta b_{21}| < 0.44$, the system is stable.

Following application of the proposed techniques, the regions which fulfill stability conditions are extended.

Thus, the final results for regions Δa_{32}, Δa_{34} and Δb_{21} are, respectively, $|\Delta a_{32}| < 0.47$, $|\Delta a_{34}| < 0.26$ and $|\Delta b_{21}| < 0.48$. These results constitute a 10% improvement of the robustness estimates.

CHAPTER 5

CONCLUSIONS

A new technique to estimate the robustness of multidimensional systems with bounded perturbations has been presented in this investigation. New stability criteria and conditions were presented and considered for selected application examples. The sign properties of the Lyapunov function derivative integrated along finite intervals of time were considered, rather than upon the sign properties of the derivative itself. Theorems were formulated to serve as a basis for both analytical and numerical procedures. The results demonstrated improvements of the bounds and global asymptotic stability for selected ranges of parameters.

An example of a two-dimensional system was investigated by Radziszewski [37], who obtained estimation results of $k = 0.866$ It was apparent that allowing $g(t,x) = -1$ results in an unstable trivial solution for a two-dimensional system. The application of the proposed method of improving robustness bounds results in a reduction in the margin of the bound estimation (i.e., the difference between the exact bound and the estimated bound). The numerical procedure based on the presented results provides $k = 0.96$, which is only 4 percent less than the maximum possible result and is 10 percent better than $k = 0.866$.

For the three-dimensional system, originally considered by Zhou and Khargonekar [54], the results of the proposed technique when compared with results from previous research for the bounds for robust stability were as follows:

a) Patel and Toda [35]: $|k_i| < 0.5207$;

b) Yedavalli and Liang [52]: $|k_i| < 0.81577$;

c) Zhou and Khargonekar [54] give three different conditions, any one of which is sufficient:

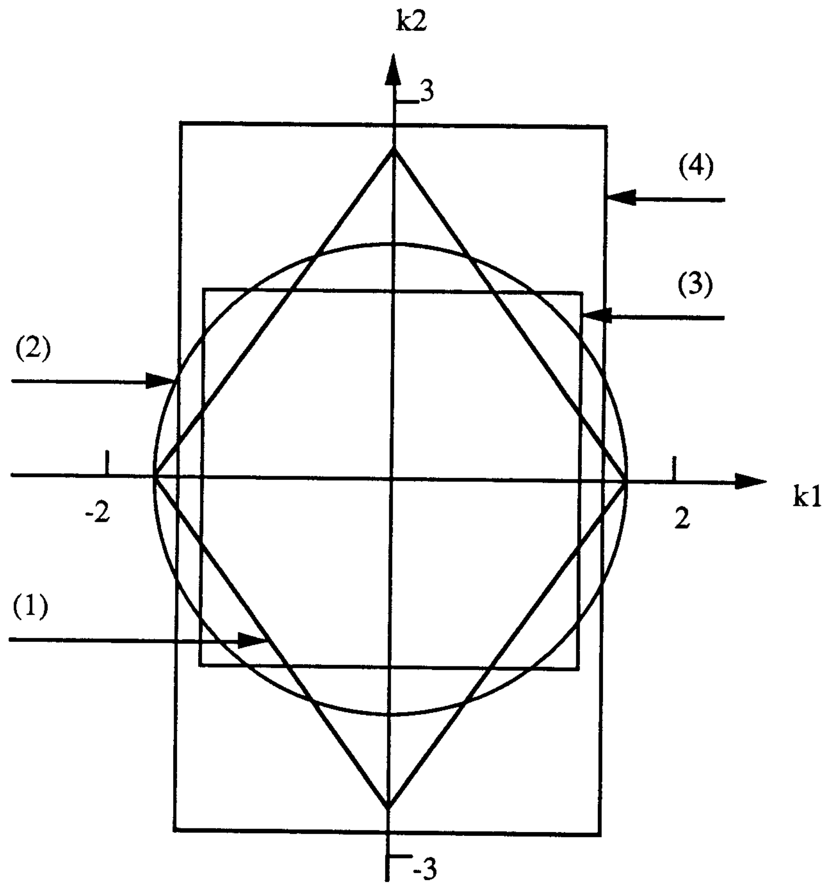
$$(1)\ k_1^2 + k_2^2 < 2.726768\ ,$$

$$(2)\ 0.60521\ |k_1| + 0.3512\ |k_2| < 1\ , \text{ and}$$

$$(3)\ |k_i| < 1.55328\ .$$

The results for the proposed method are $|k_1| < 1.60$ and $|k_2| < 2.97$, which may be compared to the regions of robustness bounds indicated in Figure 5.1. As may be seen, the proposed method allows for improvement in the k_2 region.

The proposed procedure has been applied to VTOL aircraft. Based upon a VTOL model system developed by Singh [41], the most important parameters for the control and design of the airplane controller were extended in range, assuring the stability of the original system in the context of Lyapunov stability.



Zhou and Khargonekar : (1) $0.60521|k_1| + 0.3512|k_2| < 1$

(2) $k_1^2+k_2^2 < 2.72768$

(3) $|k_i| < 1.55328$

Proposed method : (4) $|k_1| < 1.60$, $|k_2| < 2.97$

Figure 5.1 Stability Region Estimates for the Three-Dimensional Case.

Robust control design for VTOL aircraft was previously considered by Singh and Coelho [41], who obtained bounds resulting from nonlinear controls of:

$$|\Delta a_{32}| \le 0.2 \ ,$$

$$|\Delta a_{34}| \le 0.3$$

and

$$|\Delta b_{21}| \le 0.3 \ .$$

In comparison, the results obtained from the proposed technique, based upon the study of robust stability and the ability to stabilize VTOL aircraft systems with parametric (structured) uncertainties, were

$$|\Delta a_{32}| < 0.47 \ ,$$

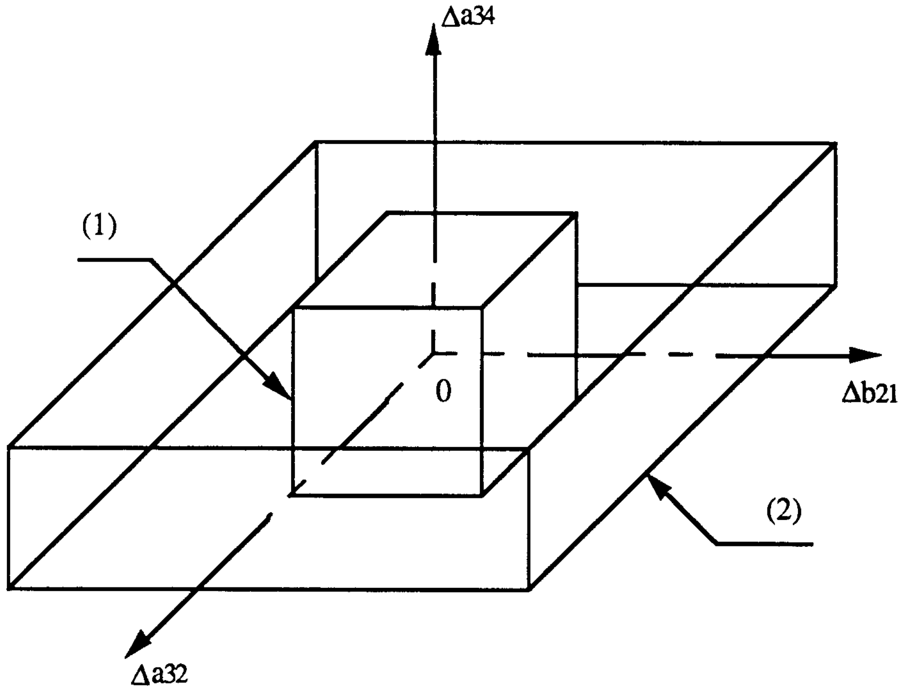
$$|\Delta a_{34}| < 0.26$$

and

$$|\Delta b_{21}| < 0.48 \ ,$$

thus an improvement upon those obtained by Singh and Coelho [41]. As indicated in Figure 5.2, the region of the proposed method (2) is more efficient in the sense of the robust control of a VTOL system.

Further research of robustness bounds based upon the Lyapunov approach should be directed toward the study of new generations of Lyapunov functions for dynamic systems, to include robot manipulators and automotive dynamic systems controlled by a wide variety of parameters. Finally,



(1) Singh and Coelho [41] : $|\Delta a_{32}| \leq 0.2$, $|\Delta a_{34}| \leq 0.3$, $|\Delta b_{21}| \leq 0.3$

(2) Proposed method : $|\Delta a_{32}| < 0.47$, $|\Delta a_{34}| < 0.26$, $|\Delta b_{21}| < 0.48$

Figure 5.2 Stability Region Estimates for VTOL Systems.

the examination of variations in bounded and unstructured perturbations will also be a fruitful area of future research.

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Appendices

APPENDIX A

Numerical Procedures and Computer Programs for Two-Dimensional System

A.1 Procedures to solve the system equation:

$\dot{x} = Ax + g(t,x)Gx$

$$= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + k \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ kx_1 \end{bmatrix} .$$

Select interval time T:

$$x(T) = x(0)\ e^{AT} + \int_0^T e^{A(T-\tau)} \begin{bmatrix} 0 \\ kx_1(\tau) \end{bmatrix} d\tau$$

$$= e^{AT} \{ x(0) + \int_0^T e^{A\tau} \begin{bmatrix} 0 \\ kx_1(\tau) \end{bmatrix} d\tau \} .$$

Introduce finite time difference λ,

$$T = n\ \lambda\ .$$

Denote the discrete time state solutions,

$$x(n) = e^{A\lambda}\left\{x(n-1) + \lambda\begin{bmatrix}0\\ kx_1(n-1)\end{bmatrix}\right\}.$$

A.2 Computer programs for obtaining robustness bounds:

A.2.1:

MACSYMA is used to obtain numerical procedures for $e^{A\lambda}$. The following list is the program and the *MACSYMA* results.

```
(C1) A:MATRIX([0,1],[-1,-1]);
(C2) I:IDENT(2);
(C3) ISA:INVERT(S*I-A);
(C4) INVISA:ILT(ISA,S,T);
```

A.2.2:

Program, written in *FORTRAN*, to find the areas which do not satisfy the condition $\dot{V} < 0$.

```
      PROGRAM TESTVFUN
      REAL*8 TPI,PIN,THETA,X1,X2,VDOT,H
      WRITE(*,*) ' ENTER |H| '
      READ(*,*) H
      TPI=8.*ATAN(1.)
      PIN=TPI/100.
      WRITE(*,100)
 100  FORMAT(5X,'THETA(RADIAN)',8X,'X1',15X,'X2',12X,
     & 'dV/dt',/)
      DO 10 I=0,100
      THETA=I*PIN
      X1=COS(THETA)
      X2=SIN(THETA)
      VDOT=-(1.+H)*X1*X1-(1.+2.*H)*X1*X2-X2*X2
      WRITE(*,101)THETA,X1,X2,VDOT
 101  FORMAT(2X,4(3X,E13.7))
  10  CONTINUE
      STOP
      END
```

A.2.3:

Program to obtain new robustness bounds with the proposed techniques.

```
      PROGRAM THESIS1
C  THIS IS THE FIRST PROGRAM TO CHECK THE ASYMPTOTICAL
C  STABILITY OF THE SYSTEM THAT HAS PERTURBATIONS.
      REAL*8 EA(2,2),X1(10,1000),X2(10,1000),RAD(5),H(2)
C       DIMENSION X1A(10000),X1B(10000)
C       REAL*8 X2A(10000),X2B(10000)
      REAL*8 T,TIN
      REAL*8 VDOT
      integer*4 id,in,i1,i2,i3,i4,im,im1,k1
C  READ THE DATA
      WRITE(*,*)'INPUT SAMPLING TIME T & ITERATION
     &         NUMBER K.'
      READ(*,*) T,IN
      WRITE(*,*) 'INPUT NO. OF INITIAL STATES '
      READ(*,*)ID
      TIN=T/IN
      WRITE(*,*) 'INPUT PERTURBATION |H|.'
      READ(*,*)H(1),H(2)
C  COMPUTE THE EXP(A*LAMDA) MATRIX.
      EA(1,1)=EXP(-TIN/2.)*(SIN(SQRT(3.)*TIN/2.)/SQRT(3.)+
     &        COS(SQRT(3.)*TIN/2.))
      EA(1,2)=2.*EXP(-TIN/2.)*SIN(SQRT(3.)*TIN/2.)/SQRT(3.)
      EA(2,1)=-EA(1,2)
      EA(2,2)=1.099807351
C  READ THE INITIAL STATES
      OPEN(UNIT = 5,FILE = 'INITIA1.DAT',STATUS = 'OLD')
      OPEN(UNIT = 6,FILE = 'VDOT1.RES',STATUS = 'NEW')
C       DELV(X1T,X2T,X10,X20)=-(X1T*X1T+X1T*X2T+X2T*X2T)
C      &                        +(X10*X10+X10*X20+X20*X20)
      DO 1 I0=1,ID
      READ(5,*)RAD(I0)
      X1(0,1)=COS(RAD(I0))
      X2(0,1)=SIN(RAD(I0))
      DO 2 I1=1,IN
      IM=2**(2**I1-1)
      IM1=2**(2**(I1-1)-1)
      I4=0
      DO 3 I2=1,IM1
      DO 4 I3=1,IM1
      DO 5 K1=1,2
      I4=I4+1
      X1(I1,I4)=EA(1,1)*X1(I1-1,I2)+EA(1,2)*(X2(I1-1,I3)
     &                             -TIN*H(K1)*X1(I1-1,I2))
      X2(I1,I4)=EA(2,1)*X1(I1-1,I2)+EA(2,2)*(X2(I1-1,I3)
     &                             -TIN*H(K1)*X1(I1-1,I2))
      WRITE(6,*)EA(1,2),EA(2,2)
```

```
      WRITE(6,*)X1(I1-1,I2),X2(I1-1,I3)
      WRITE(6,*)X1(I1,I4),X2(I1,I4)
   5  CONTINUE
   4  CONTINUE
   3  CONTINUE
      WRITE(6,100)I0,RAD(I0),X1(0,1),X2(0,1)
  100 FORMAT(/,1X,'INITIAL STATE NO.= ',I3,5X,'RADIAN= '
     &      ,E13.7,/ 9X,'X1= ',E13.7,2X,'X2= ',E13.7,//)
      DO 10 IV1=1,IM
      DO 20 IV2=1,IM
      VDOT=DEV(X1(I1,IV1),X2(I1,IV2),X1(0,1),X2(0,1))
      WRITE(6,102)I1,VDOT
  102 FORMAT(2X,'TIME STEP NO = ',I3,' VDOT= ',E20.8)
   20 CONTINUE
   10 CONTINUE
C  PRINT THE FIRST RESULT.
      WRITE(6,100)I0,RAD(I0),X1(0,1),X2(0,1)
  100 FORMAT(/,1X,'INITIAL STATE NO.= ',I3,5X,'RADIAN=
     &       ',E13.7,/9X,'X1= ',E13.7,2X,'X2= ',E13.7,//)
    WRITE(6,101)I1,XMAX1(I,1),XMIN1(I,1),XMAX2(I,1),
 &            XMIN2(I,1)
  101 FORMAT(1X,'TIME STAGE NO.= ',I5,/
     &       ' XMAX1= ',E13.7,4X,'XMIN1= ',E13.7,/
     &       ' XMAX2= ',E13.7,4X,'XMIN2= ',E13.7)
      WRITE(6,102)I1,VDMAX,VDMIN
  102 FORMAT(2X,'VDMAX = ',E20.8,5X,'VDMIN = ',E20.8)
C  FIND THE STATES FOR THE NEXT TIME STEPS.
      M=MOD(J,1000)
      IF(M.EQ.0)THEN
      WRITE(6,101)J,XMAX1(I,J),XMIN1(I,J),XMAX2(I,J),
     &            XMIN2(I,J)
      WRITE(6,103)VDMAX,VDMIN
 103  FORMAT(2X,'VDMAX= ',E20.8,5X,'VDMIN= ',E20.8)
      ENDIF
C  PRINT THE FIRST RESULT.
      WRITE(6,100)I,RAD(I),X1(I),X2(I)
  100 FORMAT(/,1X,'INITIAL STATE NO.= ',I3,5X,'RADIAN=
     &       ',E13.7,/9X,'X1= ',E13.7,2X,'X2= ',E13.7,//)
      WRITE(6,101)I1,XMAX1(I,1),XMIN1(I,1),XMAX2(I,1),
     &            XMIN2(I,1)
  101 FORMAT(1X,'TIME STAGE NO.= ',I5,/
     &       ' XMAX1= ',E13.7,4X,'XMIN1= ',E13.7,/
     &       ' XMAX2= ',E13.7,4X,'XMIN2= ',E13.7)
   2  CONTINUE
   1  CONTINUE
      STOP
      END
      FUNCTION DEV(X1,X2,X10,X20)
      REAL*8 DEV,X1,X2,X10,X20
      DEV=-(X1*X1+X1*X2+X2*X2)+
     &     (X10*X10+X10*X20+X20*X20)
      RETURN
      END
```

APPENDIX B

Numerical Procedures and Computer Programs for Three-Dimensional System

B.1 Procedures to solve the system equation:

$\dot{x} = Ax + Bu$,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix},$$

$$u = -\begin{bmatrix} 1-k_1 & 0 \\ 0 & 1-k_2 \end{bmatrix} y ,$$

$$y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x .$$

Therefore,

$$\dot{x} = \begin{bmatrix} -2+k_1 & 0 & -1+k_1 \\ 0 & -3+k_2 & 0 \\ -1+k_1 & -1+k_2 & -4+k_1 \end{bmatrix} x .$$

The state vectors are

$$x(T) = e^{AT}x(0) + \int_0^T e^{A(T-\tau)} \begin{bmatrix} k_1x_1 + k_1x_3 \\ k_2x_2 \\ k_1x_1 + k_2x_2 + k_1x_3 \end{bmatrix} d\tau .$$

Denote the finite time interval T = nλ. The discrete-time state vector is then

$$x(n) = e^{A\lambda} \left\{ x(n-1) + \lambda \begin{bmatrix} k_1x_1(n-1) + k_1x_3(n-1) \\ k_2x_2(n-1) \\ k_1x_1(n-1) + k_2x_2(n-1) + k_1x_3(n-1) \end{bmatrix} \right\} .$$

B.2 Computer programs for obtaining robustness bounds:

B.2.1:

Program list for solving $e^{A\lambda}$ in matrix form using *MACSYMA*.

```
(C1) A:MATRIX([-2,0,-1],[0,-3,0],[-1,-1,-4]);
(C2) I:IDENT(3);
(C3) ISA:INVERT(S*I-A);
(C4) INVISA:ILT(ISA,S,T);
```

B.2.2:

Program list to solve $A^TP + PA = -I$ and to obtain $\dot{V}$, using *MACSYMA*.

```
(C1) BATCH("silp1.mac\;1");
(C2) A:MATRIX([-2,0,-1],[0,-3,0],[-1,-1,-4]);
(C3) AT:TRANSPOSE(A);
(C4) P:MATRIX([P11,P12,P13],[P21,P22,P23],[P31,P32,P33]);
(C5) LHS:AT . P+P . A;
(C6) EQ1:-P31-P13-4*P11 = -1;
(C7) EQ2:-P31-P23-5*P21 = 0;
(C8) EQ3:-P33-6*P31-P11 = 0;
(C9) EQ4:-P32-P13-5*P12 = 0;
(C10) EQ5:-P32-P23-6*P22 = -1;
(C11) EQ6:-P33-7*P32-P12 = 0;
(C12) EQ7:-P33-6*P13-P11 = 0;
```

```
(C13) EQ8:-P33-7*P23-P21 = 0;
(C14) EQ9:-8*P33-P31-P13 = -1;
(C15) LIST_OF_EQS:[EQ1,EQ2,EQ3,EQ4,EQ5,EQ6,EQ7,EQ8,EQ9]$
(C16) LIST_OF_VAR:[P11,P12,P13,P21,P22,P23,P31,P32,P33]$
(C17) EV(RES:LINSOLVE(LIST_OF_EQS,LIST_OF_VAR),
GLOBALSOLVE:TRUE);

(D17)  [P11=2/7 P12=9/476 P13=-1/14 P21=9/476 P22=83/476
     P23=-11/476 P31=-1/14 P32=-11/476 P33=1/7]

(C18) EP:EV(P);
(C19) AK:MATRIX([-2+K1,0,-1+K1],[0,-3+K2,0],[-1+K1,-1+K2,-
    4+K1]);
(C20) X:MATRIX([X1],[X2],[X3]);
(C21) VDOT:EXPAND(TRANSPOSE(X).(TRANSPOSE(AK).EP+EP.AK).X);
```

B.2.3:

Program list to check the ares which do not satisfy $\dot{V} < 0$.

```
      PROGRAM TESTVSIL
      REAL*8 TPI,PIN,THETA,PHI,X1,X2,X3,VDOT,K1,K2
      OPEN(6,FILE='VDSIL.RES',STATUS='NEW')
      WRITE(*,*)'ENTER K1,K2'
      READ(*,*)K1,K2
      TPI=4.*ATAN(1.0)
      PIN=TPI/10.
      WRITE(6,100)
  100 FORMAT(2X,'THETA',3X,'PHI',3X,'X1',3X,'X2',3X,'X3',
     &       5X,'dV/dt',/)
      DO 1 I=0,10
      THETA=I*PIN
      DO 2 J=0,10
      PHI=J*PIN
      X1=SIN(PHI)*COS(THETA)
      X2=SIN(PHI)*SIN(THETA)
      X3=COS(PHI)
      VDOT=(3.*K1-7.)/7.*X1*X1-(25.*K2+2.*K1)/238.*X1*X2+
     &     4.*K1/7.*X1*X3+(36.*K2-119.)/119.*X2*X2+
     &     (57.*K2-2.*K1)/238.*X2*X3+(K1-7.)/7.*X3*X3
      WRITE(6,101)THETA,PHI,X1,X2,X3,VDOT
  101 FORMAT(2X,5(F8.5,2X),E13.7)
    2 CONTINUE
    1 CONTINUE
      STOP
      END
```

B.2.4:

Program list to obtain new robustness bounds with proposed techniques.

```
      PROGRAM THESIS2
C   THIS IS THE FIRST PROGRAM TO CHECK THE ASYMPTOTICAL
C   STABILITY OF THE SYSTEM THAT HAS PERTURBATIONS.
      REAL*8 EA(3,3),THE(100),PHI(100)
      REAL*8 X1(30,3300),X2(30,3300),X3(30,3300)
      REAL*8 VDOT
      REAL*8 T,TIN,TAU,H1(2),H2(2)
      INTEGER*4 IN,ICOUNT,ID,I1,I2,I4,IM,IM1,K1,K2
C   READ THE DATA
      WRITE(*,*) 'INPUT SAMPLING TIME T & ITERATION NUMBER
     &             IN.'
      READ(*,*) T,IN
      WRITE(*,*) 'INPUT NO. OF INITIAL STATES '
      READ(*,*)ID
      TIN=T/IN
      TAU=ABS(TIN)
      WRITE(*,*) 'INPUT PERTURBATION ¦K¦s.(K1 & K2)'
      READ(*,*)H1(1),H2(1)
      H1(2)=-H1(1)
      H2(2)=-H2(1)
C   COMPUTE THE EXP(A*LAMDA) MATRIX.
      EA(1,1)=EXP(-TIN*3.)*(SINH(SQRT(2.)*TIN)/SQRT(2.)+
     &        COSH(SQRT(2.)*TIN))
      EA(2,1)=0.
      EA(3,1)=-EXP(-3.*TIN)*SINH(SQRT(2.)*TIN)/SQRT(2.)
      EA(1,2)=EXP(-3.*TIN)*COSH(SQRT(2.)*TIN)/2.
     &        -EXP(-3.*TIN)/2.
      EA(2,2)=EXP(-3.*TIN)
      EA(3,2)=EXP(-TIN*3.)*(COSH(SQRT(2.)*TIN)/2.-
     &        SINH(SQRT(2.)*TIN)/SQRT(2.))-EXP(-3.*TIN)/2.
      EA(1,3)=-EXP(-3.*TIN)*SINH(SQRT(2.)*TIN)/SQRT(2.)
      EA(2,3)=0.
      EA(3,3)=EXP(-3.*TIN)*(COSH(SQRT(2.)*TIN)-
     &        SINH(SQRT(2.)*TIN)/SQRT(2.))
C   READ THE INITIAL STATES
      OPEN(UNIT = 5,FILE = 'INISIL1.DAT',STATUS = 'OLD')
      OPEN(UNIT = 6,FILE = 'VDSIL1.RES',STATUS = 'NEW')
C
C
      DO 1 I=1,ID
C      WRITE(*,*)' INPUT THE DATA THETA & PHI'
      READ(5,*)THE(I),PHI(I)
      X1(1,1)=SIN(PHI(I))*COS(THE(I))
      X2(1,1)=SIN(PHI(I))*SIN(THE(I))
      X3(1,1)=COS(PHI(I))
      WRITE(6,10)I,THE(I),PHI(I)
```

```
 10  FORMAT(/,1X,'INITIAL STATE NO.= ',I3,3X,'THETA=
    &        ',E15.7,3X,'PHI= ',E15.7)
     WRITE(6,100)X1(1,1),X2(1,1),X3(1,1)
 100 FORMAT(5X,'X1= ',E15.7,2X,'X2= ',E15.7,2X,'X3=
    &        ',E15.7,//)
     IM=1
     ICOUNT=0
     DO 2 I1=1,IN
     IM=2*(IM-ICOUNT)
     IM1=IM/2
     I4=0
     ICOUNT=0
     DO 3 I2=1,IM1
     DO 4 K1=1,2
C      DO 5 K2=1,2
     I4=I4+1
     X1(I1+1,I4)=EA(1,1)*(X1(I1,I2)*(1.+TAU*H1(K1))
    &                 +TAU*H1(K1)*X3(I1,I2))
    &      +EA(1,2)*(X2(I1,I2)*(1.+TAU*H2(K2)))
    &      +EA(1,3)*(X3(I1,I2)*(1.+H1(K1)*TAU)
    &                 +H1(K1)*TAU*X1(I1,I2)
    &                 +H2(K2)*TAU*X2(I1,I2))
     X2(I1+1,I4)=EA(2,1)*(X1(I1,I2)*(1.+TAU*H1(K1))
    &                 +TAU*H1(K1)*X3(I1,I2))
    &      +EA(2,2)*(X2(I1,I2)*(1.+TAU*H2(K2)))
    &      +EA(2,3)*(X3(I1,I2)*(1.+H1(K1)*TAU)
    &                 +H1(K1)*TAU*X1(I1,I2)
    &                 +H2(K2)*TAU*X2(I1,I2))
     X3(I1+1,I4)=EA(3,1)*(X1(I1,I2)*(1.+TAU*H1(K1))
    &                 +TAU*H1(K1)*X3(I1,I2))
    &      +EA(3,2)*(X2(I1,I2)*(1.+TAU*H2(K2)))
    &      +EA(3,3)*(X3(I1,I2)*(1.+H1(K1)*TAU)
    &                 +H1(K1)*TAU*X1(I1,I2)
    &                 +H2(K2)*TAU*X2(I1,I2))
     VDOT=DEV(X1(I1+1,I4),X2(I1+1,I4),X3(I1+1,I4),
    &                X1(1,1),X2(1,1),X3(1,1))
     WRITE(6,101)I1,X1(I1+1,I4),X2(I1+1,I4),X3(I1+1,I4)
    &             ,VDOT
 101 FORMAT(2X,'TIME STEP NO= ',I3,/,'  X1= ',E12.6,
    &        '  X2= ',E12.6,'  X3= ',E12.6,/,
    &        '  VDOT = ',E20.8 )
     WRITE(6,*)'  PREVIOUS TIME STEP STATES'
     WRITE(6,102)I1-1,X1(I1,I2),X2(I1,I2),X3(I1,I2)
 102 FORMAT(2X,'PREV. TIME STEP NO= ',I3,/,' X_1= ',E12.6,
    &        ' X_2= ',E12.6,' X_3= ',E12.6)
     IF(VDOT.LT.0.0)THEN
     I4=I4-1
     ICOUNT=ICOUNT+1
     ENDIF
C
C
C   5  CONTINUE
   4 CONTINUE
```

```
  3  CONTINUE
     WRITE(6,103)I4
103  FORMAT(3X,'NO. OF CALC. OF VDOT = ',I10)
     WRITE(6,*)
  2  CONTINUE
  1  CONTINUE
     STOP
     END
     FUNCTION DEV(X1,X2,X3,X10,X20,X30)
     REAL*8 DEV,X1,X2,X3,X10,X20,X30
     DEV=-(2./7.*X1*X1+9./238.*X1*X2-1./7.*X1*X3+
    &      83./476.*X2*X2-11./238.*X2*X3+1./7.*X3*X3)
    &    +(2./7.*X10*X10+9./238.*X10*X20-1./7.*X10*X30+
    &      83./238.*X20*X20-11./238.*X20*X30+1./7.*
    &      X30*X30)
     RETURN
     END
```

APPENDIX C

Numerical Procedures and Computer Programs for VTOL System

C.1 Procedures for solving the system equation:

The system equation is $\dot{x} = Ax + Bu$ and all the matrices are given as in section 4.4. The finite time is denoted as $T = n\lambda$ and the discrete time state vector is expressed as

$$x(n) = e^{A\lambda}\{x(n-1) + \lambda(\Delta A + \Delta Bk)\ x(n-1)\}\ .$$

The matrix ΔA has the elements Δa_{32} and Δa_{34} and the matrix ΔB has the element Δb_{21}.

C.2 Computer programs to obtain robustness bounds:

C.2.1:

Program to obtain the matrix form of $e^{A\lambda}$ with *MACSYMA*.

```
(C1) A:MATRIX([-0.0366,0.0271,0.0188,-0.4555],
              [0.0482,-1.01,0.0024,-4.0208],
              [0.1002,0.3681,0.707,1.42],[0,0,1,0]);
(C2) B:MATRIX([0.4422,0.1761],[3.5446,-7.5922],
              [-5.52,4.49],[0,0]);
(C3) K:MATRIX([-0.8143,-1.2207,0.266,0.826],
              [-0.2582,1.178,0.0623,-0.212]);
(C4) A1:A+B.K;
(C5) EA:IDENT(4)+A1*TIN+A1.A1*TIN^2/2+A1.A1.A1*TIN^3/6+A1.
       A1.A1.A1*TIN^4/24+A1.A1.A1.A1.A1*TIN^5/120;
```

C.2.2:

Program to solve $A^T P + PA = -I$ and to obtain $\dot{V}$ with *MACSYMA*.

```
(C1) A:MATRIX([-0.0366,0.0271,0.0188,-0.4555],
              [0.0482,-1.01,0.0024,-4.0208],
              [0.1002,0.3681,-0.707,1.42],[0,0,1,0]);
(C2) B:MATRIX([0.4422,0.1761],[3.5446,-7.5922],
              [-5.52,4.49],[0,0]);
(C3) K:MATRIX([-0.8143,-1.2207,0.266,0.826],
              [-0.2582,1.178,0.0623,-0.212]);
(C4) FLOAT:TRUE;
(C5) A1:A+B.K;
(C6) P:MATRIX([P11,P12,P13,P14],[P21,P22,P23,P24],
              [P31,P32,P33,P34],[P41,P42,P43,P44]);
(C7) LHS:P.A1+TRANSPOSE(A1).P;
(C8) EQ1:P12=P21;
(C9) EQ2:P13=P31;
(C10) EQ3:P14=P41;
(C11) EQ4:P23=P32;
(C12) EQ5:P24=P42;
(C13) EQ6:P34=P43;
(C14) EQ11:3.435818*P31-0.877862*P21+3.435818*P13
         -0.877862*P12-0.884305*P11=-1;
(C15) EQ12:3.435818*P32-0.877862*P22+12.39558*P13
         -14.72265*P12-0.305248*P11=0;
(C16) EQ13:3.435818*P33-0.877862*P23+P14-2.337745*P13
         +0.47227*P12+0.147396*P11=0;
(C17) EQ14:3.435818*P34-0.877862*P24-0.442152*P14
         -4.0914*P13+0.516586*P12-0.127576*P11=0;
(C17) EQ22:12.39558*P32+12.39558*P23-28.56101*P22
         -0.305248*P21-0.305248*P12=-1;
(C18) EQ23:12.39558*P33+P24-16.17609*P23+0.47227*P22
         +0.147396* P21-0.305248*P13=0;
(C19) EQ24:12.39558*P34-14.2805*P24-4.0914*P23+0.516586*P22
           -0.127576*P21-0.305248*P14=0;
(C20) EQ33:P43+P34-3.791186*P33+0.47227*P32+0.147396*P31
         +0.47227*P23+0.147396*P13=-1;
(C21) EQ34:P44-1.895593*P34-4.0914*P33+0.516586*P32
         -0.127576*P31+ 0.47227*P24+0.147396*P14=0;
(C22) EQ44:-4.0914*P43+0.516586*P42-0.127576*P41-4.0914*P34
           +0.516586* P24-0.127576*P14=-1;
(C23) LIST_OF_EQS:[EQ1,EQ2,EQ3,EQ4,EQ5,EQ6,EQ11,EQ12,
           EQ13,EQ14,EQ22,EQ23,EQ24,EQ33,EQ34,EQ44]$
(C24) LIST_OF_VARS:[P11,P12,P13,P14,P21,P22,P23,P24,
                  P31,P32,P33,P34,P41,P42,P43,P44]$
(C25) ELEM_P:LINSOLVE(LIST_OF_EQS,LIST_OF_VARS,
                  GLOBALSOLVE:TRUE);
(C26) P:BFLOAT(ELEM_P);
(C27) P:MATRIX([2.365108,0.19027,0.207453,-1.215635],
               [0.19027,0.348726,0.366103,0.079727],
```

```
               [0.207453,0.366103,0.460888,0.170179],
               [-1.215635,0.079727,0.170179,2.187138]);
(C28) X:MATRIX([X1],[X2],[X3],[X4]);
(C29) DELA:MATRIX([0,0,0,0],[0,0,0,0],
               [0,A32,0,A34],[0,0,0,0]);
(C30) DELB:MATRIX([0,0],[B21,0],[0,0],[0,0]);
(C31) DELA1:DELA+DELB . K;
(C32) A1_1:A1+DELA1;
(C33) VDOT:TRANSPOSE(X).(TRANSPOSE(A1_1).P+P.A1_1).X;

(D16) X3 ((0.366103 (0.826 B21 + 0.516586)

 + 0.079727 (0.266 B21 + 0.47227)

 + 0.460888 (A34 - 4.0914) + 1.658902) X4

 + (0.732206 (0.266 B21 + 0.47227) - 1.345799) X3

 + (0.348726 (0.266 B21 + 0.47227)

 + 0.366103 (- 1.2207 B21 - 14.2805)

 + 0.460888 (A32 + 12.39558) - 0.649535) X2

 + (0.19027 (0.266 B21 + 0.47227)

 + 0.366103 (- 0.8143 B21 - 0.877862)

 + 0.231528) X1) + X2 ((0.348726 (0.826 B21 + 0.516586)

 + 0.079727 (- 1.2207 B21 - 14.2805)

 + 0.366103 (A34 - 4.0914)

 + 0.170179 (A32 + 12.39558) + 0.346796) X4

 + (0.348726 (0.266 B21 + 0.47227)

 + 0.366103 (- 1.2207 B21 - 14.2805)

 + 0.460888 (A32 + 12.39558) - 0.649535) X3

 + (0.697452 (- 1.2207 B21 - 14.2805)

 + 0.732206 (A32 + 12.39558) - 0.116159) X2

 + (0.348726 (- 0.8143 B21 - 0.877862)

 + 0.19027 (- 1.2207 B21 - 14.2805)

 + 0.207453 (A32 + 12.39558) + 0.451791) X1)
```

+ X1 ((0.19027 (0.826 B21 + 0.516586)

+ 0.079727 (- 0.8143 B21 - 0.877862)

+ 0.207453 (A34 - 4.0914) + 0.820469) X4

+ (0.19027 (0.266 B21 + 0.47227)

+ 0.366103 (- 0.8143 B21 - 0.877862)

+ 0.231528) X3 + (0.348726 (- 0.8143 B21 - 0.877862)

+ 0.19027 (- 1.2207 B21 - 14.2805)

+ 0.207453 (A32 + 12.39558) + 0.451791) X2

+ (0.38054 (- 0.8143 B21 - 0.877862) - 0.665935) X1)

+ X4 ((0.159454 (0.826 B21 + 0.516586)

+ 0.340358 (A34 - 4.0914) + 0.310172) X4

+ (0.366103 (0.826 B21 + 0.516586)

+ 0.079727 (0.266 B21 + 0.47227)

+ 0.460888 (A34 - 4.0914) + 1.658902) X3

+ (0.348726 (0.826 B21 + 0.516586)

+ 0.079727 (- 1.2207 B21 - 14.2805)

+ 0.366103 (A34 - 4.0914)

+ 0.170179 (A32 + 12.39558) + 0.346796) X2

+ (0.19027 (0.826 B21 + 0.516586)

+ 0.079727 (- 0.8143 B21 - 0.877862)

+ 0.207453 (A34 - 4.0914) + 0.820469) X1)

C.2.3:

Program to check the areas which do not satisfy $\dot{V} < 0$.

```
PROGRAM TESTVAIR
REAL*8 TPI,PIN,THE1,THE2,THE3,X1,X2,X3,X4,VDOT
REAL*8 A32,A34,B21
OPEN(6,FILE='VDAIR.RES',STATUS='NEW')
```

```
      WRITE(*,*)'ENTER A32,A34,B21'
      READ(*,*)A32,A34,B21
      TPI=4.*ATAN(1.0)
      PIN=TPI/10.
      WRITE(6,100)
  100 FORMAT(2X,'THE1',3X,'THE2',3X,'THE3',3X,'X1',3X,
     & 'X2',3X,'X3',3X,'X4',5X,'dV/dt',/)
      DO 1 I=0,100
      THE1=I*PIN
      DO 2 J=0,100
      THE2=J*PIN
      DO 3 K=0,100
      THE3=K*PIN
      X1=COS(THE1)*COS(THE2)*COS(THE3)
      X2=SIN(THE1)*COS(THE2)*COS(THE3)
      X3=SIN(THE2)*COS(THE3)
      X4=SIN(THE3)
      VDOT=(0.131709*B21+0.340358*A34-1.0)*X4*X4
     &    +(0.647217*B21+0.921776*A34)*X3*X4
     &    +(0.38145*B21+0.732206*A34+0.340358*A32)*X2*X4
     &    +(0.184483*B21+0.414906*A34)*X1*X4
     &    +(0.194767*B21-1.0)*X3*X3
     &    +(-0.708282*B21+0.921776*A32)*X2*X3
     &    +(-0.495012*B21)*X1*X3
     &    +(-0.85138*B21+0.732206*A32-1.0)*X2*X2
     &    +(-1.03246*B21+0.414906*A32)*X1*X2
     &    +(-0.309874*B21-1.0)*X1*X1
      IF(VDOT.GT.0.0)THEN
      WRITE(6,101)THE1,THE2,THE3,X1,X2,X3,X4,VDOT
  101 FORMAT(1X,7(F7.4,2X),E12.5)
      ELSE
      ENDIF
    3 CONTINUE
    2 CONTINUE
    1 CONTINUE
      STOP
      END
```

C.2.4:

Program to obtain new robustness bounds with proposed techniques.

```
      PROGRAM THESIS3
C  THIS IS THE PROGRAM TO IMPROVE THE PARAMETERS
C  OF THE AIRCRAFT SYSTEM THAT COMES FROM SINGH PAPER.
      REAL*8 EA(4,4),THE1(100),THE2(100),THE3(100)
      REAL*8 X1(2,50000),X2(2,50000),X3(2,50000)
      REAL*8 X4(2,50000),VDOT,X1I,X2I,X3I,X4I
      REAL*8 T,TIN,TAU,A32(2),A34(2),B21(2)
C      REAL*8 AK11,AK12,AK13,AK14
```

```
      INTEGER*4 IN,ICOUNT,ID,IT,I1,I2,I4,IM,IM1,K1,K2,K3
C   READ THE DATA
      AK11=-0.8143
      AK12=-1.2207
      AK13=0.266
      AK14=0.826
      WRITE(*,*) 'INPUT SAMPLING TIME T & ITERATION NUMBER
     &            IN.'
      READ(*,*) T,IN
      WRITE(*,*) 'INPUT NO. OF INITIAL STATES '
      READ(*,*)ID
      TIN=T/IN
      TAU=ABS(TIN)
      WRITE(*,*) 'INPUT PURTURBATION A32,A34,B21'
      READ(*,*)A32(1),A34(1),B21(1)
      A32(2)=-A32(1)
      A34(2)=-A34(1)
      B21(2)=-B21(1)
C   COMPUTE THE EXP(A*LAMDA) MATRIX.
      EA(1,1)=4.255473*TIN**4-1.34924*TIN**3
     &        +0.484945*TIN**2-0.442152*TIN+1.
      EA(2,1)=132.3068*TIN**4-35.95812*TIN**3
     &        +7.273546*TIN**2-0.877862*TIN
      EA(3,1)=-125.5919*TIN**4+34.24123*TIN**3
     &        -9.45684*TIN**2+3.435818*TIN
      EA(4,1)=8.560309*TIN**4-3.15228*TIN**3
            +1.717909*TIN**2
      EA(1,2)=60.39193*TIN**4-16.36739*TIN**3
     &        +3.16056*TIN**2-0.305248*TIN
      EA(2,2)=1898.448*TIN**4-515.671*TIN**3
     &        +105.0274*TIN**2-14.2805*TIN+1.
      EA(3,2)=-1811.248*TIN**4+492.8055*TIN**3
     &        -100.7804*TIN**2+12.39558*TIN
      EA(4,2)=123.2013*TIN**4-33.59349*TIN**3
     &        +6.197792*TIN**2
      EA(1,3)=-2.029926*TIN**4+0.598695*TIN**3
     &        -0.308155*TIN**2+0.147396*TIN
      EA(2,3)=-64.89269*TIN**4+17.64945*TIN**3
     &        -3.626143*TIN**2+0.47227*TIN
      EA(3,3)=61.74137*TIN**4-15.89514*TIN**3
     &        +2.931178*TIN**2-1.895593*TIN+1.
      EA(4,3)=-3.973787*TIN**4+0.97706*TIN**3
     &        -0.947797*TIN**2+TIN
      EA(1,4)=-2.683134*TIN**4+0.943873*TIN**3
     &        -0.352168*TIN**2-0.127576*TIN
      EA(2,4)=-83.50298*TIN**4+22.72125*TIN**3
     &        -4.598678*TIN**2+0.516586*TIN
      EA(3,4)=78.81034*TIN**4-20.94931*TIN**3
     &        +6.860342*TIN**2-4.0914*TIN
      EA(4,4)=-5.237328*TIN**4+2.286781*TIN**3
     &        -2.0457*TIN**2+1.
C   READ THE INITIAL STATES
      OPEN(UNIT = 5,FILE = 'VDAIR1.DAT',STATUS = 'OLD')
```

```
      OPEN(UNIT = 6,FILE = 'VDAIR1.RES',STATUS = 'NEW')
C
C
      DO 1 I=1,ID
C       WRITE(*,*)' INPUT THE DATA THETA & PHI'
      READ(5,*)THE1(I),THE2(I),THE3(I)
      X1(1,1)=COS(THE1(I))*COS(THE2(I))*COS(THE3(I))
      X2(1,1)=SIN(THE1(I))*COS(THE2(I))*COS(THE3(I))
      X3(1,1)=SIN(THE2(I))*COS(THE3(I))
      X4(1,1)=SIN(THE3(I))
      X1I=X1(1,1)
      X2I=X2(1,1)
      X3I=X3(1,1)
      X4I=X4(1,1)
      WRITE(6,10)I,THE1(I),THE2(I),THE3(I)
   10 FORMAT(/,1X,'INITIAL STATE NO.= ',I3,1X,'THETA1=
     &     ',E10.5,1X,'THETA2= ',E10.5,1X,'THETA3= ',E10.5)
      WRITE(6,100)X1I,X2I,X3I,X4I
  100 FORMAT(2X,'X1= ',E12.5,2X,'X2= ',E12.5,2X,'X3=
     &        ',E12.5,2X,'X4= ',E12.5,//)
      IM=1
      ICOUNT=0
      IT=1
      DO 2 I1=1,IN
      IM=8*(IM-ICOUNT)
      IM1=IM/8
      I4=0
      ICOUNT=0
      DO 3 I2=1,IM1
      DO 4 K1=1,2
      DO 5 K2=1,2
      DO 6 K3=1,2
      I4=I4+1
      X1(IT+1,I4)=(EA(1,1)+EA(1,2)*TAU*B21(K3)*AK11)
     & *X1(IT,I2)+(EA(1,2)*(1.+TAU*B21(K3)*AK12)
     & +EA(1,3)*TAU*A32(K1))*X2(IT,I2)
     & +(EA(1,2)*TAU*B21(K3)*AK13+EA(1,3))*X3(IT,I2)
     & +(EA(1,2)*TAU*B21(K3)*AK14+EA(1,3)*TAU*A34(K2)
     & +EA(1,4))*X4(IT,I2)
       X2(IT+1,I4)=(EA(2,1)+EA(2,2)*TAU*B21(K3)*AK11)
     & *X1(IT,I2)+(EA(2,2)*(1.+TAU*B21(K3)*AK12)
     & +EA(2,3)*TAU*A32(K1))*X2(IT,I2)
     & +(EA(2,2)*TAU*B21(K3)*AK13+EA(2,3))*X3(IT,I2)
     & +(EA(2,2)*TAU*B21(K3)*AK14+EA(2,3)*TAU*A34(K2)
     & +EA(2,4))*X4(IT,I2)
       X3(IT+1,I4)=(EA(3,1)+EA(3,2)*TAU*B21(K3)*AK11)
     & *X1(IT,I2)+(EA(3,2)*(1.+TAU*B21(K3)*AK12)
     & +EA(3,3)*TAU*A32(K1))*X2(IT,I2)
     & +(EA(3,2)*TAU*B21(K3)*AK13+EA(3,3))*X3(IT,I2)
     & +(EA(3,2)*TAU*B21(K3)*AK14+EA(3,3)*TAU*A34(K2)
     & +EA(3,4))*X4(IT,I2)
```

```
      X4(IT+1,I4)=(EA(4,1)+EA(4,2)*TAU*B21(K3)*AK11)
     & *X1(IT,I2)+(EA(4,2)*(1.+TAU*B21(K3)*AK12)
     & +EA(4,3)*TAU*A32(K1))*X2(IT,I2)
     & +(EA(4,2)*TAU*B21(K3)*AK13 +EA(4,3))*X3(IT,I2)
     & +(EA(4,2)*TAU*B21(K3)*AK14+EA(4,3)*TAU*A34(K2)
     & +EA(4,4))*X4(IT,I2)
      VDOT=DEV(X1(IT+1,I4),X2(IT+1,I4),X3(IT+1,I4)
     &       ,X4(IT+1,I4),X1I,X2I,X3I,X4I)
c      WRITE(6,101)I1,X1(IT+1,I4),X2(IT+1,I4),X3(IT+1,I4)
c     &          ,X4(IT+1,I4),VDOT
c 101 FORMAT(2X,'TIME STEP NO= ',I3,/,'  X1= ',E12.6,
c     &        '  X2= ',E12.6,'  X3= ',E12.6,'  X4=
c     &        ',E12.6,/,
c     &        '  VDOT = ',E20.8 )
c      WRITE(6,*)'  PREVIOUS TIME STEP STATES'
c      WRITE(6,102)I1-1,X1(IT,I2),X2(IT,I2),
c     &              X3(IT,I2),X4(IT,I2)
c 102 FORMAT(2X,'PREV. TIME STEP NO= ',I3,/,' X_1=
c     &  ',E12.6,' X_2= ',E12.6,' X_3= ',E12.6,'
c     &  X_4= ',E12.6)
      IF(VDOT.LT.0.0)THEN
      I4=I4-1
      ICOUNT=ICOUNT+1
      ENDIF
C
C
    6 CONTINUE
    5 CONTINUE
    4 CONTINUE
    3 CONTINUE
      DO 11 IT4=1,I4
      X1(IT,IT4)=X1(IT+1,IT4)
      X2(IT,IT4)=X2(IT+1,IT4)
      X3(IT,IT4)=X3(IT+1,IT4)
      X4(IT,IT4)=X4(IT+1,IT4)
      WRITE(6,*)X1(IT,IT4),X2(IT,IT4),X3(IT,IT4),X4(IT,IT4)
   11 CONTINUE
      WRITE(6,103)I4
  103 FORMAT(3X,'NO. OF CALC OF VDOT(+) = ',I10)
C      WRITE(6,*)
    2 CONTINUE
    1 CONTINUE
      STOP
      END
      FUNCTION DEV(X1,X2,X3,X4,X10,X20,X30,X40)
      REAL*8 DEV,X1,X2,X3,X4,X10,X20,X30,X40
      DEV=-(2.369620*X1*X1+0.348741*X2*X2+0.460902*X3*X3
     &    +2.187332*X4*X4+2.*(0.190129*X1*X2+0.207394*X1*X3
     &    -1.216408*X1*X4+0.366117*X2*X3+0.079767*X2*X4
     &    +0.170208*X3*X4))
     &    +(2.369620*X10*X10+0.348741*X20*X20
     &    +0.460902*X30*X30+2.187332*X40*X40
     &    +2.*(0.190129*X10*X20+0.207394*X10*X30
```

```
&     -1.216408*X10*X40+0.366117*X20*X30
&     +0.079767*X20*X40+0.170208*X30*X40))
 RETURN
 END
```