# QUATERNIONS AND THEIR HISTORY

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# QUATERNIONS AND THEIR HISTORY

## CHAPTER I

## THE ORIGIN OF QUATERNIONS

William Rowan Hamilton was the inventor of quaternions. The first complete publication of his work with quaternions was presented in 1853. However, he had presented many lectures on quaternions prior to this time.

I believe it is appropriate to present some of the events that led up to the discovery of quaternions. Hamilton did not regard quaternions as a mathematical system, but he considered them as a new mathematical method.

Hamilton began working in this area because he felt there were many difficulties present in the presentation of negative and imaginary quantities in algebra. He was familiar with different suggestions proposed for eliminating the difficulties. Some of the suggestions were the theory of inverse quantities, the method of constructing imaginaries by lines drawn from one point with various directions in one plane, and a suggestion to explain them by algebraic operations and the properties of symbolic language.

He believed that negatives and imaginaries were not quantities, but he felt they should have a clear interpretation and meaning. He wanted this to be done in such a way that it would not be necessary to introduce geometrical ideas which would involve the idea of an angle.

Hamilton said, "It early appeared to me that these ends might be attained by our consenting to regard Algebra as being no mere Art, nor Language, nor primarily a Science of Quantity; but rather as the Science of Order in Progression. It was, however, a part of this conception, that the progression here spoken of was understood to be continuous and unidimensional; extending indefinitely forward and backward, but not in any lateral direction. And although the successive states of such a progression might (no doubt) be represented by points upon a line, yet I thought that their simple successiveness was better conceived by comparing them with moments of time, divested, however, of all reference to cause and effect; so that the 'time' here considered might be said to be abstract, ideal, or pure, like that 'space' which is the object of geometry. In this manner I was led, many years ago, to regard Algebra as the Science of pure Time." (5, pref. p.2)

Let us consider now some of the ideas expressed in Hamilton's algebra of time. Let A and B denote any two moments of time, not necessarily distinct. They could also be considered to represent dates. Then the equation, $A = B$, would be used to indicate that the moments were identical or the dates equivalent. In this representation there is no reference made to quantity nor does this express the result of comparing any two durations of time. It corresponds to the idea of synchronization.

Non-equivalence can be expressed by the formulae,

$$B > A, \text{ or } B < A.$$

The first expression would be interpreted to mean that the moment $B$ is conceived to be later than the moment $A$, and in the second expression the moment $B$ is earlier than the moment $A$. Again there is no reference to quantity or any introduction of a measure to determine how much later, or how much earlier, one moment is than the other.

Hamilton then introduced the symbol '-'. This was used to form the symbol $B - A$, which would denote the difference of two moments, or an interval of time. In this way the idea of duration, as quantity in time, was introduced. The full meaning of the symbol $B - A$, is not completely known, until there is some idea as to how long after, or how long before, if at all, $B$ is than $A$.

Extending this idea further Hamilton interpreted the equation,

$$D - C = B - A,$$

to mean that two intervals in time were equivalent. In other words the moment $D$ is related to the moment $C$ exactly as the moment $B$ is to the moment $A$. He also found, by performing a number of transformations and combinations, which could be interpreted and justified by this method of viewing the subject, the accepted rules of algebra still held. For example, if

$$C - D = A - B, \text{ then } D - B = C - A.$$

The expressions for inequality of differences were,

$$D - C > B - A \text{ and } D - C < B - A,$$

which meant in the first that D was later, relative to C, than B to A and in the second D was earlier relative to C, than B to A.

Hamilton then introduced a small letter, a. This was to indicate a step in time. In connection with this he introduced the symbol, '+'. This symbol was primarily used to indicate a combination between a step in time a, and the moment A, from which this step was to be made. Thus, the equation,

$$B = a + A,$$

would mean that the moment B could be attained by making the step a from the moment, A. A null step, o, would produce no effect,

$$o + A = A.$$

Hamilton then realized that the symbol, B - A, which represented an ordinal relation between two moments, could also be considered as denoting a step from one moment to another. Thus, he could write,

$$B - A = a,$$

where these are actually two symbols which represent the same step. We then also have,

$$a + A = B.$$

Thus we get an identity,

$$(B - A) + A = B,$$

which describes a certain connection between the operations +, and -. Namely that we first determine the difference between two moments as a relation, then apply that difference as a step. He then introduced the usual notation, +a and -a, to indicate the step itself and the opposite of that step. This proved to be consistent with the general view.

Hamilton next compared two time-steps by use of the algebraic ratio. This was to present a new relation which was the idea of a quotient. This was determined partly by their relative largeness and partly by their relative direction. This "number" found by taking the ratio of two time-steps was found to be closely related to the idea of an "algebraic number". (It appears that what Hamilton called an "algebraic number" is actually a real number.) It operates on the quantity and the direction of one step to generate or produce the quantity and direction of the other step. Thus in symbolic form, if the ratio of two time-steps is indicated by,

$$b/a = \alpha,$$

then we get the identities,

$$b/a \times a = b \text{ or } \alpha \times a = b.$$

Thus $\alpha$ is a positive or negative number, depending on whether it preserved or reversed the direction of the steps on which it operated.

Hamilton then defined operations on "algebraic numbers" and these operations were made to depend on operations of the same names on steps. Thus any operations on two algebraic numbers could be interpreted in terms of his algebra of time. His definitions were, given any two algebraic numbers $\alpha$ and $\beta$;

$(\alpha + \beta) \times a = (\alpha \times a) + (\beta \times a)$ ;

$(\alpha - \beta) \times a = (\alpha \times a) - (\beta \times \alpha)$ ;

$(\alpha \times \beta) \times a = \alpha \times (\beta \times a)$;

$(\alpha \div \beta) = (\alpha \times a) \div (\beta \times a)$.

The results derived from this were found to agree again with the accepted rules of algebra. Specifically the product of two negative numbers would be equal to a positive number. This simply meant, in his view, that two successive reversals restored the direction of a step. It is important to note here that in this view of algebra also, the square of every number is positive, and therefore no number, whether positive or negative, could be a square root of a negative number.

Since no number could be the square root of a negative number Hamilton began comparing pairs of moments. This would lead to pairs of steps and thus to pairs of numbers. Thus he was led to the expression,

$(B_1,B_2) - (A_1,A_2) = (B_1 - A_1, B_2 - A_2)$.

This expresses that the ordinal relation of one moment-pair to another moment-pair is a system of two ordinal relations

$B_1 - A_1$ and $B_2 - A_2$. Where the primary moment $B_1$ is compared with the primary moment $A_1$, and the secondary moment $B_2$ is compared with the secondary moment $A_2$.

This same relation would define a step couple such that,

$$(B_1,B_2) - (A_1,A_2) = (a_1,a_2), \quad \text{and}$$
$$(B_1,B_2) = (a_1,a_2) + (A_1,A_2), \quad \text{thus}$$
$$(B_1,B_2) = ((B_1,B_2) - (A_1,A_2)) + (A_1,A_2).$$

In this way a moment pair was generated by adding a moment pair to a step pair.

He found no difficulty in interpreting formulae for multiplication and division such as,

$$\alpha \times (a_1,a_2) = (\alpha a_1,\alpha a_2) \quad \text{and}$$
$$(\alpha a_1,\alpha a_2) \div (a_1,a_2) = \alpha \quad ;$$

where $\alpha$ is a number, positive or negative, and $a_1,a_2$ are any two steps in time. However, Hamilton ran into difficulty when he tried to interpret the division of two step-pairs, of the form;

$$(b_1,b_2) \div (a_1,a_2) \quad ,$$

where $b_1,b_2$ represented two steps which could not be derived from $a_1,a_2$ by multiplication by any single number.

Thus he was led to introduce the idea of a number-pair such as $(\alpha_1,\alpha_2)$. Any single number $\alpha$, was represented as a degenerate form, namely $(\alpha,0)$. This was necessary so that it would correspond to multiplication of a number by a step-pair, thus

$$(\alpha,0) \times (a_1,a_2) = (\alpha a_1,\alpha a_2).$$

He also wrote every step-pair as the sum of a pure primary and a pure secondary, and every number-pair as the sum of a pure primary and a pure secondary, such as;

$$(\alpha_1,\alpha_2) = (\alpha_1,0) + (0,\alpha_2).$$

He was then led to the formula for the multiplication of an arbitrary number-pair, by a primary step, such as;

$$(\alpha_1,\alpha_2)(a,0) = (\alpha_1 a,\alpha_2 a) .$$

He also defined the product of a pure secondary number-pair and a pure secondary step-pair to be,

$$(0,\alpha_2)(0,a_2) = (-\alpha_2 a_2,0) ,$$

which was found to be consistent with his interpretations.

Thus the formula for multiplication of a number-pair by a step-pair was found to be,

$$\begin{aligned}(\alpha_1,\alpha_2)(a_1,a_2) &= \{(\alpha_1,0) + (0,\alpha_2)\}\{(a_1,0) + (0,a_2)\}\\ &= (\alpha_1,0)(a_1,0) + (\alpha_1,0)(0,a_2) + (0,\alpha_2)(0,a_2)\\ &\qquad + (0,\alpha_2)(a_1,0)\\ &= (\alpha_1 a_1,0) + (0,\alpha_2 a_1) + (0,\alpha_1 a_2) + (-\alpha_2 a_2,0)\\ &= (\alpha_1 a_1 - \alpha_2 a_2 \, , \, \alpha_2 a_1 + \alpha_1 a_2) .\end{aligned}$$

With this formula the quotient of two step-pairs could always be interpreted as a number-pair.

The two factors, $(1,0)$ and $(0,1)$, were then considered to be called respectively the primary unit, and the secondary unit, of number, because;

$(1,0)(a,b) = (a,b)$ and $(0,1)(a,b) = (-b,a)$.

Also, $$(0,1)^2 (a,b) = (0,1)\{(0,1)(a,b)\}$$
$$= (0,1)(-b,a)$$
$$= (-a,-b)$$
$$= (-1,0)(a,b),$$

and thus, $(0,1)^2 = (-1,0) = -1$.

In this way then $(0,1)$ was considered to be $\sqrt{-1}$, without any notion of it being imaginary. Consequently, Hamilton was led to the conclusion that any number couple, $(\alpha_1,\alpha_2)$, could be written as, $\alpha_1 + \sqrt{-1}\,\alpha_2$.

After working with number-pairs he became very interested to find out if it might be possible to extend this to number-triads. He made this extension using a similar line of reasoning, beginning first with moment-triads, then developed step-triads, and finally to the number-triads.

Thus he found three distinct and independent unit-numbers, namely; $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ which he called respectively the primary unit, the secondary unit, and the tertiary unit. He eventually adopted the notation 1, i, and j to represent these unit-numbers.

His numerical triplet then took the form, $x + iy + jz$, where he interpreted $x, y, z$ as three rectangular co-ordinates, and the triplet itself as a line in three-dimensional space.

The theory of three-dimensional space had been previously developed and mainly began with Argand in 1806. Other writers had worked on this, but Hamilton wanted to express his theory in some new and useful way. His theory differs mainly in the concept of the product of two triples. With this in mind Hamilton began working with triplets as lines in three-dimensional space, and wanted to obtain a concise expression for the multiplication of these lines. He wanted to retain the distributive principle, with which some earlier systems had been inconsistent, and he at first assumed that he could retain the commutative principle also.

In the triplet, $x + iy + jz$, in order that this could be somewhat analogous to his development of lines in two-dimensional space, he assumed that $i^2 = -1$ and $j^2 = -1$. The interpretation of $i^2$ and $j^2$ was such that $i^2$ was a rotation through two right angles in the xy-plane and $j^2$ was a similar rotation in the xz-plane. This interpretation is seen to be valid, for if we consider any line $a + ib$ in the xy-plane, and multiply this line by $i^2$, we get

$$i^2(a+ib) = -1(a+ib)$$
$$= -(a+ib).$$

Hence the line has been rotated through two right angles. Similarly, for any line $x + jz$ in the xz-plane. Thus Hamilton regarded the unit-numbers $i$ and $j$ as operators.

Hamilton also assumed that $ij = ji$. Under these restrictions the product appeared to take the form;

$$(a + ib + jc)(x + iy + jz) = (ax - by - cz) + i(ay + bx)$$
$$+ j(az + cx) + ij(bz + cy).$$

From his theory of triplets it appeared that $ij$ should also be a triplet such that;

$ij = (e + if + jg)$, where $e,f,g,$ were three constants to be determined.

Hamilton then tried to determine these constants, so as to adopt in the best way the resulting formula of multiplication to some guiding geometrical analogies. He first considered a case where the coordinates $b,c$ were proportional to $y,z$. If we let $y = \lambda b$ and $z = \lambda c$, where $\lambda$ is any real number, then for the product of the two lines, we get

$$(a + ib + jc)(x + i\lambda b + j\lambda c) = (ax - \lambda b^2 - \lambda c^2)$$
$$+ i(ab\lambda + bx) + j(ac\lambda + cx) + ij(bc\lambda + bc\lambda)$$
$$= (ax - \lambda b^2 - \lambda c^2)$$
$$+ i[b(a\lambda + x)] + j[c(a\lambda + x)] + ij(2bc\lambda).$$

For the coefficients of $1$, $i$, and $j$ he found that,

$$(a^2 + b^2 + c^2)(x^2 + \lambda^2b^2 + \lambda^2c^2) = (a^2x^2 + \lambda^2b^4 + \lambda^2c^4 + 2b^2\lambda^2c^2) + (a^2\lambda^2b^2 + b^2x^2) + (a^2\lambda^2c^2 + c^2x^2) = (a^2x^2 - 2ax\lambda b^2 + \lambda^2b^4 - 2ax\lambda c^2 + \lambda^2c^4 + 2b^2\lambda^2c^2) + (a^2\lambda^2b^2 + 2ax\lambda b^2 + b^2x^2) + (a^2\lambda^2c^2 + 2ax\lambda c^2 + c^2x^2) = (ax - \lambda b^2 - \lambda c^2)^2 + [b(a\lambda + x)]^2 + [c(a\lambda + x)]^2 \quad .$$

Thus Hamilton concluded that in this case the triplet, $(ax - by - cz) + i(ay + bx) + j(az + cx)$ denoted a line which might, consistent with known analogies, be regarded as the product of the two lines if the fourth term $ij(bz + cy)$ did not appear. He thought that $ij = 0$ might be an answer.

Hamilton was not completely satisfied with this, so he assumed that $ij = -ji$ or that $ij = k$ and $ji = -k$ where the value of $k$ was still undetermined. In this case the product of triplets became,

$$(a + ib + jc)(x + iy + jz) = (ax - by - cz) + i(ay + bx) + j(az + cx) + k(bz - cy).$$

This representation led to the following identity in the coefficients of $1, i, j, k$;

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = a^2x^2 + a^2y^2 + a^2z^2 + b^2x^2 + b^2y^2 + b^2z^2 + c^2x^2 + c^2y^2 + c^2z^2 = (a^2x^2 - 2axby + b^2y^2 - 2axcz + c^2z^2 + 2czby) + (a^2y^2 + 2axby + b^2x^2) + (a^2z^2 + 2axcz + c^2x^2) + (b^2z^2 - 2czby + c^2y^2) = (ax - by - cz)^2 + (ay + bx)^2 + (az + cx)^2 + (bz - cy)^2 \quad .$$

This led Hamilton to believe that instead of confining himself to triplets, such as $a + ib + jc$ he should regard these as only imperfect forms of quaternions, such as $a + ib + jc + kd$. Where the symbol $k$ represents some new unit operator. Thus he found it necessary to fix the value of the square, $k^2$, and also the values of the products, $ik$, $jk$, $ki$, $kj$; so he could operate with quaternions.

Hamilton already knew that, $i^2 = j^2 = -1$ and that $ij = k$ and $ji = -k$. He then assumed that,

$ki = -ik = -i(ij) = -i^2j = j$, and $kj = -jk = j(ji) = j^2i = -i$.

To remain consistent he found that,

$k^2 = (ij)(ij) = -i(ij)j = (-i^2j^2) = -(-1)(-1) = -1$.

Therefore, Hamilton now had that,

$i^2 = j^2 = k^2 = -1$; $ij = -ji = k$; $jk = -kj = i$; $ki = -ik = j$.

From these he now had the equation,

$(a + ib + jc + kd)(a' + ib' + jc' + kd') = a'' + ib'' + jc'' + kd''$,

where;

$$\begin{aligned} a'' &= aa' - bb' - cc' - dd' \\ b'' &= (ab' + ba') + (cd' - dc') \\ c'' &= (ac' + ca') + (db' - bd') \\ d'' &= (ad' + da') + (bc' - cb') \quad . \end{aligned}$$

He also found that;

$$(a^2+b^2+c^2+d^2)(a'^2+b'^2+c'^2+d'^2) = (a''^2+b''^2+c''^2+d''^2) \quad .$$

At this point Hamilton became aware of the fact that if, instead of representing a line by $x + iy + jz$, we would represent it by $ix + jy + kz$, we could express the product of two lines in space by a quaternion which would have a very simple geometrical interpretation. Thus,

$$(ix + jy + kz)(ix' + jy' + kz') = w'' + (ix'' + jy'' + kz''),$$

where, $w'' = -xx' - yy' - zz'$; $x'' = yz' - zy'$;

$y'' = zx' - xz'$; $z'' = xy' - yx'$.

Hamilton's interpretation was, "that the part $\underline{w''}$, independent of $\underline{i}$, $\underline{j}$, $\underline{k}$, in this expression for the product, represents the <u>product of the lengths of the two factor-lines, multiplied by the cosine of the supplement of their inclination</u> to each other; and the remaining part $\underline{ix'' + jy'' + kz''}$ of the same product of the two trinomials represents a <u>line</u>, which is in <u>length</u> the <u>product of the same two lengths, multiplied by the sine of the same inclination</u>, while <u>in direction</u> it is <u>perpendicular to the plane of the factor-lines</u>, and is such that the <u>rotation round the multiplier-line</u>, from the multiplicand-line towards the product-line (or towards the <u>line-part</u> of the whole quaternion product), has the <u>same right-handed</u> (or left-handed) <u>character</u>, as the rotation round the positive semiaxis of $\underline{k}$ (or of $\underline{z}$), from the positive semiaxis of $\underline{i}$ (or of $\underline{x}$), towards that of $\underline{j}$ (or of $\underline{y}$)."
(5, Pref. p.47)

With this developed Hamilton thenfelt he had a new instrument for applying calculation to geometry.

It is important to note that Hamilton's product of two lines in three-dimensional space can be written in the following way; if $\alpha, \beta$ are two lines in space, such that,

$$\alpha = ai + bj + ck, \quad \text{and} \quad \beta = a'i + b'j + c'k, \quad \text{then}$$

$$\alpha\beta = \alpha X \beta - \alpha \cdot \beta. \quad \text{Where,}$$

$$\alpha X \beta = (bc' - cb')i + (ca' - ac')j + (ab' - ba')k,$$

is the usual vector product or outer product of $\alpha, \beta$, and $\alpha \cdot \beta = aa' + bb' + cc'$, is the scalar product or inner product of $\alpha$ and $\beta$. It was largely because of this identity, that much of the present day three-dimensional vector analysis was written in the language of quaternions in the half-century 1850-1900. (1, p.237-38)

The remaining part of this theses will be a discussion of the way quaternions fit into modern algebraic theory.

# CHAPTER II

## QUATERNIONS AS A DIVISION ALGEBRA

Real quaternions will be defined as quadruples, $(a,b,c,d)$, where $a,b,c,d$ are real numbers. Two real quaternions $x = (a,b,c,d)$ and $y = (a',b',c',d')$ are equal if and only if $a = a'$, $b = b'$, $c = c'$, and $d = d'$.

DEFINITION: If $x = (a,b,c,d)$ and $y = (a',b',c',d')$, then $x + y = (a + a', b + b', c + c', d + d')$. (7,p.60)

DEFINITION: A system $Q$ having one closed operation, '+', such that

(i) for all $x,y,z \in Q$, $x + (y+z) = (x+y) + z$,

(ii) for all $x,y \in Q$, $x + y = y + x$,

(iii) there exists an identity element $0$, such that, $0 + x = x = x + 0$, for all $x \in Q$,

(iv) for every $x \in Q$, there exists an inverse $-x$, such that,

$x + (-x) = 0 = (-x) + x$,

is called a commutative group. (4,p.18)

THEOREM 1. The set $Q$ of all real quaternions with the operation, '+', is a commutative group.

Proof. From the definition it is seen that addition is a closed operation.

(i) If $x = (a,b,c,d)$, $y = (a',b',c',d')$, and $z = (a'',b'',c'',d'')$, then

$$x + (y+z) = (a,b,c,d) + \{(a',b',c',d') + (a'',b'',c'',d'')\}$$
$$= (a,b,c,d) + (a'+a'', b'+b'', c'+c'', d'+d'')$$
$$= (a+(a'+a''), b+(b'+b''), c+(c'+c''), d+(d'+d'')).$$

But a,b, etc. are real numbers which we know are associative, thus

$$x + (y+z) = ((a+a')+a'', (b+b')+b'', (c+c')+c'', (d+d')+d'')$$
$$= (a+a', b+b', c+c', d+d') + (a'',b'',c'',d'')$$
$$= (x+y)+z.$$

Therefore the set Q of all real quaternions is associative with respect to addition.

(ii) If $x = (a,b,c,d)$ and $y = (a',b',c',d')$, then

$$x + y = (a,b,c,d) + (a',b',c',d') = (a+a',b+b',c+c',d+d').$$

But a,b, etc. are real numbers which we know are commutative, thus

$$x + y = (a'+a, b'+b, c'+c, d'+d)$$
$$= (a', b', c', d') + (a,b,c,d)$$
$$= y + x.$$

Therefore the set Q of all real quaternions is commutative with respect to addition.

(iii) Let $\theta$ be a real quaternion $(a', b', c', d')$ and let $a' = b' = c' = d' = 0$, then $\theta = (0,0,0,0)$

If $x$ is any real quaternion such that $x = (a,b,c,d)$, then $x + \theta = (a,b,c,d) + (0,0,0,0)$

$$= (a + 0, b + 0, c + 0, d + 0).$$

But since a,b,c,d,0 are real numbers

$$x + \theta = (a,b,c,d) = x.$$

In (ii) it was shown that real quaternions are commutative, thus $x + \theta = \theta + x$. Therefore, $x + \theta = x = \theta + x$ and there exists an additive identity.

(iv) If $x$ is any real quaternion such that, $x = (a,b,c,d)$, then define $-x = (-a,-b,-c,-d)$. Thus,

$$x + (-x) = (a,b,c,d) + (-a,-b,-c,-d)$$
$$= (a+(-a),\ b+(-b),\ c+(-c),\ d+(-d)).$$

But since $a,b,c,d$ are real numbers,

$$x + (-x) = (0,0,0,0)$$

$= \theta$, and since real quaternions are commutative,

$$x + (-x) = (-x) + x.$$

Therefore, $x + (-x) = \theta = (-x) + x$, and there exists for every $x \in Q$ an additive inverse $-x$.

Thus the set $Q$ of all real quaternions forms a commutative group with respect to addition.

THEOREM 2. In any group the identity element is unique.

Proof. Assume that $\theta_1$ and $\theta_2$ are both identity elements.

Since $\theta_1$ is an identity element,

$$\theta_1 + \theta_2 = \theta_2 .$$

Since $\theta_2$ is an identity element,

$$\theta_1 + \theta_2 = \theta_1$$

Therefore $\theta_1 = \theta_2$, and the identity element is unique. (8,p.49)

THEOREM 3: If $y = (a',b',c',d')$ is any real quaternion, then $y = \theta$, where $\theta$ is the additive identity element, if and only if $a' = b' = c' = d' = 0$.

Proof. If $a' = b' = c' = d' = 0$, then $y = (0,0,0,0)$ which by Theorem 1(iii) is the identity .

If $y = \theta$, then since $\theta$ is the additive identity element and by Theorem 2 this is unique, $y = (0,0,0,0)$. But this means that, $(0,0,0,0) = (a',b',c',d')$, which can only be true if $a' = b' = c' = d' = 0$. Therefore if $y = (a',b',c',d')$ is any real quaternion then $y$ is the additive identity element $(0,0,0,0)$ if and only if $a' = b' = c' = d' = 0$.

THEOREM 4. In any group if $x + y = x + z$, then $y = z$.

Proof. If $x + y = x + z$, add the inverse of $x$ to both sides, then $(-x) + (x+y) = (-x) + (x+z)$

$\{(-x)+x\} + y = \{(-x)+x\} + z$, by the associative law $\theta + y = \theta + z$,

$$y = z.$$

THEOREM 5. In any group the inverse is unique.

Proof. Assume $x_1$ and $x_2$ are two inverses of $x$, then $x + x_1 = \theta$ and $x + x_2 = \theta$. Thus $x + x_1 = x + x_2$, and by Theorem 4, $x_1 = x_2$.

Therefore in any group the inverse is unique. (8, p.50)

DEFINITION. If $m$ is any real number and $x = (a,b,c,d)$ is any real quaternion, then $mx = (ma, mb, mc, md)$. This is called scalar multiplication. (3, p.125)

DEFINITION: A system $\underline{V} = \{V, F, +, \cdot, \oplus, \odot\}$ is called a vector space over the field $\underline{F}$ if and only if,

(a) $\{F; +, \cdot\}$ is a field $\underline{F}$ whose identity elements are denoted by 0 and 1;

(b) $\{V; \oplus\}$ is a commutative group whose identity element is denoted $\theta$;

(c) for all $m, n \in F$ and all $x, y \in V$, $m \cdot x \in V$

and

(i) $(m + n) \odot x = (m \odot x) \oplus (m \odot x)$,

(ii) $m \odot (x \oplus y) = (m \odot x) \oplus (m \odot y)$,

(iii) $(mn) \odot x = m \odot (n \odot x)$,

(iv) $1 \odot x = x$. (4, p.26)

THEOREM 6. The set Q of all real quaternions is a vector space over the field F of all real numbers.

Proof. In this proof I will omit the symbols $\odot$ and $\oplus$, and there is no ambiguity.

(a) The real numbers form a field whose identity elements are 0 and 1.

(b) The set Q of all real quaternions forms a commutative group with identity element $\theta = (0,0,0,0)$, by Theorem 1.

(c) If $m,n$ are any real numbers and $x,y$ are any real quaternions, then $m + n$ and $m \cdot n$ are also real numbers.

(i) If $x = (a,b,c,d)$, then by definition of scalar multiplication,

$$(m+n)x = ((m+n)a,\ (m+n)b,\ (m+n)c,\ (m+n)d).$$

Since $m,n,a,b,c,d$ are real numbers we have the distributive law, thus

$$(m+n)x = (ma + na,\ mb + nb,\ mc + nc,\ md + nd),$$

and from the definition of addition we get

$$\begin{aligned}(m+n)x &= (ma,\ mb,\ mc,\ md) + (na,\ nb,\ nc,\ nd)\\ &= m(a,b,c,d) + n(a,b,c,d),\\ &= mx + nx.\end{aligned}$$

(ii) If $x = (a,b,c,d)$ and $y = (a',b',c',d')$, then

$m(x+y) = m(a+a',\ b+b',\ c+c',\ d+d')$, by the definition of addition. From the definition of scalar multiplication, we get

$$m(x+y) = (m(a+a'),\ m(b+b'),\ m(c+c'),\ m(d+d')).$$

Since $m,a,b,$ etc. are real numbers we have the distributive law, thus

$$\begin{aligned}m(x+y) &= (ma+ma',\ mb+mb',\ mc+mc',\ md+md')\\ &= (ma,\ mb,\ mc,\ md) + (ma',\ mb',\ mc',\ md')\\ &= m(a,b,c,d) + m(a',\ b',\ c',\ d')\\ &= mx + my.\end{aligned}$$

(iii) If $x = (a,b,c,d)$ is any real quaternion, then

$$\begin{aligned}(mn)x &= mn(a,b,c,d)\\ &= ((mn)a,\ (mn)b,\ (mn)c,\ (mn)d).\end{aligned}$$

Since $m,n,a,b,c,d$ are real numbers they are associeative, then

$$\begin{aligned}(mn)x &= (m(na),\ m(nb),\ m(nc),\ m(nd))\\ &= m(na,\ nb,\ nc,\ nd)\\ &= m(n(a,b,c,d))\\ &= m(nx).\end{aligned}$$

(iv) If $x = (a,b,c,d)$ is any real quaternion, then

$$1 \cdot x = 1 \cdot (a,b,c,d)$$
$$= (1a,\ 1b,\ 1c,\ 1d).$$

Since $1,a,b,c,d$ are real numbers, then

$$1 \cdot x = (a,b,c,d)$$
$$= x.$$

Therefore the set $Q$ of all real quaternions forms a vector space over the field of real numbers.

THEOREM 7. In any vector space if $\theta$ denotes the zero vector, and if $-x$ denotes the group inverse of $x$, then for all $x \in V$, $m \in F$,

(i) $0 \cdot x = \theta$,

(ii) $(-1)x = -x$,

(iii) $m\theta = \theta$.

Proof.

(i) $x = 1 \cdot x$, by Theorem 6 (iv),

$$= (1+0)x$$
$$= 1 \cdot x + 0 \cdot x, \text{ by Theorem 6 (i)},$$
$$= x + 0 \cdot x, \text{ by Theorem 6, (iv)}.$$

By adding $-x$ to both sides, we get

$$(-x) + x = (-x) + (x + 0 \cdot x)$$
$$= \{(-x) + x\} + 0 \cdot x$$
$$\theta = \theta + 0 \cdot x$$
$$= 0 \cdot x.$$

(ii) $x + (-1)x = 1 \cdot x + (-1)x$, by Theorem 6 (iv),

$$= \{1 + (-1\}x, \quad \text{by Theorem 6 (i)},$$

$$= 0 \cdot x$$
$$= \theta, \text{ by (i) above.}$$

Thus $(-1)x$ is an inverse of $x$ and by Theorem 5 this inverse is unique, therefore $(-1)x = -x$.

$$\begin{aligned}
\text{(iii) } m\,\theta &= m\{x + (-x)\} \\
&= mx + m(-x) \\
&= mx + m(-1x), \text{ by (ii),} \\
&= mx + \{m(-1)\}x, \text{ by associativity,} \\
&= mx + (-mx) \\
&= \theta \qquad\qquad (4, \text{ p.}27)
\end{aligned}$$

DEFINITION: A set $\{x_1,\ldots,x_n\}$ of vectors is said to be linearly independent if and only if the equation $a_1x_1 + \ldots + a_nx_n = \theta$ implies that $a_1 = \ldots = a_n = 0$. (4, p.32)

THEOREM 8. The vectors $1 = (1,0,0,0)$, $i = (0,1,0,0)$, $j = (0,0,1,0)$, and $k = (0,0,0,1)$ are linearly independent.

Proof. If $a,b,c,d$ are any real numbers then

$$\begin{aligned}
a\cdot1+b\cdot i+c\cdot j+d\cdot k &= a(1,0,0,0)+b(0,1,0,0)+c(0,0,1,0)+d(0,0,0,1) \\
&= (a,0,0,0)+(0,b,0,0)+(0,0,c,0)+(0,0,0,d) \\
&= (a,b,c,d).
\end{aligned}$$

By Theorem 3, $(a,b,c,d) = \theta$ if and only if $a=b=c=d=0$. Therefore the vectors $1,i,j,k$ are linearly independent.

DEFINITION. If an independent set cannot be extended to a larger independent set it is called a maximal independent set. (4, p.34)

THEOREM 9. The vectors $1,i,j,k$ are a maximal independent set.

Proof. Assume there exists another vector $p$ such that $1,i,j,k,p$ are linearly independent, then $a \cdot 1 + b \cdot i + c \cdot j + d \cdot k + e \cdot p = \theta$ implies $a = b = c = d = e = 0$. Let $p = (a', b', c', d')$ and from Theorem 8 $a \cdot 1 + b \cdot i + c \cdot j + d \cdot k = (a,b,c,d)$, thus $(a,b,c,d) + (ea', eb', ec', ed') = \theta$. This says that $e(a', b', c', d')$ is an additive inverse of $(a,b,c,d)$. By Theorem 5, this inverse is unique, thus $ea' = -a, eb' = -b$, $ec' = -c$, and $ed' = -d$. Therefore the expression can be equal to $\theta$ if $e \neq o$, and the vectors $1,i,j,k$ are a maximal independent set.

DEFINITION. A maximal linearly independent subset of a vector space $\underline{V}$ is called a basis of $\underline{V}$. (4, p.36)

Therefore by Theorems 8 and 9 the vectors $1,i,j,k$ form a basis for the set $Q$ of all real quaternions. Furthermore if $x = (a,b,c,d)$ is any real quaternion, we can represent $x$ as a linear combination of $1,i,j,k$ such that, $x = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k$.

The dimension of a finite-dimensional vector space is the number of vectors in any basis. Therefore the dimension of the vector space of the real quaternions is four.

DEFINITION: If $x = (a,b,c,d)$ and $y = (a',b',c',d')$ are any two real quaternions, then

$xy = (aa'-bb'-cc'-dd',\ ab'+ba'+cd'-dc',\ ac'+ca'+db'-bd',$
$ad'+da'+bc'-cb').$

DEFINITION. A linear algebra $\underline{L}$ with a multiplicative identity element over a field $\underline{F}$ is a system, $\underline{L} = \{L, F;\ +, \cdot;\ \oplus, \odot, \boxdot\}$ which satisfies the postulates:

(a) the system $\{L, F;\ +, \cdot;\ \oplus, \odot\}$ is a vector space over $\underline{F}$.

(b) $\boxdot$ is a binary operation on $\underline{L}$ such that for all $m, n \in \underline{F}$ and all $x, y, z \in \underline{L}$,

(i) $x \boxdot y \in \underline{L}$,

(ii) $x \boxdot (y \boxdot z) = (x \boxdot y) \boxdot z$,

(iii) $x \boxdot (y \oplus z) = (x \boxdot y) \oplus (x \boxdot z)$,

(iv) $(m \odot x) \boxdot y = x \boxdot (m \odot y) = m \odot (x \boxdot y)$,

(v) there exists a multiplication identity element relative to $\boxdot$. (6, p.228)

THEOREM 10. The set Q of all real quaternions is a linear algebra with a multiplicative identity element over the field F of real numbers.

Proof. (a) By Theorem 6 the real quaternions forms a vector space over the field of real numbers.

(b) By the definition of multiplication of two quaternions multiplication is a binary operation.

In the expressions that follow I will omit the symbols $\odot$, $\oplus$, and $\boxdot$, and there is no ambiguity.

If $m, n$ are any real numbers and $x, y, z$ are any real quaternions such that, $x = (a,b,c,d)$, $y = (a',b',c',d')$, and $z = (a'',b'',c'',d'')$, then

(i) $xy \in Q$, by the definition of multiplication,

(ii)
$$
\begin{aligned}
x(yz) &= (a,b,c,d)\{(a',b',c',d')(a'',b'',c'',d'')\} \\
&= (a,b,c,d)(a'a''-b'b''-c'c''-d'd'', \\
&\qquad a'b''+b'a''+c'd''-d'c'', \\
&\qquad a'c''+c'a''+d'b''-b'd'', \\
&\qquad a'd''+d'a''+b'c''-c'b'') \\
&= (a(a'a''-b'b''-c'c''-d'd'')-b(a'b''+b'a''+c'd''-d'c'') \\
&\quad -c(a'c''+c'a''+d'b''-b'd'')-d(a'd''+d'a''+b'c''-c'b''), \\
&\quad a(a'b''+b'a''+c'd''-d'c'')+b(a'a''-b'b''-c'c''-d'd'') \\
&\quad +c(a'd''+d'a''+b'c''-c'b'')-d(a'c''+c'a''+d'b''-b'd''), \\
&\quad a(a'c''+c'a''+d'b''-b'd'')+c(a'a''-b'b''-c'c''-d'd'') \\
&\quad +d(a'b''+b'a''+c'd''-d'c'')-b(a'd''+d'a''+b'c''-c'b''), \\
&\quad a(a'd''+d'a''+b'c''-c'b'')+d(a'a''-b'b''-c'c''-d'd'') \\
&\quad +b(a'c''+c'a''+d'b''-b'd'')-c(a'b''+b'a''+c'd''-d'c'')) \\
&= (aa'a''-ab'b''-ac'c''-ad'd''-ba'b''-bb'a''-bc'd'' \\
&\quad +bd'c''-ca'c''-cc'a''-cd'b''+cb'd''-da'd''-dd'a'' \\
&\quad -db'c''+dc'b'', aa'b''+ab'a''+ac'd''-ad'c''+ba'a'' \\
&\quad -bb'b''-bc'c''-bd'd''+ca'd''+cd'a''+cb'c''-cc'b'' \\
&\quad -da'c''-dc'a''-dd'b''+db'd'', aa'c''+ac'a''+ad'b'' \\
&\quad -ab'd''+ca'a''-cb'b''-cc'c''-cd'd''+da'b''+db'a'' \\
&\quad +dc'd''-dd'c''-ba'd''-bd'a''-bb'c''+bc'b'', aa'd'' \\
&\quad +ad'a''+ab'c''-ac'b''+da'a''-db'b''-dc'c''-dd'd'' \\
&\quad +ba'c''+bc'a''+bd'b''-bb'd''-ca'b''-cb'a''-cc'd''
\end{aligned}
$$

$+cd'c'')$.

The a,b,c,etc are real numbers, thus they are commutative with respect to addition. Hence,

$$
\begin{aligned}
x(yz) = \; & ((aa'-bb'-cc'-dd')a''-(ab'+ba'+cd'-dc')b'' \\
& -(ac'+ca'+db'-bd')c''-(ad'+da'+bc'-cb')d'', \\
& (aa'-bb'-cc'-dd')b''+(ab'+ba'+cd'-dc')a'' \\
& +(ac'+ca'+db'-bd')d''-(ad'+da'+bc'-cb')c'', \\
& (aa'-bb'-cc'-dd')c''+(ac'+ca'+db'-bd')a'' \\
& +(ad'+da'+bc'-cb')b''-(ab'+ba'+cd'-dc')d'', \\
& (aa'-bb'-cc'-dd')d''+(ad'+da'+bc'-cb')a'' \\
& +(ab'+ba'+cd'-dc')c''-(ac'+ca'+db'-bd')b'') \\
= \; & (aa'-bb'-cc'-dd', ab'+ba'+cd'-dc', \\
& ac'+ca'+db'-bd', ad'+da'+bc'-cb') \\
& (a'', b'', c'', d'') \\
= \; & \{(a,b,c,d)(a',b',c',d')\}(a'',b'',c'',d'') \\
= \; & (xy)z.
\end{aligned}
$$

Therefore, the set $Q$ of real quaternions is associative with respect to multiplication.

$$
\begin{aligned}
\text{(iii)}\; x(y+z) & = (a,b,c,d)\{(a',b',c',d')+(a'',b'',c'',d'')\} \\
& = (a,b,c,d)(a'+a'', b'+b'', c'+c'', d'+d'') \\
& = (a(a'+a'')-b(b'+b'')-c(c'+c'')-d(d'+d''), \\
& \quad a(b'+b'')+b(a'+a'')+c(d'+d'')-d(c'+c''), \\
& \quad a(c'+c'')+c(a'+a'')+d(b'+b'')-b(d'+d''), \\
& \quad a(d'+d'')+d(a'+a'')+b(c'+c'')-c(b'+b'')) \\
& = (aa'+aa''-bb'-bb''-cc'-cc''-dd'-dd'', \\
& \quad ab'+ab''+ba'+ba''+cd'+cd''-dc'-dc'',
\end{aligned}
$$

$$
\begin{aligned}
& ac'+ac''+ca'+ca''+db'+db''-bd'-bd'', \\
& ad'+ad''+da'+da''+bc'+bc''-cb'-cb'') \\
= {} & ((aa'-bb'-cc'-dd')+(aa''-bb''-cc''-dd''), \\
& (ab'+ba'+cd'-dc')+(ab''+ba''+cd''-dc''), \\
& (ac'+ca'+db'-bd')+(ac''+ca''+db''-bd''), \\
& (ad'+da'+bc'-cb')+(ad''+da''+bc''-cb'')) \\
= {} & (aa'-bb'-cc'-dd', ab'+ba'+cd'-dc', \\
& ac'+ca'+db'+bd', ad'+da'+bc'-cb') \\
& +(aa''-bb''-cc''-dd'', ab''+ba''+cd''-dc'', \\
& ac''+da''+db''-bd'', ad''+da''+bc''-cb'') \\
= {} & (a,b,c,d)(a',b',c',d')+(a,b,c,d)(a'',b'',c'',d'') \\
= {} & xy + xz.
\end{aligned}
$$

Therefore in the set of real quaternions multiplication is distributive over addition.

(iv) $m(xy) = m\{(a,b,c,d)(a',b',c',d')\}$

$$
\begin{aligned}
= {} & m(aa'-bb'-cc'-dd', ab'+ba'+cd'-dc', \\
& ac'+ca'+db'-bd', ad'+da'+bc'-cb') \\
= {} & (m(aa'-bb'-cc'-dd'), m(ab'+ba'+cd'-dc'), \\
& m(ac'+ca'+db'-bd'), m(ad'+da'+bc'-cb')) \\
= {} & (maa'-mbb'-mcc'-mdd', mab'+mba'+mcd'-mdc', \\
& mac'+mca'+mdb'-mbd', mad'+mda'+mbc'-mcb').
\end{aligned}
$$

$$
\begin{aligned}
x(my) = {} & (a,b,c,d)\{m(a',b',c',d')\} \\
= {} & (a,b,c,d)(ma',mb',mc',md') \\
= {} & (ama'-bmb'-cmc'-dmd', amb'+bma'+cmd'-dmc', \\
& amc'+cma'+dmb'-bmd', amd'+dma'+bmc'-cmb').
\end{aligned}
$$

Since $m,a,b$, etc. are real numbers their product is commutative, thus

$$x(my) = (maa'-mbb'-mcc'-mdd',mab'+mba'+mcd'-mdc',$$
$$mac'+mca'+mdb'-mbd',mad'+mda'+mbc'-mcd').$$

$$(mx)\,y = \{m(a,b,c,d)\}(a',b',c',d')$$
$$= (ma,mb,mc,md)(a',b',c',d')$$
$$= (maa'-mbb'-mcc'-mdd',mab'+mba'+mcd'-mdc',$$
$$mac'+mca'+mdb'-mbd',mad'+mda'+mbc'-mcb').$$

The final results of $m(xy)$, $x(my)$, and $(mx)y$ are identical. Therefore, $(mx)y = x(my) = m(xy)$.

(v) Let $x = (a,b,c,d)$ be any real quaternion. We have defined the quaternion 1 to be $(1,0,0,0)$.

Now, $$1\cdot x = (1,0,0,0)(a,b,c,d)$$
$$= (1\cdot a-0\cdot b-0\cdot c-0\cdot d,\ 1\cdot b+0\cdot a+0\cdot d-0\cdot c,$$
$$1\cdot c+0\cdot a+0\cdot b-0\cdot d,\ 1\cdot d+0\cdot a+0\cdot c-0\cdot b).$$

But $1,0,a,b,c,d$ are real numbers, thus

$$1\cdot x = (a,b,c,d)$$
$$= x.$$
$$x\cdot 1 = (a,b,c,d)(1,0,0,0)$$
$$= (a\cdot 1-b\cdot 0-c\cdot 0-d\cdot 0,\ a\cdot 0+b\cdot 1+c\cdot 0-d\cdot 0,$$
$$a\cdot 0+c\cdot 1+d\cdot 0-b\cdot 0,\ a\cdot 0+d\cdot 1+b\cdot 0-c\cdot 0).$$

But $1,0,a,b,c,d$ are real numbers, thus

$$x\cdot 1 = (a,b,c,d)$$
$$= x, \text{ and}$$
$$1\cdot x = x = x\cdot 1 .$$

Therefore $(1,0,0,0)$ is both a right and left hand identity element, and the set $Q$ of all real quaternions is a linear algebra with a multiplicative identity element over the field $F$ of all real numbers.

I have previously shown that the set $Q$ of all real quaternions, can be represented as a linear combination of the basis vectors $1,i,j,k$. The definitions for addition, scalar multiplication, and multiplication now become; if $x = a+bi+cj+dk$ and $y = a'+b'i+c'j+d'k$ are any two real quaternions, and $m$ is any real number, then (a) $x+y = (a+a') + (b+b')i + (c+c')j + (d+d')k$ ,

(b) $mx = ma + mbi + mcj + mdk$ ,

(c) $xy = (aa'-bb'-cc'-dd') + (ab'+ba'+cd'-dc')i$
$+(ac'+ca'+db'-bd')j + (ad'+da'+bc'-cb')k.$

DEFINITION. Two algebraic systems $\underline{L}$ and $\underline{L}'$ are isomorphic if there is a one-one correspondence $\alpha \leftrightarrow \alpha'$ between $\underline{L}$ and $\underline{L}'$ which preserves the operations.

THEOREM 11. The special quaternions $a\cdot 1$, where $a$ is any real number and $1$ is the basis element $(1,0,0,0)$, are isomorphic to the real numbers.

Proof. If $a$ is any real number, then for every $a$ there exists a quaternion of this form, namely $a\cdot 1$ . Conversely for every quaternion $a\cdot 1$ there corresponds a real number, namely $a$.

If $a\cdot 1 \neq b\cdot 1$ , then from the definitions of equality of two quaternions $a \neq b$. Hence the correspondence is one-one.

Now $a \cdot 1 \leftrightarrow a$ and $b \cdot 1 \leftrightarrow b$, and

$$(a+b) \cdot 1 = a \cdot 1 + b \cdot 1 \quad ,$$

$$(ab) \cdot 1 = (a \cdot 1)(b \cdot 1) \; .$$

Thus addition and multiplication is preserved, and the correspondence is an isomorphism.

DEFINITION. If $x = a + bi + cj + dk$ is any real quaternion, then the conjugate of $x$, $\bar{x} = a - bi - cj - dk$.

THEOREM 12. If $x = a+bi+cj+dk$ is any real quaternion, then $x \cdot \bar{x} = \bar{x} \cdot x = (a^2+b^2+c^2+d^2)$ .

Proof. $x \cdot \bar{x} = (a+bi+cj+dk)(a-bi-cj-dk)$

$$= \{a(a)-b(-b)-c(-c)-d(-d)\}+\{a(-b)+b(a)+c(-d)-d(-c)\}i$$
$$+\{a(-c)+c(a)+d(-b)-b(-d)\}j+\{a(-d)+d(a)+b(-c)-c(-b)\}k.$$

But $a,b,c,d$ are real numbers, thus

$$x \cdot \bar{x} = (a^2+b^2+c^2+d^2) + 0 \cdot i + 0 \cdot j + 0 \cdot k \quad ,$$

since $i,j,k$ are also quaternions, from Theorem 7(i), $0 \cdot i = 0 \cdot j = 0 \cdot k = \theta$, and $\theta$ is the additive identity of the quaternions. Thus,

$$x \cdot \bar{x} = (a^2+b^2+c^2+d^2) \; .$$

Similarly,

$$\begin{aligned} \bar{x} \cdot x &= (a-bi-cj-dk)(a+bi+cj+dk) \\ &= (a^2+b^2+c^2+d^2) + 0 \cdot i + 0 \cdot j + 0 \cdot k \\ &= (a^2+b^2+c^2+d^2) \; . \end{aligned}$$

The positive real number $x \cdot \bar{x}$ is called the norm of $x$, and we write $N(x) = x \cdot \bar{x} = \bar{x} \cdot x$ .

THEOREM 13. If $x = a+bi+cj+dk$, then $N(x) = 0$ if and only if $a = b = c = d = 0$.

Proof. If $a = b = c = d = 0$, then $x = \theta$ and $\bar{x} = \theta$. Thus $x \cdot \bar{x} = 0$ and $N(x) = 0$.

If $N(x) = 0$, then since $N(x) = a^2+b^2+c^2+d^2$, $a^2+b^2+c^2+d^2 = 0$. But $a, b, c, d$ are real numbers, thus $a^2+b^2+c^2+d^2$ is always a positive real number or zero. It can only be zero if $a = b = c = d = 0$.

Therefore $N(x) = 0$ if and only if $a = b = c = d = 0$, in which case $x = \theta$.

DEFINITION. A linear algebra with a multiplicative identity and where each element, except 0, has both a right hand and left hand inverse, is called a division algebra.

THEOREM 14. The set $Q$ of all real quaternions forms a division algebra.

Proof. Since it has previously been shown that the quaternions form a linear algebra with a unit element, it is sufficient to show that every element has a right and left hand inverse.

By Theorem 3 and Theorem 13 if $x \neq \theta$, then $N(x) \neq 0$. If $x$ is any real quaternion other than $\theta$ and $x^*$ denotes the inverse of $x$, then $x \cdot x^* = x^*x = 1$.

Let $x^* = \bar{x}/N(x)$, then

$$\begin{aligned} xx^* &= x\bar{x}/N(x) \\ &= N(x)/N(x) \end{aligned}$$

$$= 1.$$

Also, $$x^*x = \bar{x}x/N(x)$$
$$= N(x)/N(x)$$
$$= 1 \ .$$

Therefore if $x \neq \theta$, then the inverse of $x$, $x^*$, is denoted by $\bar{x}/N(x)$, which is both a right and left hand inverse. Hence, every element of $Q$ except $\theta$ has both a right and left hand inverse and $Q$ is a division algebra.

THEOREM 15. In any division algebra the right inverse and the left inverse are equal.

Proof. If $x \varepsilon D$ , then $x$ has a right inverse $x_1^*$ and a left inverse $x_2^*$ , such that $x \cdot x_1^* = 1$ and $x_2^* \cdot x = 1$. Consider $1 = x_2^* \cdot x$, multiply on the right by $x_1^*$ . Then

$$1 \cdot x_1^* = (x_2^* \cdot x)x_1^*$$
$$x_1^* = x_2^*(xx_1^*)$$
$$= x_2^* \cdot 1$$
$$= x_2^* \ .$$

Therefore the right inverse and the left inverse are equal

THEOREM 16. In any division algebra the multiplicative inverse is unique.

Proof. Let $x$ be any element in a division algebra, other than zero. Assume $x$ has two inverses $x_1^*$ and $x_2^*$ , then $xx_1^* = 1$ and $xx_2^* = 1$, hence $xx_1^* = xx_2^*$ .

Multiplying both sides of the equation by the inverse of $x$, we get

$$x^*(xx_1^*) = x^*(xx_2^*)$$

$$(x^*x)x_1^* = (x^*x)x_2^*$$

$$1x_1^* = 1x_2^*$$

$$x_1^* = x_2^* \ .$$

Therefore the multiplicative inverse element is unique.

It is appropriate, at this time, to prove that there are only three division algebras over the field of real numbers. I will begin by proving two Lemmas.

LEMMA 1. If $D$ is a division algebra, then it contains no divisors of zero.

Proof. Since $D$ is a division algebra it has a multiplicative identity $1$.

Assume that there exists $x, y \in D$ such that $xy = 0$, and that $x \neq 0$ and $y \neq 0$. Thus $y$ has a right inverse $y^*$ such that $yy^* = 1$.

Then,
$$\begin{aligned} 0 &= xy \\ 0y^* &= (xy)y^* \\ 0 &= x(yy^*) \\ &= x1 \\ &= x \ . \end{aligned}$$

But this contradicts the hypothesis that $x \neq 0$. Hence a division algebra contains no divisors of zero.

LEMMA 2. Every element of a division algebra over the field of real numbers is a root of a quadratic equation with real coefficients.

Proof. Let $D$ be a division algebra of dimension $n$ over the field $F$ of real numbers.

If $n = 1$, then the division algebra $D$ is the field of real numbers $F$.

We know that any $n+1$ elements of $D$ are linearly dependent, thus if $x$ is an element of $D$, then $1, x, x^2, \ldots, x^n$ are linearly dependent. Thus there exists real numbers $a_0, a_1, \ldots, a_n$ not all zero such that,

$$f(x) = a_0 \cdot 1 + a_1 \cdot x + \ldots + a_n \cdot x^n = 0 \quad .$$

The imaginary roots of this equation with real coefficients come in pairs of the form $a \pm bi$, where $a$ and $b$ are real.

Hence there exist real numbers which are the coefficients of the linear or quadratic factors $f_1(x), f_2(x), \ldots, f_k(x)$, where $f(x) = f_1(x) \cdot f_2(x) \ldots f_k(x)$ .

Since $f(x) = 0$ by Lemma 1, some $f_i(x) = 0$, then $x$ is a root of a linear or a quadratic equation with real coefficients. If $f_i(x)$ is linear its square is quadratic. Therefore $x$ is a root of a quadratic equation with real coefficients.

If we represent any quaternion $x$ by

$$x = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k \quad ,$$

then we get from the definition of multiplication a particular relationship between the special quaternions $1, i, j, k$. This relationship is, that $1$ is the unit element and that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k,$$
$$jk = -kj = i, \qquad ki = -ik = j.$$

These properties and the bilinear postulate completely define quaternion multiplication.

THEOREM 17. The only division algebras over the field of all real numbers are that field, the field of complex numbers, and the algebra of real quaternions.

Proof. Let $D$ be a division algebra of dimension $n$ over the field $F$ of all real numbers.

If $n = 1$, then the division algebra is the set of all real numbers.

Let $1, e_1, e_2, \ldots, e_{n-1}$ be a set of basis elements of $D$. Then from Lemma 2, each $e_i$ is a root of a quadratic equation with real coefficients, thus $e_i^2 + 2a_ie_i + b_i = 0$, where $a_i$ and $b_i$ are real numbers. By completing the square we get,

$$(e_i + a_i)^2 = a_i^2 - b_i \quad .$$

But $a_i^2 - b_i$ is a real number, hence after adding a real number to each $e_i$, the square of this new basis element $e_i$ is a real number.

If the square is $\geq 0$ , it would be the square of a real number $c_i$ , thus

$$e_i^2 = c_i^2$$

$$e_i^2 - c_i^2 = 0$$

$$(e_i - c_i)(e_i + c_i) = 0 \quad , \quad \text{hence}$$

$$e_i = \pm c_i \quad .$$

In which case $e_i$ is some scalar multiple of 1, but $e_i$ and 1 are linearly independent. Thus the square must be negative and,

$$(1) \quad e_i^2 = -d_i^2 \ , \quad \text{where } d_i \text{ is a real number.}$$

Write $E_i = e_i/d_i$, then $E_i^2 = -1$.

If $n=2$, the algebra $(1, E_1)$ is identical to the field of all complex numbers.

Let $n>2$, and denote the basis elements by $1, I, J, \ldots$, where $I = e_1/d_1$ , $J = e_2/d_2$, ... , and $e_1^2 = -d_1^2$, $e_2^2 = -d_2^2$ , ... . Thus, $I^2 = -1$, $J^2 = -1$, ... .

Since $I, J$ are elements in D, then $I + J$ and $I - J$ are also elements in D. By Lemma 2 they are the roots of some quadratic equation with real coefficients, hence

$$(I+J)^2 - a(I+J) - b = 0, \quad \text{and}$$

$$(I-J)^2 - c(I-J) - d = 0, \quad \text{where } a,b,c,d$$

are real numbers. But,

$$(I+J)^2 = -2 + IJ + JI \quad ,$$

$$(I-J)^2 = -2 - IJ - JI, \quad \text{thus}$$

$$-2 + IJ + JI = a(I+J) + b \quad ,$$

$$-2 - IJ - JI = c(I-J) + d \quad .$$

Adding the two together, we get

$$(a+c)I + (a-c)J + (b+d+4)\cdot 1 = 0 \quad .$$

But $1, I, J$ are linearly independent, hence $a + c = 0$, $a - c = 0$, and $b + d + 4 = 0$. Thus, $a = c = 0$ and $b + d = -4$. Hence, $IJ + JI = b + 2$.

Let $g = (2+b)/2$, then $g$ is a real number and $IJ + JI = 2g$. Hence,

$$(I+J)^2 = -2 + IJ + JI = 2g - 2 \quad ,$$

$$(I-J)^2 = -2 - IJ - JI = -2g - 2 \ .$$

From (1), $(I+J)^2$ and $(I-J)^2$ must both be negative, hence $-2g - 2$ and $2g - 2$ are both negative real numbers. In which case $(-1+g)$ and $(-1-g)$ are both negative, and $(-1+g)(-1-g) = 1-g^2$ is a positive real number which has a real square root.

Write $i = I$, $j = (J+gI)/\sqrt{1-g^2}$ . We know that $I^2 = -1$ and $J^2 = -1$ , hence $i^2 = -1$ and

$$\begin{aligned} j^2 &= \{(J+gI)/\sqrt{1-g^2}\}^2 \\ &= \{J^2 + g(IJ+JI) + g^2I^2\}/(1-g^2) \\ &= \{g(2g) - (1+g^2)\}/(1-g^2) \\ &= -(1-g^2)/(1-g^2) \\ &= -1 \ . \end{aligned}$$

Also,

$$ij+ji = I\{(J+gI)/\sqrt{1-g^2}\} + \{(J+gI)/\sqrt{1-g^2}\}I$$

$$= \{(IJ+JI) + gI^2 + gI^2\}/\sqrt{1-g^2}$$
$$= (2g - 2g)/\sqrt{1-g^2}$$
$$= 0 \quad .$$

The product $ij$ is linearly independent of $1,i,j$ and hence may be taken as the fourth basis element $k$. For if not it could be expressed as, $ij = a1+bi+cj$ , where $a,b,c$ are real numbers. By multiplying on the left by $i$ we get,

$$i(ij) = i(a1) + i(bi) + i(cj)$$
$$-j = ai - b1 + c(ij)$$
$$= ai - b1 + c(a+bi+cj)$$
$$= ai - b1 + ca + cbi + c^2j$$
$$0 = (ca-b)1 + (a+cb)i + (1+c^2)j.$$

But $1,i,j$ are linearly independent, thus $1 + c^2 = 0$ and $c^2 = -1$ . This contradicts the fact that $a,b,c$ are real numbers, thus $k = ij$ is linearly independent of $1,i,j$.

I have previously shown that, $i^2 = j^2 = -1$ , $ij + ji = 0$, and $ij = k$, hence $ij = -ji = k$ and

$$k^2 = (ij)(-ji) = i^2 = -1 \quad .$$

By the associative law,

$$ik = i(ij) = -j \; ; \qquad ki = (-ji)i = j \; ;$$
$$kj = (ij)j = -i \; ; \qquad jk = j(-ji) = i \; .$$

Thus $1,i,j,k$ are the basis elements of the real quaternions which was a division algebra of dimension four.

If $n > 4$, then $D$ contains a fifth basis element $p$ such that $p^2 = -1$. Since $i, p$ are in $D$, then $i + p$ and $i - p$ are also in $D$. By Lemma 2 they are the roots of some quadratic equation with real coefficients, hence

$$(i+p)^2 - a(i+p) - b = 0 \quad \text{and}$$

$$(i-p)^2 - c(i-p) - d = 0 \text{ , where } a,b,c,d \text{ are}$$

real numbers but,

$$(i+p)^2 = -2 + ip + pi \quad ,$$

$$(i-p)^2 = -2 - ip - pi \quad , \quad \text{hence}$$

$$-2 + ip + pi = a(i+p) + b \text{ ,}$$

$$-2 - ip - pi = c(i-p) + d \text{ .}$$

Adding these two equations together we get,

$$(a+c)i + (a-c)p + (b+d+4)1 = 0 \text{ .}$$ But $1, i, p$ are linearly independent, hence $a + c = 0$, $a - c = 0$, and $b + d + 4 = 0$. Thus, $a = c = 0$ and $b + d = -4$.

Hence, $ip + pi = b + 2$.

Let $g_1 = b + 2$, then $g_1$ is a real number and $ip + pi = g_1$.

In a similar way it can be found that,

$$jp + pj = g_2 \text{ , and}$$

$$kp + pk = g_3 \text{ ,}$$

where $g_2$ and $g_3$ are real numbers. Then,

$$\begin{aligned} pk &= p(ij) \\ &= (pi)j \\ &= (g_1 - ip)j \end{aligned}$$

$$= g_1 j - i(pj)$$
$$= g_1 j - i(g_2 - jp)$$
$$= g_1 j - g_2 i + kp \quad .$$

Adding $pk$ to each member of the equation, we get

$$2pk = g_1 j - g_2 i + kp + pk$$
$$= g_1 j - g_2 i + g_3 .$$

Multiplying each term on the right by $k$, we get

$$2pk^2 = g_1 jk - g_2 ik + g_3 k$$
$$-2p = g_1 i + g_2 j + g_3 k \quad .$$

Hence $p$ is a linear combination of $i, j, k$. But this is a contradiction of the assumption that $p, 1, i, j, k$ are linearly independent.

Therefore for $n > 4$ there does not exist a division algebra over the field of real numbers. (2, p.62-64)

It is interesting to note that the real numbers and the complex numbers are both commutative with respect to multiplication, but multiplication in the real quaternions is not commutative. This can easily be seen by observing the special quaternions $i, j$. From a previous statement we found that $ij = -ji$, hence $ij \neq ji$.

Therefore from Theorem 17 and the statement above, we find that the set $Q$ of real quaternions forms the only non-commutative division algebra over the field of real numbers.

# CHAPTER III

## MATRIC REPRESENTATIONS OF QUATERNIONS

DEFINITION. Two algebras $\underline{L}$ and $\underline{L}'$ over the same field $F$ are isomorphic if there is a one-one correspondence $\alpha \leftrightarrow \alpha'$ between $\underline{L}$ and $\underline{L}'$ which preserves all three operations:

$$\alpha + \beta \leftrightarrow \alpha' + \beta', \quad c\alpha \leftrightarrow c\alpha', \quad \alpha\beta \leftrightarrow \alpha'\beta'. \text{ (1,p.240)}$$

DEFINITION. A linear transformation $T: V \to W$, of a vector space $V$ to a vector space $W$ over the same field $F$, is a transformation $T$ of $V$ into $W$ which satisfies $(c\alpha + d\beta)T = c(\alpha T) + d(\beta T)$, for all vectors $\alpha$ and $\beta$ in $V$ and all scalars $c$ and $d$ in $F$. (1, p.204)

THEOREM 18. Every linear algebra of order $n$ with a unity is isomorphic to an algebra of $n$X$n$ matrices, which is a sub algebra of the algebra of all $n$X$n$ matrices with real elements.

Proof. The algebra $\underline{L}$ is a vector space of elements $\xi$. Associate with each element $\alpha$ in $\underline{L}$ the transformation $T$ obtained by right multiplication as $\xi T = \xi\alpha$, for any $\xi$ in $\underline{L}$.

If $\beta, \gamma \in \underline{L}$ and $a, b \in F$, then $a\beta + b\gamma$ is in the vector space $\underline{L}$. Consider, $(a\beta + b\gamma)T = (a\beta + b\gamma)\alpha$ . Since $\underline{L}$ is a linear algebra multiplication is bilinear, thus $\quad (a\beta + b\gamma)T = a(\beta\alpha) + b(\gamma\alpha)$

$$= a(\beta T) + b(\gamma T).$$

Hence $T$ is a linear transformation. The set of linear transformations on $\underline{L}$ also forms a linear algebra.

Since a unity $e$ is present, $\alpha \neq \beta$ implies that $e\alpha \neq e\beta$, hence distinct elements $\alpha$ and $\beta$ induce distinct transformations $T$ and $U$.

If $\alpha$ corresponds to the linear transformation $T$ and $\beta$ corresponds to the linear transformation $U$, then from the postulates of a linear algebra,

$$\begin{aligned}
\xi(T+U) &= \xi T + \xi U \\
&= \xi\alpha + \xi\beta \\
&= \xi(\alpha+\beta) \quad ; \\
\xi(cT) &= c(\xi T) \\
&= c(\xi\alpha) \\
&= \xi(c\alpha), \quad \text{where } c \text{ is any scalar;} \\
\xi(TU) &= (\xi T)U \\
&= (\xi\alpha)U \\
&= (\xi\alpha)\beta \\
&= \xi(\alpha\beta) \quad .
\end{aligned}$$

Therefore $\alpha + \beta$ corresponds to $T + U$, $c\alpha$ corresponds to $cT$, and $\alpha\beta$ corresponds to $TU$. Thus the correspondence $\alpha \longleftrightarrow T$ is an isomorphism of the given algebra to an algebra of linear transformations on $\underline{L}$.

The linear transformations of $\underline{L}$ can be represented isomorphically by $nXn$ matrices. Therefore every linear algebra of order $n$ with a unity is isomorphic to an algebra of nXn matrices. (1, p.241)

To exhibit these matrices, choose a basis $\{e_1,\ldots,e_n\}$ of $\underline{L}$. The transformation $T$ then carries each $e_i$ into some element

$$e_i\alpha = \sum_{j=1}^{n} c_{ij}e_j \quad .$$

Every element $x$ of the linear algebra can be expressed as a linear combination of the basis vectors,

$$x = \sum_{i=1}^{n} a_i e_i \quad . \text{ Hence}$$

$$\begin{aligned}
\Big(\sum_{i=1}^{n} a_i e_i\Big)T &= \Big(\sum_{i=1}^{n} a_i e_i\Big)\alpha \\
&= \sum_{i=1}^{n} a_i(e_i\alpha) \\
&= \sum_{i=1}^{n} a_i\Big(\sum_{j=1}^{n} c_{ij}e_j\Big) \\
&= \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} a_i c_{ij}\Big)e_j \quad .
\end{aligned}$$

Relative to these coordinates, the linear transformation $T$ is described by the equations

$$y_j = \sum_{i=1}^{n} a_i c_{ij} \quad ,$$

which has a corresponding matrix $C = (c_{ij})$ .

The correspondence $\alpha \leftrightarrow C$ of $\underline{L}$ to the algebra of matrices $C$ is an isomorphism, this is called the second regular representation of $\underline{L}$. The first regular representation of $\underline{L}$ is found by premultiplication with $\alpha$.

It has been shown that the set $Q$ of all real quaternions forms a linear algebra, with a unity, of dimension four, hence there must exist, by Theorem 18, an algebra of $4X4$ matrices which is isomorphic to the algebra $Q$. I have also shown that the elements $1, i, j, k$ form a basis for the algebra $Q$.

Let the quaternion $1$ correspond to the identity matrix $I$. Let $\alpha = i$, then

$$1 \cdot i = 0 \cdot 1 + 1 \cdot i + 0 \cdot j + 0 \cdot k$$
$$i \cdot i = -1 \cdot 1 + 0 \cdot i + 0 \cdot j + 0 \cdot k$$
$$j \cdot i = 0 \cdot 1 + 0 \cdot i + 0 \cdot j - 1 \cdot k$$
$$k \cdot i = 0 \cdot 1 + 0 \cdot i + 1 \cdot j + 0 \cdot k \quad .$$

Hence

$$i \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $\alpha = j$, then

$$1 \cdot j = 0 \cdot 1 + 0 \cdot i + 1 \cdot j + 0 \cdot k$$
$$i \cdot j = 0 \cdot 1 + 0 \cdot i + 0 \cdot j + 1 \cdot k$$
$$j \cdot j = -1 \cdot 1 + 0 . i + 0 \cdot j + 0 \cdot k$$
$$k \cdot j = 0 \cdot 1 - 1 \cdot i + 0 \cdot j + 0 \cdot k, \quad \text{hence}$$

$$j \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Let $\alpha = k$, then

$$1 \cdot k = 0 \cdot 1 + 0 \cdot i + 0 \cdot j + 1 \cdot k$$
$$i \cdot k = 0 \cdot 1 + 0 \cdot i - 1 \cdot j + 0 \cdot k$$

$$j \cdot k = 0 \cdot 1 + 1 \cdot i + 0 \cdot j + 0 \cdot k$$

$$k \cdot k = -1 \cdot 1 + 0 \cdot i + 0 \cdot j + 0 \cdot k, \quad \text{hence}$$

$$k \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} .$$

If $x = a1 + bi + cj + dk$ is any real quaternion and $\alpha = x$, then

$$1 \cdot x = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k$$

$$i \cdot x = -b \cdot 1 + a \cdot i - d \cdot j + c \cdot k$$

$$j \cdot x = -c \cdot 1 + d \cdot i + a \cdot j - b \cdot k$$

$$k \cdot x = -d \cdot 1 - c \cdot i + b \cdot j + a \cdot k \quad , \quad \text{hence}$$

$$x \longleftrightarrow \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} = A \quad .$$

If $y = a'1 + b'i + c'j + d'k$ is any real quaternion, then

$$y \longleftrightarrow \begin{pmatrix} a' & b' & c' & d' \\ -b' & a' & -d' & c' \\ -c' & d' & a' & -b' \\ -d' & -c' & b' & a' \end{pmatrix} = B \quad .$$

If $A$ is any matrix of the special form

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} ,$$

then there is a quaternion which corresponds to $A$, namely $a + bi + cj + dk$. If $A = B$, we know that $a = a'$, $b = b'$, $c = c'$, and $d = d'$. Hence $x = y$ and the correspondence is one-one.

To prove that this correspondence is an isomorphism it is sufficient to show that the correspondence is preserved under addition, scalar multiplication, and multiplication.

Hence if $x = a + bi + cj + dk$ and $y = a' + b'i + c'j + c'k$ are any real quaternions and $m$ is any real number we get from the definitions of addition, scalar multiplication and multiplication;

$$x+y = (a+a')1 + (b+b')i + (c+c')j + (d+d')k \ ,$$

$$mx = ma1 + mbi + mcj + mdk \ ,$$

$$xy = (aa'-bb'-cc'-dd')1 + (ab'+ba'+cd'-dc')i$$
$$+(ac'+ca'+db'-bd')j + (ad'+da'+bc'-cb')k \ .$$

If $x \longleftrightarrow A$ and $y \longleftrightarrow B$, then;

$$AB = \begin{pmatrix} (a+a') & (b+b') & (c+c') & (d+d') \\ -(b+b') & (a+a') & -(d+d') & (c+c') \\ -(c+c') & (d+d') & (a+a') & -(b+b') \\ -(d+d') & -(c+c') & (b+b') & (a+a') \end{pmatrix} .$$

Hence, $x + y \longleftrightarrow A + B$.

Also

$$mA = \begin{pmatrix} ma & mb & mc & md \\ -mb & ma & -md & mc \\ -mc & md & ma & -mb \\ -md & -mc & mb & ma \end{pmatrix}, \text{ hence}$$

$mx \longleftrightarrow mA$ .

And,

$$AB = \begin{pmatrix} a'' & b'' & c'' & d'' \\ -b'' & a'' & -d'' & c'' \\ -c'' & d'' & a'' & -b'' \\ -d'' & -c'' & b'' & a'' \end{pmatrix}, \text{ where}$$

$$a'' = aa'-bb'-cc'-dd' \ ,$$

$$b'' = ab'+ba'+cd'-dc' \ ,$$

$$c'' = ac'+ca'+db'-bd' \quad ,$$

$$d'' = ad'+da'+bc'-cb' \quad , \quad \text{hence}$$

$xy \longleftrightarrow AB$ .

Therefore this one-one correspondence of elements from Q to this matrix algebra is an isomorphism, and this is the second regular representation of the algebra Q.

The first regular representation is found by premultiplication with $\alpha$. Again we let the unit element 1 correspond to the identity matrix I.

Let $\alpha = i$, then

$$i1 = 01 + 1i + 0j + 0k$$

$$ij = -11 + 0i + 0j + 0k$$

$$ij = 01 + 0i + 0j + 1k$$

$$ik = 01 + 0i - 1j + 0k \quad , \quad \text{hence}$$

$$i \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} .$$

Let $\alpha = j$, then

$$j1 = 01 + 0i + 1j + 0k$$

$$ji = 01 + 0i + 0j - 1k$$

$$jj = -11 + 0i + 0j + 0k$$

$$jk = 01 + 1i + 0j + 0k \quad , \quad \text{hence}$$

$$j \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Let $\alpha = k$, then

$$k1 = 01 + 0i + 0j + 1k$$

$$ki = 01 + 0i + 1j + 0k$$

$$kj = 01 - 1i + 0j + 0k$$

$$kk = -11 + 0i + 0j + 0k \quad , \quad \text{hence}$$

$$k \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} .$$

If $x = a+bi+cj+dk$ and $y = a'+b'i+c'j+d'k$ are any real quaternions, in a similar way, we find that $x \longleftrightarrow A$ and $y \longleftrightarrow B$, where

$$A = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix} , \quad \text{and}$$

$$B = \begin{pmatrix} a' & b' & c' & d' \\ -b' & a' & d' & -c' \\ -c' & -d' & a' & b' \\ -d' & c' & -b' & a' \end{pmatrix} .$$

In a similar way as was done for the second regular representations, it can be shown that this correspondence is one-one.

We can see from the definitions of addition and scalar multiplication that $x + y \longleftrightarrow A + B$ and $mx \longleftrightarrow mA$. However, the product $AB$ does not correspond to the product $xy$. From the definition of matrix multiplication it is found that,

$$BA = \begin{pmatrix} a'' & b'' & c'' & d'' \\ -b'' & a'' & d'' & -c'' \\ -c'' & -d'' & a'' & b'' \\ -d'' & c'' & -b'' & a'' \end{pmatrix} , \quad \text{where}$$

$$a'' = aa'-bb'-cc'-dd'$$

$$b'' = ab'+ba'+cd'-dc'$$

$$c'' = ac'+ca'+db'-bd'$$

$$d'' = ad'+da'+bc'-cb' \quad , \quad \text{hence}$$

$xy \longleftrightarrow BA$. This type of correspondence is called an anti-isomorphism.

There is another representation of real quaternions by matrices, namely as a sub algebra of the algebra of all 2X2 matrices over the complex field.

Let 1 correspond to the identity matrix $I$, and let,

$$i \longleftrightarrow \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \quad j \longleftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \longleftrightarrow \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix},$$

where $\gamma = \sqrt{-1}$.

If $x = a+bi+cj+dk$ and $y = a'+b'i+c'j+d'k$ are real quaternions, then $x \longleftrightarrow A$ and $y \longleftrightarrow B$, where

$$A = \begin{pmatrix} a+b\gamma & c+d\gamma \\ -c+d\gamma & a-b\gamma \end{pmatrix}, \quad \text{and}$$

$$B = \begin{pmatrix} a'+b'\gamma & c'+d'\gamma \\ -c'+d'\gamma & a'-d'\gamma \end{pmatrix}.$$

If $A$ is any matrix of the special form $\begin{matrix} a+b\gamma & c+d\gamma \\ -c+d\gamma & a-b\gamma \end{matrix}$, then there is a quaternion which corresponds to $A$, namely $a+bi+cj+dk$ . If $A = B$, then $a = a'$, $b = b'$, $c = c'$, and $d = d'$. Hence $x = y$ and the correspondence is one-one.

$$A+B=\begin{pmatrix} (a+b\gamma)+(a'+b'\gamma) & (c+d\gamma)+(c'+d'\gamma) \\ (-c+d\gamma)+(-c'+d'\gamma) & (a-b\gamma)+(a'-b'\gamma) \end{pmatrix}$$

$$\begin{pmatrix} (a+a')+(b+b')\gamma & (c+c')+(d+d')\gamma \\ -(c+c')+(d+d')\gamma & (a+a')-(b+b')\gamma \end{pmatrix}, \text{ thus}$$

$x + y \longleftrightarrow A + B$ .

Let $m$ be any real number, then

$$mA = \begin{pmatrix} m(a+b\gamma) & m(c+d\gamma) \\ m(-c+d\gamma) & m(a-b\gamma) \end{pmatrix}$$

$$= \begin{pmatrix} ma + mb\gamma & mc + md\gamma \\ -mc + md\gamma & ma - mb\gamma \end{pmatrix}, \text{ hence}$$

$mx \longleftrightarrow mA$ .

$$AB = \begin{pmatrix} (a+b\gamma)(a'+b'\gamma) + (c+d\gamma)(-c'd'\gamma) & (a+b\gamma)(c'+d'\gamma) + (c+d\gamma)(a'-b'\gamma) \\ (-c+d\gamma)(a'+b'\gamma) + (a-b\gamma)(-c'+d'\gamma) & (-c+d\gamma)(c'+d'\gamma) + (a-b\gamma)(a'-b'\gamma) \end{pmatrix}$$

$$= \begin{pmatrix} a'' + b''\gamma & c''+d''\gamma \\ -c'' + d''\gamma & a''-b''\gamma \end{pmatrix}, \text{ where}$$

$$a'' = aa'-bb'-cc'-dd'$$

$$b'' = ac'+ca'+db'-bd'$$

$$d'' = ad'+da'+bc'-cb' \quad , \text{ hence}$$

$xy \longleftrightarrow AB$ .

Therefore the algebra $Q$ of real quaternions is isomorphic to an algebra of $2X2$ matrices, which is a sub algebra of an algebra of all $2X2$ matrices over the complex field.

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