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Title: NONLINEAR FREE BOUNDARY PROBLEMS ARISING FROM SOIL FREEZING IN A BOUNDED REGION

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Changes of density occur naturally in phase transition processes and introduce the bulk movement of material. It is customary in analyzing such problems to disregard this unpleasant complication and assume the densities to be equal. However, such changes are unavoidable and for one-dimensional problems the complexities introduced by this bulk movement can easily be circumvented. The key idea is posing the problem in local coordinates which are fixed in each phase. In this dissertation, we investigate freezing and thawing of soils in a bounded two-phase medium with phases whose material properties are not only distinct but their thermal dependence is also permitted.

Generally speaking, when a freezing process takes place in a cooled melt situated in contact with its solid phase, an interface boundary is formed whose movement (as the freezing proceeds) results in compression of both phases. Owing to the density differences, the density of the material will increase, movements will occur in each phase, pressures and thermal stresses will build up in the respective phases, and the freezing point will decrease. Mathematically, this results in three nonlinear free boundary problems for determining: (I) the location of the interface boundary along with the temperature distribution throughout the medium, (II) the pressure and velocity

distributions in the unfrozen phase, and (III) the displacement distribution and hence the thermal stresses in the frozen phase.

In fact, the temperature satisfies a nonlinear parabolic differential equation on each side of the interface while the temperature is continuous across the interface and equals the transition temperature, the condition of local thermodynamic equilibrium. To consider the problems from the most general point of view, mass forces are taken into account such that the pressure and velocity distributions satisfy a nonlinear couple of hyperbolic differential equations of the first order in the unfrozen phase and the pressure is related to the density through the equation of state. The displacement satisfies a nonlinear hyperbolic differential equation of the second order in the frozen phase which is related to the thermal stresses through the generalized Hooke's law. Across the interface, the pressure is equal to the negation of the normal thermal traction on the interface. Furthermore, the movement of the interface is related to the temperatures, the velocities and the material properties at the interface through conditions of dynamical compatibility for energy and mass transfer.

Based upon potential theoretic arguments, we prove existence, uniqueness and continuous dependence on the initial and boundary data of solutions to Problem I. Along with these results, explicit expressions for the densities, the specific heats and the thermal conductivities as functions of time and local coordinates in their respective phases, which fit our analysis, are also obtained. Correspondingly, the characteristic method is utilized to show existence and uniqueness of solutions to Problems II and III, and we demonstrated the continuous dependence of their solutions on the respective data. Moreover,

asymptotic estimates for the critical time of breakdown in their solutions are also obtained. Some remarks on discontinuities in general are finally discussed.

# NONLINEAR FREE BOUNDARY PROBLEMS ARISING FROM SOIL FREEZING IN A BOUNDED REGION

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Typed by Donna Moore for Fouad Abd El-Aal Mohamed

This Thesis is dedicated

to

my parents

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## CHAPTER ONE - THE PROBLEM

### 0. Introduction

In the classical two-phase Stefan problem the densities of the two phases are assumed, either explicitly or implicitily, to be equal. This allows one to neglect certain mechanical and thermo-mechanical questions, since a change in density during a phase transition inevitably gives rise to the onset of convection currents and motion in the liquid phase and of motion and thermal stresses in the solid phase, even when there are no external forces acting on the medium under consideration and the thermal expansion in both phases is negligible. Generally speaking, if one neglects convective heat transfer, motion and thermal stresses in a multi-phase medium one is actually considering the change in the phase state of the individual components of the medium to be a process dependent on the thermal conductivity only and independent of any mechanical and thermomechanical phenomena taking place in the course of the phase change. However, this leads to a serious incompatibility between the mathematical treatment of the process and its physical nature. On the other hand, taking the convective heat transfer into account forces one to consider the Stefan problem for the freezing process as belonging to the theory of convection accompanied by a change in the phase state. In other words, the heat transfer by convection must be considered. In one dimension one merely replaces $\partial T/\partial t$ by $\partial T/\partial t + v\partial T/\partial x$. To obtain a rigorous mathematical model, we have been forced to take into account the thermal expansion of both the liquid and solid phases, and consequently have to consider the temperature dependence of the physical properties (i.e., the density, the specific heat and the thermal conductivity).

Nevertheless, several authors have attempted an anaysis of one-dimensional, two-phase free boundary problems with phases of different densities. Some have assumed the relevant parameters to be temperature dependent, but others have not.

Chambré [7] studied a one-dimensional solidification problem for a two-phase system of unequal densities under the assumption that the liquid phase is a viscous incompressible fluid in a field of constant pressure. In addition, the solid phase is assumed to be immobile and of infinite thermal conductivity, so no equations were needed for this phase. Obviously, these assumptions are physically untenable. Taking the Prandtl number equal to one, Chambré derives a self-similar solution to the problem in processes with either a plane, a spherical or a cylindrical solidification front.

In [18], Horvay considered problems similar to those posed by Chambré. He pointed out that the assumption of constant pressure violates the continuity equation. He wanted to satisfy the continuity equation and to preserve the assumption of the liquid phase incompressibility in the above mentioned processes. This led him to the conclusion that the viscosity of the liquid phase does not affect the development of the process. Horvay constructed similarity solutions to the problem under investigation, assuming, like Chambré, that the freezing front advances with velocity propotional to $\sqrt{t}$. As Rubinstein [30] pointed out, Horvay did not prove the consistency of such an assumption.

Tao [33] considered a two-phase solidification problem in one space dimension with arbitrary initial and boundary conditions. The region considered was the half line $x > o$. The phase densities were constant but distinct. He made the change of variable $y(x,t) = (x + \bar{\rho}s(t))/(1+\bar{\rho})$, where $s(t)$ is the location of the phase boundary, $\bar{\rho} = (\rho_2 - \rho_1)/\rho_1$, and $\rho_i$ is the density of the phase i, and rewrote the heat transfer problem in the moving liquid using this coordinate. Tao's idea has the advantage that $y(s(t),t) = s(t)$, which simplifies the statement of the Stefan condition at the moving boundary, and since the liquid extends out to infinity, no difficulty arises at the other boundary on the right. Tao then analyzed the transformed problem using an infinite series of terms involving repeated integrals of the error functions.

In a very brief treatment indeed, Carslaw and Jaeger [6] gave what is essentially a similarity solution for a simplified two-phase Stefan problem with phases of distinct densities in an infinite medium. They made no change of variable.

Dankwerts [10] examined several two-phase problems with phases of distinct densities. These problems were posed on the entire real line, so no fixed boundary condition was imposed. The analysis is limited to what can be accomplished with similarity solutions in the local variables.

In [14], Gelder and Guy gave an analysis of practical problem involving the melting of glass. Their emphasis was on engineering applications rather than in resolving mathematical difficulties.

They suggested that a thorough mathematical formulation is needed to take the substantial change in density during the course of melting into account.

In a more recent paper, Wilson [36] defined suitable moving coordinates, Lagrangian coordinates, and used them to pose and solve a one-dimensional, multi-phase Stefan problem with phases of distinct constant densities. This problem is just a modest generalization, to the case of distinct phase densities, of a problem studied by Weiner [35]. The latter represents a model of solidification and subsequent cooling for a semi-infinite slab of a material undergoing several successive phase changes, and a constant temperature is maintained at the fixed boundary. Wilson's explicit solution, like that of Weiner, is essentially a similarity solution and is obtained in terms of error functions. In fact, this solution is very nearly the same solution given by Weiner but expressed in Lagrangian coordinates.

As a further study of similarity solutions, Andriankin [2] examined some solutions for a one-dimensional, two-phase melting problem in a medium with small thermal conductivity, but might be thermally dependent. The other properties were taken to be constant. The interval considered there was the half line $x > o$.

On the other hand, Cho and Sunderland [8] have extended the Neumann problem [6] to the case when the thermal conductivity varies linearly with temperature. However, all the other physical properties (including the density) are constant for each phase, but might be different for different phases. Unlike Anderiankin, they considered convection within the liquid phase due to the density change. Their (similarity) solution was obtained in terms of modified error functions, unlike Neumann's solution which was given in terms of ordinary error functions.

In all the above papers except for the last two, the thermal dependence of the physical properties has been neglected.

So far we have reviewed work relating to problems of the construction of exact solutions of various versions of the two-phase Stefan problem. However, comparison theorems for a one-dimensional, two-phase melting slab problem with variable thermal properties under arbitrary heat-flux conditions were proved by Boley [4]; they state the intuitively reasonable conclusion that higher temperatures and faster melting rates will always result from higher heat inputs. These theorems are useful for constructing upper and lower bounds to solutions of the problem corresponding to given heat inputs; and, in fact, form the basis of one of the available approximate methods for solution of this type of problem.

In concluding this survey, we mention the subsequent work of Boley [5]. He demonstrated the feasibility of Neumann's solution [6] pertaining to change of phase in a one-dimensional melting problem with temperature dependent properties. The region considered was $x > 0$, and the solution was expressed in terms of an auxiliary temperature distribution which was obtained by an iterative procedure.

It is worth mentioning that similarity solutions cannot be obtained in general. In fact, similarity solutions do not exist for finite domains, two phases present initially, nonuniform initial temperatures, boundary temperatures that are arbitrary functions of time, and prescribed heat-fluxes on the moving boundary. This has prompted considerable interest in further studies of a more realisitc model of a two-phase Stefan problem emphasizing two facts; namely, the mathematical description of the freezing process must take into account the convective heat transfer accompanied by a change in the phase state as well as the thermal expansion in both the solid and the liquid phases.

Therefore, in Section 1 of this chapter, we give a physical description of the problem. The mathematical formulation is presented in Section 2. In Section 3, we outline the present study, which in turn gives rise to three nonlinear free boundary problems.

Chapter 2 is made of five sections and gives a unified treatment of Problem I. In Section 4, we utilize Storm's method to transform Problem I into a version in which the governing partial differential equations become linear. This version is referred to here as Problem IV. A reduction of the resultant differential system to an equivalent system of integral equations is the aim of Section 5. In Section 6, we introduce a sequence of approximating solutions for the system of integral equations. In addition, we prove the convergence of this sequence as well as the existence of solutions to the system of integral equations for sufficiently small times. Uniqueness and stability of the solution to the system of integral equations are obtained in Section 7. In Section 8, we go back to Probelm I and deduce its well-posedness.

A treatment of Problem II constitutes the main goal of Chapter 3. Section 9 is devoted to the proof of the existence and uniqueness of its solutions. In Section 10, we derive an asymptotic estimate of the critical time, $t_c$, to breakdown of the solution of Problem II; in addition, we establish the continuous dependence on the initial and boundary data of this solution for all times up to $t_c$. In Section 11, remarks on discontinuities of solutions in general are made and extended to the case of heat-conducting media.

Chapter 4 is concerned with the displacement problem, Problem III. We begin with the reduction of the governing differential equation to a system of first order equations, and therefore introduce Problem V which is similar to Problem II. Thus, the proof of existence, uniqueness and continuous dependence on the data of the solution to Problem V is carried out in a manner analogous to that of Problem II. Furthermore, we show how to obtain related results such as thermal stresses and deformations in the frozen phase. See Sections 12 and 13.

## 1. Statement of the Physical Problem

Consider a tube insulated from the surroundings, part of which is filled with frozen soil and the other part with soil and water. The process will be considered as one-dimensional, or quasi-one-dimensional, i.e., the flow can be treated using a one-dimensional model even though the "real" flow is in fact three-dimensional, and the freezing is accomplished by the withdrawal of heat at a specific rate. The right hand side of the tube, Figure 1, is insulated and no mass is allowed to escape from the tube.

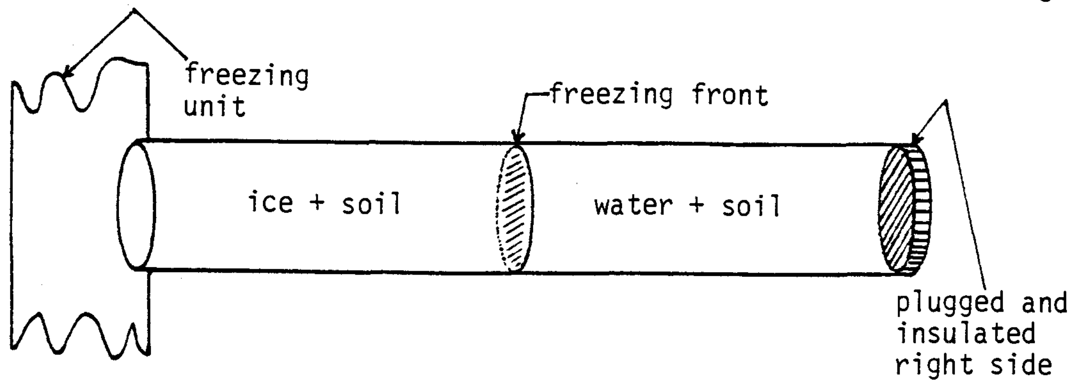


Figure 1. The Physical model.

It will be assumed that both the water-soil and ice-soil phases are compressible and the densities of the frozen and unfrozen phases are unequal. More precisely, the density of the solid phase is assumed to be strictly less than that of the liquid phase at anytime. As a consequence, a convective motion occurs in the water-soil phase having the characterisitcs of a source or sink flow because of the fact that a unit mass of the fluid occupies, on solidification, a volume differing from the volume originally occupied. It will be assumed that this convective motion obeys the Euler momentum equation for a non-viscous flow.

As the freezing proceeds, the unfrozen mass is compressed into a smaller and smaller volume. And because of the density difference, the density of the unfrozen phase will increase and pressures will build up. On the other hand, the unfrozen phase will resist the compression so that the frozen phase may also be compressed and, as a consequence, its density will increase and thermal stresses will build up. Also inertia forces will take place; and these forces will be related to the thermal stresses through Newton's law of motion. Since the frozen phase is less dense than the unfrozen one, the freezing point will decrease as the freezing proceeds. At some point in time, the pressures will become so large that either the plug on the right will be pushed back or the freezing process will come to a stop. Our concern is to model this process under certain additional assumptions.

In a recent paper [16], Guenther studied a rather simple version of this problem for a prescribed temperature boundary condition at the left side of the tube. In fact, he treated the problem as a one of thermal conductivity only, which evolved independently of mechanical and thermoelastic processes in the two-phase medium, which are generated by the change of the phase state itself. In addition, the physical properties were assumed constant, except for the liquid density which was assumed to be time dependent and linearly related to the pressure.

## 2. The Mathematical Formation of the Problem

Suppose that the length of the tube is b units. Let $x = s(t)$ denote the location of the solidification front at time t and suppose that $s(o) = a$, $o < a < b$. In addition, the thermal conductivity K, the specific heat C, and the density $\rho$ are assumed to be functions of the temperature T and therefore vary implicitly with both the coordinate x and time t. Due to the fact that the two-phase medium is being compressed, a certain amount of heat will be released in each phase. The rate q at which heat is released due to the compression per unit length appears in the energy equation as a source term.

Furthermore, since the interest is now in large pressure and stress changes, gravitational effects as well as body forces will be neglected. A partial derivative with respect to the time t at a given point on the x-axis is denoted by $\partial/\partial t$, and a total or substantial derivative describing the time change in any quantity following a moving particle in either, the liquid phase or the solid phase, is denoted by D/Dt. If v is the velocity of the particle, then

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} . \tag{2.1}$$

Let the suffixes 1 and 2 refer to quantities pertaining to the frozen and the unfrozen phases, respectively. Then under the above assumptions, the governing equations are derived in the following way.

## Equations of the Unfrozen Phase

The first equation, the continuity equation, describes the conservation of mass of the liquid,

$$\frac{D\rho_2}{Dt} + \rho_2 \frac{\partial v_2}{\partial x} = 0. \tag{2.2}$$

The second equation expresses Newton's law of motion

$$\rho_2 \frac{Dv_2}{Dt} = - \frac{\partial P}{\partial x} . \tag{2.3}$$

The third equation, the heat transfer equation, is an expression for the rate of the change of entropy of the liquid particle

$$\rho_2 T_2 \frac{DS_2}{Dt} = \frac{\partial}{\partial x} (K_2 \frac{\partial T_2}{\partial x}) + q_2 \, . \tag{2.4}$$

In these equations, $K_2$, $\rho_2$, and $C_2$ all depend on $T_2$, $q_2$ depends on x and t as well as $T_2$, and $P(x,t)$, $v_2(x,t)$, $S_2(x,t)$ and $T_2(x,t)$ are respectively the pressure, the velocity, the specific entropy and the temperature of the liquid phase at a point x and time t for $s(t) < x < b$, $t > 0$.

The fourth equation, the caloric equation of state, gives P in terms of $\rho_2$ or $\upsilon_2$, and $S_2$,

$$P = P(\rho_2, S_2) \qquad \text{or} \qquad P = P(\upsilon_2, S_2), \tag{2.5}$$

where $\upsilon_2 = \frac{1}{\rho_2}$ is the specific volume.

So far we have four equations, (2.2) to (2.5), in the four unknowns $\rho_2$, $v_2$, P, $T_2$ (or $S_2$).

It is known that for most liquids the pressure does not depend noticeably on the specific entropy. In other words, the influence of changes in entropy is negligibly small, so that P may be considered to be a function of density (or specific volume) alone. In this case the medium has separable energy, (for a definition, see Courant and Friedricks [9]), and the equation of state takes the form

$$P = P(\rho_2) \qquad \text{or } P = P(\upsilon_2) \tag{2.6}$$

The most important liquid with approximately separable energy is water. For water the caloric equation of state has some resemblence to that of a perfect gas:

$$P = A_s \rho_2^{\gamma} - B_s, \tag{2.7}$$

where the parameters $A_s$ and $B_s$ are, for almost all practical purposes, independent of the entropy, and $\gamma$ is a constant. We remark that the exponent $\gamma$ in (2.7) is not the ratio of specific heats but is denoted by this symbol for convenience because it plays the same role as the ratio of specific heats in a perfect gas undergoing isentropic processes; in particular, it will be possible to use the equations derived for the isentropic flow of a perfect gas merely by replacing P by $P + B_s$. Equation (2.7) is commonly called the Tait equation. For a brief discussion of the history of this equation, see Rowlinson [29] and Hirschfelder _et al_. [17].

Now the rate of heat release $q_2$ will be assumed proportional to the rate of change of $\rho_2$, i.e.,

$$q_2 = \text{const} \frac{D\rho_2}{Dt} = -a_2 \rho_2 \beta_2 \frac{DT_2}{Dt}, \tag{2.8}$$

where $a_2$ is a constant and $\beta_2 = -\rho_2^{-1}(\partial\rho_2/\partial T_2)p$ is the volume coefficient of thermal expansion of the liquid. Next, making use of the familiar thermodynamic relation

$$dS_2 = \frac{C_p dT_2}{T_2} + \frac{1}{\rho_2^2}\left(\frac{\partial\rho_2}{\partial T_2}\right)_p dP, \tag{2.9}$$

where $C_p$ is the specific heat at constant pressure, the heat transfer equation can be written

$$\rho_2 C_p \frac{DT_2}{Dt} = q_2 - \frac{T_2}{\rho_2}\left(\frac{\partial\rho_2}{\partial T_2}\right)_p \frac{DP}{Dt} + \frac{\partial}{\partial x}\left(K_2 \frac{\partial T_2}{\partial x}\right). \tag{2.10}$$

With the help of (2.8), Eq. (2.10) becomes

$$\rho_2 C_2^* \frac{DT_2}{Dt} = \frac{\partial}{\partial x} [K_2 \frac{\partial T_2}{\partial x}] \qquad (2.11)$$

where

$$C_2^* = C_p + a_2\beta_2 + c^2\beta_2^2 T_2 \qquad (2.12)$$

and $c^2 = dP/d\rho_2$ is the speed of sound in the liquid.

Equations of the Frozen Phase

The derivation of equations of this phase is based upon the theory of thermoelasticity. For convenience, the thermoelastic analysis will be presented in the following two steps (for simplicity, the suffix 1 pertaining to the frozen phase quantities will be dropped in Step (i)):

Step (i). Basic Equations of Thermoelasticity

Consider a perfectly elastic solid, initially unstrained, unstressed and everywhere at temperature $T_0$ on the absolute scale. Such a state free from strain and stress will be referred to as the reference state, and the temperature $T_0$ as the reference temperature. On departing from this reference state the solid in general acquires a displacement field $u_i (i = 1,2,3)$ and a non-uniform temperature distribution T. These changes give rise to a velocity field $v_i (i = 1,2,3)$ and stress and strain distributions described, respectively, by the tensors $e_{ij}$, $\sigma_{ij} (i,j = 1,2,3)$. The absolute temperature T, the vector components $u_i$, $v_i$ and the tensor elements $e_{ij}$, $\sigma_{ij}$ are functions of time t and position in the solid, as measured by rectangular cartesian coordinates $x_i (i = 1,2,3)$.

For a homogeneous isotropic material, we have, corresponding to (2.2) - (2.4), the equations

$$\frac{D\rho}{Dt} + \rho \frac{\partial v_j}{\partial x_j} = 0; \quad v_j = \frac{\partial u_j}{\partial t}, \tag{2.13}$$

$$\rho \frac{Dv_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{2.14}$$

$$\rho T \frac{DS}{Dt} = \frac{\partial}{\partial x_j} (K \frac{\partial T}{\partial x_j}) + q, \tag{2.15}$$

where $D/Dt = \partial/\partial t + v_j \partial/\partial x_j$, (see References [3] and [13]).

The displacement $u_i$ of each particle in the instantaneous state from its position in the reference state will be assumed to be small, so that the infinitesimal strain tensor is

$$e_{ij} = \frac{1}{2} (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), \tag{2.16}$$

On the other hand, the stress tensor is given by the generalized Hooke's law

$$\sigma_{ij} = (\lambda e - \varepsilon(T-T_o))\delta_{ij} + 2\mu e_{ij}; \quad e = e_{kk} \tag{2.17}$$

where

$$\varepsilon = (\lambda + \frac{2}{3}\mu)\beta, \quad \beta = (\frac{\partial e_{kk}}{\partial T})_\sigma . \tag{2.18}$$

Here $\lambda$ and $\mu$ are the (isothermal) Lamé elastic constants of the solid, and $\beta$ is its volume expansion coefficient. In this work, the coefficients $\lambda$ and $\mu$ are taken as material constants but $\beta$ is assumed to be a function of temperature. Relations (2.17) are the equations of state (or the constitutive equations) of the class of solids under investigation. If, in addition, the velocity $v_i$ is small, the substantial derivative D/Dt can be replaced by $\partial/\partial t$, and Eqs. (2.13) - (2.15) become

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v_j}{\partial x_j} = 0 , \tag{2.19}$$

$$\rho \frac{\partial v_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x_j} , \tag{2.20}$$

$$\rho T \frac{\partial s}{\partial t} = \frac{\partial}{\partial x_j} (K \frac{\partial T}{\partial x_j}) + q . \tag{2.21}$$

Equation (2.19) gives

$$\rho = \rho_o \exp(-e_{kk}) \simeq \rho_o(1 - e_{kk}) \tag{2.22}$$

where $\rho_o$ is the initial (uniform) density.

Now, introducing the Gibbs equation,

$$ds = \frac{\varepsilon}{\rho} de_{kk} + C_\upsilon \frac{dT}{T} , \tag{2.23}$$

where $C_\upsilon$ is the specific heat at constant deformation. We note here that $C_p$ and $C_\upsilon$ are related by the equation

$$C_p = C_\upsilon + \frac{\beta^2 T}{\kappa \rho} , \tag{2.24}$$

in which $\kappa = (\lambda + \frac{2}{3}\mu)^{-1}$ is the isothermal compressibility.

Next, combining in turn the two sets of equations (2.16), (2.17), (2.20) and (2.21), (2.23) and expressing the results in vector notation we arrive at the thermoelastic equations

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \mu \nabla^2 \vec{u} + (\lambda + \mu) \nabla(\mathrm{div}\ \vec{u}) - \varepsilon \nabla T \ . \tag{2.25}$$

$$\rho C_\upsilon \frac{\partial T}{\partial t} + \varepsilon T \frac{\partial}{\partial t} (\mathrm{div}\ \vec{u}) = \nabla \cdot (K \nabla T) + q \ . \tag{2.26}$$

Weiner [34] has proved that the solutions of Eqs. (2.25), (2.26) in a region free from body forces and heat sources are unique when the initial distributions of $T$, $\vec{u}$, $\vec{v}$ are given and $T$, $\vec{u}$ are specified on the boundary of this region. His proof extends to the other boundary conditions, i.e., the Neumann and Robin conditions.

Finally, the basic equations given above combine the theory of elasticity with heat conduction under transient conditions. Boundary value problems involving these equations are of considerable difficulty to solve. Fortunately, in most practical applications it is possible to omit the mechanical coupling term in the energy equation (2.26) and the inertia term in the equation of motion (2.25) without significant error. It is customary to refer to the thermoelastic theory based upon none of these simplifying assumptions as the coupled theory, upon the first only as the uncoupled theory, and upon both of them as the uncoupled quasi-static theory.

<u>Step (ii)</u>. <u>The Thermoelastic Equations Applied to the Frozen Phase</u>

The problem considered in this phase is that of a finite medium, initially at a uniform temperature, subject to a heat source intensity $q_1(x,t,T_1)$ per unit volume due to its compressibility:

$$q_1 = \text{const}\ \left(\frac{\partial e_{kk}}{\partial t}\right)_\sigma = -a_1\beta_1 \frac{\partial T_1}{\partial t}\ , \qquad (2.27)$$

where $a_1$ is a constant and

$$\beta_1 = \left(\frac{\partial e_{kk}}{\partial T_1}\right)_\sigma \qquad (2.28)$$

is the volume expansion coefficient of the frozen phase.

Suitable constraints are imposed so that the displacement components $u_x$, $u_y$, $u_z$ in the $(x,y,z)$ directions may be taken as

$$u_x \equiv u = u(x,t); \quad u_y = u_z = o\ . \qquad (2.29)$$

If we write $(x,y,z)$ instead of $(x_1,x_2,x_3)$, the equations to be solved follow directly from Step (i):

$$\rho_1 \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial T_1}{\partial x}\ , \qquad (2.30)$$

$$\rho_1 C_{1\upsilon} \frac{\partial T_1}{\partial t} + \varepsilon T_1 \frac{\partial^2 u}{\partial t \partial x} = \frac{\partial}{\partial x}\left(K_1 \frac{\partial T_1}{\partial x}\right) + q_1\ , \qquad (2.31)$$

$$\sigma_{xx} = (\lambda + 2\mu)e_{xx} - \varepsilon(T_1 - T_o)\ , \qquad (2.32)$$

$$\sigma_{yy} = \sigma_{zz} = \lambda e_{xx} - \varepsilon(T_1 - T_o)\ , \qquad (2.33)$$

$$\sigma_{xy} = \sigma_{yz} = \sigma_{xz} = e_{xy} = e_{yz} = e_{xz} = o; \quad e_{yy} = e_{zz} = o\ , \qquad (2.34)$$

$$e_{xx} = \frac{\partial u}{\partial x}\ , \qquad (2.35)$$

in which $K_1$, $\rho_1$ and $C_{1\upsilon}$ all depend on $T_1$; and $T_1(x,t)$ is the temperature of the frozen phase at a point x at time t for $o < x < s(t)$, $t > o$.

Again, equations (2.30) and (2.31), forming the equations of the coupled thermoelastic theory, are to be solved for u and $T_1$ simultantously so that the strain and stress components can be readily found from Eqs. (2.32) to (2.35). However, the following analysis is based upon the uncoupled theory of thermal stresses. Thus, on neglecting the mechanical coupling term in (2.31), the frozen phase problem degenerates into heat conduction and thermoelasticity as two separate problems. With this in mind and the use of (2.22) and (2.27), the above system of equations can be rewritten

$$u_{tt} = \omega^2(1 + u_x)u_{xx}; \quad \omega^2 = \frac{4\mu}{\rho_o} , \tag{2.36}$$

$$\rho_1 C_1^* \frac{\partial T_1}{\partial t} = \frac{\partial}{\partial x} [K_1 \frac{\partial T_1}{\partial x}] , \tag{2.37}$$

$$\sigma_{xx} = \rho_1(v_T^2 e - \overline{k}\theta) , \tag{2.38}$$

$$\sigma_{yy} = \sigma_{zz} = \frac{1}{1-\nu} (\nu\sigma_{xx} - \frac{1}{3}\beta_1 E\theta) , \tag{2.39}$$

$$e = \frac{\partial u}{\partial x} , \tag{2.40}$$

in which

$$\begin{gathered} e_{xx} = e(x,t), \ \theta = T_1 - T_o, \ C_1^* = C_{1\upsilon} + a_1\beta_1 , \\ \overline{k} = \frac{\varepsilon}{\rho_1} , \ v_T = \sqrt{\frac{\lambda+2\mu}{\rho_1}} , \end{gathered} \tag{2.41}$$

where $\nu$ is Poisson's ratio and $v_T$ is the isothermal velocity of the dilational waves. We emphasize here that the thermal stresses $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{zz}$ in the directions of x, y and z are given in terms of one component of the strain, the x-component e(x,t).

After the position of the interface s(t) is known, the displacement u(x,t) satisfies the mixed boundary value problem consisting of (2.36) under the conditions:

$$u(x,o) = o \tag{2.42}$$

$$u_t(x,o) = f(x) \tag{2.43}$$

$$u(o,t) = o \tag{2.44}$$

$$u_x(s(t),t) = \frac{1}{\rho_1 v_T^2}[\varepsilon\theta(P)-P] \tag{2.45}$$

Here $\theta(P) = T(P)-T_o$, and T(P), the transition temperature, is a prescribed function of the pressure P. We will assume that T(P) is a twice continuously differentiable and that is decreasing. The condition (2.45) states that at the boundary x = s(t) the stress should be minus the pressure, i.e.,

$$\sigma_{xx} = -P = (\lambda + 2\mu)e - \varepsilon(T(P)-T_o) \; . \tag{2.46}$$

However, Eq. (2.46) implies that

$$\varepsilon = \frac{1}{(T'(P))_e} \; . \tag{2.47}$$

Since $T_o$ = o, by (I.9) in the next section, the condition (2.45) can be rewritten

$$u_x = \frac{T(p)[(T'(p))_e]^{-1} - P}{\lambda + 2\mu} \quad \text{at} \quad x = s(t) \ . \tag{2.48}$$

## 3. Outline of the Present Study

Once s(t) is determined, we can solve the mixed problem defined above to obtain the displacement u(x,t). Then using (2.40), (2.38), (2.39), and (2.34) the strain and stress components can be readily specified. Also a knowledge of s(t) is sufficient to find a pair (P(x,t), $v_2(x,t)$) satisfying (2.2) and (2.3) together with (2.6), or its alternative form

$$\rho_2 = \rho_2(P) \tag{3.1}$$

(See Problem II). However to get s(t), one has to solve (2.11) and (2.37) under suitable conditions to be specified in the following:

The initial and boundary conditions are

$$v_2(x,o) = r(x); \quad T_1(x,o) = T_o; \quad T_2(x,o) = f_o(x) \ , \tag{3.2}$$

$$\lim_{x\to b} v_2(x,t) = o; \quad \frac{\partial T_2}{\partial x}(b,t) = o; \quad [K_1 \frac{\partial T_1}{\partial x}]_{x=o} = \ell(t) \tag{3.3}$$

At the solidification front, the conditions of local thermodynamic equilibrium and conditions of dynamical compatibility for heat and mass are necessarily satisfied. These are respectively (see, for example, Rubinstein [30])

$$T_1(s(t),t) = T_2(s(t),t) = T(P) \ , \tag{3.4}$$

$$[\rho_1 L + (\rho_1 C_1 - \rho_2 C_2)T(P)] \frac{ds}{dt} = K_1 \frac{\partial T_1}{\partial x} - K_2 \frac{\partial T_2}{\partial x} +$$

$$(\rho_1 C_1 v_1 - \rho_2 C_2 v_2)T(P) \quad \text{at} \quad x = s(t) \; , \tag{3.5}$$

$$(\rho_1 - \rho_2) \frac{ds}{dt} = \rho_1 v_1 - \rho_2 v_2 \quad \text{at} \quad x = s(t) \; , \tag{3.6}$$

where $v_1 = \partial u/\partial t$. Here L is the specific latent heat, in general a function of the pressure, (see, for example, Morse [27]; however, L will be treated here as a constant), and $C_i (i = 1,2)$ are the specific heats of the same type (using Eq. (2.24) to convert one to another; if necessary).

The discussion presented above now motivates the statements of the following three problems:

Problem I. Find a triple $(s(t), T_i(x,t), i = 1,2)$ satisfying the following contitions:

(I.1) $R_1 \frac{\partial T_1}{\partial t} = \frac{\partial}{\partial x} [K_1 \frac{\partial T_1}{\partial x}]$ in $\Omega_1 = \{(x,t): o < x < s(t), o < t < \tilde{t}\}$

with $s(o) = a$

(I.2) $R_2 \frac{DT_2}{DT} = \frac{\partial}{\partial x} [K_2 \frac{\partial T_2}{\partial x}]$ in $\Omega_2 = \{(x,t): s(t) < x < b, o < t < \tilde{t}\}$

(I.3) $T_1(x,o) = T_o \le o, \; o \le x \le a$

(I.4) $T_2(x,o) = f_o(x) \ge o, \; a \le x \le b$

(I.5) $[K_1 \frac{\partial T_1}{\partial x}]_{x=o} = \ell(t), \; o \le t \le \tilde{t}$

(I.6) $\frac{\partial T_2}{\partial x} (b,t) = o, \; o \le t \le \tilde{t}$

(I.7) $T_1(s(t),t) = T_2(s(t),t) = T(P), \; o \le t \le \tilde{t}$

$$(I.8)\quad [\rho_1 L + \rho_2(C_1 - C_2)T(P)]\,\frac{ds}{dt} = K_1\,\frac{\partial T_1}{\partial x} - K_2\,\frac{\partial T_2}{\partial x} + \rho_2 v_2(C_1 - C_2)T(P) \quad \text{at} \quad x = s(t)$$

and the compatibility conditions

$$(I.9)\quad \ell(o) = o,\ T_o = f(a) = [T(P)]_{t=o} = o,\ f'(b) = o,$$

where $R_i \equiv R(\rho_i, C_i) = \rho_i C_i^*$ $(i = 1,2)$, $D/Dt \equiv \partial/\partial t + v_2\partial/\partial x$ , and the prime represents the differentiation with respect to x. Note also that (I.8) is obtained by multiplying (3.6) by $C_1T(P)$ and subtracting the result from (3.5). In these equations, $\tilde{t}$ is a fixed value of t, a and b are given positive constants with $b > a$, and the functions $f_o$, $\ell$ and $T(P)$ are given functions of their respective arguments. The function $f_o(x)$ is defined and three times continuously differentiable for $o \le x \le \infty$. In addition the functions $R_i$ (and so $\rho_i$ and $C_i$), $K_i$, $v_i$ are sufficiently smooth functions of $T_i$ $(i = 1,2)$ over the range $-\infty < T_1 \le T(P) \le T_2 < \infty$; and $K_i$ and $\rho_i$ $(i = 1,2)$ are positive functions with $\rho_1 < \rho_2$.

Problem II. Find a pair $(P(x,t),\ v_2(x,t))$ satisfying the following conditions:

$$\left.\begin{aligned} &(II.1)\quad \frac{D\rho_2}{Dt} + \rho_2\,\frac{\partial v_2}{\partial x} = o \\ &(II.2)\quad \rho_2\,\frac{Dv_2}{Dt} + \frac{\partial P}{\partial x} = o \end{aligned}\right\} \begin{aligned} &\text{in } \Omega_2 = \{(x,t):\ s(t) < x < b,\ 0 < t < \tilde{t}\} \\ &\text{with } \rho_2 = \rho_2(P) \end{aligned}$$

$$\left.\begin{aligned} &(II.3)\quad P(x,o) = P_o(x) \\ &(II.4)\quad v_2(x,o) = r(x) \end{aligned}\right\} \text{for } a \le x \le b$$

$$\left.\begin{array}{ll}\text{(II.5)} & v_2(b,t) = o \\ \text{(II.6)} & P(s(t),t) = g(s(t))\end{array}\right\} \text{for } o \le t \le \tilde{t}$$

together with the compatibility conditions

(II.7) $P_o(a) = g(a)$ and $r(b) = o$.

Here s, $P_o$ and r are given functions of their respective arguments, which are assumed continuously differentiable. We further remark that the condition (II.6) expresses our assumption that the pressure of the phase change is a known function of the location of the phase boundary.

<u>Problem III</u>. Find a function $u(x,t)$ satisfying the conditions:

(III.1) $$\frac{\partial^2 u}{\partial t^2} = \omega^2\left(1 + \frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2} \quad \text{in } \Omega_1 = \{(x,t)\colon o < x < s(t),\ o < t < \tilde{t}\}$$

$$\left.\begin{array}{ll}\text{(III.2)} & u(x,o) = 0 \\ \text{(III.3)} & \frac{\partial u}{\partial t}(x,o) = f(x)\end{array}\right\} \text{for } o \le x \le a$$

$$\left.\begin{array}{ll}\text{(III.4)} & u(o,t) = o \\ \text{(III.5)} & \frac{\partial u}{\partial x}(s(t),t) = \chi(s(t))\end{array}\right\} \text{for } o \le t \le \tilde{t}$$

and the compatibility conditions

(III.6) $f(o) = o$ and $\chi(a) = o$.

Here s, f and $\chi$ are given functions of their respective arguments. The coefficient $\omega^2 = 4\mu/\rho_o$ is a constant. The function $\chi(s(t))$ is just the right hand side of (2.48) combined with (II.6).

# CHAPTER TWO - ANALYTICAL TREATMENT OF THE NONLINEAR PARABOLIC FREE BOUNDARY PROBLEM

We are concerned in this chapter with the study of Problem I stated in the previous section.

## 4. Linearization Procedure

In the following, we shall describe a method of linearization of the one-dimensional, nonlinear, nonsteady heat diffusion equation by successive transformation of the dependent and independent variables. It will be shown that when the thermal properties depend on the temperature in a certain way, the linearization is possible when the heat flux is prescribed at $x = o$. This linearization procedure was developed by Storm [32], and re-interpreted by Knight and Philip [23]. An extension to Storm's method when the time derivative of temperature is replaced by its substantial derivative is discussed below. Under a certain condition, the Kirchhoff transformation [22] will be proved to be a special case of the transformation used by Storm.

We are first concerned with the equations (I.1) and (I.5),

$$\frac{\partial}{\partial x}\left[K_1 \frac{\partial T_1}{\partial x}\right] = R_1 \frac{\partial T_1}{\partial t} , \tag{4.1}$$

$$\left[K_1 \frac{\partial T_1}{\partial x}\right]_{x=o} = \ell(t) , \tag{4.2}$$

where $K_1 = K_1(T_1)$, $R_1 = R_1(T_1)$, and $T_1 = T_1(x,t)$. The condition (4.2) expresses the heat outflux from the two-phase medium through the boundary surface $x = o$.

Introduce a new dependent variable $Q_1(x,t)$ by the equation

$$Q_1 = \int^{T_1} [K_1(T)R_1(T)]^{1/2} dT \; . \tag{4.3}$$

Then Eq. (4.1) becomes

$$\alpha_1^{1/2} \frac{\partial}{\partial x} (\alpha_1^{1/2} \frac{\partial Q_1}{\partial x}) = \frac{\partial Q_1}{\partial t} \; , \tag{4.4}$$

where $\alpha_1 \equiv K_1/R_1$ is a function of $T_1$.

Now let

$$Q_1(x,t) = Q_1^*[X(x,t),t] \; , \tag{4.5}$$

where $X(x,t)$ is a new independent variable defined by

$$X = \int_0^x \alpha_1^{-1/2} dx \; . \tag{4.6}$$

Utilizing Eqs. (4.5) and (4.6), the variables $x$ and $t$ are transformed to the variables $X$ and $t$ as follows:

$$\alpha_1^{1/2} \frac{\partial Q_1}{\partial x} = \alpha_1^{1/2} \frac{\partial Q_1^*}{\partial X} \frac{\partial X}{\partial x} = \frac{\partial Q_1^*}{\partial X}$$

and the left-hand side of Eq. (4.4) can be written as

$$\alpha_1^{1/2} \frac{\partial}{\partial x} (\alpha_1^{1/2} \frac{\partial Q_1}{\partial x}) = \alpha_1^{1/2} \frac{\partial}{\partial X} (\frac{\partial Q_1^*}{\partial X}) \frac{\partial X}{\partial x} = \frac{\partial^2 Q_1^*}{\partial X^2} \; . \tag{4.7}$$

Hence, Eq. (4.4) yields

$$\frac{\partial^2 Q_1^*}{\partial X^2} = \frac{\partial Q_1}{\partial t} \; . \tag{4.8}$$

The right-hand side of Eq. (4.8) is now expressed in terms of $Q_1^*$, as

$$\frac{\partial Q_1}{\partial t} = \frac{\partial Q_1^*}{\partial X} \left[\int_0^X \frac{\partial \alpha_1^{-1/2}}{\partial t} dx\right] + \frac{\partial Q_1^*}{\partial t}, \qquad (4.9)$$

where

$$\frac{\partial \alpha_1^{-1/2}}{\partial t} = \left(\frac{d\alpha_1^{-1/2}}{dQ_1}\right) \frac{\partial Q_1}{\partial t} = \frac{d\alpha_1^{-1/2}}{dQ_1} \frac{\partial^2 Q_1^*}{\partial X^2}, \qquad (4.10)$$

since $\partial Q_1/\partial t$ is given by Eq. (4.8). Inserting Eq. (4.10) into (4.9) and using the fact $dx = \alpha_1^{1/2} dX$, we get

$$\frac{\partial Q_1}{\partial t} = \frac{\partial Q_1^*}{\partial X} \left\{\int_0^X \left[\frac{d}{dQ_1} (\ln \alpha_1^{-1/2})\right] \frac{\partial^2 Q_1^*}{\partial X^2} dX\right\} + \frac{\partial Q_1^*}{\partial t}. \qquad (4.11)$$

Combining Eqs. (4.8) and (4.11) gives

$$\frac{\partial^2 Q_1^*}{\partial X^2} = \frac{\partial Q_1^*}{\partial t} + \frac{\partial Q_1^*}{\partial X} \left\{\int_0^X \left[\frac{d}{dQ_1} (\ln \alpha_1^{-1/2})\right] \frac{\partial^2 Q_1^*}{\partial X^2} dX\right\}. \qquad (4.12)$$

Now the integral in Eq. (4.12) can be evaluated in terms of $\partial Q_1^*/\partial X$ if and only if

$$\frac{d}{dQ_1} (\ln \alpha_1^{-1/2}) = (K_1 R_1)^{-1/2} \frac{d}{dT_1} (\ln \alpha_1^{-1/2}) \equiv A = \text{constant}. \qquad (4.13)$$

Under this condition Eq. (4.12) reduces to

$$\frac{\partial^2 Q_1}{\partial X^2} = \frac{\partial Q_1^*}{\partial t} + A \left(\frac{\partial Q_1^*}{\partial X}\right)^2 - A \frac{\partial Q_1^*}{\partial X} \left[\frac{\partial Q_1^*}{\partial X}\right]_{X=o}. \qquad (4.14)$$

The term $[\partial Q_1^*/\partial X]_{X=o}$ can be evaluated from the prescribed heat flux boundary condition (4.3) as

$$\ell(t) = K_1 \frac{\partial T_1}{\partial Q_1} \left[\frac{\partial Q_1^*}{\partial X}\right]_{X=o} \frac{\partial X}{\partial x} = \left[\frac{\partial Q_1^*}{\partial X}\right]_{X=o}. \qquad (4.15)$$

Therefore, Eq. (4.14) becomes

$$\frac{\partial^2 Q_1^*}{\partial X^2} = \frac{\partial Q_1^*}{\partial t} + A\left(\frac{\partial Q_1^*}{\partial X}\right)^2 - A\ell(t)\frac{\partial Q_1^*}{\partial X} \tag{4.16}$$

This equation is still nonlinear. However, introducing a new dependent variable $\Phi(X,t)$ by

$$Q_1^* = -\frac{1}{A}\ln\Phi , \tag{4.17}$$

reduces Eq. (4.16) to a linear partial differential equation for $\Phi$:

$$\frac{\partial^2\Phi}{\partial X^2} = \frac{\partial\Phi}{\partial t} - A\ell\,\frac{\partial\Phi}{\partial X} . \tag{4.18}$$

In particular, if $A = o$ in Eq. (4.13), $\alpha_1$ = constant, say $\alpha_*$. For such a case, the transformation (4.3) reduces to the Kirchhoff transformation [22],

$$Q_1 = \alpha_*^{-1/2}\int^{T_1} K_1(T)dT ,$$

and the equation (4.16), using (4.7) and (4.11), to

$$\alpha_*\,\frac{\partial^2 Q_1}{\partial x^2} = \frac{\partial Q_1}{\partial t} ,$$

which is linear.

It is easily seen that the most general forms for $K_1$ and $R_1$ which satisfy (4.13) are

$$K_1(T_1) = G_1(T_1)\exp[-A\int^{T_1} G_1(T)dT] \tag{4.19}$$

and

$$R_1(T_1) = G_1(T_1)\exp[A\int^{T_1} G_1(T)dT] , \tag{4.20}$$

or, by the definition $R_1 = \rho_1 C_1^*$ ,

$$\rho_1(T_1)C_1^*(T_1) = G_1(T_1)\exp[A\int^{T_1} G_1(T)dT] , \tag{4.21}$$

where $G_1(T_1)$ is an arbitrary function of $T_1$. On the substitution from (4.21) in the relation $C_1^* = C_{1\upsilon} + a_1\beta_1$, we obtain

$$\rho_1 C_{1\upsilon} - a_1 \frac{d\rho_1}{dT_1} = G_1(T_1)\exp[A\int^{T_1} G_1(T)dT] , \tag{4.22}$$

since $\beta_1 = -\rho_1^{-1}(d\rho_1/dT_1)_\sigma$ which is an alternative form to that of (2.18); this is evident from (2.19).

We note here that $G_1(T_1)$, which depends upon the thermal properties of the medium under consideration, must be known a priori. For simple metals, Storm [32] has shown that $G_1(T_1)$ is essentially constant, its variation with temperature being much less than that of either $K_1$ or $R_1$ considered separately. In fact, this discovery was the motivation for an investigation of the relations between the thermal parameters of simple metals on the bases of the theory of solids and available experimental data. Therefore, solving Eq. (4.22) gives the functional $\rho_1$ in terms of $C_{1\upsilon}$ and $T_1$, from which $C_{1\upsilon}$ is obtained in terms of $\rho_1$ and $T_1$, and the functions $\rho_1$ and $T_1$ will be obtained from the solutions of Problems I and II. Indeed, this is the value of $C_{1\upsilon}$ which is applicable to our analysis.

We remark that Eqs. (4.19) and (4.20) can be simplified if the product

$$K_1 R_1 = \text{constant} , \tag{4.23}$$

in which case $G_1(T_1) \equiv D$ = constant, and the variation of $K_1$ and $R_1$ with temperature is given by

$$K_1(T_1) = D\exp[-ADT_1] , \qquad (4.24)$$

$$R_1(T_1) = D\exp[ADT_1] . \qquad (4.25)$$

If the exponentials in these expressions can be linearized, Eqs. (4.24) and (4.25) reduce to

$$K_1(T_1) = D(1 - ADT_1) , \qquad (4.26)$$

$$R_1(T_1) = D(1 + ADT_1) . \qquad (4.27)$$

We now summarize the results of the above analysis: The nonlinear heat-conduction equation (4.1) subject to the prescribed heat-flux boundary condition (4.2) is transformed into the linear equation (4.18) on the assumption that the physical properties $K_1$ or $R_1$ depend on the temperature in the form specified by equations (4.19) and (4.20), (4.24) and (4.25), or (4.26) and (4.27).

Storm [32] applied this method to the solution of the problem of a nonstationary temperature distribution in a semi-infinite medium subject to an initial uniform temperature condition and a constant flux boundary condition at the surface x = o.

We note also that if we define a new dependent variable $V(X,t)$ by means of

$$\Phi(X,t) = V(X,t)\exp[-\frac{A\ell}{2}X + (\frac{A\ell}{2})^2 t] , \qquad (4.28)$$

equation (4.18) transforms to the more convenient form

$$\frac{\partial^2 V}{\partial X^2} = \frac{\partial V}{\partial t} . \tag{4.29}$$

Equations (4.17) and (4.28) when combined give

$$V(X,t) = \exp[-AQ_1^*(X,t) + \frac{A\ell}{2} X + (\frac{A\ell}{2})^2 t] . \tag{4.30}$$

We next consider the equations (I.2) and (I.6),

$$\frac{\partial}{\partial x} [K_2 \frac{\partial T_2}{\partial x}] = R_2 \frac{DT_2}{Dt} , \tag{4.31}$$

$$[\frac{\partial T_2}{\partial x}]_{x=b} = o , \tag{4.32}$$

where $K_2 = K_2(T_2)$, $R_2 = R_2(T_2)$, $T_2 = T_2(x,t)$, and $D/Dt = \partial/\partial t + v_2 \partial/\partial x$. In contrast to (4.2), the condition (4.32) expresses the fact that the boundary surface x = b is insulated.

Since the same method applied above can be used to linearize (4.31) subject to (4.32), only the salient steps will be given.

We define the new dependent variable $Q_2(x,t)$ by

$$Q_2 = \int^{T_2} [K_2(T) R_2(T)]^{1/2} dT , \tag{4.33}$$

so that Eq. (4.31) becomes

$$\alpha_2^{1/2} \frac{\partial}{\partial x} (\alpha_2^{1/2} \frac{\partial Q_2}{\partial x}) = \frac{DQ_2}{Dt} , \tag{4.34}$$

where $\alpha_2 \equiv K_2/R_2$ is a function of $T_2$.

A new independent variable $Y(x,t)$ is defined by

$$Y = \int_x^b \alpha_2^{-1/2} dx \tag{4.35}$$

and the corresponding dependent variable is dented by

$$Q_2(x,t) = Q_2^*[Y(x,t),t] \ . \tag{4.36}$$

Then Eq. (4.34) has the form

$$\frac{\partial^2 Q_2^*}{\partial Y^2} = \frac{DQ_2}{Dt} \tag{4.37}$$

The right-hand side of this equation can be expressed in terms of $Q_2^*$ as

$$\frac{DQ_2}{Dt} = \frac{\partial Q_2^*}{\partial Y}\frac{DY}{Dt} + \frac{\partial Q_2^*}{\partial t} \ , \tag{4.38}$$

where

$$\frac{DY}{Dt} = \int_x^b \frac{\partial \alpha_2^{-1/2}}{\partial t}\, dx - v_2\alpha_2^{-1/2} \ . \tag{4.39}$$

Since $v_2(b,t) = o$ by condition (II.5), then

$$-v_2\alpha_2^{-1/2} = \int_x^b v_2 \frac{\partial \alpha_2^{-1/2}}{\partial x}\, dx + \int_x^b \alpha_2^{-1/2} \frac{\partial v_2}{\partial x}\, dx \tag{4.40}$$

and, by equation (II.1),

$$\frac{\partial v_2}{\partial x} = - \frac{1}{\rho_2}\frac{d\rho_2}{dT_2}\frac{DT_2}{Dt} = - \frac{1}{\rho_2}\frac{d\rho_2}{dQ_2}\frac{DQ_2}{Dt} \ . \tag{4.41}$$

Combining (4.40), (4.41), and (4.39) yields

$$\frac{DY}{Dt} = \int_x^b \left[\frac{D\alpha_2^{-1/2}}{Dt} - \frac{\alpha_2^{-1/2}}{\rho_2}\frac{d\rho_2}{dQ_2}\frac{DQ_2}{Dt}\right]dx$$

$$= \int_x^b \left[\frac{d\alpha_2^{-1/2}}{dQ_2} - \frac{\alpha_2^{-1/2}}{\rho_2}\frac{d\rho_2}{dQ_2}\right]\frac{DQ_2}{Dt}\,dx\ . \tag{4.42}$$

Putting (4.42) in (4.38) yields, after using (4.37) and the fact that $dx = -\alpha_2^{1/2}\,dY$, the equation

$$\frac{\partial^2 Q_2^*}{\partial Y^2} = \frac{\partial Q_2^*}{\partial t} + \frac{\partial Q_2^*}{\partial Y}\left\{\int_o^Y \left[\frac{d}{dQ_2}\ln\left(\frac{\alpha_2^{-1/2}}{\rho_2}\right)\right]\frac{\partial^2 Q_2^*}{\partial Y^2}\,dY\right\}\ . \tag{4.43}$$

Now the integral in Eq. (4.43) can be evaluated in terms of $\partial Q_2^*/\partial Y$ if and only if

$$\frac{d}{dQ_2}\ln\left(\frac{\alpha_2^{-1/2}}{\rho_2}\right) = (K_2R_2)^{-1/2}\frac{d}{dT_2}\ln\left(\frac{\alpha_2^{-1/2}}{\rho_2}\right) \equiv B = \text{constant} \tag{4.44}$$

under which condition Eq. (4.43) reduces to

$$\frac{\partial^2 Q_2^*}{\partial Y^2} = \frac{\partial Q_2^*}{\partial t} + B\left(\frac{\partial Q_2^*}{\partial Y}\right)^2\ . \tag{4.45}$$

In Eq. (4.45), we employed the condition (4.32), which in terms of $Q_2^*$ has the form

$$\left[\frac{\partial Q_2^*}{\partial Y}\right]_{Y=o} = o\ . \tag{4.46}$$

Finally, a new dependent variable $W(Y,t)$ is introduced by means of the equation

$$Q_2^*(Y,t) = -\frac{1}{B}\ln W(Y,t) \tag{4.47}$$

so that Eq. (4.45) transforms to

$$\frac{\partial^2 W}{\partial Y^2} = \frac{\partial W}{\partial t}\ , \tag{4.48}$$

which is a linear parabolic differential equation for W.

In this case, the most general forms for $K_2$ and $R_2$ which satisfy (4.44) are

$$K_2(T_2) = G_2(T_2)\exp[-B\int^{T_2} G_2(T)dT] \qquad (4.49)$$

and

$$R_2(T_2) = \rho_2^2(T_2)G_2(T_2)\exp[B\int^{T_2} G_2(T)dT] \ , \qquad (4.50)$$

or, by the definition $R_2 = \rho_2 C_2^*$ ,

$$\frac{C_2^*(T_2)}{\rho_2(T_2)} = G_2(T_2)\exp[B\int^{T_2} G_2(T)dT] \ , \qquad (4.51)$$

where $G_2(T_2)$ is an arbitrary function of $T_2$. Again, the function $G_2(T_2)$ must be specified a priori. Furthermore, the value of $C_p$, which is being used in the above analysis, is obtained by combining (2.12) and (4.51) and the definition of $\beta_2$. The result is

$$T_2(\frac{d\rho_2}{dT_2})^2 - a_2\rho_2 \frac{d\rho_2}{dT_2} + \rho_2^2 C_p = \rho_2^3 G_2(T_2)\exp[B\int^{T_2} G_2(T)dT] \qquad (4.52)$$

with $\rho_2$ obtained from the solution of Problem II.

Summarizing the above results, we see that equation (4.1) subject to the condition (4.2) is transformed to

$$\frac{\partial^2 V}{\partial X^2} = \frac{\partial V}{\partial t} \qquad (4.29)$$

subject to

$$[\frac{\partial V/\partial X}{V}]_{X=o} + \frac{A\ell(t)}{2} = o \ , \qquad (4.53)$$

or, in an alternative form,

$$\left[\frac{\partial V}{\partial X} - \Lambda(t,V)\right]_{X=o} = o \tag{4.53*}$$

with

$$V(X,t) = \exp\left[-AQ_1^*(X,t) + \frac{A\ell(t)}{2} X + \left(\frac{A\ell(t)}{2}\right)^2 t\right] \tag{4.30}$$

and

$$\Lambda(t,V(X,t)) = -\frac{A\ell(t)}{2} V(X,t) \ . \tag{4.54}$$

Similarly, Eq. (4.31) subject to (4.32) is transformed to

$$\frac{\partial^2 W}{\partial Y^2} = \frac{\partial W}{\partial t} \tag{4.48}$$

subject to

$$\left[\frac{\partial W}{\partial Y}\right]_{Y=o} = o \tag{4.55}$$

with

$$W(Y,t) = \exp\left[-BQ_2^*(Y,t)\right] \ . \tag{4.56}$$

The conditions (4.53) and (4.55) are just Eqs. (4.15) and (4.46) written in terms of V and W, respectively

We close this section by pointing out that the conditions (I.3) and (I.7),

$$T_1(x,o) = T_o = o, \quad T_1(s(t),t) = T(P)$$

can be respectively expressed in terms of V given by (4.30) as

$$V(X_o,o) \equiv \phi(X_o), \quad V(X_s,t) \equiv V_m(X_s) \tag{4.57}$$

with $X_o \equiv X_o(x) = X(x,o)$ and $X_s \equiv X_s(t) = X(s(t),t)$. Similarly, the conditions (I.4) and (I.7),

$$T_2(x,o) = f_o(x), \quad T_2(s(t),t) = T(P)$$

are respectively expressed in terms of W given by (4.56) as

$$W(Y_o,o) \equiv \psi(Y_o), \; W(Y_s,t) \equiv W_m(Y_s) \tag{4.58}$$

with $Y_o \equiv Y_o(x) = Y(x,o)$ and $Y_s \equiv Y_s(t) = Y(s(t),t)$.

As previously mentioned, $C_i$ can be expressed as functions of $T_i$ and $\rho_i$ $(i = 1,2)$. On combining (II.6) and (3.1), we obtain $\rho_2 = \rho_2(g(s(t)))$ and hence $C_2$ is obtained as a function of $s(t)$ on the phase boundary. Also from solutions to Problem II, we get $v_2 = v_2(x,t|s)$ or $v_2 = v_2(t|s)$ at $x = s(t)$. On the other hand, we combine (III.5) and (2.22) to get $\rho_1 = \rho_o(1-\chi(s(t)))$ on s. Therefore, we have $C_1$ as a function of $s(t)$ on the phase boundary. Finally, the quantity L, the specific latent heat, was assumed to be constant. Thus, the quantities $[\rho_1 L + \rho_2(C_1 - C_2)T(P)]$ and $[\rho_2 v_2(C_1 - C_2)T(P)]$ appearing in (I.8) are functions of t and s(t).

The quantity $K_1(\partial T_1/\partial x) - K_2(\partial T_2/\partial x)$ is transformed under (4.3), (4.6), and (4.33), (4.35) to

$$\left[\frac{\partial Q_1^*}{\partial X}\right]_{X=X_s} + \left[\frac{\partial Q_2^*}{\partial Y}\right]_{Y=Y_s} .$$

By virtue of (4.30) and (4.56), this can be expressed as

$$\frac{\ell(t)}{2} - \frac{1}{A}\left[\frac{\partial V/\partial X}{V}\right]_{X=X_s} - \frac{1}{B}\left[\frac{\partial W/\partial Y}{W}\right]_{Y=Y_s}$$

Therefore, the condition (I.8) may be written in the form

$$k_1(s(t),t)\,\frac{ds}{dt} = k_2(s(t),t) + \frac{\ell(t)}{2} - \frac{1}{A}\left[\frac{\partial V/\partial X}{V}\right]_{X=X_s} - \frac{1}{B}\left[\frac{\partial W/\partial Y}{W}\right]_{Y=Y_s}, \tag{4.59}$$

where

$$k_1(s(t),t) \equiv \rho_1 L + \rho_2(C_1 - C_2)T(P) \tag{4.60}$$

and

$$k_2(s(t),t) \equiv \rho_2 v_2(C_1 - C_2)T(P)\ , \tag{4.61}$$

or, in a more compact form,

$$\frac{ds}{dt} = Z(t,x,V,W,\frac{\partial V}{\partial X},\frac{\partial W}{\partial Y}) \text{ for } x = s(t),\ X = X_s(t),\ Y = Y_s(t);\ t > o, \tag{4.62}$$

where the meaning of the funcitonal Z is clear from equation (4.59).

## 5. Reduction to a System of Integral Equations

By collecting the results of the previous section together, we are led to a version of Problem I, where the governing partial differential equations are linear. This problem is referred to as

Problem IV. Find functions $V(X,t)$, $W(Y,t)$ and $s(t)$ such that a triple $(V,W,s)$ satisfy:

The Solid Phase Equations

(IV.1) $\dfrac{\partial^2 V}{\partial X^2} = \dfrac{\partial V}{\partial t}$ , $(X,t)\varepsilon\Omega_1^*$

(IV.2) $\dfrac{\partial V}{\partial X} + \dfrac{A\ell}{2} V = o$ for $X = o,\ t > o$

(IV.3) $V = \phi(X)$ for $t = o,\ o \leq X \leq X_s(o)$

(IV.4) $V = V_m(X)$ for $X = X_s(t),\ t > o$

The Liquid Phase Equations

(IV.1)' $\dfrac{\partial^2 W}{\partial Y^2} = \dfrac{\partial W}{\partial t}$ , $(Y,t)\varepsilon\Omega_2^*$

(IV.2)' $\dfrac{\partial W}{\partial Y} = o$ for $Y = o,\ t > o$

(IV.3)' $W = \psi(Y)$ for $t = o,\ Y_s(o) \geq Y \geq o$

(IV.4)' $W = W_m(Y)$ for $Y = Y_s,\ t > o$

and

(IV.5) $\dfrac{ds}{dt} = Z(t,x,V,W,\frac{\partial V}{\partial X},\frac{\partial W}{\partial Y})$ for $x = s(t),\ X = X_s(t),\ Y = Y_s(t);\ t > o$

(IV.6) $s(o) = o$

where $\Omega_1^*$ and $\Omega_2^*$ are defined as follows:

Given $\tilde{t} > o$ as a fixed value of time, and recall the definitions

$$\Omega_1 = \{(x,t):\ o < x < s(t),\ o < t < \tilde{t}\}$$
$$\Omega_2 = \{(x,t):\ s(t) < x < b,\ o < t < \tilde{t}\} \qquad (5.1)$$

which are tranformed under (4.6) and (4.35), respectively, to

$$\Omega_1^* = \{(X,t):\ o < X < X_s(t),\ o < t < \tilde{t}\}$$
$$\Omega_1^* = \{(Y,t):\ Y_s(t) > Y > o,\ o < t < \tilde{t}\} \tag{5.2}$$

In these equations, it is assumed that:

(IV.7) V is defined and uniformly bounded in $\Omega_1^*$; V and $\partial V/\partial X$ are continuous in $\overline{\Omega_1^*} = \Omega_1^* \cup \partial\Omega_1^*$ everywhere, with the possible exception of the point (o,o) where

$$\partial\Omega_1^* = \{X{=}o,\ X{=}X_s \text{ for } 0 \le t < \tilde{t};\ o \le X \le X_s(o) \text{ for } t = o\} \tag{5.3}$$

(IV.8) W is defined and uniformly bounded in $\Omega_2^*$; W and $\partial W/\partial Y$ are continuous in $\overline{\Omega_2^*} = \Omega_2^* \cup \partial\Omega_2^*$ everywhere, with the possible exception of the point (o,o) where

$$\partial\Omega_1^* = \{Y{=}o,\ Y{=}Y_s \text{ for } o \le t \le \tilde{t};\ Y_s(o) \ge Y \ge o \text{ for } t = o\} \tag{5.4}$$

(IV.9) $s = s(t)$ is continuously differentiable for $o < t < \tilde{t}$ and continuous for $o \le t \le \tilde{t}$, $s(o) = a$, and $o < s(t) < b$.

Conditions (IV.1) with (IV.7) and (IV.1)' with (IV.8) imply that V is parabolic in $\Omega_1^*$ and W is parabolic in $\Omega_2^*$ respectively.

Sometimes it is convenient to designate (IV.1) to (IV.4) and (IV.1)' to (IV.4)' as an auxiliary problem for a given Lipschitz continuous function $s(t)$. By a solution to the auxiliary problem, we mean a pair of functions $V = V(X,t)$ and $W = W(Y,t)$ such that

1°. The derivatives appearing in the equations exist and are continuous in their respective domains of definition,

2°. (IV.7) and (IV.8) are satisified,

3°. V and W satisfy (IV.1) to (IV.4) and (IV.1)' to (IV.4)' respectively.

Classical results in the theory of parabolic equations [15] assert that the solution of auxiliary problem exits and is unique under the assumptions given above.

By a solution (V,W,s) of Problem IV, we mean that

1°. s satisfies (IV.9),

2°. The pair V and W is the solution of the auxiliary problem for this $s = s(t)$ in the sense specified above,

3°. V, W and s satisfy (IV.5).

In fact, Problem IV has the same form as that of Problem II described in Fasano and Primicerio [11], a generalized two-phase Stefan problem with the flux prescribed for boundary conditions. They have proved, however, that Problem I, the same as Problem II but with the temperature prescribed for boundary conditions, is well-posed by proving the well-posedness of the differential system involved. However, we will prove the well-posedness of an equivalent system of integral equations. This approach was, in fact, used by Rubinstein [31] in solving one-phase Stefan problem.

We shall now derive a set of integral equations equivalent to the above differential system.

The fundamental solution for the heat equation

$$\frac{\partial^2\psi}{\partial z^2} = \frac{\partial\psi}{\partial t} \tag{5.5}$$

will be denoted by $\Gamma(z,t)$, and

$$\Gamma(z,t) = (4\pi t)^{-1/2}\exp(-z^2/4t). \tag{5.6}$$

The Green's and Neumann's functions on the half-space $z > o$ (or $z < o$) are denoted by $G(z,t;\ \xi,\tau)$ and $N(z,t;\ \xi,\tau)$ respectively. These are given by

$$\begin{aligned} G(z,t;\ \xi,\tau) &= \Gamma(z-\xi,\ t-\tau) - \Gamma(z+\xi,\ t-\tau) \\ N(z,t;\ \xi,\tau) &= \Gamma(z-\xi,\ t-\tau) + \Gamma(z+\xi,\ t-\tau) \end{aligned} \tag{5.7}$$

where $z$ is to be replaced by either $X$ or $Y$ for $z > o$ or $(z < o)$.

Suppose first that a triple $(V,W,s)$ form a solution to Problem IV in $(o,\tilde{t})$, and write

$$\begin{aligned} &V_o(t) = V(o,t);\ V_X(X,t) = \frac{\partial V}{\partial X}(X,t),\ o < X < X_s; \\ &W_Y(Y,t) = \frac{\partial W}{\partial Y}(Y,t),\ Y_s > Y > o;\ v(t) = \frac{\partial}{\partial X} V(X_s(t),t); \\ &w(t) = \frac{\partial}{\partial Y} W(Y_s(t),t);\ \frac{ds}{dt} = \eta(t). \end{aligned} \tag{5.8}$$

Then integrating Green's identity

$$\frac{\partial}{\partial \xi}\left(N\frac{\partial V}{\partial \xi} - V\frac{\partial N}{\partial \xi}\right) - \frac{\partial}{\partial \tau}(VN) = o \tag{5.9}$$

over the domain $\Omega_1^*$ yields, upon using (IV.2) to (IV.4), the integral representation

$$V(X,t) = \int_o^{X_s(o)} \phi(\xi)N(X,t;\ \xi,o)d\xi + \frac{A}{2}\int_o^t \ell(\tau)V_o(\tau)N(X,t;\ o,\tau)d\tau$$

$$+ \int_o^t [v(\tau) + X_s'(\tau)V_m(X_s(\tau))]N(X,t;\ X_s(\tau),\tau)d\tau$$

$$- \int_o^t V_m(X_s(\tau))N_\xi(X,t;\ X_s(\tau),\tau)d\tau \equiv \Delta^1(X,t|V_o,v,s,\eta)\ , \quad (5.10)$$

where

$$V_o(t) = \Delta^1|_{X=o} \equiv \Phi(t|V_o,v,s,\eta)\ . \quad (5.11)$$

Assuming that $V_m$ and $\phi$ have bounded derivatives up to and including the second and third orders, respectively, and that

$$V_m(X_s(o)) = \phi(X_s(o)),\ 2\phi'(o) + A\ell(o)V_o(o) = o\ . \quad (5.12)$$

Then differentiating both sides of Eq. (5.10) with respect to X, we obtain, upon using the conjugacy of G and N together with integration by parts when necessary,

$$V_X(X,t) = -\frac{A}{2}\int_o^t \ell(\tau)V_o(\tau)G_\xi(X,t;\ o,\tau)d\tau + \int_o^{X_s(o)} \phi'(\xi)G(X,t;\ \xi,\tau)d\xi$$

$$+ \int_o^t v(\tau)N_X(X,t;\ X_s(\tau),\tau)d\tau + \int_o^t X_s'(\tau)V_m'(X_s(\tau))G(X,t;\ X_s(\tau),\tau)d\tau$$

$$\equiv U^1(X,t|V_o,v,s,\eta)\ , \quad (5.13)$$

where, from now on, primes denote differentiation of the function under investigation with respect to its argument.

Now letting $X \to X_s(t) - o$ in (5.13) and using the theorem of Holmgren on the discontinuity of the heat potential of a double layer (see Friedman [12]), according to which

$$\lim_{X\to X_s(t)-o}\int_o^t v(\tau)N_X(X,t;\ X_s(\tau),\tau)d\tau = \frac{1}{2}v(t) +$$

$$+\int_o^t v(\tau)N_X(X_s(t),t;\ X_s(\tau),\tau)d\tau\ , \qquad (5.14)$$

we find that

$$v(t) = 2U^1|_{X=X_s} \equiv F^1(t|V_o,v,s,\eta)\ . \qquad (5.15)$$

To obtain a similar representation for $W(Y,t)$ and hence for $W_Y(Y,t)$ and $w(t)$, we start by integrating (5.9), with V replaced by W, over $\Omega_2^*$ to get the formula

$$W(Y,t) = -\int_o^{Y_s(o)}\psi(\xi)N(Y,t;\ \xi,o)d\xi -$$

$$-\int_o^t[w(\tau)+Y_s'(\tau)W_m(Y_s(\tau))]N(Y,t;Y_s(\tau),\tau)d\tau$$

$$+\int_o^t W_m(Y_s(\tau))N_\xi(Y,t;Y_s(\tau),\tau)d\tau$$

$$\equiv \Delta^2(Y,t|w,s,\eta)\ . \qquad (5.16)$$

Again, differentiating (5.16) with respect to Y, and using the conjugacy of G and N together with integration by parts result in

$$W_Y(Y,t) = -\int_o^{Y_s(o)}\psi'(\xi)G(Y,t;\ \xi,o)d\xi - \int_o^t w(\tau)N_Y(Y,t;Y_s(\tau),\tau)d\tau$$

$$-\int_o^t Y'(\tau)W_m'(Y_s(\tau))G(Y,t;Y_s(\tau),\tau)d\tau \equiv U^2(Y,t|w,s,\eta)\ . \quad (5.17)$$

Here it is assumed that $W_m$ and $\psi$ have bounded derivatives up to and including the second and third orders, respectively, and that

$$W_m(Y_s(o)) = \psi(Y_s(o)), \ \psi'(o) = o \ . \tag{5.18}$$

Let $Y \to Y_s(t) + o$ in (5.17) and employing the theorem

$$\lim_{Y \to Y_s(t)+o} \int_o^t w(\tau) N_Y(Y,t;\ Y_s(\tau),\tau) d\tau = \frac{1}{2} w(t) + $$

$$+ \int_o^t w(\tau) N_Y(Y_s(t),t;\ Y_s(\tau),\tau) d\tau, \tag{5.19}$$

we find that

$$w(t) = 2U^2 |_{Y=Y_s} \equiv F^2(t|w,s,\eta) \ . \tag{5.20}$$

Making use of the above results, the condition (IV.5) gives

$$s(t) = a + \int_o^t \eta(\tau) d\tau \equiv S(t|\eta) \ , \tag{5.21}$$

where

$$\eta(t) = Z(t,s,V_m(X_s),W_m(Y_s),v,w)$$

$$= \frac{1}{k_1(s(t),t)} [k_2(s(t),t) + \frac{\ell(t)}{2} - \frac{1}{A} \frac{v(t)}{V_m(X_s)} - \frac{1}{B} \frac{w(t)}{W_m(Y_s)}] \ . \tag{5.22}$$

This means that if the triple $(V,W,s)$ is a solution of Problem IV, the functions $V$, $V_o, V_X, v, W, W_Y, w, s$ and $\eta$ are solutions of the system of integral relationships (5.10), (5.11), (5.13), (5.15), (5.16), (5.17), (5.20), (5.21) and (5.22).

Conversely, we shall prove that if the functions $V, V_o, V_X, v, W,$ $W_Y, w, s$ and $\eta$ are continuous solutions of the above system of ingegral equations over $(o,\tilde{t})$, then V, W and s constitute a solution to Problem IV.

It is easily seen that V and W given by (5.10) and (5.16) satisfy (IV.1) and (IV.1)' respectively. Letting $t \to o$ in (5.10) and (5.16), we get $V(X,o) = \phi(X)$, $o \le X \le X_s(o)$ and $W(Y,o) = \psi(Y)$, $Y_s(o) \ge Y \ge o$. If we let $X \to o$ in (5.13) and $Y \to o$ in (5.17), then it is easily proved that $V_X(o,t) = -\frac{A\ell(t)}{2} V_o(t)$ or $V_X(o,t) = -\frac{A\ell(t)}{2} V(o,t)$ and $W_Y(o,t) = o$. Next we let $X \to X_s(t) - o$ and $Y \to Y_s(t) + o$ in (5.13) and (5.17) and apply (5.14) and (5.19), we find that

$$\lim_{X \to X_s(t)-o} V_X = \frac{1}{2} v(t) + U^1(X_s(t),t) \ , \tag{5.23}$$

$$\lim_{Y \to Y_s(t)+o} W_Y = \frac{1}{2} w(t) + U^2(Y_s(t),t) \ , \tag{5.24}$$

where $U^1$ and $U^2$ are respectively the right hand sides of (5.13) and (5.17). Comparing (5.23) with (5.15) and (5.24) with (5.20) shows us that in fact

$$\begin{aligned} V_X(X_s(t),t) &= \lim_{X \to X_s(t)-o} V_X = v(t) \ , \\ W_Y(Y_s(t),t) &= \lim_{Y \to Y_s(t)+o} W_Y = w(t) \ . \end{aligned} \tag{5.25}$$

Differentiating (5.21), combining the result with (5.22) and using (5.25), we get the relationship

$$s'(t) = \frac{1}{k_1(s(t),t)} [k_2(s(t),t) + \frac{\ell(t)}{2} - \frac{1}{A} \frac{V_X(X_s(t),t)}{V_m(X_s(t))} - \frac{1}{B} \frac{W_Y(Y_s(t),t)}{W_m(Y_s(t))}] \ . \tag{5.26}$$

Comparing (5.26) with (IV.5) or (4.59), we have only to prove that

$$V(X_s(t),t) = V_m(X_s(t)) \quad \text{and} \quad W(Y_s(t),t) = W_m(Y_s(t)) \ . \tag{5.27}$$

With this aim in mind, we integrate the identity (5.9) over $\Omega_1^*$ and use the initial and boundary properties of V just proved. Then, subtracting (5.10) from this result, we get

$$\int_0^t \Theta(\tau)[X_s'(\tau) - \frac{\partial}{\partial\xi}]N(X,t;\ X_s(\tau),\tau)d\tau = o \tag{5.28}$$

in which

$$\Theta(t) \equiv V(X_s(t),t) - V_m(X_s(t)) \ . \tag{5.29}$$

Letting $X \to X_s(t)$-o, and applying (5.14), we obtain

$$\Theta(t) = 2\int_0^t \Theta(\tau)[\frac{\partial}{\partial\xi} - X_s'(\tau)]N(X_s(t),t;\ X_s(\tau),\tau)d\tau = o \tag{5.30}$$

The expression in the square brackets in (5.30) has absolute value $\leq c(t-\tau)^{-1/2}$, where c is a constant depending on t. Consequently, (5.30) is a homogeneous integral equation of Volterra type of the second kind with polar kernel. From the general theory of such equations, we know that the solution is unique. Thus $\Theta \equiv o$ and the first assertion of (5.27) is proved. In exactly the same manner, we can prove that $W(Y_s(t),t) = W_m(Y_s(t))$.

In summary, we have proved

Theorem 1. Problem IV under the conditions (5.12) and (5.18) which are simply the conditions (I.9) in terms of V and W, is equivalent to the problem of finding a continuous solution to the system of integral equations (5.10), (5.11), (5.13), (5.15), (5.16), (5.17), (5.20), (5.21) and (5.22).

Remark. Equations (5.12) and (5.18) are again the compatibility conditions for the continuity of the boundary and initial conditions. If these conditions are not valid, but V, W and s form a solution to Problem IV, then v and w are determined by the equations (5.15) and (5.20) if to their right sides we add the term

$$2[V_m(X_s(o)) - \phi(X_s(o))]G(X_s(t),t;\ X_s(o),o)$$

and correspondingly

$$2[\psi(Y_s(o)) - W_m(Y_s(o))]G(Y_s(t),t;\ Y_s(o),o)\ .$$

Simultaneously, we must add the terms

$$[V_m(X_s(o)) - \phi(X_s(o))]G(X_s(t),t;\ X_s(o),o)$$

$$[\psi(Y_s(o)) - W_m(Y_s(o))]G(Y_s(t),t;\ Y_s(o),o)$$

to the right sides of equations (5.13) and (5.17), respectively.

In view of Theorem 1, it is, therefore, sufficient to prove that the problem of finding a solution to the system (5.10), (5.11), (5.13), (5.15), (5.16), (5.17), (5.20), (5.21) and (5.22) is well-posed.

## 6. Local Existence Theorem

We first recall the definition of X(x,t) and Y(x,t) given by (4.6) and (4.35),

$$X(x,t) = \int_0^x \alpha_1^{-1/2}(x',t)dx'\ ,\quad Y(x,t) = \int_x^b \alpha_2^{-1/2}(x',t)dx'\ . \qquad (6.1)$$

According to the mean value theorem for integrals, we have

$$X(b,t) = \int_0^b \alpha_1^{-1/2}(x',t)dx' = b\alpha_1^{-1/2}(x_0,t) \quad \text{for some } x_0 \varepsilon [o,b] \ ,$$
$$Y(o,t) = \int_0^b \alpha_2^{-1/2}(x',t)dx' = b\alpha_2^{-1/2}(\bar{x}_0,t) \quad \text{for some } \bar{x}_0 \varepsilon [o,b] \ . \tag{6.2}$$

then we write

$$\bar{X} \equiv \max_{o \le t \le \tilde{t}} X(b,t) \quad \text{and} \quad \bar{Y} \equiv \max_{o \le t \le \tilde{t}} Y(o,t) \ . \tag{6.3}$$

We now begin with a list of the assumptions needed for the existence theorem.

(i) If

$$o \le X, X_s \le \bar{X}; \ \bar{Y} \ge Y, Y_s \ge o; \ o \le t \le \tilde{t}; \ o \le s(t) \le b \ ;$$
$$|V_o| \le N_o; \ |V| \le N_1; \ |v| \le N_2; \ |V_X| \le N_3 \ ; \tag{6.4}$$
$$|\eta| \le N_4; \ |W| \le N_1^*; \ |w| \le N_2^*; \ |W_Y| \le N_3^*$$

Then

$$|\Lambda| \le M_o; \ |\phi|, |\phi'| \le M_1; \ |V_m|, |V_m'| \le M_2 \ ;$$
$$|Z| \le M_3; \ |\psi|, |\psi'| \le M_1^*; \ |W_m|, |W_m'| \le M_2^* \tag{6.5}$$

and

$$|\frac{\partial \Lambda}{\partial t}| < M_{o,1}; \ |\frac{\partial \Lambda}{\partial V_o}| < M_{o,2}; \ |V_m''| < M_{2,1}; \ |W_m''| < M_{2,1}^* \ ;$$
$$|\phi''|, |\phi'''| < M_1; \ |\psi''|, |\psi'''| < M_1^*; \ |\frac{\partial Z}{\partial t}| < N_{4,1} \ ;$$
$$|\frac{\partial Z}{\partial X}| < N_{4,2}; \ |\frac{\partial Z}{\partial Y}| < N_{4,2}^*; \ |\frac{\partial Z}{\partial V}| < N_{4,3}; \ |\frac{\partial Z}{\partial W}| < N_{4,3}^* \ ; \tag{6.6}$$
$$|\frac{\partial Z}{\partial V_X}| < N_{4,4}; \ |\frac{\partial Z}{\partial W_Y}| < N_{4,4}^* \ ;$$

where $M_i$, $M_i^*$, $N_i$, $N_i^*$, $M_{i,j}$, $N_{4,j}$ and $N_{4,j}^*$ are positive constants.

(ii) Let $V_{o,o}(t)$, $V_o(X,t)$, $W_o(Y,t)$, $V_{o,X}(X,t)$, $W_{o,Y}(y,t)$, $v_o(t)$, $w_o(t)$, $s_o(t)$ and $\eta_o(t)$ be arbitrary differentiable functions which have bounded partial derivatives with respect to all their arguments. More precisely, these functions satisfy for $o \le X \le \bar{X}$, $\bar{Y} \ge Y \ge o$, $o \le t \le \tilde{t}$ the inequalities (6.4) and

$$
\begin{gathered}
|\sqrt{t}\, V'_{o,o}(t)| < L_o; \; |\sqrt{t}\, \frac{\partial V_o}{\partial t}| < L_1; \; |\sqrt{t}\, v'_o(t)| < L_2; \; |\sqrt{t}\, \frac{\partial V_{o,X}}{\partial t}| < L_3 , \\
|\sqrt{t}\, \frac{\partial W_o}{\partial t}| < L^*_1; \; |\sqrt{t}\, w'_o(t)| < L^*_2; \; |\sqrt{t}\, \frac{\partial W_{o,Y}}{\partial t}| < L^*_3; \; |\frac{\partial V_{o,X}}{\partial X}| < L_4; \\
|\frac{\partial W_{o,Y}}{\partial Y}| < L^*_4; \; |\sqrt{t}\, \eta'_o(t)| < L_5 ,
\end{gathered}
\tag{6.7}
$$

in addition to the compatibility conditions

$$
\begin{gathered}
s'_o(t) = \eta_o(t); \; V_{o,X}(X_{s_o}(t),t) = v_o(t); \; V_o(o,t) = V_{o,o}(t); \; V_o(X,o) = \phi(X); \\
V_{o,X}(X,o) = \phi'(X); \; V_{o,o}(o) = \phi(o); \; W_{o,Y}(Y_{s_o}(t),t) = w_o(t), W_o(Y,t) = \psi(Y); \\
W_{o,Y}(Y,o) = \psi'(Y); \; s_o(o) = a ,
\end{gathered}
\tag{6.8}
$$

where $N_i$, $\bar{N}_i$, $L_i$, $\bar{L}_i$ are positive constants, $X_{s_o} = X(s_o(t),t)$ and $Y_{s_o} = Y(s_o(t),t)$. In particular, the continuous agreement of the boundary and initial conditions are from (6.8)

$$
\begin{gathered}
2\phi'(o) + A\ell(o)V_{o,o}(t) = o; \; \phi'(X_s(o)) = V_{o,X}(X_s(o),o) = v_o(o); \\
\psi'(o) = o; \; \psi'(Y_s(o)) = W_{o,Y}(Y_s(o),o) = w_o(o); \\
s'_o(t) = \eta_o(t) .
\end{gathered}
\tag{6.9}
$$

In what follows, we will employ the Picard method of successive approximation to construct a solution to the system of integral equations obtained in the previous section.

A sequence of approximating solutions to the above system of integral equations can be defined recurssively by the scheme

$$V_{n,o} = \Phi_n;\ V_n = \Delta_n^1;\ v_n = F_n^1;\ V_{n,X} = U_n^1;\ W_n = \Delta_n^2;$$
$$w_n = F_n^2;\ W_{n,Y} = U_n^2;\ \eta_n = Z_n;\ s_n = S_n\ , \tag{6.10}$$

where the functions on the right sides of (6.10) are

$$\begin{cases} \Phi_n = \Phi(t|V_{n-1,o}, v_{n-1}, s_{n-1}, \eta_{n-1}) \\ \Delta_n^1 = \Delta^1(X,t|V_{n,o}, v_{n-1}, s_{n-1}, \eta_{n-1}) \\ F_n^1 = F^1(t|V_{n,o}, v_{n-1}, s_{n-1}, \eta_{n-1}) \\ U_n^1 = U^1(X,t|V_{n,o}, v_n, s_{n-1}, \eta_{n-1}) \\ \Delta_n^2 = \Delta^2(Y,t|w_{n-1}, s_{n-1}, \eta_{n-1}) \\ F_n^2 = F^2(t|w_{n-1}, s_{n-1}, \eta_{n-1}) \\ U_n^2 = U^2(Y,t|w_n, s_{n-1}, \eta_{n-1}) \\ Z_n = Z(t, s_{n-1}, V_m(X_{s_{n-1}}), W_m(Y_{s_{n-1}}), v_n, w_n) \\ S_n = S(t|\eta_n) \end{cases} \tag{6.11}$$

We now turn to the proof of the following lemma which plays a major role in the convergence proof of approximating solutions:

Lemma 1. There exists a constant $t_o > o$ such that if $V$, $V_o$, $V_X$, $v$, $W$, $W_Y$, $w$ and $\eta$ satisfy (6.4) and (6.9) for $o \le X \le \overline{X}$, $\overline{Y} \ge Y \ge o$, $o \le t \le t_o$, then at these points the same conditions will be satisfied by the values of the linear operators $\Delta^i$, $U^i$, $F^i$, $i=1,2$, $\Phi$ and $Z$.

Proof. It is convenient to introduce the following notation:

$$I_{1,i}(X,t,\xi,\tau|\Theta(X,t,\xi,\tau)) = \int_0^t \Theta(X,t,\xi,\tau)\Gamma(X+(-1)^i\xi,t-\tau)d\tau ,$$

$$I_{2,i}(X,t,\xi,\tau|\Theta(X,t,\xi,\tau)) = \int_0^t \Theta(X,t,\xi,\tau)\Gamma_\xi(X+(-1)^i\xi,t-\tau)d\tau , \quad (6.12)$$

$$I_{3,i}(X,t,\xi,\alpha,\beta|\Theta(X,t,\xi)) = \int_\alpha^\beta \Theta(X,t,\xi)\Gamma(X+(-1)^i\xi,t)d\xi .$$

In this notation, the operators $\Delta^i$, $U^i$ for i=1,2 appearing on the rignt sides of (5.10), (5.13), (5.16) and (5.17) become

$$\Delta^1(X,t|\ell,V_o,\phi,V_m,v,s,\eta)$$

$$= \sum_{i=1}^{2}[I_{1,i}(X,t,o,\tau|\frac{A\ell(\tau)}{2}V_o(\tau))+I_{1,i}(X,t,X_s(\tau),\tau|\{v(\tau)+X_s'(\tau)V_m(X_s(\tau))\})$$

$$+ I_{2,i}(X,t,X_s(\tau),\tau|-V_m(X_s(\tau)))+I_{3,i}(X,t,\xi,o,X_s(o)|\phi(\xi))] , \quad (6.13)$$

$$\Delta^2(Y,t|\psi,W_m,w,s,\eta)$$

$$= \sum_{i=1}^{2}[I_{1,i}(Y,t,Y_s(\tau),\tau|-\{w(\tau)+Y_s'(\tau)W_m(Y_s(\tau))\})$$

$$+ I_{2,i}(Y,t,Y_s(\tau),\tau|W_m(Y_s(\tau))) + I_{3,i}(Y,t,\xi,o,Y_s(o)|-\psi(\xi))] , \quad (6.14)$$

$$U^1(X,t|\ell,V_o,\phi',V_m',v,s,\eta)$$

$$= \sum_{i=1}^{2}(-1)^{i+1}[I_{1,i}(X,t,X_s(\tau),\tau|X_s'(\tau)V_m'(X_s(\tau)))$$

$$+ I_{2,i}(X,t,o,\tau|- \frac{A\ell(\tau)}{2}V_o(\tau)) + I_{2,i}(X,t,X_s(\tau),\tau|-v(\tau))$$

$$+ I_{3,i}(X,t,\xi,o,X_s(o)|\phi'(\xi))] , \quad (6.15)$$

$$U^2(Y,t\,|\,\psi,W_m,w,s,\eta)$$

$$= \sum_{i=1}^{2}(-1)^{i+1}[I_{1,i}(Y,t,Y_s(\tau),\tau\,|\,-Y_s'(\tau)W_m(Y_s(\tau))+I_{2,i}(Y,t,Y_s(\tau),\tau\,|\,w(\tau))$$

$$+I_{3,i}(Y,t,\xi,o,Y_s(o)\,|\,-\psi'(\xi))] \;. \qquad (6.16)$$

By virtue of (5.11), (5.15) and (5.20), the expressions for $\Phi$, $F^1$ and $F^2$ in the above notation can be trivally obtained from (6.13), (6.15) and (6.16).

The values of $I_{i,j}(i=1,2,3;\ j=1,2)$ are estimated by means of the inequalities (6.4). Let $t_o^* > o$ be so small that

$$o < \sigma < a - N_4 t_o^* < a + N_4 t_o^* \leq b \;, \qquad (6.17)$$

where $\sigma$ is some constant. Then we automatically have

$$o < \sigma < s = a + \int_o^t \eta(\tau)d\tau \leq b;\ o \leq t \leq t_o^* \;. \qquad (6.18)$$

Let X(t), Y(t) and $\xi(\tau)$ be arbitrary differentiable functions such that

$$o \leq X(t),\xi(\tau) \leq \overline{X};\ |X'(t)|,|\xi'(\tau)| \leq N_5$$

or $\qquad (6.19)$

$$\overline{Y} \geq Y(t),\xi(\tau) \geq o;\ |Y'(t)|,|\xi'(\tau)| \leq N_5^*$$

for $o \leq \tau < t \leq t_o^*$ according as the medium under question is the frozen phase or the unfrozen phase. Suppose also that the function $\Theta$ appearing in the definitions (6.12) satisfies the inequality

$$|\Theta| < M \text{ for } o \leq X;\ \xi \leq \overline{X} \text{ or } \overline{Y} \geq Y,\xi \geq o;\ o \leq \tau \leq t \leq t_o^* \;, \qquad (6.20)$$

but is otherwise arbitrary. It obviously suffices to estimate $I_{i,j}$

in one phase, say the frozen phase, $o \le X,\xi \le \overline{X}$; $o \le \tau \le t \le t_o^*$. Then it is easily seen that

$$|I_{1,i}| \le \frac{M}{2\sqrt{\pi}} \int_o^t \frac{d\tau}{\sqrt{t-\tau}} = M\sqrt{\frac{t}{\pi}} \ , \tag{6.21}$$

$$|I_{3,i}| \le \frac{M}{\sqrt{\pi}} \int_c^d \exp(-\lambda^2)d\lambda \le M \ , \tag{6.22}$$

where

$$c = \frac{X(t)+(-1)^i\alpha}{2\sqrt{t}} \ , \ d = \frac{X(t)+(-1)^i\beta}{2\sqrt{t}} \ . \tag{6.23}$$

Moreover,

$$I_{2,1}(X(t),t,\xi(\tau),\tau|\Theta) = \frac{1}{4\sqrt{\pi}} \int_o^t \Theta \, \frac{X(t)-\xi(\tau)}{(t-\tau)^{3/2}} \exp[- \frac{(X(t)-\xi(\tau))^2}{4(t-\tau)}]d\tau$$

$$= \frac{1}{4\sqrt{\pi}} \int_o^t \Theta \, \frac{X(t)-\xi(t)}{(t-\tau)^{3/2}} \exp[- \frac{(X(t)-\xi(t))^2}{4(t-\tau)}] \cdot$$

$$\exp[- \frac{(\xi(t)-\xi(\tau))^2}{4(t-\tau)} - \frac{(X(t)-\xi(t))(\xi(t)-\xi(\tau))}{2(t-\tau)}]d\tau$$

$$+ \frac{1}{4\sqrt{\pi}} \int_o^t \Theta \, \frac{\xi(t)-\xi(\tau)}{(t-\tau)^{3/2}} \exp [- \frac{(X(t)-\xi(\tau))^2}{4(t-\tau)}]d\tau \ . \tag{6.24}$$

It follows from (6.19) and (6.20) that

$$|I_{2,1}| < \frac{M}{4\sqrt{\pi}} [ \int_o^t \frac{|X(t)-\xi(t)|}{(t-\tau)^{3/2}} \exp \{ - \frac{(X(t)-\xi(t))^2}{4(t-\tau)} \} \cdot$$

$$\exp \{ \frac{(\xi(t)-X(t))(\xi(t)-\xi(\tau))}{2(t-\tau)} \}d\tau + \int_o^t \frac{|\xi'|d\tau}{(t-\tau)^{3/2}}]$$

$$< \frac{M}{2\sqrt{\pi}} [C \text{ erfc } \frac{|X(t)-\xi(t)|}{2\sqrt{t}} + |\xi'|\sqrt{t} \ ]$$

$$< M \ [C + \frac{N_5}{2}\sqrt{\frac{t}{\pi}} \ ] \ , \tag{6.25}$$

where

$$C = \frac{1}{2}\exp(N_5\overline{X}/2) \ . \tag{6.26}$$

Finally, to estimate $I_{2,2}$ we use the result obtained for $I_{2,1}$, replacing $\xi$ by $-\xi$. Thus

$$|I_{2,2}| < M[C + \frac{N_5}{2}\sqrt{\frac{t}{\pi}}\,] \ . \tag{6.27}$$

Hence

$$|I_{2,i}| < M[C + \frac{N_5}{2}\sqrt{\frac{t}{\pi}}\,] \ . \tag{6.28}$$

Employing (6.21), (6.22), (6.28) and taking (6.4) and (6.5) into consideration, we obtain

$$|\Delta^1| < \sqrt{\frac{t}{\pi}}\,[2M_o + 2\,N_2 + 3\,M_2N_5] + 2(M_2C + M_1) \ , \tag{6.29}$$

$$|\Delta^2| < \sqrt{\frac{t}{\pi}}\,[2N_2^* + 3M_2^*N_5^*\,] + 2(M_2^*C^* + M_1^*) \ , \tag{6.30}$$

$$|U^1| < \sqrt{\frac{t}{\pi}}\,[N_5(M_o + 2M_2 + N_2)] + 2C(M_o + N_2) + 2M_1 \ , \tag{6.31}$$

$$|U^2| < \sqrt{\frac{t}{\pi}}\,[N_5^*(2M_2^* + N_2^*)] + 2C^*N_2^* + 2M_1^* \tag{6.32}$$

with

$$C^* = \frac{1}{2}\exp(N_5^*\,\overline{Y}/2) \ . \tag{6.26*}$$

We now require the inequality condition

$$N_2 > 3M_1 \tag{6.33$_1$}$$

and thence fix $N_2$. In addition, we require

$$N_4 > M_3 \tag{6.33$_2$}$$

and thence fix $N_4$. This is possible, since $M_1$ is independent of $V$, $V_o$, $V_X$, $v$, $W$, $W_Y$, $w$ and $\eta$ but $M_3$ depends only upon $M_2$ and is independent of $V$, $V_o$,---, and $\eta$, since $M_3 \geq |Z|$. We will also assume the validity of the inequalities

$$N_o; \; N_1 > 3(M_1 + M_2C); \; N_3 > 3(M_1 + M_oC) + 2N_2C \tag{6.33$_3$}$$

and we then fix the values of $N_o$, $N_1$ and $N_3$. This can be done for fixed $N_2$ and $N_4$, since $M_1$ and $M_2$ are not dependent on $N_i$ ($i = o,---,4$). A similar argument applies to $N_i^*$ ($i = 1,\ldots,4$).

Having fixed $N_o$, $N_i$, $N_i^*$($i = 1,---,4$), it is obviously possible to choose $t_o > o$ such that for $o \leq t \leq t_o \leq t_o^*$ we have simultaneously

$$\begin{aligned} &|V_o| < N_o; \; |V| < N_1; \; |v| < N_2; \; |V_X| < N_3 \; ; \\ &|\eta| < N_4; \; |W| < N_1^*; \; |w| < N_2^*; \; |W_Y| < N_3^* \; , \end{aligned} \tag{6.34}$$

which completes the proof of the lemma.

Lemma 2. Let $V$, $V_o$, $W$, $V_X$, $W_Y$, $v$, $w$, $\Phi$ and $\eta$ satisfy not only (6.4) and (6.9) but also (6.7). Then the functionals $\Delta^i$, $U^i$, $F^i$, $i = 1,2,\Phi$ and $Z$ satisfy (6.7) for $o \leq X$, $\xi \leq \overline{X}$, $\overline{Y} \geq Y$, $\xi \geq o$, $o \leq \tau \leq t \leq \overline{t} \leq t_o$, where $\overline{t}$ is sufficiently small.

Proof: From (5.10) and (5.13), we obtain

$$\frac{d\Delta^1}{dt} = \frac{A}{2}\int_0^t \frac{d}{d\tau}\left[\ell(\tau)V_o(\tau)\right]N(X,t;\, o,\tau)d\tau + \int_0^t v'(\tau)N(X,t;\, X_s(\tau),\tau)d\tau$$

$$+ \int_0^t X_s'(\tau)\left[v(\tau)-V'(X_s(\tau))\right]N_\xi(X,t;\, X_s(\tau),\tau)d\tau$$

$$+ \int_0^{X_s(o)} \phi''(\xi)N(X,t;\ \xi,o)d\xi \equiv \Delta^{1*}\ , \tag{6.35}$$

$$\begin{aligned}
\frac{dU^1}{dt} = &\{X_s'(o)[V_m'(X_s(o))-v(o)] - \phi''(X_s(o))\}G(X(t),t;\ X_s(o),o) \\
&+ \int_0^{X_s(o)} \{\phi'''(\xi)G(X(t),t;\ \xi,o) + X'(t)\phi''(\xi)N(X(t),t;\ \xi,o)\}d\xi \\
&- \frac{A}{2}\int_0^t \frac{d}{d\tau}[\ell(\tau)V_o(\tau)]G_\xi(X(t),t;\ o,\tau)d\tau \\
&+ \frac{A}{2}\int_0^t X'(t)\frac{d}{d\tau}[\ell(\tau)V_o(\tau)]N(X(t),t;\ o,\tau)d\tau \\
&+ \int_0^t \{\frac{d}{d\tau}[X_s'(\tau)(V_m'(X_s(\tau))-v(\tau))]G(X(t),t;\ X_s(\tau),\tau) \\
&\qquad + X'(t)v'(\tau)N(X(t),t;\ X_s(\tau),\tau)\}d\tau \\
&+ \int_0^t \{X_s'^2(\tau)(V_m'(X_s(\tau))-v(\tau))-v'(\tau)\}G_\xi(X(t),t;\ X_s(\tau),\tau)d\tau \\
&+ \int_0^t X'(t)[X_s'(\tau)(v(\tau)-V_m'(X_s(\tau)))]N_\xi(X(t),t;\ \xi,\tau)d\tau \equiv U^{1*}\ .
\end{aligned} \tag{6.36}$$

Here

$$o < X(t) < X_s(t) \text{ for } X(t) \not\equiv X_s(t), \text{ or } X(t) \equiv X_s(\tau)\ . \tag{6.37}$$

Using (5.11) and (5.15), we get

$$\frac{d\Phi}{dt} \equiv \Phi^* = \Delta^{1*}\Big|_{X=o}\ , \tag{6.38}$$

$$\frac{dF^1}{dt} \equiv F^{1*} = 2U^{1*}\Big|_{X=X_s(t)}\ . \tag{6.39}$$

Also,

$$\frac{dU^1}{dX} = \int_0^{X_s(o)} \phi''(\xi)N(X,t;\ \xi,o)d\xi + \frac{A}{2}\int_0^t \frac{d}{d\tau}[\ell(\tau)V_o(\tau)]N(X,t;\ o,\tau)d\tau$$

$$+\int_0^t v'(\tau)N(X,t;\ X_s(\tau),\tau)d\tau\ +$$

$$+\int_0^t X_s'(\tau)[v(\tau)-V_m'(X_s(\tau))]N_\xi(X,t;\ X(\tau),\tau)d\tau \equiv U^{1**}\ . \tag{6.40}$$

Analogously, we have from (5.16) and (5.17) the results

$$\begin{aligned}\frac{d\Delta^2}{dt} = &-\int_0^t w'(\tau)N(Y,t;\ Y_s(\tau),\tau)d\tau \\ &-\int_0^t Y_s'(\tau)[w(\tau)-W_m'(Y_s(\tau))]N_\xi(Y,t;\ Y_s(\tau),\tau)d\tau \\ &-\int_0^{Y_s(o)} \psi''(\xi)N(Y,t;\ \xi,o)d\xi \equiv \Delta^{2*}\ ,\end{aligned} \tag{6.41}$$

$$\begin{aligned}\frac{dU^2}{dt} = &-\{Y_s'(o)[W_m'(Y_s(o))-w(o)]-\psi''(Y_s(o))\}G(Y,t;\ Y_s(o),o) \\ &-\int_0^{Y_s(o)}\{\psi'''(\xi)G(Y(t),t;\ \xi,o)+Y'(t)\psi''(\xi)N(Y(t),t;\ \xi,o)\}d\xi \\ &-\int_0^t\{\frac{d}{dt}[Y_s'(\tau)(W_m'(Y_s(\tau))-w(\tau))]G(Y(t),t;\ Y_s(\tau),\tau)\ + \\ &\qquad + Y'(t)w'(\tau)N(Y(t),t;\ Y_s(\tau),\tau)\}d\tau \\ &-\int_0^t\{Y_s'^2(\tau)(W_m'(Y_s(\tau))-w(\tau))-w'(\tau)\}G_\xi(Y(t),t;\ Y_s(\tau),\tau)d\tau \\ &-\int_0^t Y'(t)[Y_s'(\tau)(w(\tau)-W_m'(Y_s(\tau)))]N_\xi(Y(t),t;\ \xi,\tau) \equiv U^{2*}\ .\end{aligned} \tag{6.42}$$

And

$$Y_s(t) > Y(t) > o \quad \text{for} \quad Y(t) \not\equiv Y_s(t),\ \text{or}\ Y(t) \equiv Y_s(\tau)\ . \tag{6.43}$$

Using (5.20), we deduce

$$\frac{dF^2}{dt} \equiv F^{2*} = 2U^{2*}\Big|_{Y=Y_s(t)} \; . \tag{6.44}$$

Finally

$$\frac{dU^2}{dY} = -\int_0^{Y_s(o)} \psi''(\xi)N(Y,t;\ \xi,o)d\xi - \int_0^t w'(\tau)N(Y,t;\ Y_s(\tau),\tau)d\tau$$

$$-\int_0^t Y'(\tau)[w(\tau)-W_m'(Y_s(\tau))]N_\xi(Y,t;\ Y_s(\tau),\tau)d\tau \equiv U^{2**} \; . \tag{6.45}$$

We now introduce the notation

$$\begin{cases} \Theta^1_{1,i,o} = \frac{A}{2}\frac{d}{d\tau}[\ell(\tau)V_o(\tau)] \; ; \quad \Theta^1_{1,i,1} = v'(\tau) \\ \Theta^1_{2,i,1} = X_s'(\tau)[v(\tau)-V_m'(X(\tau))] \\ \Theta^1_{3,i} = \phi''(\xi) \end{cases} \tag{6.46}$$

$$\begin{cases} \Theta_o = X_s'(o)[V_m'(X_s(o))-v(o)]-\phi''(X_s(o)) \\ \Theta^2_{1,i,o} = (-1)^{i+1}\frac{A}{2}X'(t)\frac{d}{d\tau}[\ell(\tau)V_o(\tau)];\Theta^2_{2,i,o} = -\frac{A}{2}\frac{d}{d\tau}[\ell(\tau)v_o(\tau)] \\ \Theta^2_{1,i,1} = \frac{d}{d\tau}[X_s'(\tau)(V_m'(X_s(\tau))-v(\tau))] + (-1)^{i+1}X'(t)v'(\tau) \\ \Theta^2_{2,i,1} = [X_s'(\tau)+(-1)^iX'(t)]\cdot[X_s'(\tau)(V_m'(X_s(\tau))-v(\tau))]-v'(\tau) \\ \Theta^2_{3,i} = \phi'''(\xi) + (-1)^{i+1}X'(t)\phi''(\xi) \end{cases} \tag{6.47}$$

$$\begin{cases} \Theta^3_{1,i,o} = \frac{A}{2}\frac{d}{d\tau}[\ell(\tau)V_o(\tau)] \; ; \quad \Theta^3_{1,i,1} = v'(\tau) \\ \Theta^3_{2,i,1} = X_s'(\tau)[v(\tau)-V_m'(X_s(\tau))]; \quad \Theta^3_{3,i} = \phi''(\xi) \end{cases} \tag{6.48}$$

A similar notation in terms of Y is introduced analogously. Using (6.12), (6.46)-(6.48) as well as (5.7), we can reduce (6.35), (6.36) and (6.40) to the form

$$\frac{d\Delta^1}{dt} \equiv \Delta^{1*}(X,t,\ldots) = \sum_{i=1}^{2}[I_{1,i}(X,t,o,\tau|\Theta^1_{1,i,o})+I_{1,i}(X,t,X_s(\tau),\tau|\Theta^1_{1,i,1})$$

$$+ I_{2,i}(X,t,X_s(\tau),\tau|\Theta^1_{2,i,1})+I_{3,i}(X,t,\xi,o,X_s(o)|\Theta^1_{3,i})] \; , \qquad (6.49)$$

$$\frac{dU^1}{dt} \equiv U^{1*}(X(t),t,\ldots) =$$

$$= \sum_{i=1}^{2}(-1)^{i+1}[\Theta_o\Gamma(X(t)+(-1)^i X_s(o),t)+I_{1,i}(X(t),t,o,\tau|\Theta^2_{1,i,o})$$

$$+ I_{1,i}(X(t),t,X_s(\tau),\tau|\Theta^2_{1,i,1})+I_{2,i}(X(t),t,o,\tau|\Theta^2_{2,i,o})+$$

$$+ I_{2,i}(X(t),t,X_s(\tau),\tau|\Theta^2_{2,i,1})+I_{3,i}(X(t),t,\xi,o,X_s(o)|\Theta^2_{3,i})] \; , \qquad (6.50)$$

$$\frac{dU^1}{dX} \equiv U^{1**}(X,t,\ldots) =$$

$$= \sum_{i=1}^{2}[I_{1,i}(X,t,o,\tau|\Theta^3_{1,i,o})+I_{1,i}(X,t,X_s(\tau),\tau|\Theta^3_{1,i,1})$$

$$+ I_{2,i}(X,t,X_s(\tau),\tau|\Theta^3_{2,i,1})+I_{3,i}(X,t,\xi,o,X_s(o)|\Theta^3_{3,i})] \; . \qquad (6.51)$$

Similarly, (6.41), (6.42) and (6.45) are reduced to the form

$$\frac{d\Delta^2}{dt} \equiv \Delta^{2*}(Y,t,\ldots) =$$

$$= \sum_{i=1}^{2}[I_{1,i}(Y,t,Y_s(\tau),\tau|-\Theta^1_{1,i,1})+I_{2,i}(Y,t,Y_s(\tau),\tau|-\Theta^1_{2,i,1})$$

$$+ I_{3,i}(Y,t,\xi,o,Y_s(o)|-\Theta^1_{3,i})] \; , \qquad (6.52)$$

$$\frac{dU^2}{dt} \equiv U^{2*}(Y(t),t,\ldots) =$$

$$= \sum_{i=1}^{2}(-1)^i[\Theta_0\Gamma(Y(t)+(-1)^iY_s(o),t)+I_{1,i}(Y(t),t,Y_s(\tau),\tau|\Theta^2_{1,i,1})$$

$$+ I_{2,i}(Y(t),t,Y_s(\tau),\tau|\Theta^2_{2,i,1})+I_{3,i}(Y(t),t,\xi,o,Y_s(o)|\Theta^2_{3,i})] , \tag{6.53}$$

$$\frac{dU^2}{dY} \equiv U^{2**}(Y,t,\ldots) = \sum_{i=1}^{2}[I_{1,i}(Y,t,Y(\tau),\tau|-\Theta^3_{1,i,1})$$

$$+ I_{2,i}(Y,t,Y_s(\tau),\tau)|-\Theta^3_{2,i,1})$$

$$+ I_{3,i}(Y,t,\xi,o,Y_s(o)|-\Theta^3_{3,i})] . \tag{6.54}$$

We now estimate the integrals $I_{i,j}$ assuming that the argument function $\Theta$ satisfies the inequality

$$|\Theta| < \frac{L}{\sqrt{t}} . \tag{6.55}$$

Therefore, we have

$$|I_{1,i}(X,t,\xi,\tau|\Theta)| < \frac{L}{2\sqrt{\pi}}\int_0^t \frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}} = \frac{L\sqrt{\pi}}{2} , \tag{6.56}$$

$$|I_{3,i}(X,t,\xi,o,X_s(o)|\Theta)| \le \frac{L}{\sqrt{\pi t}}\int_c^d \exp(-\lambda^2)d\lambda = \frac{L}{\sqrt{t}} \tag{6.57}$$

with c and d given by (6.23).

We now estimate $I_{2,i}$. Using (6.24), we find that

$$|I_{2,1}(X(t),t,\xi(\tau),\tau|\Theta)| < \frac{LC}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}}\frac{|X(t)-\xi(t)|}{(t-\tau)^{3/2}}\exp[-\frac{(X(t)-\xi(t))^2}{4(t-\tau)}]d\tau$$

$$+ \frac{L}{4\sqrt{\pi}}\int_0^t \frac{|\xi(t)-\xi(\tau)|}{\sqrt{\tau}\,(t-\tau)^{3/2}}\exp[-\frac{(\xi(t)-\xi(\tau))^2}{4(t-\tau)}]d\tau$$

$$= I' + I'' .$$

Since $|\xi(t)-\xi(\tau)| < N_5(t-\tau)$, then

$$|I''| < \frac{LN_5}{4\sqrt{\pi}} \int_0^t \frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}} = \frac{LN_5\sqrt{\pi}}{4} .$$

Also, on setting

$$\mu^2 = \frac{[X(t)-\xi(t)]^2}{4} [\frac{1}{t-\tau} - \frac{1}{t}]; \; |X(t)-\xi(t)| = G$$

in I', we find that

$$|I'| < \frac{2LC}{\sqrt{\pi}} \frac{\exp(-G^2/4t)}{\sqrt{t}} \int_0^\infty \exp[-\mu^2]d\mu = \frac{LC}{\sqrt{t}} \exp(- \frac{G^2}{4t}) .$$

Hence,

$$|I_{2,1}| < \frac{LC}{\sqrt{t}} \exp\ (- \frac{G^2}{4t}) + \frac{LN_5\sqrt{\pi}}{4} . \tag{6.58}$$

In exactly the same way, we obtain

$$|I_{2,2}(X(t),t,\xi,\tau|\Theta)| < L[\frac{C}{\sqrt{t}} \exp(- \frac{G^{*2}}{4t}) + \frac{N_5\sqrt{\pi}}{4}] . \tag{6.59}$$

Here

$$G^* = |X(t)+\xi(t)| .$$

From (6.58) and (6.59), we may write

$$|I_{2,i}(X(t),t,\xi,\tau|\Theta)| < L[\frac{C}{\sqrt{t}} + \frac{N_5\sqrt{\pi}}{4}] . \tag{6.60}$$

Using (6.56), (6.57), (6.60), the definitions (6.46)-(6.48) and taking the inequalities (6.5), (6.6) and (6.9) into account, we find that

$$|\Delta^{1*}| < \frac{\sqrt{\pi}}{2}\,[2M_{o,1}+N_5^2(N_2+M_2)] + \frac{1}{\sqrt{t}}\,[L_2\sqrt{\pi} + 2N_5(N_2+M_2)+2M_1], \tag{6.61}$$

$$|\Delta^{2*}| < \frac{\sqrt{\pi}}{2}\,[N_5^{*2}(N_2^*+M_2^*)] + \frac{1}{\sqrt{t}}\,[L_2^*\sqrt{\pi} + 2N_5^*(N_2^*+M_2^*)+2M_1^*] , \tag{6.62}$$

$$\begin{aligned}|U^{1*}| < \sqrt{\pi}\,&[\tfrac{3}{2}N_5M_{o,1}+N_5^2M_{2,1}+N_5^3(N_2+M_2)]\\ &+ \frac{1}{\sqrt{t}}\,[\frac{\sqrt{\pi}}{2}\,(5L_2N_5+2L_5(N_2+M_2)) + \frac{1}{\sqrt{\pi}}\,(M_1+N_5(N_2+M_2))\\ &+ 2C(M_{o,1}+2N_5^2(N_2+M_2))+2M_1(1+N_5)] + \frac{2L_2C}{t} ,\end{aligned} \tag{6.63}$$

$$\begin{aligned}|U^{2*}| < \sqrt{\pi}\,&[M_{2,1}^*N_5^{*2}+N_5^{*3}(N_2^*+M_2^*)]\\ &+ \frac{1}{\sqrt{t}}\,[\frac{\sqrt{\pi}}{2}\,(5L_2^*N_5^*+2L_5^*(N_2^*+M_2^*)) + \frac{1}{\sqrt{\pi}}\,(M_1^*+N_5^*(N_2^*+M_2{}^*))\\ &+ 4CN_5^{*2}(N_2^*+M_2^*)+2M_1^*(1+N_5^*)] + \frac{2L_2^*C^*}{t} ,\end{aligned} \tag{6.64}$$

$$|U^{1**}| < \frac{\sqrt{\pi}}{2}\,[2M_{o,1}+N_5^2(N_2+M_2)] + \frac{1}{\sqrt{t}}\,[L_2\sqrt{\pi} + 2N_5(N_2+M_2)+2M_1] , \tag{6.65}$$

$$|U^{2**}| < \frac{\sqrt{\pi}}{2}\,[N_5^{*2}(N_2^*+M_2^*)] + \frac{1}{\sqrt{t}}\,[L_2^*\sqrt{\pi} + 2N_5^*(N_2^*+M_2^*)+2M_1^*] . \tag{6.66}$$

We see that $|U^{1**}|$ and $|U^{2**}|$ are estimated by the same bounds on the right sides of (6.61) and (6.62) respectively.

A similar procedure to that in the proof of Lemma 1 can be applied here to choose the numbers $L_o$, $L_i$, $L_i^* > o(i=1,2,3,4)$ and $L_5 > o$ so large, and $\bar{t} \le t_o$ sufficiently small that for $o \le t \le \bar{t}$ the inequalities

$$\left|\frac{d\Phi}{dt}\right| < \frac{L_o}{\sqrt{t}}; \quad \left|\frac{\partial\Delta^1}{\partial t}\right| < \frac{L_1}{\sqrt{t}}; \quad \left|\frac{dF^1}{dt}\right| < \frac{L_2}{\sqrt{t}}; \quad \left|\frac{\partial U^1}{\partial t}\right| < \frac{L_3}{\sqrt{t}}; \quad \left|\frac{\partial U^1}{\partial X}\right| < L_4 ; \tag{6.67}$$

$$\left|\frac{dZ}{dt}\right| < \frac{L_5}{\sqrt{t}};\ \left|\frac{\partial\Delta^2}{\partial t}\right| < \frac{L_1^*}{\sqrt{t}};\ \left|\frac{dF^2}{dt}\right| < \frac{L_2^*}{\sqrt{t}};\ \left|\frac{\partial U^2}{\partial t}\right| < \frac{L_3^*}{\sqrt{t}};\ \left|\frac{\partial U^2}{\partial Y}\right| < L_4^* .$$

are simultaneously fulfilled. Hence our assertion is proved.

From Lemmas 1 and 2, we deduce that there exists a small value $\overline{t} > o$ such that for $o \le X \le \overline{X}$, $\overline{Y} \ge Y \ge o$, $o \le t \le \overline{t}$ the sequences $\{V_n(X,t)\}$, $\{V_{n,o}(t)\}$, $\{W_n(Y,t)\}$, $\{V_{n,X}(X,t)\}$, $\{W_{n,Y}(Y,t)\}$, $\{v_n(t)\}$, $\{w_n(t)\}$, $\{\eta_n(t)\}$ are uniformly bounded and equicontinuous. Hence, by the Arzelà-Ascoli theorem and the uniform continuity of the operators $\Delta^1, \ldots, Z$ there exist subsequences of $\{V_n(X,t)\}, \ldots, \{\eta_n(t)\}$ which converge. In other words, a solution to the system of integral equations exists. Furthermore, by virtue of the uniform boundedness of the sequences

$$\begin{aligned}&\{\sqrt{t}\,\frac{\partial V_n}{\partial t}\};\ \{\sqrt{t}\,\frac{dV_{n,o}}{dt}\};\ \{\sqrt{t}\,\frac{\partial W_n}{\partial t}\};\ \{\sqrt{t}\,\frac{\partial V_{n,X}}{\partial t}\};\ \{\sqrt{t}\,\frac{\partial W_{n,Y}}{\partial t}\};\\ &\{\sqrt{t}\,\frac{dv_n}{dt}\};\ \{\sqrt{t}\,\frac{dw_n}{dt}\};\ \{\sqrt{t}\,\frac{d\eta_n}{dt}\};\ \{\frac{\partial V_{n,X}}{\partial X}\};\ \{\frac{\partial W_{n,Y}}{\partial Y}\}\ ,\end{aligned} \tag{6.68}$$

we easily infer that this solution satisfies the Lipschitz condition of the form

$$|\Theta(t)-\Theta(\tau)| < A^*(\sqrt{t}+\sqrt{\tau})^{-1}\,|t-\tau| \tag{6.69}$$

with respect to t, where the constant $A^*$ is defined uniformly for all X and Y. This completes the proof of the following theorem of "local" existence.

<u>Theorem 2</u>. There exists a certain small value of time $\overline{t} > o$ depending on bounds of $\Lambda$, Z, $\phi$, $\psi$, $V_m$ and $W_m$, on bounds of all partial derivatives of $\Lambda, \ldots, W_m$ arising in the conditions of the problem, and

on $a = s(o)$ such that for $o < t < \bar{t}$, there exists a solution to the problem for the system of integral equations and hence to Problem IV. This solution can be constructed by the Picard method of successive approximation if we begin the iteration process with arbitrary functions $V_o$, $V_{o,o}$, $W_o$, $V_{o,X}$, $W_{o,Y}$, $v_o$, $w_o$, $s_o$, $\eta_o$ which have bounded partial derivatives with respect to each of their arguments; moreover, the solution satisfies the conditions (6.9) and (6.69).

We take note here that in Section 7 we will sharpen these estimates somewhat by restricting ourselves to even smaller time intervals if necessary and so obtain local uniqueness.

In addition to the uniform boundedness and equicontinuity of the sequences $\{V_n\},\ldots,\{\eta_n\}$ that have been established above, a stronger assertion can be yet proved. In fact, we will prove

Lemma 3. The sequences $\{V_n\}$, $\{V_{n,o}\}$, $\{W_n\}$, $\{V_{n,X}\}$, $\{W_{n,Y}\}$, $\{v_n\}$, $\{w_n\}$ and $\{\eta_n\}$ are uniformly convergent.

This result will then imply convergence of the entire iteration sequence to a solution of the system of integral equations of Problem IV.

Proof. It suffices to prove uniform convergence of the series

$$\sum_{m=o}^{\infty} (V_{m+1}-V_m),\ldots, \sum_{m=o}^{\infty} (\eta_{m+1}-\eta_m)\,. \tag{6.70}$$

This is guaranteed for $o \le t \le t_o < \frac{1}{L}$ if

$$|V_{m+1}-V_m| < M(Lt_1)^{m/2},\ldots, |\eta_{m+1}-\eta_m| < M(Lt_1)^{m/2};\; m = o,1,2,\ldots, \tag{6.71}$$

where M and L are positive constants independent of m. The proof of (6.71) will be by mathematical induction.

By virtue of the uniform boundedness of $\{V_n\}, \ldots, \{\eta_n\}$, (6.71) holds for $m=o$. Hence it suffices to show that if (6.71) is valid for a general case $m=n$, then it is also valid for $m=n+1$. To this end we examine the family of functions, considering only the case of solid phase variables,

$$\{\Theta(X,t,\xi,\tau)\},\ \{X(t)\},\ \{\xi(t)\},\ \{X_s(t)\}$$

having the following properties:

(i) $X(t)$, $\xi(t)$ and $X_s(t)$ are well defined and twice continuously differentiable functions for $o \le t \le \bar{t}$; moreover,

$$o \le X(t),\ X_s(t),\ \xi(t) \le \bar{X} \quad \text{for} \quad o \le t \le \bar{t} \tag{6.72}$$

$$|X'(t)|,\ |X_s'(t)|,\ |\xi'(t)| \le N;\ |\sqrt{t}\, X''(t)|,\ |\sqrt{t}\, X_s''(t)|,\ |\sqrt{t}\, \xi''(t)| \le L$$

(ii) $\Theta(X,t,\xi,\tau)$ is defined for $o \le \tau \le t \le \bar{t}$, and continuous together with its partial derivatives with respect to all its arguments for $o \le \tau \le t \le \bar{t}$; moreover,

$$|\Theta| < M;\ |\sqrt{t}\, \Theta_X|,\ |\sqrt{t}\, \Theta_t|,\ |\sqrt{\tau}\, \Theta_\tau|,\ |\sqrt{\tau}\, \Theta_\xi| < M^* , \tag{6.73}$$

where $M$ and $M^*$ are some constants, and the inequalities are uniform for the entire family under investigation.

We will examine variations of the integrals $I_{i,j}$ within the class of functions being considered. Let $\Theta$, $X(t)$, $\xi(\tau)$, $X_s(\tau)$ and $\Theta^*$, $X^*(t)$, $\xi^*(\tau)$, $X_s^*(t)$ be two groups of admissible functions. We will write

$$I_{i,j}(X^*(t),t,\xi^*(\tau),\tau,\ldots|\Theta^*(X^*(t),t,\xi^*(\tau),\tau))$$
$$- I_{i,j}(X(t),t,\xi(\tau),\tau,\ldots|\Theta(x(t),t,\xi(\tau),\tau)) \tag{6.74}$$
$$= \delta I_{i,j}(X(t),t,\xi(\tau),\tau,\ldots|\Theta(X(t),t,\xi(\tau),\tau))$$

and

$$\max|\delta I_{i,j}| = \delta^* I_{i,j};\ o \le \tau \le t \le \overline{t},\ o \le X, \xi \le \overline{X}\ ;$$
$$i=1,2,3;\ j=1,2. \tag{6.75}$$

We will use anologous notation for all functions in the class of functions under investigation.

For any i and j, we have

$$\delta I_{i,j} = I_{i,j}(X^*(t),t,\xi^*(\tau),\tau,\ldots|\delta\Theta) + \Delta I_{i,j} \tag{6.76}$$

where

$$\Delta I_{i,j} = I_{i,j}(X^*(t),t,\xi^*(\tau),\tau,\ldots|\Theta(X(t),t,\ldots))$$
$$-I_{i,j}(X(t),t,\xi(\tau),\tau,\ldots|\Theta(X(t),t,\ldots))\ . \tag{6.77}$$

We find that

$$\Delta I_{1,i} = \int_0^t \Theta d\tau \int_{X(t)+(-1)^i\xi(\tau)}^{X^*(t)+(-1)^i\xi^*(\tau)} \Gamma_\zeta(\zeta,t-\tau)d\zeta\ ,$$

or, equivalently,

$$\Delta I_{1,i} = \int_0^t \Theta d\tau \int_0^{\delta[X(t)+(-1)^i\xi(t)]} \Gamma_\zeta(X(t)+(-1)^i\xi(\tau)+\zeta,t-\tau)d\zeta$$
$$+ \int_0^t \Theta d\tau \int_0^{(-1)^{i+1}\delta[\xi(t)-\xi(\tau)]} \Gamma_\zeta(X^*(t)+(-1)^i\xi(\tau)+(-1)^i\delta\xi(t)+\zeta,t-\tau)d\zeta$$
$$= \Delta I'_{1,i} + \Delta I''_{1,i} \tag{6.78}$$

We now set

$$\gamma_m = \max_{0 \le \lambda < \infty} \lambda^m \exp(-\lambda^2) \ . \tag{6.79}$$

By virtue of (6.72), we have

$$|\Gamma_\zeta| < \frac{\gamma_1}{2\sqrt{\pi}} \frac{1}{(t-\tau)} \ ; \ |\delta[\xi(t)-\xi(\tau)]| < \delta^*\xi'(t-\tau) \ . \tag{6.80}$$

Consequently

$$|\Delta I''_{1,i}| < \frac{M\gamma_1}{2\sqrt{\pi}} t\delta^*\xi' \ . \tag{6.81}$$

Next, changing the order of integraion for $X(t) \neq \xi(t)$ in the term $\Delta I'_{1,i}$, we get

$$\Delta I'_{1,i} = (-1)^i \int_0^{\delta[X(t)+(-1)^i\xi(t)]} d\zeta \int_0^t \Theta\Gamma_\xi(X(t)+(-1)^i\xi(t)+\zeta, t-\tau)d\tau \ ,$$

or, in the notation of (6.12),

$$\Delta I'_{1,i} = (-1)^i \int_0^{\delta[X(t)+(-1)^i\xi(t)]} I_{2,i}(X(t)+\zeta, t, \xi(t), \tau | \Theta(X(t), t, \xi(\tau), \tau))d\zeta \ . \tag{6.82}$$

Using the estimates (6.28) and observing that the constants in these estimates do not depend on $\delta X$ and $\delta\xi$, we find that

$$|\Delta I'_{1,i}| < M'[|\delta X(t)|+|\delta\xi(t)|] \ , \tag{6.83}$$

where M' depends only on M, C, N and $\bar{t}$. Combining (6.81) and (6.83), we obtain

$$|\Delta I_{1,i}| < M_{1,1}t\delta^*\xi' + M_{1,2}(\delta^*X+\delta^*\xi) \ . \tag{6.84}$$

We now estimate $\Delta I_{2,i}$. On writing

$$\nu^*(t,\tau) = X^*(t) + (-1)^i \xi^*(\tau) \; ;$$
$$\nu(t,\tau) = X(t) + (-1)^i \xi(\tau) \; , \tag{6.85}$$

we see that

$$\frac{\nu(t,\tau)}{4\sqrt{\pi}\,(t-\tau)^{3/2}} \exp\left[-\frac{\nu^2(t,\tau)}{4(t-\tau)}\right] = -\frac{1}{2}\left[\frac{d}{d\tau} \operatorname{erfc} \frac{\nu(t,\tau)}{2\sqrt{t-\tau}} + \right.$$
$$\left. + 2\Gamma(\nu(t,\tau),t-\tau) \frac{\partial\nu}{\partial\tau}\right] . \tag{6.86}$$

Accordingly

$$\Delta I_{2,i} = -\frac{1}{2}\int_0^t \Theta \frac{d}{d\tau} \delta \operatorname{erfc} \frac{\nu(t,\tau)}{2\sqrt{t-\tau}}\, d\tau - \int_0^t \Theta\delta\left(\frac{\partial\nu(t,\tau)}{\partial\tau}\right)\Gamma(\nu^*(t,\tau),t-\tau)d\tau$$
$$- \int_0^t \Theta \frac{\partial\nu(t,\tau)}{\partial\tau}\, \delta\Gamma(\nu(t,\tau),t-\tau)d\tau \tag{6.87}$$
$$= \Delta I'_{2,i} + \Delta I''_{2,i} + \Delta I'''_{2,i} \; .$$

We have

$$\Delta I''_{2,i} = - I_{1,i}(X^*(t),t,\xi^*(\tau),\tau|\Theta\delta(\frac{\partial\nu(t,\tau)}{\partial\tau})) \; ,$$
$$\Delta I'''_{2,i} = - \Delta I_{1,i}(X(t),t,\xi(\tau),\tau|\Theta \frac{\partial\nu(t,\tau)}{\partial\tau}) \; . \tag{6.88}$$

Using (6.21), (6.84), (6.85), (6.72) and (6.73), we find that

$$|\Delta I''_{2,i}| < \frac{M}{\sqrt{\pi}} \sqrt{t}\; \delta^*\xi' \; ; \tag{6.89}$$

$$|\Delta I'''_{2,i}| < M_1' t\delta^*\xi' + M_2'(\delta^*X+\delta^*\xi) \; , \tag{6.90}$$

where $M_1'$ and $M_2'$ are suitably chosen constants; independent of $\delta^*\xi'$, $\delta^*X$ and $\delta^*\xi$.

Moreover, for $\nu(t,o) \not\equiv o$,

$$\Delta I'_{2,i} = \frac{1}{2}\,\Theta(X(t),t,\xi(o),o)\delta \text{ erfc } \frac{\nu(t,o)}{2\sqrt{t}} \tag{6.91}$$
$$+ \frac{1}{2}\int_0^t \frac{d}{d\tau}\,\Theta(X(t),t,\xi(\tau),\tau)\delta \text{ erfc } \frac{\nu(t,\tau)}{2\sqrt{t-\tau}}\, d\tau \ .$$

But

$$\frac{1}{2}\,\delta \text{ erfc } \frac{\nu(t,\tau)}{2\sqrt{t-\tau}} = \int_0^{-\delta\nu(t,\tau)} \Gamma(\zeta+\nu(t,\tau),t-\tau)d\zeta \ . \tag{6.92}$$

then

$$\frac{1}{2}\left|\delta \text{ erfc } \frac{\nu(t,\tau)}{2\sqrt{t-\tau}}\right| < \frac{1}{2\sqrt{\pi}}\,\frac{|\delta\nu(t,\tau)|}{\sqrt{t-\tau}} \ . \tag{6.93}$$

Employing (6.91), (6.92), (6.93), (6.85) and (6.73), we find that

$$|\Delta I'_{2,i}| < \frac{M}{2\sqrt{\pi t}}\,[\delta^* X+|\delta\xi(o)|] + \frac{M^*\sqrt{\pi}}{2}\,(\delta^* X+\delta^*\xi) \ . \tag{6.94}$$

Combining (6.89), (6.90) and (6.94), we get

$$|\Delta I_{2,i}| < M_{2,1}\sqrt{t}\ \delta^*\xi' + M_{2,2}t^{-1/2}[\delta^* X+|\delta\xi(o)|] \ , \tag{6.95}$$

where $M_{2,i}(i=1,2)$ are independent constants of $\delta^*\xi$, $\delta X$ and $\delta^*\xi'$.

We finally estimate $\Delta I_{3,i}$. We have

$$\Delta I_{3,i} = \int_\alpha^\beta \Theta(X(t),t,\xi)d\xi \int_0^{\delta X(t)} (-1)^i\,\Gamma_\xi(X(t)+\zeta+(-1)^i\xi,t)d\zeta \ . \tag{6.96}$$

By changing the order of integration and taking absolute values, we obtain

$$|\Delta I_{3,i}| < \frac{M\delta^{*}X}{\sqrt{\pi t}} . \tag{6.97}$$

Using the estimates (6.84), (6.95) and (6.97) and noting that the first term of (6.76) is estimated by means of (6.21), (6.22) and (6.28), we obtain, upon replacing M by $\delta^{*}\Theta$, the inequalities

$$\begin{aligned}
&\delta^{*}I_{1,i} < \tilde{K}_1\sqrt{t}\,\delta^{*}\Theta_{1,i} + M_{1,1}t\delta^{*}\xi' + M_{1,2}(\delta^{*}X+\delta^{*}\xi) \ ; \\
&\delta^{*}I_{2,i} < \tilde{K}_2[1+\sqrt{t}]\,\delta^{*}\Theta_{2,i} + M_{2,1}\sqrt{t}\,\delta^{*}\xi' + \frac{M_{2,2}}{\sqrt{t}}[\delta^{*}X+\delta^{*}\xi(o)] \ ; \\
&\delta^{*}I_{3,i} < \delta^{*}\Theta_{3,i} + M_{3,i}\,\frac{\delta^{*}X}{\sqrt{t}} ,
\end{aligned} \tag{6.98}$$

where $\tilde{K}_i$, $M_{i,j}$ and $M_i$ are constants independent of $\delta X, \ldots, \delta\Theta_{3,i}$. Here we assign the same indices to $\Theta$ as we assigned to the operators $I_{i,j}$.

Assuming now that

$$\begin{aligned}
&X(t) \equiv X_s(t), \text{ or } X(t) \equiv X = \text{constant } \varepsilon(o,X_s) \ ; \\
&\xi(t) \equiv X_s(t), \text{ or } \xi(t) \equiv o \ ; \\
&\delta X_s(o) = o ,
\end{aligned} \tag{6.99}$$

then for $o \le t \le \bar{t}$, $|\eta(t)| < N$ and $|\alpha_1^{-1/2}| < \frac{\bar{X}}{b}$, we obtain

$$\delta^{*}X,\ \delta^{*}\xi,\ \delta^{*}X_s \le \frac{t\delta^{*}\eta\bar{X}}{b} . \tag{6.100}$$

In what follows the symbols K and $\bar{K}$ denote arbitrary constants entering into the estimates, which are independent of the variation. Consequently, (6.98) can be written in the form

$$\delta^* I_{i,j} < K\sqrt{t}\,(\delta^*\Theta_{1,i} + \delta^*\eta) \;;$$

$$\delta^* I_{2,i} < K\{(1+\sqrt{t})\delta^*\Theta_{2,i} + \sqrt{t}\,\delta^*\eta\} \;; \tag{6.101}$$

$$\delta^* I_{3,i} < \delta^*\Theta_{3,i} + K\sqrt{t}\,\delta^*\eta \;.$$

We now estimate variations of the functionals $\Delta^i$, $U^i$, $F^i$, $i=1,2$, $\Phi$ and Z which are induced by the variations of V, W, $V_X$, $W_Y$, v, w, $V_o$ and $\eta$. On observing that the expressions $I_{3,j}$ contain the arguments $\phi$ or $\phi'$ ($\psi$ or $\psi'$ for the unfrozen phase case) which are not being varied, and writing $Z(t, s, V_m(X_s), W_m(Y_s), v, w) = Z(t, x, v, w)$ in the calculation of the variation of Z, then by means of (6.13)-(6.16), (6.6) and (6.101), we find after a lengthy but straightforward computation that

$$\begin{aligned}
\delta^*\Phi &< K\sqrt{t}\,(\delta^*V_o+\delta^*v+\delta^*\eta) \;;\\
\delta^*\Delta^1 &< K\sqrt{t}\,(\delta^*V_o+\delta^*v+\delta^*\eta) \;;\\
\delta^*\Delta^2 &< K\sqrt{t}\,(\delta^*w+\delta\eta^*) \;;\\
\delta^*U^1 &< K\sqrt{t}\,(\delta^*\eta) + \overline{K}(\delta^*V_o+\delta^*v) \;;\\
\delta^*U^2 &< K\sqrt{t}\,(\delta^*\eta) + \overline{K}(\delta^*w)\\
\delta^*F^1 &< K\sqrt{t}\,(\delta^*V_o+\delta^*v+\delta^*\eta) \;;\\
\delta^*F^2 &< K\sqrt{t}\,(\delta^*w+\delta^*\eta) \;;\\
\delta^*Z &< K\sqrt{t}\,(\delta^*\eta) + \overline{K}(\delta^*v+\delta^*w) \;.
\end{aligned} \tag{6.102}$$

Let $o \le \tau \le t$, $o \le \xi \le \overline{X}$ or $\overline{Y} \ge \xi \ge o$ and write

$$\delta^*V_o = \delta_n V_o = \max|V_{n,o}(\tau)-V_{n-1,o}(\tau)| \;;$$

$$
\begin{aligned}
\delta^{*}V &= \delta_n V = \max|V_n(\xi,\tau)-V_{n-1}(\xi,\tau)| \;; \\
\delta^{*}W &= \delta_n W = \max|W_n(\xi,\tau)-W_{n-1}(\xi,\tau)| \;; \\
\delta^{*}v &= \delta_n v = \max|v_n(\tau)-v_{n-1}(\tau)| \;; \\
\delta^{*}w &= \delta_n w = \max|w_n(\tau)-w_{n-1}(\tau)| \;; \\
\delta^{*}V_X &= \delta_n V_X = \max|V_{n,X}(\xi,\tau)-V_{n-1,X}(\xi,\tau)| \;; \\
\delta^{*}W_Y &= \delta_n W_Y = \max|W_{n,Y}(\xi,\tau)-W_{n-1,Y}(\xi,\tau)| \;; \\
\delta^{*}\eta &= \delta_n \eta = \max|\eta_n(\tau)-\eta_{n-1}(\tau)| \;.
\end{aligned}
\tag{6.103}
$$

Then the definitions (6.10) and estimates (6.102) imply

$$
\begin{aligned}
\delta_{n+1}V_o &< K\sqrt{t}\,(\delta_n V_o+\delta_n v+\delta_n \eta) \;; \\
\delta_{n+1}V &< K\sqrt{t}\,(\delta_{n+1}V_o+\delta_n v+\delta_n \eta) \;; \\
\delta_{n+1}W &< K\sqrt{t}\,(\delta_n w+\delta_n \eta) \;; \\
\delta_{n+1}v &< K\sqrt{t}\,(\delta_{n+1}V_o+\delta_n v+\delta_n \eta) \;; \\
\delta_{n+1}w &< K\sqrt{t}\,(\delta_n w+\delta_n \eta) \;; \\
\delta_{n+1}V_X &< K\sqrt{t}\,(\delta_n \eta) + \overline{K}(\delta_{n+1}V_o+\delta_{n+1}v) \;; \\
\delta_{n+1}W_Y &< K\sqrt{t}\,(\delta_n \eta) + \overline{K}(\delta_{n+1}w) \;; \\
\delta_{n+1} &< K\sqrt{t}\,(\delta_n \eta) + \overline{K}(\delta_{n+1}v+\delta_{n+1}w) \;.
\end{aligned}
\tag{6.104}
$$

Hence, there obviously exists some $t_{*}(t_{*} \le \overline{t})$ sufficiently small and L sufficiently large, so that if

$$
\delta_n = \max\{\delta_n V_o;\ \delta_n v;\ \delta_n w;\ \delta_n \eta\}
\tag{6.105}
$$

Then for $o \leq t \leq t_*$, we have

$$\delta_{n+1} < \sqrt{Lt} \cdot \delta_n , \tag{6.106}$$

which is equivalent to (6.71). This completes the proof of our assertion.

Now letting

$$\begin{aligned} &V = \lim_{n\to\infty} V_n;\ V_o = \lim_{n\to\infty} V_{n,o};\ W = \lim_{n\to\infty} W_n;\ V_X = \lim_{n\to\infty} V_{n,X}\ ; \\ &W_Y = \lim_{n\to\infty} W_{n,Y};\ v = \lim_{n\to\infty} v_n;\ w = \lim_{n\to\infty} w_n;\ \eta = \lim_{n\to\infty} \eta_n;\ s = \lim_{n\to\infty} s_n\ . \end{aligned} \tag{6.107}$$

Again by the uniform boundedness of the sequences (6.68), it follows that $V, \ldots, \eta$ satisfy the Lipschitz condition of the form (6.69) for $o \leq t \leq t_*$. Thus, it is possible to modify the assertion of Theorem 1 to read

Theorem 1'. The solution constructed for the system of integaral equations is simultaneously the solution to the original Problem IV satisfying in addition the Lipschitz condition of the form (6.69) with respect to each of their arguments.

## 7. Uniqueness Theorem. Stability of the Solution

(i) We now start with the proof of the uniqueness theorem. Let $(V, \ldots, \eta)$ and $(V^*, \ldots, \eta^*)$ be two sets of solutions to the system of integral equations belonging to the class of functions in which solutions have been shown to exist. Furthermore, let $(\Delta^1, \ldots, Z)$ and $(\Delta^{1*}, \ldots, Z^*)$ be the values of the functionals $\Delta^1, \ldots, Z$ corresponding to them. We then set

$$\bar{s}(t) = \min\,(s(t),\ s^*(t)) \tag{7.1}$$

and claim that the following inequalities are actually true:

$$\begin{aligned}
|\Delta^1(X,t,\ldots) - \Delta^{1*}(X,t,\ldots)| &< N(\sigma)L(t)\delta\ ;\\
|\Delta^2(Y,t,\ldots) - \Delta^{2*}(Y,t,\ldots)| &< N(\sigma)L(t)\delta\ ;\\
|U^1(X,t,\ldots) - U^{1*}(X,t,\ldots)| &< N(\sigma)L(t)\delta\ ;\\
|U^2(Y,t,\ldots) - U^{2*}(Y,t,\ldots)| &< N(\sigma)L(t)\delta\ ;\\
|\Phi^1(t,\ldots) - \Phi^{1*}(t,\ldots)| &< M\,L(t)\delta\ ;\\
|F^1(t,\ldots) - F^{1*}(t,\ldots)| &< M\,L(t)\delta\ ;\\
|F^2(t,\ldots) - F^{2*}(t,\ldots)| &< M\,L(t)\delta\ ;\\
|Z(t,\ldots) - Z^*(t,\ldots)| &< M\,L(t)\delta\ ,
\end{aligned} \tag{7.2}$$

where N and L are defined for $o \le t \le \bar{t}$, such that

$$\lim_{\sigma\to o} N(\sigma) = \infty\ ; \qquad \lim_{t\to o} L(t) = o\ , \tag{7.3}$$

M is a constant independent of $\delta$, and

$$\begin{aligned}\delta = \max\{&|V-V^*|\ ,\ |W-W^*|\ ,\ |V_X-V_X^*|\ ,\ |W_Y-W_Y^*|\ ,\ |V_o-V_o^*|\ ,\\ &|v-v^*|\ ,\ |w-w^*|\ ,\ |\eta-\eta^*|\}\end{aligned} \tag{7.4}$$

for

$$o < \sigma \le X \le X_{\bar{s}}(t) - \sigma \le X_{\bar{s}}(t);\ Y_{\bar{s}}(t) \ge Y_{\bar{s}}(t) - \sigma \ge Y \ge \sigma;\ o \le t \le \bar{t}\ ,$$

and $\sigma$ is a sufficiently small number.

Next fix $\sigma < o$. Then for $o \le t \le \bar{t}(\sigma)$ and $\sigma \le X \le X_{\bar{s}} - \sigma$, $Y_{\bar{s}} - \sigma \ge Y \ge \sigma$, we assert that

$$V = V^*;\ W = W^*;\ V_X = V_X^*;\ W_Y = W_Y^*;\ V_o = V_o^*;\ v = v^*;\ w = w^*;\ \eta = \eta^* . \tag{7.5}$$

For, since

$$\begin{aligned} &\delta\Delta^1 = \delta V;\ \delta\Delta^2 = \delta W;\ \delta U^1 = \delta V_X;\ \delta U^2 = \delta W_Y\ ; \\ &\delta F^1 = \delta v;\ \delta F^2 = \delta w;\ \delta\Phi = \delta V_o;\ \delta Z = \delta\eta\ , \end{aligned} \tag{7.6}$$

then Eq. (7.2) can be written in the form

$$\delta V;\ \delta W;\ \delta V_X;\ \delta W_Y;\ \delta v;\ \delta w;\ \delta\eta < \overline{M} L(t)\delta\ , \tag{7.7}$$

where $\overline{M} > o$ is a suitably chosen constant. Comparing (7.4) and (7.7), we see that for $\overline{t}(\sigma)$ sufficiently small and $\delta > o$ we have $\delta < \overline{M}\ L(t)\delta < \delta$, which is impossible. This proves our assertion.

Now considering $\overline{t}(\sigma)$ as a new initial moment in time, we prove the validity of (7.5) in the interval $(o, 2\overline{t}(\sigma))$. If we continue with this reasoning we see that (7.5) is valid on the interval $(o,\overline{t})$ of definition of the solutions $(V,\ldots, \eta)$ and $(V^*,\ldots, \eta^*)$ for $\sigma \le X \le X_{\underline{s}} - \sigma;\ Y_{\underline{s}} - \sigma \ge Y \ge \sigma$. Since $\sigma > o$ is small but artibrary and $V$, $V_X$ are continuous for $0 \le X \le X_s$, and $W$, $W_Y$ are continuous for $Y_s \ge Y \ge o$, we see that the uniqueness conditions are established in every interval $(o,\overline{t})$ in which the solution is defined.

It now remains to establish the validity of the estimate (7.2) in the class of functions satisfying a Lipschitz condition of the form (6.69) with respect to t. To this end, we first note that differentiability of the functions $V$, $V_o$, $W$, $V_X$, $W_Y$, $v$, $w$ and $\eta$ with respect to t is needed to estimate the integral $I_{2,i}$. Therefore, in this case the estimates (6.84) and (6.97) remain valid. We first note that

$$\Delta I_{2,i}(X,t,o,\tau|\Theta) = o \quad \text{for } X = \text{const.} \tag{7.8}$$

Furthermore, it is easily seen that for $o \le X \le X_s(t) - \sigma \le X_s(t)$ and the use of the fact that $\alpha \exp(-\alpha) <$ const.,

$$|\Delta I_{2,i}(X,t,X_s(\tau),\tau|\Theta)| < \frac{\text{const } \delta^* X_s}{\sigma} = N(\sigma)\delta^* X_s \ . \tag{7.9}$$

By the same reasoning, we obtain for $X_s(\tau) \ge \sigma_o > o$ and $o \le \tau \le t$ ,

$$|\Delta I_{2,2}(X_s(t),t,X_s(\tau),\tau|\Theta)| < M\delta^* X_s \ . \tag{7.10}$$

We now estimate $\Delta I_{2,1}(X_s(t),t,X_s(\tau),\tau|\Theta)$. We write

$$\begin{aligned}\Delta I_{2,1}(X_s(t),t,X_s(\tau),\tau|\Theta(\tau)) &= \Theta(t)\Delta I_{2,1}(X_s(t),t,X_s(\tau),\tau|1) \\ &\quad + \Delta I_{2,1}(X_s(t),t,X_s(\tau),\tau|\Theta(\tau)-\Theta(t)) \\ &= \Delta I'_{2,1} + \Delta I''_{2,1} \ .\end{aligned} \tag{7.11}$$

Using (6.95) with $\xi(\tau) \equiv X_s(\tau)$ and $X(t) \equiv X_s(t)$, we estimate $\Delta I'_{2,1}$ as

$$\begin{aligned}|\Delta I'_{2,1}| &\le M_{2,1}\sqrt{t}\ \delta^*\eta|\alpha_1^{-1/2}| + M_{2,2}t^{-1/2}[t\delta^*\eta|\alpha_1^{-1/2}|] \\ &\equiv M\ \sqrt{t}\ \delta^*\eta \ ,\end{aligned} \tag{7.12}$$

since $|\alpha_1^{-1/2}| < \overline{X}/b$.

As for $\Delta I''_{2,1}$, we have

$$\Delta I''_{2,1} = \int_o^t \frac{\Theta(\tau)-\Theta(t)}{4\sqrt{\pi(t-\tau)}}\ \frac{\nu^*(t,\tau)\exp(-\frac{\nu^{*2}(t,\tau)}{4(t-\tau)}) - \nu(t,\tau)\exp(-\frac{\nu(t,\tau)}{4(t-\tau)})}{t-\tau}\, d\tau \ , \tag{7.13}$$

where

$$\nu(t,\tau) = X_s(t) - X_s(\tau) = \int_{s(\tau)}^{s(t)} \alpha_1^{-1/2}(\lambda,t)d\lambda \ ,$$

$$\nu^*(t,\tau) = X_s^*(t) - X_s^*(\tau) = \int_{s^*(\tau)}^{s^*(t)} \alpha_1^{-1/2}(\lambda,t)d\lambda \ . \tag{7.14}$$

Let

$$|\Theta(\tau) - \Theta(t)| < A^*\sqrt{t-\tau} \ ; \ |\eta|,|\eta^*| < N \ . \tag{7.15}$$

Then

$$\Delta I''_{2,1} = I' + I''$$

$$= \int_0^t \frac{\Theta(\tau)-\Theta(t)}{4\sqrt{\pi(t-\tau)}} \frac{\nu^*(t,\tau)-\nu(t,\tau)}{t-\tau} \exp\left(- \frac{\nu^{*2}(t,\tau)}{4(t-\tau)}\right)d\tau \tag{7.16}$$

$$+ \int_0^t \frac{\Theta(\tau)-\Theta(t)}{4\sqrt{\pi(t-\tau)}} \frac{\nu(t,\tau)d\tau}{t-\tau} \int_{\nu(t,\tau)}^{\nu^*(t,\tau)} \frac{\zeta}{2(t-\tau)} \exp\left(- \frac{\zeta^2}{4(t-\tau)}\right)d\zeta$$

With the use of (7.14) and (7.15), the absolute values of I' and I" are estimated as

$$|\Delta I'| < \frac{A^*\overline{X}}{4\sqrt{\pi}} t\delta^*\eta, \quad |I''| < \frac{A^*\overline{X}\gamma_1}{4\sqrt{\pi}} t^{3/2}\delta^*\eta \ .$$

Therefore

$$|\Delta I''_{2,1}| < Mt\delta^*\eta \ , \tag{7.17}$$

where M is a constant, independent of $\delta^*\eta$.

From (7.10) and (7.11) together with (7.12) and (7.17), we deduce that

$$|\Delta I_{2,i}(X(t),t,X_s(\tau),\tau|\Theta)| < M\sqrt{t}\ \delta^*\eta \tag{7.18}$$

and

$$|\delta^* I_{2,i}| < K\{(1+\sqrt{t})\delta^*\Theta_{2,i} + \sqrt{t}\,\delta^*\eta\}, \tag{7.19}$$

which agrees with $(6.101_2)$. Thus we have the same estimate (6.101) and hence the validity of (7.2) follows immediately from the discussion leading to (6.106).

Finally, since the uniform boundedness of the sequences $\{\sqrt{t}\,\frac{\partial V_n}{\partial t}\},\ldots,\{\sqrt{t}\,\frac{d\eta_n}{dt}\}$ implies that the totality of the sequences $\{V_n\},\ldots,\{\eta_n\}$ uniformly satisfies the conditions (6.69), the uniqueness theorem is valid in every class of functions in which the existence of the solution is established.

The above discussion leads to the following result:

Theorem 3. There exists a certain value of time $\overline{t} > o$ depending on the data such that for $o < t < \overline{t}$, Problem IV is solved uniquely by the triple (V, W, s) as given by (6.107).

(ii) We now turn to investigate the stability of the solution under small variations of all known quantities appearing in Problem IV. Let $a_i$, $\Lambda_i$, $V_{i,m}$, $W_{i,m}$, $\phi_i$, $\psi_i$ and $Z_i$ for $i=1,2$ be two different sets of data, having the same differentiability properties. For definitness, we refer to the former as the unperturbed system and to the latter as the perturbed one. In addition, the perturbed data are assumed to satisfy the conditions arising from (6.5), (6.6) and (6.9) on the replacement of the unperturbed data by the perturbed ones. Furthermore, let the corresponding solutions $V_1,\ldots, s_1$ and $V_2,\ldots, s_2$ of the system of integral equations be defined simultaneously for $o \le t \le \overline{t}$, where for $o \le t \le \overline{t}$ and $o \le X \le \overline{X}$, $\overline{Y} \ge Y \ge o$,

$$|V_i|\ ,\ |V_{i,o}|\ ,\ |W_i|\ ,\ |V_{i,X}|\ ,\ |W_{i,Y}|\ ,\ |v_i|\ ,\ |w_i|\ ,\ |\eta_i| < N\ ;$$
$$o \le s_i(t) \le b_i;\ i = 1,2\ . \tag{7.20}$$

We can now prove the following continuous dependence theorem.

<u>Theorem 4</u>. For every $\varepsilon > o$ and for each small interval $o \le t \le t_*$ (with $t_* \le \bar{t}$ independent of $\varepsilon$), there exist a $\delta(\varepsilon, t_*) > o$ such that the inequalities

$$\Delta V;\ \Delta V_o;\ \Delta W;\ \Delta V_X;\ \Delta W_Y;\ \Delta v;\ \Delta w;\ \Delta\eta;\ \Delta s < \varepsilon \tag{7.21}$$

are fulfilled, provided the absolute values of the differences of $a_1$, $\Lambda_1$, $V_{1,m}$, $W_{1,m}$, $\phi_1$, $\psi_1$, $Z_1$ and respectively $a_2$, $\Lambda_2$, $V_{2,m}$, $W_{2,m}$, $\phi_2$, $\psi_2$, $Z_2$, and all their partial derivatives of the first order, taken at the same values of the argument satisfying the inequalities (7.20), do not exceed $\delta$. Here

$$\Delta V = \max_{\substack{o\le X\le X_S \\ o\le t\le t_*}} |V_1(X,t)-V_2(X,t)|;\ \Delta W = \max_{\substack{Y_S\ge Y\ge o \\ o\le t\le t_*}} |W_1(Y,t)-W_2(Y,t)|\ ;$$

$$\Delta V_X = \max_{\substack{o\le X\le X_S \\ o\le t\le t_*}} |V_{1,X}(X,t)-V_{2,X}(X,t)|;\ \Delta W_Y = \max_{\substack{Y_S\ge Y\ge o \\ o\le t\le t_*}} |W_{1,Y}(y,t)-W_{2,Y}(Y,t)|\ ;$$

$$\Delta v = \max_{o\le t\le t_*} |v_1(t)-v_2(t)|;\ \Delta w = \max_{o\le t\le t_*} |w_1(t)-w_2(t)|\ ; \tag{7.22}$$

$$\Delta\eta = \max_{o\le t\le t_*} |\eta_1(t)-\eta_2(t)|;\ \Delta s = \max_{o\le t\le t_*} |s_1(t)-s_2(t)|\ ;$$

$$\Delta V_o = \max_{o\le t\le t_*} |V_{1,o}(t)-V_{2,o}(t)|$$

and

$$s = s(t) = \min(s_1(t), s_2(t))\ . \tag{7.23}$$

Proof. Two cases must be considered.

Case 1. Assume that

$$a_1 = a_2 \,. \tag{7.24}$$

To estimate $\Delta V, \ldots, \Delta s$, we will use the estimates (6.98) and (6.100) of the integrals $I_{i,j}$. We may remark here that the estimate (6.98) remains valid for a nondifferentiable solution satisfying (6.75).

Let $\Theta_1(X, t, \gamma_1, \ldots, \gamma_n)$ and $\Theta_2(X, t, \gamma_1, \ldots, \gamma_n)$ be defined and differentiable for $o \le X \le \overline{X}$, $o \le t \le t_*$ and $|\gamma_k| < N (k=1,\ldots,n)$. Set

$$E_\Theta = \max_{\substack{o \le X \le \overline{X} \\ o \le t \le \overline{t}}} \left\{ |\Theta_1 - \Theta_2| ; \left|\frac{\partial\Theta_1}{\partial X} - \frac{\partial\Theta_2}{\partial X}\right| ; \left|\frac{\partial\Theta_1}{\partial t} - \frac{\partial\Theta_2}{\partial t}\right| ; \left|\frac{\partial\Theta_1}{\partial\gamma_k} - \frac{\partial\Theta_2}{\partial\gamma_k}\right| \right\} ,$$

$$k=1,\ldots,n \; ; \; |\gamma_k| < N \,. \tag{7.25}$$

In addition, let

$$|\Theta_i| ; \left|\frac{\partial\Theta_i}{\partial X}\right| ; \left|\frac{\partial\Theta_i}{\partial t}\right| ; \left|\frac{\partial\Theta_i}{\partial\gamma_k}\right| \le K ; \tag{7.26}$$
$$k=1,\ldots,n \; ; \; i=1,2,$$

where $K$ is a constant independent of $E_\Theta$ . We have

$$\begin{aligned} &|\Theta_1(X,t,V_1,V_{1,o},W_1,V_{1,X},W_{1,Y},v_1,w_1,\eta_1,s_1) - \\ &\quad - \Theta_2(X,t,V_2,V_{2,o},W_2,V_{2,X},W_{2,Y},v_2,w_2,\eta_2,s_2)| \\ &\quad \le E_\Theta + K(\Delta V + \Delta V_o + \Delta V_X + \Delta W_Y + \Delta v + \Delta w + \Delta\eta + \Delta s) \,. \end{aligned} \tag{7.27}$$

We now examine the variations of the functionals $\Delta^i$, $U^i$, $F^i$, $i = 1,2,\Phi$ and $Z$ induced by variations of the data of Problem IV and consequent variations of $V$, $V_o$, $W$, $V_X$, $W_Y$, $v$, $w$, $\eta$, and $s$.

According to (6.13)-(6.16) and the use of (6.101), (7.27) and the above notation, we obtain, as in the derivation of (6.102),

$$\begin{aligned}
\delta^*\Phi &< K\sqrt{t}\,(\Delta V_0+\Delta v+\Delta\eta) + K^*E \;; \\
\delta^*\Delta^1 &< K\sqrt{t}\,(\Delta V_0+\Delta v+\Delta\eta) + K^*E \;; \\
\delta^*\Delta^2 &< K\sqrt{t}\,(\Delta w+\Delta\eta) + K^*E \;; \\
\delta^*U^1 &< K\sqrt{t}\,(\Delta\eta) + \overline{K}(\Delta V_0+\Delta v) + K^*E \;; \\
\delta^*U^2 &< K\sqrt{t}\,(\Delta\eta) + \overline{K}(\Delta w) + K^*E \;; \\
\delta^*F^1 &< K\sqrt{t}\,(\Delta V_0+\Delta v+\Delta\eta) + K^*E \;; \\
\delta^*F^2 &< K\sqrt{t}\,(\Delta w+\Delta\eta) + K^*E \;; \\
\delta^*Z &< K\sqrt{t}\,(\Delta\eta) + \overline{K}(\Delta v+\Delta w) + K^*E \;.
\end{aligned} \tag{7.28}$$

Here K, $\overline{K}$ and $K^*$ are constants which are independent of $\Delta V$, $\Delta V_0$, $\Delta W$, $\Delta V_X$, $\Delta W_Y$, $\Delta v$, $\Delta w$, $\Delta\eta$ and E, and

$$E = \max\{E_\Lambda,\ E_Z,\ E_{V_m},\ E_{W_m},\ E_\phi,\ E_\psi\}\,. \tag{7.29}$$

Since $V_1,\ldots,s_1$ and, respectively, $V_2,\ldots,s_2$ are the solutions to the problems corresponding to comparable values of the data, it is possible to replace $\delta^*\Phi,\ldots,\delta^*Z$ by $\Delta V_0,\ldots,\Delta\eta$ in (7.28). Consequently, we find that

$$\Delta V;\ \Delta V_0;\ \Delta W;\ \Delta V_X;\ \Delta W_Y;\ \Delta v;\ \Delta w;\ \Delta\eta;\ \Delta s < HR \equiv \delta\ , \tag{7.30}$$

where H is a suitably chosen constant, growing with t but independent of $\Delta V,\ldots,\Delta s$. This proves the assertion of Theorem 4 for this case.

Case 2. Suppose that (7.24) is not true. Let $X_{10}$ and $X_{20}$ (and correspondingly $Y_{10}$ and $Y_{20}$) be the values of X (and Y) corresponding to $a_1$ and $a_2$ as defined in (6.1). We set

$$X^* = \frac{X_{10}}{X_{20}} X; \; Y^* = \frac{Y_{10}}{Y_{20}} Y; \; \xi^* = \frac{X_{10}}{X_{20}} \xi \text{ or } \xi^* = \frac{Y_{10}}{Y_{20}} \xi \;;$$

$$t^* = \frac{a_1^2}{a_2^2} t; \; \tau^* = \frac{a_1^2}{a_2^2} \tau; \; s^*(t^*) = \frac{a_1}{a_2} s_2(t); \; \eta^*(t^*) = \frac{a_2}{a_1} \eta_2(t) \;;$$

$$V^*(X^*,t^*) = V_2(X,t) \;;\; W^*(Y^*,t^*) = W_2(Y,t) \;; \tag{7.31}$$

$$V^*_{X^*}(X^*,t^*) = \frac{X_{20}}{X_{10}} V_{2,X}(X,t); \; W^*_{Y^*}(Y^*,t^*) = \frac{Y_{20}}{Y_{10}} W_{2,Y}(Y,t) \;;$$

$$V^*_o(t^*) = V_{2,o}(t); \; v^*(t^*) = \frac{X_{20}}{X_{10}} v_2(t); \; w^*(t^*) = \frac{Y_{20}}{Y_{10}} w_2(t) \;.$$

Moreover, set

$$V^*_m(X^*) = V_{2,m}(X); \; W^*_m(Y^*) = W_{2,m}(Y); \; \Lambda^*(t^*,V^*_o) = \frac{X_{20}}{X_{10}} \Lambda_2(t,V_{2,o})$$

$$\phi^*(X^*) = \frac{X_{20}}{X_{10}} \phi_2(X); \; \psi^*(Y^*) = \frac{Y_{20}}{Y_{10}} \psi_2(Y) \;; \tag{7.32}$$

$$Z^*(t^*,V^*_m,W^*_m,v^*,w^*,s^*) = \frac{a_2}{a_1} Z_2(t,V_{2,m},W_{2,m},v_2,w_2,s_2) \;.$$

Then we get the equations

$$V^*(X^*,t^*) = \Delta^1(X^*,t^*,V^*_o,v^*,\eta^*,s^*) \;;$$

$$V^*_o(t^*) = \Phi(t^*,V^*_o,v^*,\eta^*,s^*) \;;$$

$$W^*(Y^*,t^*) = \Delta^2(Y^*,t^*,w^*,\eta^*,s^*) \;;$$

$$V^*_{X^*}(X^*,t^*) = U^1(X^*,t^*,V^*_o,v^*,\eta^*,s^*) \;;$$

$$W^*_{Y^*}(Y^*,t^*) = U^2(Y^*,t^*,w^*,\eta^*,s^*) \;; \qquad (7.33)$$

$$v^*(t^*) = F^1(t^*,V^*_o,v^*,\eta^*,s^*) \;;$$

$$w^*(t^*) = F^2(t^*,w^*,\eta^*,s^*) \;;$$

$$\eta^*(t^*) = Z(t^*,V^*_m,W^*_m,v^*,w^*,s^*) \;;$$

$$s^*(t^*) = S(t^*,\eta^*) \;,$$

where, as above, $\Delta^i$, $U^i$, $F^i$, $i=1,2,\Phi$ and $Z$ are operators as in Section 5. We now have

$$\Delta V = \max|V_1(X,t) - V^*(X^*,t^*)| \;;$$

$$\Delta V_o = \max|V_1(t) - V^*_o(t^*)| \;;$$

$$\Delta W = \max|W_1(Y,t) - W^*(Y^*,t^*)| \;;$$

$$\Delta V_X = \max|V_{1,X}(X,t) - \frac{X_{10}}{X_{20}} V^*_{X^*}(X^*,t^*)| \;;$$

$$\Delta W_Y = \max|W_{1,Y}(Y,t) - \frac{Y_{10}}{Y_{20}} W^*_{Y^*}(Y^*,t^*)| \;; \qquad (7.34)$$

$$\Delta v = \max|v_1(t) - \frac{X_{10}}{X_{20}} v^*(t^*)| \;;$$

$$\Delta w = \max|w_1(t) - \frac{Y_{10}}{Y_{20}} w^*(t^*)| \;;$$

$$\Delta \eta = \max|\eta_1(t) - \frac{a_1}{a_2} \eta^*(t^*)| \;;$$

$$\Delta s = \max|s_1(t) - \frac{a_2}{a_1} s^*(t^*)| \;.$$

From (7.32), it follows that $V_1,\ldots, s_1$ and $V^*,\ldots, s^*$ correspond to equal values of a. Therefore the variations

$$\Delta\hat{}V = \max|V_1(x,t) - V^*(x,t)| \; ;$$
$$\cdots\cdots\cdots\cdots \qquad (7.35)$$
$$\Delta\hat{}s = \max|s_1(t) - s^*(t)|$$

can be estimated by (7.30). But the differences $|\Delta V - \Delta\hat{}V|,\ldots,$ $|\Delta s - \Delta\hat{}s|$ tend to zero together with $\tilde{\gamma} = |a_1 - a_2|$ by virtue of the continuity of $V_1,\ldots, s_1$ and $V_2,\ldots, s_2$ with respect to X, Y and t on the one hand, and the fact that $|X^*-X|$, $|Y^*-Y|$ and $|t^*-t|$ tend to zero together with $\gamma$ on the other. Thus, our assertion is valid for this case.

Generally speaking, the stability of the solution to the system of integral equations was proved in a certain small time interval $(o,t_*)$, rather than in an interval $(o,\bar{t})$ in which the solutions of the perturbed and unperturbed problems are simultaneously defined. However, since $t_*$ does not depend on $\varepsilon$, but only on the bounds of the data $\Lambda$, $V_m$, $W_m$, $\phi$, $\psi$, Z and their derivatives, and, moreover, on the bounds of the solutions $V_1,\ldots, s_1$ and $V_2,\ldots, s_2$, it is possible to extend the conclusions of smallness of $\Delta V,\ldots, \Delta s$ to the entire interval $(o,\bar{t})$.

Indeed, let $t_*$ be the greatest length of a time interval on which the inequalities

$$|a_1-a_2| < \delta; \; E_\Lambda^{t_o,t_*}, E_Z^{t_o,t_*} < \delta; \; E_{V_m}, E_{W_m}, E_{V(X,t_o)}, E_{W(Y,t_o)} < \delta \qquad (7.36)$$

imply the inequalities

$$\Delta_{t_o}V;\ \Delta_{t_o}V_o;\ \Delta_{t_o}W;\ \Delta_{t_o}V_X;\ \Delta_{t_o}W_Y;\ \Delta_{t_o}v;\ \Delta_{t_o}w;\ \Delta_{t_o}\eta;\ \Delta_{t_o}s < \varepsilon \qquad (7.37)$$

Here $E_\Theta^{t_o,t_*}$ is defined exactly as $E_\Theta$ above but on the interval $t_o \leq t \leq t_* + t_o$ instead of on the interval $o \leq t \leq \overline{t}$, and

$$\Delta_{t_o}V = \max_{\substack{t_o \leq t \leq t_o + t_* \\ o \leq X \leq X_{\overline{s}}}} |V_1(X,t) - V_2(X,t)| \ ;$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \qquad (7.38)$$

$$\Delta_{t_o}s = \max_{t_o \leq t \leq t_o + t_*} |s_1(t) - s_2(t)| \ .$$

Again, since $t_*$ is independent of $\varepsilon$, $\delta$ and $t_o$, then we can take $\varepsilon(\delta)$ to be a monotonically increasing function such that (7.21) holds when $\varepsilon$ is replaced by $\varepsilon(\delta)$. Let $\delta(\varepsilon)$ be the inverse of this function. In addition, let n be a positive integer such that

$$n-1 < \frac{\overline{t}}{t_*} \leq n \ . \qquad (7.39)$$

Put $t_k = kt_*$ $(k=o,\ldots, n-1)$. Next, choose $\varepsilon > o$ arbitrarily small and define the sequence

$$\varepsilon_{n-1} = \varepsilon;\ \varepsilon_{n-k} = \delta(\varepsilon_{n-k+1}),\ (k=1,\ldots, n-1);\ \delta = \varepsilon_o \ . \qquad (7.40)$$

With this in mind, the inequalities

$$|a_1 - a_2| < \delta;\ E_\Lambda^{o,t_*},\ E_Z^{o,t_*} < \delta;\ E_{V_m}, E_{W_m},\ E_{V(X,o)=\phi},\ E_{W(Y,o)=\psi} < \delta \qquad (7.41)$$

imply the validity of (7.37) on each of the intervals $(t_k, t_{k+1})$ $(k=o,\ldots, n-1)$, and at the same time on the entire interval $(o,\overline{t})$. Hence the proof of Theorem 4 is complete.

## 8. The Nonlinear Problem

In this section, we go back to the original Problem I. First of all, Eq. (5.10) contains the variables X and t, while Eq. (5.16) contains the variables Y and t. It is now desirable to express x in terms of X and Y in the respective domains of definition, since the original problem was formulated in terms of x and t. From Eq. (4.6), the relationship between x and X is given by

$$x = \int_0^X \alpha_1^{1/2}\, dX \tag{8.1}$$

or, by Eqs. (4.13), (4.5) and (4.30),

$$\begin{aligned} x &= \int_0^X \alpha_1^{1/2}\, dX = \int_0^X \exp(-AQ_1^*)dX \\ &= \int_0^X V(X,t)\exp[-\tfrac{A\ell}{2}X-(\tfrac{A\ell}{2})^2 t]dX \quad \text{for } o \le X \le X_s \; . \end{aligned} \tag{8.2}$$

Similarly, the relationship between x and Y can be obtained from Eq. (4.35):

$$x = b - \int_0^Y \alpha_2^{1/2}\, dY \tag{8.3}$$

and, by Eqs. (4.44), (4.36) and (4.56),

$$\begin{aligned} x &= b - \int_0^Y \alpha_2^{1/2}\, dY = b - \int_0^Y \frac{1}{\rho_2(Q^*)} \exp(-BQ_2^*)dY \\ &= b - \int_0^Y \frac{1}{\rho_2(W)} W(Y,t)dY \quad \text{for } Y_s \ge Y \ge o \; . \end{aligned} \tag{8.4}$$

As a consequence of Eqs. (8.1) and (8.3), we have at the freezing front

$$s(t) = \int_o^{X_s} \alpha_1^{1/2}\, dX = b - \int_o^{Y_s} \alpha_2^{1/2}\, dY\ . \tag{8.5}$$

The expressions $V = V(X,t)$ and $x = x(X,t)$, given, respectively, by (5.10) and (8.2), represent the solution to Problem I for $o \le X \le X_s$, while the expressions $W = W(Y,t)$ and $x = x(Y,t)$, given, respectively, by (5.16) and (8.4), represent its solution in $Y_s \ge Y \ge o$. In fact, for a given instant of time, this solution of Problem I is given until now in parametric form, where the parameter is X in $o \le X \le X_s$, and is Y in $Y_s \ge Y \ge o$.

Once V is calculated as a function of X and t, $Q_1^*$ can be calculated as a function of X and t by means of the expression $Q_1^* = -\frac{1}{A}\ln V + \frac{\ell}{2}X + \frac{A\ell^2}{2}t$ . Also, having obtained W as a function of Y and t, $Q_2^*$ can be evaluated as a function of Y and t using the relation $Q_2^* = -\frac{1}{B}\ln W$. But x can be calculated for a given X and t by means of (8.2) and for a given Y and t by (8.4); which means that $Q_1$ and $Q_2$ are then known as functions of x and t, (see Eqs. (4.5) and (4.36)). Once $Q_i$ is known, $T_i$ can be obtained from the relation

$$Q_i = \int^{T_i} [K_i(T)R_i(T)]^{1/2}\, dT \qquad (i=1,2)\ .$$

Therefore, although the calculation of $T_i (i=1,2)$ as a function of x and t is tedious, the process is straightforward and can be carried out.

The discussion and the results of Sections 6 and 7 yield the following fundamental result of this chapter.

Theorem 5. Problem I is locally well-posed.

Remark. Having obtained the unique solution $(T_1, T_2, s)$ to Problem I, we substitute for $s = s(t)$ in the expressions of $\rho_i(x,t|s)$, $v_i(x,t|s)$, $C_i(x,t|s)$, $P(x,t|s)$ and $T(P)$ described in the previous sections. However, the calculation of $P(x,t|s)$ and $v_2(x,t|s)$ solving Problem II uniquely is the core of Chapter 3. In the same way, the calculation of the displacement $u(x,t|s)$, and hence the thermal stresses, $v_1 = \partial u/\partial t$ and $\rho_1$, (see Eq. (2.22)), which solves Problem III uniquely represents the main goal of Chapter 4.

# CHAPTER THREE - HYDRODYNAMICS OF THE UNFROZEN PHASE

Throughout this chapter, all physical quantities that we will encounter, belong to the liquid phase. Therefore, we will drop the suffix 2 characterizing quantities of this phase.

## 9. The Method of Characteristics Applied to the One-Dimensional Liquid Phase Flow

In this section, we shall investigate the existence and uniqueness of solutions to the second problem stated in Section 2. For convenience, it is restated here:

Problem II. Find a pair $(P(x,t), v(x,t))$ satisfying the following conditions:

$$\left.\begin{aligned} &\text{(II.1)} \quad \frac{D\rho}{Dt} + \rho\frac{\partial v}{\partial x} = o \\ &\text{(II.2)} \quad \rho\frac{Dv}{Dt} + \frac{\partial P}{\partial x} = o \end{aligned}\right\} \quad \begin{aligned} &\text{in } \Omega_2 = \{(x,t)\colon s(t) < x < b,\ o < t < \tilde{t}\} \\ &\text{with } \rho = \rho(P) \end{aligned}$$

$$\left.\begin{aligned} &\text{(II.3)} \quad P(x,o) = P_o(x) \\ &\text{(II.4)} \quad v(x,o) = r(x) \end{aligned}\right\} \quad \text{for } a \le x \le b$$

$$\left.\begin{aligned} &\text{(II.5)} \quad v(b,t) = o \\ &\text{(II.6)} \quad P(s(t),t) = g(s(t)) \end{aligned}\right\} \quad \text{for } o \le t \le \tilde{t}$$

and the compatibility conditions

$$\text{(II.7)} \quad r(b) = o\ , \quad P_o(a) = g(a)\ .$$

Here $P_o$, r and g are known functions of their respective arguments, which are assumed to be continuously differentiable. We remark again that in this problem, $s(t)$ is assumed to be known.

As stated in Section 2, the pressure in liquids does not depend noticeably upon the entropy. In other words, the influence of changes in entropy is negligibly small, and so in many practical applications, the specific entropy can be everywhere approximated without significant error by a fixed value, say $S_0$. In particular, this approximation is very good for liquid water. In gas dynamics, a flow satisfying this idealization is called homentropic flow. In this case, the liquid flow can be described by two functions: the density $\rho(x,t)$, the pressure $P(x,t)$, or the sound speed $c(x,t)$. These variables are uniquely related to each other at every point by the purely thermodynamic relations:

$$\rho = \rho(P),\ c = c(\rho),\ \text{or } P = P(\rho),\ c = c(\rho)\ , \tag{9.1}$$

where in both cases $c^2 = dP/d\rho$. If we choose the first set in (9.1), then (II.1) becomes

$$\frac{1}{\rho c}\frac{DP}{Dt} + c\,\frac{\partial v}{\partial x} = o\ . \tag{9.2}$$

To put the equations of motions (II.2) and (9.2) in characteristic form, we add (9.2) to (II.2) and subtract (9.2) from (II.2) to obtain the two equations

$$\left(\frac{\partial v}{\partial t} \pm \frac{1}{\rho c}\frac{\partial P}{\partial t}\right) + (v \pm c)\left(\frac{\partial v}{\partial x} \pm \frac{1}{\rho c}\frac{\partial P}{\partial x}\right) = o \tag{9.3}$$

or

$$\left[\frac{\partial}{\partial t} + (v \pm c)\,\frac{\partial}{\partial x}\right](v \pm F) = o\ , \tag{9.4}$$

where the new thermodynamic function $F(p,S_0)$ is defined by

$$F \equiv \int^{P} \frac{dP}{\rho c} \ . \tag{9.5}$$

In this form, each equation contains only one differential operator. By definition, an equation is in characteristic form if it contains only one differential operator; equations (9.4) are thus the equations of motion in characteristic form.

Equations (9.4) state that $v \pm F$ are constant along the curves $C^{\pm}$ defined as the solutions to the differential equations

$$\frac{dx}{dt} = v \pm c \ . \tag{9.6}$$

Evidently, $C^+$ ( or also $C^-$) is the locus of a point moving forward, i.e., in the positive x-direction (respectively, backwards, i.e., in the negative x-direction) at the local wave speed c relative to the local fluid velocity v. These curves are called characteristics and are physically identified with sound waves, which may be thought of as infinitesimal signals of the pressure disturbance. The $C^{\pm}$ characteristics serve as carriers of information. The paths of characteristics are not known in advance (except in the acoustic case) because the defining relations for the characteristics, equations (9.6), depend on the solution yet to be found.

By (9.4), the quantities $(v+F)$ and $(v-F)$ do not vary along their respective characteristics and are called the Riemann invariants. These invariants will be labeled

$$J^+ \equiv v + F \ , \ J^- \equiv v - F \ . \tag{9.7}$$

In general, the value of $J^{\pm}$ will vary from one $C^{\pm}$ characteristic to another. Equations (9.4) are rewritten in this notation as

$$dJ^{+} = o \ , \ J^{+} = \text{const} \quad \text{along} \quad C^{+}: \frac{dx}{dt} = v + c \ ;$$
$$dJ^{-} = o \ , \ J^{-} = \text{const} \quad \text{along} \quad C^{-}: \frac{dx}{dt} = v - c \ . \tag{9.8}$$

This statement can be regarded as a generalization of relations which hold for the case of acoustic waves propagating through a gas with constant velocity, density, and pressure. In fact, these relations may be obtained from the general expressions for the invariants as a first approximation.

We note that (9.5) can be written, alternatively, as

$$F = \int^{\rho} c \, \frac{d\rho}{\rho} \ . \tag{9.9}$$

For a liquid satisfying Tait's equation (see Eq. (2.7)),

$$P = A_s \rho^{\gamma} - B_s \ , \tag{2.7}$$

we have $F = 2c/(\gamma - 1)$ and the Riemann invariants are just

$$J^{+} = v + \frac{2c}{\gamma - 1} \ , \ J^{-} = v - \frac{2c}{\gamma - 1} \ . \tag{9.10}$$

We remark here that the Riemann invariants are determined to within an arbitrary constant, which can always be dropped for convenience, as was done above in (9.10).

It is now convenient to introduce the following notation:

$J_{\alpha}^{\pm}(x,t)$, where $\alpha$ is a positive number not necessarily an integer, means the initial value of the Riemann invariant $J^{\pm}$ at the point

$(x_\alpha, o)$ on the initial line MQ where $x_\alpha$ depends on the point $(x,t)$ at which we seek a solution to Problem II. That is, we write

$$J_\alpha^\pm(x,t) \equiv J^\pm(x_\alpha, o) \quad \text{with} \quad x_\alpha = x_\alpha(x,t) \ , \ \alpha > o \ . \tag{9.11}$$

Thus, for example, $J_1^+(x,t) = J^+(x_1,o)$ with $x_1 = x_1(x,t)$, and $J_2^+(s(t),t) \equiv J^+(x_2,o)$ with $x_2 = x_2(s(t),t)$. In fact, we have upon the application of the initial conditions

$$J_\alpha^\pm(x,t) = r(x_\alpha) \pm \int^{P_o(x_\alpha)} \frac{dP}{\rho c} \ , \ \alpha > o \ . \tag{9.12}$$

It is now desired to find a solution to Problem II at a general point $(x,t)$ in $\Omega_2$. (See Figure 2). Choose first a typical point $(x,t) \equiv D_1$ in the region 1, the region of determinacy. Obviously, the solution at $D_1$ is determined completely by the initial data on the line MQ. Indeed, use of (9.8) gives

$$v(D_1) + F(D_1) = J_1^+(x,t), \ v(D_1) - F(D_1) = J_6^-(x,t) .$$

Solving these relations for the unknowns $v(D_1)$ and $F(D_1)$ gives

$$v(D_1) = \tfrac{1}{2}(J_1^+(x,t) + J_6^-(x,t)), \ F(D_1) = \tfrac{1}{2}(J_1^+(x,t) - J_6^-(x,t)) \tag{9.13}$$

The value of F in (9.13) with the help of (9.5) gives the corresponding value of P. This value together with the value of v in (9.13) constitute the desired solution.

Next the $C^-$ characteristic emanating from any $M_\alpha$ for some $\alpha$ and intersecting the phase boundary $x = s(t)$ carries the initial value $J_\alpha^-(x,t)$. We then have, on the interface s,

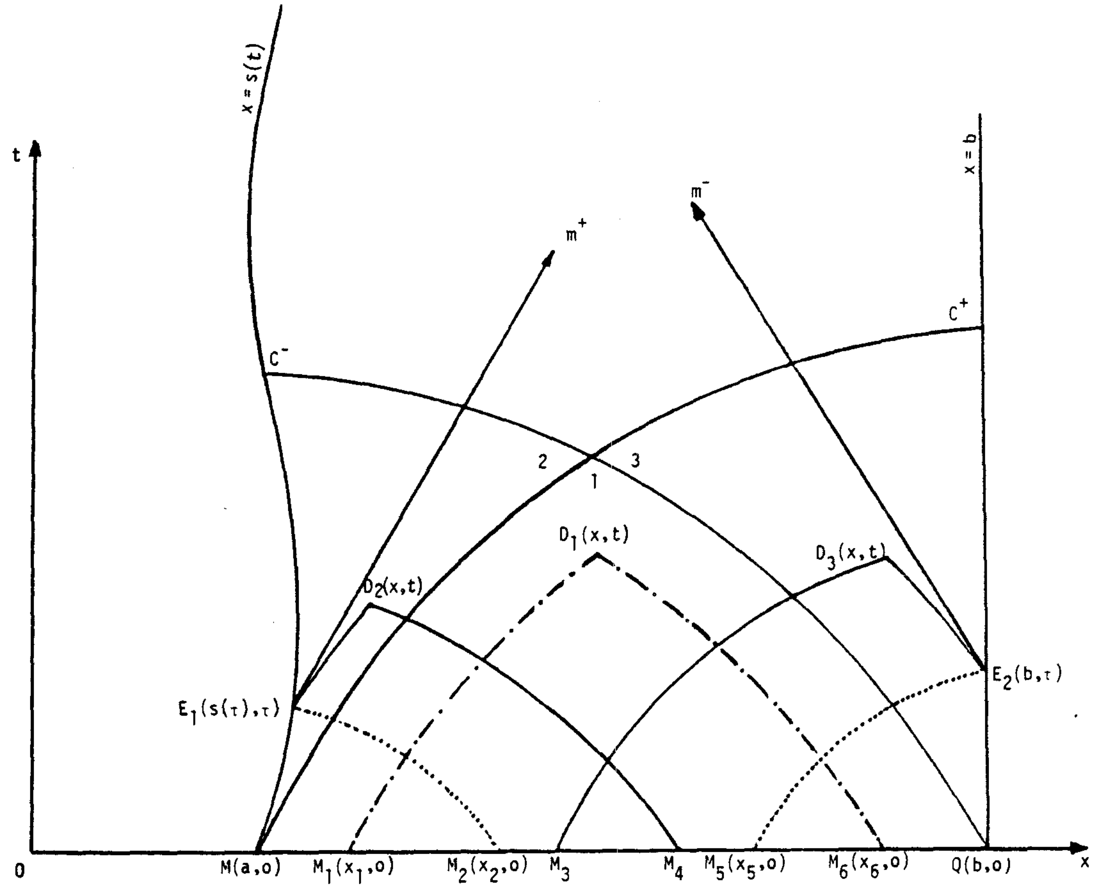


Figure 2. Sketch of characteristics for the unfrozen phase.

$$J_{\alpha}^{-}(s(t),t) = v(s(t),t) - F(s(t),t)$$

so that

$$v(s(t),t) = F(s(t),t) + J_{\alpha}^{-}(s(t),t) \ , \tag{9.14}$$

where

$$F(s(t),t) = \int^{g(s(t))} \frac{dP}{\rho c} \ . \tag{9.14*}$$

We remark here that (9.1) gives at $E_1$

$$\rho(s(t),t) = \rho(g(s(t))) \ . \tag{9.15}$$

In particular, for a liquid satisfying (2.7), the pressure is given explicitly by

$$\rho(s(t),t) = \left[\frac{B_s + g(s(t))}{A_s}\right]^{1/\gamma} . \tag{9.16}$$

We might call the attention to the fact that equation (9.15), or in particular (9.16), was used in Problem I to express the quantities $[\rho_1 L + \rho_2(C_1 - C_2 T(P)]$ and $[\rho_2 v_2(c_1 - c_2)T(P)]$ as functions of the phase boundary $s(t)$. At the same time, (II.6) was employed in Section 2 to express the free boundary condition for the displacement, (III.5), in terms of $s(t)$.

From Eqs. (9.14) and (9.14*), we see that both $v$ and $F$ are known at the freezing front. To find a solution at a typical point $(x,t) \equiv D_2$ near the phase boundary $x = s(t)$ in the region 2, (see again Figure 2), we have to know $J^+$ and $J^-$ at this point. Since $J^-(D_2) = J_4^-(x,t)$, it remains to determine $J^+(D_2)$. In doing so, we consider the $C^+$ characteristic issuing from $E_1$ after reflection on $s$ and passing through the point $D_2$. We then define the tangent line at $E_1$ to that $C^+$ characteristic by

$$m^+: \quad x = s(\tau) + [F(s(\tau),\tau) + J_2^-(s(\tau),\tau) + c(s(\tau),\tau)](t-\tau) , \tag{9.17}$$

and prescribe the Riemann invariant $J^+(s(\tau),\tau)$ along that $C^+$ characteristic by

$$J^+(s(\tau),\tau) \equiv v(s(\tau),\tau) + F(s(\tau),\tau) = 2F(s(\tau),\tau) + J_2^-(s(\tau),\tau) \tag{9.18}$$

with $\tau(x,t)$ determined implicitly from (9.17).

Now, on setting

$$v(D_2) \equiv \frac{1}{2}\{J^+[s(\tau(x,t)),\ \tau(x,t)] + J_4^-(x,t)\}\ ,$$
$$F(D_2) \equiv \frac{1}{2}\{J^+[s(\tau(x,t)),\ \tau(x,t)] - J_4^-(x,t)\}\ , \tag{9.19}$$

we readily see that the Riemann invariants $J^-(D_2) = J_4^-(x,t)$ and $J^+(D_2) = J^+[s(\tau(x,t)),\tau(x,t)]$ at $D_2(x,t)$ satisfy (9.4). Using (9.5) to find $P(D_2)$ corresponding to $F(D_2)$ in ($9.19_2$), then the value of P just obtained and v given by ($9.19_1$) form a solution to the couple of equations (II.1) and (II.2) satisfying the boundary condition (II.6). Furthermore, this solution is unique since Eq. (9.17) has only one root $\tau = \tau(x,t)$.

On the fixed end, we apply a similar procedure. The $C^+$ characteristic issuing from $M_\alpha$ on the initial line MQ for some $\alpha$ will carry the initial value $J_\alpha^+(x,t)$, and on boundary $x = b$, we have

$$v(b,t) = o \quad , \quad F(b,t) = J_\alpha^+(b,t)\ . \tag{9.20}$$

Again at a given point $(x,t) \equiv D_3$ near the fixed boundary $x = b$ in the region 3, at which we seek the solution of (II.1) and (II.2), we want to evaluate $J^+$ and $J^-$. We have $J^+(D_3) = J_3^+(x,t)$. Next we consider the $C^-$ characteristic emanating from $E_2(b,\tau)$ after reflection on the boundary $x = b$ and passing through the point $D_3$. Then we define the tangent line at $E_2$ to that $C^-$ characteristic by

$$m^-:\quad x = b - [c(b,\tau)](t-\tau) \tag{9.21}$$

and define the Riemann invariant $J^-(b,\tau)$ along that $C^-$ characteristic by

$$J^-(b,\tau) \equiv -F(b,\tau) = -J_5^-(b,\tau) \tag{9.22}$$

with $\tau(x,t)$ given implicitly by (9.21).

Then, on letting

$$\begin{aligned} v(D_3) &\equiv \frac{1}{2}\{J_3^+(x,t) - J_5^-[b,\tau(x,t)]\} , \\ F(D_3) &\equiv \frac{1}{2}\{J_3^+(x,t) + J_5^-[b,\tau(x,t)]\} , \end{aligned} \tag{9.23}$$

we see at once that the Riemann invariants $J^+(D_3) = J_3^+(x,t)$ and $J^-(D_3) = - J_5^-[b, (x,t)]$ constitute a unique solution to the system (9.4). Again use (9.5) to find $P(D_3)$ corresponding to $F(D_3)$ in $(9.23_2)$ so that this value of P together with the value of v in $(9.23_1)$ satisfy (II.1), (II.2) and (II.5). Moreover, this solution is unique because Eq. (9.21) possesses only one root $\tau = \tau(x,t)$.

Naturally, for nonlinear systems one must remain aware of the fact that statements are always meant to be valid merely for sufficiently small regions. With this in mind, we can summarize the above discussion in the following main result of this section.

Theorem 6. Assume that $r(x)$, $P_0(x)$ and $g(x)(o \le x \le b)$ are continuously differentiable functions. Suppose also the compatibility conditions (II.7) hold. Then there exists a unique solution $v(x,t)$, $P(x,t)$ of the system (II.1)-(II.6) for all $t < t^*$ for some positive value $t^* \le \tilde{t}$. Moreover, this solution is continuously differentiable.

## 10. The Evolution of Discontinuities in the Solution of Problem II

The development of discontinuities in solutions of nonlinear hyperbolic equations possessing discontinuous initial data has been the subject of research for many years. (See, for example, Jeffrey [19] and the references cited there). On the other hand, the evolution of discontinuities from smooth initial data was apparently first examined in a simple problem by Riemann [28]. His conjecture in the context of the initial-value problem for unsteady isentropic perfect gas flow was re-examined by Ludford [26]. The latter developed a variant of the hodograph method which basically is unfolding process for initial curves in the hodograph plane, and then applied his method to obtain an asymptotic estimate for the earliest time of breakdown in the solution of the original set of equations. At this time the derivative of the solution becomes unbounded. The work of Lax [25] and Jeffrey [19] employ the Riemann invariants to develop comparison theorems which provide upper and lower bounds for the critical time of singularity occurrence. Finally, an ingenious approach to calculate the critical time for singularity formation was developed by Ames [1], which is useful for a certain class of problems. The advantages of this method, when applicable, are its simplicity and the exactness of the results as opposed to asymptotic estimates or bounds. However, Jeffrey's work [19] is more general, and he has given several examples utilizing his theorems. Consequently, we shall apply the Jeffrey-Lax method to examine Problem II for wave breakdown, and as correspondingly was done in Ludford's paper, convert the original problem to a pure initial value problem by a suitable smooth extension outside the interval $a \le x \le b$.

The governing equations for the liquid phase problem [see Eqs. (II.1) and (II.2) of Sec. 9] are

$$\overline{V}_t + \overline{M}\,\overline{V}_x = o \tag{10.1}$$

in which

$$\overline{V} = \begin{bmatrix} v \\ P \end{bmatrix}, \ \overline{M} = \begin{bmatrix} v & \frac{1}{\rho} \\ \rho c^2 & v \end{bmatrix} . \tag{10.1*}$$

The characteristic values of $\overline{M}$ are

$$\lambda^+ = v + c \ , \ \lambda^- = v - c \tag{10.2}$$

and the corresponding left eigenvectors are

$$[1 \, , \frac{1}{\rho c}] \ , \ [1 \, , - \frac{1}{\rho c}] \ , \tag{10.2*}$$

respectively. Then from Reference [20], we have the Riemann invariant relationships

$$v + \int^P \frac{dP}{\rho c} = -J^+ \quad \text{along the } C^+ \text{ characteristics,} \tag{10.3}$$

and

$$v - \int^P \frac{dP}{\rho c} = -J^- \quad \text{along the } C^- \text{ characteristics.} \tag{10.4}$$

The Riemann invariants in (10.3) and (10.4) are the same as in (9.7), except for the minus signs which are introduced to make $(\partial\lambda^+/\partial J^+)$ and $(\partial\lambda^-/\partial J^-)$ negative as required in Jeffrey's analysis. From (10.3) and (10.4), we obtain

$$J^+ + J^- = -2v \ , \ J^+ - J^- = -2 \int^P \frac{dP}{\rho c} . \tag{10.5}$$

Let $t_{inf}$ ($t_{sup}$) denote the best estimate obtainable by Jeffrey's method of the lower (upper) bound for the time of existence of a solution to the original system of equations (10.1). Then the actual value $t_c$ of the time of existence of a solution to the original system (10.1) must satisfy the inequality

$$t_{inf} < t_c < t_{sup} \; . \tag{10.6}$$

The numbers $t_{inf}$ and $t_{sup}$ are to be interpreted in the sense that the solution is certainly bounded for $t < t_{inf}$, while the solution is certainly unbounded for $t > t_{sup}$. Note that $t_c \le t^*$, where $t^*$ is given in Theorem 6.

In order to utilize the estimates for the critical time provided in Jeffrey's paper, we need to determine $(\partial\lambda^+/\partial J^+)$, $(\partial\lambda^+/\partial J^-)$, $(\partial\lambda^-/\partial J^+)$ and $(\partial\lambda^-/\partial J^-)$, which will be denoted by $(\lambda_{+,+})$, $(\lambda_{+,-})$ $(\lambda_{-,+})$ and $(\lambda_{-,-})$, respectively.

From (10.2) and (10.5), we obtain

$$\lambda_{+,+} = \frac{\partial v}{\partial J^+} + \left(\frac{dc}{d\rho}\right)\left(\frac{d\rho}{dP}\right)\left(\frac{\partial P}{\partial J^+}\right)$$

$$= -\frac{1}{2}(1+\phi) \quad ,$$

where $\phi \equiv d \ln c/d \ln \rho$, and by similar reasoning

$$\lambda_{+,-} = \frac{1}{2}(-1+\phi) = \lambda_{-,+} \; ; \; \lambda_{-,-} = -\frac{1}{2}(1+\phi) \; .$$

The initial values of $v$ and $P$, specified on $a < x < b$, determine the initial values

$$J_0^{\pm}(x) \equiv J^{\pm}(x,o) = -r(x) \mp \int^{P_0(x)} \frac{dP}{\rho c} \quad \text{on} \quad a < x < b \; .$$

Furthermore, the boundary conditions that are to be imposed are $v(b,t) = o$ and $P(s(t),t) = g(s(t))$ for all $t \ge o$. So, by virtue of (10.5), we may write

$$J_o^+(b) + J_o^-(b) = o \ , \ J_o^+(a) - J_o^-(a) = -\int^{g(a)} \frac{dP}{\rho c} \ .$$

Hence, by a suitable smooth extension of $J_o^+(x) \pm J_o^-(x)$ everywhere, the original problem becomes a pure initial value problem, and the boundary conditions may be disregarded.

Considering the pure initial value problem just defined and using Jeffrey's estimates for $t_{inf}$ and $t_{sup}$ we see that when $\max(\partial J^+/\partial x)_{t=o}$ and $\max(\partial J^-/\partial x)_{t=o}$ are both positive, $t_{inf}$ is the smaller of the two numbers

$$\frac{2}{\max\limits_{J^+,J^-}[(1+\phi)\exp\{\frac{(J_*^- - J^-)(-1+\phi)}{4c_o(x)}\}]\max(\frac{\partial J^+}{\partial x})_{t=o}} \ , \tag{10.7}$$

$$\frac{2}{\max\limits_{J^+,J^-}[(1+\phi)\exp\{\frac{(J_*^+ - J^+)(1-\phi)}{4c_o(x)}\}]\max(\frac{\partial J^-}{\partial x})_{t=o}} \ . \tag{10.8}$$

Similarly, $t_{sup}$ is the smaller of the two numbers

$$\frac{2}{\min\limits_{J^+,J^-}[(1+\phi)\exp\{\frac{(J_*^- - J^-)(-1+\phi)}{4c_o(x)}\}]\max(\frac{\partial J^-}{\partial x})_{t=o}} \ , \tag{10.9}$$

$$\frac{2}{\min\limits_{J^+,J^-}[(1+\phi)\exp\{\frac{(J_*^+ - J^+)(1-\phi)}{4c_o(x)}\}]\max(\frac{\partial J^-}{\partial x})_{t=o}} \ . \tag{10.10}$$

In these expressions, it is assumed, as was done by Ludford [26], that $J^+(x)$ and $J_o^-(x)$ (and so also $J^+$ and $J^-$, since they are constant along their respective characterisitcs) differ only slightly from their constant values $J_*^+$ and $J_*^-$.

A much simplified version of these results was obtained by Lax [25] using different comparison theorems in which he assumed that $J^+ = J_*^+$, $J^- = J_*^-$ in order to study the existence of solutions of a certain nonlinear string equation.

For a liquid satisfying Tait's equation (2.7), $\phi = (\gamma-1)/2$ so that $(1+\phi) = (1+\gamma)/2$ and $(1-\phi) = (3-\gamma)/2$. Therefore, $t_{inf}$ becomes the smaller of the two numbers

$$\frac{4}{(1+\gamma)\max\limits_{J^-}[\exp\{\frac{(\gamma-3)(J_*^- - J^-)}{8c_o(x)}\}]\max(\frac{\partial J^+}{\partial x})_{t=o}}, \tag{10.11}$$

$$\frac{4}{(1+\gamma)\max\limits_{J^+}[\exp\{\frac{(3-\gamma)(J_*^+ - J^+)}{8c_o(x)}\}]\max(\frac{\partial J^-}{\partial x})_{t=o}}. \tag{10.12}$$

and $t_{sup}$ becomes the smaller of the two numbers

$$\frac{4}{(1+\gamma)\min\limits_{J^-}[\exp\{\frac{(\gamma-3)(J_*^- - J^-)}{8c_o(x)}\}]\max(\frac{\partial J^+}{\partial x})_{t=o}}, \tag{10.13}$$

$$\frac{4}{(1+\gamma)\min\limits_{J^+}[\exp\{\frac{(3-\gamma)(J_*^+ - J^+)}{8c_o(x)}\}]\max(\frac{\partial J^-}{\partial x})_{t=o}}. \tag{10.14}$$

If $J^+$ and $J^-$ in these results are replaced by their constant values $J_*^+$ and $J_*^-$, then $t_{inf}$ and $t_{sup}$ coincide, i.e., (10.6) reduces to equality, and we obtain the simplest asymptotic estimate

$$t_c = \frac{4}{(\gamma+1)N}, \quad \text{with} \quad N = \max\{\max(\partial J_o^+/\partial x), \max(\partial J_o^-/\partial x)\}, \tag{10.15}$$

for the time of breakdown in the solution.

This is precisely the result obtained by Ludford when allowance is made for the fact that his definition of the Riemann invariants $J^+$ and $J^-$ differ by a numerical factor two from those given in equations (10.3) and (10.4). Equation (10.15) also has a similar form to that obtained by Ames [1], although the notation is somewhat different. His result must be multiplied by two to compare exactly with (10.15). Stated alternatively, the initial data in the treatment of Ludford and Ames must be taken to be one-half those in the work of Jeffrey and Lax.

Remark. Assume that if

$$o \le x, s(t) \le b; \quad o \le t \le t_c; \quad \left|\frac{1}{\rho c}\right| \le N, \tag{10.16}$$

then

$$|P_o| \le M_1; \quad |r| \le M_2; \quad |g|, |g'| \le M_3, \tag{10.17}$$

where N, $M_1$, $M_2$, $M_3$ are positive constants. Hence, the continuous dependence of the solution on the initial and boundary data follows immediately from (9.13), (9.18) and (9.19), and (9.23). Indeed, let $(v^i, P^i)$ $(i=1,2)$ be the solutions of Problem II corresponding to the two different sets of data $(P_o^i, r^i, g^i)$ $(i=1,2)$. In addition, we assume that

$$|P_o^1(x) - P_o^2(x)| < \delta; \quad |r^1(x) - r^2(x)| < \delta; \quad |g^1(s(t)) - g^2(s(t))| < \delta. \tag{10.18}$$

where $\delta > o$ is a small number. Then equation (9.12) for any $\alpha$ implies that

$$|J_\alpha^{1\pm}(x,t) - J_\alpha^{2\pm}(x,t)| \le |r^1(x) - r^2(x)| + N|P_0^1(x) - P_0^2(x)|$$
$$< \delta(1+N) . \tag{10.19}$$

However, it is easily seen from (9.13) that

$$\begin{aligned} |v^1(D_1) - v^2(D_1)| &< \delta(1+N) , \\ |F^1(D_1) - F^2(D_1)| &< \delta(1+N) \end{aligned} \tag{10.20}$$

so that the application of the mean value theorem for integrals gives

$$|P^1(D_1) - P^2(D_1)| < \frac{\delta(1+N)}{N_0} , \tag{10.21}$$

where

$$N_0 = (\rho c)^{-1}(\tilde{P}) \text{ for some } \tilde{P} \text{ between } P^1 \text{ and } P^2. \tag{10.22}$$

In a similar way, we find from (9.19) and (9.23) that

$$|v^1(D_2) - v^2(D_2)| < \delta(1+2N); \quad |P^1(D_2) - P^2(D_2)| < \frac{\delta(1+2N)}{N_0} \tag{10.23}$$

and

$$|v^1(D_3) - v^2(D_3)| < \delta(1+N); \quad |P^1(D_3) - P^2(D_3)| < \frac{\delta(1+N)}{N_0} , \tag{10.24}$$

respectively. Let

$$\varepsilon = \max \{ \delta(1+2N) , \frac{\delta(1+2N)}{N_0} \} . \tag{10.25}$$

Then

$$|v^1(D) - v^2(D)| < \varepsilon ; \quad |P^1(D) - P^2(D)| < \varepsilon , \tag{10.26}$$

whenever (10.19) is defined. Here $D(x,t)$ is either $D_1(x,t)$, $D_2(x,t)$

or $D_3(x,t)$. This proves our assertion.

The results of Sections 9 and 10 may be gathered together as follows:

Theorem 7. There exists a value of time $t_c > o$ depending on the initial and boundary data such that for $o \le t < t_c$, Problem II is well-posed.

## 11. Remarks on Discontinuities and Shock Waves

Let $V(t)$ be a material volume of a continuum model bounded by a material surface $\Sigma(t)$. By definition, $V(t)$ is an arbitrary collection of matter made up of the same type of particles as time progresses which is enclosed by a material surface (or boundary), every point of which moves with the local fluid velocity. If $V(t)$ is shrunk to a point, the resulting material point is called a fluid particle. Let, however, $V^*(t)$ be a moving volume in space (not necessarily a material volume) with boundary surface $\Sigma^*(t)$. Such an arbitrary moving volume is often called a control volume; the boundary need not in general be identified with any physical boundaries.

Specialized to one dimension and disregarding the body forces (although it would not affect the jump conditions), in which case $V(t)$ is bounded by the surfaces $x_1(t) < x_2(t)$ and $V^*(t)$ by the surface $x_1^*(t) < x_2^*(t)$. In other words, the $x^i(t)$, i=1,2, denote the positions of the moving particles that form the bounding surfaces of the material volume $V(t)$. A similar statement applies to $x_i^*(t)$, i=1,2. Then the integral equations for $V^*(t)$ (see, for example, Jeffrey [21]) are

Conservation of Mass (continuity):

$$\frac{d}{dt}\int_{x_1^*(t)}^{x_2^*(t)} \rho dx + [\rho w]_{x_1^*(t)}^{x_2^*(t)} = o \ , \tag{11.1}$$

Balance of Linear Momentum (Newton's second law):

$$\frac{d}{dt}\int_{x_1^*(t)}^{x_2^*(t)} \rho v dx + [P + \rho v w]_{x_1^*(t)}^{x_2^*(t)} = o \ , \tag{11.2}$$

Conservation of Energy (first law of thermodynamics):

$$\frac{d}{dt}\int_{x_1^*(t)}^{x_2^*(t)} \rho(\Xi + \frac{1}{2}v^2) dx + [\rho w(\Xi + \frac{1}{2}v^2) + Pv + q]_{x_1^*(t)}^{x_2^*(t)} = o \ , \tag{11.3}$$

Production of Entropy (second law of thermodynamics):

$$\frac{d}{dt}\int_{x_1^*(t)}^{x_2^*(t)} \rho S dx + [\rho w S + \frac{1}{2}\, q]_{x_1^*(t)}^{x_2^*(t)} \geq o \ , \tag{11.4}$$

where $\Xi$ is the specific internal energy, $q = -K\ \partial T/\partial x$ is the heat flux ($K$ is the thermal conductivity), and $w = v-U$ is the fluid velocity relative to the control volume with $dx_i^*(t)/dt = v(x_i^*,t)$, $i=1,2$.

Often the above integral equations are applied to $V(t)$, in which case the boundary velocity $U$ is just the fluid velocity $v(x_i,t)$ and the balance statements respectively become

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho dx = o \ , \tag{11.5}$$

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho v dx + [P]_{x_1(t)}^{x_2(t)} = o \ , \tag{11.6}$$

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho(\Xi + \frac{1}{2}\, v^2) dx + [Pv + q]_{x_1(t)}^{x_2(t)} = o \ , \tag{11.7}$$

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho S dx + [\frac{1}{T}\, q]_{x_1(t)}^{x_2(t)} \geq o \ , \tag{11.8}$$

By definition, a shock wave is a relatively thin region of rapid variation in the variables governing the behaviour of the system across which there is a flow of matter. Because the region is thin, it is usually idealized as a surface of discontinuity in space. This surface propagates into the fluid and is not necessarily stationary. In general, all the fluid flow variables v, $\rho$, P and S are discontinuous across the shock surface.

The treatment of shock waves as discontinuities, or surfaces of zero thickness, in fact, is an idealization of inviscid gas dynamics. Physically, shocks are found to have a finite and measurable thickness, commonly of the order of $10^{-6}$ m. In the following, shocks are treated, however, as true discontinuities.

Since the shock wave is treated as a true discontinuity, we can make the control volume $V^*(t)$ arbitrarily thin and still enclose a portion of the shock. This can be achieved by letting $x_2 - x_1$ be sufficiently small (for example, $x_2(t) - x_1(t) = \bar{\varepsilon}(t)$ with $\bar{\varepsilon}(t) \to o$).

On the other hand, in applying the integral balance statements (11.1) to (11.4), the condition that the control volume be thin allows us to neglect the volume integrals (because there can be no storage of mass, momentum, etc. in essentially zero volume). Then the balance statements (11.1) to (11.4) for mass, momentum, energy, and entorpy become respectively, per unit area of the shock surface,

$$[\rho w] = o \tag{11.9}$$

$$[P + \rho v w] = o \ , \tag{11.10}$$

$$[\rho w(\Xi + \frac{1}{2} v^2) + Pv + q] = o \ , \tag{11.11}$$

$$[\rho s w + \frac{1}{2} q] \geq o \ , \tag{11.12}$$

in which $[\psi] = \psi_2 - \psi_1$ is the discontinuous jump of $\psi$ across $\Sigma^*(t)$, and the suffixes 1 and 2 denote the value of $\psi$ on adjacent sides of and arbitrarily close to $\Sigma^*(t)$, according as the discontinuity surface is approached from the side 1 or the side 2, respectively. These are called the jump conditions across the discontinuity surface $\Sigma^*(t)$.

Rearranging Eq. (11.9) yields

$$w_1[\rho] + \rho_2[v] = o \ . \tag{11.13}$$

Similarily, using Eq. (11.9), Eqs. (11.10) and (11.11) may be written respectively,

$$\rho_1 w_1[v] + [P] = o \ , \tag{11.14}$$

$$\rho_1 w_1[\Xi + \frac{1}{2} v^2] + [Pv + q] = o \ . \tag{11.15}$$

Properties of the Jump Conditions

Equation (11.9) says that the mass flux density $\rho w$ is continuous across $\Sigma^*(t)$, say $m = \rho w$. If, now, $w_1 = o$, $m = o$ and also $w_2 = o$ since the densities on adjacent sides of $\Sigma^*(t)$ may be assumed to be non-zero. Then no fluid crosses $\Sigma^*(t)$ (i.e., it moves with the fluid) and Eqs. (11.13) to (11.15) become

$$[v] = o \ , \tag{11.16}$$

$$[P] = o \ , \tag{11.17}$$

$$[q] = o \ . \tag{11.18}$$

This type of discontinuity is called a contact discontinuity and has the property that the pressure, the fluid velocity and the heat flux are continuous across the contact discontinuity.

If, however, $[\rho] = o$ and $w_1 \neq o$, it follows from Eqs. (11.13) and (10.14) that v and P are continuous and so

$$U - v_1 = - w_1 = \frac{[q]}{\rho_1 [\Xi]} \tag{11.19}$$

This type of discontinuity is called a phase front. Clearly a phase front cannot exist if there is no heat conduction since Eq. (11.19) implies that $[\Xi] = o$ if $[q] = o$ and so all variables then become continuous across the phase front; Eq. (11.15) becomes an identity, and thus a phase front ceases to exist.

A stronger form of discontinuity, the shock, occurs when $[\rho] \neq o$ and $w_1 \neq o$, in which case Eqs. (11.13) and (11.14) gives

$$[v] = -\frac{[P]}{\rho_1 w_1} = -\frac{[P]}{m} , \tag{11.20}$$

$$w_1 = \pm \left(\frac{\rho_2 [P]}{\rho_1 [\rho]}\right)^{1/2} . \tag{11.21}$$

Because of the square root in Eq. (11.21), the liquid pressure and density can only either increase or decrease together, i.e. $[P]$ and $[\rho]$ are both of the same sign. Up to this point, the mathematical solution to the shock conditions is not physically realizable because a liquid shock wave is known from physical observation always to be compressive, in the sense that liquid density increases after the passage of the shock wave. The difficulty is resolved in this case by appeal to a mathematical principle which has not so far been used in connection with (11.9) to (11.11). At the end of the present section, we will find that, with the help of (11.12), only the compression shock is possible.

The selection principle utilized here for identifying a physically realizable liquid shock wave is called the entorpy condition which comes in from outside the framework of the equations of fluid dynamics. In general, without this selection principle the mathematical shock solution would not be unique.

A further investigation for the shock discontinuity now follows:

Subtracting $U[\rho w] = o$ from Eq. (11.10) gives

$$[P + \rho w^2] = o , \tag{11.22}$$

which states that the total momentum flux density is the same in both sides of the shock.

On the other hand, Eq. (11.11) can be rewritten as

$$[\rho w(h + \frac{1}{2} v^2) + UP + q] = o \ ,$$

where $h = \Xi + P/\rho$ is the specific enthalpy. Using Eqs. (11.9) and (11.10), and the result $[\frac{1}{2} v^2 - Uv] = [\frac{1}{2} w^2]$, this becomes

$$[\rho w(h + \frac{1}{2} w^2) + q] = o \ . \tag{11.23}$$

This means that the total energy flux density is the same on both sides of the shock.

Finally, it follows from (11.12) that there is an increase in the total entropy flux density across the shock and a corresponding production within the shock due to thermal and mechanical and relaxation processes.

This completes the reduction of the balance or conservation statements to the elementary shock conditions. For convenience, the main shock conditions are rewritten from (11.9), (11.22), (11.23) and (11.12) for continuity, momentum, energy, and entropy, respectively,

$$[\rho w] = o \tag{11.24}$$

$$[P + \rho w^2] = o \tag{11.25}$$

$$[\rho w(h + \frac{1}{2} w^2) + q] = o \tag{11.26}$$

$$[\rho w S + \frac{1}{T} q] = o \tag{11.27}$$

We observe here that the shock conditions may be written with the relative velocity w only. Such a result is a consequence of

Galilean relatively; for the laws of mechanics must have the same expression in any intertial frame. (That means they must be invariant if v is replaced by $v + v_0$ and U by $U + v_0$ where $v_0$ is a constant velocity).

In what follows we shall always take m positive, with the liquid going from side 1 to side 2. That is, we call liquid 1 the one into which the shock wave moves, and liquid 2 that which remains behind the shock. We name the side of the shock wave towards liquid 1 the front of the shock, and that towards liquid 2 the back. (See Figure 3).

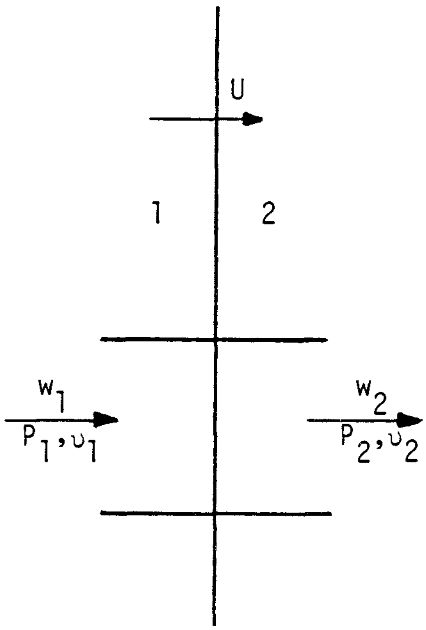


Figure 3. Flow quantities for moving discontinuity.

The usual situation is that the flow in the front of the shock is known and we ask whether the shock conditions can be used to determine uniquely the flow behind in terms of the shock velocity or to determine uniquely the shock velocity and the remaining flow quantities in terms of one of the flow quantities behind.

It is assumed that the fluids ahead of and behind the shock are essentially the same fluid and satisfy the same equation of state (this assumption precludes certain chemical relations). Then the thermodynamic state of the fluid is fixed by two variables, say P and $\upsilon$. The unknowns are any three of $w_2$, $P_2$, $\upsilon_2$, U, and the other one is fixed as a parameter. To determine these unknowns in a unique manner, three equations, (11.24) to (11.26), are available.

As a matter of completeness, we shall establish four important properties of the liquid state on either side of a shock wave, namely:

1°. Only compression shocks, $[P] > o$, are possible, assuming that $(\partial^2\upsilon/\partial P^2)_s > o$. Correspondingly, $[\rho] > o$ and $[w] < o$.

2°. The increase of the total entropy flux density across a weak shock is at most of the third order in the shock strength, or the pressure jump [P].

3°. The flow velocity relative to the shock wave is supersonic at the front side, subsonic at the back side.

To prove these statements, we first use the shock conditions to derive some useful relations.

Combining the continuity and momentum conditions (Eqs. (11.24) and (11.25)) yields

$$w_1 w_2 = \frac{[P]}{[\rho]} \,. \tag{11.28}$$

Further manipulation of Eqs. (11.24) and (11.25) leads to the non-dimensional relation

$$\frac{[P]}{\rho_1 c_1^2} = - M_{1n} \frac{[w]}{c_1} = - M_{1n}^2 \frac{[\upsilon]}{\upsilon_1} \,, \tag{11.29}$$

where $M_{1n} = w_1/c_1$ is the shock Mach number. This result is valid for all types of fluid dynamic discontinuities and shows that a decrease in w is associated with an increase in P and an increase in $\rho$. Conversely, an increase in w is associated with a decrease in P and a decrease in $\rho$. This allows the immediate classification of all discontinuities into those of compression (deceleration) types and those of expansion (acceleration) types. If the three quantities equated in Eq. (10.29) are zero, there is no discontinuity. If they are small the discontinuity is called weak.

It will be convenient to introduce the nondimensional pressure jump $\Pi$,

$$\Pi \equiv \frac{[P]}{\rho_1 c_1^2} \,. \tag{11.30}$$

By definition, the numerical value of $\Pi$ is a measure of the strength of the shock. Two extreme cases exist:

$$\begin{aligned} &\Pi \ll 1 \quad \text{weak shock,} \\ &\Pi \gg 1 \quad \text{strong shock.} \end{aligned} \tag{11.31}$$

Using (10.28), the energy condition can be expressed in the form

$$[h] = \upsilon_1[P] + \frac{1}{2}[\upsilon][P] - [\frac{q}{m}] . \tag{11.32}$$

For non-heat conducting fluids, $q = o$, this is the famous Rankine-Hugoniot equation which has the useful property that it contains only thermodynamic quantities.

In general, shocks are classified according to the sign of the pressure jump [P]:

$[P] > o$ compression shocks,

$[P] < o$ rarefaction shocks.

To calculate the entropy change in terms of the pressure change for a weak shock, we expand $h(S,P)$ and $\upsilon(S,P)$ of Eq. (11.32) in a Taylor series. Then using the identities $T = (\partial h/\partial S)_P$ and $\upsilon = (\partial h/\partial P)_S$ and retaining terms up to third order in [P] (because the first- and second-order terms in [P] are cancelled) and of the first-order only in [S], Eq. (11.32) gives the important result

$$[mS + \frac{q}{T_1}] = \frac{m}{12\, T_1} (\frac{\partial^2 \upsilon}{\partial P^2})_{S,1} [P]^3 \tag{11.33}$$

Rewritten in nondimensional form

$$[mS + \frac{q}{T_1}] = \frac{1}{6} C_1^2 \Upsilon_1 \Pi^3 , \tag{11.34}$$

where

$$\Upsilon = \frac{C^4}{2\upsilon^3} (\frac{\partial^2 \upsilon}{\partial P^2})_S . \tag{11.35}$$

Since the temperature behind the shock wave must be higher than that in front of it, (i.e., $T_2 > T_1$) it follows that

$$[\frac{q}{T}] = \frac{q_2}{T_2} - \frac{q_1}{T_1} < [\frac{q}{T_1}] ,$$

and Eq. (11.33) leads to

$$[mS + \frac{q}{T}] < \frac{m}{12\ T_1} (\frac{\partial^2 \upsilon}{\partial P^2})_{S,1} [P]^3 . \tag{11.36}$$

The entropy condition (11.27) implies that the right-hand side of (11.36) must be positive; thus the pressure jump [P] necessarily has the same sign as $(\frac{\partial^2 \upsilon}{\partial P^2})_{S,1}$ (or $\mathrm{T}$). Except in bizarre cases, $\mathrm{T} > o$. In particular, for a liquid satisfying the Tait equation (2.7), we have

$$\mathrm{T} = \frac{\gamma + 1}{2} , \tag{11.37}$$

in which $\gamma$ is not the ratio of specific heats, see Section 2, but satisfies $\gamma \geq 1$; e.g. water has $\gamma \simeq 7$, giving $\mathrm{T} \simeq 4$. Values of $\mathrm{T}$ for various liquids can be found in the literature. Roughly, we can take $\mathrm{T}$ to be of order unity for fluids in general. In any case, assuming $\mathrm{T} > o$ it follows that $[P] > o$, and only compression shocks are possible for normal fluids. Correspondingly, $[w] < o$ and $[\rho] > o$ from Eq. (11.29). This proves assertion 1°. At the same time, assertion 2° follows from (11.36).

Finally, the first part of property 3° follows immediately from 1°. For, since $S_2 > S_1$ and $\partial P/\partial S > o$, Eq. (11.28) yields

$$w_1 w_2 = \frac{P(\rho_2, S_2) - P(\rho_1, S_1)}{\rho_2 - \rho_1} > \frac{P(\rho_2, S_1) - P(\rho_1, S_1)}{\rho_2 - \rho_1} =$$

$$= \frac{\partial P(\bar{\rho}, S_1)}{\partial \rho} > \frac{\partial P(\rho_1, S_1)}{\partial \rho} = c^2(\rho_1, S_1) = c_1^2 ,$$

in which $\bar{\rho}$ is a properly chosen intermediate value between $\rho_1$ and $\rho_2$. Hence $w_1^2 > c_1^2$ by property 1°, and so $w_1 > c_1$.

Alternatively, using the entropy condition and the Hugoniot diagram, both parts of assertion 3° can be shown to be true; see, for example, Landau and Lifshitz [24] or Zel'dovich and Raizer [38].

# CHAPTER FOUR - THERMAL STRESSES IN THE FROZEN PHASE

This chapter is devoted to the discussion of the displacement problem which was formulated in Section 3 and was referred to as Problem III. Find a function $u(x,t)$ satisfying the following conditions:

(III.1) $u_{tt} = \omega^2(1+u_x)u_{xx}$ in $\Omega_1 = \{(x,t): o < x < s(t),\ o < t < \tilde{t}\}$

$$\left.\begin{aligned} &(\text{III.2})\quad u(x,o) = o \\ &(\text{III.3})\quad u_t(x,o) = f(x) \end{aligned}\right\}\quad o \le x \le s(o) = a$$

$$\left.\begin{aligned} &(\text{III.4})\quad u(o,t) = o \\ &(\text{III.5})\quad u_x(s(t),t) = \chi(s(t)) \end{aligned}\right\}\quad o \le t \le \tilde{t}$$

and the compatibility conditions

(III.6) $f(o) = o$ , $\chi(a) = o$ .

Here $\omega^2$ is a constant and, s, f and $\chi$ are given functions of their respective arguments. We shall assume that in $\Omega_1$ the functions f and $\chi$ are continuously differentiable, while the function u is twice continuously differentiable.

As mentioned in Section 3, the knowledge of the value $u(x,t)$ of the wave function u at a general point $(x,t)$, i.e., the solution of Problem III, is sufficient to determine the stress and strain components by using Eqs. (2.38), (2.39) and (2.40).

## 12. The Method of Characteristics Applied to the Displacement Problem

If we define

$$w = u_x \ , \ v = u_t \ , \tag{12.1}$$

then (III.1) can be written as two coupled partial differential equations of the first order:

$$v_t - H^2(w)w_x = o \ , \tag{12.2}$$

$$w_t - v_x = o \ , \tag{12.3}$$

where

$$H^2(w) = \omega^2(1+w) \ . \tag{12.4}$$

Equation (12.3) is a consistency condition, whereas Eq. (12.2) describes the nonlinear behavior given by (III.1). In terms of the unknowns w and v, Problem III is transformed into

Problem V. Find a pair of functions $(w(x,t), v(x,t))$ satisfying the following conditions:

$$\left.\begin{array}{ll} \text{(V.1)} & v_t - H^2(w)w_x = o \\ \text{(V.2)} & w_t - v_x = o \end{array}\right\} \quad (x,t)\omega\Omega_1$$

$$\left.\begin{array}{ll} \text{(V.3)} & w(x,o) = o \\ \text{(V.4)} & v(x,o) = f(x) \end{array}\right\} \quad o \le x \le a$$

$$\left.\begin{array}{ll} \text{(V.5)} & v(o,t) = o \\ \text{(V.6)} & w(s(t),t) = \chi(s(t)) \end{array}\right\} \quad o \le t \le \tilde{t}$$

where $H(w)$ is defined by (12.4).

The solution to Problem III is clearly recovered from that of Problem V by integration.

If we now multiply (12.3) by $H(w)$ and add and subtract the result to (12.2), we obtain the pair of equations:

$$v_t \pm Hw_t \mp Hv_x - H^2w_x = o \tag{12.5}$$

or

$$J^-_t - H(w)J^-_x = o \ , \tag{12.6}$$

$$J^+_t + H(w)J^+_x = o \ , \tag{12.7}$$

where

$$J^{\mp} = v \pm \int^w H(w')dw' = v \pm R(w) \ . \tag{12.8}$$

Equations (12.6) and (12.7) are the characteristic equations for equations (12.2) and (12.3). Here the variables $J^-$ and $J^+$ are the Riemann invariants; $J^-$ is invariant along the characteristic $\Gamma^-$: $dx = -Hdt$ and $J^+$ is invariant along the characteristic $\Gamma^+$: $dx = +Hdt$ (see Section 9).

From Eq. (12.4), we get

$$R(w) = \int^w H(w')dw' = \frac{2\omega}{3}(1 + w)^{3/2} \ . \tag{12.9}$$

Then Eqs. (12.8) become

$$J^{\mp} = v \pm \frac{2\omega}{3}(1 + w)^{3/2} \ . \tag{12.10}$$

Again, the Riemann invariants are determined up to an arbitrary constant, which can always be dropped for convenience. This was actually done in Eq. (12.9) and hence also in (12.10). Thus, subtracting Eqs. (12.10) from each other and taking the inverse yield

$$w = -1 + [\frac{3}{4\omega}\ (J^- - J^+)]^{2/3} \ . \tag{12.11}$$

Also, adding the two equations in (12.10), we obtain

$$v = \frac{1}{2}(J^- + J^+) \ . \tag{12.12}$$

The equations in Eqs. (12.11) and (12.12) are evaluated for the same values of their arguments.

It is desirable now to obtain the solution to Problem V at a typical point (x,t) in $\Omega_1$. This point will be called either $D_1$, $D_2$ or $D_3$ whenever it belongs to either the region 1, 2 or 3. (See Figure 4).

For convenience, we will employ the same notation introduced in Section 9. Thus

$$J^{\pm}_{\alpha}(x,t) \equiv J^{\pm}(x_{\alpha},o) \quad \text{with } x_{\alpha} = x_{\alpha}(x,t), \tag{12.13}$$

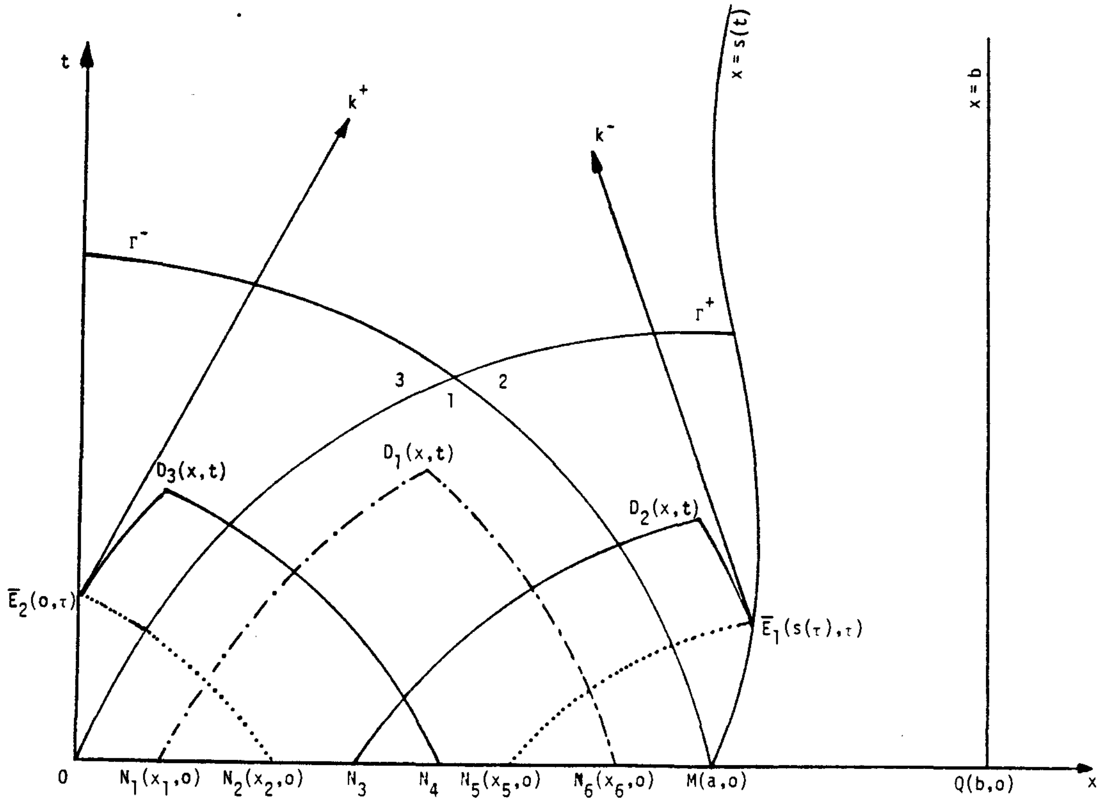


Figure 4. Sketch of characteristics for the frozen phase.

where again $\alpha$ is a positive real number, not necessarily an integer. Then, according to (V.3) and (V.4), we obtain

$$J_{\alpha}^{\mp}(x,t) = f(x_{\alpha}) \pm \frac{2\omega}{3}, \; \alpha > 0 \tag{12.14}$$

Clearly, the solution at $D_1$ is determined completely by the initial data on the line OM. In fact, by the invariance properties of $J^+$ and $J^-$ along their respective characteristic curves $\Gamma^+$ and $\Gamma^-$, we obtain

$$J^+(D_1) = J_1^+(x,t) \;, \; J^-(D_1) = J_6^-(x,t) \;. \tag{12.15}$$

Hence, on employing the formulas (12.11) and (12.12), we get

$$w(D_1) = -1 + [1 + \frac{3}{4\omega}\{f(x_6) - f(x_1)\}]^{2/3} \tag{12.16}$$

$$v(D_1) = \frac{1}{2}\{f(x_6) + f(x_1)\} \;. \tag{12.17}$$

where $x_i = x_i(x,t)$ for i=1,6. This is the solution of Problem V at a general point $D_1 = (x,t)$ in the region 1.

Now the $\Gamma^+$ characteristic issuing from $N_5$ and impinging on the freezing front at $\bar{E}_1$ carries the initial value $J_5^+(x,t)$. On s, we have

$$J_5^+(s(\tau),\tau) = v(s(\tau),\tau) - \frac{2\omega}{3}(1 + u(s(\tau),\tau))^{3/2}$$

so that

$$v(s(\tau),\tau) = J_5^+(s(\tau),\tau) + \frac{2\omega}{3}(1 + \chi(s(\tau))^{3/2} \;. \tag{12.18}$$

In order to find the solution of Problem V at a general point $D_2$ near the moving boundary x = s(t) in the region 2, one has to determine the Riemann invariants at this point. Since $J^+(D_2) = J_3^+(x,t)$, it remains to find $J^-(D_2)$. In doing so, we consider the $\Gamma^-$ characteristic emanating from $\overline{E}_1$ after refelction on s and passing through the point $D_2$. We then define the tangent line at $\overline{E}_1$ to that $\Gamma^-$ characteristic by

$$k^-: x = s(\tau) - \omega(1+\chi(s(\tau)))^{1/2}(t-\tau) \qquad (12.19)$$

and prescribe the Riemann invariant $J^-$ along that $\Gamma^-$ characteristic by

$$\begin{aligned} J^-[s(\tau),\tau] &\equiv v(s(\tau),\tau) + R(w(s(\tau),\tau)) \\ &= J_5^+(s(\tau),\tau) + \frac{4\omega}{3}(1+\chi(s(\tau)))^{3/2} \end{aligned} \qquad (12.20)$$

with $\tau(x,t)$ determined implicitly from (12.19).
Then, on setting

$$R(D_2) \equiv \frac{1}{2}\{J^-[s(\tau(x,t)) \; , \; \tau(x,t)] - J_3^+(x,t)\} \; , \qquad (12.21)$$

$$v(D_2) \equiv \frac{1}{2} \; J^-[s(\tau(x,t)) \; , \; \tau(x,t)] + J_3^+(x,t)\} \; , \qquad (12.22)$$

we see at once that the Riemann invariant $J^+(D_2) = J_3^+(x,t)$ and $J^-(D_2) = J^-[s(\tau(x,t)),\tau(x,t)]$ at the point $D_2 = (x,t)$ satisfy the system of equations (12.7) and (12.6). Using (12.9) to calculate $w(D_2)$ corresponding to $R(D_2)$ in (12.21), we obtain

$$w(D_2) = -1 + [\frac{3}{4\omega}\{J^-[s(\tau(x,t)),\tau(x,t)] - J_3^+(x,t)\}]^{2/3} \; . \qquad (12.23)$$

Therefore, the values $v(D_2)$ and $w(D_2)$ of the functions v and w given by (12.22) and (12.23) form a solution to the system of equations (V.1) and (V.2). This solution satisfies the boundary condition (V.6). Moreover, it is unique since Eq. (12.19) has only one root $\tau = \tau(x,t)$.

On the left end $x = o$, we apply a similar procedure. The $\Gamma^-$ characteristic carrying the initial value $J^-(x_2,o)$ will impinge on the fixed boundary $x = o$ after some time $\tau$, i.e., at the point $\bar{E}_2$ in Figure 4. On the boundary $x = o$, we have upon using (V.5) the result

$$J_2^-(o,\tau) = \frac{2\omega}{3}(1 + w(o,\tau))^{3/2} . \tag{12.24}$$

Again, assume that we wish to determine the values $v(x,t)$ and $w(x,t)$ of the functions $v$ and $w$ at a point $D_3$ near the boundary $x = o$ with coordinates $(x,t)$. Then as mentioned above, we have to evaluate the Riemann invariants at $D_3$. Since $J^-(D_3) = J_4^-(x,t)$, we only have to specify $J^+(D_3)$. To this end, we consider the $\Gamma^+$ characteristic issuing from $\bar{E}_2$ after reflection on the boundary $x = o$ and passing through the point $D_3$. Then we define the tangent line at $\bar{E}_2$ to that $\Gamma^+$ characteristic by

$$k^+ : \; x = \omega[\frac{3}{2\omega} J_2^-(o,\tau)]^{1/3}(t - \tau) \tag{12.25}$$

and define the Riemann invariant $J^+(o,\tau)$ along that $\Gamma^+$ characteristic by

$$J^+(o,\tau) \equiv v(o,\tau) - R(o,\tau) = -J_2^-(o,\tau) \tag{12.26}$$

with $\tau(x,t)$ found implicitly by (12.25).

Thus, on writing

$$R(D_3) \equiv \frac{1}{2}\{J_4^-(x,t) - J_2^-[o,\tau(x,t)]\} \tag{12.27}$$

$$v(D_3) \equiv \frac{1}{2}\{J_4^-(x,t) + J_2^-[o,\tau(x,t)]\} , \tag{12.28}$$

we readily see that the Riemann invariants $J^-(D_3) = J_4^-(x,t)$ and $J^+(D_3) = J_2^-[o,\tau(x,t)]$ form a unique solution to the equations (12.6) and (12.7), since Eq. (12.25) has only one root $\tau = \tau(x,t)$.

Using the relation (12.9) to evaluate $w(D_3)$ corresponding to $R(D_3)$ in (12.27), we get

$$w(D_3) = -1 + [\frac{3}{4\omega}\{J_4^-(x,t) - J_2^-[o,\tau(x,\tau)]\}]^{2/3} \ . \qquad (12.29)$$

Hence, the values of $v(D_3)$ and $w(D_3)$ of the functions v and w given by (12.29) and (12.28) constitute a unique solution to the system (V.1) and (V.2) satisfying the boundary condition (V.5).

As remarked in Section 9, any statement regarding nonlinear equations is meant to be valid only for sufficiently small regions. We then summarize the above discussion and state a theorem similar to that of Section 9:

Theorem 8. Let $f(x)$ and $\chi(x)$ $(o \leq x \leq b)$ be continuously differentiable functions and satisfy the conditions (III.6). Then there exists a unique solution $v(x,t)$, $w(x,t)$ to Problem V for all $t \leq t^{**}$ for some positive constant $t^{**} \leq \tilde{t}$. Furthermore, the solution is continuously differentiable.

By the previous remarks, a further quadrature to the solution of Problem V is needed before the solution of Problem III can be determined. The following result then follows at once from Theorem 8:

Theorem 9. Problem III possesses a unique, twice continuously differentiable solution $u(x,t)$ for all $t \leq t^{**}$.

## 13. The Development of Discontinuities in the Solution of the Displacement Problem

We now turn to the application of the Jeffrey-Lax theory to study Probelm III for wave breakdown. We further developed this theory in Section 10 to calculate an asymptotic estimate for the earliest time of breakdown in the solution of Problem II. At this critical time the derivative of the solution becomes unbounded. As Problem V is a mixed initial and boundary value problem it will require conversion to a pure initial value problem before Jeffrey's estimates [19] can be used. This will be accomplished later by a suitable extension of the solution from the fundamental interval [o,a] to the entire initial line.

The system of equations involved (see Eqs. (12.2) and (12.3)) is

$$\tilde{U}_t + \tilde{M}\tilde{U}_x = o \tag{13.1}$$

in which

$$\tilde{U} = \begin{bmatrix} v \\ w \end{bmatrix}, \quad \tilde{M} = \begin{bmatrix} o & -H^2 \\ -1 & o \end{bmatrix} \tag{13.2}$$

and

$$H(w) = \omega(1+w)^{1/2} . \tag{13.3}$$

The characteristic values of $\tilde{M}$ are

$$\lambda^- = -H \quad \text{and} \quad \lambda^+ = +H \tag{13.4}$$

and the corresponding left eigenvectors are

$$[1,H] \quad \text{and} \quad [1,-H] . \tag{13.5}$$

Therefore, we have along the characteristic curves

$$\Gamma^-: \frac{dx}{dt} = -H \quad \text{and} \quad \Gamma^+: \frac{dx}{dt} = +H \ , \tag{13.6}$$

the Riemann invariant relationships

$$v + \frac{2\omega}{3}(1 + w)^{3/2} = J^- \tag{13.7}$$

and

$$v - \frac{2\omega}{3}(1 + w)^{3/2} = J^+ \ , \tag{13.8}$$

respectively.

Subtracting Eq. (13.8) from Eq. (13.7) and taking the inverse gives again

$$w = -1 + [\frac{3}{4\omega}(J^- - J^+)]^{2/3} \ . \tag{12.11}$$

In Jeffrey's paper, we need $(\partial\lambda^-/\partial J^-)$, $(\partial\lambda^-/\partial J^+)$, $(\partial\lambda^+/\partial J^-)$ and $(\partial\lambda^+/\partial J^+)$. These quantities will be respectively denoted by $\lambda_{-,-}$, $\lambda_{-,+}$, $\lambda_{+,-}$ and $\lambda_{+,+}$.

In terms of $J^-$ and $J^+$, the defining relation of H using (12.11) becomes

$$H \equiv H(J^-, J^+) = \omega[\frac{3}{4\omega}(J^- - J^+)]^{1/3} \tag{13.9}$$

from which

$$\frac{\partial H}{\partial J^-} = \frac{1}{4}\phi \ , \quad \frac{\partial H}{\partial J^+} = -\frac{1}{4}\phi \tag{13.10}$$

where

$$\phi \equiv [\frac{3}{4\omega}(J^- - J^+)]^{-2/3} \ . \tag{13.11}$$

Hence, we have

$$\lambda_{-,-} = \lambda_{+,+} = -\frac{1}{4}\phi \;, \quad \lambda_{-,+} = \lambda_{+,-} = +\frac{1}{4}\phi \;. \tag{13.12}$$

We may further notice from (13.12) that $\lambda_{-,-}$ and $\lambda_{+,+}$ are both negative, as required in Jeffrey's analysis, provided $\phi$ is positive. (Otherwise the signs of $J^-$ and $J^+$ must be changed.) Initially (t=o), we have

$$\lambda^- = -c,\ \lambda^+ = c\ ;\ \lambda_{-,-} = \lambda_{+,+} = -\frac{1}{4},\ \lambda_{-,+} = \lambda_{+,-} = \frac{1}{4}\ . \tag{13.13}$$

It is at this point in the argument that the problem requires conversion to an equivalent initial value problem. We may first note that the specification of the initial values of v and w on [o,a] determines the initial values of $J_0^{\pm}(x) \equiv J_0^{\pm}(x,o)$ on this interval. The boundary conditions that are to be imposed on v and w are $v(o,t) = o$ and $w(s(t),t) = \chi(s(t))$ for all $t \geq o$. So, by virtue of (12.11) and (12.12) and the compatibility condition $\chi(a) = o$, we obtain

$$J_0^-(o) + J_0^+(o) = o \quad \text{and} \quad \frac{\partial J_0^-(a)}{\partial x} - \frac{\partial J_0^+(a)}{\partial x} = o\ .$$

Therefore, by a suitable smooth extension of $J_0^-(x) \pm J_0^+(x)$ from [o,a] to the entire initial line, the boundary conditions may be disregarded since the problem becomes then a pure initial value problem.

Considering the pure initial value problem just defined and using Jeffrey's estimates for the $t_{inf}$ and $t_{sup}$ (see Sec. 10 for their precise definitions) show that when $\max(\partial J^-/\partial x)_{t=o}$ and $\max(\partial J^+/\partial x)_{t=o}$ are positive, $t_{inf}$ is the smaller of the two quantities

$$\frac{4}{\max\limits_{J^+,J^-}[\phi\ \exp(\frac{J^+-J_*^+}{8\omega})]\max f'(x)}\ ,\ \frac{4}{\max\limits_{J^+,J^-}[\phi\ \exp(\frac{J^--J_*^-}{8\omega})]\max f'(x)}\ . \tag{13.14}$$

Similarly, $t_{sup}$ is the smaller of the two quantities

$$\frac{4}{\min\limits_{J^+,J^-}[\phi\ \exp(\frac{J^+-J_*^+}{8\omega})]\max f'(x)}\ ,\ \frac{4}{\min\limits_{J^+,J^-}[\phi\ \exp(\frac{J_*^--J^-}{8\omega})]\max f'(x)}\ . \tag{13.15}$$

It is to be remembered, as remarked in Sec. 10, that $J_0^-(x)$ and $J_0^+(x)$ (and so $J^-$ and $J^+$, since they are constant along their respective characteristics) differ only slightly from their constant values $J_*^-$ and $J_*^+$. Here $f'(x)$ means $df(x)/dx$.

When $J^-$ and $J^+$ in these expressions are replaced by their associated constant values $J_*^-$ and $J_*^+$, as was done by Lax [25], so that $J^- = J_*^-$ and $J^+ = J_*^+$, the estimates $t_{inf}$ and $t_{sup}$ coincide with $t_c$:

$$t_c = \frac{4}{\beta[\phi]_{J^\pm = J_*^\pm}}\ ,\ \text{where } \beta = \max f'(x)\ , \tag{13.16}$$

which is the simplest asymptotic estimate for the time of breakdown in the solution of Problem V and hence of Problem III.

This is precisely the result obtained by Lax when account is taken for the fact that his initial conditions are differently described from those considered here. In his paper, Lax [25] used different comparison theorems from those applied by Jeffrey [19] in which the former assumed that $J^- = J_*^-$ , $J^+ = J_*^+$ in order to study the existence of solutions of the nonlinear string equation, Eq. (III.1). This equation was previously studied by Zabusky [37], who utilized the hodograph method to develop both a solution and an estimate for the time of breakdown in this solution.

Remark. A similar discussion to that at the end of Section 10 may be carried out here to deduce the main result of this chapter.

Theorem 10. There exists a value $t_c > o$ depending on the data such that for $o \le t < t_c$, Problem III is well-posed.

We finally point out that $t_c \le t^{**}$, where $t^{**}$ is given in Theorem 8.

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