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The principal objective of this study was to compare the results of a proposed method based upon the response surface model to the Taguchi method. To modify the Taguchi method, the proposed model was developed to encompass the following objectives. The first, with the exception of the Taguchi inner array, was obtain optimal design variable settings with minimum variations, at the same time achieving the target value of the *nominal-the best* performance quality characteristics. The second was to eliminate the need for the use of a noise matrix (that is, the Taguchi outer array), resulting in the significant reduction of the number of experimental runs required to implement the model. The final objective was to provide a method whereby signal-to-noise ratios could be eliminated as performance statistics.

To implement the proposed method, a central composite design (CCD) experiment was selected as a second-order response surface design for the estimation of mean response functions. A Taylor's series expansion was applied to obtain estimated variance expressions for a fitted second-order model. Performance measures, including mean squared error, bias and variance, were obtained by simulations at optimal settings.

Nine test problems were developed to test the accuracy of the proposed CCD method. Statistical comparisons of the proposed method to the Taguchi method were performed. Experimental results indicated that the proposed response surface model can be used to provide significant improvement in product quality. Moreover, by the reduction of the number of experimental runs required for use of the Taguchi method, lower cost process design can be achieved by use of the CCD method.

Comparison of Response Surface Model and Taguchi Methodology for Robust Design

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CHAPTER 1

INTRODUCTION

1.1 Manufacturing Quality

In recent years, for reason of strong competitiveness throughout international markets, manufacturing quality has become a major concern of worldwide importance. Since it is believed that one means to increase market share is to provide high quality products at low costs, continuous quality improvements and cost reductions are regarded as the essential tools to remain in business. In other words, consumers want both high quality and low prices (Kackar, 1986). Since the clientele for manufactured goods have significantly increased their quality requirements and competitive pressures have intensified within numerous business organizations, the need for quality improvements have become increasingly and readily apparent.

Nonetheless, the specific meaning of the word "quality" is difficult to define since there is no single word which can be used to describe all of the possible aspects of quality. These aspects may include, for example, performance, features, reliability, conformance, durability, and serviceability. In addition, the most important aspects of quality change with the nature or characteristics of both the product and customer requirements. Historically, quality has been defined as the ability to perform according to specifications (or a targeted ideal) that satisfy and meet customer requirements. For example, customers want high speed computers, but only at the lowest possible levels of pricing. Genechi Taguchi, who with Edward Deming and others, is considered one of the principal authorities with respect to quality control issues, has drawn considerable attention to an otherwise neglected dimension of quality: the degree to which societal loss could be attributed to a given product. According to Taguchi, "quality is the loss imparted to society from the time a product is shipped" (Taguchi and Wu, 1980). In this sense, losses were due to product performance characteristic deviations from targeted values. Examples of societal losses thus include: failure to meet customer fitness for use requirements; product deterioration during shipping time; and product failure to meet performance ideals. Taguchi's standard of quality measurement was based upon the assertion: the smaller the loss, the more desirable the product. Furthermore, he suggested that the product deviations from ideal standards should be minimized, and considered to be the key to quality improvement. Therefore, only minor production variations from targeted goals (i.e., the ideal) was the preferred standard of quality.

The Taguchi approach was illustrated by a study of customer preferences with respect to Sony televisions (Phadke, 1989a). Investigators reported that the color density distributions for sets made at two different factories were the key factor in customer perceptions of quality (that is, color density distributions were the primary quality characteristic upon which customers based their purchasing decisions). As demonstrated in Figure 1.1, m is the target color density and $m \pm 5$ are the tolerance limits. The distributions for the Sony–Japan factory were approximately normal with the mean in relation to the target at a standard deviation of 1.67. The distributions of the Sony–USA sets were approximately uniform in the range of $m \pm 5$. Among the sets shipped by Sony–Japan, approximately 0.3% were outside of the tolerance limits, whereas practically all of the sets shipped by the Sony–USA were within the limits. From this comparison, it was obvious that the fraction of defective sets was not the key to customer preferences.

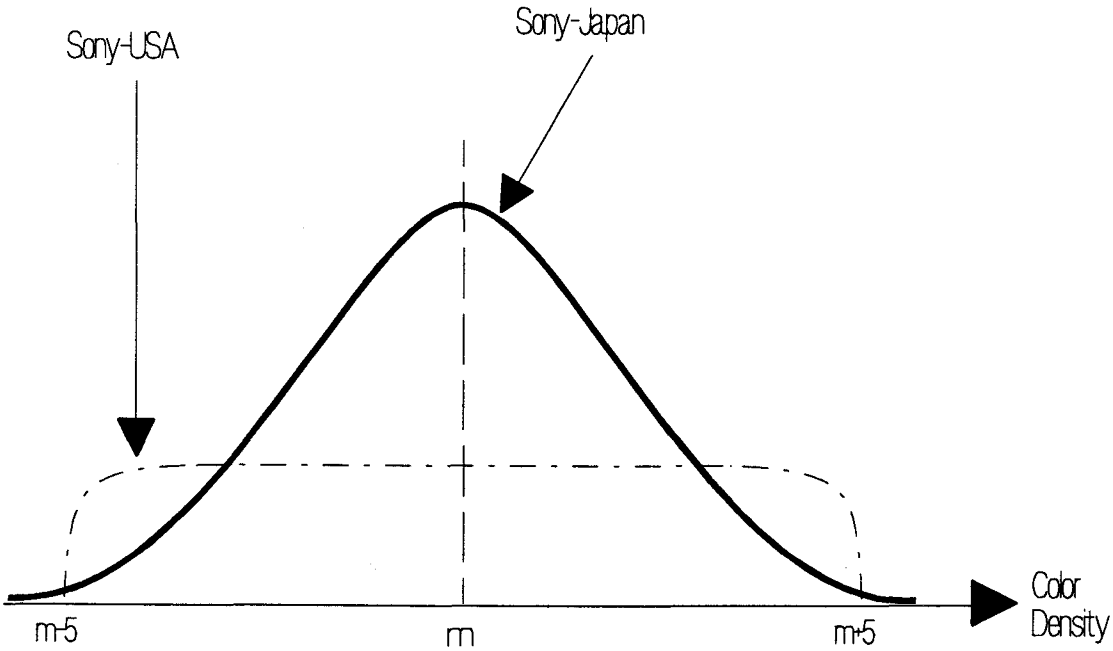


Figure 1.1. Distribution of color density in television sets (Phadke, 1989a).

According to manufacturer specifications, the sets with color density in closest relation to the target (m) performed the best, and were therefore classified as grade A; as color densities deviated from m, performance was regarded as increasingly substandard. The Sony–Japan plant produced a higher percentage of sets which approached target color densities, whereas the Sony–USA factory concentrated upon producing sets that were within the tolerance limits. Thus, the sets produced by Sony–Japan earned the better average grades and were accorded higher preferences from Sony customers in the U.S. The study supported the principal point established by Taguchi, that the effort to minimize deviations of product performance characteristics from an ideal target could be defined as the most important key to quality improvement.

Thus, Taguchi sought to minimize deviations from targets by the introduction and use of the loss function. The objective of this approach was to determine the combinations of values for controllable design variables which minimized expected losses (that is, minimized the mean squares of product performance characteristic deviations from the targets) with respect to an uncontrollable noise space. However, the actual form of the loss function for any given performance characteristic was difficult to express. Therefore, a quadratic approximation of the loss function was recommended as a meaningful approach for most situations. Let y be a response vector variable (i.e., a performance quality characteristic) and τ a target value of y; then y is a random variable with some probability distribution, and may be observed for the quantification of quality level and for the optimization of robust design. Variations in y cause losses to consumers and to producers. Thus, let l(y) represent the loss in dollars due to the deviation of y from τ. For practical purposes, the quadratic loss function which represents economic losses due to performance variation, as suggested by Taguchi (1980), is of the form:

$$l(y) = k^*(y - \tau)^2 , \tag{1}$$

where k is the constant, *quality loss coefficient.* The unknown constant k can thus be determined if l(y) is known for any value of y (Kackar, 1985). Then suppose that

$(\tau-\Delta, \tau+\Delta)$ is the customer's tolerance interval and

$(\tau-\delta, \tau+\delta)$ is the manufacturer's tolerance interval, where

\$A is the cost of lost customers and

\$B is the cost of repair/rework.

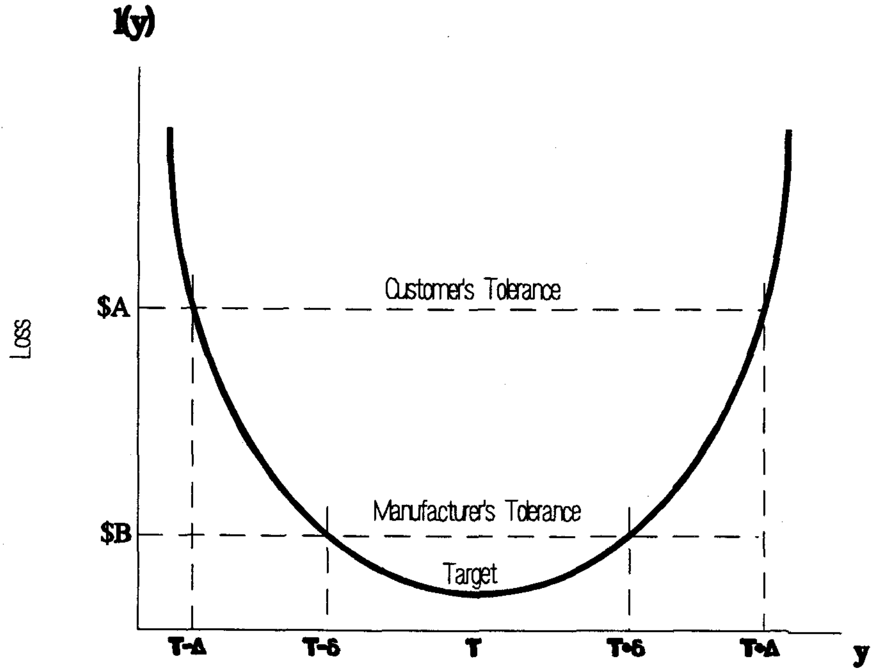
An example of the symmetric quadratic loss function is given in Figure 1.2. However, the loss function can be either symmetric or asymmetric (Kackar, 1985). Variations of the quadratic loss functions (Phadke, 1989b) are as shown in Figure 1.3. In addition, the expected loss can easily be defined for the distribution of y during both the product life span and across different users of product units as:

$$L(y) = E[l(y)] = k * E[(y - \tau)^2] . \tag{2}$$

It is clear that since the mean square deviation is proportional to the expected loss, as shown in (2), minimizing the expected loss is equivalent to minimizing the mean square of the deviation of the product performance characteristic from the target.

Furthermore, the expected loss, or the mean square deviation from the target, can be decomposed into two main parts, including (1) the mean (the location effect) and (2) the variance (the dispersion effect), which can be easily shown by expressing the term $E[(y - \tau)^2]$ as:

$$\begin{aligned} E[(y-\tau)^2] &= E[(y - E(y)) + (E(y) - \tau)]^2 \\ &= E[y - E(y)]^2 + E[E(y) - \tau]^2 + 2 * E[y - E(y)] * E[E(y) - \tau] \\ &= \sigma_y^2 + [E(y)-\tau]^2. \end{aligned}$$



Performance Characteristics
Figure 1.2. Loss function.

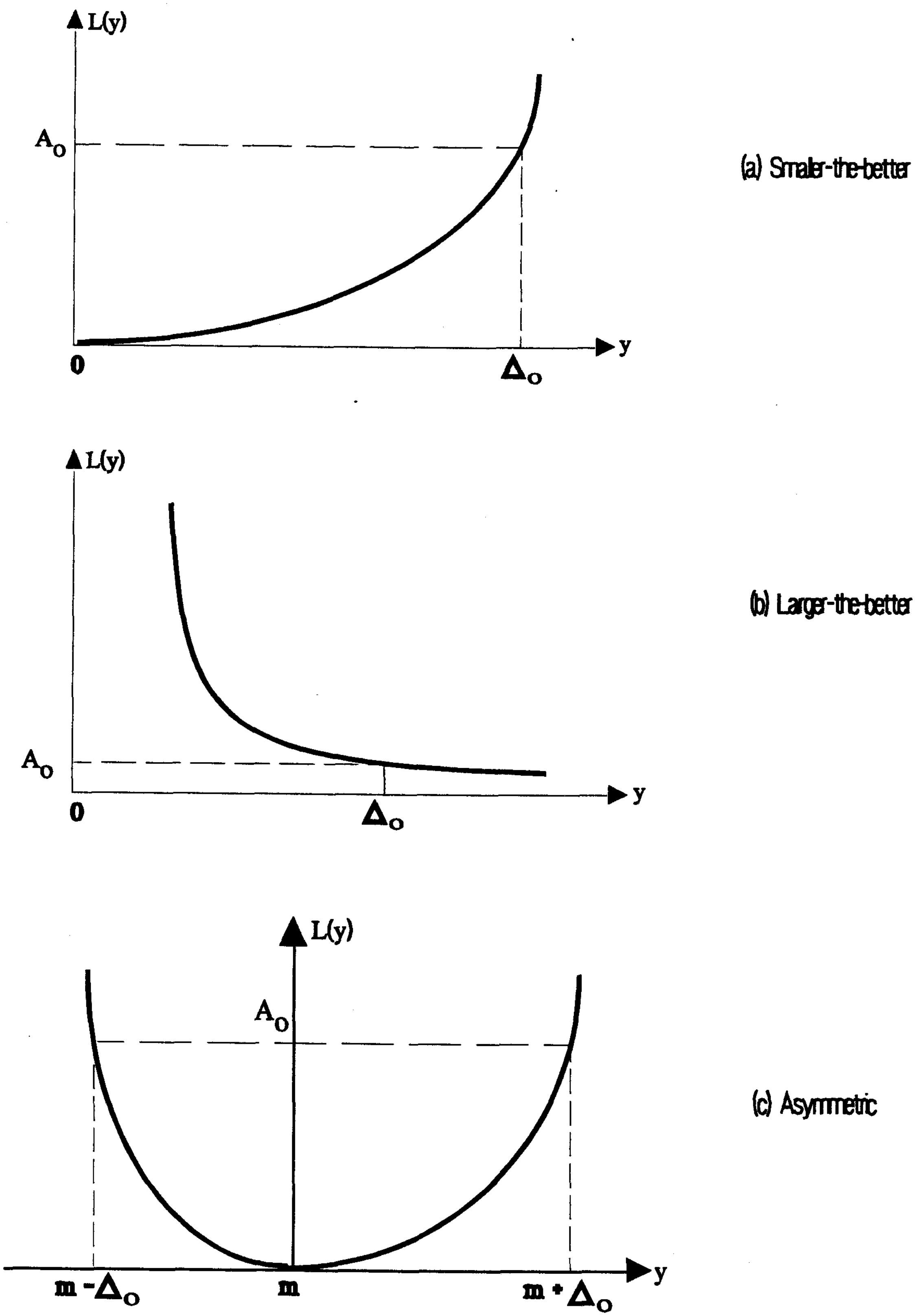


Figure 1.3. Variations of the quadratic loss function.

As a result, for his experimental parameter design, Taguchi (1980) emphasized the application of "orthogonal arrays," recommending the use of *inner* and *outer* arrays for the purpose of accommodating the mean effect and variance effects within the design. The design variables that were determined to influence the mean within the inner array were used to adjust the mean to the target, and were called the *adjustment variables*. The purpose of the outer array was to obtain variance estimates for each design point within the inner array.

1.2 Summary of the Taguchi Approach

The Taguchi approach can be described from the two aspects of strategy and tactics. Recall that the objective of the Taguchi approach is to minimize an expected loss, or to minimize mean square deviations from the target. Taguchi strategy is focused upon finding the design that best minimizes the expected loss (or mean square deviation) over an uncontrollable noise space. The source of noise is classified in the two categories of external and internal sources of noise (Kackar, 1985). The external sources of noise normally are environmental variables, whereas the internal sources of noise include product deterioration or manufacturing imperfections. Due to physical limitations and/or lack of knowledge, not all sources of noise can be included in a parameter design experiment. Those which cannot be included are referred to as uncontrollable noise space. In turn, Taguchi tactics consist of the specific methods and techniques, including design considerations and the signal-to noise ratios, used to accomplish the objectives of the approach.

As noted in section 1.1, Taguchi experimental design consists of the use of an inner array (or a design matrix) and an outer array (or a noise matrix). The inner array, or design matrix, consists of a sample that is controllable from the design variable space. The outer array, or noise matrix, consists of a sample from the noise space. The columns of the noise matrix represent noise factors, whereas the matrix rows represent different combinations of noise factor levels. A complete Taguchi experimental parameter design consists of a combination of design and noise matrices in which the *n* rows of the design matrix represent *n* test runs for *p* observations in each run, as indicated in Table 1.1.

Table 1.1. Taguchi experimental design plan.

Test run	Design matrix (parameters)	Noise matrix (noise factors)	PC[a]	PS[b]
1	$x_{11},x_{12},\ldots,x_{1p}$	$\begin{matrix} w_{11} & w_{12} & \cdots & w_{1q} \\ \vdots & \vdots & & \vdots \\ w_{k1} & w_{k2} & \cdots & w_{kq} \\ \vdots & \vdots & & \vdots \\ w_{r1} & w_{r2} & \cdots & w_{rq} \end{matrix}$	$\begin{matrix} y_{11} \\ \vdots \\ y_{1k} \\ \vdots \\ y_{1r} \end{matrix}$	$\{S/N\}_1$
2	$x_{21},x_{22},\ldots,x_{2p}$	$\vdots$	$\vdots$	$\vdots$
3	$x_{31},x_{32},\ldots,x_{3p}$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n-1	$x_{n-1,1},x_{n-1,2},\ldots,x_{n-1,p}$	$\vdots$	$\vdots$	$\vdots$
n	$x_{n1},x_{n2},\ldots,x_{np}$	$\begin{matrix} w_{11} & w_{12} & \cdots & w_{1q} \\ \vdots & \vdots & & \vdots \\ w_{k1} & w_{k2} & \cdots & w_{kq} \\ \vdots & \vdots & & \vdots \\ w_{r1} & w_{r2} & \cdots & w_{rq} \end{matrix}$	$\begin{matrix} y_{n1} \\ \vdots \\ y_{nk} \\ \vdots \\ y_{nr} \end{matrix}$	$\{S/N\}_n$

Notes: [a] = performance characteristics; [b] = performance statistics.

The use of an orthogonal array was recommended for both matrices and, for data collection plans, the system was tested for a three-level fractional-factorial experiment , based upon signal-to-noise ratios as the sole source of performance statistics (Kackar, 1985). The signal-to-noise ratio statistics (S/N) can be categorized as follows for one of three types of fixed targets for given quality characteristics:

1. "The smaller the better": $S/N_S = -10*\log(\sum_{i=1}^{n} y_i^2/n)$,

2. "The larger the better": $S/N_L = -10*\log(\sum_{i=1}^{n} (1/y_i^2)/n)$, and

3. "Target value is the best": $S/N\tau = 10*\log(\bar{y}^2/s^2)$,

where $\bar{y} = 1/n\sum_{i=1}^{n} y_i$, and $s^2 = 1/(n-1)\sum_{i=1}^{n} (y_i-\bar{y})^2$. It should be noted that the Taguchi signal-to-noise ratio has been regarded as one of the principal weaknesses of the experimental parameter design plan (Pignatiello, 1988).

The steps to identify optimal design parameter settings for the maximization of performance statistics, or the "signal-to-noise ratios," can be addressed as follows:

1. Identify the design parameters and the noise factors, including their ranges.
2. Construct the design matrix and the noise matrix.
3. Plan and conduct the design experiment. (Taguchi recommended use of a fractional-factorial experiment based upon an orthogonal array for physical experimentation, or the collection of real data from the design. Note that the experiment can be based upon either physical trials or a computer-based simulation.)
4. Calculate the performance statistics (the signal-to-noise ratios) for each test run of the design parameter matrix (Table 1.1).
5. Based upon analyses of variance (ANOVA) of the signal-to-noise ratios and the means, identify the controllable variables which influence both

means and variances and those which influence only the means (i.e., the "adjustment" variables), respectively.

6. Determine the optimal settings for the design parameters by determining the combinations of the controllable variables (as identified in step 5) that maximize the signal-to-noise ratios. (First, to fine-tune the solution toward the target, select the controllable variables which exercise an effect upon the mean. The remaining controllable variables, which influence neither means nor variances are set at their most economical condition of performance.)

7. Confirm that the new optimal settings serve to improve the performance statistics.

The Taguchi method has been successfully applied to a number of industrial processes, including the automotive industry (McElroy, 1985; Ealey, 1988), robotics processing (Wu et al., 1991; Jiang et al., 1991), plastics industries (Warner and O'Connor, 1989), and computer-aided design/electrical engineering tasks (Liu et al., 1990; Young et al, 1991). When this method is carefully considered, it may be observed that Taguchi limited possible choices of values for the design variables to those values contained within the design matrix (i.e., the inner array). However, any combination of just these specified values may not be the best to minimize expected losses (i.e., mean square deviations from the target). Moreover, since the design matrix and the noise matrix are crossed, the Taguchi method requires an excessive number of experimental test runs. Thus, it has been hypothesized that if the noise matrix could be eliminated, the number of experimental test runs could be reduced substantially. Given the expense of experimental test runs, as well as the fact that it is often impossible to conduct them during experiments based upon physical experimentation (Kackar, 1985), this approach is considered in Chapter 3. Moreover, designs based upon a three-level fractional-factorial design may be extremely complicated, producing situations in which the experimenters cannot obtain correct answers for reason of the large two-factor interactions between the controllable design variables.

1.3 Research Objectives

Intensive competitive pressures within international markets have necessitated the application of parameter design concepts to production processes. Lin and Kackar (1985), Pao et al., (1985), Phadke et al., (1983), Prasad (1982), and Taguchi and Wu (1980) have demonstrated significant uses of these concepts for the improvement of manufacturing quality. In recent years, attention has been directed primarily at the Taguchi methodology. The principal objective of the current investigation is to compare the results of a proposed approach based upon a response surface model to the Taguchi method. The proposed approach was developed to encompass and test the following objectives as modifications of the Taguchi method:

1. To obtain a set of values, other than those within the inner array, with minimum variations while achieving the target value of *a nominal-the best* type performance quality characteristic;
2. To eliminate the need for the use of the noise matrix (i.e., the outer array); and
3. To eliminate the use of signal-to-noise ratios for the generation of performance statistics.

To implement this approach, a central composite design experiment was chosen as the response surface design for obtaining a fitted second-order response model. According to Lucas (1976), Draper (1982), and Myers et al. (1992), the central composite design, from among all possible second-order response surface designs, has been used to generate the most favorable results. A Taylor's series expansion was applied to obtain estimated variance expressions for the fitted second-order model. Statistical comparisons of the proposed approach to the results obtained from the application of the Taguchi method have been performed. These experimental comparisons have indicated that the proposed response surface model can be used to provide significant improvements in product quality as well as lower cost process designs by reducing the number of experimental test runs required by application of the Taguchi method. Thus, the proposed approach may serve to reduce manufacturing costs and quality loss, resulting in the production of increased numbers of the high-quality products at lower cost factors.

However, due to unknown performance characteristic functions in cases in which the first-order derivatives, or the gradient of the true performance functions, are approximately zero, the proposed model encompasses certain limitations . According to Poston and Stewart (1976), if all derivatives vanish at zero during application of true smoothing functions, approximations based upon a Taylor's series expansion will result in substandard performance. Thus, further research, based upon the removal of this flaw or the use of alternative second-order response surface model designs in place of a central composite design, is suggested. In addition, explorations of the use of weighted least-squares to obtain the improved quadratic response function estimations should be conducted.

CHAPTER 2

LITERATURE REVIEW

The Taguchi approach to parameter design provides an excellent starting point for further research in the statistical analyses of product and process design improvements (Kackar, 1985). As reviewed in Chapter 1, the Taguchi method can be approached from both the strategic and the tactical points of view. Taguchi strategy provides a conceptual framework for planning product and process design experimentation, directed at the determination of designs that are robust with respect to the uncontrollable variables within the product manufacturing and use environments. The quadratic loss function, $l(y) = k*(y - \tau)^2$, is applied to the minimization of performance quality characteristic variations from target goals. Thus, the best design is one which serves to minimize expected losses (i.e., mean square deviations of product performance characteristics from the target) with respect to uncontrollable noise space such as those represented by environmental conditions (e.g., temperature, humidity, or human skill levels).

Tactics consist of the specific methods and techniques that can be applied to Taguchi strategies. As such, these tactics include signal-to-noise ratios and the design process. Box (1985) has stated that it is important for engineers to absorb these Taguchi concepts and then to apply them to processes of quality improvement. However, it was also noted that the Taguchi tactics, as represented by proposed statistical procedures, were often unnecessarily complicated and inefficient. For example, the most common design choices advocated by Taguchi were limited to 2^k factorial, 2^{k-p} fractional-factorial, and three-level fractional-factorial experiments, as well as the use of complicated orthogonal arrays composed of both inner and outer levels. Moreover, most of the

Taguchi design experiments were not capable of accommodating interactions, and thus could provide misleading interpretations of the nature of the experiment or of the interested model. Reviewing considerations of Taguchi strategies and tactics, Pignatiello (1988) observed that much of the controversy surrounding Taguchi methodology has been focused more upon the tactics than the strategies. Therefore, to the end of improving the efficiency and the statistical sufficiency of the Taguchi method, it has become apparent that alternative strategic designs and performance measures should be applied within the context of Taguchi methodology.

2.1 Considerations of Taguchi Performance Statistics

The performance statistics that Taguchi implemented for the determination of the settings of product and process design parameters are called "the signal-to-noise ratios." In Taguchi methodology, the connections between the signal-to-noise ratios and the quadratic loss functions lead to a general principle for the selection of performance measures. As summarized by Kackar (1985), when the performance characteristics Y are continuous variables, the loss functions *l(y)* are usually presented as one of three forms, dependent upon whether smaller is better, larger is better, or a specific target value is best. For the first two cases, it was demonstrated that the connections in question lead to the following performance statistics, respectively: signal-to-noise ratios of ($S/N_S = -10 * \log(\sum_{i=1}^{n} y_i^2/n)$) and ($S/N_L = -10 * \log(\sum_{i=1}^{n} (1/y_i^2)/n)$). However, for the third case, it was demonstrated that two different engineering situations could lead to two different sets of performance statistics, including one which had been recommended in the Taguchi method and one which had not been recommended. Thus, if product performance characteristic variances were linked to the means (i.e., variances and means were functionally dependent upon each other), then the most appropriate performance statistics were those recommended by Taguchi for the target. However, if the performance characteristic variances were not linked to the mean, the most reasonable performance statistics were $\log(s^2)$, which was not included in the Taguchi methodology.

At nearly the same time, Lucas (1985) observed that the emphasis in the Taguchi method was only upon the signal-to-noise ratios performance statistics, and expressed his opinion that it was easier to separately analyze and explain the various responses, including both the signal and noise responses. Lucas noted that with readily available computer capabilities, multiple responses could be analyzed with little effort beyond that required for a single response. However, Lucas did not develop any new methodologies for the separate analysis of the signal and noise responses.

Hunter (1985) presented a new approach to the determination of the design variable settings for "*the target–the best.*" When the signal-to-noise ratios for "the target–the best" were defined as $S/N\tau = 10 * \log(\bar{y}^2/s^2)$, the term $\bar{y}^2/s^2$ was obviously recognizable as the reciprocal of the square of the coefficient of variation, $s/\bar{y}$. To maximize the signal-to-noise ratios (to obtain the best parameter design settings, as advocated within the Taguchi methodology) was the equivalent of minimizing the coefficient of variation, $s/\bar{y}$. The methodology recommended by Hunter was to consider the logarithms of the observations, and then to determine the design variable settings that would yield the minimum s^2 computed from the logarithms of Y.

Leon et al. (1987) also illustrated an inappropriate use of the signal-to-noise ratio for *the target–the best*, $S/N\tau$, for a problem in which the signal-to-noise ratio was dependent upon the adjustment parameters (i.e., the controllable variables that exercise an effect upon the location effects of the mean). In other words, as recommended previously by Kackar (1985), the $S/N\tau$ should not be applied as a performance statistic when the performance characteristic variances were not independent from the means. In addition, Leon et al. (1987) demonstrated that *if certain models for the product or process* *responses were assumed, then the maximization of the signal-to-noise ratio led to the minimization of average squared error loss.* Furthermore, it was stated that when the parameters existed, the use of the signal-to-noise ratio allowed the parameter design optimization procedure to be decomposed into two smaller steps reflecting a division of the design parameters into two groups, one affecting locations and the other affecting dispersions (or both locations and dispersions). Based upon the assumption of a quadratic loss and a particular multiplicative transfer-function (performance characteristics) model, it was further observed that the Taguchi signal-to-noise ratio and the two-step procedure was valid. However, exposed to different types of transfer-functions (e.g., additive models), the validity of using the signal-to-noise ratio for the *target–the best*, $S/N\tau$, was not justified. Therefore, performance measures independent of adjustment (PerMIA) were introduced as new performance measures, in which approach the Taguchi signal-to-noise ratio, $S/N\tau$, was considered to be a special case of the performance measure, PerMIA.

Box et al. (1988) commented on the unnecessary and inefficient use of the signal-to-noise ratios for *the target–the best*, $S/N\tau$, an integral part of the Taguchi method when the experimental analysis for both the dispersion and location effects was under study. With respect to $S/N\tau$, it was stated that the Taguchi analysis implied that the elimination of unnecessarily coupling of dispersion effects and location effects could be effected by application of a log-transformation to the data. The signal-to-noise ratio for *the target–the best* could then be addressed as

$$S/N\tau = 10 * \log(\bar{y}^2/s^2) = 20 * (\log(\bar{y}) - \log(s)) .$$

It was noted that in some situations, either no transformation or some other form of transformation was needed to produce the uncoupling.

Examples of the use of a "*lambda plot*" for determination of an appropriate transformation was presented by Box (1988) and Fung (1986). A *lambda plot* (Box and

Fung, 1983) was a practical tool used for the selection of an appropriate scale for data transformation, based upon the following: Consider a class of transformation y^{λ} indexed by the scalar parameter λ. To construct a plot, data are transformed with respect to Y = ln(y); $\lambda = 0$, and $Y = y^{\lambda}$, where λ is not equal to zero. The main effects and interactions among the variables are calculated for each set of transformed data (using different values of λ). The *t*-ratios or *F*-ratios for these effects are calculated and used as suitably relevant statistics for both the dispersion effects and the location effects. The plot of these ratios against the λ values is obtained as an aid for the selection of an appropriate transformation. The best scale, λ, for the data transformation is at the location of the maximum simplification and the separation of the *t*-ratios or the *F*-ratios. The fitted model thus consists of the effects that have the largest *t*-ratios or *F*-ratios, and that simultaneously reflect a minimum number of main effects and interaction terms (i.e., the simplest model yields).

In addition, Box and Fung (1986) demonstrated that the Taguchi procedure did not necessarily yield an optimal solution, and that the use of the signal-to-noise ratio was therefore without value. Rather, information obtained from experimental data, both expected and unexpected, could be reviewed by simple data analytical methods based upon means and standard deviations in place of the signal-to-noise ratios, which were not only unnecessarily complicated, but which were also inefficient (Box, 1988). This study had indicated that the signal-to-noise ratios for "the smaller the better" (S/N_S) and "the larger the better" (S/N_L) were completely ineffective for identification of the dispersion effects. The use of the signal-to-noise ratio as a performance measure for the response variable S/N_S served to confound the location and the dispersion effects since

$$S/N_S = -10 * \log(\sum_{i=1}^{n} y_i^2/n)$$

and

$$\sum_{i=1}^{n} y_i^2/n = \bar{y}^2 + 1/n(\sum_{i=1}^{n} y_i^2 - n\bar{y}^2) = \bar{y}^2 + (n-1)s^2/n\,.$$

This example supported the previous recommendation to separately analyze the location and the dispersion effects (Lucas, 1985). Box (1988) used an experimental example reported by Quinlan (1985) to demonstrate a simpler means for the separate analysis of the dispersion and location effects, based upon the conduct of normal plots. It was noted that the function $\sum_{i=1}^{n} (1/y_i^2)$ in the expression of the signal-to-noise ratio for *the larger-the better*, S/N_L, which was in turn dependent upon the squared reciprocals of the data, was likely to be exceptionally non-robust with respect to the effects of outlying observations. Moreover, it was observed that the data in *the larger-the better* case may require the reciprocal transformation, Y^{-1}, to induce approximate properties of constant variance, normality, and additivity. Therefore, it was determined that S/N_L was not a wholly appropriate performance statistic. Finally, since the S/N_S and S/N_L ratios involved y^2 and $1/y^2$, both of which were sensitive to either extraordinary values (outliers) or values near zero, Montgomery (1991) provided a strong recommendation against the use of signal-to-noise ratios for *the smaller-the better* (S/N_S) and *the larger-the better* (S/N_L). It was noted that these ratios were not invariant to the linear transformation of the original response.

2.2 Considerations of Taguchi Design

The Taguchi experimental design recommendations were also subject to careful criticism. According to Kackar (1985), experimental designs tested through physical experimentation based upon Taguchi methodology may be impossible to conduct, or may contribute to an excessively large number of experimental runs at considerable expense. Box and Meyer (1986) have stated if the dispersion effects of several factors of influence are investigated using replications of a design, the number of experimental runs based upon use of the Taguchi method could prove excessive. To solve the problem of an excessive number of experimental runs, Box and Meyer (1986) introduced the use of two-level fractional-factorial experiments for the identification of those factors that affect variances as well as those that affect the means. This experimental approach has been recommended for screening as many as 16 factors with an equal number of experimental runs, whereas four replications of the factorial approach could be used to screen only three factors.

Box et al. (1988) traced the origin of the Taguchi designs and indicated that some of the orthogonal designs based upon this approach reflected very complex alias structures. In particular, the Plackett-Burman (1946) design, a saturated resolution III two-level design, and all of the other designs based upon three-level factors, involved partial aliasing of two-factor interactions with the main effects. In cases where the two-factor interactions were large, the experimenter may not have been able to obtain the correct response with respect to the design objectives.

According to Montgomery (1991), Taguchi had argued that explicit considerations of two-factor interactions were not required, stating that it was possible to eliminate these interactions either by correctly specifying the response and design factors or by using a sliding setting approach to choose the factor levels. These two approaches are particularly difficult to implement since they require a high level of process knowledge, which is rarely the case for most experimental situations. Hence, the lack of adequate method for accommodating potential interactions between controllable factors and the noise variables is one of the weak points in the Taguchi parameter design. A safer means is to identify the potential effects and the interactions that may be of importance among the concerned factors, and then provide further consideration only to those which are important. This would lead to the need for fewer experimental runs, simpler interpretation of the data, and better understanding of the process.

Montgomery (1991) observed that there were several alternative experimental designs which could provide results superior to those generated within the inner and outer arrays of the Taguchi parameter design. It was also stated that the use of both arrays was not often necessary and, in any event, the use of this technique would contribute substantially to the size of experiments. Montgomery (1991) demonstrated these criticisms by proposing an alternative design requiring a smaller number of experimental runs and which demonstrated greater statistical efficiency for the pull-force problem. For this problem, Byrne and Taguchi (1987) had used 72 test runs to investigate only seven factors (four of which were controllable factors). However, estimates of the two-factor interactions among the four controllable factors could not be obtained. Montgomery (1991) suggested the use of an experiment that ran all seven factors at two levels. This approach proved to be a superior design for the pull-force problem, and was based upon a one-fourth fractional-factorial design (2^{7-2}) at resolution IV. At 32 test runs, this alternative required fewer than half as many runs as had been conducted by Byrne and Taguchi (1987). The alias relationships for this design have been considered by Montgomery (1991).

Montgomery (1991) introduced two alternative schemes for the assignment of process controllable and noise variables. Each encompassed techniques that allowed experimenters to investigate the interactions between both types of variables, illustrating cleaner relationships among all factors than had been presented in the Byrne and Taguchi (1987) design. Montgomery (1991) concluded that a superior strategy for the improvement of the basic Taguchi design should be based upon a single design inner array which incorporated both the controllable and the noise factors. It was suggested that the design have sufficient resolution, at least resolution IV or higher, to allow for the estimation of all interactions of interest. (The design of resolution k implies that no r factors are aliased with another effect containing less than $k - r$ factors.)

2.3 Alternative Second-Order Designs

Response surface methodology (RSM) consists of a collection of tools for the determination of optimum operating conditions, and is commonly used for the improvement of the basic Taguchi design as well as in the construction of a number of industrial applications. In a review of RSM techniques, Myers et al. (1989) observed that RSM was affected by technological advances effected in other and associated fields of inquiry, including the engineering sciences, the food sciences, and the biological and clinical sciences. The conclusion was that RSM constituted the most favorable means to determine an optimal set of conditions throughout a broad expanse of otherwise unrelated areas of research.

Based upon prior research by Myers and Carter (1973), and Vining and Myers (1990) developed the dual response technique as an implementation of the Taguchi methodology. In this sense, RSM was applied to a dual response problem and an appropriate second-order response surface experiment was conducted. In the area of inquiry of the current investigation, second-order response surface designs are always referred as alternative designs used for the improvement of the Taguchi method. In the dual response problem, two quadratic response functions were fitted, representing the responses of primary interest and secondary interest, respectively. The objective of this approach was to optimize the primary response subject as an appropriate constraint upon the values of the secondary response, to the end of determining appropriate primary and secondary responses. For example, if the objective of the experiment was "the target is the best," or minimizing variance while achieving a target value, the primary quadratic response would be the appropriate function of the variance and the secondary quadratic response would be the mean value or the target value. The Lagrangian multiplier was applied to optimize as well as to determine the set of design variables that would best satisfy the experimental objective. Based upon this approach, repeated experimental runs were required to obtain the quadratic response function for the standard deviation.

Recently, Myers et al. (1992) have sought to determine appropriate second-order response surface design by the use of a variance dispersion graph (VDG) for the prediction of standard second-order design variance properties. As previously developed by Giovannitti-Jensen and Myers (1989), the VDG can be used for the instrumental assessment of the predictive capabilities of design properties for given regions of interest. The VDG "footprint" provides a two-dimensional plot of average prediction variances (APV) with respect to the distances that design points lie from the design center (i.e., the radius values), allowing users to identify both maximum and minimum prediction variances throughout the region of interest.

The second-order designs investigated by Myers et al. (1992) were the central composite design (CCD), the Box-Behnken design (BBD), and the small composite design (SCD) developed by, respectively, Box and Wilson (1951), Box and Behnken (1960), and Hartley (1959). These designs were analyzed over both their spherical and cubodial regions, as follows: Where $x_1, x_2, \ldots, x_k$ represent design variables that have been coded and scaled for use in modeling the response, a spherical region is defined by $\sum_{i=1}^{k} x_i^2 \leq k$, and thus consists of all points on or inside a hypersphere of radius k; a cubodial region is defined by $-1 \leq x_i \leq 1$, for $i = 1,2,\ldots,k$, and thus consists of all points on or inside the hypercube. The standard second-order response surface model is then given by :

$$\hat{y}(x) = \hat{\beta}_o + \sum_{i=1}^{k} \hat{\beta}_i x_i + \sum_{i=1}^{k} \hat{\beta}_{ii} x_i^2 + \sum\sum_{i<j}^{k} \hat{\beta}_{ij} x_i x_j \,.$$

At this point, with the use of the VDG, it was clearly indicated that adding the number of center points (runs) for the central composite design experiment resulted in an improvement of design support for the model at or near its center, while adversely affecting model performance at the perimeters of the spherical regions. Therefore, it was concluded that if the area of interest was design support for the model at the perimeters of the region, the expense of the additional center runs was not justified. However, when the VDG readings were considered, it was determined that the CCD provided the probability of being the most efficient standard second-order response surface model design for experimentally obtaining the estimates $\hat{\beta}$ over both spherical and cubodial regions. (Note that Lucas (1976) had previously suggested central composite experimentation provided one of the most favorable designs for a quadratic response surface model.) For the current investigation, the CCD was employed for the conduct of experimental designs.

Despite the obvious drawbacks of the Taguchi methodology, as previously reviewed, the Taguchi parameter design procedures have gained widespread support as a useful basis for the estimation of manufacturing and process quality improvements. To summarize, Taguchi methods are frequently statistically inefficient with respect to the use of the "signal-to-noise ratios" and the excessive number of experimental runs necessitated by crosses between the inner and outer arrays. Moreover, most of the Taguchi designs consist of alias structures which are excessively complicated. Thus, considerable research efforts have been devoted to the purpose of providing necessary improvements to the basic Taguchi methodology. This is also true of the current investigation, which consists of an analysis of an alternative approach to the determination of settings for the design variables that will contribute to quality improvements.

The research problem is formulated in the following chapter, including an explanation of the means to reduce the excessive number of experimental runs required by the

Taguchi method, as well as the means to eliminate the use of the signal-to-noise ratios as the basis for the Taguchi performance statistics.

CHAPTER 3

RESEARCH APPROACH

The working principle of the Taguchi theory is to minimize the deviation of product performance characteristic from ideal target values through consideration of a quadratic loss function. The specific objective is thus to determine that combination of controllable design variables which best serves to minimize expected losses (that is, the mean square deviation of the product performance characteristics from their targeted goals) over an uncontrollable noise space. The fundamentals of robust design are employed to accomplish this goal. However, the Taguchi experimental design, based upon both inner and outer orthogonal arrays, requires an excessive number of experimental runs, is excessively complicated, and incapable of dealing with interactions (Box, 1985). The present research study, through the introduction of a probable best second-order response surface design, identified as the central composite design (CCD), presents an approach which reduces the excessive number of runs to a significant degree. To clarify this research approach, the influence factors in product or process design experimentation are classified for further consideration.

In Figure 3.1, a block diagram representing a manufacturing process illustrates the involvement of various types of influence factors. Response variables or performance characteristics are denoted by the symbol "Y." The factors that influence the performance characteristics, and which are of concern to the present investigation, may be categorized in two mutually exclusive groups as follows:

1) *Control factors* (X): factors that can be specified freely by the design experimenter. Each control factor can assume different values or levels.

The changing levels of some control factors may increase total manufacturing costs, while those of other control factors may not. The experimenter is responsible for determining the best values for the robust design parameters, generally identified as the "design variables."

2) *Noise factors* (W): factors that cannot be controlled by the experimenter. The levels of noise factors are either difficult and/or expensive to control. For reason of physical limitations and lack of system knowledge, not all of the noise factor sources can be identified, and only the statistical characteristics (e.g., means and variances) of the noise factors can be known and/or specified. In addition, noise factors can cause response deviations of Y from target values. Thus, noise factors may contribute to quality loss.

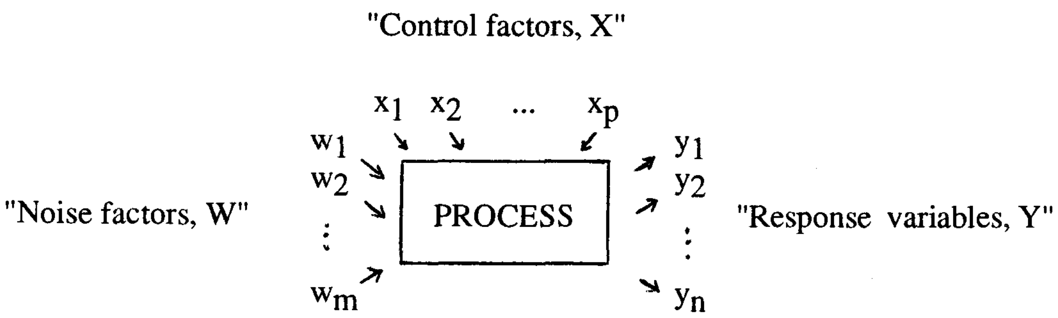


Figure 3.1. Manufacturing process block diagram.

The task of design experimenters and manufacturing engineers is to correctly identify responses, noise factors, and control factors in the process prior to the implementation of statistical analysis procedures. Since cost reduction is one of the essential tools of survival in business competitions, it is also important to recognize which of these factors will affect manufacturing costs and which will not. At the same time, as an equally important requirement to remain in business, the company in question must continue to pursue quality improvement. The principal objective of both the CCD approach and the Taguchi method is to manage both the control factors and the noise factors, but both methods can be used equally as tools in product or process design for the purpose quality improvement. Both methods seek to determine combinations of design variables for an uncontrollable noise space. Thus, the purpose of the current investigation was to compare sets of design variable values (control factors) obtained from the application of the two approaches. In the following sections, problem formulations for the Taguchi method and the proposed CCD approach are compared.

3.1 Problem Formulation

The objective of Taguchi parameter design is to select a set of values for the controllable design variables that will minimize the expected loss function over an uncontrollable noise space. The mathematical formulation of the problem is described as follows:

Let $Y = [y_1\ y_2 \dots y_n]^T$ be an *nx1* vector of response variables (performance quality characteristics), where $\tau = [\tau_1\ \tau_2 \dots \tau_n]^T$ represents the target values of Y. Then let $X = [x_1\ x_2 \dots x_p]^T$ be a *px1* vector for the quantitative, continuous, and controllable design variables, where x_i are stochastic random variables with either known or unknown distributions, $E[X] = [\mu_1\ \mu_2 \dots \mu_p]^T$ is controllable, and $Cov(X) = V_x$ is the variance–covariance matrix of X. Then let $W = [w_1\ w_2 \dots w_m]^T$ be an *mx1* vector of the noise variables where the variance–covariance matrix of W is V_w. Assume that Y is related to X and to W as

$$Y = g(X,W) + \bar{e}\ ,$$

where $\bar{e} = [e_1\ e_2 \dots e_n]^T$ is an nx1 vector of either pure or measurement errors. Then assume that $E[\bar{e}] = \underline{0}$, and that the variance–covariance matrix of $\bar{e}$ is $Cov(\bar{e}) = \sigma_e^2 I$. (Note that the relationship of the performance characteristics and the product parameters can be either an additive model ($Y = g(X,W) + \bar{e}$) or a multiplicative model ($Y = g(X,W) \cdot \bar{e}$). However, for the purposes of the present investigation, concern is directed only to the additive model.) Furthermore, assume that $\Sigma_Y = \mathrm{Cov}(\bar{e}, X, W)$ is the variance–covariance matrix of Y and that V_{XW} is the covariance matrix for x and w, as follows:

$$\mathrm{Cov}(\bar{e}, X, W) = \begin{bmatrix} \sigma_e^2 I & 0 & 0 \\ 0 & V_X & V_{XW} \\ 0 & V_{XW} & V_W \end{bmatrix} \text{ where, in general, } V_{XW} = \underline{0},$$

and the variance–covariance matrix of X and W is $\Sigma_{X,W} = \begin{bmatrix} V_X & V_{XW} \\ V_{XW} & V_W \end{bmatrix}$.

Thus, the Taguchi quadratic loss function can be denoted by

$$l(X,W) = k^*(Y-\tau)^2 = k^*[g(X,W) + \bar{e} - \tau]^2,$$

where k is a constant.

The objective of the Taguchi approach is to determine a set of design variables that minimize expected losses over an uncontrollable noise space. Expected losses are clearly in proportion to the mean square deviation of the product performance characteristics. Therefore, minimization of the expected losses is equivalent to minimization of the mean square deviation of the product performance characteristics. Furthermore, since the $\mathrm{Cov}(\bar{e}, g(X,W)) = \underline{0}$, and $E[\bar{e}] = \underline{0}$, the mean square deviation of the product performance characteristics can be expressed as:

$$\begin{aligned} L(X,W) = E[Y-\tau]^2 &= E[g(X,W) + \bar{e} - \tau]^2 \\ &= E[(g(X,W) - \tau) + \bar{e}]^2 \\ &= E[\{g(X,W) - \tau\}^2 + 2\bar{e}(g(X,W) - \tau) + \bar{e}^2] \\ &= E[\{g(X,W) - \tau\}]^2 + 2E[\bar{e}[g(X,W)] - 2\tau\bar{e} + \bar{e}^2] \\ &= E[g(X,W) - \tau]^2 + \sigma_e^2 I. \end{aligned}$$

Thus,

$$L(X,W) = \sigma^2{}_y(X,W) + [\mu_y(X,W) - \tau]^2 + \sigma_e{}^2 I \,.$$

As a result, the objective of the Taguchi method is then to determine the optimal setting for the design variables, X, to minimize L(X,W). This problem may be defined as:

$$\mathrm{Min}_X\, Z = f_1(X,W) + f_2(X,W) \,,$$

where $f_1(X,W) = [\mu_y(X,W) - \tau]^2$ and $f_2(X,W) = \sigma^2{}_y(X,W) + \sigma_e{}^2 I$. Note that $f_1(X,W)$ is the squared bias of the product performance characteristics, and that $f_2(X,W)$ is the variance. However, the functions $f_1(X,W)$ and $f_2(X,W)$ are usually unknowns with regard to unknown performance characteristics function, Y. The question is then how the estimates $f_1(X,W)$ and $f_2(X,W)$ can be obtained?. For instance, are the first-order models (linear functions) the best estimates for $f_1(X,W)$ and $f_2(X,W)$? Are the second-order models (quadratic functions) the best estimates of $f_1(X,W)$ and $f_2(X,W)$? These two queries form the basis for the current research investigation.

3.2 Central Composite Design Approach

Assume that the product parameters and that performance (quality) characteristics are related in an unknown but possibly non-linear function. The principle goal of central composite design is to exploit the quadratic model, approximating the nonlinear relationship of the product parameters and the performance characteristics. Moreover, the emphasis of CCD applications is to determine a set of values for the product parameters which minimize variations in the product performance characteristics while at the same time achieving the target values. The Taguchi method uses an "outer" array to obtain the estimated variances (s^2), where (s^2) is the product of small changes effected in selected factors (that is, the product control factors and the noise factors) within each point of an "inner" array. To the contrary, the CCD employs a Taylor's series expansion to obtain estimated means and variances for the quadratic approximating response function.

As demonstrated in Figure 3.2, a Taylor's series expansion can be used to approximate a function.

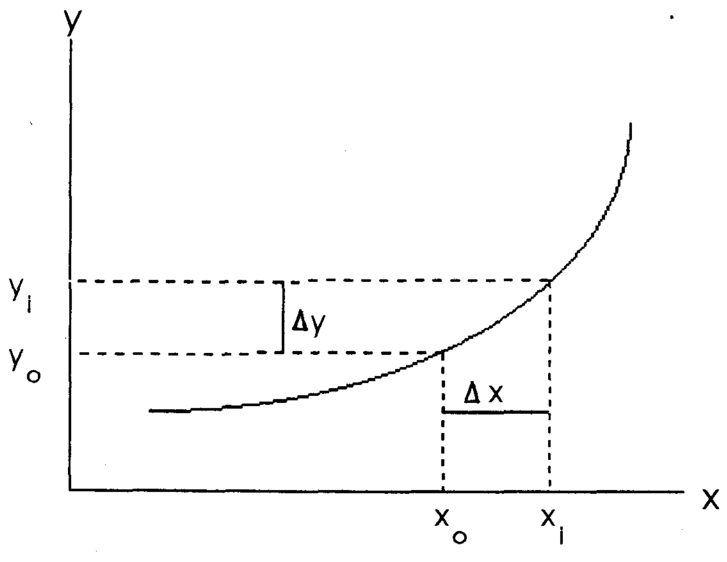


Figure 3.2. Relationships of y to x.

Let $y = f(\cdot)$ be a performance characteristic function. If the value of the function $f(x_o)$ is known at a point x_o, the estimated function of $f(\cdot)$, applying a Taylor's series expansion, is:

$$y = f(x_i) \approx f(x_o) + (\frac{df}{dx}|x_o)^* (x_i - x_o) + o(|x_i - x_o|)^2 .$$

As $x_i \rightarrow x_o$, the term $o(|x_i-x_o|)^2$ vanishes to zero. The estimated mean ($\bar{y}$) and the estimated variance ($s^2{}_y$) can then be derived as

$$\begin{aligned} E(y) &= E[f(x_i)] \approx E[f(x_o)] + (\frac{df}{dx}|x_o)^* E(x_i - x_o) \\ &= f(x_o) + (\frac{df}{dx}|x_o)^* [E(x_i) - x_o] \\ &= f(x_o) + (\frac{df}{dx}|x_o)^* (\mu_x - x_o) , \end{aligned}$$

where, if $x_o = \mu_x$, then $E[y] \approx f(x_o)$. Since $Cov(x_i,x_o) = 0$, and $Var(x_o) = 0$, it follows that

$$\sigma^2_y = Var(y) \approx Var[f(x_o) + (\frac{df}{dx}|x_o)^* (x_i-x_o)]$$

$$= Var[f(x_o)] + Var[(\frac{df}{dx}|x_o)^* (x_i-x_o)]$$

$$= 0 + (\frac{df}{dx}|x_o)^2 \, Var[(x_i-x_o)]$$

$$= (\frac{df}{dx}|x_o)^2[Var(x_i) - 2Cov(x_i,x_o) + Var(x_o)]$$

$$= (\frac{df}{dx}|x_o)^2 \, [Var(x_i)] \, .$$

Thus,

$$\sigma^2_y \approx (\frac{df}{dx}|x_o)^2\sigma^2_x. \qquad (1)$$

Now, consider the functions, $f_1(X,W)$ and $f_2(X,W)$, where

$$f_1(X,W) = [\mu_y(X,W) - \tau]^2 \text{ and } f_2(X,W) = \sigma^2_y(X,W) + \sigma_e^2 I \, .$$

A second-order response surface design, the central composite design, is used for the determination of the estimated quadratic mean response function for the product performance characteristics. This function is expressed in the form

$$E[Y] = \mu_y(X,W) = \beta_o + \beta'(X,W) + (X,W)' \boldsymbol{B} (X,W) + \text{error}.$$

Then, let $D = [z_o : X : W]$ be a data matrix of the size $nx(p+m+1)$, where

1) z_o is an $nx1$ unit vector for which $(X,W) = [x_1 \; x_2 \; ... \; x_p : w_1 \; w_2 \; ... \; w_m]$;
2) x_i is an $nx1$ vector of the design variables; $i = 1,2,..., p$; and
3) w_j is an $nx1$ vector of the noise variables; $j = 1,2,..., m$
4) $\boldsymbol{B}$ is a symmetric $(p+m) \times (p+m)$ matrix of the quadratic coefficients and the cross-product coefficients terms of (X,W), defined as:

$$
\boldsymbol{B} = \begin{bmatrix} \beta_{11} & \frac{\beta_{12}}{2} & \cdots & \beta_{1,(p+m)} \\ \frac{\beta_{12}}{2} & \beta_{22} & \ddots & \vdots \\ \vdots & \vdots & & \\ \beta_{1,(p+m)} & \frac{\beta_{2,(p+m)}}{2} & \cdots & \beta_{(p+m),(p+m)} \end{bmatrix},
$$

where $\beta = [\beta_1\ \beta_2 \ldots \beta_{(p+m)}]^T$ is the *(p+m)x1* vector of the linear coefficients of the term (X,W). Thus,

$$
f_1(X,W) = [\mu_y(X,W) - \tau]^2 \approx [\{\beta_o + \beta'(X,W) + (X,W)'\boldsymbol{B}(X,W)\} - \tau]^2
$$

Furthermore, a polynomial expression of degree d can be thought of as a Taylor's series expansion of the true underlying theoretical function,

$$
f(\xi) = E[Y] = E[g(X,W)],
$$

truncated after the terms of d^{th} order (Box & Draper, 1987). The estimated variance of the estimated quadratic mean response function is then obtained by expanding the variance formula in equation (1) for the univariate case ($\sigma^2_y \approx (\frac{df}{dx}|x_o)^2\sigma^2_x$). The variance function of the product performance characteristics,

$$
f_2(X,W) = \sigma^2_y(X,W) + \sigma_e^2 I,
$$

is approximately equal to $[\beta + 2\boldsymbol{B}(X,W)]'\,\Sigma_{X,W}[\beta + 2\boldsymbol{B}(X,W)]$. This is because

$$
\frac{d\mu(X,W)}{d(X,W)} = [\beta + 2\boldsymbol{B}(X,W)].
$$

Therefore,

$$
\begin{aligned}
\sigma^2_y(X,W) &\approx \left(\frac{d\mu(X,W)}{d(X,W)}\Big|X_o,W_o\right)'\ \Sigma_{X,W}\left(\frac{d\mu(X,W)}{d(X,W)}\Big|X_o,W_o\right) \\
&= [\beta + 2\boldsymbol{B}(X_o,W_o)]'\ \Sigma_{X,W}\,[\beta + 2\boldsymbol{B}(X_o,W_o)],
\end{aligned}
$$

where $\Sigma_{X,W} = \begin{bmatrix} V_X & V_{XW} \\ V_{XW} & V_W \end{bmatrix}$, in which $\Sigma_{X,W}$ is the variance–covariance matrix of X and W, and in general $V_{XW} = \underline{0}$.

Thus, utilizing a second-order response surface model to estimate g(X,W) by means of linear least-squares regression, the estimates, $f_1(X,W)$ and $f_2(X,W)$ are

$$f_1(X,W) \approx [\{\beta_o + \beta'(X,W) + (X,W)'\boldsymbol{B}(X,W)\} - \tau]^2$$

and

$$f_2(X,W) \approx [\beta + 2\boldsymbol{B}(X,W)]' \; \Sigma_{X,W} \; [\beta + 2\boldsymbol{B}(X,W)] \, .$$

Recall that the objective of the Taguchi method is to identify the combinations of design variables values that best minimize expected losses over an uncontrollable noise space, defined as:

$$\mathrm{Min}_X \, Z = f_1(X,W) + f_2(X,W) \, ,$$

where $f_1(X,W) = [\mu_y(X,W) - \tau]^2$ and $f_2(X,W) = \sigma^2{}_y(X,W) + \sigma_e{}^2 I$.

The CCD alternative approach pursues similar goals, utilizing estimates of $f_1(X,W)$ and $f_2(X,W)$ to determine optimal settings for the design variables X that best minimize estimated variances from product performance characteristics, while achieving the target values, τ. Hence, the problem, as formulated in section 3.1, may be defined for the CCD approach as

$$\mathrm{Min}_X \, Z = \sigma^2{}_y(X,W)$$

where $|\mu_y(X,W) - \tau| \leq a$, $x \geq 0$, $w \geq 0$, and $a \geq 0$; or

$$\mathrm{Min}_X \, Z_o = [\beta + 2\boldsymbol{B}\,(X,W)]' \; \Sigma_{X,W}[\beta + 2\boldsymbol{B}\,(X,W)]$$

where $|\{\beta_o + \beta'(X,W) + (X,W)'\boldsymbol{B}(X,W)\} - \tau| \leq a$. Note that the values for w are fixed at their mean prior to optimization of the problem. The value a is the width of the specification limits, $a \geq 0$. The values for x, obtained by optimizing the non-linear program above, are the optimal set of the variables which minimize variations of product performance characteristics, while achieving the target values.

3.2.1 Conceptualization of the Central Composite Design

The purpose of the development of the CCD was to eliminate the excessive number of experimental runs required by applications of the Taguchi method. The underlying concepts for the alternative model are described in the following steps.

STEP 1: Define the design variables (X) and the noise variables (W), obtaining their respective ranges and corresponding means and variances.

STEP 2: Plan the experiment, utilizing the central composite design approach as the basis for a second-order response surface design.

The design matrix for this approach includes both the design variables and the noise variables. Note that the noise matrix (i.e., the "outer" array of the Taguchi method) is not utilized. Useful central composite design experiments should at the least be "rotatable." According to Hunter (1985), in addition to requirements of orthogonality, rotatability and robustness to biases due to unestimated higher order terms are the essential keys to good design. Thus, rotatability, assures that the variances and covariances of the second-order design effects remain unaffected by rotation. (On the other hand, note that orthogonality implies that the design variables may be varied independently.)

STEP 3: Conduct the experiment and obtain values for the performance characteristics (response variables).

STEP 4: Estimate the second-order polynomial used to approximate means for the system ($\mu_y(X,W)$) via linear least-squares regression.

STEP 5: Apply a Taylor's series expansion to obtain estimated variances for the response variables.

Recall the noise matrix ("outer" array) used in the Taguchi method for determination of estimated variances is not required. Given that the use of an outer array crossed with an inner array has often resulted in an unnecessary large number of experimental runs in the Taguchi design, this omission contributes to a reduction in the number of experimental runs . For example, for a 2-variable design problem, the Taguchi method would require the use of a 3^2 factorial experiment for the inner array as well as a 2^2 factorial experiment for the outer array. This would require a total of 36 experimental runs. In comparison, a design based upon the CCD would require only 9 runs. The CCD experiment for k variables problem, as demonstrated in Table 3.1, consists of 2^k runs (i.e., the maximum possible number of runs) for the cubed part, 2*k runs for the star points, and one center point. Therefore, application of the CCD alternative would result in a 75% reduction in the number of experimental runs required.

Table 3.1. Comparison of number of experimental runs, Taguchi method vs. CCD.

No. of design variables	Taguchi inner array	Taguchi outer array	No. of runs/Taguchi	No. of runs/CCD	Percentage reduction
2	3^2	2^2	9 x 4 = 36	9	75
5*	3^{4-2}	2^{5-2}	9 x 8 = 72	43	40.28
10*	3^{9-4}	2^{10-5}	7,776	1,045	86.56
20*	3^{19-8}	2^{20-10}	181,398,528	1,048,597	99.42

* implies that one of the total number of design variables is a noise variable.

Since the exact formula for the design of the inner and outer arrays was not provided in the Taguchi method, specifications provided in Table 3.1 are based upon estimates derived from the implications of the Taguchi approach. In addition, for experimental purposes, the number of experimental runs for those arrays can added to or decreased upon the initiative of the experimenter.

STEP 6: Prior to problem optimization, substitute the values of the means for the noise variables in the estimated quadratic mean response function, and the estimated variance function of the product performance characteristics obtained by applying the Taylor's series expansion.

This step is undertaken since the levels of the noise variables are both difficult and expensive to control. Nonetheless, these variables are required to obtain closer approximation of the slope used in a Taylor's series expansion. Thus, the noise variables are included in the design matrix and their levels are varied only to obtain the slopes of the estimated quadratic mean response function.

STEP 7: Utilizing non-linear programming optimization, determine optimum values for the design variables to minimize variations in production processes while achieving the target values.

STEP 8: Reoptimize the problem around the optimum set determined in Step 7 by repeating Steps 2 to 7, as required. The difference between the two optimum sets (based upon the percentage error allowed in the experimental design) provide the criterion for termination of the central composite design approach.

Note that since it is convenient to avoid the use of actual numerical measures for the variables, *standardized* variables can be developed prior to the application of the CCD method. This is because distinctive design variable scales may lead to a numerically unstable during the optimization problem in Step 7. The *standardized* variables will help solving this difficulty.

In summary, the proposed method is less complicated, and decreasing the number of experimental runs that would have been required from the application of the Taguchi method. In addition, the use of unsatisfactory performance statistics (i.e., the "signal-to-noise ratios") of the Taguchi method can be eliminated in the CCD method. However, due to unknown performance quality characteristic functions, there may be limitations to the utility of the CCD approach. If the first-order derivatives, or the gradient of the true performance function, are approximately zero (i.e., the function is very flat), then the CCD method may not work as well as the Taguchi method. Moreover, the quadratic function may not approximate the true performance characteristics function with accuracy when the true function consists of very high-order degrees for the polynomial terms. This is because the slopes of the true performance characteristics function and the estimated quadratic function are quite different.

For the present investigation, the CCD method was tested using response surface design as provided in *STATGRAPHICS*, version 5.0 (Statistical Graphics Corporation), to obtain the design matrix outline in Step 2. Experimentation, as outlined in Step 3, has not been conducted. The estimated quadratic mean response function of the product performance characteristics was also obtained by the application of regression analysis techniques provided in *STATGRAPHICS.* Note that this program has been used to perform statistical analyses throughout the current investigation. In additions, *GAMS* (developed by Brooke, Kendrick, and Meeraus, 1988) was employed as the nonlinear programming optimization software for the determination of optimal design variable settings based upon application of the CCD. In the following section, an example is provided to demonstrate how the design variable settings are obtained using, respectively, the Taguchi method and the CCD.

3.2.2 Example Problem

A force problem* is used to illustrate the application of the CCD to obtain the optimal design variable settings in comparison to the method of application of the Taguchi approach. The problem is given as:

$$y = (300 + 16x_5) * (140/x_1 - 1) + x_3 * (x_2 + (x_5-20) * (280/x_1 - 1) - x_4) * (280/x_1 - 1)$$

where $y =$ force (grams),

$x_1 =$ front edge of the paper to pivot,

---

* As developed by Dr. David Ullman, Associate Professor of Mechanical Engineering, Oregon State University, personal communication to Dr. Edward McDowell, Associate Professor of Industrial and Mechanical Engineering, Oregon State University, March, 1988.

$x_2 =$ spring connection point,

$x_3 =$ spring stiffness,

$x_4 =$ spring free length,

$x_5 =$ paper thickness,

$x_1 \in$ (100, 180) mm,

$x_2 \in$ (35, 75) mm,

$x_3 \in$ (5, 15) mm,

$x_4 \in$ (20, 50) mm,

$\sigma_{x1} = \sigma_{x2} = 1$ mm,

$\sigma_{x3} = \sigma_{x4} = 2$ mm,

x_5 is distributed Uniform(0, 50), and is a noise variable. The target value of force is 400 grams.

Hence, the performance (quality) characteristic (or a response variable) in this problem is "force (y).". The design variables are the front edge of the paper (x_1), the spring connection point (x_2), the spring stiffness (x_3), and the spring free length (x_4). The noise variable is the paper thickness (x_5).

A 3^{4-2} or 1/9 replicate factorial experiment for the Taguchi method design matrix or inner array was employed (Table 3.2). A 2^{5-2} or 1/4 replicate was employed for the noise matrix or outer array (Table 3.3). The data matrix was obtained by crossing the design and noise matrices. Therefore, 72 (9 × 8) observations resulted, and signal-to-noise ratios for the target, the best case ($S/N_T = 10 * \log(\bar{y}^2/s^2)$, were calculated (Table 3.4). Analyses of variances for the means and for the signal-to-noise ratios were constructed to determine the *adjustment variables* (i.e., the design variables that affected only the means of the performance characteristics) and the design variables that affected both the means and the variances of the performance characteristics.

Table 3.2. Force problem design matrix for the the Taguchi method.

run	x_1	x_2	x_3	x_4	x_5
1	100 (0)	35 (0)	5 (0)	20 (0)	25
2	140 (1)	35 (0)	15 (2)	35 (1)	25
3	180 (2)	35 (0)	10 (1)	50 (2)	25
4	100 (0)	55 (1)	10 (1)	35 (1)	25
5	140 (1)	55 (1)	5 (0)	50 (2)	25
6	180 (2)	55 (1)	15 (2)	20 (0)	25
7	100 (0)	75 (2)	15 (2)	50 (2)	25
8	140 (1)	75 (2)	10 (1)	20 (0)	25
9	180 (2)	75 (2)	5 (0)	35 (1)	25

* represents level codes within the design matrix

Table 3.3. Force problem noise matrix for the Taguchi method.

x_1	x_2	x_3	x_4	x_5
-1	-1	-2	+2	39
+1	-1	-2	-2	11
-1	+1	-2	-2	39
+1	+1	-2	+2	11
-1	-1	+2	+2	11
+1	-1	+2	-2	39
-1	+1	+2	-2	11
+1	+1	+2	+2	39

Table 3.4. Data matrix obtained via the Taguchi method.

run	x_1	x_2	x_3	x_4	x_5	y	mean	S/N_τ
	99	34	3	22	39	639.01		
	101	34	3	18	11	184.07		
	99	36	3	18	39	671.92		
1	101	36	3	22	11	173.43	493.34	2.82
	99	34	7	22	11	140.12		
	101	34	7	18	39	973.04		
	99	36	7	18	11	216.91		
	101	36	7	22	39	948.22		
	139	34	13	37	39	221.25		
	141	34	13	33	11	-104.26		
	139	36	13	33	39	300.37		
2	141	36	13	37	11	-129.90	74.27	-9.76
	139	34	17	37	11	-205.74		
	141	34	17	33	39	324.11		
	139	36	17	33	11	-102.28		
	141	36	17	37	39	290.59		
	179	34	8	52	39	-234.18		
	181	34	8	48	11	-190.62		
	179	36	8	48	39	-207.09		
3	181	36	8	52	11	-199.37	-223.71	19.43
	179	34	12	52	11	-259.97		
	181	34	12	48	39	-232.98		
	179	36	12	48	11	-219.35		
	181	36	12	52	39	-246.11		
	99	54	8	37	39	1139.39		
	101	54	8	33	11	255.39		
	99	56	8	33	39	1227.15		
4	101	56	8	37	11	227.04	799.42	2.60
	99	54	12	37	11	209.10		
	101	54	12	33	39	1519.55		
	99	56	12	33	11	340.73		
	101	56	12	37	39	1477.01		
	139	54	3	52	39	71.39		
	141	54	3	48	11	-11.87		
	139	56	3	48	39	89.64		
5	141	56	3	52	11	-17.79	49.25	-4.32
	139	54	7	52	11	-47.20		
	141	54	7	48	39	164.10		
	139	56	7	48	11	-4.60		
	141	56	7	52	39	150.30		

Table 3.4 (continued).

run	x_1	x_2	x_3	x_4	x_5	y	mean	S/N$_\tau$
	179	54	13	22	39	112.05		
	181	54	13	18	11	113.15		
	179	56	13	18	39	156.06		
6	181	56	13	22	11	98.93	159.04	10.25
	179	54	17	22	11	154.53		
	181	54	17	18	39	222.07		
	179	56	17	18	11	212.08		
	181	56	17	22	39	203.47		
	99	74	13	52	39	1731.18		
	101	74	13	48	11	415.34		
	99	76	13	48	39	1873.79		
7	101	76	13	52	11	369.26	1195.51	3.14
	99	74	17	52	11	369.49		
	101	74	17	48	39	2154.67		
	99	76	17	48	11	555.97		
	101	76	17	52	39	2094.41		
	139	74	8	52	39	585.04		
	141	74	8	48	11	368.30		
	139	76	8	48	39	633.73		
8	141	76	8	52	11	352.52	599.31	9.80
	139	74	12	52	11	525.27		
	141	74	12	48	39	877.49		
	139	76	12	48	11	598.31		
	141	76	12	52	39	853.83		
	179	74	3	37	39	-120.54		
	181	74	3	33	11	-48.62		
	179	76	3	33	39	-110.38		
9	181	76	3	37	11	-51.91	-36.97	-4.0
	179	74	7	37	11	22.37		
	181	74	7	33	39	-12.54		
	179	76	7	33	11	46.07		
	181	76	7	37	39	-20.19		

In Table 3.5, by ranking the sum of the squares, the design variables that affect the means were, respectively, in descending order x_1, x_2, x_3, and x_4. From Table 3.6, the variables that affected the signal-to-noise ratios (both means and variances) were, respectively, in descending order x_3, x_4, x_1, and x_2. The Taguchi method first calls for the selection of the variables that have the least effect on the means and variances and which have the greatest effect on the means. In this approach,whether x_1 or x_2 should be selected first for the adjustment of the target is an unknown. The ratios of the sums of the squares of x_1 and x_2 were calculated from Tables 3.5 and Table 3.6.

Table 3.5. Analysis of variance for the means, Taguchi approach.

Sources of variation	Sums of squares	d.f.	Mean squares
x_1	1,167,129.2	2	583,564.61
x_2	333,623.1	2	166,811.57
x_3	151,645.4	2	75,822.72
x_4	28,819.2	2	14,409.58
residual	-1.42×10^{-10}	0	
Total (corrected)	1,681,217.0	8	

Table 3.6. Analysis of variance for the mean and variances, Taguchi approach.

Source of variation	Sums of squares	d.f.	Mean square
x_1	150.62	2	75.31
x_2	3.16	2	1.58
x_3	252.46	2	126.23
x_4	227.15	2	113.57
residual	-5.68×10^{-14}	0	
Total (corrected)	633.3859	8	

Though x_1 had an effect upon the mean that was approximately triple that of x_2, the x_2 effect upon the signal-to-noise ratios was smaller than that of x_1. by a ratio of approximately 1 to 50. Therefore, x_2 was selected first. Similarly, the ratios of the sum of the squares for x_3 and x_4 were calculated. The order of the design variables selected to adjust the values to meet the target values was x_2, x_1, x_3, and x_4, respectively. The means table for the signal-to-noise ratios (Table 3.7) was then used to determine the levels of the design variables for maximization of the signal-to-noise ratios. The starting values of the design variables before adjusting to meet the target were x_2 =35 mm, x_1 = 180 mm, x_3 = 10 mm, and x_4 = 20 mm (recall x_5 was fixed at a mean level of 25 mm).

Table 3.7. Estimates of the means for the signal-to-noise ratios.

Level	Count	Average
x_1		
0 (100)	3	2.8533
1 (140)	3	-1.4267
2 (180)	3	8.5600*
x_2		
0 (35)	3	4.1633*
1 (55)	3	2.8433
2 (75)	3	2.9800
x_3		
0 (5)	3	-1.8333
1 (10)	3	10.6100*
2 (15)	3	1.2100
x_4		
0 (20)	3	7.6233*
1 (35)	3	-3.7200
2 (50)	3	6.0833

* represents levels of the variable for which the signal-to-noise ratios yielded a maximum.

Substuting the starting values of the design variables into the force equation resulted in a negative value of force (-56.79 grams), the value of x_2 was first adjusted by effecting a maximum increase in x_2 in order to meet the target. The result was x_2 = 75 mm. and the force (y) = 165.432 grams. Second, the values of x_1 was adjusted to meet the target value, resulting in x_1 = 156 mm. and a force of (y) = 396.976 grams, values which were reasonably close to those for target. Thus, the optimal settings resulting from application of the Taguchi approach were:

x_1 = 156 mm,

x_2 = 75 mm,

x_3 = 10 mm,

x_4 = 20 mm, and

x_5 = 25 mm.

The estimated mean response of the force (y) was 396.976 grams.

For the CCD experimental design, a 2^5 rotatable force problem is used as an example to demonstrate how the optimal settings for the design variables are obtained. Based upon a 2^5 factorial experiment for the cubed part of the rotatable CCP design, with two central points for the force problem, a design matrix was constructed with the use of *STATGRAPHICS*. Based upon response surface design, the data matrix was arranged so that the design points were in given variable ranges, adjusting the low and high values accordingly. The resultant low and high values that used as the inputs for the design variables in all cases are presented in Appendix A. Based upon the data given in Table 3.8, approximated quadratic mean response functions for the force problem was obtained through application of mulitiple linear regression . The estimated quadratic mean response function was :

$$\hat{\mu}_Y(x) = \hat{\beta}_0 + \sum_{i=1}^{5} \hat{\beta}_i x_i + \sum_{i=1}^{5} \hat{\beta}_{ii} x_i^2 + \sum\sum_{i<j}^{5} \hat{\beta}_{ij} x_i x_j ,$$

where $\hat{\beta}_0 = 507.055564$, $\hat{\beta}_{12} = -0.144516$, $\hat{\beta}_{25} = -3.299\text{x}10^{-14} \approx 0$,

$\hat{\beta}_1 = -15.338159$, $\hat{\beta}_{13} = -0.436905$, $\hat{\beta}_{33} = -0.01481$,

$\hat{\beta}_2 = 20.287442$, $\hat{\beta}_{14} = 0.144516$, $\hat{\beta}_{34} = -1.023226$,

$\hat{\beta}_3 = 39.339175$, $\hat{\beta}_{15} = -0.411358$, $\hat{\beta}_{35} = 1.093982$,

$\hat{\beta}_4 = -20.005715$, $\hat{\beta}_{22} = -0.001053$, $\hat{\beta}_{44} = -0.00237$,

$\hat{\beta}_5 = 57.511553$, $\hat{\beta}_{23} = 1.023226$, $\hat{\beta}_{45} = 7.456\text{x}10^{-14} \approx 0$,

$\hat{\beta}_{11} = 0.083098$, $\hat{\beta}_{24} = -2.83\text{x}10^{-13} \approx 0$, $\hat{\beta}_{55} = -0.000592$.

Table 3.8. Data matrix for a 2^5 rotatable CCD for the force problem.

run	x_1	x_2	x_3	x_4	x_5
1	140	55	14.7568	35	25
2	175.676	55	10	35	25
3	125	47.5	8	30	35
4	125	47.5	12	40	35
5	140	55	10	35	25
6	125	47.5	12	40	15
7	140	55	10	35	1.21586
8	140	37.1619	10	35	25
9	155	47.5	12	40	15
10	125	62.5	8	40	35
11	140	55	10	46.8921	25
12	155	62.5	12	40	35
13	155	62.5	8	30	35
14	125	47.5	12	30	35
15	125	62.5	12	40	35
16	155	47.5	12	30	35
17	125	62.5	12	30	35
18	155	62.5	12	40	15
19	155	47.5	12	30	15
20	155	47.5	8	40	35
21	155	62.5	8	40	35
22	155	62.5	12	30	35
23	155	62.5	8	40	15
24	125	47.5	8	30	15
25	155	47.5	8	40	15
26	125	62.5	12	40	15
27	155	47.5	12	40	35
28	155	62.5	8	30	15
29	125	47.5	8	40	35
30	155	62.5	12	30	15
31	140	55	10	35	48.7841
32	125	62.5	8	30	15
33	155	47.5	8	30	35
34	140	55	10	35	25
35	155	47.5	8	30	15
36	125	62.5	12	30	15
37	125	47.5	8	40	15
38	140	72.8381	10	35	25
39	140	55	5.24317	35	25
40	125	62.5	8	30	35
41	125	47.5	12	30	15
42	140	55	10	23.1079	25
43	104.324	55	10	35	25
44	125	62.5	8	40	15

Since the purpose was to obtain the best estimates possible for slopes from the estimated quadratic mean response function, statistical significance tests are not performed for all coefficient terms to distinguish them from zero. In this case, the results revealed that the best estimates of slope resulted in improved approximations of variance for the response (y) from the application of the Taylor's series.

From the previous section, the nonlinear programming problem for the CCD method was defined as:

$$\mathrm{Min}_X\, Z_o = [\beta + 2\boldsymbol{B}(X,W)]'\, \Sigma_{x,w}\, [\beta + 2\boldsymbol{B}(X,W)]\,,$$

where s.t. $|\{\beta_o + \beta'(X,W) + (X,W)'\,\boldsymbol{B}(X,W)\} - \tau| \le a$. For the 2^5 rotatable CCP case, the nonlinear programming problem could be defined as:

$$\mathrm{Min}_X\, Z_o = [\hat{\beta} + 2\hat{\boldsymbol{B}}\, X]'\, \Sigma_{x,w}\, [\hat{\beta} + 2\hat{\boldsymbol{B}}X]\,,$$

where s.t. $|\{\hat{\beta}_o + \hat{\beta}'X + X'\hat{\boldsymbol{B}}X\} - 400| \le 0$,

and where

$$\hat{\beta}_o = 507.055564,$$

$$\hat{\beta} = [-15.338159\;\; 20.287442\;\; 39.339175\;\; -20.005715\;\; 57.511553]^T,$$

$$\hat{\boldsymbol{B}} = \begin{bmatrix} 0.083098 & -0.144516/2 & \cdots & -0.411358 \\ -0.144516/2 & -0.001053 & & \vdots \\ \vdots & \vdots & \ddots & \\ -0.411358 & 0 & \cdots & -0.000592 \end{bmatrix},$$

$$X = [\,x_1\;\; x_2\;\; x_3\;\; x_4\;\; 25]^T.$$

(Note: x_5 is a noise variable, thus its value is set at mean level prior to optimization of the nonlinear programming problem. $\sigma_{x1} = \sigma_{x2} = 1$ mm., $\sigma_{x3} = \sigma_{x4} = 2$ mm., $\sigma_{x5} =$ 14.434 mm, and the covariances for x are zero.) Then, recall that V_x, V_w, and V_{xw} are the variance–covariance matrices for the control variables, x_i, i = 1,2,3,4, and for the noise variables, x_5, and for the covariance matrices of the control variables and the noise variable, respectively. Thus,

$$V_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \text{ and } V_w = [208.33] \ (V_{xw} = \underline{0}).$$

Hence,

$$\Sigma_{x,w} = \begin{bmatrix} V_x & V_{xw} \\ V_{xw} & V_w \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 208.33 \end{bmatrix}.$$

Substituting the values of $\hat{\beta}_o$, $\hat{\beta}$, $\hat{\boldsymbol{B}}$, X, and $\Sigma_{x,w}$ in the nonlinear program list, provided in Appendix C, and applying nonlinear programming optimization software (*GAMS*), the optimal set of design variables for the 2^5 rotatable CCD force problem case is solved with values of

$x_1 = 176.48$ mm,

$x_2 = 75$ mm,

$x_3 = 15$ mm,

$x_4 = 20.72$ mm, and

$x_5 = 25$ mm.

The force (y), obtained by substituting optimal values for x_i in the force function, is 358.7047 grams. (Note that the force function is given further consideration in section 3.4.2.)

3.3 Research Questions

From application of the CCD to problem situations, a number of questions remain at issue. First, the CCD approach is based upon a second-order response surface model, considered as a Taylor's series expansion, to obtain estimated variance functions for the response, Y. Experimenters must remain aware of differences between the Taylor's series expansion and the second-order response surface model. Note that the approximation possibilities for a polynomial function of a given degree are improved in relation to reductions in the size of the region R over which the approximations are made (Box & Draper, 1987). This can be demonstrated based upon examples of a univariate case, as given in Figures 3.3 and 3.4, each of which provides a comparison of estimated quadratic functions to Taylor's series expansion. Figure 3.3 indicates, when the quadratic function is obtained from data consisting of points around x_o (i.e., within the small region around the point of interest ,x_o), that the Taylor's series expansion at x_o yields approximately the same result as the quadratic approximation of a true function, $f(x_o)$. In contrast, Figure 3.4 indicates, when the quadratic function is obtained from data consisting of points distant from x_o (i.e., from a wider region around the point of interest ,x_o), how the Taylor's series expansion at x_o yields a different result from the quadratic approximation of a true function, $f(x_o)$. Thus, in comparison to results obtained by application of the Taguchi method, reasonable doubts may be raised with respect to the accuracy with which the CCD approach can be used to estimate variance response functions for the performance characteristics. In other words, the principal issue of concern is the performance of the CCD in comparison to Taguchi method performance.

Second, the second-order response design of the central composite design experiment can be formed in various ways. The CCD consists of a cubed part (i.e., a 2^{k-p} or 2^k factorial experiment), star points, and center points. The design can be rotatable, orthogonal, or both rotatable and orthogonal. The research question which arises is whether rotatability is a necessary property of the design required for the implementation of this approach. Or is it necessary to use a 2^{k-p} or 2^k factorial experiment for the cubed part of the design experiment? Moreover, what are the *best* designs to use for the approximation of $\mu_y(X, W)$?

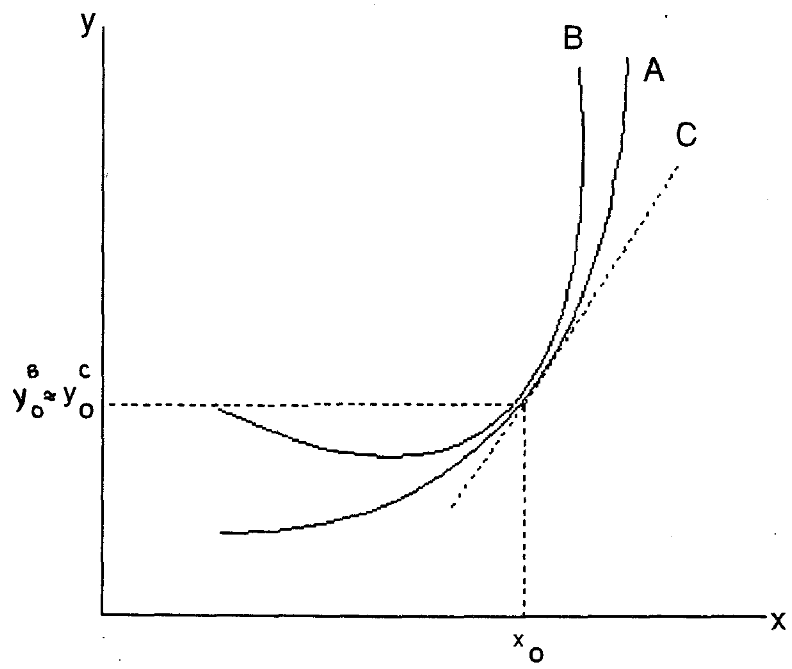


Figure 3.3. Quadratic functions identical to Taylor's series approximations.

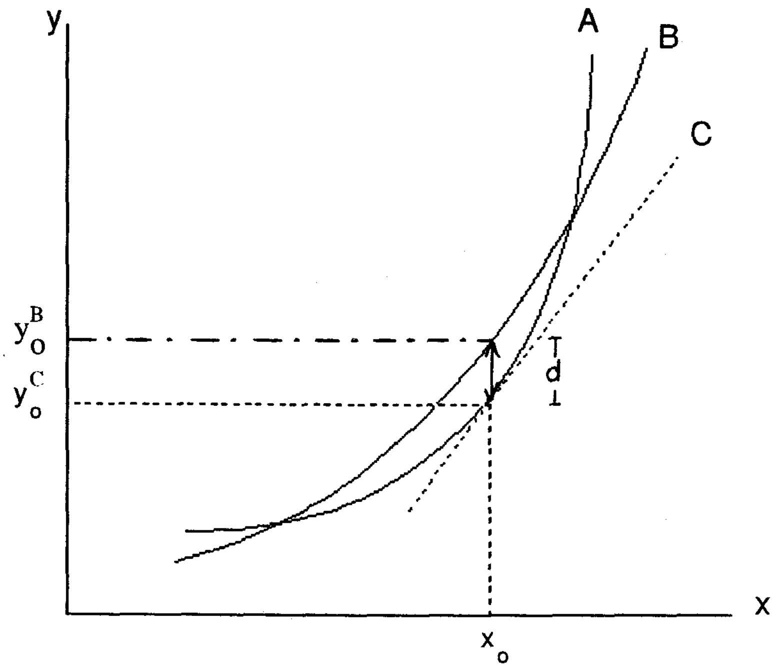


Figure 3.4. Quadratic functions differing from Taylor's series approximations.

Notes: A = true function f(·), B = estimated quadratic function approximation of true function, C = estimate from Taylor's series approximation).

Finally, the use of linear least-squares regression for the estimation of the quadratic mean response functions of the product performance characteristic, $\mu_y(X, W)$, may be questioned. What are the problems in the use of weighted least-squares regression in conjunction with the CCD? It may be speculated that accurate weights could serve to improve estimations of the quadratic mean function, as well as the estimations of variance functions for the product performance characteristics. Thus, following implementation of the research approach, the questions which remain at issue for the current investigation are as follows:

1) How does CCD performance compare to results achieved by application of the Taguchi method?
2) What are the "best" designs for the approximation of the quadratic function, $\mu_y(X, W)$?
3) Should weighted least-squares regression be used to obtain the estimated quadratic function?

Therefore, the evaluation procedures developed for the current investigation, including the means to generate test problems, are considered in the following section.

3.4 Evaluation of Solution Methods

Since the principal objective of the current investigation was to compare the results obtained by application of the CCD method, based upon the use of response surface techniques, to those obtained by application of the Taguchi method, nine test problems were considered, including the "force problem" developed in section 3.2.2 to demonstrate how design variable settings were obtained using both the CCD and the Taguchi approach. In addition, for the CCD approach, various types of central composite design experiments were considered for the approximation of the quadratic mean response function. Design matrices were varied by employing either fractional-factorial or full-factorial experiments for the cubed part and such properties as rotatability and orthogonality were also considered. Furthermore, in response to question 3 (section 3.3), five different weights were applied to obtain estimated quadratic mean response functions.

Finally, simulation studies were conducted at the design variable settings obtained, respectively, from application of the CCD and the Taguchi approach. For each approach, 10 simulation runs were performed, each of which in turn consisted of 500 observations. From the simulation, the means and the variances for the product performance characteristics (y) were then used to calculate the performance measures for each approach for all of the test problems. The performance measures considered included estimated (absolute) biases, variances, and the mean square errors of the product performance characteristics. With the objective of determining whether the CCD approach yielded statistically significant differences in the performance measure mean values from comparable measures obtained by application of the Taguchi method, statistical analyses were implemented. Detailed descriptions of the evaluation procedures as well as the generation of test problems are considered below.

3.4.1 Evaluation Procedures

For comparison of the results of the CCD approach to those obtain from consideration of the Taguchi method, the following procedural steps were developed:

A) Taguchi method:

1) Define a design matrix based upon a three-level fractional-factorial experiment (using means for the values of the noise variables).
2) Define a noise matrix based upon a two-level fractional-factorial experiment.
3) Obtain a data matrix from consideration of the matrices developed in 1) and 2).

4) Calculate the signal-to-noise ratios (as described in Chapter 1), then determine the initial set of the design variables for the maximization of the signal-to-noise ratios.

5) Adjust the design variable values to meet the target.

B) Central composite design method:

1) Define a design matrix based upon a central composite design experiment, encompassing both design variables and noise variables. To determine the values of the experimental variables, arrange the design matrix based upon *STATGRAPHICS* analysis; adjust all design points to exist within defined ranges.

2) Obtain the approximated quadratic mean response function using multiple regression analysis in *STATGRAPHICS* , and estimate variance functions from the application of Taylor's series expansions.

3) Based upon *GAMS* analysis, obtain optimum values for the design variables. Use these values to minimize the estimated variance function and to meet the specified target based upon the estimated quadratic mean response function, as described in 2); prior to the optimization of the estimation functions, values for the noise variables are fixed at mean estimates.

During the application of the CCD approach, various types of the designs were investigated to determine the best design for the approximation of the quadratic mean response function. Designs were varied by the use of either a fractional-factorial experiment or a full-factorial experiment for the cubed part. In addition, design properties such as orthogonality, rotatability, and both orthogonality and rotatability, were employed for the consideration of six possible cases for the cubed part of the problem:

i) orthogonal CCD based upon a fractional-factorial experiment,

ii) rotatable CCD based upon a fractional-factorial experiment,

iii) both orthogonal and rotatable CCD based upon a fractional-factorial experiment,

iv) orthogonal CCD based upon a full-factorial experiment,

v) rotatable CCD based upon a full-factorial experiment, and

vi) both orthogonal and rotatable CCD based upon a full-factorial experiment.

In addition, weighted least-squares regression was applied to the test problems to determine whether this means of analysis would result in the improvement of the estimations of the quadratic mean functions to the target as well as the estimations of the variance functions. Five different weights (wt) were used, as follows:

1. $wt = 1$,
2. $wt = 1/|y-\tau|$,
3. $wt = 1/(|y-\tau|)^2$,
4. $wt = 1/(\Sigma x_i^2)^{1/2}$, and
5. $wt = 1/(\Sigma x_i^2)$.

The reason for consideration of the weights 2 and 3 is that they provide higher weights to the points that are the closest to the target. Thus, the fitted quadratic mean function obtained from the use of these weights may yield estimated means that are closer to the target values. Since the information function (I_x) for a rotatable design, which is the inverse of the variance function (i.e., $I_x = V_x^{-1}$), is dependent upon distance from the origin, $(\Sigma x_i^2)^{1/2}$, weights 4 and 5 were used (Box & Draper, 1987). This was based upon the assumption that the term (Σx_i^2) may serve to improve the estimations of the variance functions or the solution capabilities of this methodology. The results of the application of the weights considered are presented in Appendix B.

The designs which resulted from development of the two approaches were tested by the simulation of optimum set of values for the design variables. Simulation programs, based upon 10 simulation runs of 500 observations for each approach, were prepared in *BASIC* (program listings are provided in Appendix C). The mean response values and the variances for the product performance characteristics at optimal sets of design variable values for each approach were calculated, in addition to biases (absolute) and mean square errors. The absolute biases were equal to the absolute values of the differences between the means for the simulation and for the target values. The mean square errors were the sum of the variances and the squared biases. Examples of simulation results, based upon a 2^5 rotatable CCD design for the force problem (section 3.2.2) are shown in Table 3.9. Finally, the performance statistics for the (absolute) biases, the variances, and the mean square errors were used to indicate which approach was superior, based upon statistical analyses.

Table 3.9. Comparison of simulated results for the force problem.

Approach	x_1	x_2	x_3	x_4	x_5	Estimated mean (|bias|)	Estimated variance	Mean square error
Taguchi	156	75	10	20	25	396.3571 (3.6429)	14,390.3 3	14,403.6 0
2^5 rotatable CCD	176.48	75	15	20.72	25	358.38 (41.62)	5,769.86	7,502.08

3.4.2 Test Problems

Since the product parameters and the performance characteristic are usually related in a complicated non-linear function, which is normally unknown, the scope of the test problems was focused upon the high-order power functions. Thus, both the additive and multiplicative high-order functions were generated. Multiplicative high-order functions were of the form: $y = x_1^{\beta 1} * x_2^{\beta 2} * x_3^{\beta 3} * x_4^{\beta 4} + \varepsilon$. Four additive and four multiplicative high-order functions were tested for the CCD to investigate its limitations with respect to the order degree polynomial of the true response function. It was hypothesized that a second-order response surface model approximation of the true response function would not be efficient insofar as the true function consisted of a very high-order degree polynomial. If the order polynomial of the true function was too high (e.g., a fourth- or fifth-degree polynomial), then the slopes of the quadratic approximation of the function would differ significantly from those of the true function.

For example, consider $x = \{1, 2, 2.5, 3, 3.2, 4, 5\}$. If the true function was of the form $y = x^3$, then the estimated quadratic function obtained via *STATGRAPHICS* regression would be $\hat{y} = 16.81 - 23.39x + 8.95x^2$. Hence, the slope of the true function evaluated at $x = 1$ would be equal to 3, whereas that of the slope of the estimated quadratic function would be 2.37. Obviously, there is little difference in the values of the slopes obtained from either the true function or the estimated quadratic function. However, if the true function was $y = x^4$, then the estimated quadratic function obtained via regression would be $\hat{y} = 157.32 - 197.986x + 57.63x^2$. The slope of the true function at $x = 1$ would then be 4 and that for the estimated quadratic function would be 16.96, or approximately quadruple the slope of the true function. The difference in slopes between the true function (with a very high-order degree polynomial) and the estimated quadratic function approximating the true function would then contribute to poor variance function estimates when the Taylor's series expansion was applied. The question then becomes: "What is the highest degree polynomial of the true function which allows for accurate functioning of the CCD approach?"

The order of degree polynomials of the multiplicative high-order function were selected using a random number generator in the range (0,d), where d was the highest order of interest. The set of order degree polynomials for the variables x_i was chosen systematically to consist of the highest order d. The additive higher order functions were randomly selected from the list of the functions provided in (Bazaraa & Shetty, 1979). The eight test problems were as follow:

model 1: $y = 2x_1^2 - 3x_2^2 + 3x_1x_2 + 1.5x_3^2 + 4x_2^4 - 6x_2x_3 + 5x_4^2 + 0.5x_1x_4 + \varepsilon$,

model 2: $y = 5x_1^2x_2^5x_3 + 10x_1^3x_2^2x_3^2 + 6x_1x_2x_4 + x_1x_3^3x_4^4 + \varepsilon$,

model 3: $y = 4x_1^3 + 2x_2 - 3x_3^2 + \exp(x_4/2) + \varepsilon$,

model 4: $y = x_1 + 2x_1^2 + 3x_1x_2 + 5x_1^{1.6}\, x_2^{1.8} - 1.5x_3^{0.5}\, x_4^2 + \varepsilon$,

model 5: $y = x_1^3x_2^{0.5}\, x_3x_4^2 + \varepsilon$,

model 6: $y = x_1^{1/2}\, x_2^{3/4}\, x_3^{1/3}\, x_4^{2/3} + \varepsilon$,

model 7: $y = x_1^{5/2}\, x_2x_3^3\, x_4^4 + \varepsilon$, and

model 8: $y = x_1^3\, x_2^5\, x_3^{3/2}\, x_4^2 + \varepsilon$.

Note that the range of design variables and noise variables were limited in (0,1), thus none of the problems were optimized with the *GAMS* software (i.e., standardized variables into the (0,1) scale were used). Furthermore, each model had a specified target and the means and standard deviations for x_i were randomly selected based upon round-up (only two significant digits) of the random number (0,1). The results were:

$$\sigma_{x1} = \sigma_{x2} = 0.1\ ,$$

$$\sigma_{x3} = 0.15\ ,$$

and

$$\sigma_{x4} = 0.09$$

(where x_4 is a noise variable) for $\mu_{x4} = 0.5$.

However, there were nine test problems considered for the current investigation. The final problem, the "force problem," was previously used (section 3.2.2) to demonstrate how the optimal design variable values were obtained using the CCD method. Furthermore, the objective of consideration of the force problem was to determine the settings for the system parameters, to the end of minimizing the mean square errors of the force target while achieving the target values. The force problem was given as shown in Figure 3.5.

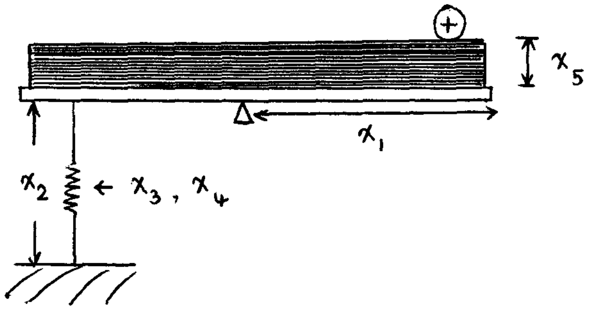


Figure 3.5. Diagram of the force problem.

Results were as follows:

$$y = (300 + 16x_5) * (140/x_1 - 1)$$
$$+ x_3 * (x_2 + (x_5 - 20) * (280/x_1 - 1) - x_4) * (280/x_1 - 1)\,,$$

where $y =$ force (grams)

$x_1 =$ front edge of the paper to pivot,

$x_2 =$ spring connection point,

$x_3 =$ spring stiffness,

$x_4 =$ spring free length, and

$x_5 =$ paper thickness,

$x_1 \; \varepsilon \; (100, 180)$ mm,

$x_2 \; \varepsilon \; (35, 75)$ mm,

$x_3 \; \varepsilon \; (5, 15)$ mm,

$x_4 \; \varepsilon \; (20, 50)$ mm,

$\sigma_{x1} = \sigma_{x2} = 1$ mm,

$\sigma_{x3} = \sigma_{x4} = 2$ mm.

x_5 is distributed Uniform(0, 50) and is a noise variable. The target value of the force was 400 grams.

The results of all nine test problems, as solved by the various methods considered (section 3.4.1), are presented in Chapter 4, accompanied by statistical analyses of the results.

CHAPTER 4

RESULTS AND DISCUSSIONS

4.1 Introduction

The principal objective of this investigation was to compare the results of the central composite design, based upon response surface methodology, to the Taguchi approach. The CCD was developed to:

1. Obtain a set of values, other than those within the inner array, by minimization of variations in product performance characteristics, while achieving target values of the *nominal-the best type*;
2. Eliminate the need for consideration of a noise matrix (the outer array); and
3. Eliminate the use of signal-to-noise ratios as the basis for performance statistics.

Nine performance characteristic function test problems (as developed in section 3.4.2) were investigated and solved for the two contrasting approaches. The data for each test problem were derived from the application of *STATGRAPHICS*, version 5.0, for response surface design. Note that real experiments were not conducted. In addition, the inputs, consisting of the lower and upper bounds of the design variables (Appendix A), were controlled to the extent that points within the data matrix (the design matrix) were within given ranges for the design variables and covered the greater parts of regions of interest. The results of the design variable settings and the statistical test results for all test problems are presented in Appendix B. Listings for the simulation program and the nonlinear optimization program are presented in Appendix C.

Given the principal objective of this study, performance measures were represented by the mean square errors for the two approaches, consisting of the biases and variances. Which parts of the mean square errors provided the greatest contributions to differences between the two approaches was examined. Therefore, the final performance measures were considered to be the mean square errors, biases, and variances in the product performance characteristics, (y). Results for the three performance measures were obtained by simulating optimal settings for the design variables yielded from consideration of the two approaches. Since research interest was concentrated upon the magnitude (i.e., the deviation of the estimated mean response from the target value) rather than the direction (overestimated or underestimated) of mean responses from the specified target, for purposes of statistical analyses absolute biases were determined.

The CCD approach was investigated for each of the test problems. Initially, seven contrasting design matrices were to be considered, but they could be applied only to the force problem. The seven design matrix problems were identified as follows:

1) TAGUCHI (0) for the Taguchi approach,
2) FRAC-ORTH (1) for the CCD based upon fractional-factorial orthogonal design,
3) FRAC-ROTATE (2) for the CCD based upon fractional-factorial rotatable design,
4) FRAC-ORRO (3) for the CCD based upon fractional-factorial orthogonal and rotatable design,
5) FULL-ORTH (4) for the CCD based upon full-factorial orthogonal design,
6) FULL-ROTATE (5) for the CCD based upon full-factorial rotatable design, and

7) FULL-ORRO (6) for the CCD based upon full-factorial orthogonal and rotatable design.

For models 1–8, the four-variable problems, only four of the above approaches could be employed. Since, due to limitations upon design resolution, the fractional-factorial design could not be considered for the four-variable problems. This is because the CCD method employs a central composite design experiment which is a resolution IV design or higher. (Note that the resolution design k implies that no r factors are aliased with another effect containing less than k-r factors.) Thus, the problems considered for the investigation of models 1-8 were limited to: TAGUCHI, FULL-ORTH, FULL-ROTATE, and FULL-ORRO. Nevertheless, the results obtained from FULL-ORRO were either identical or equivalent to those for the CCD based upon full-factorial rotatable design (FULL-ROTATE) for models 1–8. With the exception of model 7, the results for the case FULL-ORRO were identical to those obtained from the case FULL-ROTATE (Appendix B, Table B15). Moreover, the FULL-ORRO case required a greater number of experimental runs than the FULL-ROTATE case. The additional number of experimental runs required implies greater expenditure. Thus, for purposes of statistical analysis, consideration of models 1–8 did not include the FULL-ORRO case. Therefore, only three approaches, TAGUCHI, FULL-ORTH, and FULL-ROTATE, were compared for purposes of the discussion of the results of this investigation.

To compare statistically significant differences between the results for the Taguchi method and the CCD approach, statistical analyses for the present study were performed with the nonparametric Kruskal-Wallis analysis of variance (ANOVA) for all three performance measures. The test statistics were distributed Chi-squares for k-1 degrees of freedom, where k represents the number of samples. This method of analysis was thus equivalent to the approaches used for each of the test problems. The null hypotheses tested were that there would be no differences in the performance statistic averages, including variances, absolute biases, and mean square errors, between the CCD method and the Taguchi method at a 5% level of significance for all test problems. The test results are presented in Appendix B (Tables B19–B26).

As noted above, statistical analyses were conducted for the TAGUCHI, FULL-ORTH, and FULL-ROTATE test cases. In addition, a weighting function of one was assigned to a linear least-squares regression analysis to obtain estimated mean response functions for the CCD method for purposes of comparison with the results obtain for the Taguchi method. For example, comparing biased results for the force problem yielded by the FULL-ORTH, FULL-ROTATE, and TAGUCHI test problems, the number of samples (k) is equal to 3 as determined by Kruskal-Wallis ANOVA. (Note that the biases consisted of the differences between mean responses and the target values.) Biases were then obtained from the problem simulations, each run 10 times at design variable settings determined for each approach. For example, the test statistics considered were distributed as Chi-squares with two degrees of freedom, where each sample was of size 10. The results of this method of analysis indicated that, with the exception of model 4, all of the null hypotheses were rejected at the 5% level of significance. Overall, these test results imply that the CCD method is to be preferred to Taguchi methods of analysis.

The balance of this chapter is organized as follows. Statistical test results for the three performance measures for variances, biases, and mean square errors are presented, respectively, in sections 4.2–4.4. In each section, overall comparisons of results for the CCD and Taguchi methods for the additive models (models 1-4), the multiplicative models (models 5-8), and the force problem are presented, including comparisons of the design properties of each case considered (i.e., orthogonality vs. rotatability). The results of comparisons for two types of experimental designs for the cubed part of the CCD design (i.e., full-factorial vs. fractional-factorial experiments), as well as the results of investigations for five different weighting functions were presented previously and discussed in section 3.4.2. All of the results of this investigation are summarized in section 4.5, whereas a comparison of the results obtained for all test problems for each of the two methods under consideration is presented in section 4.6.

4.2 Analysis of Results for Variances

The results analyzed in this section were obtained by applying a weighting function of one to obtain the estimated quadratic mean response function.

4.2.1 CCD Method vs. Taguchi Method

The results from three of four additive high-order models (models 1–3), all of the four multiplicative high-order models (models 5–8), and the force problem indicated statistically significanct differences (p-ranges from 0.000 to 0.0412) between the variance averages obtained for the CCD (either FULL-ORTH or FULL-ROTATE) and the Taguchi methods (Appendix B, Table B19). The exception was model 4, for which no statistically significant differences were demonstrated. However, the settings obtained by the use of either FULL-ORTH or FULL-ROTATE resulted in performance characteristic (y) variations that were smaller than those obtained by use of TAGUCHI for all test problems (Appendix B, Table B20). Specifically, for models 1 and 2, FULL-ORTH yielded the smallest variances, whereas for models 3, 6–8 and the force problem, FULL-ROTATE yielded the smallest variances. For models 4 and 5, FULL-ORTH and FULL-ROTATE yielded variances that were approximately equal. Thus, with the exception of model 4, results obtained with the CCD method were superior to those obtained with the Taguchi method. In addition, as shown in Figures 4.1–4.3, variance averages for the Taguchi method were somewhat larger than variance averages for the CCD method for all the test problems (including model 4).

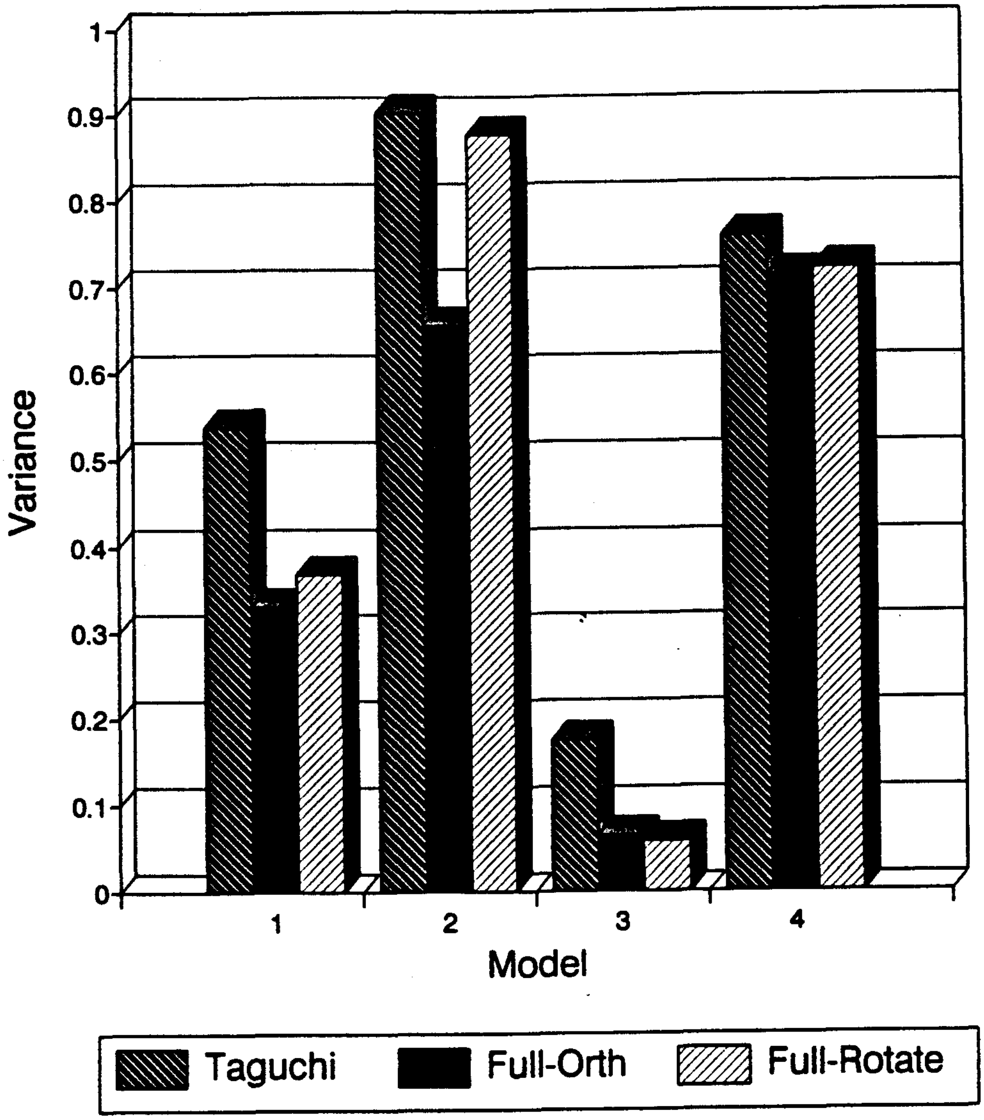


Figure 4.1. Variances for additive models (1–4).

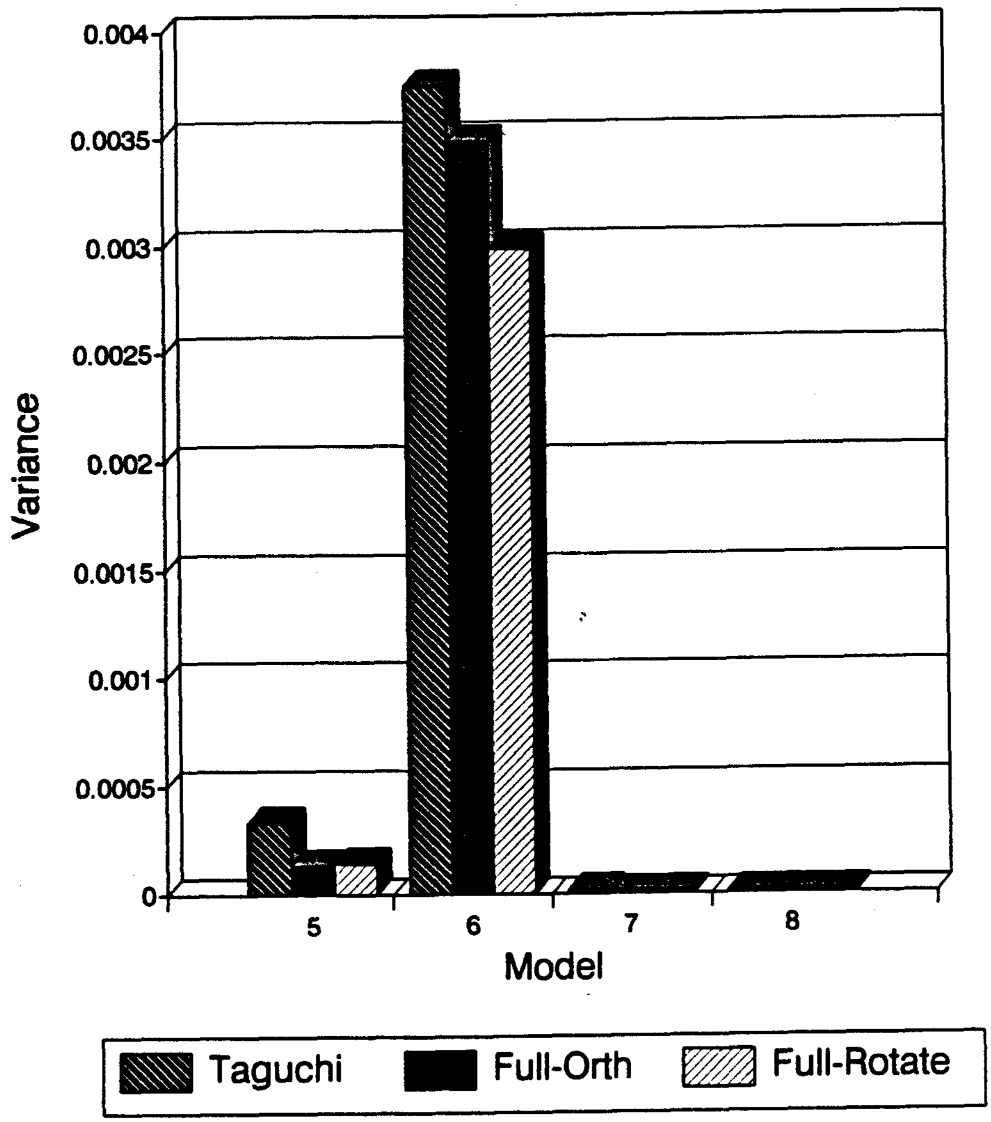


Figure 4.2. Variances for multiplicative models (5–8).

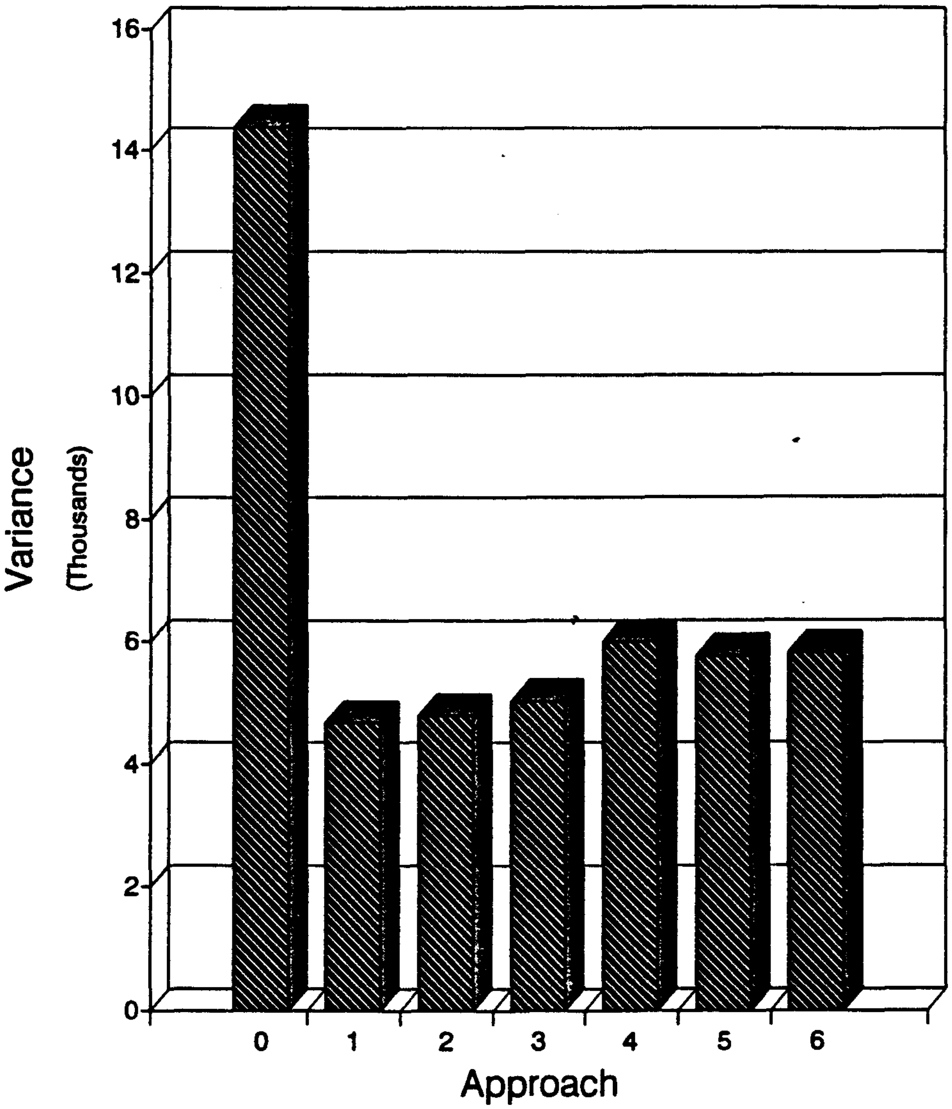


Figure 4.3. Variances for the force problem, approaches 0–6.

The average variance for FULL-ORTH models 1 and 2, were, respectively, 0.3293 with a standard error of 0.0078 and 0.6552 with a standard error of 0.0378, in comparison to, respectively, 0.5369 with a standard error of 0.0198 and 0.9019 with a standard error of 0.0313 for the Taguchi method. The average variance for FULL-ROTATE models 3, 6, 7, 8 and the force problem were, respectively, 0.0574 with a standard error of 0.0037, 2.98×10^{-3} with a standard error of 5.37×10^{-5}, 1.24×10^{-7} with a standard error of 1.51×10^{-8}, 1.29×10^{-6} with a standard error of 1.44×10^{-7}, and 5,769.87 with a standard error of 109.45 in comparison to, respectively, 0.1744 with a standard error of 0.0084, 3.74×10^{-3} with a standard error of 1.30×10^{-4}, 1.52×10^{-5} with a standard error of 1.68×10^{-6}, 1.91×10^{-6} with a standard error of 2.03×10^{-7}, and 14,390.30 with a standard error of 323.67 for the Taguchi method. Moreover, the average variance for FULL-ORTH (FULL-ROTATE) models 4 and 5 were, respectively, 0.7107 (0.7208) with a standard error of 0.025 (0.0246) and 1.33×10^{-4} (1.38×10^{-4}) with a standard error of 7.19×10^{-6} (4.87×10^{-6}), in comparison to, respectively, 0.7574 with a standard error of 0.0203 and 3.28×10^{-4} with a standard error of 1.68×10^{-5} for the Taguchi method.

Furthermore, the test results for orthogonal and rotatable design properties favored the CCD method. Though results for five of nine test problems failed to indicate statistically significant differences in variance averages for models 3–5, 8 and the force problem (p-values from 0.1988 to 0.8798) between FULL-ORTH and FULL-ROTATE, it was determined that results for seven of nine FULL-ROTATE test problems yielded smaller or equivalent variances in comparison to those for FULL-ORTH, as discussed above. (Note that the exceptions were models 1 and model 2, for which FULL-ORTH yielded the smaller variances.) Moreover, from thorough investigation of model 8, use of rotatable design resulted in statistically significant differences between the proposed method and the Taguchi method for variance averages when there were no signals. That is, though there were no statistically significant differences among FULL-ORTH, FULL-ROTATE, and TAGUCHI for variance averages at p-value † 0.1383, between FULL-ORTH and FULL-ROTATE for variance averages at p-value † 0.2899, or between FULL-ORTH and TAGUCHI for variance averages at p-value † 0.4495, test results did demonstrate statistically significant differences between FULL-ROTATE and TAGUCHI for variance averages at p-value † 0.0412. As a result, rotatability was determined to be a significant property for the proposed CCD design.

4.2.2 Design Experiments

Force problem results indicated that the CCD method, employing either the fractional-factorial experiments or the full-factorial experiments (models 4–6), was superior to the Taguchi method. From Table B25 (Appendix B), all six designs employing the CCD method yielded variance average results which were statistically significant improvements upon Taguchi method at p-values less than 0.05. Additional analysis was performed for the full-factorial vs. fractional-factorial experimental designs. Results indicated that the full-factorial designs were provided results which were statistically significant improvements upon variance averages for the fractional-factorial designs at p-values less than 0.05. However, within either the full or fractional factorial groups, there were no statistically significant differences for average variances in the force problem with respect to orthogonal or rotatable design properties. At the same time, the full-factorial experiments resulted in larger average variances than the averages for the fractional-factorial experiments (Figure 4.3). Thus, for the determination of variances, the fractional-factorial approach was preferred to the full-factorial approach.

4.2.3 Weighting Functions

Since it may be speculated that assurance of the accuracy of weighting functions could serve to improve the estimation of the quadratic mean response functions as well as of variance function estimation for the product performance characteristics, consider one of the research questions introduced in section 3.3: "Should weighted least-squares regression be used to obtain the estimated quadratic function?" The weighting functions employed for the CCD method based upon full-factorial rotatable design were:

1) $w1 = 1$,
2) $w2 = 1/|y-\tau|$,
3) $w3 = 1/(y-\tau)^2$,
4) $w4 = 1/(\Sigma x^2)^{1/2}$, and
5) $w5 = 1/(\Sigma x^2)$.

The bases for the choice of these five different weighting functions were discussed in section 3.4.1.

As indicated in Figures 4.4–4.6, for the application of a weighting function of one, variances for the additive high-order functions (models 1–4) and the multiplicative high-order functions (model 5–8) were decreased, respectively, by 22.4% and 4.24% for consideration of the inverse of the squared distance from the origin of the variables x (w5). However, the weighting function of one (i.e., ordinary least-squares regression) was sufficient for the force problem. The average of variances of weighting function w5 for the additive models was 0.3908, with a standard error of 0.1329, whereas the average of weighting function of one (w1) for the same models was 0.5064, with a standard error of 0.1827. For the multiplicative models, the average of variances of weighting function w5 was 0.000746, with a standard error of 6.94×10^{-4}, whereas the average of weighting function one was 0.000774, with a standard error of 7.35×10^{-4}. For the force problem, the average variance based upon ordinary least-squares regression (i.e., a weighting function of one) was 5,769.86, in comparison to 5,908.28 for the weighting function w4 (i.e., the next smallest weighting function).

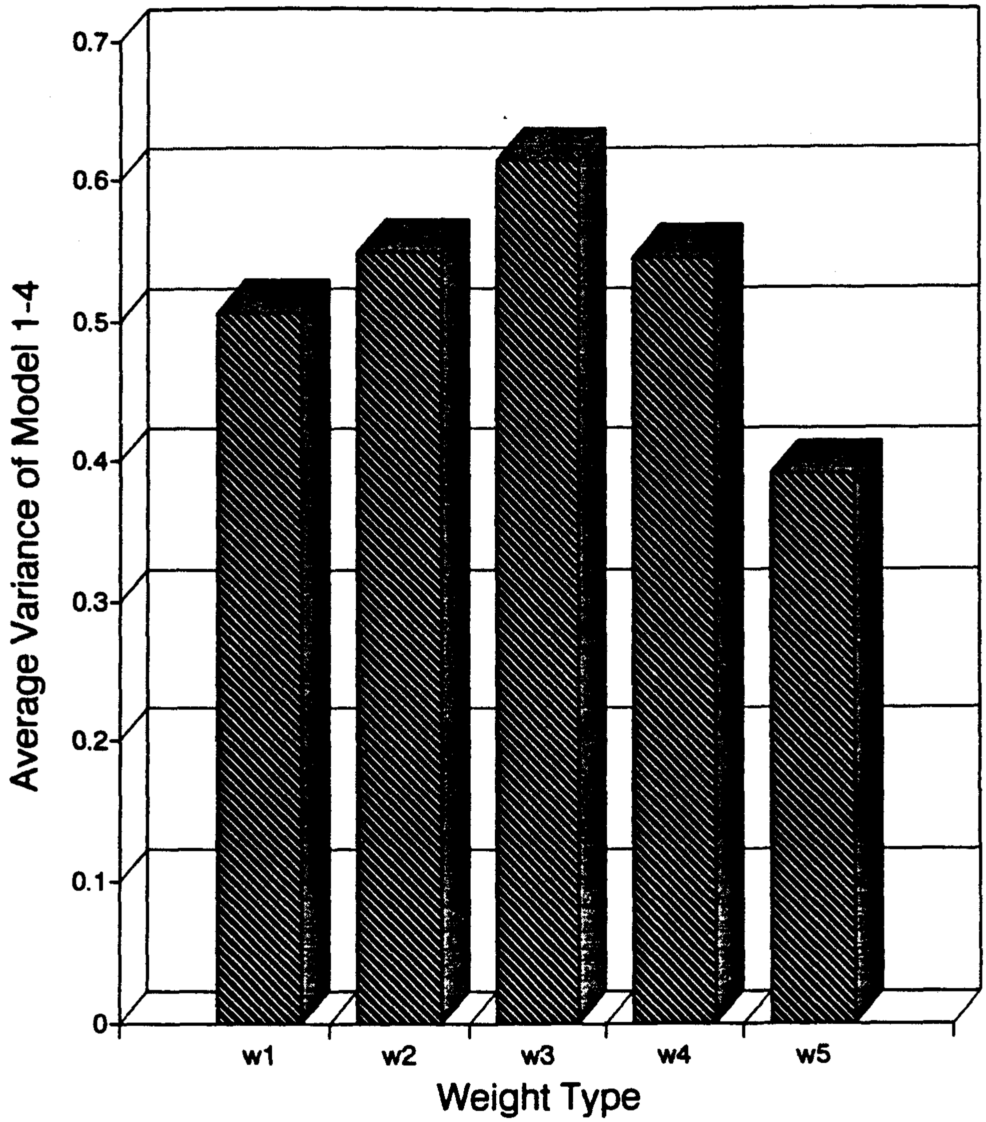


Figure 4.4. Average variances for additive models (1–4).

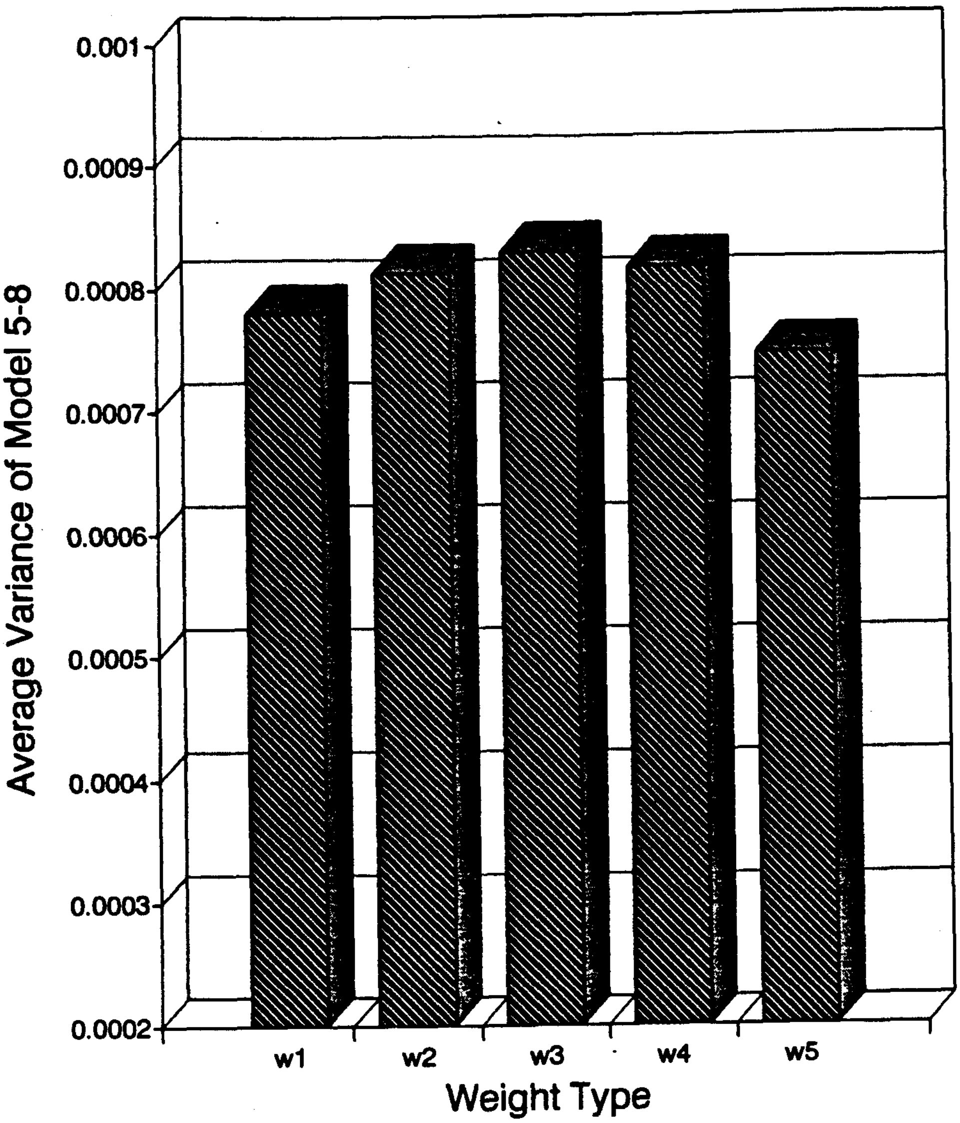


Figure 4.5. Average variances for multiplicative models (5–8).

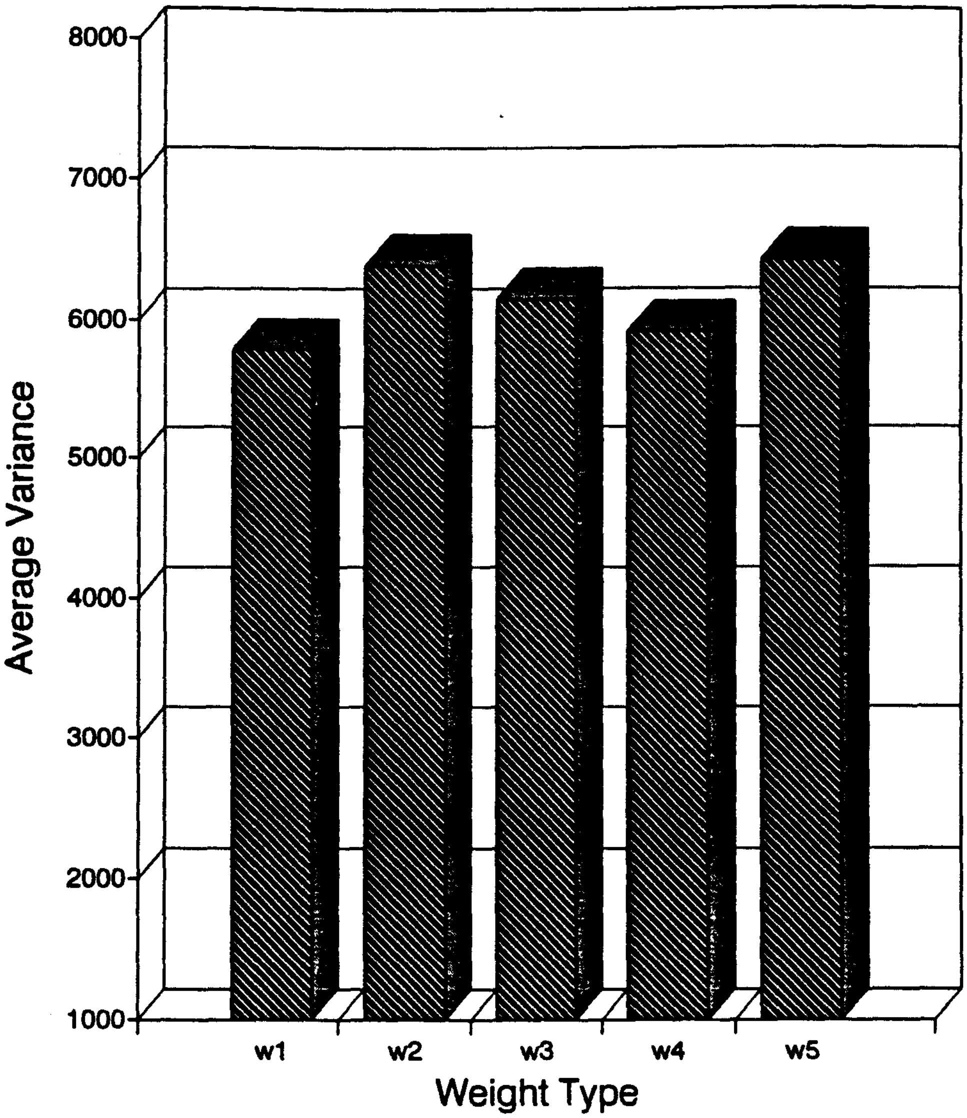


Figure 4.6. Average variances for the force problem (FULL-ROTATE).

Hence, it was determined that weighted least-squares regression improved estimation of the variances functions for the product performance characteristics of the high-order functions (models 1–8). In addition, the appropriate and reasonable weighting function for the high-order functions was the inverse of the squared distance from the origin of the variables x ($w5 = 1/(\Sigma x^2)$). This was because the information function (I_x), which is the inverse of the variance function (i.e., $I_x = V_x^{-1}$), was dependent only upon the distance from the origin for the rotatable design, and the weighting function w5 was the inverse of the square of that distance. The next point of interest was whether the application of weighted least-squares regression would result in an improvement of estimates of the quadratic mean functions, and which of the five distinctive weights would provide an appropriate choice in that context.

4.3 Analysis of Results for Biases

As in the previous section, the results discussed in sections 4.3.1–4.3.2 were obtained by applying a weighting function of one to the proposed method.

4.3.1 CCD Method vs. Taguchi Method

Statistical analysis based upon Kruskal-Wallis ANOVA (Appendix B, Table B21) demonstrated statistically significant different average absolute biases existed between the CCD method (FULL-ORTH and FULL-ROTATE) and the Taguchi method (TAGUCHI). Results for three of four of the additive high-order models (models 1–3), all of the multiplicative high-order models (models 5–8), and the force problem demonstrated statistically significant differences at p-value ranges from 0.0000 to 0.0043. Again, the exception was that there were no significant differences between CCD FULL-ORTH and FULL-ROTATE and the Taguchi method for model 4. However, the FULL-ROTATE CCD (full-factorial rotatable design) yielded the smallest absolute bias for model 4 (Figure 4.7). In addition, with the exception of model 1 and the force problem, results from seven of nine test problems indicated that the CCD methods (either FULL-ORTH or FULL-ROTATE) yielded average absolute biases which were smaller than those for TAGUCHI (Figs 4.7-4.9). Thus, with the exception of model 4, the CCD method was determined to be superior to the Taguchi method for the determination of average absolute bias. In addition, as demonstrated in Figures 4.7-4.9, for three of four additive models (2–4) and all of the multiplicative models (5–8), the use of either the FULL-ROTATE or FULL-ORTH CCD method resulted in the smallest average absolute biases. Specifically, for models 2–5 and 8, FULL-ROTATE yielded the smallest absolute biases, whereas for model 6–7, FULL-ORTH yielded the smallest absolute biases.

The average absolute biases for model 2–5 FULL-ROTATE were, respectively, 2.99%, 0.4%, 1.07%, and 5.34% in comparison to, respectively, 7.26%, 0.96%, 1.96%, 54.9% for the Taguchi method. The average absolute biases of model 2 for FULL-ROTATE was 50.57%, in comparison to 83.91% for the Taguchi method. In turn, the average absolute biases of models 6-7 for FULL-ORTH were, respectively, 1.45% and 33.01%, in comparison to, respectively, 3.61% and 75.36% for the Taguchi method. Though the Taguchi method resulted in smaller absolute biases by percentages for model 1 and the force problem, the results from other test problems, in particular those for the multiplicative high-order functions, indicated that the performance of the Taguchi method was inferior to CCD method performances.

From the statistical analyses of the design properties, results indicated that for three of four of the additive models (models 1–3 at p-value ranges from 0.0002 to 0.0032) and one of the four multiplicative models (model 7, p-value = 0.0002) there were statistically significant differences for average absolute biases which favored rotatability vs. orthogonality. The exception was model 7, the results for which favored FULL-ORTH. Thus, rotatability was determined to be a significant property for the proposed

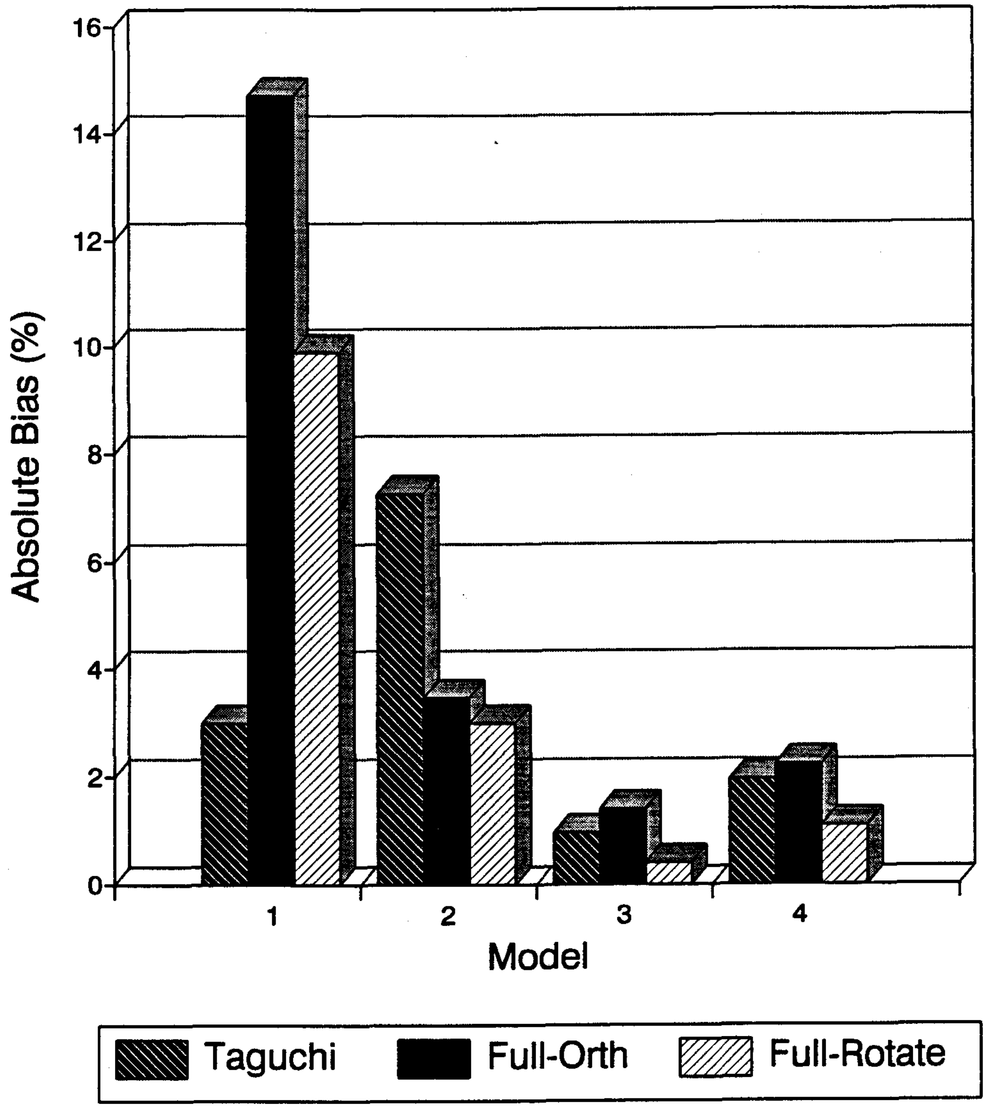


Figure 4.7. Absolute bias for additive models (1–4)
(in percentages).

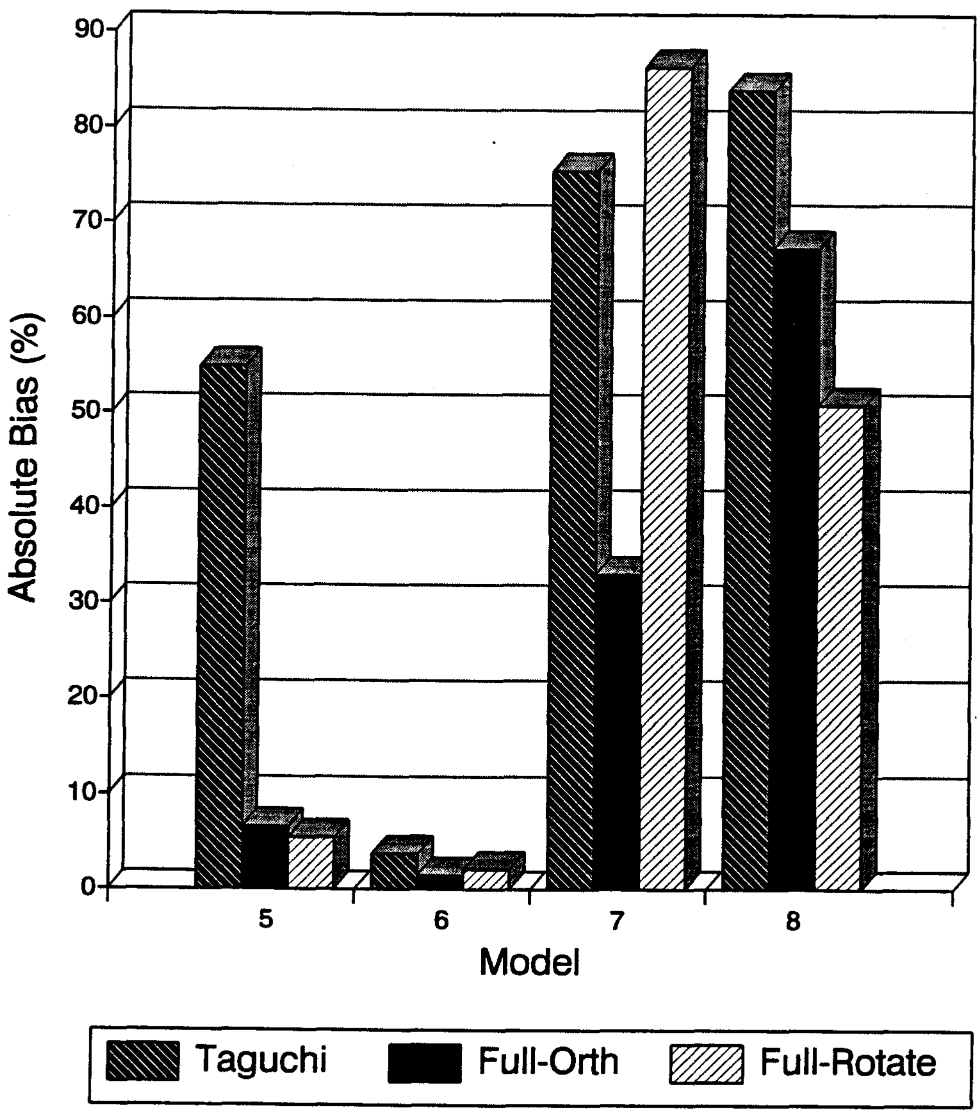


Figure 4.8. Absolute bias for multiplicative models (5–8) (in percentages).

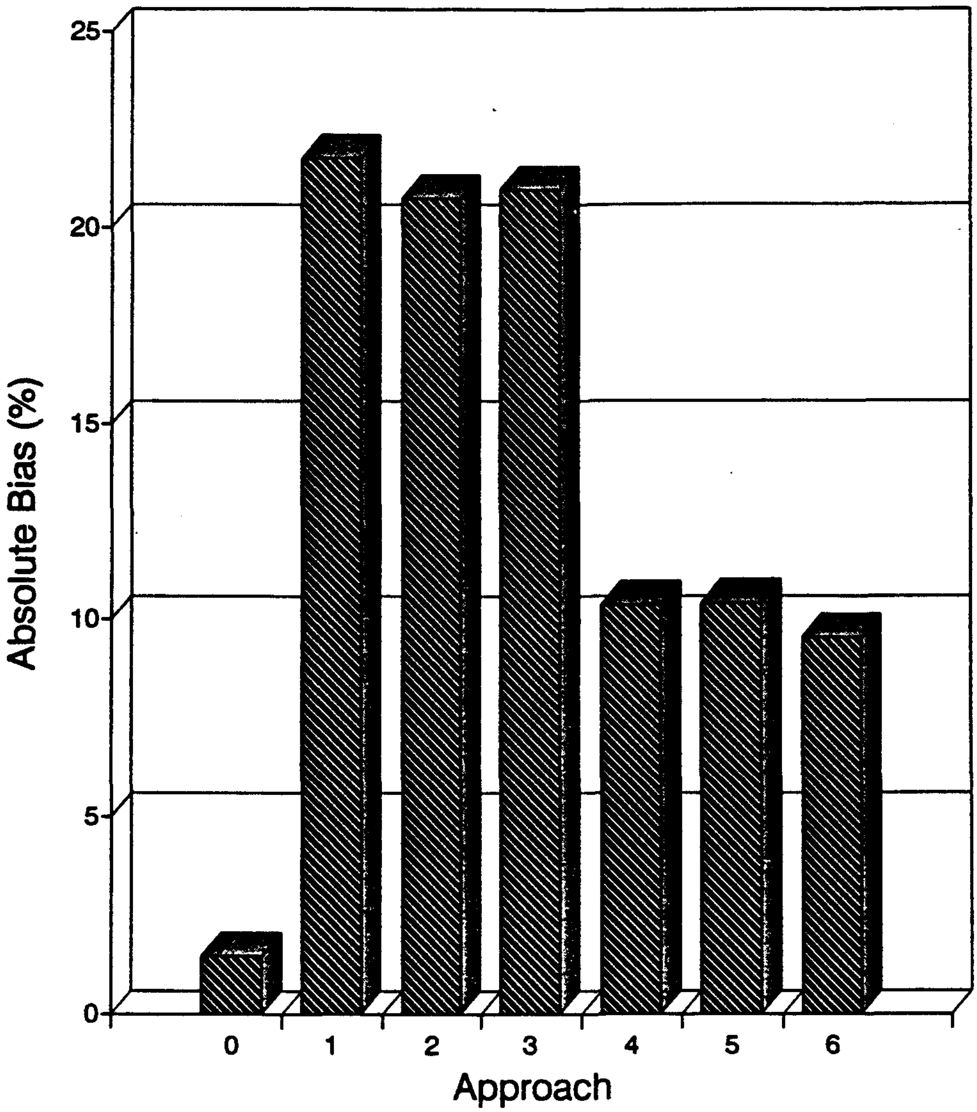


Figure 4.9. Absolute bias for the force problem, approaches 0–6 (in percentages).

CCD design for the bias case as well as for the variance case. Moreover, the bias results (Appendix B, Table B22) demonstrated that the CCD methods tended to result in the underestimation of mean responses (i.e., negative bias), whereas the Taguchi method tended to result in the overestimation of mean response (i.e., positive bias).

In addition, absolute biases for the multiplicative high-order functions (models 7 and 8) obtained via the CCD method were substantial, at a range from 33% to 87% of the target value. From Table B22 (Appendix B), model 7 FULL-ROTATE resulted in an 86.67% underestimation of mean response from the target value of 0.0015 (i.e., estimated bias = -0.0013), whereas model 8 FULL-ROTATE resulted in a 50% overestimation of mean response from the target value of 0.0005 (i.e., estimated bias = 0.00034). For model 7, FULL-ORTH resulted in a 33.33% underestimation of mean response from the target value, whereas the model 8 results revealed a 68% overestimation of mean response from the target value. In contrast, estimated biases for the remaining models obtained from the proposed CCD method were rather small, reflecting estimations of mean response which could be as little as 10% less than or greater than the target values.

4.3.2 Design Experiments

Though the results from the force problem (Figure 4.9, and Tables B25-B26, Appendix B) did not generally favor the CCD method (including both the full-factorial and the fractional-factorial designs) with respect to bias, the results did indicate that the full-factorial design yielded statistically significant improvements, at p-values less than 0.05, with respect to the fractional-factorial design. Moreover, the absolute bias for the full-factorial design was approximately twice as small as that for the fractional-factorial design (Appendix B, Table B26). For example, the average absolute bias for FULL-ROTATE was 10.41%, whereas that for FRAC-ROTATE was 20.76%; the average absolute bias for FULL-ORTH was 10.36%, whereas that for FRAC-ORTH was

21.75%; finally, the average absolute bias for FULL-ORRO was 9.57%, whereas that for FRAC-ORRO was 20.96%. Thus, the full-factorial design provided superior performance with respect to bias.

4.3.3 Weighting Functions

Since the target values for each test problem differed, percentages of absolute bias from the target were investigated, based upon consideration of weighted least-squares regression to obtain the quadratic mean response functions. Graphs for the average percentages of absolute bias for the additive high-order functions (models 1–4), the multiplicative high-order functions (models 5–8), and the force problem are presented in Figures 4.10–4.12, respectively. From these results, no clear evidence was provided which favored a specific type of weighting function. For the additive high-order models, ordinary least-squares regression (w1), since it resulted in the smallest absolute percentage of bias (i.e., approximately 32.49% smaller than that of the next smallest yielded by applying the inverse of absolute differences of the responses from the target (w2)), was appropriate. For multiplicative high-order models, the inverse of the absolute differences of responses from the target (w2) yielded the next smallest percentages of absolute bias among all of the weighting functions. However, for this model type, the inverse of the distance from the origin (w4) yielded approximately the same values as the weighting function (w2); that is, approximately a 2.4% difference from that of the next smallest (w2).

For the force problem, the inverse of the squared distance from the origin (w5) yielded the smallest percentage of absolute bias (i.e., the square of the inverse of absolute response difference from the target (w3), approximately 28.87% smaller than for the next smallest). Nevertheless, since the absolute distance of the performance characteristics from the target (w2) yielded the smallest average absolute biases (e.g., absolute bias,

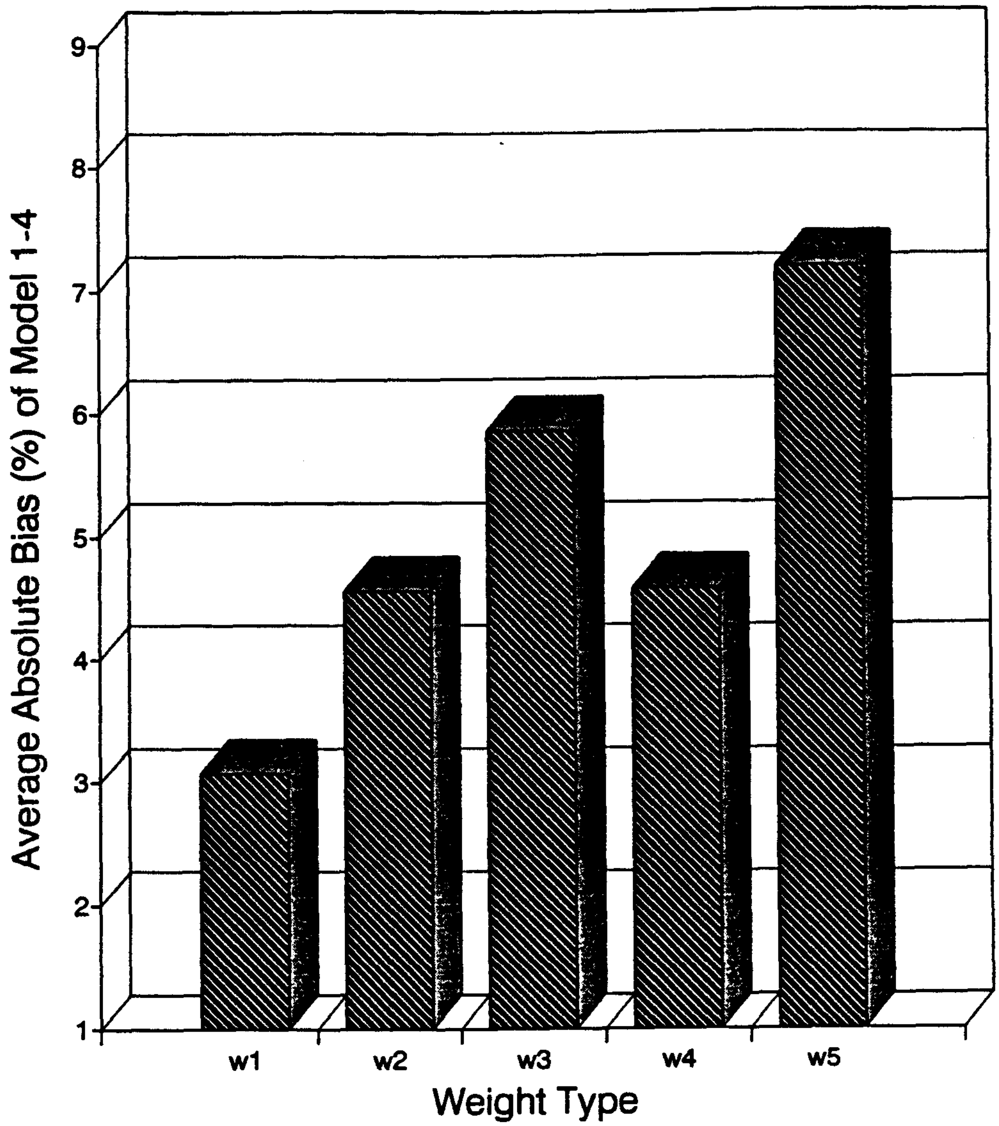


Figure 4.10. Average absolute bias for additive models (1–4).

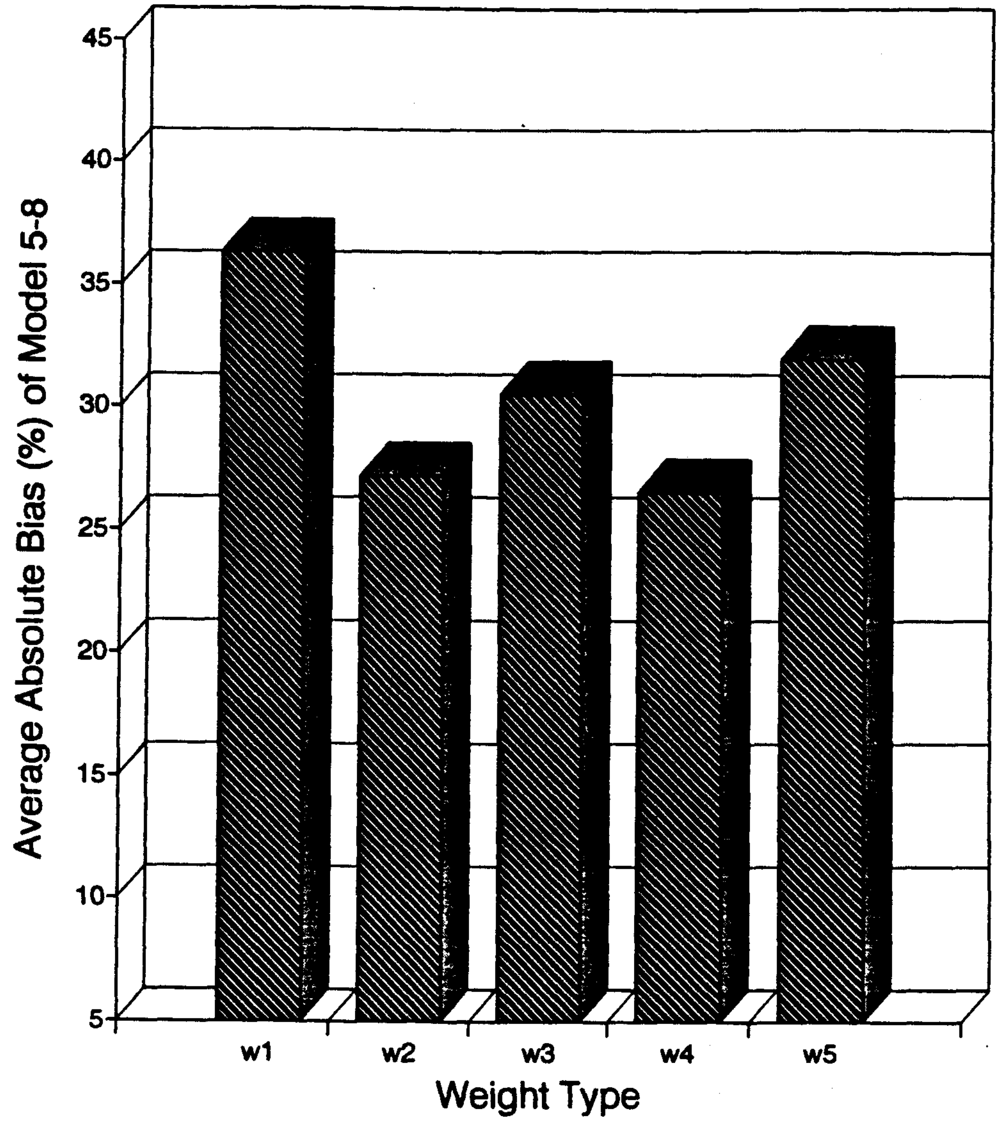


Figure 4.11. Average absolute bias for multiplicative models (5–8).

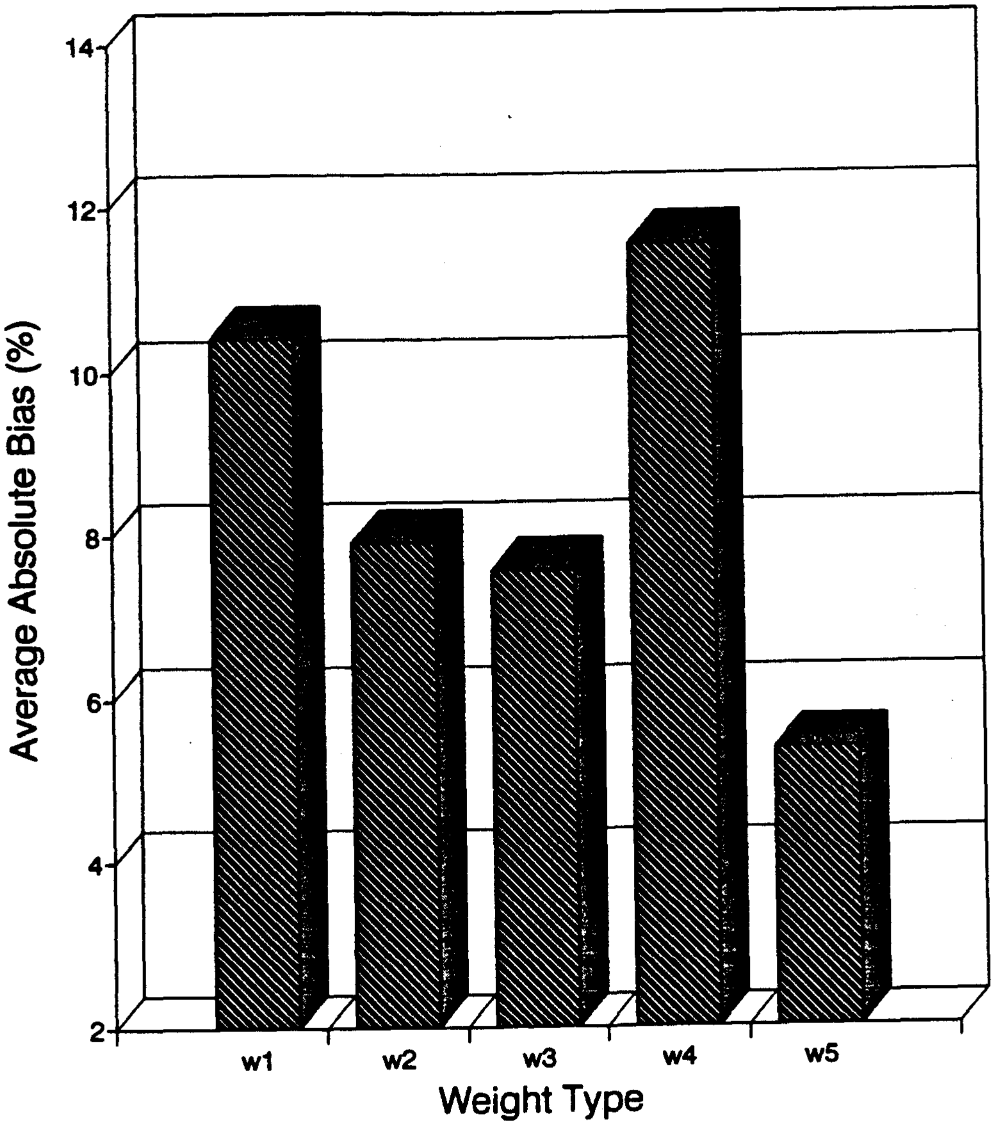


Figure 4.12. Average absolute bias for the force problem (FULL-ROTATE).

14.99%), averaging all models resulted in an appropriate alternative because this approach provided superior performance for the estimation of mean response with respect to target values. Applying the weight function w2 provided higher weights to the points that were closer to the target value. The fitted quadratic mean function resulting from the use of this weighting function yielded the estimated means that were closest to the target. Thus, the absolute distance of the performance characteristics from the target (w2) is an appropriate alternative when selecting a weighting function for the improvement of estimates of mean response with respect to target values.

4.4 Overall Analysis of Mean Square Errors

In sections 4.4.1–4.4.2, as for previously reported results, a weighting function of one was applied to the results for mean squared errors.

4.4.1 CCD method vs. Taguchi Method

Mean square errors for the performance characteristics (y) were represented by the sum of the squared biases and variances. This measure provides the most comprehensive means of analysis among all the measures considered. For three of four additive models (models 1–3), all of the multiplicative models (model 5–8), and the force problem, there were statistically significant differences for mean squared error averages between the CCD methods (FULL-ORTH and FULL-ROTATE) and the Taguchi method (TAGUCHI) at a p-value range from 0.0000 to 0.0233. Statistical analysis performed with the Kruskal-Wallis ANOVA (Appendix B, Table B23) indicated that for all models, with the exception of model 4, the CCD method was superior to the Taguchi method. As indicated in Figures 4.13–4.15, for all nine test problems the CCD methods (FULL-ORTH or FULL-ROTATE) yielded the smallest mean squared errors. In addition, even

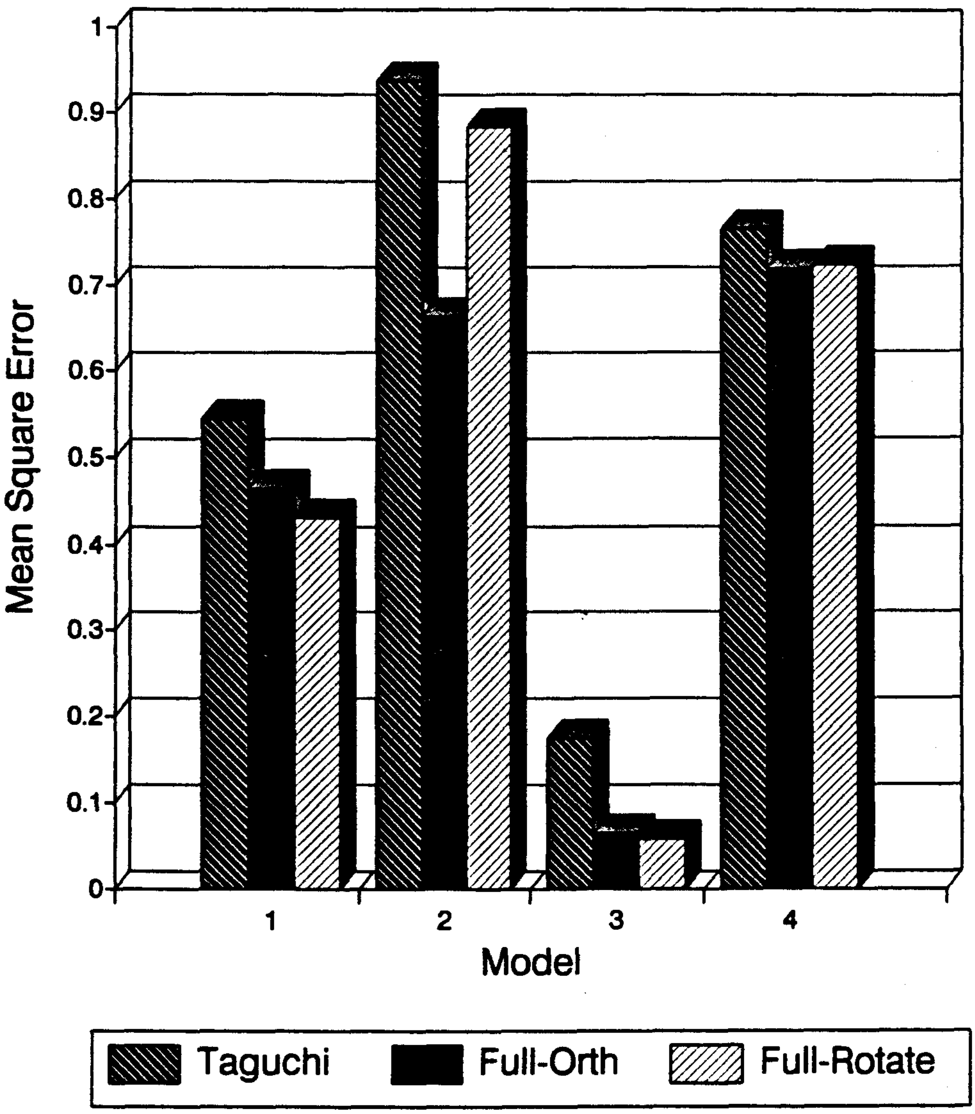


Figure 4.13. Mean squared errors for additive models (1–4).

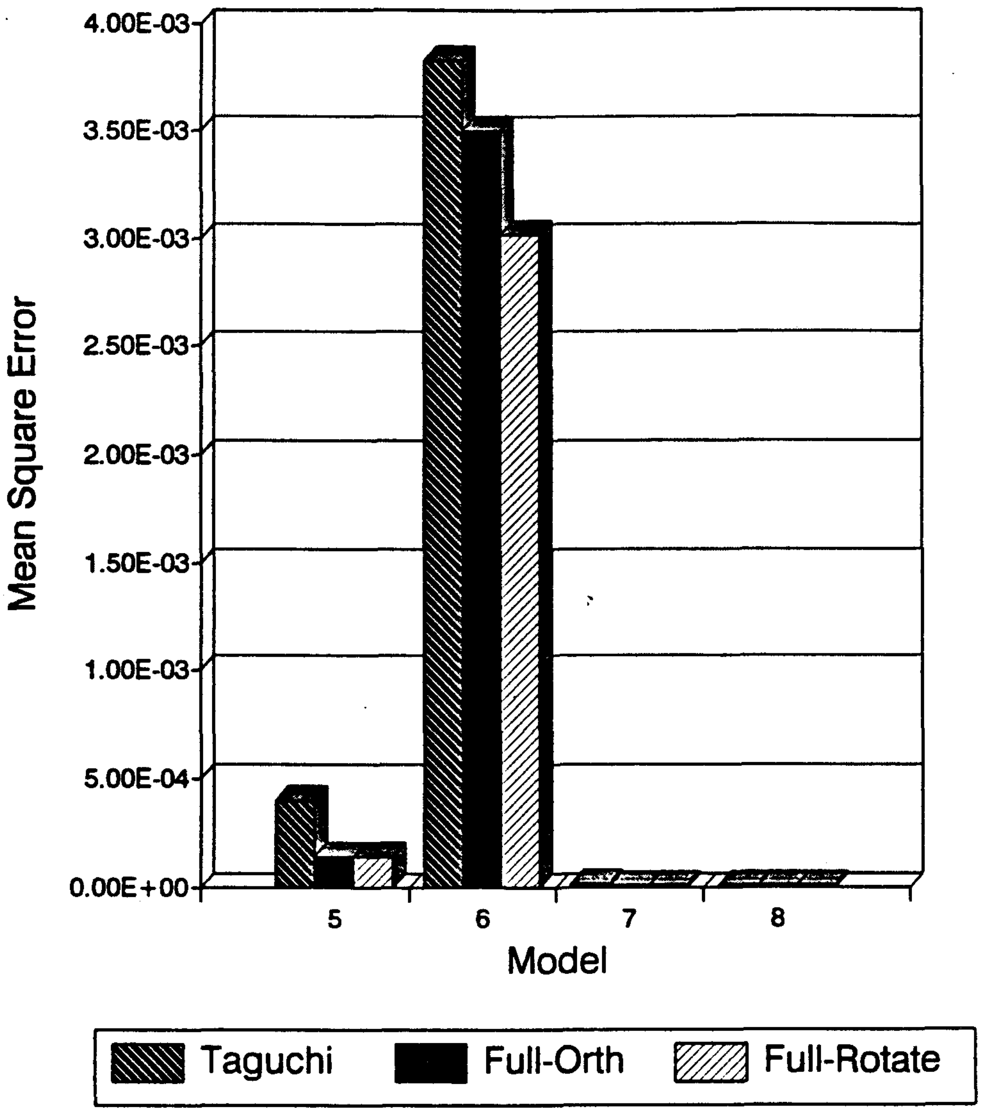


Figure 4.14. Mean squared errors for multiplicative models (5–8).

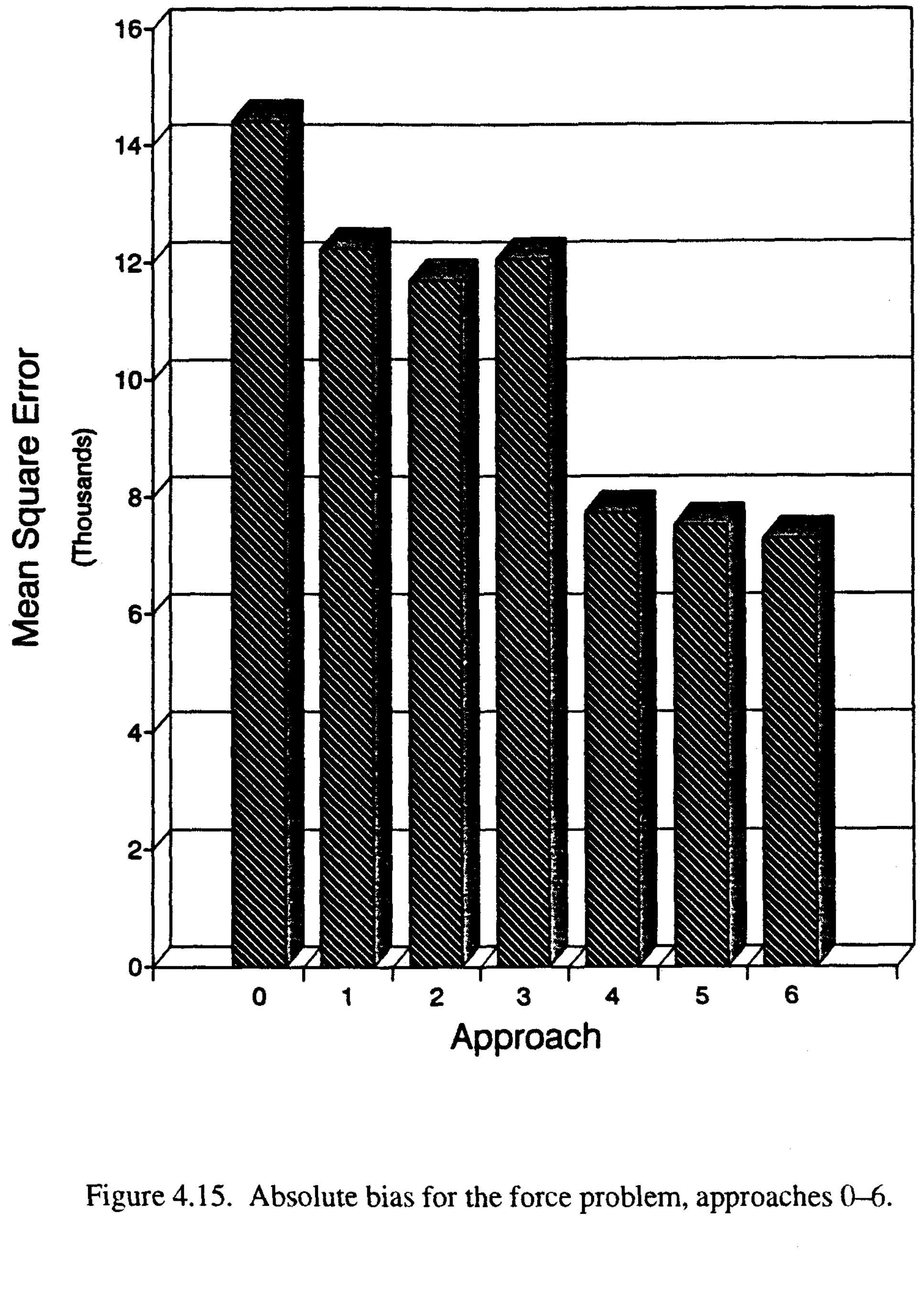


Figure 4.15. Absolute bias for the force problem, approaches 0–6.

the smallest mean squared error for model 4 was achieved based upon a CCD method (FULL-ORTH).

The smallest mean squared errors for models 1–8 and the force problem FULL-ROTATE were, respectively, 0.4315 with a standard error of 0.0103, 0.6645 with a standard error of 0.0358, 0.0576 with a standard error of 0.0037, 0.7187 with a standard error of 0.0245, 0.000135 with a standard error of 7.61×10^{-6}, 0.00302 with a standard error of 5.93×10^{-5}, 1.36×10^{-6} with a standard error of 1.51×10^{-7}, 1.36×10^{-6} with a standard error of 1.51×10^{-7}, and 7,513.92 with a standard error of 109.77 in comparison to, respectively, 0.5445 with a standard error of 0.0205, 0.9738 with a standard error of 0.3432, 0.1753 with a standard error of 0.0084, 0.7643 with a standard error of 0.0202, 0.000383 with a standard error of 2.24×10^{-5}, 0.000383 with a standard error of 0.000135, 1.66×10^{-5} with a standard error of 1.8×10^{-6}, 2.1×10^{-6} with a standard error of 2.23×10^{-7}, and 14,431.4 with a standard error of 318.32 for the Taguchi method. Thus, for two of four additive models (models 1 and 3), for three of four multiplicative models (models 6–8), and for the force problem, FULL-ROTATE yielded mean squared errors which were smaller than those for FULL-ORTH (Figs. 4.13–4.15). For five of nine test problems, there were no statistically significant differences for averaged mean squared errors between the two design properties, whereas four of nine test problems indicated statistically significant differences for averaged mean squared errors at p-value ranges from 0.0004 to 0.0343. Specifically, for models 1–3 and 7, statistically significant differences were indicated between the two design properties in favor of rotatability. Therefore, for mean squared errors, the rotatable design yielded the most favorable results when compared to the orthogonal design. This result was similar to those for variance and bias.

4.4.2 Design Experiments

The results for the force problem with respect to experimental design indicated that the CCD methods, based upon either the full-factorial or fractional-factorial experiments, were superior to the Taguchi method. Between the two methods, there were statistically significant differences for average mean squared errors at p-values less than 0.05 (Appendix B, Table B25). In addition, the full-factorial CCD design experiments yielded statistically significant differences for average mean squared errors at p-values less than 0.05. Results, as indicated in Tables B25–B26 (Appendix B) and Figure 4.15, demonstrated the superiority of the full-factorial CCD design when compared to the fractional-factorial CCD design. In comparison to the use of the fractional-factorial CCD design, the reduction of mean squared errors was approximated one-third more often when employing the full-factorial CCD design. For example, the mean squared error for FULL-ROTATE was 7,513.92, whereas that for FRAC-ROTATE was 11,705.48. However, if costs were a major consideration, then the use of the fractional-factorial CCD design would be appropriate in applicable situations.

4.4.3 Weighting Functions

The results for weighting functions, as shown in Figures 4.16–4.18, favored the use of the inverse of the squared distance from the origin (w5) to decrease mean squared errors for all test problems. However, the next smaller values for the mean squared errors, yielded by alternative weighting functions, were only from 2.45% to 4.42% larger than the mean squared errors obtained from the use of weighting function w5. Moreover, since it resulted in very little difference in the values of mean squared errors, a weighting function of one was seemingly sufficient for the CCD method when compared to the use of the weighting function w5 (i.e., a range from 3.5% to 16.99% larger than for the weighting function w5). In addition, the results of the use of a weighting function of one (w1) for the CCD method were superior to the results obtained from the use of

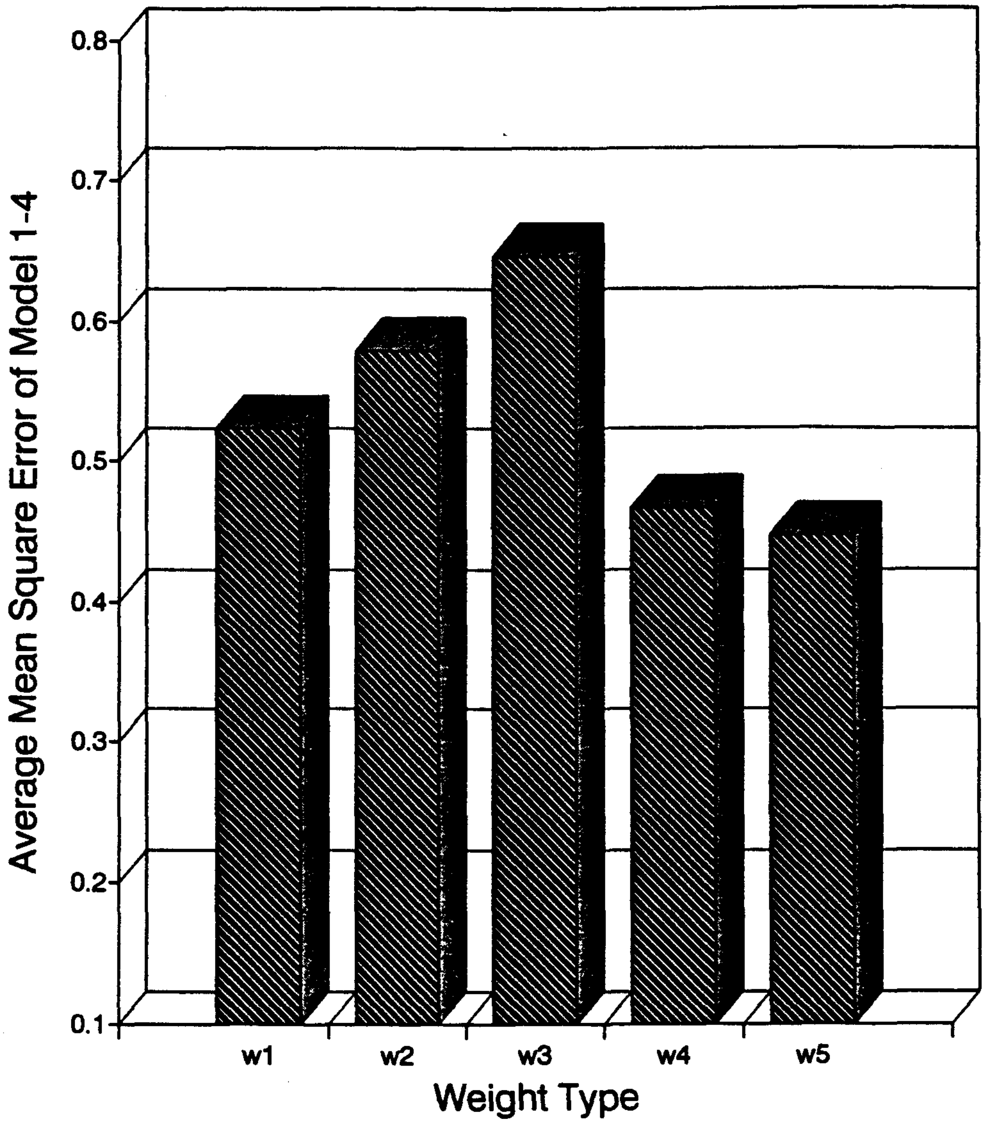


Figure 4.16. Average MSE for additive models (1–4).

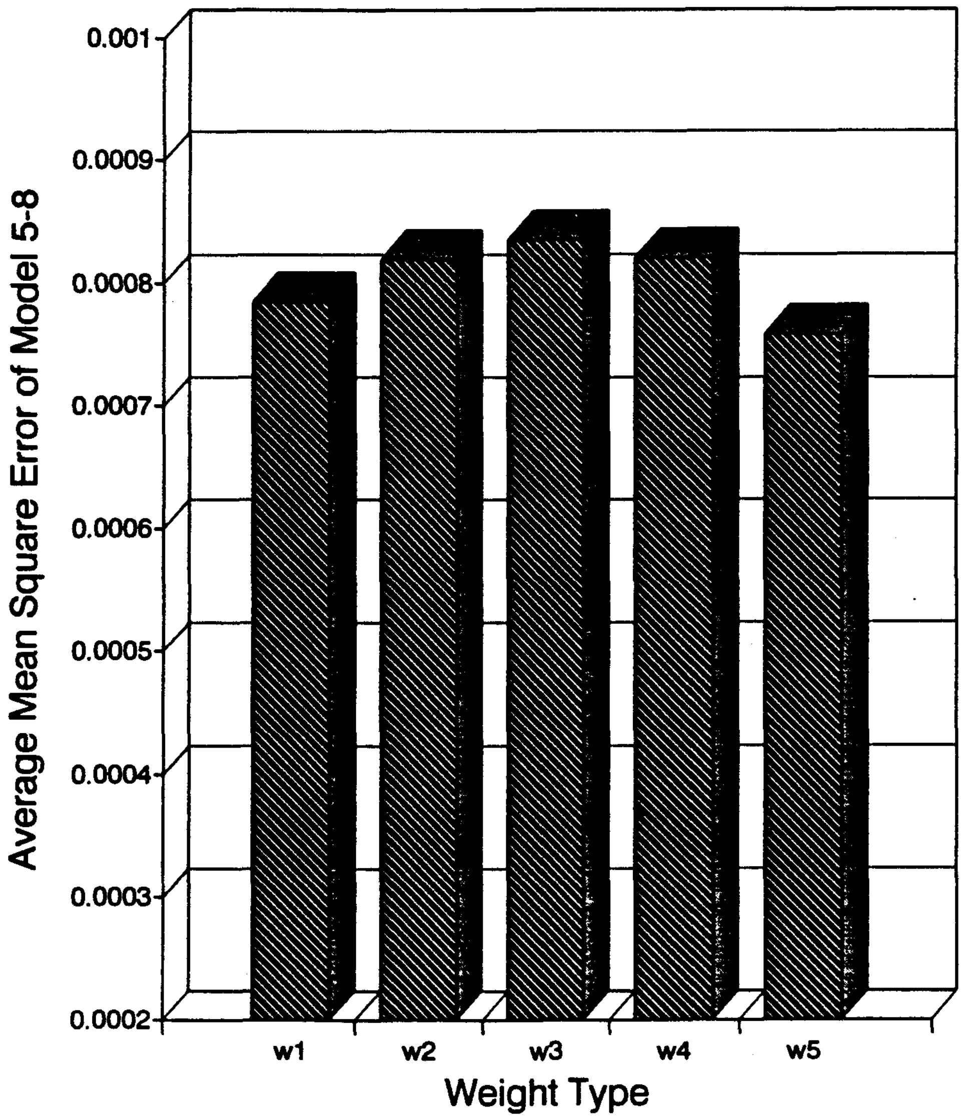


Figure 4.17. Average MSE for multiplicative models (5–8).

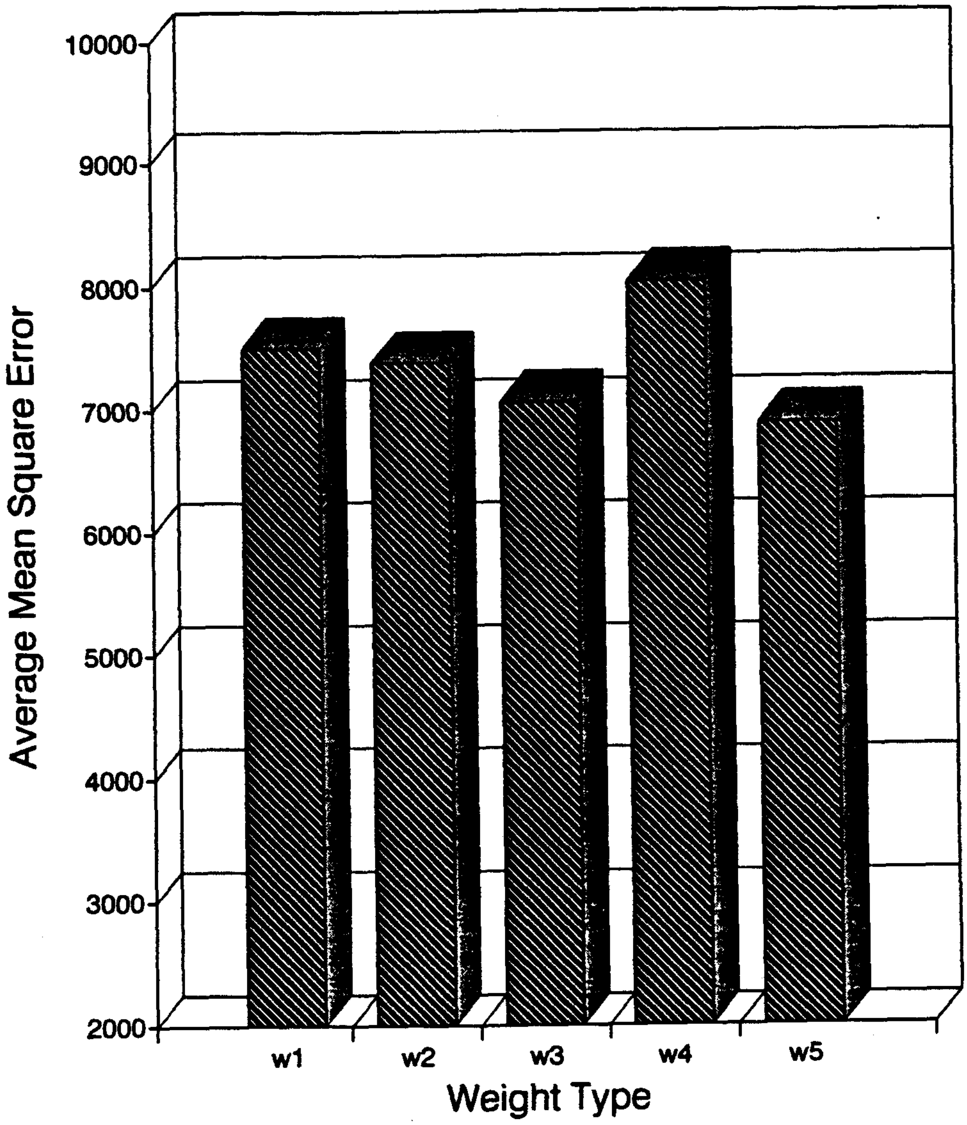


Figure 4.18. Average MSE for the force problem.

the Taguchi method. Therefore, if bias or variance are not major considerations, then the ordinary least-squares method (i.e., weighting function of one) remains appropriate for the CCD method with respect to the overall performance measures (MSE's).

4.5 Summary of the Results

Overall, the results of the current investigation favored the use of the proposed CCD methods. Results for the three performance statistics, including variance, bias, and mean squared error, revealed statistically significant differences between the CCD methods and the Taguchi method in location (i.e., averages) at the 5% level of significance for all the test problems, with the exception of model 4. The CCD methods tended to provide underestimations for mean responses, whereas the Taguchi method tended to overestimate the mean response. Differences in design properties tended toward differing results. Moreover, for the force problem, the statistically significant differences for the performance measure averages between the two types of design experiments, full-factorial and fractional-factorial, were yielded at p-values less than 0.05. Optimal settings for the design variables obtained via the CCD approach favored the full-factorial, rotatable design. The introduction of the weighting function served to decrease variance, bias, and mean squared error. Discussion of the comparison of results for the CCD and Taguchi methods is presented in the following section.

4.6 Discussion of the Findings

Overall, the optimal settings of the design variables obtained with the CCD method yielded the smallest mean squared errors for all test problems. There were differences in terms of biases for some of the test problems (i.e., models 1 and the force problem), the use of the CCD method provided statistically inferior results to the Taguchi method. However, for all test problems, estimated variances for the estimated mean response from the CCD method were smaller than for the Taguchi method.

As noted in the previous section, consideration of design properties yielded statistical differences in the results obtained with the use of the alternative CCD approaches. However, for some of the test problems, there were no differences in the performance measure locations (averages) between the two designs. For example, statistical analyses at the 5% level of significance for estimated variance (models 3–5 and 8) and for estimated bias as well as mean squared error (model 3–5, 8, and the force problem) indicated there were no statistically significant differences between the two design properties. At the same time, the CCD design with rotatability was preferred to the design with orthogonality since the former yielded smaller or approximately equivalent mean squared errors for all test problems, with the exception of model 2. As stated by Box and Draper (1987), "rather than attempt to generalize the property of orthogonality to second order, we shall instead generalize the property of rotatability" (p. 484). Thus, the findings seemingly favor selection, where possible, of a second-order design in which the variances and covariances of the estimates tend to remain constant as the design is rotated. Moreover, for all test problems, the CCD design with rotatabilty frequently yielded results which were identical or approximately equivalent to those obtained from the use of CCD design with both orthogonality and rotatability. The exception was model 7, for which the former case was to be preferred. Thus, at a minimum, CCD design with rotatability is to be preferred when considering alternatives among the proposed CCD methods.

The results also indicated that differences in the design experiments for the cubed part of the proposed CCD method yielded statistical differences for the performance measures. However, the full-factorial experiment for the cubed part of the CCD design was superior to the fractional-factorial experiment since the former more often resulted in smaller overall performance measures (i.e., mean squared errors). Consider the results of the force problem. The average mean squared error yielded by the employment of full-factorial experiments provided a range approximately 35.8% to 39.46% smaller than for the fractional-factorial experiments (that is, when compared for the same design properties: orthogonality, rotatability or both orthogonality and rotatability). This effect occurred on some occasions as a result of the exclusion of some design points that were parts of the region of optimal settings for the test problem, based upon fractional-factorial experimentation. Though the full-factorial experiment provided superior mean response estimates (i.e., smaller biases) than the fractional-factorial experiment, the former case tended to generate higher variances. This was because the wider ranges for x in the fractional-factorial experiments resulted in improved estimations of slope. Thus, the estimated variance function provided better performance and results for smaller variances than for the full-factorial cases (i.e., refer to 1–3 vs. 4–6, Tables B25 and B26, Appendix B). However, it should be noted that among all approaches (0–6), the full-factorial experiments yielded the smallest mean squared errors.

When the multiplicative high-order functions (model 5–8) are considered, the results indicated that the estimated mean responses for models 7 and 8 differed significantly from the target value. This was because the slope differences between the true function (with very high-order degree polynomials) and the estimated quadratic mean response function approximating the true function were quite large. For instance, the first-order derivative with respect to x_1 for models 7 and 8 were, respectively, 0 and 0.00274, when taken from the true function at optimal settings obtained from, respectively, FULL-ORTH (which yielded smaller variances than FULL-ROTATE) and FULL-ROTATE (which yielded smaller variance than the FULL-ORTH), whereas those taken from the estimated quadratic mean response function of, respectively, FULL-ORTH and FULL-ROTATE, at identical settings was, respectively, 0.01168 and 0.00643 (i.e., almost triple the gradient from the true function). The differences for both cases were considered to be large in comparison to the target values of both models (i.e. the target value of models 7 and 8 were 0.0015 and 0.0005, respectively). It may be observed that large slope differences will contribute to poor estimations of the variance function when a Taylor's series is applied to the estimated mean response function. Moreover, model 7 consisted of a fourth-degree polynomial and model 8 consisted of a fifth-degree polynomial. Hence, the results implied that the highest-order of polynomials for multiplicative high-order functions should be the third-order. Beyond third-order degree polynomials, the results obtained from the CCD method yielded larger bias (i.e., in excess of 10% of the target value, or even approaching 90% of the target value, as noted in section 4.3.1). However, the results for the additive high-order functions (models 1–4) did not reflect this problem.

Nevertheless, from further investigation of model 4, the performance characteristics function was found to be rather flat around the region of the optimal settings for the design variables. Thus, the Taylor's series expansion did not perform as well. According to Poston and Stewart (1976), if all derivatives vanish at zero during application of true smoothing functions, approximations based upon a Taylor's series expansion will result in substandard performance. In addition, a polynomial expression of the degree d can be thought of as a Taylor's series expression of the true underlying theoretical function $f(\xi) = E[Y]$, truncated following the terms of d^{th} order (Box & Draper, 1987). Comparing Taylor's series expansion to the fitted quadratic mean response function, the derivative terms can be thought of as the coefficient terms for the fitted quadratic mean response function (i.e., the terms $\frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i x_j}$, and $\frac{\partial^2 f}{\partial x_i^2}$ can be regarded as coefficients, respectively, of the first-order term x_i (β_i), the cross-product terms x_{ij} (β_{ij}, where $i < j$), and the second-order terms x_i^2 (β_{ii})). Since most of the coefficient terms of the fitted quadratic mean response function for model 4 were rather small (close to zero) or zero, especially for the cross-product terms in both the full-factorial orthogonal and the full-factorial rotatable designs for the CCD methods, the estimated variance function of model 4 obtained by applying a Taylor's series to the fitted quadratic mean response function was somewhat substandard in performance. Therefore, the proposed CCD method encompassed certain limitations when the performance characteristics function was flat around the region of the optimal settings.

Finally, the results obtained from weighted least squares regression analysis indicated that applying particular weights for the proposed method could serve to decrease mean squared error, bias, and variance. However, there were no clear conclusions with regard to the most favorable weighting. Even investigating the test problems themselves did not reveal any pattern. To improve mean squared error findings, the inverse of the square of the distance from the point of origin of x (w5) was seemingly appropriate to all test problems. For bias, the absolute distance of the performance characteristics from the target (w2) would be appropriate, which in any case yielded better results than alternative approaches (i.e., the estimated mean responses were closer to the target values). For variance, ordinary least-squares regression (w1) analysis was seemingly sufficient for the force problem. However, the use of the inverse of the squared distance from the point of origin could decrease estimated variances for the high-order functions (models 1–8). Nevertheless, bias was the most significant contribution to mean squared errors for the CCD method. Bias was an exceptional case in that the results yielded statistically significant differences which favored the Taguchi method, especially with respect to the force problem. The absolute bias (%) of the Taguchi method was 1.45%, in comparison to 9.57% for full-factorial with both orthogonal and rotatable design. As a result, the absolute distance of the performance characteristics from the target ($w2 = 1/|y - \tau|$)

provide the most favorable alternative since this approach serves to improve the mean response with respect to the target value.

In summary, the proposed CCD approach was superior to the Taguchi approach for all test problems developed and considered for the current investigation. However, the CCD method encompassed certain limitations when the performance characteristics function was flat around the region of the optimal settings. The full-factorial CCD design with rotatability, among all CCD design alternatives, appeared to be the most favorable design for the approximation of the quadratic mean response function. The limit of the highest-order polynomial degree for multiplicative high-order functions was determined to be the third-order, at which level the CCD method continued to provide acceptable bias values (i.e., bias less than 10% of the target value). No conclusions were reached with respect to an appropriate weight for the CCD method. However, the inverse of the squared distance from the point of origin would be the correct choice when the objective is to improve (decrease) mean squared errors for functions similar to those considered in the test problems. However, since bias was the most significant contributor to mean square error, the absolute distance of the performance characteristics from the target ($w2 = 1/|y - \tau|$) provides the most favorable alternative. In the final chapter, the principal accomplishments of this research investigation are summarized and suggestions for further research are provided.

CHAPTER 5

CONCLUSIONS

The purpose of this chapter is to summarize the principal objectives accomplished by the conduct of this research study and to suggest areas of further investigation in related research.

5.1 Principal Accomplishments

This research investigation has presented a new methodology for obtaining optimal design variable settings for product/process design. The principal objective was to compare results obtained with a proposed central composite design approach to those obtained by application of the Taguchi method. However, it should be noted that for consideration of these two methods there was a minor difference with respect to the definition of the optimization problem. The objective of the Taguchi method is to minimize the mean square error (MSE), whereas the objective of the CCD method is to minimize product performance characteristic variations while achieving the target value. In addition, the CCD method employed only design matrices based upon central composite design experimentation, thus requiring fewer experimental runs than the Taguchi method.

The results of the statistical analyses presented in Chapter 4 indicated that the CCD approach, with the exception of a single model (model 4), was superior to the Taguchi approach for all test problems at the 5% level of significance. The reason for the exception was that the performance characteristics function for problem model 4 was rather flat in the areas of the optimal settings for the design variables. Therefore, CCD performance superiority was not so marked as for the remainder of the test problems. However, it should be observed that the CCD method results for model 4 still yielded smaller performance measure values, including mean square errors, biases and variances, in comparison to similar measure for the Taguchi method. In addition, though the product performance characteristic variations for the CCD were minimized (in comparison to those for the Taguchi method), the optimal settings for the CCD method tended to yield the underestimations of mean responses,

It should be noted that the development of the full-factorial rotatable central composite design is significant to the success of the proposed CCD method with respect to the shortcomings of the Taguchi method. Applying the CCD method based upon this approach to multiplicative high-order functions, the results illustrated the limitations of the order degree polynomial. Thus, the estimated quadratic mean response function will be less appropriate when multiplicative high-order functions consist of fourth-degree polynomials or higher. The substantial slope differences between the estimated quadratic function and the true multiplicative function of the fourth degree or higher will result in a large bias and/or poor estimations of the variance function. Therefore, this approach would serve to reduce the level of performance of the proposed CCD method, the performance of which is adequate when the highest-order degree polynomials of the multiplicative high-order functions are not extended beyond the third order.

Investigation on the use of weighted least-squares regression analysis to obtain the estimated mean response function with the CCD method indicated that the use of weighted least-squares is of some assistance in decreasing mean square errors. In addition, the inverse of the squared distance from the point of origin of the design variables ($1/\Sigma x^2$), since its use resulted in the smallest average mean square errors for the additive models (models 1–4), the multiplicative models (models 5–8), and the force problem, was an appropriate weight for decreasing mean square errors in all of the test problems considered for the current investigation. In addition, mean square errors yielded by applying the $1/\Sigma x^2$ weighting function to the CCD method were approximately from 3.43% to 14.5% smaller than those yielded by applying a weighting function of one (ordinary least-squares regression) using the same method. However, since the most contributing part of the mean square error is the bias, the absolute distance of the performance characteristics from the target ($w2 = 1/|y - \tau|$) would be the best alternative.

Thus, the proposed CCD method based upon response surface methodology has been successfully developed and can be used to improve product or production design quality. A complete response surface methodology for the determination of optimal design variable settings which minimize the product performance characteristic variations while achieving specified target values has been presented. Results from all test problems, with the single exception of model 4, indicated that the CCD method was superior to the Taguchi method. However, the proposed response surface model can be used to provide significant improvements in product quality as well as lower cost process designs when compared to applications of the Taguchi method. In addition, the proposed CCD method also requires significantly fewer experimental runs than the Taguchi method. For example, consider the force problem examined in this study. The CCD method based upon full-factorial rotatable design required approximately 40.28% fewer experimental runs than the Taguchi method (i.e., 44 runs vs. 72 runs). In turn, the CCD method based upon fractional-factorial rotatable design required approximately 61.11% fewer experimental runs than the Taguchi method (i.e., 28 runs vs. 72 runs). However, due to unknown performance characteristic functions in those cases for which the gradients of the performance functions are approximately zero, the proposed CCD model encompasses certain limitations.

5.2 Recommendations for Further Study

For practical purposes, the influence factors in the manufacturing process can be categorized in three mutually exclusive groups: control factors, noise factors, and signal factors (Phadke, 1989b). The first two of these groups were considered in Chapter 3, and solutions for the problems they posed have been encompassed within the proposed method. However, the signal factors, M, which are normally selected by the design engineer based upon engineering knowledge of the product being developed, have not been considered in the proposed method. Since this study has disregarded signal factors in the formulation and analysis of the proposed CCD method, further research should consider the inclusion of signal factors into the problem, thus modifying the objective function of the CCD approach to minimize product performance characteristic variations for the worst signal factors cases, as follows:

$$\mathrm{Min}_X\{\mathrm{Max}_M\ \sigma^2{}_y(X,W,M)\}\ ,$$

s.t. $|\mu_y(X,W,M) - \tau| \leq a$, where $x \geq 0$, $w \geq 0$, $m \geq 0$, and $a \geq 0$, or

$$\mathrm{Min}_X\{\mathrm{Max}_M\ [\beta + 2B(X,W,M)]'\ \Sigma_{X,W,M}\ [\beta + 2B(X,W,M)]\ ,$$

s.t. $|\mu_y(X,W,M) - \tau| \leq a$, where $x \geq 0$, $w \geq 0$, $m \geq 0$, and $a \geq 0$.

Note that the values for w are fixed at their means prior to problem optimization with the proposed CCD method. The values of the signal factors are set by the user or the operator to express the intended values for product response.

Since estimated mean responses obtained via the proposed CCD method tend to be underestimated, additional explorations of the use of weighted least-squares regression analysis would also be an appropriate area for further study. Determining an appropriate weight constitutes an interesting problem. The introduction of different weighting functions to obtain estimations of the quadratic mean function and the variance function is one possibility. Moreover, since study of the CCD method is based upon the use of the central composite design throughout the process, other types of second-order response surface design should be considered. For example, the Box-Behnken design, a minimum bias design, or a small composite design could be investigated with respect to the proposed CCD method. Finally, an additional area for possible investigation would be the relation of the proposed approach to other types of functions. This could be accomplished in the context of central composite design experiments, or it could encompass alternative second-order response designs.

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APPENDICES

APPENDIX A

Inputs for Data Matrices

This appendix demonstrates the low and high values for the design variables, as inputs to *STATGRAPHICS* response surface designs to obtain data matrices for all test problems considered for the CCD method.

Table A1. Inputs for *STATGRAPHICS*.

Test Problem	Variable names	low	high
models 1-8	x_1	0.3	0.7
	x_2	0.3	0.7
	x_3	0.3	0.7
	x_4	0.3	0.7
"force problem" for all full-factorial cases	x_1	125	155
	x_2	47.5	62.5
	x_3	8	12
	x_4	30	40
	x_5	15	35
"force problem" for all fractional-factorial cases	x_1	120	160
	x_2	45	65
	x_3	8	12
	x_4	27.5	42.5
	x_5	15	35

Note: For three-level factorial experiments via the Taguchi method, the levels of x for models 1-8 are represented as follows: level 0, x = 0.25, level 1, x = 0.5, and level 2, x = 0.75. Furthermore, the 3^{3-1} fractional-factorial experiment is used for the design matrix (an inner array) and the 2^{4-2} fraction-factorial experiment is used for the noise matrix (an outer array) for models 1-8 for the Taguchi method.

APPENDIX B

Simulation Results

In the pages which follow, simulation results for all test problems are provided in Tables B1–B26.

Table B1. Simulation results for the force problem ($\tau = 400$).

Approach	Taguchi	2^{5-1} orthogonal CCP	2^{5-1} rotatable CCP	2^{5-1} rotatable & orthogonal CCP	2^5 orthogonal CCP	2^5 rotatable CCP	2^5 rotatable & orthogonal CCP
Mean	396.36	312.99	316.97	316.16	358.55	358.38	361.73
Estimated variance	14,390.33	4,667.53	4,801.29	5,018.46	5,996.25	5,769.86	5,828.22
Mean square error	14,403.60	12,238.86	11,695.24	12,048.11	7,714.35	7,502.08	7,292.46
x_i^*, s (x_5^*=25)	$x_1^* = 156$ $x_2^* = 75$ $x_3^* = 10$ $x_4^* = 20$	$x_1^* = 180$ $x_2^* = 73.83$ $x_3^* = 15$ $x_4^* = 20.42$	$x_1^* = 180$ $x_2^* = 74.08$ $x_3^* = 15$ $x_4^* = 20$	$x_1^* = 180$ $x_2^* = 73.89$ $x_3^* = 15$ $x_4^* = 20.06$	$x_1^* = 176.67$ $x_2^* = 75$ $x_3^* = 15$ $x_4^* = 20.58$	$x_1^* = 176.48$ $x_2^* = 75$ $x_3^* = 15$ $x_4^* = 20.72$	$x_1^* = 176.39$ $x_2^* = 75$ $x_3^* = 15$ $x_4^* = 20.57$

Table B2. Simulation results for the force problem with weight (2^5 rotatable CCD).

Approach	Taguchi	CCP w/o weight (w =1)	CCP $w = 1/\lvert y - \tau \rvert$	CCP $w = 1/(y - \tau)^2$	CCP $w = 1/(\Sigma x^2)^{1/2}$	CCP $w = 1/(\Sigma x^2)$
Mean	396.36	358.38	368.35	369.80	353.87	378.52
Estimated variance	14,390.33	5,769.86	6,376.74	6,142.47	5,908.28	6,424.23
Mean square error	14,403.60	7,502.08	7,378.26	7,054.42	8,036.42	6,885.51
x_i^*, s ($x_5^* = 25$)	$x_1^* = 156$ $x_2^* = 75$ $x_3^* = 10$ $x_4^* = 20$	$x_1^* = 176.48$ $x_2^* = 75$ $x_3^* = 15$ $x_4^* = 20.72$	$x_1^* = 176.19$ $x_2^* = 75$ $x_3^* = 15$ $x_4^* = 20$	$x_1^* = 176.06$ $x_2^* = 75$ $x_3^* = 15$ $x_4^* = 20$	$x_1^* = 176.78$ $x_2^* = 75$ $x_3^* = 15$ $x_4^* = 20.94$	$x_1^* = 175.33$ $x_2^* = 75$ $x_3^* = 15$ $x_4^* = 20$

Table B3. Simulation results for model 1 ($\tau = 2.5$).

Approach	Taguchi	2^5 orthogonal CCP	2^5 rotatable CCP	2^5 rotatable & orthogonal CCP
Mean	2.5618	2.1327	2.2519	2.2519
Estimated variance	0.5373	0.3293	0.3679	0.3679
Mean square error	0.5411	0.4642	0.4294	0.4294
x_i^*, s (x_5^*=25)	$x_1^* = 0.7075$ $x_2^* = 0.25$ $x_3^* = 0.75$ $x_4^* = 0.5$	$x_1^* = 0.56506$ $x_2^* = 0.02883$ $x_3^* = 0$ $x_4^* = 0.5$	$x_1^* = 0.60380$ $x_2^* = 0.03431$ $x_3^* = 0$ $x_4^* = 0.5$	$x_1^* = 0.60380$ $x_2^* = 0.03431$ $x_3^* = 0$ $x_4^* = 0.5$

Table B4. Simulation results for model 1 with weight (2^4 rotatable CCD).

Approach	Taguchi	CCP w/o weight ($w = 1$)	CCP $w = 1/\lvert y - \tau \rvert$	CCP $w = 1/(y - \tau)^2$	CCP $w = 1/(\Sigma x^2)^{1/2}$	CCP $w = 1/(\Sigma x^2)$
Mean	2.5618	2.2519	2.1638	2.1964	2.1444	2.1387
Estimated variance	0.5373	0.3679	0.3329	0.7250	0.3492	0.3441
Mean square error	0.5411	0.4294	0.4460	0.8171	0.4757	0.4747
x_i^*'s ($x_5^* = 25$)	$x_1^* = 0.7075$ $x_2^* = 0.25$ $x_3^* = 0.75$ $x_4^* = 0.5$	$x_1^* = 0.6038$ $x_2^* = 0.03431$ $x_3^* = 0$ $x_4^* = 0.5$	$x_1^* = 0.60342$ $x_2^* = 0$ $x_3^* = 0$ $x_4^* = 0.5$	$x_1^* = 0.9352$ $x_2^* = 0.6921$ $x_3^* = 1.0$ $x_4^* = 0.5$	$x_1^* = 0.57863$ $x_2^* = 0$ $x_3^* = 0$ $x_4^* = 0.5$	$x_1^* = 0.5833$ $x_2^* = 0$ $x_3^* = 0$ $x_4^* = 0.5$

Table B5. Simulation results for model 2 ($\tau = 2.5$).

Approach	Taguchi	2^5 orthogonal CCP	2^5 rotatable CCP	2^5 rotatable & orthogonal CCP
Mean	2.6815	2.4132	2.4681	2.5093
Estimated variance	0.9019	0.6552	0.8763	0.8725
Mean square error	0.9349	0.6627	0.8773	0.8725
x_i^*, s ($x_5^*=25$)	$x_1^* = 0.895$ $x_2^* = 0.75$ $x_3^* = 0.25$ $x_4^* = 0.5$	$x_1^* = 0.78544$ $x_2^* = 0.84771$ $x_3^* = 0.15641$ $x_4^* = 0.5$	$x_1^* = 0.74577$ $x_2^* = 0.80347$ $x_3^* = 0.31302$ $x_4^* = 0.5$	$x_1^* = 0.74589$ $x_2^* = 0.80333$ $x_3^* = 0.31346$ $x_4^* = 0.5$

Table B6. Simulation results for model 2 with weight (2^4 rotatable CCD).

Approach	Taguchi	CCP w/o weight (w =1)	CCP w = 1/|y- τ|	CCP w =1/(y - τ)^2	CCP w = 1/(Σx^2)$^{1/2}$	CCP w = 1/(Σx^2)
Mean	2.6815	2.4681	2.5564	2.6368	2.2453	2.2045
Estimated variance	0.9019	0.8763	1.1008	1.0944	0.5369	0.4758
Mean square error	0.9349	0.8773	1.1040	1.1131	0.6017	0.5632
x_i^*, s (x_5^*=25)	x_1^* = 0.895 x_2^* = 0.75 x_3^* = 0.25 x_4^* = 0.5	x_1^* = 0.74577 x_2^* = 0.80347 x_3^* = 0.31302 x_4^* = 0.5	x_1^* = 0.69412 x_2^* = 0.74172 x_3^* = 0.55382 x_4^* = 0.5	x_1^* = 0.72148 x_2^* = 0.76432 x_3^* = 0.45928 x_4^* = 0.5	x_1^* = 0.79781 x_2^* = 0.89346 x_3^* = 0 x_4^* = 0.5	x_1^* = 0.79489 x_2^* = 0.87586 x_3^* = 0 x_4^* = 0.5

Table B7. Simulation results for model 3 ($\tau = 2.5$).

Approach	Taguchi	2^5 orthogonal CCP	2^5 rotatable CCP	2^5 rotatable & orthogonal CCP
Mean	2.4986	2.4650	2.5038	2.5038
Estimated variance	0.17438	0.06326	0.06074	0.06074
Mean square error	0.17438	0.06448	0.06076	0.06076
x_i^*, s (x_5^*=25)	$x_1^* = 0.4655$ $x_2^* = 0.5$ $x_3^* = 0.25$ $x_4^* = 0.5$	$x_1^* = 0.22378$ $x_2^* = 0.58688$ $x_3^* = 0.00001$ $x_4^* = 0.5$	$x_1^* = 0.20833$ $x_2^* = 0.61321$ $x_3^* = 0.00002$ $x_4^* = 0.5$	$x_1^* = 0.20833$ $x_2^* = 0.61321$ $x_3^* = 0.00002$ $x_4^* = 0.5$

Table B8. Simulation results for model 3 with weight (2^4 rotatable CCD).

Approach	Taguchi	CCP w/o weight (w =1)	CCP $w = 1/\lvert y - \tau \rvert$	CCP $w = 1/(y - \tau)^2$	CCP $w = 1/(\Sigma x^2)^{1/2}$	CCP $w = 1/(\Sigma x^2)$
Mean	2.4986	2.5038	2.4596	2.4418	2.5289	2.5161
Estimated variance	0.17438	0.06074	0.05691	0.05492	0.05615	0.05391
Mean square error	0.17438	0.06076	0.05855	0.05830	0.05633	0.05462
x_i^*, s (x_5^*=25)	$x_1^* = 0.4655$ $x_2^* = 0.5$ $x_3^* = 0.25$ $x_4^* = 0.5$	$x_1^* = 0.20833$ $x_2^* = 0.61321$ $x_3^* = 0.00002$ $x_4^* = 0.5$	$x_1^* = 0.20953$ $x_2^* = 0.57451$ $x_3^* = 0$ $x_4^* = 0.5$	$x_1^* = 0.22128$ $x_2^* = 0.57451$ $x_3^* = 0$ $x_4^* = 0.5$	$x_1^* = 0.20270$ $x_2^* = 0.60578$ $x_3^* = 0$ $x_4^* = 0.5$	$x_1^* = 0.19719$ $x_2^* = 0.60070$ $x_3^* = 0$ $x_4^* = 0.5$

Table B9. Simulation results for model 4 ($\tau = 3.5$).

Approach	Taguchi	2^5 orthogonal CCP	2^5 rotatable CCP	2^5 rotatable & orthogonal CCP
Mean	3.5396	3.4621	3.4670	3.4670
Estimated variance	0.7574	0.71073	0.72079	0.72079
Mean square error	0.7590	0.7122	0.7219	0.7219
x_i^*, s ($x_5^*=25$)	$x_1^* = 0.75$ $x_2^* = 0.46$ $x_3^* = 0.25$ $x_4^* = 0.5$	$x_1^* = 0.82381$ $x_2^* = 0.33488$ $x_3^* = 0.0$ $x_4^* = 0.5$	$x_1^* = 0.82213$ $x_2^* = 0.33790$ $x_3^* = 0.0$ $x_4^* = 0.5$	$x_1^* = 0.82213$ $x_2^* = 0.33790$ $x_3^* = 0.0$ $x_4^* = 0.5$

Table B10. Simulation results for model 4 with weight (2^4 rotatable CCD).

Approach	Taguchi	CCP w/o weight (w =1)	CCP w = 1/|y- τ|	CCP w =1/(y - τ)^2	CCP w = 1/(Σx^2)$^{1/2}$	CCP w = 1/(Σx^2)
Mean	3.5396	3.4670	3.4688	3.6222	3.4433	3.4489
Estimated variance	0.7574	0.72079	0.7021	0.5829	0.7312	0.6894
Mean square error	0.7590	0.7219	0.7031	0.5978	0.7344	0.6920
x_i^*, s (x_5^*=25)	x_1^* = 0.75 x_2^* = 0.46 x_3^* = 0.25 x_4^* = 0.5	x_1^* = 0.82213 x_2^* = 0.3379 x_3^* = 0 x_4^* = 0.5	x_1^* = 0.79883 x_2^* = 0.36341 x_3^* = 0 x_4^* = 0.5	x_1^* = 1.0 x_2^* = 0.15720 x_3^* = 0 x_4^* = 0.5	x_1^* = 0.82612 x_2^* = 0.33277 x_3^* = 0 x_4^* = 0.5	x_1^* = 0.82998 x_2^* = 0.32763 x_3^* = 0 x_4^* = 0.5

Table B11. Simulation results for model 5 ($\tau = 0.015$).

Approach	Taguchi	2^5 orthogonal CCP	2^5 rotatable CCP	2^5 rotatable & orthogonal CCP
Mean	0.02323	0.01578	0.01594	0.01594
Estimated variance	0.0003281	0.0001376	0.00013352	0.00013352
Mean square error	0.000396	0.0001382	0.0001344	0.0001344
x_i^*, s (x_5^*=25)	$x_1^* = 0.75$ $x_2^* = 0.03596$ $x_3^* = 0.75$ $x_4^* = 0.5$	$x_1^* = 0.5166$ $x_2^* = 0.5111$ $x_3^* = 0.57369$ $x_4^* = 0.5$	$x_1^* = 0.52876$ $x_2^* = 0.49994$ $x_3^* = 0.54208$ $x_4^* = 0.5$	$x_1^* = 0.52876$ $x_2^* = 0.49994$ $x_3^* = 0.54208$ $x_4^* = 0.5$

Table B12. Simulation results for model 5 with weight (2^4 rotatable CCD).

Approach	Taguchi	CCP w/o weight (w =1)	CCP $w = 1/\lvert y - \tau \rvert$	CCP $w = 1/(y - \tau)^2$	CCP $w = 1/(\Sigma x^2)^{1/2}$	CCP $w = 1/(\Sigma x^2)$
Mean	0.02323	0.015941	0.017079	0.016801	0.016287	0.016344
Estimated variance	0.0003281	0.0001335	0.000184	0.0001773	0.0001643	0.000157
Mean square error	0.000396	0.0001344	0.0001885	0.00018058	0.0001659	0.0001588
x_i^*, s (x_5^*=25)	$x_1^* = 0.75$ $x_2^* = 0.03596$ $x_3^* = 0.75$ $x_4^* = 0.5$	$x_1^* = 0.52876$ $x_2^* = 0.49994$ $x_3^* = 0.54208$ $x_4^* = 0.5$	$x_1^* = 0.54109$ $x_2^* = 0.51713$ $x_3^* = 0.52145$ $x_4^* = 0.5$	$x_1^* = 0.61771$ $x_2^* = 0.28221$ $x_3^* = 0.48816$ $x_4^* = 0.5$	$x_1^* = 0.53523$ $x_2^* = 0.7262$ $x_3^* = 0.54115$ $x_4^* = 0.5$	$x_1^* = 0.51471$ $x_2^* = 0.53599$ $x_3^* = 0.56014$ $x_4^* = 0.5$

Table B13. Simulation results for model 6 ($\tau = 0.25$).

Approach	Taguchi	2^5 orthogonal CCP	2^5 rotatable CCP	2^5 rotatable & orthogonal CCP
Mean	0.24097	0.24922	0.24507	0.24507
Estimated variance	0.003741	0.0034831	0.002983	0.002983
Mean square error	0.0038225	0.003484	0.003007	0.003007
x_i^*, s (x_5^*=25)	$x_1^* = 0.29373$ $x_2^* = 0.75$ $x_3^* = 0.75$ $x_4^* = 0.5$	$x_1^* = 0.50297$ $x_2^* = 0.62699$ $x_3^* = 0.52084$ $x_4^* = 0.5$	$x_1^* = 0.50209$ $x_2^* = 0.57256$ $x_3^* = 0.61556$ $x_4^* = 0.5$	$x_1^* = 0.50209$ $x_2^* = 0.57256$ $x_3^* = 0.61556$ $x_4^* = 0.5$

Table B14. Simulation results for model 6 with weight (2^4 rotatable CCD).

Approach	Taguchi	CCP w/o weight (w =1)	CCP $w = 1/\lvert y-\tau \rvert$	CCP $w = 1/(y-\tau)^2$	CCP $w = 1/(\Sigma x^2)^{1/2}$	CCP $w = 1/(\Sigma x^2)$
Mean	0.24097	0.24507	0.24285	0.24425	0.24522	0.24552
Estimated variance	0.003741	0.002983	0.003044	0.003140	0.003097	0.002824
Mean square error	0.0038225	0.003007	0.003095	0.003174	0.003120	0.002844
x_i^*, s ($x_5^*=25$)	$x_1^* = 0.29373$ $x_2^* = 0.75$ $x_3^* = 0.75$ $x_4^* = 0.5$	$x_1^* = 0.50209$ $x_2^* = 0.57256$ $x_3^* = 0.61556$ $x_4^* = 0.5$	$x_1^* = 0.48076$ $x_2^* = 0.56695$ $x_3^* = 0.67238$ $x_4^* = 0.5$	$x_1^* = 0.41710$ $x_2^* = 0.57078$ $x_3^* = 0.80585$ $x_4^* = 0.5$	$x_1^* = 0.50248$ $x_2^* = 0.57360$ $x_3^* = 0.61231$ $x_4^* = 0.5$	$x_1^* = 0.50283$ $x_2^* = 0.57472$ $x_3^* = 0.60890$ $x_4^* = 0.5$

Table B15. Simulation results for model 7 ($\tau = 0.0015$).

Approach	Taguchi	2^5 orthogonal CCP	2^5 rotatable CCP	2^5 rotatable & orthogonal CCP
Mean	0.0026305	0.00100485	0.00020517	0.0015574
Estimated variance	0.0000152	0.00000219	0.000000124	0.00000392
Mean square error	0.0000165	0.00000244	0.000001801	0.000003924
x_i^*, s (x_5^*=25)	$x_1^* = 0.75$ $x_2^* = 0.75$ $x_3^* = 0.40349$ $x_4^* = 0.5$	$x_1^* = 0.65628$ $x_2^* = 0$ $x_3^* = 0.74668$ $x_4^* = 0.5$	$x_1^* = 0.34846$ $x_2^* = 0.53782$ $x_3^* = 0.35135$ $x_4^* = 0.5$	$x_1^* = 0.52161$ $x_2^* = 0.39891$ $x_3^* = 0.59993$ $x_4^* = 0.5$

Table B16. Simulation results for model 7 with weight (2^4 rotatable CCD).

Approach	Taguchi	CCP w/o weight (w =1)	CCP $w = 1/\lvert y-\tau\rvert$	CCP $w = 1/(y-\tau)^2$	CCP $w = 1/(\Sigma x^2)^{1/2}$	CCP $w = 1/(\Sigma x^2)$
Mean	0.00263	0.00020517	0.0007808	0.000000152	0.0007031	0.00020465
Estimated variance	0.00001521	0.0000000124	0.000001997	4.8×10^{-13}	0.00000151	0.0000002967
Mean square error	0.00001649	0.000001801	0.000002514	0.00000225	0.000002145	0.000001975
x_i^*, s ($x_5^*=25$)	$x_1^* = 0.75$ $x_2^* = 0.75$ $x_3^* = 0.40349$ $x_4^* = 0.5$	$x_1^* = 0.34846$ $x_2^* = 0.53782$ $x_3^* = 0.35135$ $x_4^* = 0.5$	$x_1^* = 0.18108$ $x_2^* = 1.0$ $x_3^* = 0.76567$ $x_4^* = 0.5$	$x_1^* = 0$ $x_2^* = 0$ $x_3^* = 0.09141$ $x_4^* = 0.5$	$x_1^* = 0.33335$ $x_2^* = 1.0$ $x_3^* = 0.46255$ $x_4^* = 0.5$	$x_1^* = 0.76036$ $x_2^* = 1.0$ $x_3^* = 0$ $x_4^* = 0.5$

Table B17. Simulation results for model 8 ($\tau = 0.0005$).

Approach	Taguchi	2^5 orthogonal CCP	2^5 rotatable CCP	2^5 rotatable & orthogonal CCP
Mean	0.0009196	0.0005359	0.000753	0.000753
Estimated variance	0.00000191	0.000001629	0.00000129	0.00000129
Mean square error	0.00000208	0.000001546	0.000001355	0.000001355
x_i^*, s (x_5^*=25)	$x_1^* = 0.75$ $x_2^* = 0.37381$ $x_3^* = 0.75$ $x_4^* = 0.5$	$x_1^* = 0.52523$ $x_2^* = 0.5382$ $x_3^* = 0.47527$ $x_4^* = 0.5$	$x_1^* = 0.48167$ $x_2^* = 0.55268$ $x_3^* = 0.45373$ $x_4^* = 0.5$	$x_1^* = 0.48167$ $x_2^* = 0.55268$ $x_3^* = 0.45373$ $x_4^* = 0.5$

Table B18. Simulation results for model 8 with weight (2^4 rotatable CCD).

Approach	Taguchi	CCP w/o weight (w =1)	CCP $w = 1/\lvert y-\tau \rvert$	CCP $w = 1/(y-\tau)^2$	CCP $w = 1/(\Sigma x^2)^{1/2}$	CCP $w = 1/(\Sigma x^2)$
Mean	0.00091958	0.000753	0.0007206	0.0008918	0.0007128	0.0006953
Estimated variance	0.00000191	0.000001291	0.000001243	0.00000143	0.0000011398	0.000001122
Mean square error	0.000002086	0.000001355	0.0000012917	0.0000015835	0.000001185	0.00000116
x_i^*, s (x_5^*=25)	x_1^* = 0.75 x_2^* = 0.37381 x_3^* = 0.75 x_4^* = 0.5	x_1^* = 0.48167 x_2^* = 0.55268 x_3^* = 0.45373 x_4^* = 0.5	x_1^* = 0.52673 x_2^* = 0.50845 x_3^* = 0.51460 x_4^* = 0.5	x_1^* = 0.51749 x_2^* = 0.53630 x_3^* = 0.51368 x_4^* = 0.5	x_1^* = 0.48686 x_2^* = 0.54025 x_3^* = 0.47050 x_4^* = 0.5	x_1^* = 0.50943 x_2^* = 0.49761 x_3^* = 0.53261 x_4^* = 0.5

Table B19. Kruskal-Wallis analyses of variance.

Model / Comparing Approaches	1	2	3	4	5	6	7	8	Force problem
0 & 4 & 5	21.8348 * (0.0000)	14.1523 * (0.0008)	20.1006 * (0.0000)	2.2607 (0.3229)	19.4856 * (0.0001)	16.5239 * (0.0003)	25.8065 * (0.0000)	3.9570 (0.1383)	20.1006 * (0.0000)
0 & 4	14.2857 * (0.0002)	12.0914 * (0.0005)	14.2857 * (0.0002)	1.4629 (0.2265)	14.2857 * (0.0002)	1.8514 (0.1736)	14.2857 * (0.0002)	0.5719 (0.4495)	14.2857 * (0.0002)
0 & 5	14.2857 * (0.0002)	1.4629 (0.2265)	14.2857 * (0.0002)	1.8514 (0.1736)	14.2965 * (0.0002)	13.1657 * (0.0003)	14.2857 * (0.0002)	4.1689 * (0.0412)	14.2857 * (0.0002)
4 & 5	5.49143 * (0.0191)	7.4057 * (0.0065)	1.6514 (0.1988)	0.0229 (0.8798)	0.2802 (0.5966)	9.6057 * (0.0019)	14.2857 * (0.0002)	1.1200 (0.2899)	1.6514 (0.1988)

xx..xxxx represents the value of test statistics for comparing approaches.
* shows the significant difference at 0.5 level.
(x.xxxx) represents the p-value (smallest probability of rejecting the null hypothesis).

0 = TAGUCHI, 4 = FULL-ORTH, 5 = FULL-ROTATE

Table B20. Average variances (sample size for each approach = 10).

Model / Approach	1	2	3	4	5	6	7	8	force problem
TAGUCHI	0.53692 (0.019842)	0.901948 (0.031268)	0.174377 (0.008357)	0.757434 (0.020265)	3.28×10^{-4} (1.68×10^{-5})	3.74×10^{-3} (1.30×10^{-4})	1.52×10^{-5} (1.68×10^{-6})	1.91×10^{-6} (2.03×10^{-7})	14,390.3 (323.665)
FULL-ORTH	0.329295 (0.007844)	0.655156 (0.037769)	0.063256 (0.002144)	0.710729 (0.024974)	1.33×10^{-4} (7.19×10^{-6})	3.48×10^{-3} (1.11×10^{-4})	2.19×10^{-6} (1.89×10^{-7})	1.63×10^{-6} (2.29×10^{-7})	5,996.25 (123.437)
FULL-ROTATE	0.367876 (0.009099)	0.87632 (0.070925)	0.057406 (0.003748)	0.720795 (0.024584)	1.38×10^{-4} (4.87×10^{-6})	2.98×10^{-3} (5.37×10^{-5})	1.24×10^{-7} (1.51×10^{-8})	1.29×10^{-6} (1.44×10^{-7})	5,769.86 (109.451)

(*.******) represents the standard error of the average of the estimated variances.

Table B21. Kruskal-Wallis ANOVA for absolute bias.

Model Comparing Approaches	1	2	3	4	5	6	7	8	force problem
0 & 4 & 5	25.8065* (0.0000)	14.2684* (0.0008)	11.0168* (0.0041)	3.2284 (0.1991)	19.3574* (0.0001)	10.5953* (0.0050)	20.4929* (0.0000)	10.911* (0.0427)	19.3574* (0.0001)
0 & 4	14.2857* (0.0002)	10.08* (0.0015)	1.12 (0.2899)	0.1429 (0.7055)	14.2857* (0.0002)	10.08 (0.0015)	14.2857* (0.0002)	2.7657 (0.0963)	14.2857* (0.0002)
0 & 5	14.2857* (0.0002)	0.5714 (0.4497)	6.6057* (0.0102)	1.12 (0.2899)	14.2857* (0.0002)	5.3197* (0.0211)	2.52* (0.1124)	10.0800* (0.0015)	14.2857* (0.0002)
4 & 5	14.2857* (0.0002)	10.5657* (0.0012)	8.6914* (0.0032)	3.5714 (0.0588)	0.0057 (0.9397)	0.28 (0.5967)	14.2857* (0.0002)	3.2914 (0.0696)	0.0057 (0.9397)

xx..xxxx represents the value of test statistics for comparing approaches.
* shows the significant difference at 0.5 level.
(x.xxxx) represents the p-value (smallest probability of rejecting the null hypothesis).
0 = TAGUCHI, 4 = FULL-ORTH, 5 = FULL-ROTATE

Table B22. Average bias (sample size for each approach = 10).

Model / Approach	1 ($\tau = 2.5$)	2 ($\tau = 2.5$)	3 ($\tau = 2.5$)	4 ($\tau = 3.5$)	5 ($\tau = 0.015$)	6 ($\tau = 0.25$)	7 ($\tau = 0.0015$)	8 ($\tau = 0.0005$)	force problem ($\tau = 400$)
TAGUCHI	0.0618 (0.0206)	0.1815 (0.0179)	-0.0014 (0.0098)	0.0396 (0.0243)	0.0082 (0.00045)	0.0090 (0.00093)	0.0011 (8.2x10-5)	0.00042 (2.87x10-5)	-3.6429 (1.7578)
FULL-ORTH	-0.3673 (0.0057)	-0.0868 (0.0143)	-0.0350 (0.0068)	-0.0379 (0.0269)	0.0009 (0.00026)	-0.0008 (0.0014)	-0.0005 (2.4x10-5)	0.00034 (3.72x10-5)	-41.4501 (1.0564)
FULL-ROTATE	-0.2481 (0.0152)	-0.0319 (0.0264)	0.0038 (0.0042)	-0.0330 (0.0092)	0.0008 (0.00023)	-0.0049 (0.0011)	-0.0013 (7.63x10-6)	0.00025 (1.84x10-5)	-41.6209 (1.1430)

(*.******) represents the standard error of the average of the estimated biases.

Table B23. Kruskal-Wallis ANOVA for mean squared error.

Model / Comparing Approaches	1	2	3	4	5	6	7	8	force problem
0 & 4 & 5	16.7097* (0.0002)	14.2632* (0.0008)	20.4929* (0.0000)	2.5729 (0.2762)	19.52* (0.0001)	16.88* (0.0002)	25.0555* (0.0000)	4.9007 (0.0863)	19.6129* (0.0001)
0 & 4	8.2514* (0.0041)	12.0914* (0.0005)	14.2857* (0.0002)	1.4629 (0.2265)	14.2857* (0.0002)	2.52 (0.1124)	14.2857* (0.0002)	0.8229 (0.3643)	14.2857* (0.0002)
0 & 5	12.6229* (0.0004)	1.6514 (0.1988)	14.2857* (0.0002)	2.2857 (0.1306)	14.2857* (0.0002)	13.1657* (0.0003)	14.2857* (0.0002)	5.1429* (0.0233)	14.2857* (0.0002)
4 & 5	4.48* (0.0343)	7.4057* (0.0065)	2.52 (0.1124)	0.0514 (0.8206)	0.3657 (0.5454)	9.6057* (0.0019)	12.6229* (0.0004)	1.2857 (0.2568)	0.5714 (0.4497)

xx..xxxx represents the value of test statistics for comparing approaches.
* shows the significant difference at 0.5 level.
(x.xxxx) represents the p-value (smallest probability of rejecting the null hypothesis).
0 = TAGUCHI, 4 = FULL-ORTH, 5 = FULL-ROTATE

Table B24. Average mean squared error (sample size for each approach = 10).

Model / Approach	1	2	3	4	5	6	7	8	force problem
TAGUCHI	0.54454 (0.02049)	0.973777 (0.034317)	0.17525 (0.008358)	0.76434 (0.020212)	0.000383 (2.24×10^{-5})	0.00383 (0.000135)	1.655×10^{-5} (1.8×10^{-6})	2.096×10^{-6} (2.23×10^{-7})	14,431.4 (318.32)
FULL-ORTH	0.464524 (0.00976)	0.664512 (0.035762)	0.064898 (0.002483)	0.71867 (0.02446)	0.000135 (7.61×10^{-6})	0.003501 (0.00011)	2.444×10^{-6} (1.74×10^{-7})	1.754×10^{-6} (2.51×10^{-7})	7,724.41 (155.41)
FULL-ROTATE	0.43149 (0.010313)	0.88359 (0.07154)	0.05758 (0.003743)	0.72265 (0.0243)	0.000139 (4.98×10^{-6})	0.003018 (5.93×10^{-5})	1.358×10^{-6} (1.51×10^{-7})	1.36×10^{-6} (1.51×10^{-7})	7,513.92 (109.77)

(*.******) represents the standard error of the estimated mean square errors.

Table B25. Kruskal-Wallis test for the force problem.

Approach	Absolute Biases		Estimated Variances		Mean Squared Errors	
	Average Rank	Test of Homogenous Group	Average Rank	Test of Homogeneous Group	Average Rank	Test of Homogeneous Group
TAGUCHI (0)	5.50	x	65.50	x	65.50	x
FRAC-ORTH (1)	62.10	x	13.30	x	50.40	x
FRAC-ROTATE (2)	50.60	x	14.80	x	39.40	x
FRAC-ORRO (3)	53.80	x x	21.90	x	46.70	x x
FULL-ORTH (4)	28.10	x	47.30	x	18.70	x
FULL-ROTATE (5)	28.20	x	41.90	x	16.00	x
FULL-ORRO (6)	20.20	x	43.80	x	11.80	x

(test significant at .0.05 level)

* Approach results that aligned vertically are not statistically different.

Table B26. Averages for performance measures, force problem (τ = 400 grams).

Approach	Absolute Bias (%)	Estimated Variance	Mean Square Error
0	1.45	14,390.33	14,431.41
1	21.75	4,667.53	12,242.7
2	20.76	4,801.29	11,705.48
3	20.96	5,018.46	12,057.65
4	10.36	5,996.25	7,724.41
5	10.41	5,769.86	7,513.92
6	9.57	5,828.22	7,299.18

0	represents	TAGUCHI.
1	represents	FRAC-ORTH.
2	represents	FRAC-ROTATE.
3	represents	FRAC-ORRO.
4	represents	FULL-ORTH.
5	represents	FULL-ROTATE.
6	represents	FULL-ORRO.

APPENDIX C

Computer Programs

The programs used for this study included a simulation program (written in *BASIC*) and a nonlinear optimization program (*GAMS*). The program listings are as follows:

C1: Simulation program for the additive high-order model (models 1-4)

```
5       REM  Define variables:     Y     = response variable
                                   XI    = x1
                                   XII   = x2
                                   XIII  = x3
                                   XIV =  x4
                                   XBAR1 = the optimal values of x1
                                   XBAR2 = the optimal values of x2
                                   XBAR3 = the optimal values of x3
                                   XBAR4 = the optimal values of x4
                                   VAR1 = the variance of x1
                                   VAR 2 = the variance of x2
                                   VAR 3 = the variance of x3
                                   VAR4 = the variance of x4

10      DIM  Y(800)
20      DIM  XI(800)
30      DIM  XII(800)
35      DIM  XIII(800)
40      DIM  XIV(800)
50      PRINT  "ENTER MEAN OF X1"
60      INPUT XBAR1
70      PRINT  "ENTER VARIANCE OF X1"
80      INPUT VAR1
90      PRINT  "ENTER MEAN OF X2"
100     INPUT XBAR2
110     PRINT  "ENTER VARIANCE OF X2"
120     INPUT VAR2
130     PRINT  "ENTER MEAN OF X3"
140     INPUT XBAR3
150     PRINT  "ENTER VARIANCE OF X3"
160     INPUT VAR3
```

```
170    PRINT  "ENTER MEAN OF X4"
180    INPUT XBAR4
190    PRINT  "ENTER VARIANCE OF X4"
200    INPUT VAR4
300    WW  =  0
305    QQ  =  0
310    FOR  L = 1 TO 10
320    REM  GENERATE X1
330    FOR I = 1 TO 500
340    GOSUB 2000
350    XI(I) = XBAR1 + (VAR1^0.5)*Z
360    NEXT I
370    REM  GENERATE X2
380    FOR II = 1 TO 500
390    GOSUB 2000
400    XII(II) = XBAR2 + (VAR2^0.5)*Z
410    NEXT II
420    REM  GENERATE X3
430    FOR J = 1 TO 500
440    GOSUB 2000
450    XIII(J) = XBAR3 + (VAR3^0.5)*Z
460    NEXT J
470    REM  GENERATE X4
480    FOR JJ = 1 TO 500
490    GOSUB 2000
500    XIV(JJ) = XBAR4 + (VAR4^0.5)*Z
510    NEXT JJ
520    REM  COMPUTE RESPONSE Y
530    SUMY = 0
540    SUMSQ = 0
550    FOR K = 1 TO 500
555    REM    for example Y = 4x1^3+2x2-3x3^2+exp(x4/2)
560*   Y1  =  4*XI(K)*XI(K)*XI(K)
570*   Y2  =  2*XII(K)
580*   Y3  =  -3*XIII(K)*XIII(K)
590*   Y4  =  EXP((XVI(K)/2))
600    Y(K)  =  Y1+ Y2+ Y3+ Y4
610    SUMY = SUMY + Y(K)
620    SUMSQ = SUMSQ + Y(K)^2
630    NEXT K
640    WW = WW + SUMY
650    MEAN = SUMY/500
660    QQ  =  QQ + (SUMSQ-500*(MEAN^2))
670    VAR  =  (SUMSQ - 500*(MEAN^2))/499
680    LPRINT
690    LPRINT
700    LPRINT  "MEAN = "; MEAN; "  VARIANCE Y =  ";VAR
710    LPRINT
720    LPRINT
730    LPRINT "B1  =  ";B1; " B2 = "; B2; " B3 = "; B3; " B4 = ";B4
740    LPRINT "X1* = ";XBAR1; "  X2* = ";XBAR2; " X3* = "; XBAR3;" X4* =
";XBAR4
750    AVEMEAN = WW/(10*500)
```

```
760     AVEVAR = QQ/(499*10)
770     LPRINT "AVEMEAN  =  ";AVEMEAN;"    AVEVAR =  ";AVEVAR
780     END
2000    REM SUBROUTINE GENERATE RANDOM NUMBER
2100    NR = 0
2200    RANDOMIZE TIMER
2300    FOR LL = 1 TO 12
2400    NR = NR + RND
2500    NEXT LL
2600    Z = NR - 6
2700    RETURN
```

<u>Note</u>: * implies that the command lines must be changed in correspondence with the function of a response, Y.

C2: <u>Simulation program for the multiplicative high-order model</u> (models 5-8)

```
5       REM  Define variables:    Y     = response variable
                                  XI    = x1
                                  XII   = x2
                                  XIII  = x3
                                  XIV   = x4
                                  B1    = the degree polynomial of x1
                                  B2    =  the degree polynomial of x2
                                  B3    = the degree polynomial of x3
                                  B4    =  the degree polynomial of x4
                                  XBAR1 = the optimal values of x1
                                  XBAR2 = the optimal values of x2
                                  XBAR3 = the optimal values of x3
                                  XBAR4 = the optimal values of x4
                                  VAR1 = the variance of x1
                                  VAR2 = the variance of x2
                                  VAR3 = the variance of x3
                                  VAR4 = the variance of x4

10      DIM  Y(800)
20      DIM  XI(800)
30      DIM  XII(800)
35      DIM  XIII(800)
40      DIM  XIV(800)
50      PRINT  "ENTER B1"
60      INPUT  B1
70      PRINT  "ENTER B2"
80      INPUT  B2
90      PRINT  "ENTER B3"
100     INPUT  B3
110     PRINT  "ENTER B4"
120     INPUT  B4
130     PRINT  "ENTER MEAN OF X1"
140     INPUT XBAR1
```

```
150    PRINT  "ENTER VARIANCE OF X1"
160    INPUT VAR1
170    PRINT  "ENTER MEAN OF X2"
180    INPUT XBAR2
190    PRINT  "ENTER VARIANCE OF X2"
200    INPUT VAR2
210    PRINT  "ENTER MEAN OF X3"
220    INPUT XBAR3
230    PRINT  "ENTER VARIANCE OF X3"
240    INPUT VAR3
250    PRINT  "ENTER MEAN OF X4"
260    INPUT XBAR4
270    PRINT  "ENTER VARIANCE OF X4"
280    INPUT VAR4
290    WW  =  0
300    QQ  =  0
310    FOR  L = 1 TO 10
320    REM  GENERATE X1
330    FOR I = 1 TO 500
340    GOSUB 2000
350    XI(I) = XBAR1 + (VAR1^0.5)*Z
360    NEXT I
370    REM  GENERATE X2
380    FOR II = 1 TO 500
390    GOSUB 2000
400    XII(II) = XBAR2 + (VAR2^0.5)*Z
410    NEXT II
420    REM  GENERATE X3
430    FOR J = 1 TO 500
440    GOSUB 2000
450    XIII(J) = XBAR3 + (VAR3^0.5)*Z
460    NEXT J
470    REM  GENERATE X4
480    FOR JJ = 1 TO 500
490    GOSUB 2000
500    XIV(JJ) = XBAR4 + (VAR4^0.5)*Z
510    NEXT JJ
520    REM  COMPUTE RESPONSE Y
530    SUMY = 0
540    SUMSQ = 0
550    FOR K = 1 TO 500
555    REM  make sure x's is greater than or equal to zero before raising the
power
560    Y1  =  (XI(K)*XI(K))^0.5)^B1
570    Y2  =  (XII(K)*XII(K))^0.5)^B2
580    Y3  =  (XIII(K)*XIII(K))^0.5)^B3
590    Y4  =  ((XIV(K)*XIV(K))^0.5)^B4
600    Y(K)  =  Y1* Y2* Y3* Y4
610    SUMY = SUMY + Y(K)
620    SUMSQ = SUMSQ + Y(K)^2
630    NEXT K
640    WW = WW + SUMY
650    MEAN = SUMY/500
```

```
660     QQ  =  QQ + (SUMSQ-500*(MEAN^2))
670     VAR  =  (SUMSQ - 500*(MEAN^2))/499
680     LPRINT
690     LPRINT
700     LPRINT  "MEAN = "; MEAN; "  VARIANCE Y =  ";VAR
710     LPRINT
720     LPRINT
730     LPRINT "B1  =  ";B1; " B2 = "; B2; " B3 = "; B3; " B4 = ";B4
740     LPRINT "X1* = ";XBAR1; "  X2* = ";XBAR2; " X3* = "; XBAR3;" X4* =
";XBAR4
750     AVEMEAN = WW/(10*500)
760     AVEVAR = QQ/(499*10)
770     LPRINT "AVEMEAN  =  ";AVEMEAN;"   AVEVAR =  ";AVEVAR
780     END
2000    REM SUBROUTINE GENERATE RANDOM NUMBER
2100    NR = 0
2200    RANDOMIZE TIMER
2300    FOR LL = 1 TO 12
2400    NR = NR + RND
2500    NEXT LL
2600    Z = NR - 6
2700    RETURN
```

C3: Simulation program for the force problem

```
5       REM  Define variables:     Y     = response variable
                                   XI    = x1
                                   XII   = x2
                                   XIII  = x3
                                   XIV =  x4
                                   XV  =  x5
                                   XBAR1 = the optimal values of x1
                                   XBAR2 = the optimal values of x2
                                   XBAR3 = the optimal values of x3
                                   XBAR4 = the optimal values of x4
                                   XBAR5 = the optimal values of x5
                                          SD1 = the standard deviation of x1
                                   SD2 = the standard deviation of x2
                                   SD3 = the standard deviation of x3
                                               SD4 = the standard deviation of
x4                                                     SD5 = the standard
deviation of x5
10      DIM  Y(800)
20      DIM  XI(800)
30      DIM  XII(800)
35      DIM  XIII(800)
40      DIM  XIV(800)
45      DIM  XV(800)
50      PRINT  "ENTER MEAN OF X1"
60      INPUT XBAR1
70      PRINT  "ENTER STANDARD DEVIATION OF X1"
```

```
80      INPUT SD1
90      PRINT  "ENTER MEAN OF X2"
100     INPUT XBAR2
110     PRINT  "ENTER STANDARD DEVIATION OF X2"
120     INPUT SD2
130     PRINT  "ENTER MEAN OF X3"
140     INPUT XBAR3
150     PRINT  "ENTER STANDARD DEVIATION OF X3"
160     INPUT SD3
170     PRINT  "ENTER MEAN OF X4"
180     INPUT XBAR4
190     PRINT  "ENTER STANDARD DEVIATION OF X4"
200     INPUT SD4
210     PRINT  "ENTER MEAN OF X5"
220     INPUT XBAR5
230     PRINT  "ENTER STANDARD DEVIATION OF X5"
240     INPUT SD5
250     WW  =  0
260     QQ  =  0
270     FOR  L = 1 TO 10
280     REM  GENERATE X1
290     FOR I = 1 TO 500
300     GOSUB 2000
310     XI(I) = XBAR1 + SD1*Z
320     NEXT I
330     REM  GENERATE X2
340     FOR II = 1 TO 500
350     GOSUB 2000
360     XII(II) = XBAR2 + SD2*Z
370     NEXT II
380     REM  GENERATE X3
390     FOR J = 1 TO 500
400     GOSUB 2000
410     XIII(J) = XBAR3 + SD3*Z
420     NEXT J
430     REM  GENERATE X4
440     FOR JJ = 1 TO 500
450     GOSUB 2000
460     XIV(JJ) = XBAR4 + SD4*Z
470     NEXT JJ
480     REM  GENERATE X5
490     FOR M = 1 TO 500
500     XV(M) = 50*RND(112233!)
510     NEXT M
520     REM  COMPUTE RESPONSE Y
522     A = 140
524     C1 = 16
526     C2 = 300
530     SUMY = 0
540     SUMSQ = 0
550     FOR K = 1 TO 500
560     K1  =  (C1*XV(K)) + C2
565     K2  =  (A/(XI(K)) - 1
```

```
570    Y1  =  K1*K2
580    K3  =  (2*A - XI(K))/XI(K)
585    K4  =  XV(K) - 20
590    K5  =  XII(K) + K3*K4 - XIV(K)
595    Y2  =  K5*XIII(K)*K3
600    Y(K)  =  Y1 + Y2
610    SUMY = SUMY + Y(K)
620    SUMSQ = SUMSQ + Y(K)^2
630    NEXT K
640    WW = WW + SUMY
650    MEAN = SUMY/500
660    QQ  =  QQ + (SUMSQ-500*(MEAN^2))
670    VAR  =  (SUMSQ - 500*(MEAN^2))/499
680    LPRINT
690    LPRINT
700    LPRINT  "MEAN = "; MEAN; "  VARIANCE Y =  ";VAR
710    LPRINT
720    LPRINT
730    LPRINT "X1* = ";XBAR1; "  X2* = ";XBAR2; " X3* = "; XBAR3;" X4* =
";XBAR4
740    LPRINT "X5* = ";XBAR5
750    AVEMEAN = WW/(10*500)
760    AVEVAR = QQ/(499*10)
770    LPRINT "AVEMEAN  =  ";AVEMEAN;"   AVEVAR =  ";AVEVAR
780    END
2000   REM SUBROUTINE GENERATE RANDOM NUMBER
2100   NR = 0
2200   RANDOMIZE TIMER
2300   FOR LL = 1 TO 12
2400   NR = NR + RND
2500   NEXT LL
2600   Z = NR - 6
2700   RETURN
```

C4: Nonlinear optimizing program using *GAMS* software

The program listed below is an example of the optimizing program written in GAMS format for the force problem.

```
$  TITLE  A PROGRAM FOR OPTIMIZING THE PROBLEM VIA THE NEW METHOD
$  OFFUPPER

VARIABLES
        X1      DECISION VARIABLE
        X2      DECISION VARIABLE
        X3      DECISION VARIABLE
        X4      DECISION VARIABLE
        X5      DECISION VARIABLE
        A       RESPONSE FUNCTION
```

```
      Z       OBJECTIVE FUNCTION;

POSITIVE VARIABLE X;

EQUATIONS
        EQN1
        EQN2
        EQN3
        EQN4
        EQN5
        EQN6
        EQN7
        EQN8
        EQN9
        EQN10
        RESPONSE
        OBJFTN;

X1.L  = 154;
X2.L  = 60.5;
X3.L  = 8.0;
X4.L  = 28.0;

RESPONSE..  A =E= B0+B1*X1+ B2*X2 +B3*X3+B4*X4+B5*X5
                  + B11*X1*X1+B22*X2*X2+B33*X3*X3+B44*X4*X4+B55*X5*X5
                  + B12*X1*X2+B13*X1*X3+B14*X1*X4+B15*X1*X5
                  + B23*X2*X3+B24*X2*X4+B25*X2*X5
                  + B34*X3*X4+B35*X3*X5+B45*X4*X5;

EQN1.. A =L= 400.01;
EQN2.. A =G= 399.99;
EQN3.. X1 =L= 180;
EQN4.. X1 =G= 100;
EQN5.. X2 =L= 75;
EQN6.. X2 =G= 35;
EQN7.. X3 =L= 15;
EQN8.. X3 =G= 5;
EQN9.. X4 =L= 50;
EQN10.. X4 =G= 20;

OBJTFN..  Z =E=  V1*(B1+2*B11*X1+B12*X2+B13*X3+B14*X4+B15*X5)
                   *(B1+2*B11*X1+B12*X2+B13*X3+B14*X4+B15*X5)
              +V2*(B2+2*B22*X2+B12*X1+B23*X3+B24*X4+B25*X5)
                   *(B2+2*B22*X2+B12*X1+B23*X3+B24*X4+B25*X5)
              +V3*(B3+2*B33*X3+B13*X1+B23*X2+B34*X4+B35*X5)
                   *(B3+2*B33*X3+B13*X1+B23*X2+B34*X4+B35*X5)
              +V4*(B4+2*B44*X4+B14*X1+B24*X2+B34*X3+B45*X5)
                   *(B4+2*B44*X4+B14*X1+B24*X2+B34*X3+B45*X5)
              +V5*(B5+2*B55*X5+B15*X1+B25*X2+B35*X3+B45*X4)
                   *(B5+2*B55*X5+B15*X1+B25*X2+B35*X3+B45*X4);

MODEL  QP /ALL/;
SOLVE  QP USING NLP MINIMIZING Z;
```

```
OPTION DECIMALS = 6;
DISPLAY Z.L, A.L;
OPTION DECIMALS = 5;
DISPLAY X1.L, X2.L, X3.L, X4.L;
```

__Note__: Since **X5** is a noise variable, the value of **X5** is substituted by its mean prior to optimization. Therefore, **X5** in the program must be placed by 25 for the force problem. Moreover, for B0, B1, B2,...,B5, B11, B22, ...,B55, B12, B13,..., and B45, values must be replaced by the coefficient terms of the fitted quadratic mean response function. For example, B0 must be placed by the constant term of the estimated quadratic response function, B1 must be placed by the coefficient of x_1, B11 must be placed by the coefficient of x_1^2, and B12 must be placed by the coefficient of x_1 and x_2, and so on.

Also note that A represents the fitted quadratic (mean) response function, and `OBJFTN` represents the estimated variance function.

The optimization programs for models 1-8 are obtained by modifying the program above for the four variables problem, treating x_4 in each model as the noise variable.