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Title THE DIFFRACTION OF CYLINDRICAL WAVES BY TWO PARALLEL HALF PLANES

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Abstract approved ___________
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The electromagnetic field produced by a line current oriented parallel to the edges of two perfectly conducting parallel half planes is considered. Maxwell's equations reduce to a single wave equation involving only one component of the electric field. Moreover the value of the field is zero on the two half planes. The problem is reduced to the determination of the current distributions on the two half planes. This leads to two integral equations of the Wiener-Hopf type.

For the solution of these integral equations standard techniques are used.

THE DIFFRACTION OF CYLINDRICAL WAVES BY
TWO PARALLEL HALF PLANES

by

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THE DIFFRACTION OF CYLINDRICAL WAVES BY TWO PARALLEL HALF PLANES

1. INTRODUCTION

Maxwell's equations for free space are

$$\left.\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{H} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\mu_0 \frac{\partial \vec{H}}{\partial t} \\ \vec{\nabla} \times \vec{H} &= \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}\right\} \qquad \text{------I}$$

When the sources are line currents parallel to z-axis, the electric and magnetic fields are such that $E_x = E_y = H_z = 0$. The equations (I) then reduce to

$$\nabla^2 U = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} \qquad \text{------II}$$

where

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}, \quad U(x, y, t) = E_z (x, y, t)$$

With the time dependency $c^{i\omega t}$, (II) reduces to (IIa)

$$\nabla^2 u + k^2 u = 0 \qquad \text{------IIa}$$

where

$$k = \frac{\omega}{c}$$

The boundary conditions are

$$\frac{\partial u}{\partial n} = 0$$

or

$$u = 0 .$$

$\frac{\partial u}{\partial n} = 0$ on the boundary represents acoustic excitations in the presence of a rigid surface whereas $u = 0$ on the boundary represents electromagnetic excitations in the presence of a perfectly conducting surface.

In the present investigation we assume the solution of (IIa) to be of the form $u(x, y) = U_i + U_s$.

$$U_i = \frac{i}{4} H_0^{(2)}\left(k \sqrt{(x-x')^2 + (y-y')^2} \right)$$

where U_i is the field incident at the point $A(x, y)$ due to a line source of unit strength at $B(x', y')$ and U_s is the secondary field which is regular in the entire region under consideration.

The results of plane wave diffraction by a perfectly reflecting half plane were published in the late nineteenth century using multivalent solutions of the wave equation. These were followed by those of line source (cylindrical excitations) diffraction by the half plane. The solution to the corresponding wedge problem (for both the plane wave and the cylindrical source) was published by Carslaw (6). For sometime the interest in diffraction problems slackened due to the extraordinary difficulties presented by this method in its application to new problems.

The publication of (33) in 1931 revived the interest of research workers in this field as can be seen from the number of papers published since then. Magnus (17) expressed the solution of the half plane problem with incident plane waves in terms of an integral equation of the Wiener-Hopf type. He then solved the integral equation by computing coefficients in terms of Bessel functions. Copson (7) solved the same integral equation by using the Weiner-Hopf technique. The solutions for complicated gratings (2, 3, 4, 5, 10, 11, 14) demonstrates the power of this technique in solving diffraction problems. A typical Wiener-Hopf homogeneous equation is

(III)------ $$\int_{-\infty}^{\infty} f(\lambda)\ell(x-\lambda)d\lambda + g(x) = h(x) \qquad -\infty < x < \infty$$

where $f(\lambda)$ and $h(x)$ are unknown functions. It is, however, known that $f(\lambda) = 0, \lambda < 0$ and $h(x) = 0, \ x > 0$. Multiplying both sides by $e^{-i\alpha x}$ and integrating with respect to x, one obtains

(IV)------ $$\bar{f}(\alpha)\,\bar{\ell}(\alpha) + \bar{g}(\alpha) = \bar{h}(\alpha)$$

where $\bar{\ell}(\alpha) = \int_{-\infty}^{\infty} \ell(x)e^{-i\alpha x}dx$ and $\bar{f}(\alpha)$, $\bar{g}(\alpha)$, $\bar{h}(\alpha)$ are defined in a similar manner.

The equation (IV) holds in a common strip of regularity of the functions $\bar{g}(\alpha)$, $\bar{h}(\alpha)$, $\bar{\ell}(\alpha)$, $\bar{f}(\alpha)$ considered as functions of α (provided such a strip exists). To determine the regions of regularity of these functions one assumes the behavior of the functions $f(X)$, $h(x)$, $g(x)$, $\ell(x)$ for large x. For example, let $f(\lambda) \approx e^{ik\lambda}$ for large λ.

$$k = k_1 + ik_2, \quad \text{where } k_2 > 0.$$

Then
$$\bar{f}(\alpha) = \int_0^{\infty} f(\lambda)\, e^{-i\alpha\lambda}\, d\lambda \; .$$

We note that $\bar{f}(\alpha)$ will be regular in the lower half plane. Im $\alpha < k_2$.

Similarly if

$$h(x) \approx e^{-ikx} \quad \text{as } x \to -\infty,$$

$$\bar{h}(\alpha) = \int_{-\infty}^{0} h(x)\, e^{-i\alpha x}\, dx,$$

then $\bar{h}(\alpha)$ is regular in the upper half plane Im $\alpha > -k_2$.

These half planes have a common strip $|\text{Im } \alpha| < k_2$. If the functions $\bar{\ell}(\alpha)$, $\bar{g}(\alpha)$ are also regular in this strip, the equation (IV) then has a strip of validity and can be rewritten as

$$\text{(V)} \qquad \bar{h}_+(\alpha) = \bar{f}_-(\alpha)\, \bar{\ell}(\alpha) + \bar{g}(\alpha).$$

The subscripts +, -, denote regularity in the upper and lower half planes respectively.

$\bar{\ell}(\alpha)$ is now expressed in the form $\dfrac{\ell_-(\alpha)}{\ell_+(\alpha)}$. Then (IV) becomes

(VI) $\quad \bar{h}_+(\alpha)\ \ell_+(\alpha) = \bar{g}(\alpha)\ \ell_+(\alpha) + \bar{f}_-(\alpha)\ \ell_-(\alpha).$

$P(\alpha) = \bar{g}(\alpha)\,\ell_+(\alpha)$ is now expressed in the form $P_+(\alpha) + P_-(\alpha)$. Using this we get

(VII) $\quad \bar{h}_+(\alpha)\ \ell_+(\alpha) - P_+(\alpha) = P_-(\alpha) + \bar{f}_-(\alpha)\ \ell_-(\alpha) = F(\alpha).$

The left hand side of (VII) is a function regular in an upper half plane whereas the right hand side is regular in a lower half plane. These half planes overlap. By analytic continuation they define a function $F(\alpha)$ regular in the entire α-plane. The growths of $\ell_+(\alpha)$, $\ell_-(\alpha)$, $P_+(\alpha)$, $P_-(\alpha)$, $\bar{f}_-(\alpha)$, $\bar{h}_+(\alpha)$ are studied in the proper half planes. As it turns out, the analytic function $F(\alpha)$ is a constant in the problem under investigation.

Thus $\bar{h}_+(\alpha)$ and $\bar{f}_-(\alpha)$ are known in terms of the already determined functions. Then

$$f(\lambda) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \bar{f}_-(\alpha)\, e^{i\alpha\lambda}\, d\alpha, & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}$$

$$h(x) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \bar{h}_+(\alpha)\, e^{i\alpha x}\, d\alpha, & x < 0 \\ 0 & x > 0 \end{cases}$$

In the problem of plane wave diffraction by two parallel half planes one gets a single integral equation of the type III provided that the direction of propogation is perpendicular to the edges of the half planes and parallel to the planes. For the cylindrical source placed symmetrically with respect to the two half planes, one again gets a single integral equation. However, when the line source is not symmetric with respect to the two half planes (the line source is still parallel to the edges of the halfplanes) we get two integral equations of the type III. These involve four unknown functions whose behavior at large distances is assumed to be known. The technique used on (III) has to be applied to both the integral equations separately. Chapter 2 is devoted to the problem of the two parallel planes with a line source between them. In Chapter 3 we investigate the diffraction of a line source excitation by two parallel half planes. Chapter 4 contains the proofs of asymptotic estimates used. Finally in Chapter 5 we obtain the results of two parallel planes problem from the results of Chapter 3.

2. FIELD OF A LINE SOURCE PLACED BETWEEN TWO INFINITE, PARALLEL PLANES

This chapter is primarily devoted to the derivation of the induced current distributions in the two conducting planes at large distances from the source. This forms a basis of the assumptions made about the induced currents in the next chapter. If the source of incident field is oriented in such a way that $E_x = E_y = H_z = 0$, the secondary currents in the two conducting planes flow parallel to the z-axis. These secondary currents act as sources of new radiation and re-radiate the energy incident upon the two planes to create a secondary field. Thus the secondary field can be constructed in terms of the distributions of the induced currents in the two planes.

Consider a cylindrical source (of unit strength) which is parallel to the z-axis, passes through the point $Q(x', y')$ and is radiating on two infinite parallel and perfectly conducting planes of zero thickness,

$$y = b \qquad -\infty < x < \infty$$

$$y = -b \qquad -\infty < x < \infty \qquad \text{(Figure 1)}$$

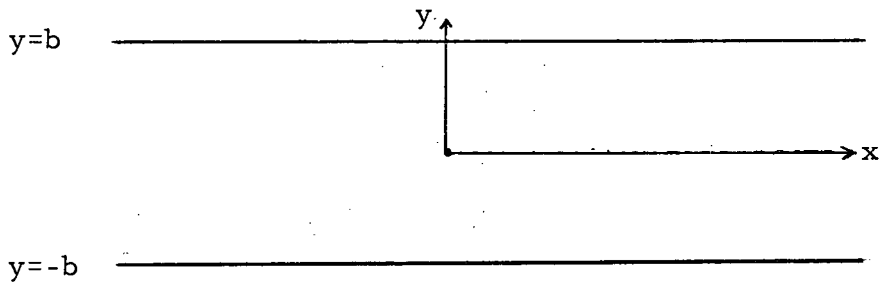

Figure 1

We may assume, without any loss of generality, that $x' = 0$. We further assume that $y' = 0$. There are two distinct cases for the possible values of y'.

(i) $|y'| > b$.

(ii) $|y'| < b$.

In (i) the source lies on the same side of both the planes and the one nearer to the source acts like a shield. This reduces to the problem of a line source radiating on an infinite conducting plane. It has already been worked out by Amos (1). In (ii) the source lies between the two planes. The assumption $y' = 0$ is justified due to the fact that the main purpose of this chapter is to obtain an estimate the strength of the induced currents for $x \gg 1$. The source is symmetrically located with respect to the planes and hence the strengths of the induced currents at the same distance in the x-direction are equal. Let $I(\lambda)$ be the strength of the induced currents at a distance λ in the x-direction. The total field $U(x, y)$ at the point $P(x, y)$ can be written as

$$U(x, y) = U_i + U_s,$$

where

$$U_i = \frac{i}{4} H_o^{(2)}\left(k \sqrt{x^2 + y^2}\right)$$

and

$$U_s = \frac{i}{4}\int_{-\infty}^{\infty} I(\lambda) \left[H_o^{(2)}\left(k \sqrt{(x-\lambda)^2 + (y-b)^2}\right)\right.$$

$$\left. + H_o^{(2)}\left(k \sqrt{(x-\lambda)^2 + (y+b)^2}\right)\right] d\lambda .$$

Clearly $U(x,y) = U(x, -y)$.

$$\text{(2.1)} \qquad U(x, y) = \frac{i}{4} H_o^{(2)}(k \sqrt{x^2 + y^2})$$

$$+ \frac{i}{4}\int_{-\infty}^{\infty} I(\lambda) \left[H_o^{(2)} (k \sqrt{(x-\lambda)^2 + (y-b)^2})\right.$$

$$\left. + H_o^{(2)}(k \sqrt{(x-\lambda)^2 + (y+b)^2})\right] d\lambda .$$

Inserting the boundary condition $U(x, \pm b) = 0$ in (2.1), we get

$$\text{(2.2)} \qquad \int_{-\infty}^{\infty} I(\lambda) \left[H_o^{(2)}(k \sqrt{(x-\lambda)^2 + (2b)^2}) + H_o^{(2)} (k|x-\lambda|)\right] d\lambda$$

$$= - H_o^{(2)}(k \sqrt{x^2 + b^2})$$

$$-\infty < x < \infty .$$

Integral equation (2.2) can be solved easily by the Fourier transform methods. Multiplying both sides of (2.2) by $\exp(-i\alpha x)$ and integrating with respect to x from $-\infty$ to ∞ we have (24a, pp. 77, 80)

$$(2.3)\quad \frac{2}{\sqrt{k^2-a^2}}\left[1+\exp\left(-2ib\sqrt{k^2-a^2}\right)\right]\bar{I}(a)$$
$$= \frac{-2}{\sqrt{k^2-a^2}}\exp\left(-ib\sqrt{k^2-a^2}\right)$$

provided that the exchange of the order of integration is permitted.

$$\bar{I}(a) = \int_{-\infty}^{\infty} I(\lambda)\exp(-ia\lambda)\,d\lambda$$

From (2.3)

$$(2.4)\qquad \bar{I}(a) = -\frac{\exp\left(-ib\sqrt{k^2-a^2}\right)}{1+\exp\left(-2ib\sqrt{k^2-a^2}\right)} = -\frac{1}{2}\operatorname{sech}\left(b\sqrt{a^2-k^2}\right)$$

The arguments of $(k+a)$ and $(k-a)$ in the a-plane are subject to the following restrictions.

$$(2.5)\qquad \begin{aligned} -2\pi &\leqslant \arg(k-a) < 0 \\ -\pi &\leqslant \arg(k+a) < \pi \end{aligned}$$

We further assume that $bk \leqslant \pi/2 - \mu$, $\mu > 0$.

Taking the inversion of $\bar{I}(a)$, we get

$$I(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{I}(a)\exp(ia\lambda)\,d\lambda$$
$$= -\frac{1}{2\pi}\left(\frac{\pi}{b}\right)^2\sum_{n=0}^{\infty}\frac{(-1)^n\,(n+\frac{1}{2})}{\sqrt{\frac{(n+\frac{1}{2})^2\pi^2}{b^2}-k^2}}\exp\left[-\lambda\left\{\frac{(n+\frac{1}{2})^2\pi^2}{b^2}-k^2\right\}^{\frac{1}{2}}\right].$$

Thus $\quad I(\lambda) \approx e^{ik\lambda} \quad$ as $\quad \lambda \to \infty$.

3. DIFFRACTION BY TWO PARALLEL HALF PLANES

In its formulation, the problem is similar to that of Chapter 2. However, in the present case, the cylindrical source is not located symmetrically with respect to the two half planes. This leads to different strenghts of induced currents $I_0(\lambda)$, $I_1(\lambda)$ in these two conducting half planes. Furthermore, in Chapter 2, $I(\lambda)$ was determined for all λ with $U(x, \pm b)$ known for all x. In the present case, $I_0(\lambda) = I_1(\lambda) = 0$ for $\lambda < 0$ and $U(x, b) = U(x, -b) = 0$ for $x > 0$. The problem is to determine $I_0(\lambda)$, $I_1(\lambda)$ for $\lambda > 0$ and $U(x, b)$, $U(x, -b)$ for $x < 0$.

Consider a cylindrical source (of unit strength), which is parallel to the z-axis and passes through $Q(x', y')$, radiating on two parallel half planes of perfectly conducting material and zero thickness.

$$\begin{aligned} y &= b \qquad 0 < x < \infty \\ y &= -b \qquad 0 < x < \infty. \end{aligned} \qquad \text{(Figure 2)}$$

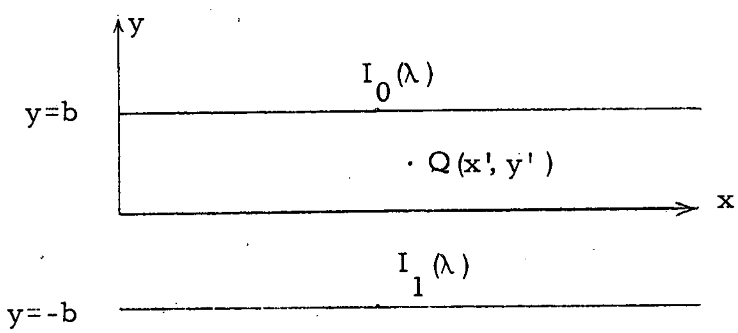

Figure 2

$$(3.1)\qquad U_i = \frac{i}{4} H_0^{(2)}\left(k\sqrt{(x-x')^2+(y-y')^2}\right)$$

$$(3.2)\qquad U_s = \frac{i}{4}\int_0^\infty I_0(\lambda) H_0^{(2)}\left(k\sqrt{(x-\lambda)^2+(y-b)^2}\right)d\lambda + \frac{i}{4}\int_0^\infty I_1(\lambda) H_0^{(2)}\left(k\sqrt{(x-\lambda)^2+(y+b)^2}\right)d\lambda$$

$$(3.3)\qquad U(x,y) = \frac{i}{4} H_0^{(2)}\left(k\sqrt{(x-x')^2+(y-y')^2}\right) + \frac{i}{4}\int_0^\infty \Big[I_0(\lambda) H_0^{(2)}\left(k\sqrt{(x-\lambda)^2+(y-b)^2}\right) + I_1(\lambda) H_0^{(2)}\left(k\sqrt{(x-\lambda)^2+(y+b)^2}\right)\Big] d\lambda$$

Inserting the boundary conditions $U(x, \pm b) = 0$ for $x > 0$ in (3.3), we have

$$(3.4)\qquad \frac{i}{4}\int_0^\infty \Big[I_0(\lambda) H_0^{(2)}(k|x-\lambda|) + I_1(\lambda) H_0^{(2)}\left(k\sqrt{(x-\lambda)^2+(2b)^2}\right)\Big] d\lambda + \frac{i}{4} H_0^{(2)}\left(k\sqrt{(x-x')^2+(b-y')^2}\right) = 0$$

$$0 < x < \infty$$

$$(3.5)\qquad \frac{i}{4}\int_0^\infty \Big[I_0(\lambda) H_0^{(2)}\left(k\sqrt{(x-\lambda)^2+(2b)^2}\right) + I_1(\lambda) H_0^{(2)}(k|x-\lambda|)\Big] d\lambda + \frac{i}{4} H_0^{(2)}\left(k\sqrt{(x-x')^2+(b+y')^2}\right) = 0$$

$$0 < x < \infty.$$

Furthermore, we note that the left hand side of (3.4) should give the total field $U(x, b)$ for $x < 0$ and the left hand side of (3.5) the total field $U(x, -b)$ for $x < 0$.

Equations (3.4) and (3.5) can, therefore, be written in the form

$$(3.6) \qquad \int_0^\infty \left[I_0(\lambda) H_0^{(2)}(k|x-\lambda|) + I_1(\lambda) H_0^{(2)}\left(k\sqrt{(x-\lambda)^2 + (2b)^2}\right)\right] d\lambda$$

$$+ H_0^{(2)}\left(k\sqrt{(x-x')^2 + (b-y')^2}\right) = \begin{cases} 0 & x > 0 \\ \frac{4}{i}\, U(x,b) & x < 0 \end{cases}$$

$$(3.7) \qquad \int_0^\infty \left[I_0(\lambda) H_0^{(2)}\left(k\sqrt{(x-\lambda)^2 + (2b)^2}\right) + I_1(\lambda) H_0^{(2)}(k|x-\lambda|)\right] d\lambda$$

$$+ H_0^{(2)}\left(k\sqrt{(x-x')^2 + (b+y')^2}\right) = \begin{cases} 0 & x > 0 \\ \frac{4}{i}\, U(x,-b) & x < 0 \end{cases}$$

Adding and subtracting (3.7) from (3.6), we have

$$(3.8) \qquad \int_0^\infty p_0(\lambda) \left[H_0^{(2)}(k|x-\lambda|) + H_0^{(2)}\left(k\sqrt{(x-\lambda)^2 + (2b)^2}\right)\right] d\lambda$$

$$+ H_0^{(2)}\left(k\sqrt{(x-x')^2 + (b-y')^2}\right)$$

$$+ H_0^{(2)}\left(k\sqrt{(x-x')^2 + (b+y')^2}\right)$$

$$= \begin{cases} 0 & x > 0 \\ \frac{4}{i}\left[U(x,b) + U(x,-b)\right] & x < 0 \end{cases}$$

$$(3.9)\qquad \int_0^\infty p_1(\lambda)\left[H_0^{(2)}(k|x-\lambda|) - H_0^{(2)}\left(k\sqrt{(x-\lambda)^2+(2b)^2}\right)\right]d\lambda$$

$$+\ H_0^{(2)}\left(k\sqrt{(x-x')^2+(b-y')^2}\right)$$

$$-H_0^{(2)}\left(k\sqrt{(x-x')^2+(b+y')^2}\right)$$

$$= \begin{cases} 0 & x > 0 \\ \frac{4}{i}\left[U(x,b)-U(x,-b)\right] & x < 0 \end{cases}$$

where

$$p_0(\lambda) = I_0(\lambda) + I_1(\lambda)$$

$$p_1(\lambda) = I_0(\lambda) - I_1(\lambda)$$

The equations (3.8) and (3.9) can be expressed as integral equations of the Wiener-Hopf type

$$(3.10)\qquad \int_{-\infty}^{\infty} f(\lambda)\,\ell(x-\lambda)\,d\lambda = g(x) + h(x) \qquad -\infty < x < \infty$$

by the following relations

$$f_0(\lambda) = \begin{cases} p_0(\lambda) & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}$$

$$f_1(\lambda) = \begin{cases} p_1(\lambda) & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}$$

$$\ell_0(x-\lambda) = H_0^{(2)}(k|x-\lambda|) + H_0^{(2)}(k\sqrt{(x-\lambda)^2 + (2b)^2}) \qquad -\infty < x < \infty$$

$$\ell_1(x-\lambda) = H_0^{(2)}(k|x-\lambda|) - H_0^{(2)}(k\sqrt{(x-\lambda)^2 + (2b)^2}) \qquad -\infty < x < \infty$$

$$g_0(x) = -\Big[H_0^{(2)}(k\sqrt{(x-x')^2 + (b-y')^2}) + H_0^{(2)}(k\sqrt{(x-x')^2 + (b+y')^2})\Big] \qquad -\infty < x < \infty$$

$$g_1(x) = -\Big[H_0^{(2)}(k\sqrt{(x-x')^2 + (b-y')^2}) - H_0^{(2)}(k\sqrt{(x-x')^2 + (b+y')^2})\Big] \qquad -\infty < x < \infty$$

$$h_0(x) = \begin{cases} 0 & x > 0 \\ \frac{4}{i}[U(x,b) + U(x,-b)] & x < 0 \end{cases}$$

$$h_1(x) = \begin{cases} 0 & x > 0 \\ \frac{4}{i}[U(x,b) - U(x,-b)] & x < 0 \end{cases}$$

Thus obtaining

$$\text{(3.11)} \qquad \int_{-\infty}^{\infty} f_0(\lambda)\,\ell_0(x-\lambda)\,d\lambda = g_0(x) + h_0(x) \qquad -\infty < x < \infty$$

$$\text{(3.12)} \qquad \int_{-\infty}^{\infty} f_1(\lambda)\,\ell_1(x-\lambda)\,d\lambda = g_1(x) + h_1(x) \qquad -\infty < x < \infty$$

Multiplying both sides of (3.11) and (3.12) by $\exp(-i\alpha x)$ and integrating from $-\infty$ to ∞ we get (24a, pp. 77, 80)

$$(3.13)\qquad \bar{f}_0(\alpha)\frac{2}{\sqrt{k^2-\alpha^2}}\left[1+\exp(-2ib\sqrt{k^2-\alpha^2})\right]$$

$$= -\frac{2\exp(-i\alpha x')}{\sqrt{k^2-\alpha^2}}\left[\exp(-i|b-y'|\sqrt{k^2-\alpha^2})\right.$$

$$\left.+\exp(-i|b+y'|\sqrt{k^2-\alpha^2})\right] + \bar{h}_0(\alpha)$$

$$(3.14)\qquad \frac{2\bar{f}_1(\alpha)}{\sqrt{k^2-\alpha^2}}\left[1-\exp(-2ib\sqrt{k^2-\alpha^2})\right]$$

$$= \frac{-2\exp(-i\alpha x')}{\sqrt{k^2-\alpha^2}}\left[\exp(-i|b-y'|\sqrt{k^2-\alpha^2})\right.$$

$$\left.-\exp(-i|b+y'|\sqrt{k^2-\alpha^2})\right] + \bar{h}_1(\alpha)$$

where

$$\bar{f}_{\substack{0\\1}}(\alpha) = \int_0^\infty p_{\substack{0\\1}}(\lambda)\exp(-i\alpha\lambda)\,d\lambda$$

and

$$\bar{h}_{\substack{0\\1}}(\alpha) = \frac{4}{i}\int_{-\infty}^{0}[U(x,b)\pm(x,-b)]\exp(-i\alpha x)\,dx$$

At this point certain assumptions about the forms of $p_0(\lambda)$, $p_1(\lambda)$, $U(x,b)$ and $U(x,-b)$ are necessary before we can proceed further. First, we require that $I_0(\lambda)$, $I_1(\lambda)$ be absolutely integrable over any finite length. Furthermore, we assume that the behavior for large distances from the source be like that of $I(\lambda)$ in Chapter 2. i. e.

$$\left| I_{\substack{0\\1}}(\lambda) \right| \leqslant C_1 \exp(ik\lambda) \text{ as } \lambda \to \infty$$ and that k has a small positive imaginary part.

Also

$$\left| h_{\substack{0\\1}}(x) \right| \leqslant C_2 \exp(-ikx) \quad \text{as} \quad x \to -\infty$$

Thus

$$|\bar{f}_0(\alpha)| \leqslant \int_0^G |p_0(\lambda)| \, d\lambda + C_1 \int_G^\infty \exp(-i\alpha\lambda + ik\lambda) \, d\lambda$$

We notice that $\bar{f}_0(\alpha)$ represents an analytic function which is regular in the lower half plane $\text{Im } \alpha < \text{Im } k$. We also note that $\bar{f}_0(\alpha)$ is bounded in the proper half plane $\text{Im } \alpha \leqslant \text{Im } k - \epsilon$, $\text{Im } k > \epsilon > 0$. Similarly $\bar{f}_1(\alpha)$ represents an analytic function, regular and bounded in $\text{Im } \alpha \leqslant \text{Im } k - \epsilon$.

A similar analysis for $\bar{h}_0(\alpha)$, $\bar{h}_1(\alpha)$ shows that these represent analytic functions, regular and bounded in the upper half plane $\text{Im } \alpha \geqslant -\text{Im } k + \epsilon$. Let us re-write (3.13) and (3.14) as

$$\text{(3.15)} \qquad \bar{H}_0(\alpha) = \frac{i \exp(-i\alpha x')}{2\sqrt{k^2-\alpha^2}} \left[\exp\left(-i|b-y'|\sqrt{k^2-\alpha^2}\right) + \exp\left(-i|b+y'|\sqrt{k^2-\alpha^2}\right) \right] + \frac{i}{2\sqrt{k^2-\alpha^2}} \left[1 + \exp\left(-2ib\sqrt{k^2-\alpha^2}\right) \right] \bar{f}_0(\alpha)$$

$$\bar{H}_1(\alpha) = \frac{i\exp(-i\alpha x')}{2\sqrt{k^2-\alpha^2}}\Big[\exp(-i|b-y'|\sqrt{k^2-\alpha^2}) - \exp(-i|b+y'|\sqrt{k^2-\alpha^2})\Big] + \frac{i}{2\sqrt{k^2-\alpha^2}}\Big[1-\exp(-2ib\sqrt{k^2-\alpha^2})\Big]\bar{f}_1(\alpha) \tag{3.16}$$

where

$$\bar{H}_0(\alpha) = \frac{i}{4}\bar{h}_0(\alpha)$$

$$\bar{H}_1(\alpha) = \frac{i}{4}\bar{h}_1(\alpha)$$

$\bar{\ell}_0(\alpha)$, $\bar{\ell}_1(\alpha)$, $\bar{g}_0(\alpha)$ and $\bar{g}_1(\alpha)$ in the integral forms represent functions regular in the strip $|\text{Im }\alpha| < \text{Im } k$. However, the closed forms give analytic continuations into the whole α-plane when it is cut from k to ∞ and $-k$ to $-\infty$ along lines parallel to the real axis. (Figure 3).

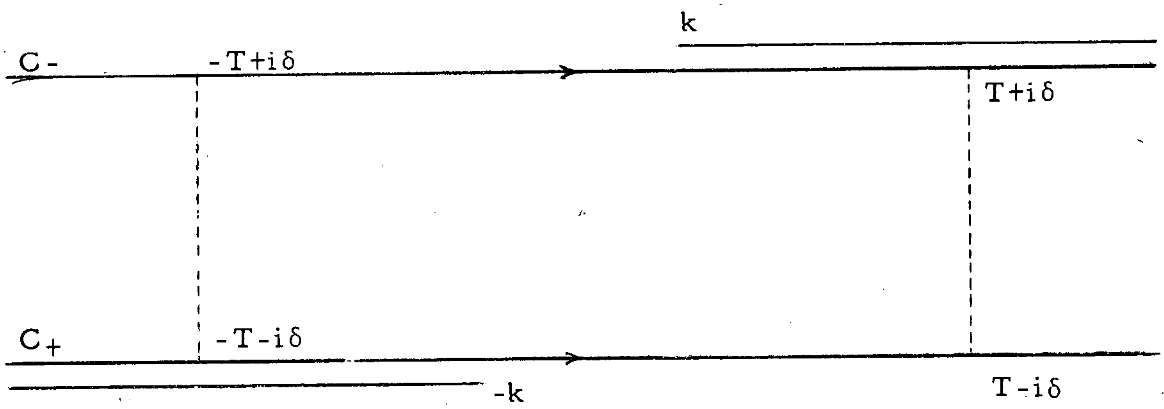

Figure 3

That branch of $\sqrt{k^2 - \alpha^2}$ is considered in which

$$-2\pi \leqslant \arg (k-\alpha) < 0$$

$$-\pi \leqslant \arg (k+\alpha) < \pi$$

Using this information, we find that (3.15) and (3.16) are valid only in the strip $|\mathrm{Im}\ \alpha| < \mathrm{Im}\ k$ which is the overlapping region of regularity of $\bar{f}_{\substack{0\\1}}(\alpha)$ and $\bar{h}_{\substack{0\\1}}(\alpha)$.

We decompose $L(\alpha) = \dfrac{i\left[1 - \exp\left(-2ib\sqrt{k^2-\alpha^2}\right)\right]}{2\sqrt{k^2-\alpha^2}}$

as

$$L(\alpha) = \frac{L_-(\alpha)}{L_+(\alpha)}^{*}$$

where $L_-(\alpha)$ and $L_+(\alpha)$ are regular in the lower and upper, overlapping half planes. If we apply Cauchy's theorem to $\log L(\alpha)$ on a rectangular contour of Figure 3 and take the limit as $T \to \infty$, we obtain

$$\log L(\alpha) = \frac{1}{2\pi i}\int_{C_+} \frac{\log L(t)}{t-\alpha}\,dt - \frac{1}{2\pi i}\int_{C_-} \frac{\log L(t)}{t-\alpha}\,dt$$

since the contributions due to the vertical sides of the rectangular contour vanish as $T \to \infty$

$$\text{(3.17)} \qquad \log L_-(\alpha) - \log L_+(\alpha) = \frac{1}{2\pi i}\int_{C_+} \frac{\log L(t)}{t-\alpha}\,dt - \frac{1}{2\pi i}\int_{C_-} \frac{\log L(t)}{t-\alpha}\,dt$$

---

* The signs - or + attached to the functions will denote its half plane of regularity as defined here.

$$\text{(3.18)}\qquad \log L_+(a) = -\frac{1}{2\pi i}\int_{C_+}\frac{\log L(t)}{t-a}\,dt$$

$$= \lim_{T\to\infty} -\frac{1}{2\pi i}\int_{-T-i\delta}^{T-i\delta}\frac{\log L(t)}{t-a}\,dt$$

$$\text{(3.19)}\qquad \log L_-(a) = -\frac{1}{2\pi i}\int_{C_-}\frac{\log L(t)}{t-a}\,dt$$

$$= \lim_{T\to\infty} -\frac{1}{2\pi i}\int_{-T+i\delta}^{T+i\delta}\frac{\log L(t)}{t-a}\,dt$$

where that branch of the logrithm function is taken on which $\log 1 = 0$. In fact both $\log L_+(a)$ and $\log L_-(a)$ are given by the same integral when $\delta \to 0$. Moreover,

$$L_+(-a) = \exp\left\{-\frac{1}{2\pi i}\lim_{T\to\infty}\int_{-T-i\delta}^{T-i\delta}\frac{\log L(t)}{t+a}\,dt\right\}$$

put $-t = z$

$$L_+(-a) = \exp\left\{\frac{1}{2\pi i}\lim_{T\to\infty}\int_{T+i\delta}^{-T+i\delta}\frac{\log L(-z)}{-z+a}\,dz\right\}$$

$$= \exp\left\{\frac{1}{2\pi i}\lim_{T\to\infty}\int_{-T+i\delta}^{T+i\delta}\frac{\log L(z)}{z-a}\,dz\right\}$$

$$\text{(3.20)}\qquad L_+(-a) = \frac{1}{L_-(a)}$$

Since $\log L(-z) = \log L(z)$.

To evaluate $L_+(\alpha)$ in a closed form, we replace C_+ by the contour shown in Figure 4. The contour must be in the lower half plane since α in the function $L_+(\alpha)$ lies in the upper half plane.

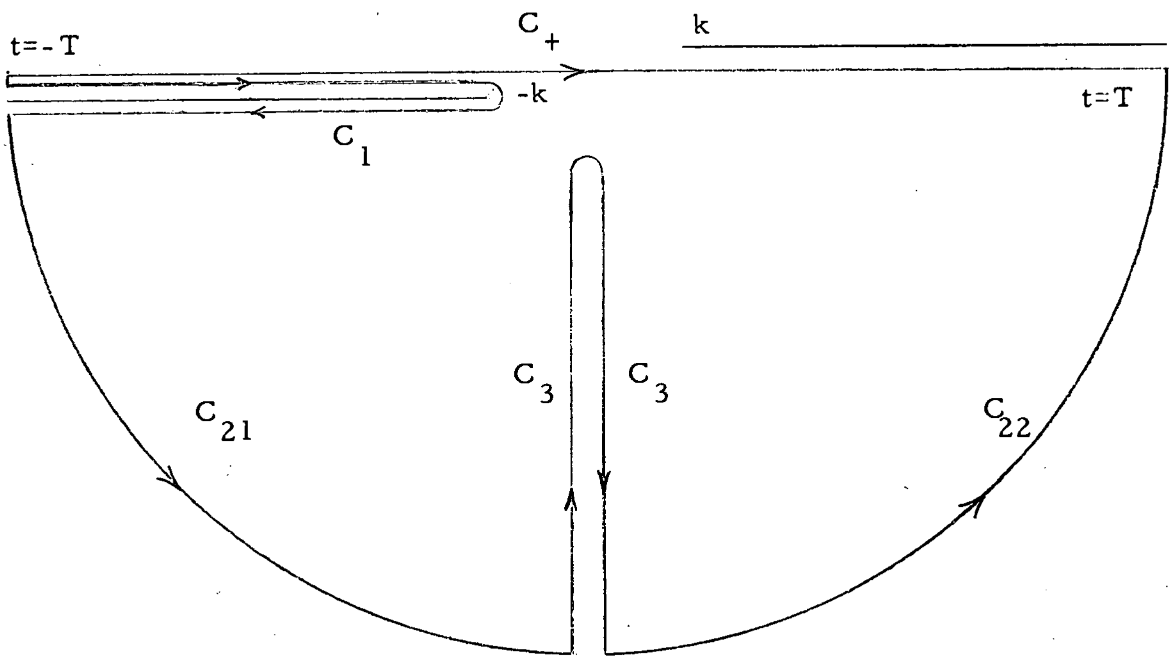

Figure 4

Applying Cauchy's theorem we have

$$\log L_+(\alpha) = -\frac{1}{2\pi i}\int_{C_+} \log\left\{ i\,\frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}}\right\}\frac{dt}{t-\alpha}$$

(3.21) $$\frac{d}{d\alpha}\log L_+(\alpha) = -\frac{1}{2\pi i}\int_{C_+} \log\left\{ i\,\frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}}\right\}\frac{dt}{(t-\alpha)^2}$$

(3.22) $$\frac{d}{d\alpha}\log L_+(\alpha) = -\frac{1}{2\pi i}\int_{C_1+C_{21}+C_3+C_{22}} \log\left\{ i\,\frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}}\right\}\frac{dt}{(t-\alpha)^2}$$

We evaluate the contributions to the right hand side of (3.22) by C_1, C_{21}, C_3, C_{22} separately.

On C_1, the argument of $\sqrt{k^2-t^2}$ changes from $-\frac{\pi}{2}$ to $-\frac{3\pi}{2}$ from the upper to the lower side of the cut.

$$-\frac{1}{2\pi i}\int_{C_1}\log\left[i\,\frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}}\right]\frac{dt}{(t-a)^2}$$

$$=-\frac{1}{2\pi i}\int_{-k}^{-T}\log\left[i\,\frac{1-\exp(2b\sqrt{t^2-k^2})}{2i\sqrt{t^2-k^2}}\right]\frac{dt}{(t-a)^2}$$

$$+\frac{1}{2\pi i}\int_{-k}^{-T}\log\left[i\,\frac{1-\exp(-2b\sqrt{t^2-k^2})}{-2i\sqrt{t^2-k^2}}\right]\frac{dt}{(t-a)^2}$$

$$=\frac{1}{2\pi i}\int_{-k}^{-T}\log\left[\frac{1-\exp(-2b\sqrt{t^2-k^2})}{-\left(1-\exp(+2b\sqrt{t^2-k^2})\right)}\right]\frac{dt}{(t-a)^2}$$

$$=\frac{1}{2\pi i}\int_{-k}^{-T}\frac{-2b\sqrt{t^2-k^2}}{(t-a)^2}\,dt$$

$$=\frac{b}{\pi i}\int_{k}^{T}\frac{\sqrt{t^2-k^2}}{(t+a)^2}\,dt$$

$$=\frac{b}{\pi i}\left(-\frac{\sqrt{T^2-k^2}}{T+a}+\cosh^{-1}\frac{T}{k}-a\int_{k}^{T}\frac{dt}{(t+a)\sqrt{t^2-k^2}}\right)$$

In $\eta=\int_{k}^{T}\frac{dt}{(t+a)\sqrt{t^2-k^2}}$ put $t = k\cosh u$

$$\eta = \int_0^{\cosh^{-1}\frac{T}{k}} \frac{du}{a+k\cosh u}$$

$$= \frac{2}{\sqrt{k^2-a^2}} \tan^{-1}\left[\left(\frac{k-a}{k+a}\right)^{\frac{1}{2}} \tanh\frac{u}{2}\right]\Bigg|_0^{\cosh^{-1}\frac{T}{k}}$$

At the upper limit, $\cosh u = \frac{T}{k}$

$$\tanh\frac{u}{2} = \sqrt{\frac{T-k}{T+k}}$$

As $T \to \infty$ $\tanh\frac{u}{2} \to 1$ at the upper limit. At the lower limit $\tanh 0 = 0$ and taking arctan on the principal branch we get,

$$\lim_{T\to\infty} \int_k^T \frac{dt}{(t+a)\sqrt{t^2-k^2}} = \frac{2}{\sqrt{k^2-a^2}} \tan^{-1}\left(\frac{k-a}{k+a}\right)^{\frac{1}{2}}$$

(3. 23) $$\lim_{T\to\infty}\left[\frac{1}{2\pi i}\int_{C_1} - \frac{b}{\pi i}\cosh^{-1}\frac{T}{k}\right]$$

$$= -\frac{b}{\pi i}\left\{1 + \frac{2a}{\sqrt{k^2-a^2}}\tan^{-1}\left(\frac{k-a}{k+a}\right)^{\frac{1}{2}}\right\}$$

On C_{21}, $-\pi < \arg t < -\frac{\pi}{2}$

$$-\frac{3\pi}{2} < \arg (k^2-t^2)^{\frac{1}{2}} < -\pi$$

The integrand behaves asympotically as

$$\frac{\log\left[\exp(2bt)\right]}{t^2} = \frac{2bt}{t^2} = \frac{2b}{t}, \quad |t| \gg 1.$$

$$- \frac{1}{2\pi i} \int_{C_{21}} \log \left[i \;\; \frac{1-\exp\,(2ib\,\sqrt{k^2-t^2}\,)}{2\sqrt{k^2-t^2}} \right] \frac{dt}{(t-a)^2}$$

$$\approx - \frac{1}{2\pi i} \int_{Te^{-i\pi}}^{Te^{-i\pi/2}} \frac{2b}{t}\, dt \;=\; - \frac{b}{\pi i} \int_{-\pi}^{-\pi/2} i d\theta \;=\; - \frac{b}{2}$$

(3.24) $\therefore \lim_{T\to\infty} - \frac{1}{2\pi i} \int_{C_{21}} = - \frac{b}{2}$ in the limit as $T \to \infty$

On C_{22}, $-\frac{\pi}{2} < \arg t < 0$

$$- \pi < \arg (k^2-t^2)^{\frac{1}{2}} < - \frac{\pi}{2}$$

The integrand behaves as $\approx \frac{\log t}{t^2}$ in this quadrant. As $T \to \infty$, the contribution due to this part vanishes.

(3.25) $\therefore \lim_{T\to\infty} - \frac{1}{2\pi i} \int_{C_{22}} = 0$

The contirubtion from C_3 is due to the phase difference of the logrithm term in the integrand on the two sides of C_3. The zeros of the function $\left[1-\exp\,(-2ib\,\sqrt{k^2-t^2}\,)\right]$ are located on the negative imaginary axis. The points $t = -i \left[\frac{n^2\pi^2}{b^2} - k^2 \right]^{\frac{1}{2}}$ are branch points of the logrithm function in the integrand. The term $\log \left\{ i \, \frac{1-\exp\,(-2ib\sqrt{k^2-t^2}\,)}{2\sqrt{k^2-t^2}} \right\}$ in the integrand, for $|\mathrm{Im}\; t| < T$, has the same phase as

$$\log \prod_{n=1}^{N} \left\{ t + i \left(\frac{n^2\pi^2}{b^2} - k^2 \right)^{\frac{1}{2}} \right\}$$

where N is the largest integer such that $\frac{N\pi}{b} < T$

$$-\frac{1}{2\pi i}\int_{C_3} = -\frac{1}{2\pi i}\int_{C_3} \log \prod_{n=1}^{N} \left\{ t+i \left(\frac{n^2\pi^2}{b^2} - k^2 \right)^{\frac{1}{2}} \right\} \frac{dt}{(t-a)^2}$$

$$= \sum_{n=1}^{N} \int_{-i\left(\frac{n^2\pi^2}{b^2} - k^2\right)^{\frac{1}{2}}}^{-iT} \frac{1}{(t-a)^2}\, dt = \sum_{n=1}^{N} \frac{1}{a-t} \Bigg|_{-i\left(\frac{n^2\pi^2}{b^2} - k^2\right)^{\frac{1}{2}}}^{-iT}$$

$$= \sum_{n=1}^{N} \left\{ \frac{1}{a+iT} - \frac{1}{a+i\left(\frac{n^2\pi^2}{b^2} - k^2\right)^{\frac{1}{2}}} \right\}$$

$$(3.26) \qquad \therefore -\frac{1}{2\pi i}\int_{C_3} = \sum_{n=1}^{N} \left\{ \frac{1}{a+iT} - \frac{1}{a+i\left(\frac{n^2\pi^2}{b^2} - k^2\right)^{\frac{1}{2}}} \right\}$$

Using (3.23), (3.24), (3.25), and (3.26) we have

$$(3.27) \qquad \frac{d}{da} \log L_+(a) = -\frac{b}{2} - \frac{b}{\pi i}\left[1 + \frac{2a}{\sqrt{k^2-a^2}} \tan^{-1} \left(\frac{k-a}{k+a} \right)^{\frac{1}{2}} \right]$$

$$+ \lim_{T\to\infty} \left[\sum_{n=1}^{N} \left\{ \frac{1}{a+iT} - \frac{1}{a+i\left(\frac{n^2\pi^2}{b^2} - k^2\right)^{\frac{1}{2}}} \right\} + \frac{b}{\pi i}\cosh^{-1}\frac{T}{k} \right]$$

But $\displaystyle\sum_{n=1}^{N} \frac{1}{n} \approx \log N + \gamma \quad , \qquad N \to \infty$

$$\cosh^{-1} x \approx \log 2x \quad , \qquad x \to \infty$$

$$T \approx \frac{N\pi}{b} \qquad \text{for} \quad T \gg 1$$

(3.27) can be written as

$$\frac{d}{d\alpha} \log L_+(\alpha) = -\frac{b}{2} - \frac{b}{\pi i}\left[1 + \frac{2\alpha}{\sqrt{k^2-\alpha^2}} \tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{\frac{1}{2}}\right] + \frac{b}{\pi i}$$

$$+ \lim_{T\to\infty}\left\{\sum_{n=1}^{N}\left[\frac{b}{\pi i n} - \frac{1}{\alpha + i\left(\frac{n^2\pi^2}{b^2} - k^2\right)^{\frac{1}{2}}}\right] - \frac{b\gamma}{\pi i} - \frac{b}{\pi i}\log N + \frac{b}{\pi i}\log\frac{2T}{k}\right\}$$

$$= \frac{-2b\alpha}{\pi i\sqrt{k^2-\alpha^2}} \tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{\frac{1}{2}} - \frac{b}{2} - \frac{b\gamma}{\pi i} + \frac{b}{\pi i}\log\frac{2\pi}{bk}$$

$$+ \sum_{n=1}^{\infty}\left[\frac{b}{\pi i n} - \frac{1}{\alpha + i\left(\frac{n^2\pi^2}{b^2} - k^2\right)^{\frac{1}{2}}}\right]$$

$$(3.28)\qquad \frac{d}{d\alpha}\log L_+(\alpha) = \frac{-2b\alpha}{\pi i\sqrt{k^2-\alpha^2}}\tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{\frac{1}{2}} - \frac{b}{2} - \frac{b\gamma}{\pi i} + \frac{b}{\pi i}\log\left(\frac{2\pi}{bk}\right)$$

$$+\sum_{n=1}^{\infty}\left\{\frac{b}{\pi i n} - \frac{-i\frac{b}{n\pi}}{\frac{-i\alpha b}{n\pi} + \left(1-\frac{b^2k^2}{n^2\pi^2}\right)^{\frac{1}{2}}}\right\}.$$

Integration of (3.28) yields

$$\log L_+(\alpha) = \frac{-2ib}{\pi}\sqrt{k^2-\alpha^2}\ \tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{\frac{1}{2}} - \frac{ib\alpha}{\pi} - \frac{b\alpha}{2}$$

$$+\frac{b\alpha}{\pi i}\log\left(\frac{2\pi}{bk}\right) - \frac{b\gamma\alpha}{\pi i} + \sum_{n=1}^{\infty}\left\{-\frac{ib\alpha}{n\pi}\right.$$

$$\left.+\log\left[\left(1-\frac{b^2k^2}{n^2\pi^2}\right)^{\frac{1}{2}} - \frac{ib\alpha}{n\pi}\right]^{-1}\right\} + \text{const.}$$

$$(3.29)\qquad L_+(\alpha) = K\ \exp\left[-\frac{2ib}{\pi}\sqrt{k^2-\alpha^2}\ \tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{\frac{1}{2}} - \frac{ib\alpha}{\pi} - \frac{b\alpha}{2}\right.$$

$$\left.+\frac{b\alpha}{\pi i}\log\left(\frac{2\pi}{bk}\right) + \frac{ib\gamma\alpha}{\pi}\right]$$

$$\times\prod_{n=1}^{\infty}\left[\left(1-\frac{b^2k^2}{n^2\pi^2}\right)^{\frac{1}{2}} - \frac{ib\alpha}{n\pi}\right]^{-1}\exp\left(-\frac{ib\alpha}{n\pi}\right).$$

Using (3. 20)

$$L_-(\alpha) = \frac{1}{L_+(-\alpha)} = K' \exp\left[\frac{2ib}{\pi}\sqrt{k^2-\alpha^2}\ \tan^{-1}\left(\frac{k+\alpha}{k-\alpha}\right)^{\frac{1}{2}}\right. \tag{3.30}$$

$$\left. - \frac{ib\alpha}{\pi} - \frac{b\alpha}{2} - \frac{ib\alpha}{\pi}\log\left(\frac{2\pi}{bk}\right) + \frac{ib\gamma}{\pi}\alpha\right]$$

$$\cdot \prod_{n=1}^{\infty}\left[\left(1-\frac{b^2k^2}{n^2\pi^2}\right)^{\frac{1}{2}} + \frac{ib\alpha}{n\pi}\right] \exp\left(-\frac{ib\alpha}{n\pi}\right).$$

$$L(\alpha) = \frac{L_-(\alpha)}{L_+(\alpha)} = L_-(\alpha)\, L_-(-\alpha)$$

Thus

$$i\,\frac{1-\exp(-2ib\sqrt{k^2-\alpha^2})}{2\sqrt{k^2-\alpha^2}} = (K')^2 \prod_{n=1}^{\infty}\left[1-\frac{b^2(k^2-\alpha^2)}{n^2\pi^2}\right]$$

$$\cdot \exp\left[\frac{2ib}{\pi}\sqrt{k^2-\alpha^2}\left(\tan^{-1}\left(\frac{k+\alpha}{k-\alpha}\right)^{\frac{1}{2}}\right.\right.$$

$$\left.\left. + \tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{\frac{1}{2}}\right)\right]$$

$$= (K')^2\,\frac{\exp(ib\sqrt{k^2-\alpha^2}) - \exp(-ib\sqrt{k^2-\alpha^2})}{2ib\sqrt{k^2-\alpha^2}} \tag{3.31}$$

$$\cdot \exp\left[-ib\sqrt{k^2-\alpha^2}\right]^{*}$$

---

* See Appendix

$$(K')^2 = -b$$

$$K' = \sqrt{-b}$$

$$(3.32)\quad L_-(\alpha) = \sqrt{-b}\ \exp\left[-\frac{b\alpha}{2} - \frac{ib\alpha}{\pi}\left(1-\gamma+\log\frac{2\pi}{bk}\right)\right.$$

$$\left.+\frac{2ib}{\pi}\sqrt{k^2-\alpha^2}\ \tan^{-1}\left(\frac{k+\alpha}{k-\alpha}\right)^{\frac{1}{2}}\right]\prod_{n=1}^{\infty}\left\{\left(1-\frac{b^2k^2}{n^2\pi^2}\right)^{\frac{1}{2}}\right.$$

$$\left.+\frac{ib\alpha}{n\pi}\right\}\exp\left(-\frac{ib\alpha}{n\pi}\right)$$

$$= \sqrt{-b}\left[\frac{\sin\left(b\sqrt{k^2-\alpha^2}\right)}{b\sqrt{k^2-\alpha^2}}\right]^{\frac{1}{2}}\exp\left[-\frac{b\alpha}{2}-\frac{ib\alpha}{\pi}\left(1-\gamma+\log\frac{2\pi}{bk}\right)\right.$$

$$+\frac{2ib}{\pi}\sqrt{k^2-\alpha^2}\quad\tan^{-1}\left(\frac{k+\alpha}{k-\alpha}\right)^{\frac{1}{2}}$$

$$\left.+\ i\sum_{n=1}^{\infty}\left(\tan^{-1}\frac{\alpha}{\sqrt{\frac{n^2\pi^2}{b^2}-k^2}}-\frac{b\alpha}{n\pi}\right)\right]$$

$$(3.33)\quad L_+(\alpha) = \frac{1}{\sqrt{-b}}\left[\frac{b\sqrt{k^2-\alpha^2}}{\sin\left(b\sqrt{k^2-\alpha^2}\right)}\right]^{\frac{1}{2}}\quad\exp\left[-\frac{b\alpha}{2}\right.$$

$$-\frac{ib\alpha}{2}\left(1-\gamma+\log\frac{2\pi}{bk}\right)-\frac{2ib}{\pi}\sqrt{k^2-\alpha^2}\ \tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{\frac{1}{2}}$$

$$\left.+\ i\sum_{n=1}^{\infty}\left(\tan^{-1}\frac{\alpha}{\sqrt{\frac{n^2\pi^2}{b^2}-k^2}}-\frac{b\alpha}{n\pi}\right)\right]$$

As $|\alpha| \to \infty$ in the proper half plane

$$
(3.34) \quad \begin{cases} |L_+(\alpha)| = O(|\alpha|^{\frac{1}{2}}) , & \operatorname{Im} \alpha \geqslant -\delta + \epsilon . \\ \\ |L_-(\alpha| = O(|\alpha|^{-\frac{1}{2}}) , & \operatorname{Im} \alpha \leqslant -\delta + \epsilon . \end{cases}
$$

Let $P(\alpha) = \frac{1}{2} [1 + \exp (-2ib \sqrt{k^2 - \alpha^2})]$

We decompose $P(\alpha)$ in the form $P(\alpha) = \dfrac{P_-(\alpha)}{P_+(\alpha)}$ where $P_-(\alpha)$, $P_+(\alpha)$ are regular in lower and upper half planes with a common strip.

The analysis for this decomposition is essentially the same as in the case of function $L(\alpha)$. We will, therefore, give only a sketch of this analysis.

$$
\log P(\alpha) = \frac{1}{2\pi i} \int_{C_+} \frac{\log P(t)}{t-\alpha} dt - \frac{1}{2\pi i} \int_{C_-} \frac{\log P(t)}{t-\alpha} dt
$$

$$
= \log P_-(\alpha) - \log P_+(\alpha)
$$

$$
(3.35) \quad \log P_+(\alpha) = - \frac{1}{2\pi i} \int_{C_+} \frac{\log P(t)}{t-\alpha} dt .
$$

$$
(3.36) \quad \log P_-(\alpha) = - \frac{1}{2\pi i} \int_{C_-} \frac{\log P(t)}{t-\alpha} dt
$$

Furthermore

$$
(3.37) \quad P_+(-\alpha) = \frac{1}{P_-(\alpha)}
$$

$$\text{(3.38)} \quad \frac{d}{da} \log P_+(a) = -\frac{1}{2\pi i} \int_{C_+} \frac{\log P(t)}{(t-a)^2} dt.$$

Replacing C_+ by the contour of Figure 4, we get

$$\text{(3.39)} \quad \frac{d}{da} \log P_+(a) = -\frac{1}{2\pi i} \int_{C_1+C_{21}+C_3+C_{22}} \log \left[\frac{1+\exp(-2ib\sqrt{k^2-t^2})}{2} \right] \frac{dt}{(t-a)^2}$$

On C_1, the argument of $\sqrt{k^2-t^2}$ changes from $-\frac{\pi}{2}$ to $-\frac{3\pi}{2}$ from the upper side to the lower side of the cut.

$$-\frac{1}{2\pi i} \int_{C_1} = -\frac{1}{2\pi i} \int_{-k}^{-T} \log \left[\frac{1+\exp(2b\sqrt{t^2-k^2})}{2} \right] \frac{dt}{(t-a)^2}$$

$$+ \frac{1}{2\pi i} \int_{-k}^{-T} \log \left[\frac{1+\exp(-2b\sqrt{t^2-k^2})}{2} \right] \frac{dt}{(t-a)^2}$$

$$= -\frac{1}{2\pi i} \int_{-k}^{-T} \log \left[\frac{1+\exp(2b\sqrt{t^2-k^2})}{1+\exp(-2b\sqrt{t^2-k^2})} \right] \frac{dt}{(t-a)^2}$$

$$= -\frac{1}{2\pi i} \int_{-k}^{-T} \frac{(2b\sqrt{t^2-k^2})}{(t-a)^2} dt$$

$$= \frac{b}{\pi i} \int_{k}^{T} \frac{\sqrt{t^2-k^2}}{(t+a)^2} dt$$

$$(3.40) \quad \lim_{T\to\infty} \left[-\frac{1}{2\pi i} \int_{C_1} -\frac{b}{\pi i} \cosh^{-1} \frac{T}{k} \right]$$

$$= -\frac{b}{\pi i} \left[1 + \frac{2a}{\sqrt{k^2-a^2}} \tan^{-1} \left(\frac{k-a}{k+a}\right)^{\frac{1}{2}} \right]$$

On C_{21}, $\quad -\pi < \arg t < -\frac{\pi}{2}$

$$-\frac{3\pi}{2} < \arg \left(\sqrt{k^2-t^2}\right) < -\pi$$

The integrand is $\approx \dfrac{\log[\exp(2bt)]}{t^2} = \dfrac{2bt}{t^2} = \dfrac{2b}{t}$

$$(3.41) \quad \therefore \; -\frac{1}{2\pi i} \int_{C_{21}} = -\frac{b}{2} \quad \text{as } T \to \infty$$

On C_{22}, $\quad -\frac{\pi}{2} < \arg t < 0$

$$-\pi < \arg\left(\sqrt{k^2-t^2}\right) < -\frac{\pi}{2}$$

The integrand is $\approx \dfrac{\log(1)}{t^2}$.

The contribution for this segment of the contour vanishes as $T \to \infty$.

$$(3.42) \quad -\frac{1}{2\pi i} \int_{C_{22}} = 0 \qquad \text{as } T \to \infty .$$

The contribution due to C_3 is again due to the phase difference on the two sides of C_3. The zeros of the function

$[1+\exp(-2ib\sqrt{k^2-t^2})]$ are located at

$$t = -i\left[\frac{(n+\frac{1}{2})^2\pi^2}{b^2} - k^2\right]^{\frac{1}{2}} \quad , \quad n = 0, 1, 2 \ldots .$$

For $|\mathrm{Im}\ t| < T$, the logrithm term in the integrand has the same phase as

$$\log \prod_{n=0}^{N}\left[t+i\left\{\frac{(n+\frac{1}{2})^2\pi^2}{b^2} - k^2\right\}^{\frac{1}{2}}\right]$$

where N is the largest integer such that $\dfrac{(N+\frac{1}{2})\pi}{b} < T$

$$-\frac{1}{2\pi i}\int_{C_3} = -\frac{1}{2\pi i}\int_{C_3} \log \prod_{n=0}^{N}\left[t+i\left\{\frac{(n+\frac{1}{2})^2\pi^2}{b^2} - k^2\right\}^{\frac{1}{2}}\right]\frac{dt}{(t-a)^2}$$

$$= \sum_{n=0}^{N}\left[\frac{1}{ai+T} - \frac{1}{a+i\left\{\frac{(n+\frac{1}{2})^2\pi^2}{b^2} - k^2\right\}^{\frac{1}{2}}}\right]$$

$$(3.43) \qquad -\frac{1}{2\pi i}\int_{C_3} = \sum_{n=0}^{N}\left[\frac{1}{a+iT} - \frac{1}{a+i\left[(n+\frac{1}{2})^2\pi^2/b^2 - k^2\right]^{\frac{1}{2}}}\right]$$

Adding up all the contributions, we have

$$\frac{d}{da}\log P_+(a) = -\frac{b}{2} - \frac{b}{\pi i}\left[1 + \frac{2a}{\sqrt{k^2-a^2}}\tan^{-1}\left(\frac{k-a}{k+a}\right)^{\frac{1}{2}}\right.$$

$$+ \lim_{T\to\infty}\left[\sum_{n=0}^{N}\left[\frac{1}{a+iT} - \frac{1}{a+i\left\{\frac{(n+\frac{1}{2})^2\pi^2}{b^2} - k^2\right\}^{\frac{1}{2}}}\right]\right.$$

$$\left. + \frac{b}{\pi i}\cosh^{-1}\frac{T}{k}\right]$$

We note the following asymptotic relations

$$\sum_{n=0}^{N} \frac{1}{n+\frac{1}{2}} \approx \log 4N + \gamma \quad , \; N \to \infty$$

$$\cosh^{-1} x \approx \log 2x \quad , \; x \to \infty$$

$$T \approx \frac{(N+\frac{1}{2})\pi}{b} \qquad \text{as } T \to \infty$$

$$\frac{d}{da} \log P_+(a) = -\frac{b}{2} - \frac{b}{\pi i}\left[1 + \frac{2a}{\sqrt{k^2-a^2}} \tan^{-1}\left(\frac{k-a}{k+a}\right)^{\frac{1}{2}}\right] + \frac{b}{\pi i}$$

$$+ \lim_{T\to\infty}\left\{ \sum_{n=0}^{N}\left[\frac{b}{i\pi(n+\frac{1}{2})} - \frac{1}{a+i\left(\frac{(n+\frac{1}{2})^2 \pi^2}{b^2} - k^2\right)^{\frac{1}{2}}}\right] - \frac{b\gamma}{\pi i}\right.$$

$$\left. - \frac{b}{\pi i} \log 4N + \frac{b}{\pi i} \log \frac{2T}{k}\right\}$$

$$\frac{d}{da} \log P_+(a) = \frac{-2ba}{\pi i \sqrt{k^2-a^2}} \tan^{-1}\left(\frac{k-a}{k+a}\right)^{\frac{1}{2}} - \frac{b}{2} - \frac{b\gamma}{\pi i} + \frac{b}{\pi i}\log\left(\frac{\pi}{2bk}\right)$$

$$+ \sum_{n=0}^{\infty}\left\{\frac{b}{\pi i(n+\frac{1}{2})}\right.$$

$$\left. - \frac{-i\,\dfrac{b}{\pi(n+\frac{1}{2})}}{\dfrac{-iab}{(n+\frac{1}{2})\pi} + \left(1 - \dfrac{b^2k^2}{(n+\frac{1}{2})^2\pi^2}\right)^{\frac{1}{2}}}\right\}$$

$$\log P_+(a) = -\frac{2ib}{\pi}\sqrt{k^2-a^2}\tan^{-1}\left(\frac{k-a}{k+a}\right)^{\frac{1}{2}} - \frac{iba}{\pi} - \frac{ba}{2}$$

$$+\frac{ba}{\pi i}\log\left(\frac{\pi}{2bk}\right) - \frac{b\gamma}{\pi i}a + \log\prod_{n=0}^{\infty}\left[\left(1-\frac{b^2k^2}{(n+\frac{1}{2})^2\pi^2}\right)^{\frac{1}{2}}\right.$$

$$\left.-\frac{iba}{(n+\frac{1}{2})\pi}\right]^{-1} + \sum_{n=0}^{\infty} -\frac{iba}{(n+\frac{1}{2})\pi} + \text{const.}$$

(3.44) $$P_+(a) = C\exp\left\{-\frac{2ib}{\pi}\sqrt{k^2-a^2}\tan^{-1}\left(\frac{k-a}{k+a}\right)^{\frac{1}{2}} - \frac{ba}{2} - \frac{iba}{\pi}\right.$$

$$\left.+\frac{ba}{\pi i}\log\left(\frac{\pi}{2bk}\right) + \frac{ib\gamma}{\pi}a\right\}\prod_{n=0}^{\infty}\left[\left(1-\frac{b^2k^2}{(n+\frac{1}{2})^2\pi^2}\right)^{\frac{1}{2}}\right.$$

$$\left.-\frac{iba}{(n+\frac{1}{2})\pi}\right]^{-1}\exp\left(-\frac{iba}{(n+\frac{1}{2})\pi}\right)$$

But $P_-(a) = \dfrac{1}{P_+(-a)}$

(3.45) $$P_-(a) = C'\exp\left\{\frac{2ib}{\pi}\sqrt{k^2-a^2}\tan^{-1}\left(\frac{k+a}{k-a}\right)^{\frac{1}{2}} - \frac{ba}{2} - \frac{iba}{\pi}\right.$$

$$\left.+\frac{ba}{\pi i}\log\left(\frac{\pi}{2bk}\right) + \frac{ib\gamma}{\pi}a\right\}\prod_{n=0}^{\infty}\left[\left(1-\frac{b^2k^2}{(n+\frac{1}{2})^2\pi^2}\right)^{\frac{1}{2}} + \frac{iba}{(n+\frac{1}{2})\pi}\right]$$

$$\exp\left(-\frac{iba}{(n+\frac{1}{2})\pi}\right)$$

$$P(\alpha) = \frac{P_-(\alpha)}{P_+(\alpha)} = P_-(\alpha)\ P_-(-\alpha)$$

(3.45a) $$\frac{1 + \exp(-2ib\sqrt{k^2-\alpha^2})}{2} = (C')^2 \prod_{n=0}^{\infty}\left[1 - \frac{b^2(k^2-\alpha^2)}{(n+\frac{1}{2})^2\pi^2}\right] \cdot$$

*

$$\cdot \exp\left\{\frac{2ib}{\pi}\sqrt{k^2-\alpha^2}\ (-\pi/2)\right\}$$

$$(C')^2 = 1$$

$$C' = 1$$

(3.46) $$P_-(\alpha) = \exp\left\{\frac{2ib}{\pi}\sqrt{k^2-\alpha^2}\ \tan^{-1}\left(\frac{k+\alpha}{k-\alpha}\right)^{\frac{1}{2}} - \frac{b\alpha}{2} - \frac{ib\alpha}{\pi}\right.$$

$$\left. + \frac{ib\gamma}{\pi}\alpha + \frac{b\alpha}{\pi i}\log\left(\frac{\pi}{2bk}\right)\right\} \prod_{n=0}^{\infty}\left[\left(1 - \frac{b^2k^2}{(n+\frac{1}{2})^2\pi^2}\right)^{\frac{1}{2}}\right.$$

$$\left. + \frac{ib\alpha}{(n+\frac{1}{2})\pi}\right] \exp\left[-\frac{ib\alpha}{(n+\frac{1}{2})\pi}\right]$$

(3.47) $$P_+(\alpha) = \exp\left\{-\frac{2ib}{\pi}\sqrt{k^2-\alpha^2}\ \tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{\frac{1}{2}} - \frac{b\alpha}{2} - \frac{ib\alpha}{\pi}\right.$$

$$\left. + \frac{ib\gamma}{\pi}\alpha + \frac{b\alpha}{\pi i}\log\left(\frac{\pi}{2bk}\right)\right\} \prod_{n=0}^{\infty}\left[\left(1 - \frac{b^2k^2}{(n+\frac{1}{2})^2\pi^2}\right)^{\frac{1}{2}}\right.$$

$$\left. - \frac{ib\alpha}{(n+\frac{1}{2})\pi}\right]^{-1} \exp\left(-\frac{ib\alpha}{(n+\frac{1}{2})\pi}\right)$$

---

* See Appendix

$$(3.48)\quad \begin{cases} |P_+(\alpha)| = O(1) & \operatorname{Im}\alpha \geqslant -\delta+\epsilon \\ |P_-(\alpha)| = O(1) & \operatorname{Im}\alpha \leqslant \delta-\epsilon \end{cases}$$

in their respective half planes.

Equations (3.15) and (3.16) can now be rewritten as

$$(3.49)\quad \sqrt{k+\alpha}\;\bar{H}_0(\alpha)\,P_+(\alpha) = \frac{ie^{-i\alpha x'}}{2\sqrt{k-\alpha}}\,[\exp(-i|b-y'|\sqrt{k^2-\alpha^2})$$

$$+\exp(-i|b+y'|\sqrt{k^2-\alpha^2})]\cdot P_+(\alpha) + i\,\frac{P_-(\alpha)}{\sqrt{k-\alpha}}\,\bar{f}_0(\alpha),$$

$$(3.50)\quad \bar{H}_1(\alpha)\,L_+(\alpha) = \frac{i\exp(-i\alpha x')}{2\sqrt{k^2-\alpha^2}}\,[\exp(-i|b-y'|\sqrt{k^2-\alpha^2})$$

$$-\exp(-i|b+y'|\sqrt{k^2-\alpha^2})]\cdot L_+(\alpha) + L_-(\alpha)\,\bar{f}_1(\alpha).$$

Let

$$(3.51)\quad A(\alpha) = \frac{i\exp(-i\alpha x')}{2\sqrt{k^2-\alpha^2}}\,[\exp(-i|b-y'|\sqrt{k^2-\alpha^2})$$

$$+\exp(-i|b+y'|\sqrt{k^2-\alpha^2})]\cdot P_+(\alpha)$$

We decompose $A(\alpha)$ as the difference of two analytic functions $A_-(\alpha)$, $A_+(\alpha)$ which are regular in half planes with a strip in common.

$$(3.52)\quad A(\alpha) = A_-(\alpha) - A_+(\alpha)$$

Applying Cauchy's theorem to the contour of Figure 3 we have

$$A(\alpha) = \frac{1}{2\pi i} \int \frac{A(t)}{t-\alpha}\, dt.$$

The integrand tends to zero on the vertical segments of the contour and thus the contribution due to these segments tends to zero as $T \to \infty$

(3.53) $$A(\alpha) = \frac{1}{2\pi i} \int_{C_+} \frac{A(t)}{t-\alpha}\, dt - \frac{1}{2\pi i} \int \frac{A(t)}{t-\alpha}\, dt.$$

(3.54) $$\begin{cases} A_+(\alpha) = \lim\limits_{T\to\infty} - \dfrac{1}{2\pi i} \displaystyle\int_{-T-i\delta}^{T-i\delta} \dfrac{A(t)}{t-\alpha}\, dt \\ \\ A_-(\alpha) = \lim\limits_{T\to\infty} - \dfrac{1}{2\pi i} \displaystyle\int_{-T+i\delta}^{T+i\delta} \dfrac{A(t)}{t-\alpha}\, dt . \end{cases}$$

(3.55) $|A_+(\alpha) = O(|\delta + \mathrm{Im}\,\alpha|^{-\frac{1}{2}})$ as $|\alpha| \to \infty$ in the upper half plane $\mathrm{Im}\,\alpha \geqslant -\delta + \epsilon$.

(3.56) $|A_-(\alpha) = O(|\delta - \mathrm{Im}\,\alpha|^{-\frac{1}{2}})$ as $|\alpha| \to \infty$ in the lower half plane $\mathrm{Im}\,\alpha \leqslant \delta - \epsilon$.

Let

$$(3.57) \qquad B(\alpha) = \frac{i\exp(-i\alpha x')}{2\sqrt{k^2-\alpha^2}} \left[\exp(-i|b-y'|\sqrt{k^2-\alpha^2}) - \exp(-i|b+y'|\sqrt{k^2-\alpha^2})\right] \cdot L_+(\alpha)$$

We decompose $B(\alpha)$ as the difference of two analytic functions $B_-(\alpha)$, $B_+(\alpha)$ such that $B(\alpha) = B_-(\alpha) - B_+(\alpha)$. In a manner similar to that of $A(\alpha)$

$$(3.58) \qquad \begin{cases} B_+(\alpha) = \lim\limits_{T\to\infty} -\dfrac{1}{2\pi i}\displaystyle\int_{C_+} \frac{B(t)}{t-\alpha}\, dt \\[2ex] B_-(\alpha) = \lim\limits_{T\to\infty} -\dfrac{1}{2\pi i}\displaystyle\int_{C_-} \frac{B(t)}{t-\alpha}\, dt. \end{cases}$$

$$(3.59) \qquad \begin{cases} |B_+(\alpha)| = O(|\delta + \mathrm{Im}\,\alpha|^{-\frac{1}{2}}), & |\alpha| \to \infty,\ \mathrm{Im}\,\alpha \geqslant -\delta + \epsilon \\[2ex] B_-(\alpha) = O(|\delta - \mathrm{Im}\,\alpha|^{-\frac{1}{2}}), & |\alpha| \to \infty,\ \mathrm{Im}\,\alpha \leqslant \delta - \epsilon \end{cases}$$

Equations (3.49) and (3.50) take the form

$$(3.60) \qquad \sqrt{k+\alpha}\ \bar{H}_0(\alpha) P_+(\alpha) + A_+(\alpha) = A_-(\alpha) + \frac{i P_-(\alpha)}{\sqrt{k-\alpha}}\ \bar{f}_0(\alpha)$$

$$(3.61) \qquad \bar{H}_1(\alpha) L_+(\alpha) + B_+(\alpha) = B_-(\alpha) + L_-(\alpha)\,\bar{f}_1(\alpha)$$

The left hand of (3.60) is regular in an upper half plane and the right hand side in a lower half plane. The two half planes have a common strip. By analytic continuation they define an analytic function $F(\alpha)$ in the entire α-plane.

Furthermore $|F(\alpha)| = O(|\alpha|^{-\frac{1}{2}})$ as $|\alpha| \to \infty$ in the region $\operatorname{Im} \alpha \leqslant \delta - \epsilon$ and

$$|F(\alpha)| = O(|\alpha|^{\frac{1}{2}}) \text{ as } |\alpha| \to \infty \text{ for } \operatorname{Im} \alpha \geqslant -\delta + \epsilon .$$

According to an extension of Liouville's theorem if $F_1(\alpha)$ is analytic for all finite α and $F_1(\alpha) = O(|\alpha|^m)$ as $|\alpha| \to \infty$ then $F_1(\alpha)$ is a polynomial of degree $\leqslant m$. Hence the function $F(\alpha)$ defined by (3.60) is a polynomial of degree $\leqslant \frac{1}{2}$. It follows that $F(\alpha)$ is a constant. Moreover as $|\alpha| \to \infty$ along the negative imaginary axis we note that $F(\alpha) \to 0$. This implies that $F(\alpha) \equiv 0$.

Thus

$$i \bar{f}_0(\alpha) \frac{P_-(\alpha)}{\sqrt{k-\alpha}} + A_-(\alpha) = 0$$

$$\sqrt{k+\alpha}\, \bar{H}_0(\alpha)\, P_+(\alpha) + A_+(\alpha) = 0$$

$$\bar{f}_0(\alpha) = \frac{i\sqrt{k-\alpha}\, A_-(\alpha)}{P_-(\alpha)} \tag{3.62}$$

$$\bar{H}_0(\alpha) = -\frac{A_+(\alpha)}{\sqrt{k+\alpha}\, P_+(\alpha)} \tag{3.63}$$

Similarly (3.61) defines an analytic function $G(\alpha)$ in the entire a-plane and $G(a) \equiv 0$. It leads to

(3.64) $$\bar{f}_1(a) = -\frac{B_-(a)}{L_-(a)}$$

(3.65) $$\bar{H}_1(a) = -\frac{B_+(a)}{L_+(a)}$$

$$\bar{f}_{\substack{0\\1}}(a) = \int_{-\infty}^{\infty} p_{\substack{0\\1}}(\lambda) \exp(-ia\lambda)\, d\lambda$$

$$\bar{f}_0(a)+\bar{f}_1(a) = \int_{-\infty}^{\infty} [p_0(\lambda) + p_1(\lambda)] \exp(-ia\lambda)\, d\lambda$$

$$= 2\int_{-\infty}^{\infty} I_0(\lambda) \exp(-ia\lambda)\, d\lambda$$

Similarly

$$\bar{f}_0(a) - \bar{f}_1(a) = 2\int_{-\infty}^{\infty} I_1(\lambda) \exp(-ia\lambda)\, d\lambda$$

Thus

(3.66) $$I_0(\lambda) = \frac{1}{4\pi}\int_{-\infty}^{\infty} [\bar{f}_0(a) + \bar{f}_1(a)] \exp(ia\lambda)\, da \qquad \lambda > 0$$

(3.67) $$I_1(\lambda) = \frac{1}{4\pi}\int_{-\infty}^{\infty} [\bar{f}_0(a) - \bar{f}_1(a)] \exp(ia\lambda)\, da \qquad \lambda > 0$$

where $\bar{f}_0(a)$, $\bar{f}_1(a)$ are given by (3.62) and (3.64) respectively.

$$\bar{H}_{\substack{0\\1}}(a) = \frac{i}{4}\ \bar{h}_{\substack{0\\1}}(a) = \int_{-\infty}^{\infty} [U(x,b) \pm U(x,-b)]\ \exp(-iax)\ dx$$

$$(3.68)\quad U(x,b) = \frac{1}{4\pi}\int_{-\infty}^{\infty} [\bar{H}_0(a) + \bar{H}_1(a)]\ \exp(iax)\ da \qquad x < 0$$

$$(3.69)\quad U(x,-b) = \frac{1}{4\pi}\int_{-\infty}^{\infty} [\bar{H}_0(a) - \bar{H}_1(a)]\ \exp(iax)\ dx \qquad x < 0$$

where $\bar{H}_0(a)$, $\bar{H}_1(a)$ are given by (3.63) and (3.65) respectively.

We have, therefore, evaluated all the unknown functions in the form of integrals (3.66) to (3.69).

4. ASYMPTOTIC ESTIMATES

In this chapter we justify the assumptions of Chapter 3 regarding the asymptotic behavior of various functions. In particular we establish the validity of (3.34), (3.48), (3.55), (3.56) and (3.59).

$$L_+(\alpha) = \exp\left\{ \lim_{T \to \infty} - \frac{1}{2\pi i} \int_{-T-i\delta}^{T-i\delta} \log \left[i \frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}} \right] \frac{dt}{t-\alpha} \right\}$$

Let $k = k_1 + ik_2$

$$L_+(\alpha) = \exp\left\{ -\frac{1}{2\pi i} \int_{-k_1}^{k_1} \log \left[i \frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}} \right] \frac{dt}{t-\alpha} \right.$$

$$+ \frac{1}{2\pi i} \int_{-k_1}^{-\infty} \log \left[i \frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}} \right] \frac{dt}{t-\alpha}$$

$$\left. - \frac{1}{2\pi i} \int_{k_1}^{\infty} \log \left[i \frac{i-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}} \right] \frac{dt}{t-\alpha} \right\}$$

Here we have put $\delta = 0$. This is justified since contour C_+ can be replaced by the contour through the origin

$$\frac{1}{2\pi i}\int_{-k_1}^{-\infty} \log\left[i\ \frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}}\right] \frac{dt}{t-a}$$

$$= -\frac{1}{2\pi i}\int_{k_1}^{\infty} \log\left[i\ \frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}}\right] \frac{dt}{-t-a}$$

$$= \frac{1}{2\pi i}\int_{k_1}^{\infty} \log\left[i\ \frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}}\right] \frac{dt}{t+a}$$

$$L_+(a) = \exp\left\{ -\frac{1}{2\pi i}\int_{-k_1}^{k_1} \log\left[i\ \frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}}\right] \frac{dt}{t-a}\right.$$

$$\left. -\frac{1}{2\pi i}\int_{k_1}^{\infty} \log\left[i\ \frac{1-\exp(-2b\sqrt{t^2-k^2})}{-2i\sqrt{t^2-k^2}}\right] \left(\frac{1}{t-a}-\frac{1}{t+a}\right)dt\right\},$$

since $\arg(\sqrt{k^2-t^2}) = -\frac{\pi}{2}$ as t varies from k_1 to ∞.

$$(4.1)\qquad L_+(a) = \exp\left\{ -\frac{1}{2\pi i}\int_{-k_1}^{k_1} \log\left[i\ \frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}}\right] \frac{dt}{t-a}\right.$$

$$\left. -\frac{a}{\pi i}\int_{k_1}^{\infty} \log\left[\frac{\exp(-2b\sqrt{t^2-k^2})-1}{-2\sqrt{t^2-k^2}}\right] \frac{dt}{t^2-a^2}\right\}.$$

$$(4.2)\qquad \int_{-k_1}^{k_1} \log\left[i\ \frac{1-\exp(-2ib\sqrt{k^2-t^2})}{2\sqrt{k^2-t^2}}\right] \frac{dt}{t-a} = O\left(\frac{1}{a}\right)$$

since the integrand remains bounded for $-k_1 < t < k_1$.

Also $k^2 - t^2 \neq 0$. An additive constant does not make any difference as long as it is bounded.

$$(4.3) \qquad \int_{k_1}^{\infty} \log\left[\frac{\exp(-2b\sqrt{t^2-k^2})-1}{2\sqrt{t^2-k^2}}\right]\frac{dt}{t^2-a^2}$$

$$= \int_{k_1}^{\infty} \log\left[t\,\frac{\exp(-2b\sqrt{t^2-k^2})-1}{2\sqrt{t^2-k^2}}\right]\frac{dt}{t^2-a^2}$$

$$+ \int_{k_1}^{\infty} \log\left(\frac{1}{t}\right)\frac{dt}{t^2-a^2} = O\left(\frac{1}{a}\right) + \int_{k_1}^{\infty} \log\left(\frac{1}{t}\right)\frac{dt}{t^2-a^2}$$

Since the logrithm term tends to $\log\left(\frac{1}{2}\right)$ as $t \to \infty$,

$$(4.4) \qquad \left|\int_{k_1}^{\infty} \log\left[t\,\frac{\exp(-2b\sqrt{t^2-k^2})-1}{2\sqrt{t^2-k^2}}\right]\frac{dt}{t^2-a^2}\right| \approx \int_{k_1}^{\infty}\frac{dt}{t^2-a^2} = O\left(\frac{1}{a}\right)$$

In $\displaystyle\int_{k_1}^{\infty} \log\left(\frac{1}{t}\right)\frac{dt}{t^2-a^2}$ put $t = -iav$

$$\text{Then} \quad \int_{k_1}^{\infty} \log\left(\frac{1}{t}\right)\frac{dt}{t^2-a^2} = \frac{i}{a}\int_{-\frac{k_1}{ia}}^{\infty} \log\left(\frac{1}{-iav}\right)\frac{1}{1+v^2}\,dv$$

$$= \frac{i}{a}\int_{-\frac{k_1}{ia}}^{\infty} \log\left(\frac{1}{-ia}\right)\frac{dv}{1+v^2} + \frac{i}{a}\int_{-\frac{k_1}{ia}}^{\infty} \log\left(\frac{1}{v}\right)\frac{dv}{1+v^2}$$

But

$$\frac{i}{a}\int_{-\frac{k_1}{ia}}^{\infty} \log\left(-\frac{1}{ia}\right)\frac{dv}{1+v^2} \approx \frac{i}{a}\log\left(-\frac{1}{ia}\right)\frac{\pi}{2},$$

and

$$-\frac{i}{a}\int_{-\frac{k_1}{ia}}^{\infty}\log(v)\frac{dv}{1+v^2} = -\frac{i}{a}\left[\int_{-\frac{k_1}{ia}}^{1}\log(v)\frac{dv}{1+v^2} + \int_{1}^{\infty}\log(v)\frac{dv}{1+v^2}\right]$$

$$\approx -\frac{i}{a}\left[\int_{-\frac{k_1}{ia}}^{1}\log(v)\,dv + O(1)\right]$$

$$\approx -\frac{i}{a}\left[\frac{k_1}{ia}\log\left(-\frac{k_1}{ia}\right) + O(1)\right] \approx O\left(\frac{1}{a}\right)$$

Hence

$$\int_{k_1}^{\infty}\log\left(\frac{1}{t}\right)\frac{dt}{t^2-a^2} \approx \frac{i\pi}{2a}\log\left(-\frac{1}{ia}\right) + O\left(\frac{1}{a}\right) \tag{4.5}$$

Combining the results of (4.1), (4.2), (4.3), (4.4) and (4.5) we get

$$L_+(a) \approx \exp\left[O\left(\frac{1}{a}\right) + \left(-\frac{a}{\pi i}\right)\frac{i\pi}{2a}\log\left(-\frac{1}{ia}\right)\right]$$

$$\approx \exp\left[O\left(\frac{1}{a}\right) - \frac{1}{2}\log\left(-\frac{1}{ia}\right)\right]$$

$$L_+(a) \approx (-ia)^{\frac{1}{2}} \tag{4.6}$$

According to (3.20)

$$L_-(\alpha) = \frac{1}{L_+(-\alpha)}$$

(4.7) $$L_-(\alpha) \approx (i\alpha)^{-\frac{1}{2}}$$

Equations (4.6) and (4.7) justify the assumptions made regarding the behavior of the functions $L_+(\alpha)$ and $L_-(\alpha)$ in (3.34).

$$P_+(\alpha) = \exp\left\{-\frac{1}{2\pi i} \lim_{T\to\infty} \int_{-T-i\delta}^{T-i\delta} \log\left[\frac{1+\exp(-2ib\sqrt{k^2-t^2})}{2}\right] \frac{dt}{t-\alpha}\right\}$$

We again let $\delta \to 0$.

$$P_+(\alpha) = \exp\left\{-\frac{1}{2\pi i} \int_{-k_1}^{k_1} \log\left[\frac{1+\exp(-2b\sqrt{k^2-t^2})}{2}\right] \frac{dt}{t-\alpha}\right.$$

$$\left. -\frac{1}{2\pi i} \int_{k_1}^{\infty} \log\left[\frac{1+\exp(-2b\sqrt{t^2-k^2})}{2}\right] \left(\frac{1}{t-\alpha} - \frac{1}{t+\alpha}\right) dt\right\}$$

$$= \exp\left\{-\frac{1}{2\pi i} \int_{-k_1}^{k_1} \log\left[\frac{1+\exp(-2ib\sqrt{k^2-t^2})}{2}\right] \frac{dt}{t-\alpha}\right.$$

$$\left. -\frac{\alpha}{\pi i} \int_{k_1}^{\infty} \log\left[\frac{1+\exp(-2b\sqrt{t^2-k^2})}{2}\right] \frac{dt}{t^2-\alpha^2}\right\}$$

But

(4.8) $$-\frac{1}{2\pi i} \int_{-k_1}^{k_1} \log\left[\frac{1+\exp(-2ib\sqrt{k^2-t^2})}{2}\right] \frac{dt}{t-\alpha} = O\left(\frac{1}{\alpha}\right)$$

and

$$(4.9) \qquad -\frac{a}{\pi i}\int_{k_1}^{\infty}\log\left[\frac{1+\exp(-2b\sqrt{t^2-k^2})}{2}\right]\frac{dt}{t^2-a^2} = O(1) \text{ by}$$

analysis similar to that for (4.3)

Hence

$$(4.10) \qquad |P_+(a)| \approx \exp\ \{O(\tfrac{1}{a}) + O(1)\} = O(1)$$

in the proper half plane.

Similarly

$$(4.11) \qquad |P_-(a)| = O(1)$$

$$A_+(a) = -\frac{1}{2\pi i}\int_{-\infty - i\delta}^{\infty - i\delta}\frac{A(t)}{t-a}\,dt = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{A(x-i\delta)}{x-i\delta-a}\,dx$$

$$(4.12) \qquad |A_+(a)|^2 \leqslant \frac{1}{4\pi^2}\int_{-\infty}^{\infty}|A(x-i\delta)|^2\,dx\int_{-\infty}^{\infty}\frac{dx}{|x-i\delta-a|^2}$$

by Cauchy-Schwarz inequality

$$(4.13) \qquad |A_+(a)|^2 \leqslant B_1^{\,2}\int_{-\infty}^{\infty}\frac{dx}{|x-i\delta-a|^2}$$

where $B_1^{\,2} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}|A(x-i\delta)|^2\,dx.$

As $|x| \to \infty$, $\arg(\sqrt{k^2-t^2}) = -\frac{\pi}{2}$, $t = x - i\delta$

$A(t)$ contains the terms $\exp(-i|b-y'|\sqrt{k^2-t^2})$ and

$\exp(-i|b+y'|\sqrt{k^2-t^2})$ which decay exponentially as $|t| \to \infty$.

Hence $\int_{-\infty}^{\infty} |A(t)|^2 \, dt$ converges and is bounded

$$\therefore \quad B_1^2 < \infty$$

$$\int_{-\infty}^{\infty} \frac{dx}{|x-i\delta-\alpha|^2} = \int_{-\infty}^{\infty} \frac{dx}{(x-u)^2+(\delta+v)^2} \qquad \text{where } \alpha = u + iv$$

(4.14) $$\int_{-\infty}^{\infty} \frac{dx}{|x-i\delta-\alpha|^2} = \frac{\pi}{|\delta+v|} = \frac{\pi}{|\delta+\operatorname{Im}\alpha|}$$

Hence $$|A_+(\alpha)|^2 \leqslant \frac{\pi B_1^2}{|\delta+\operatorname{Im}\alpha|} = O(|\delta+\operatorname{Im}\alpha|^{-1})$$

(4.15) $|A_+(\alpha)| = O(|\delta + \operatorname{Im}\alpha|^{-\frac{1}{2}})$ as $|\alpha| \to \infty$ in the upper half plane $\operatorname{Im}\alpha \geqslant -\delta+\epsilon$

(4.16) $$A_-(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+i\delta}^{+\infty+i\delta} \frac{A(t)}{t-\alpha}\, dt = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{A(x+i\delta)}{x+i\delta-\alpha}\, dx$$

By Cauchy-Schwarz inequality we have

(4.17) $$|A_-(\alpha)|^2 \leqslant \frac{1}{4\pi^2}\int_{-\infty}^{\infty} |A(x+i\delta)|^2\, dx \int_{-\infty}^{\infty} \frac{dx}{|x+i\delta-\alpha|^2}$$

$$\leqslant B_2^2 \int_{-\infty}^{\infty} \frac{dx}{|x+i\delta-\alpha|^2}$$

where $B_2^{\,2} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |A(x+i\delta)|^2 \, dx < \infty$ due to reasons similar to that for B_1.

$$\int_{-\infty}^{\infty} \frac{dx}{|x+i\delta-\alpha|^2} = \int_{-\infty}^{\infty} \frac{dx}{(x-u)^2 + (\delta-v)^2} \qquad \text{where} \quad \alpha = u+iv$$

$$= \frac{\pi}{|\delta - v|} = \frac{\pi}{|\delta - \mathrm{Im}\,\alpha|}$$

Hence $\quad |A_-(\alpha)|^2 \leqslant \dfrac{\pi B_2^{\,2}}{|\delta - \mathrm{Im}\,\alpha|} = O(|\delta - \mathrm{Im}\,\alpha|^{-1})$

(4.18) $\quad |A_-(\alpha)| = O(|\delta - \mathrm{Im}\,\alpha|^{-\frac{1}{2}})$ as $|\alpha| \to \infty$ in the lower half plane $\mathrm{Im}\,\alpha \leqslant \delta - \epsilon$.

The analysis for the asymptotic behavior of the functions $B_+(\alpha)$, $B_-(\alpha)$ is similar to that of $A_+(\alpha)$ and $A_-(\alpha)$. By a similar reasoning we get

(4.19) $\quad |B_+(\alpha)| = O(|\delta + \mathrm{Im}\,\alpha|^{-\frac{1}{2}})$, $\quad |\alpha| \to \infty$, $\quad \mathrm{Im}\,\alpha \geqslant -\delta + \epsilon$

(4.20) $\quad |B_-(\alpha)| = O(|\delta - \mathrm{Im}\,\alpha|^{-\frac{1}{2}})$, $\quad |\alpha| \to \infty$, $\quad \mathrm{Im}\,\alpha \leqslant \delta - \epsilon$

in their proper half planes.

5. TWO INFINITE, PARALLEL PLANES

As a special case we consider a line source of unit strength located at Q (0, 0) and oriented parallel to the z-axis, radiating on two conducting planes of zero thickness.

$$-x' < x < \infty \qquad y = b$$

$$-x' < x < \infty \qquad y = -b$$

Then the total field $U(x, y)$ at $P(x, y)$ is given by

$$(5.1) \qquad U(x,y) = \frac{i}{4} H_0^{(2)}\left(k\sqrt{x^2+y^2}\right) + \frac{i}{4}\int_{-x'}^{\infty} I(\lambda) \cdot \left[H_0^{(2)}\left(k\sqrt{(x-\lambda)^2+(y-b)^2}\right) + H_0^{(2)}\left(k\sqrt{(x-\lambda)^2+(y+b)^2}\right)\right] d\lambda$$

The secondary currents have the same strength due to symmetry.

Boundary conditions $U(x, \pm b) = 0$ for $x > -x'$ give

$$(5.2) \qquad 0 = \frac{i}{4} H_0^{(2)}\left(k\sqrt{x^2+b^2}\right) + \frac{i}{4}\int_{-x'}^{\infty} I(\lambda)\left[H_0^{(2)}(k|x-\lambda|) + H_0^{(2)}\left(k\sqrt{(x-\lambda)^2+(2b)^2}\right)\right] d\lambda$$

Also the right hand side of (5.2) should represent the total field for $x < -x'$.

This gives

$$(5.3) \qquad \int_{-x'}^{\infty} I(\lambda)\,[H_0^{(2)}(k|x-\lambda|) + H_0^{(2)}(k\sqrt{(x-\lambda)^2 + (2b)^2})]\,d\lambda$$

$$+ H_0^{(2)}(k\sqrt{x^2+b^2}) = \begin{cases} 0 & x > -x' \\ \frac{4}{i} U(x,b) & x < -x' \end{cases}$$

Multiplying both sides by exp (-iαx) and integrating with respect to x from $-\infty$ to ∞

$$(5.4) \qquad 2\,\bar{I}(\alpha)\,\frac{[1+\exp(-2ib\sqrt{k^2-\alpha^2})]}{\sqrt{k^2-\alpha^2}} + \frac{2\exp(-ib\sqrt{k^2-\alpha^2})}{\sqrt{k^2-\alpha^2}} = \frac{4}{i}\bar{h}(\alpha)$$

where

$$\bar{I}(\alpha) = \int_{-x'}^{\infty} I(\lambda)\exp(-i\alpha\lambda)\,d\lambda$$

$$\bar{h}(\alpha) = \int_{-\infty}^{-x'} U(x,b)\exp(-i\alpha x)\,dx$$

As $x' \to \infty$, $\bar{h}(\alpha) \to 0$

In the limiting case as $x' \to \infty$

$$\bar{I}^*(\alpha) = \int_{-\infty}^{\infty} I(\lambda)\exp(-i\alpha\lambda)\,d\lambda$$

Hence in this case (5.4) takes the form

$$(5.5) \quad \bar{I}^*(\alpha)\,\frac{[1+\exp(-2ib\sqrt{k^2-\alpha^2})]}{\sqrt{k^2-\alpha^2}} + \frac{\exp(-ib\sqrt{k^2-\alpha^2})}{\sqrt{k^2-\alpha^2}} = 0$$

which is the same as (2.3).

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APPENDIX

APPENDIX

In the derivation of (3. 31) and (3. 45a) we used the following result

$$\tan^{-1}\left(\frac{k-\alpha}{k+\alpha}\right)^{\frac{1}{2}} + \tan^{-1}\left(\frac{k+\alpha}{k-\alpha}\right)^{\frac{1}{2}} = -\frac{\pi}{2}$$

Let

$$z_1 = \left(\frac{k-\alpha}{k+\alpha}\right)^{\frac{1}{2}}, \quad z_2 = \left(\frac{k+\alpha}{k-\alpha}\right)^{\frac{1}{2}}$$

First of all we note that in the common strip of validity of $L_+(\alpha)$ and $L_-(\alpha)$, (this strip reduces to the real axis when Im k is arbitrarily small), arg $(k-\alpha)$ and arg $(k+\alpha)$ differ by π

$$\arg(k-\alpha) = -\pi$$

$$\arg(k+\alpha) = 0$$

$$\arg\left(\frac{k-\alpha}{k+\alpha}\right)^{\frac{1}{2}} = -\frac{\pi}{2}$$

$$\arg\left(\frac{k+\alpha}{k-\alpha}\right)^{\frac{1}{2}} = \frac{\pi}{2}$$

Define $\quad r = \left|\sqrt{\left(\frac{k-\alpha}{k+\alpha}\right)}\right| < 1$

then $\quad z_1 = -ir$

$$z_2 = \frac{i}{r}$$

Restricting the argument to the principal branch we have (18, p. 34-35)

$$\tan^{-1} z = \frac{1}{2i} \text{Log} \left(\frac{1+iz}{1-iz}\right)$$

$$\tan^{-1} z_1 = \frac{1}{2i} \text{Log} \left(\frac{1+r}{1-r}\right) = \frac{1}{2i} \log \left(\frac{1+r}{1-r}\right)$$

$$\tan^{-1} z_2 = \frac{1}{2i} \text{Log} \left(\frac{1-\frac{1}{r}}{1+\frac{1}{r}}\right) = \frac{1}{2i} \text{Log} \left(\frac{r-1}{r+1}\right)$$

$$= \frac{1}{2i} \left[\text{Log}(-1) + \text{Log} \left(\frac{1-r}{1+r}\right) \right]$$

$$= \frac{1}{2i} \left[-i\pi + \log \left(\frac{1-r}{1+r}\right) \right]$$

where the logrithm function is taken on the principal branch such that

$$-\pi \leqslant \arg (\text{Log } z) < \pi$$

Thus

$$\tan^{-1} z_1 + \tan^{-1} z_2 = \frac{1}{2i} \log \left(\frac{1+r}{1-r}\right)$$

$$+ \frac{1}{2i} \left[-i\pi + \log \left(\frac{1-r}{1+r}\right) \right] = -\frac{\pi}{2}$$

when $r > 1$

$$\tan^{-1} z_1 = \frac{1}{2i} \text{Log} \left(\frac{1+r}{1-r}\right)$$

$$= \frac{1}{2i} \left[\text{Log } (-1) + \text{Log} \left(\frac{r+1}{r-1}\right) \right]$$

$$= \frac{1}{2i} \left[-i\pi + \log \left(\frac{r+1}{r-1}\right) \right]$$

$$\tan^{-1} z_2 = \frac{1}{2i} \text{Log} \left(\frac{1-\frac{1}{r}}{1+\frac{1}{r}}\right) = \frac{1}{2i} \text{Log} \left(\frac{r-1}{r+1}\right)$$

$$= \frac{1}{2i} \log \left(\frac{r-1}{r+1}\right)$$

Again we have

$$\tan^{-1} z_1 + \tan^{-1} z_2 = -\frac{\pi}{2} .$$

Furthermore $r \neq 0$ since the result $L(\alpha) = \dfrac{L_-(\alpha)}{L_+(\alpha)} = L_-(\alpha) L_-(-\alpha)$ is valid only in the common strip of regularity of both $L_-(\alpha)$ and $L_+(\alpha)$. The points $\alpha = \pm k$ which could make $r=0$ are outside this strip.

The result remains valid in the limiting cases $r \to 0$ and $r \to 1$.