CONSTRUCTION OF THE REAL NUMBERS BY SEQUENCES

by

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INTRODUCTION

In Edmund Landau's book, Foundations of Analysis, we find a complete and rigorous development of the real number system from Peano's axioms. One of the major steps in this development is the defining of the real numbers in terms of "cuts" of rational numbers. This method of defining real numbers is paralleled by a method of defining real numbers in terms of Cauchy sequences of rational numbers. The usage of Cauchy sequences will be pursued by this paper.

By a Cauchy sequence of rational numbers, we mean a sequence $A_1, \ldots, A_n, \ldots$ of rationals such that if E is any fixed positive rational number, then there is an integer $p(E)$ such that for all $m,n>p(E)$

$$|A_n - A_m| < E.$$

But since we shall use Landau's development of the rational number system as a foundation, the condition involving the absolute value becomes,

(1) $A_n = A_m$; or

(2) $A_n > A_m$ and $A_n - A_m < E$; or

(3) $A_n < A_m$ and $A_m - A_n < E$.

For this paper, since the rational numbers which we have are only the positive rationals, we will attach the additional condition as a part of being a Cauchy sequence: there exists a (positive) rational number N such that $A_n \geq N$ for all $n = 1, 2, \ldots$.

To complete a rigorous development of the real numbers by Cauchy sequences, we shall presume that the reader is familiar with Landau's book, Foundations of Analysis, for it is with this basis that we are able to continue. We shall presume that the (positive) rational number system has already been developed by a rigorous development from Peano's axioms. To this end, we shall presume that pages 1-43 of Landau have already been done. Assuming this, we shall refer to Landau whenever a reference to the rational number system is needed.

Our Definition and Theorem numbers will correspond with Landau. That is, our Definition and Theorem numbers will begin where Landau left off on page 43. Thus pages 1-43 of Landau and this paper will form a complete construction of the real numbers from Peano's axioms.

We shall use Landau's notation whenever possible; that is, we use small italic letters for integers, capital italic letters for rational numbers, etc. And whenever a theorem from Landau is used we shall say, "Landau's Theorem 100", etc.

Also we shall assume that the reader is familiar with the meaning of a one-to-one correspondence, which plays a very important part in the completion of this paper.

To aid the reader with the notation used in the text, we shall give a complete table of notation in the introduction as an easy reference. The notation given in the table below will be used consistantly throughout the paper; that is, m and n will always stand for integers, etc.

The symbols listed below are followed by a brief statement of their meaning and by the number of the page on which they first appear.

A_1, A_2, A_m, A_n, N, E; rational number, p. 5

B_n, C_n, D_n; rational number, p. 6

M; rational number, p. 10

X, Y, R; rational number, p. 70

Z, A_{n_0}; rational number, p. 73, 74

A_{nm}; rational number, p. 93

$A_{p_1(E_1),1}$; rational number, p. 95

m, n, $p_1(E/2)$; integer, p. 5, 6, 7

x; integer, p. 70

ξ, η, ζ, υ; Cauchy sequence of rational numbers, p. 6

$\{A_n\}, \{B_n\}, \{C_n\}, \{D_n\}$; Cauchy sequence of rational numbers, p. 5, 6

$\{\alpha_n\}, \{\beta_n\}$; Cauchy sequence of secs, p. 87

$\sim$; tantamount, p. 6

$\not\sim$; not tantamount, p. 6

$\gtrsim$; greater than or tantamount to, p. 22

$\lesssim$; less than or tantamount to, p. 22

α, β, γ, δ ; sec, p. 52

α_n, β_n; sec, p. 87

α_0; limit sec, p. 89

$\alpha_n \to \alpha_0$; α_n converges to α_0, p. 89

R^o; constant sequence of rational numbers, p. 47

X^o, Y^o; constant sequences of rational numbers, p. 70

x^*, y^*; integral secs, p. 70, 71

X^*, Y^*; rational secs, p. 70

$\mathfrak{Z}^*$, $\mathfrak{M}^*$; sets of integral secs, p. 71,72

σ, τ; cut, p. 80

CHAPTER I

CAUCHY SEQUENCES

1

Definition and Tantamount

We shall be interested in defining the real numbers in terms of sequences of rational numbers. A sequence of rational numbers is an ordered set of rational numbers $A_1, A_2, \dots, A_n, \dots$ arranged in an ordered one-to-one correspondence with the natural numbers. The terms of the sequence are $A_1, A_2, \dots$. The nth term of the sequence is A_n. The sequence is frequently indicated by $\{A_n\}$.

Definition 28: A sequence of rational numbers $\xi = \{A_n\} = \{A_1, \dots, A_n, \dots\}$ is called a Cauchy sequence, if there is a rational number N such that $A_n \geq N$ $(n = 1, 2, \dots)$, and for every rational number E (E arbitrarily small), there is an integer $p(E)$ such that for every pair of integers $m, n > p(E)$ one of the following is true.

(1) $A_n = A_m$; or

(2) $A_n > A_m$ and $A_n - A_m < E$; or

(3) $A_n < A_m$ and $A_m - A_n < E$.

For the purposes of notation; let us assign the small Greek letters ξ, η, ζ, and ν to denote Cauchy sequences. The terms in the Cauchy sequences ξ, η, ζ, and ν, respectively, will always be denoted as follows:

$$\xi = \{A_n\} = \{A_1, \dots, A_n, \dots\},$$

$$\eta = \{B_n\} = \{B_1, \dots, B_n, \dots\},$$

$$\zeta = \{C_n\} = \{C_1, \dots, C_n, \dots\},$$

$$\nu = \{D_n\} = \{D_1, \dots, D_n, \dots\}.$$

Definition 29: Two Cauchy sequences ξ and η are tantamount (in symbols, $\xi \sim \eta$) if for every E, there is a $p(E)$ such that for all $n > p(E)$ one of the following is true.

(1) $A_n = B_n$; or

(2) $A_n > B_n$ and $A_n - B_n < E$; or

(3) $A_n < B_n$ and $B_n - A_n < E$.

Otherwise, $\xi \nsim \eta$ ($\nsim$ to be read "is not tantamount to").

Theorem 116: $\xi \sim \xi$.

Proof: Given any E, take $p(E)=1$, then for all $n > p(E)$

(1) $A_n = A_n$.

Theorem 117: If $\xi \sim \eta$, then $\eta \sim \xi$.

Proof: Suppose $\xi \sim \eta$; given any E, there exists a $p(E)$ such that for all $n>p(E)$

(1) $A_n = B_n$; or

(2) $A_n > B_n$ and $A_n - B_n<E$; or

(3) $A_n < B_n$ and $B_n - A_n<E$.

But these cases are equivalent, respectively, to

(1) $B_n = A_n$; or

(2) $B_n < A_n$ and $A_n - B_n<E$; or

(3) $B_n > A_n$ and $B_n - A_n<E$.

Therefore, $\eta\sim\xi$.

<u>Theorem 118</u>: If $\xi\sim\eta$, $\eta\sim\zeta$, then $\xi\sim\zeta$.

Proof: By hypothesis; $\xi\sim\eta$, $\eta\sim\zeta$; given any E there exists a $p_1(E/2)$ such that, for all $n>p_1(E/2)$

(1) $A_n = B_n$; or

(2) $A_n > B_n$ and $A_n - B_n<E/2$; or

(3) $A_n < B_n$ and $B_n - A_n<E/2<E$.

And there exists a $p_2(E/2)$ such that, for all $n>p_2(E/2)$

(1) $B_n = C_n$; or

(2) $B_n > C_n$ and $B_n - C_n<E/2<E$; or

(3) $B_n < C_n$ and $C_n - B_n<E/2<E$.

With the same E, pick $p(E)$ to be the maximum of $p_1(E/2)$ and $p_2(E/2)$, then for all $n>p(E)$ we have one of the following cases:

Case 1: $A_n = B_n$, $B_n = C_n$. By logical equality, $A_n = C_n$, satisfying Definition 29, (1).

Case 2: $A_n = B_n$, $B_n > C_n$ and $B_n - C_n < E$. $A_n = B_n$ gives $A_n > C_n$ and $A_n - C_n < E$, satisfying Definition 29, (2).

Case 3: $A_n = B_n$, $B_n < C_n$ and $C_n - B_n < E$. $A_n = B_n$ gives $A_n < C_n$ and $C_n - A_n < E$, satisfying Definition 29, (3).

Case 4: $A_n > B_n$ and $A_n - B_n < E$, $B_n = C_n$. $B_n = C_n$ gives $A_n > C_n$ and $A_n - C_n < E$, satisfying Definition 29, (2).

Case 5: $A_n < B_n$ and $B_n - A_n < E$, $B_n = C_n$. $B_n = C_n$ gives $A_n < C_n$ and $C_n - A_n < E$, satisfying Definition 29, (3).

Case 6: $A_n > B_n$ and $A_n - B_n < E/2$, $B_n > C_n$ and $B_n - C_n < E/2$. By Landau's Theorem 86, $A_n > C_n$.

$A_n = [\ (A_n - B_n) + (B_n - C_n)\ ] + C_n < (E/2 + E/2) + C_n = E + C_n$. Therefore, $A_n > C_n$ and $A_n - C_n < E$, satisfying Definition 29, (2).

Case 7: $A_n < B_n$ and $B_n - A_n < E/2$, $B_n < C_n$ and $C_n - B_n < E/2$. By Landau's Theorem 86, $A_n < C_n$.

$C_n = [(B_n - A_n) + (C_n - B_n)] + A_n < (E/2 + E/2) + A_n = E + A_n$. Therefore, $A_n < C_n$ and $C_n - A_n < E$, satisfying Definition 29, (3).

Case 8: $A_n > B_n$ and $A_n - B_n < E$, $B_n < C_n$ and $C_n - B_n < E$. The proof is by subcases;

1.) Suppose $A_n > C_n > B_n$. Since $A_n - B_n < E$, $C_n > B_n$; by adding and simplifying; $A_n = (A_n - B_n) + B_n < E + C_n$;

Therefore, $A_n > C_n$ and $A_n - C_n < E$, satisfying Definition 29, (2).

2.) Suppose $A_n = C_n > B_n$. Satisfies Definition 29, (1).

3.) Suppose $C_n > A_n > B_n$. Since $C_n - B_n < E$, $A_n > B_n$; by adding and simplifying; $C_n = (C_n - B_n) + B_n < E + A_n$;

Therefore, $A_n < C_n$ and $C_n - A_n < E$, satisfying Definition 29, (3).

Case 9: $A_n < B_n$ and $B_n - A_n < E$, $B_n > C_n$ and $B_n - C_n < E$. The proof is the same as for Case 8 with the inequality signs of the subcases reversed.

In every case, $\xi \sim \zeta$; thus the theorem is proved.

By Theorems 116 through 118, all Cauchy sequences fall into classes, in such a way that

$$\xi \sim \eta$$

if and only if ξ and η belong to the same class.

<u>Definition 30</u>: A sequence is said to be <u>bounded</u> if and only if there exist M, N such that $N \leq A_n \leq M$ for $n=1, 2, \ldots$. N is called a <u>lower bound</u> and M is called an <u>upper bound</u> for the sequence.

<u>Theorem 119</u>: Every Cauchy sequence is bounded.

<u>Proof</u>: Let $\{A_n\}$ be a Cauchy sequence, then if we choose $E=1$, there is a $p(1)$ such that for all $m,n>p(1)$ one of the following is true.

(1) $A_n = A_m$; or

(2) $A_n > A_m$ and $A_n - A_m < 1$; or

(3) $A_n < A_m$ and $A_m - A_n < 1$.

Then, for all $m,n>p(1)$ one of the following is true.

(1) $A_n = A_m$. Then $A_n < A_m < A_m + 1$; or

(2) $A_n > A_m$ and $A_n - A_m < 1$. Then

$A_n = (A_n - A_m) + A_m < A_m + 1$; or

(3) $A_n < A_m$ and $A_m - A_n < 1$. Then $A_n < A_m < A_m + 1$.

Hence all terms for $m,n>p(1)$ have $A_m + 1$ as an upper bound. N exists as a part of the definition of a Cauchy Sequence. Since the terms of the sequence are bounded for all $m,n>p(1)$, and since there are only a finite number of terms A_q with $q\leq p(1)$, the entire sequence has A_m+1 as an upper bound.

2

Ordering

Definition 31: ξ is greater than η (in symbols, $\xi>\eta$) if there exist numbers E and $p(E)$ such that for all $n>p(E)$, $A_n - B_n \geq E$.

Definition 32: ξ is less than η (in symbols, $\xi<\eta$) if there exist numbers E and $p(E)$ such that for all $n>p(E)$, $B_n - A_n \geq E$.

Theorem 120: If $\xi > \eta$, then $\eta < \xi$.

Proof: Suppose $\xi>\eta$; there exist E and $p(E)$ such that for all $n>p(E)$, $A_n - B_n \geq E$; which means, by Definition 32, that $\eta < \xi$.

Theorem 121: If $\xi < \eta$, then $\eta > \xi$.

Proof: Suppose $\xi<\eta$; there exist E and $p(E)$ such that for all $n>p(E)$, $B_n - A_n \geq E$; which means, by Definition 31, that $\eta > \xi$.

Theorem 122: For any given ξ, η, exactly one of $\xi>\eta$, $\xi\sim\eta$, $\xi<\eta$ is the case.

Proof: I.) $\xi>\eta$, $\xi\sim\eta$ are incompatible by Definition 29 and Definition 31.

II.) $\xi<\eta$, $\xi\sim\eta$ are incompatible by Definition 29 and Definition 32.

III.) If $\xi > \eta$, $\xi < \eta$, then there exist E_1 and $p_1(E_1)$ such that for all $n > p_1(E_1)$ it is true that $A_n - B_n \geq E_1$. Also there exist E_2 and $p_2(E_2)$ such that for all $n > p_2(E_2)$ it is true that $B_n - A_n \geq E_2$. Let E be the minimum of E_1 and E_2 and let $p(E)$ be the maximum of $p_1(E_1)$ and $p_2(E_2)$.

If $n > p(E)$, then $A_n - B_n \geq E$ and $B_n - A_n \geq E$. Thus $A_n > B_n$ and $B_n > A_n$; but this contradicts the Tricotomy Law (Landau's Theorem 81). Therefore we can have at most one of the three cases.

To show that at least one of $\xi > \eta$, $\xi \sim \eta$, or $\xi < \eta$ happens, suppose $\xi \not\sim \eta$; we will show that $\xi > \eta$ or $\xi < \eta$. Since $\xi \not\sim \eta$, there is an E_0 such that for every value of $p(E_0)$ there is an $n_0 > p(E_0)$ such that all of the following are true.

(1) $A_{n_0} \neq B_{n_0}$

(2) if $A_{n_0} > B_{n_0}$, then $A_{n_0} - B_{n_0} \geq E$

(3) if $A_{n_0} < B_{n_0}$, then $B_{n_0} - A_{n_0} \geq E$.

If $A_{n_0} > B_{n_0}$ happens an infinite number of times, we will show that there exist E and $p(E)$ such that for all $n > p(E)$ it is true that $A_n - B_n \geq E$. Similarly, if $A_{n_0} < B_{n_0}$ happens an infinite number of times, we will show that there exist E and $p(E)$ such that for all $n > p(E)$

it is true that $B_n - A_n \geq E$. First, if $A_{n_0} > B_{n_0}$, then since ξ and η are Cauchy sequences, with $E = E_0/3$ there exists a $p_1(E)$ such that for all $m, n > p_1(E)$ one of the following is true.

(1) $A_n = A_m$; or

(2) $A_n > A_m$ and $A_n - A_m < E$; or

(3) $A_n < A_m$ and $A_m - A_n < E$.

But since $A_{n_0} > B_{n_0}$ happens an infinite number of times, there must be a value $n_1 > p_1(E)$ for which $A_{n_1} > B_{n_1}$. With $m = n_1$, the Cauchy conditions for ξ become

(1) $A_n = A_{n_1}$; or

(2) $A_n > A_{n_1}$ and $A_n - A_{n_1} < E$; or

(3) $A_n < A_{n_1}$ and $A_{n_1} - A_n < E$.

Similarly there exists a $p_2(E)$ such that for all $m, n > p_2(E)$ one of the following is true.

(1) $B_n = B_m$; or

(2) $B_n > B_m$ and $B_n - B_m < E$; or

(3) $B_n < B_m$ and $B_m - B_n < E$.

But since $A_{n_0} > B_{n_0}$ happens an infinite number of times, there must be a value $n_2 > p_2(E)$ for which $A_{n_2} > B_{n_2}$. With $m = n_2$, the Cauchy conditions for η become

(1) $B_n = B_{n_2}$; or

(2) $B_n > B_{n_2}$ and $B_n - B_{n_2} < E$; or

(3) $B_n < B_{n_2}$ and $B_{n_2} - B_n < E$.

With n_0 the maximum of n_1 and n_2 and $p(E)$ the maximum of $p_1(E)$ and $p_2(E)$, then for all $n > p(E)$ the following cases arise.

Case 1: $A_n = A_{n_0}$, $B_n = B_{n_0}$. Certainly, $A_n - B_n \geq E$.

Case 2: $A_n = A_{n_0}$, $B_n > B_{n_0}$ and $B_n - B_{n_0} < E$. From the inequalities and the hypothesis: $A_n = A_{n_0} > B_n > B_{n_0}$. Adding and simplifying;

$$(A_{n_0} - B_{n_0}) + E_0/3 > E_0 + (B_n - B_{n_0}),$$

$$(A_{n_0} - B_{n_0}) > E_0 - E_0/3 + (B_n - B_{n_0}) = 2E_0/3 + (B_n - B_{n_0}),$$

Adding B_{n_0} and simplifying; $A_{n_0} > 2E_0/3 + B_n > E + B_n$, therefore, $A_n - B_n \geq E$.

Case 3: $A_{n_0} = A_n$, $B_n < B_{n_0}$ and $B_{n_0} - B_n < E$. From the hypothesis and the inequalities:

$$A_n = A_{n_0} > B_{n_0} > B_n.$$

Therefore, $A_n - B_n \geq E$.

Case 4: $A_n > A_{n_0}$ and $A_n - A_{n_0} < E$, $B_n = B_{n_0}$. From the hypothesis and the inequalities; $A_n > A_{n_0} > B_{n_0} = B_n$. Therefore, $A_n - B_n \geq E$.

Case 5: $A_n < A_{no}$ and $A_{no} - A_n < E$, $B_n = B_{no}$. From the hypothesis and the inequalities; $A_{no} > A_n > B_n = B_{no}$.

Adding;

$$(A_{no} - B_{no}) + 2E_0/3 > E_0 + (A_{no} - A_n),$$

Adding B_{no}, A_n and simplifying,

$$A_{no} + A_n \geq 2E_0/3 + A_{no} + B_{no} > E + A_{no} + B_{no},$$

Cancelling A_{no} and simplifying;

$$A_n > B_{no} + E = B_n + E,$$

Therefore, $A_n - B_n \geq E$.

Case 6: $A_n > A_{no}$ and $A_n - A_{no} < E$, $B_n > B_{no}$ and $B_n - B_{no} < E$.

From the hypothesis and the inequalities; $A_n > A_{no} > B_n > B_{no}$.

Adding;

$$(A_{no} - B_{no}) + E_0/3 > E_0 + (B_n - B_{no}),$$

Adding B_{no} and simplifying;

$$A_n > A_{no} > 2E_0/3 + B_n > E + B_n,$$

therefore, $A_n - B_n \geq E$.

Case 7: $A_n < A_{no}$ and $A_{no} - A_n < E$, $B_n < B_{no}$ and $B_{no} - B_n < E$. From the hypothesis and the inequalities; $A_{no} > A_n > B_{no} > B_n$.

Adding;

$$(A_{n_0} - B_{n_0}) + E_0/3 > (A_{n_0} - A_n) + E_0,$$

Adding A_n, B_{n_0} and simplifying;

$$(A_{n_0} + A_n) > 2E_0/3 + B_{n_0} + A_{n_0},$$

Cancelling A_{n_0} and simplifying;

$$A_n > E + B_{n_0} > E + B_n;$$

Therefore, $A_n - B_n > E$.

Case 8: $A_n > A_{n_0}$ and $A_n - A_{n_0} < E$, $B_n < B_{n_0}$ and $B_{n_0} - B_n < E$. From the hypothesis and the inequalities; $A_n > A_{n_0} > B_{n_0} > B_n$. Therefore, $A_n - B_n \geq E$.

Case 9: $A_n < A_{n_0}$ and $A_{n_0} - A_n < E$, $B_n > B_{n_0}$ and $B_n - B_{n_0} < E$. From the hypothesis and the inequalities; $A_{n_0} > A_n > B_n > B_{n_0}$.

Adding; $(A_{n_0} - B_{n_0}) + E_0/3 + E_0/3 > E_0 + (B_n - B_{n_0}) + (A_{n_0} - A_n)$, adding A_n, B_{n_0} and simplifying;

$$(A_{n_0} - B_{n_0}) + B_{n_0} + A_n + 2E_0/3 > E_0 + (B_n - B_{n_0}) + B_{n_0} + (A_{n_0} - A_n) + A_n,$$

$A_{n_0} + A_n > E_0/3 + B_n + A_{n_0}$; Therefore cancelling A_{n_0}, $A_n - B_n \geq E$.

In every case $A_n - B_n \geq E$, hence $\xi > \eta$.

By a similar proof, if $A_{n_0} < B_{n_0}$ happens an infinite number of times, then $\xi<\eta$. Hence for all $n>p(E)$ at least one of $\xi>\eta$, $\xi\sim\eta$, or $\xi<\eta$ is the case.

<u>Theorem 123</u>: If $\xi>\eta$, $\xi\sim\zeta$, $\eta\sim\nu$, then $\zeta>\nu$.

<u>Remark</u>: Thus if a Cauchy sequence of one class is greater than a Cauchy sequence of another class, then the same will be true for all pairs of representations of the two classes.

<u>Proof</u>: Suppose $\xi>\eta$, $\xi\sim\zeta$, $\eta\sim\nu$. Since $\xi>\eta$, there exist E_1 and $p_1(E_1)$ such that for all $n>p_1(E_1)$, $A_n - B_n \geq E_1$. Since $\xi\sim\zeta$, there exists a $p_2(E_1/3)$ such that for all $n>p_2(E_1/3)$ it is true that

(1) $A_n = C_n$; or

(2) $A_n > C_n$ and $A_n - C_n < E_1/3$; or

(3) $A_n < C_n$ and $C_n - A_n < E_1/3$.

And since $\eta\sim\nu$, there exists a $p_3(E_1/3)$ such that for all $n>p_3(E_1/3)$ it is true that

(1) $B_n = D_n$; or

(2) $B_n > D_n$ and $B_n - D_n < E/3$; or

(3) $B_n < D_n$ and $D_n - B_n < E/3$.

Let $p(E)$ be the maximum of $p_1(E_1)$, $p_2(E_1/3)$, and $p_3(E_1/3)$ and let $E = E_1/3$. Then for all $n>p(E)$ one of the following cases arises.

Case 1: $A_n = C_n$, $B_n = D_n$. Since $\xi>\eta$, $A_n - B_n \geq E$, thus substituting equals, certainly $C_n - D_n \geq E$.

Case 2: $A_n = C_n$, $B_n > D_n$ and $B_n - D_n < E$. From the hypothesis and the inequalities; $A_n = C_n > B_n > D_n$; therefore, $C_n - D_n \geq E$.

Case 3: $A_n = C_n$, $B_n < D_n$ and $D_n - B_n < E$. From the hypothesis and the inequalities: $A_n = C_n > D_n > B_n$;

Adding and simplifying;

$$(C_n - B_n) + E_1/3 \geq E_1 + (D_n - B_n),$$

$$(C_n - B_n) \geq E_1 - E_1/3 + (D_n - B_n) = 2E_1/3 + (D_n - B_n),$$

Adding B_n; $C_n \geq D_n + 2E_1/3 > E + D_n$,

therefore, $C_n - D_n > E$.

Case 4: $A_n > C_n$ and $A_n - C_n < E$, $B_n = D_n$. From the hypothesis and the inequalities: $A_n > C_n > B_n = D_n$.

Adding,

$$(A_n - B_n) + E_1/3 > E_1 + (A_n - C_n),$$

Adding B_n, C_n and simplifying;

$$(A_n + C_n) > 2E_1/3 + A_n + B_n > E + A_n + B_n,$$

Cancelling A_n: $C_n > E + B_n = E + D_n$,

therefore, $C_n - D_n < E$.

Case 5: $A_n < C_n$ and $C_n - A_n < E$, $B_n = D_n$. From the hypothesis and the inequalities: $C_n > A_n > B_n = D_n$.

Certainly $C_n - D_n \geq E$.

Case 6: $A_n > C_n$ and $A_n - C_n < E$, $B_n > D_n$ and $B_n - D_n < E$. From the hypothesis and the inequalities:

$A_n > C_n > B_n > D_n$.

Adding,

$$(A_n - B_n) + E_1/3 > E_1 + (A_n - C_n),$$

Adding B_n, C_n and simplifying,

$$A_n + C_n > 2E_1/3 + A_n + B_n > E + A_n + B_n,$$

Cancelling A_n and simplifying;

$$C_n > B_n + E > D_n + E,$$

therefore, $C_n - D_n > E$.

Case 7: $A_n < C_n$ and $C_n - A_n < E$, $B_n < D_n$ and $D_n - B_n < E$.

From the hypothesis and the inequalities; $C_n > A_n > D_n > B_n$.

Adding,

$$(A_n - B_n) + E_1/3 > E_1 + (D_n - B_n) > E + (D_n - B_n),$$

Add B_n; $A_n > E + D_n$, and from the inequalities,

$$C_n > A_n > E + D_n.$$

Therefore, $C_n - D_n < E$.

Case 8: $A_n > C_n$ and $A_n - C_n < E$, $B_n < D_n$ and $D_n - B_n < E$. From the hypothesis and the inequalities; $A_n > C_n > D_n > B_n$.

Adding, $(A_n - B_n) + E_1/3 + E_1/3 > E_1 + (D_n - B_n) + (A_n - C_n)$,

adding B_n, C_n and simplifying;

$(A_n - B_n) + B_n + C_n + 2E_1/3 > E_1 + (D_n - B_n) + B_n + (A_n - C_n) + C_n$.

$A_n + C_n < (E_1/3 = E) + D_n + A_n$; therefore cancelling A_n,

$$C_n - D_n > E.$$

Case 9: $A_n < C_n$ and $C_n - A_n < E$, $B_n > D_n$ and $B_n - D_n < E$.

From the hypothesis and the inequalities: $C_n > A_n > B_n > D_n$.

Certainly $C_n - D_n \geq E$.

In every case $C_n - D_n \geq E$, thus by Definition 31, $\xi > \upsilon$ and the theorem is proved.

<u>Theorem 124</u>: If $\xi < \eta$, $\xi \sim \zeta$, $\eta \sim \upsilon$, then $\zeta < \upsilon$.

<u>Remark</u>: Thus if a Cauchy sequence of one class is less than a Cauchy sequence of another class, then the same will be true for all pairs of representatives of the two classes.

<u>Proof</u>: By Theorem 121, we have

$$\eta > \xi;$$

since $\eta \sim \upsilon$, $\xi \sim \zeta$

we then have by Theorem 123 that $\upsilon > \zeta$

so that, by Theorem 120, $\zeta < \upsilon$.

<u>Definition</u> <u>33</u>: $\xi \gtrsim \eta$ means $\xi > \eta$ or $\xi \sim \eta$. ($\gtrsim$ to be read "<u>is greater than or tantamount to</u>".)

<u>Definition</u> <u>34</u>: $\xi \lesssim \eta$ means $\xi < \eta$ or $\xi \sim \eta$. ($\lesssim$ to be read "<u>is less than or tantamount to</u>".)

<u>Theorem</u> <u>125</u>: If $\xi \gtrsim \eta$, $\xi \sim \zeta$, $\eta \sim \upsilon$; then $\zeta \gtrsim \upsilon$.

<u>Proof</u>: Theorem 123 if $>$ holds in the hypothesis; otherwise, we have $\xi \sim \eta \sim \zeta \sim \upsilon$.

<u>Theorem</u> <u>126</u>: If $\xi \lesssim \eta$, $\xi \sim \zeta$, $\eta \sim \upsilon$; then $\zeta \lesssim \upsilon$.

<u>Proof</u>: Theorem 124 if $<$ holds in the hypothesis; otherwise, we have $\xi \sim \eta \sim \zeta \sim \upsilon$.

<u>Theorem</u> <u>127</u>: If $\xi \gtrsim \eta$, then $\eta \lesssim \xi$.

<u>Proof</u>: Theorem 117 for tantamount, Theorem 120 for "greater than".

<u>Theorem</u> <u>128</u>: If $\xi \lesssim \eta$, then $\eta \gtrsim \xi$.

<u>Proof</u>: Theorem 117 for tantamount, Theorem 121 for "less than".

<u>Theorem</u> <u>129</u>: If $\xi < \eta$, $\eta < \xi$, then $\xi < \zeta$. (Transitivity of Ordering.)

<u>Proof</u>: $\xi < \eta$ means there exists an E_1 and a $p_1(E_1)$ such that for all $n > p_1(E_1)$, $B_n - A_n \geq E_1$. $\eta > \zeta$ means there exists an E_2 and a $p_2(E_2)$ such that for all $n > p_2(E_2)$, $C_n - B_n \geq E_2$. Let E be the minimum of E_1 and E_2, and let $p(E)$ be the maximum of $p_1(E_1)$ and $p_2(E_2)$.

For all $n > p(E)$, we find by adding inequalities:

$$(B_n - A_n) + (C_n - B_n) \geq E + E = 2E,$$

$$(B_n - A_n) + A_n + (C_n - B_n) \geq 2E + A_n.$$

Simplifying;

$C_n \geq A_n + 2E$, which gives $C_n - A_n \geq 2E > E$, thus $\zeta > \xi$, and the theorem is proved.

<u>Theorem 130</u>: If $\xi \lesssim \eta$, $\eta < \zeta$ or $\xi < \eta$, $\eta \lesssim \zeta$ then $\xi < \zeta$.

<u>Proof</u>: Follows from Theorem 124 if a tantamount sign holds in the hypothesis; otherwise from Theorem 129.

<u>Theorem 131</u>: If $\xi \lesssim \eta$, $\eta \lesssim \zeta$, then $\xi \lesssim \zeta$.

<u>Proof</u>: Theorem 118 if two tantamount signs hold in the hypothesis, otherwise Theorem 130 does it.

3

Addition

Theorem 132: Let ξ and η be Cauchy sequences. Then the sequence whose nth term is $(A_n + B_n)$, is itself a Cauchy sequence; that is, there is an N such that $(A_n + B_n) \geq N$ for all n; and given any E, there exists a $p(E)$ such that for all $m,n>p(E)$ it is true that

(1) $(A_n+B_n) = (A_m+ B_m)$; or

(2) $(A_n+ B_n) > (A_m+ B_m)$ and

$(A_n+ B_n) - (A_m+ B_m) < E$; or

(3) $(A_n+ B_n) < (A_m+ B_m)$ and

$(A_m+ B_m) - (A_n+ B_n) < E$.

Proof: Let ξ and η be the given Cauchy sequences. There exists an N such that $(A_n+ B_n) \geq N$ for all n, for since $\{A_n\}$ is a Cauchy sequence, there exists an N_1 such that $A_n \geq N_1$ for all n and certainly $(A_n+ B_n) \geq N_1 = N$ for all n. Then for every E there exists a $p_1(E/2)$ such that for all $m,n>p_1(E/2)$ it is true that

(1) $A_n = A_m$; or

(2) $A_n > A_m$ and $A_n- A_m< E/2$; or

(3) $A_n < A_m$ and $A_m- A_n< E/2$;

and there exists a $p_2(E/2)$ such that for all $m,n>p_2(E/2)$ it is true that

(1) $B_n = B_m$; or

(2) $B_n > B_m$ and $B_n - B_m < E/2$; or

(3) $B_n < B_m$ and $B_m - B_n < E/2$.

Let $p(E)$ be the maximum of $p_1(E/2)$ and $p_2(E/2)$, then for all $m,n>p(E)$ one of the following cases arises:

Case 1: $A_n = A_m$, $B_n = B_m$. Then $(A_n + B_n) = (A_m + B_m)$, satisfying Definition 28, (1).

Case 2: $A_n = A_m$, $B_n > B_m$ and $B_n - B_m < E/2 < E$. Adding inequalities we find, $(A_n + B_n) > (A_m + B_m)$.

$$(A_n + B_n) = \{[(A_n + B_m) - (A_m + B_m)]+[(A_m + B_m)+(B_n - B_m)]\}$$
$$= \{[(A_n + B_m) - (A_m + B_m)+(A_m + B_m)] + (B_n - B_m)\} = (A_n + B_m) + (B_n - B_m) = (A_n + B_n) < E + (A_m + B_m).$$

Therefore, $(A_n + B_n) - (A_m + B_m) < E$, satisfying Definition 28, (2).

Case 3: $A_n = A_m$, $B_n < B_m$ and $B_m - B_n < E/2 < E$. We find, adding inequalities, that $(A_n + B_n) < (A_m + B_m)$.

$$(A_m + B_m) = \{[(A_m + B_n) - (A_n + B_n)]+[(A_n + B_n)+(B_m - B_n)]\}$$
$$= \{[(A_m + B_n) - (A_n + B_n) + (A_n + B_n)] + (B_m - B_n)\} =$$
$$= (A_m + B_n) + (B_m - B_n) = (A_m + B_m) < E + (A_n + B_n).$$

Therefore, $(A_m + B_m) - (A_n + B_n) < E$, satisfying Definition

28, (3).

Case 4: $A_n > A_m$ and $A_n - A_m < E/2 < E$, $B_n = B_m$. The proof of Case 4 is exactly the proof of Case 2 with the A's and B's interchanged.

Case 5: $A_n < A_m$ and $A_m - A_n < E/2 < E$, $B_n = B_m$. The proof of Case 5 is exactly the proof of Case 3 with the A's and B's interchanged.

Case 6: $A_n > A_m$ and $A_n - A_m < E/2$, $B_n > B_m$ and $B_n - B_m < E/2$. We find, adding inequalities, that

$(A_n + B_n) > (A_m + B_m)$.

Adding; $(A_n - A_m) + A_m + (B_n - B_m) + B_m = (A_n + B_n) < E/2 +$

$$E/2 + (A_m + B_m) = E + (A_m + B_m).$$

Therefore, $(A_n + B_n) - (A_m + B_m) < E$, satisfying Definition 28, (2).

Case 7: $A_n < A_m$ and $A_m - A_n < E/2$, $B_n < B_m$ and $B_m - B_n < E/2$. We find, adding inequalities, that

$$(A_n + B_n) < (A_m + B_m).$$

Adding and simplifying;

$$(A_m - A_n) + A_n + (B_m - B_n) + B_n = (A_m + B_m) < E/2 + E/2 +$$

$$(A_n + B_n) = E + (A_n + B_n).$$

Therefore, $(A_m + B_m) - (A_n + B_n) < E$, satisfying Definition 28, (3).

Case 8: $A_n > A_m$ and $A_n - A_m < E/2 < E$, $B_n < B_m$ and $B_m - B_n < E/2 < E$; from the Tricotomy Law it is true that

1.) $(A_n + B_n) = (A_m + B_m)$; or

2.) $(A_n + B_n) > (A_m + B_m)$; or

3.) $(A_n + B_n) < (A_m + B_m)$.

We must satisfy the inequalities of Definition 28 for subcases 2 and 3 above.

Subcase 1: $(A_n + B_n) = (A_m + B_m)$, satisfies Definition 28, (1).

Subcase 2: $(A_n + B_n) > (A_m + B_m)$, hence adding and simplifying; $(A_n + B_n) < (A_n + B_m) =$

$$=\left\{[(A_n + B_n) - (A_m + B_m)] + [(A_m + B_m) + (B_m - B_n)]\right\} =$$

$$=\left\{[(A_n + B_n) - (A_m + B_m) + (A_m + B_m)] + (B_m - B_n)\right\} =$$

$$= (A_n + B_n) + (B_m - B_n) = (A_n + B_m) < E + (A_m + B_m);$$

Therefore, $(A_n + B_n) - (A_m + B_m) < E$, satisfying Definition 28, (2).

Subcase 3: $(A_n + B_n) < (A_m + B_m)$, hence adding and simplifying; $(A_m + B_m) < (A_n + B_m) =$

$$=\left\{[(A_m + B_m) - (A_n + B_n)] + [(A_n + B_n) + (A_n - A_m)]\right\} =$$

$$=\left\{[(A_m + B_m) - (A_n + B_n) + (A_n + B_n)] + (A_n - A_m)\right\} =$$

$= (A_m + B_m) + (A_n - A_m) = (A_n + B_m) < E + (A_n + B_n)$;

therefore, $(A_m+B_m)-(A_n+B_n)<E$, satisfying Definition 28,(3).

Case 9: $A_n < A_m$ and $A_m - A_n < E/2 < E$, $B_n > B_m$ and $B_n - B_m < E/2 < E$; the Tricotomy Law again gives three subcases for which the proof is analogous to the proof of Case 8 and its three subcases.

In each case, $(A_n + B_n)$ is a Cauchy sequence; thus the theorem is proved.

<u>Definition 35</u>: The Cauchy sequence constructed in Theorem 132 is denoted by $\xi + \eta$ and is called the <u>sum</u> of ξ and η, or the Cauchy sequence obtained by addition of ξ to η.

<u>Theorem 133</u>: If $\xi \sim \eta$, $\zeta \sim \nu$, then $\xi + \zeta \sim \eta + \nu$.

<u>Remark</u>: The class of the sum thus depends only on the classes to which the "summands" belong.

<u>Proof</u>: Suppose $\xi \sim \eta$, $\zeta \sim \nu$; Theorem 132 and Definition 35 show that each of $\xi + \zeta$ and $\eta + \nu$ is a Cauchy sequence. We must show that given any E, there exists a $p(E)$ such that for all $n>p(E)$ it is true that

(1) $(A_n + C_n) = (B_n + D_n)$; or

(2) $(A_n + C_n) > (B_n + D_n)$ and

$(A_n + C_n) - (B_n + D_n) < E$; or

(3) $(A_n + C_n) < (B_n + D_n)$ and

$(B_n + D_n) - (A_n + C_n) < E$.

Given any E, there exists a $p_1(E/2)$ such that for all $n > p_1(E/2)$ it is true that

(1) $A_n = B_n$; or

(2) $A_n > B_n$ and $A_n - B_n < E/2$; or

(3) $A_n < B_n$ and $B_n - A_n < E/2$;

and there exists a $p_2(E/2)$ such that for all $n > p_2(E/2)$ it is true that

(1) $C_n = D_n$; or

(2) $C_n > D_n$ and $C_n - D_n < E/2$; or

(3) $C_n < D_n$ and $D_n - C_n < E/2$.

Let $p(E)$ be the maximum of $p_1(E/2)$ and $p_2(E/2)$, then for all $n > p(E)$ one of the following cases arises;

Case 1: $A_n = B_n$, $C_n = D_n$.

Case 2: $A_n = B_n$, $C_n > D_n$ and $C_n - D_n < E/2$.

Case 3: $A_n = B_n$, $C_n < D_n$ and $D_n - C_n < E/2$.

Case 4: $A_n > B_n$ and $A_n - B_n < E/2$, $C_n = D_n$.

Case 5: $A_n < B_n$ and $B_n - A_n < E/2$, $C_n = D_n$.

Case 6: $A_n > B_n$ and $A_n - B_n < E/2$, $C_n > D_n$ and $C_n - D_n < E/2$.

Case 7: $A_n < B_n$ and $B_n - A_n < E/2$, $C_n < D_n$ and $D_n - C_n < E/2$.

Case 8: $A_n > B_n$ and $A_n - B_n < E/2$, $C_n < D_n$ and $D_n - C_n < E/2$.

Case 9: $A_n < B_n$ and $B_n - A_n < E/2$, $C_n > D_n$ and $C_n - D_n < E/2$.

By a proof analogous to the proof of Theorem 132, we could show that one of the three conditions of Definition 29 for tantamount is satisfied, which would complete the proof; the details are omitted.

__Theorem 134:__ (Commutative Law of Addition):

$$\xi + \eta \sim \eta + \xi.$$

__Proof:__ Given any E, take $p(E) = 1$, then for all $n > p(E)$ it is true that

(1) $A_n + B_n = B_n + A_n$.

Hence, $\xi + \eta \sim \eta + \xi$.

__Theorem 135:__ (Associative Law of Addition):

$$(\xi + \eta) + \zeta \sim \xi + (\eta + \zeta).$$

__Proof:__ Given any E, take $p(E)=1$, then for all $n > p(E)$ it is true that

(1) $(A_n + B_n) + C_n = A_n + (B_n + C_n)$.

Hence, $(\xi + \eta) + \zeta \sim \xi + (\eta + \zeta)$.

<u>Theorem 136</u>: $\xi + \eta > \xi$.

<u>Proof</u>: Choose $E \leq B_n$ $(n = 1, 2, \ldots)$ and let $p(E)=1$, then for all $n>p(E)$ it is true that $(A_n + B_n) - A_n \geq E$.

<u>Theorem 137</u>: If $\xi > \eta$, then $\xi + \zeta > \eta + \zeta$.

<u>Proof</u>: There exist E and $p(E)$ such that for all $n>p(E)$ it is true that $A_n - B_n \geq E$. Hence

$A_n \geq E + B_n$.

Add C_n obtaining;

$$(A_n + C_n) \geq E_n + (B_n + C_n), \text{ so that}$$

$$(A_n + C_n) - (B_n + C_n) \geq E.$$

Therefore, $\xi + \zeta > \eta + \zeta$.

<u>Theorem 138</u>: If $\xi > \eta$, or $\xi \sim \eta$, or $\xi < \eta$ then $\xi + \zeta > \eta + \zeta$, or $\xi + \zeta \sim \eta + \zeta$, or $\xi + \zeta < \eta + \zeta$, respectively.

<u>Proof</u>: The first part is Theorem 137; the second is contained in Theorem 133; the third follows from the first since, if $\xi < \eta$, we find successively

$$\eta > \xi,$$
$$\eta + \zeta > \xi + \zeta,$$
$$\xi + \zeta < \eta + \zeta.$$

<u>Theorem 139</u>: If $\xi + \zeta > \eta + \zeta$, or $\xi + \zeta \sim \eta + \zeta$,

or $\xi + \zeta < \eta + \zeta$, then $\xi > \eta$, or $\xi \sim \eta$, or $\xi < \eta$, respectively.

<u>Proof</u>: Follows from Theorem 138, since the three cases, in both instances, are mutually exclusive and exhaust all possibilities.

<u>Theorem 140</u>: If $\xi > \eta$, $\zeta > \upsilon$, then $\xi + \zeta > \eta + \upsilon$.

<u>Proof</u>: By Theorem 137 we have

$$\xi = \zeta > \eta + \zeta \text{ and}$$

$$\eta + \zeta \sim \zeta + \eta > \upsilon + \eta \sim \eta + \upsilon, \text{ hence } \xi + \zeta > \eta + \upsilon.$$

<u>Theorem 141</u>: If $\xi \gtrsim \eta$, $\zeta > \upsilon$, or $\xi > \eta$, $\zeta \gtrsim \upsilon$, then $\xi + \zeta > \eta + \upsilon$.

<u>Proof</u>: Follows from Theorem 133 and 137 if the tantamount signs hold in the hypothesis; otherwise from Theorem 140.

<u>Theorem 142</u>: If $\xi \gtrsim \eta$, $\zeta \gtrsim \upsilon$, then $\xi + \zeta \gtrsim \eta + \upsilon$.

<u>Proof</u>: Follows from Theorem 133 if two tantamount signs hold in the hypothesis; otherwise from Theorem 141.

<u>Theorem 143</u>: If $\xi > \eta$

then $\eta + \upsilon \sim \xi$

has a solution υ. If υ_1 and υ_2 are solutions, then

$$\upsilon_1 \sim \upsilon_2.$$

<u>Remark</u>: If $\xi \lesssim \eta$

there does not exist a solution, by Theorem 136.

<u>Proof</u>: The second assertion of Theorem 143 is an immediate consequence of Theorem 139; for if

$$\eta + \nu_1 \sim \eta + \nu_2$$

then, by Theorem 139, $\nu_1 \sim \nu_2$.

The existence of a ν (the first assertion of Theorem 143) is proved as follows. Since $\xi > \eta$, there exist E_1 and $p(E_1)$ such that, for all $n>p(E_1)$, $A_n - B_n \geq E_1$. Define a sequence $\nu = \{D_n\}$ by

$D_n = 1$ for all $n \leq p(E_1)$,

$D_n = A_n - B_n$ for all $n>p(E_1)$.

We must show that ν is a Cauchy sequence; that is, there is an N such that $D_n \geq N$ for all n and given any E, there exists a $p(E)$ such that for all $m,n>p(E)$ it is true that

(1) $D_n = D_m$; or

(2) $D_n > D_m$ and $D_n - D_m < E$; or

(3) $D_n < D_m$ and $D_m - D_n < E$.

To find an N such that $D_n \geq N$ for all n, consider 1 and E_1. Since $A_n - B_n \geq E_1$ for all $n>p(E_1)$, and $D_n = 1$ for all $n \leq p(E_1)$, we may take N to be the minimum of 1 and E.

Since ξ and η are Cauchy sequences, given any E, there exists a $p_1(E)$ such that for all $m,n>p_1(E)$ it is true that

(1) $A_n = A_m$; or

(2) $A_n > A_m$ and $A_n - A_m < E$; or

(3) $A_n < A_m$ and $A_m - A_n < E$;

and there exists a $p_2(E)$ such that for all $m,n > p_2(E)$ it is true that

(1) $B_n = B_m$; or

(2) $B_n > B_m$ and $B_n - B_m < E$; or

(3) $B_n < B_m$ and $B_m - B_n < E$.

Pick $p(E)$ to be the maximum of $p_1(E)$, $p_2(E)$, and $p(E_1)$. Thus with $D_n = (A_n - B_n)$, $D_m = (A_m - B_m)$, the following cases arise.

Case 1: $A_n = A_m$, $B_n = B_m$. Since $p(E) \geq p(E_1)$, $A_n - B_n \geq E$, hence $A_n > B_n$ and therefore, $(A_n - B_n) = (A_m - B_m)$, satisfying Definition 28, (1).

Before going on to Case 2, let us prove the following Lemma which will be useful in proving Cases 2 through 9.

<u>Lemma 1</u>: If $X > Y$, $Y > Z$, then $(X-Y)+(Y-Z)=(X-Z)$.

<u>Proof</u>: $X+Y = Y+(X-Y) + [(Y-Z) + Z] =$

$= [(X-Z) + Z] + Y = X + Y;$

$Y + (X-Y) + [(Y-Z) + Z] = [(X-Z) + Z] + Y,$

Cancelling Y, Z, $(X-Y) + (Y-Z) = (X-Z)$.

Case 2: $A_n = A_m$, $B_n > B_m$ and $B_n - B_m < E$, $A_n - B_n \geq E$.

Since $A_n + B_n > A_m + B_m$,

$$A_m + B_m = \left\{[(A_m - B_n) + B_n] + B_m\right\} < \left\{[(A_n - B_m) + B_m] + B_n\right\} =$$
$$= A_n + B_n,$$

then $[(A_m - B_n) + (B_n + B_m)] < [(A_m - B_m) + (B_m + B_n)]$,

cancelling $(B_n + B_m)$; $(A_m - B_n) = (A_n - B_n) < (A_m - B_m)$.

By Lemma 1, we find

$$(A_n - B_n) + (B_n - B_m) = A_n - B_m = A_m - B_m < (A_n - B_n) + E.$$

Therefore, $(A_m - B_m) > (A_n - B_n)$ and $(A_m - B_m) - (A_n - B_n) < E$,

satisfying Definition 28, (3).

Case 3: $A_n = A_m$, $B_n < B_m$ and $B_m - B_n < E$, $A_n - B_n \geq E$.

Since $A_n + B_n < A_m + B_m$,

$$(A_n + B_n) + \left\{[(A_n - B_m) + B_m] + B_n\right\} < \left\{[(A_n - B_n) + B_n] + B_m\right\} = A_m + B_m,$$

then $[(A_n - B_m) + (B_m + B_n)] < [(A_n - B_n) + (B_n + B_m)]$,

Cancelling $(B_m + B_n)$; $(A_m - B_m) = (A_m - B_m) < (A_n - B_n)$.

By Lemma 1, we find

$$(A_n - B_m) + (B_m - B_n) = A_n - B_n < (A_n - B_m) + E = (A_m - B_m) + E.$$

Therefore, $(A_n - B_n) > (A_m - B_m)$ and $(A_n - B_n) - (A_m - B_m) < E$, satisfying Definition 28, (2).

Case 4: $A_n > A_m$ and $A_n - A_m < E$, $B_n = B_m$, $A_n - B_n \geq E$.

The proof of Case 4 is exactly the proof of Case 2 with the A's and B's interchanged.

Case 5: $A_n < A_m$ and $A_m - A_n < E$, $B_n = B_m$, $A_n - B_n \geq E$.

The proof of Case 5 is exactly the proof of Case 3 with the A's and B's interchanged.

Case 6: $A_n > A_m$ and $A_n - A_m < E$, $B_n > B_m$ and $B_n - B_m < E$, $A_n - B_n \geq E$. From the Tricotomy Law it follows that

(1) $(A_n - B_n) = (A_m - B_m)$; or

(2) $(A_n - B_n) > (A_m - B_m)$; or

(3) $(A_n - B_n) < (A_m - B_m)$.

We must show that the inequalities of Definition 28 are satisfied in the subcases 2 and 3 above.

Subcase 1: $(A_n - B_n) = (A_m - B_m)$. Definition 28, (1) is satisfied.

Subcase 2: $(A_n - B_n) > (A_m - B_m)$.

By Lemma 1, we find

$$(A_n - A_m) + (A_m - B_m) = (A_n - B_m) < E + (A_m - B_m).$$

But from the proof of Case 2, $A_n - B_n < A_n - B_m$.

Therefore, $(A_n - B_n) > (A_m - B_m)$ and $(A_n - B_n)-(A_m - B_m) < E$, satisfying Definition 28, (2).

Subcase 3: $(A_n - B_n) < (A_m - B_m)$.

By Lemma 1, we find

$$(A_m - B_m) \leq (A_n - B_m)=(A_n - B_n)+(B_n - B_m) < E + (A_n - B_n).$$

Therefore, $(A_n - B_n)<(A_m - B_m)$ and $(A_m - B_m) - (A_n - B_n) < E$, satisfying Definition 28, (3).

Case 7: $A_n < A_m$ and $A_m - A_n < E$, $B_n < B_m$ and $B_m - B_n < E$, $A_n - B_n \geq E$.

Case 8: $A_n > A_m$ and $A_n - A_m < E$, $B_n < B_m$ and $B_m - B_n < E$, $A_n - B_n \geq E$.

Case 9: $A_n < A_m$ and $A_m - A_n < E$, $B_n < B_m$ and $B_m - B_n < E$, $A_n - B_n \geq E$.

The proofs of Cases 7, 8, and 9 will be omitted since they are analogous to the proof of Case 6.

This completes the proof that ν is a Cauchy sequence. We must show that ν satisfies the relation $\eta + \nu \sim \xi$. That is, given any E, there exists a $p(E)$ such that for all $n>p(E)$,

(1) $B_n + D_n = A_n$; or

(2) $B_n + D_n > A_n$ and $(B_n + D_n) - A_n < E$; or

(3) $B_n + D_n < A_n$ and $A_n - (B_n + D_n) < E$.

Given any E, let $p(E)=p(E_1)$ [$p(E_1)$ is described at the beginning of the proof of this theorem], then for all $n>p(E)$ it is true that

(1) $B_n + D_n = B_n + (A_n - B_n) = A_n$.

Thus we have $\mathcal{V}$ as a solution of our relation and the theorem is proved.

Definition 36: The $\mathcal{V}$ of Theorem 143 is denoted by $\xi - \eta$, or the Cauchy sequence obtained by subtraction of η from ξ.

Thus if $\xi \sim \eta + \mathcal{V}$, then $\mathcal{V} \sim \xi - \eta$.

4

Multiplication

Theorem 144: Let ξ and η be Cauchy sequences. Then the sequence whose nth term is A_nB_n, is a Cauchy sequence; that is, there is an N such that $A_nB_n \geq N$ for all n; and given any E, there exists a p(E) such that for all $m,n>p(E)$ it is true that

(1) $A_nB_n = A_mB_m$; or

(2) $A_nB_n > A_mB_m$ and $A_nB_n - A_mB_m < E$; or

(3) $A_nB_n < A_mB_m$ and $A_mB_m - A_nB_n < E$.

Proof: Let ξ and η be the Cauchy sequences. There exists an N such that $A_nB_n \geq N$ for all n; for since ξ, η are Cauchy sequences, it is true that $A_n \geq N_1$, $B_n \geq N_2$ for all n; multiplying inequalities, $A_nB_n \geq N_1N_2 = N$ for all n. Also ξ and η have upper bounds by Theorem 119, hence $A_n \leq M_1$, $B_n \leq M_2$ for all n. Let M be the maximum of M_1, M_2, and M_1M_2 such that $A_n \leq M$, $B_n \leq M$ for all n.

Then for every E, there exists a $p_1(E/2M)$ such that for all $m,n>p_1(E/2M)$ it is true that

(1) $A_n = A_m$; or

(2) $A_n > A_m$ and $A_n - A_m < E/2M$; or

(3) $A_n < A_m$ and $A_m - A_n < E/2M$;

and there exists a $p_2(E/2M)$ such that for all $m,n > p_2(E/2M)$ it is true that

(1) $B_n = B_m$; or

(2) $B_n > B_m$ and $B_n - B_m < E/2M$; or

(3) $B_n < B_m$ and $B_m - B_n < E/2M$.

Let $p(E)$ be the maximum of $p_1(E/2M)$ and $p_2(E/2M)$, then for all $m,n > p(E)$ one of the following cases arises;

Case 1: $A_n = A_m$, $B_n = B_m$. Then $A_nB_n = A_mB_m$, satisfying Definition 28, (1).

Before going on to Case 2, let us prove the following Lemma which will be useful in proving Cases 2 through 9.

Lemma 2: If $Y > Z$, then $X(Y-Z) = XY - XZ$.

Proof: If $Y > Z$, then $(Y-Z) + Z = Y$, so that, by Landau's Theorem 104,

$$X(Y-Z) + XZ = XY,$$

$$X(Y-Z) = XY - XZ.$$

Case 2: $A_n = A_m$, $B_n > B_m$ and $B_n - B_m < E/2M$.

By multiplying inequalities, $A_nB_n > A_mB_m$.

Since $A_n \leq M$ for all n, by Lemma 2 we find

$$A_n(B_n - B_m) = A_nB_n - A_mB_m < ME/2M = E/2 < E.$$

Therefore, $A_nB_n > A_mB_m$ and $A_nB_n - A_mB_m < E$, satisfying Definition 28, (2).

Case 3: $A_n = A_m$, $B_n < B_m$ and $B_m - B_n < E/2M$.

By multiplying inequalities, $A_nB_n < A_mB_m$.

Since $A_n \leq M$ for all n, by Lemma 2 we find

$$A_m(B_m - B_n) = A_mB_m - A_mB_n = A_mB_m - A_nB_n < ME/2M = E/2 < E.$$

Therefore, $A_nB_n < A_mB_m$ and $A_mB_m - A_nB_n < E$, satisfying Definition 28, (3).

Case 4: $A_n > A_m$ and $A_n - A_m < E$, $B_n = B_m$.

The proof is exactly the proof of Case 2 with the A's and the B's interchanged.

Case 5: $A_n > A_m$ and $A_m - A_n < E/2M$, $B_n = B_m$.

The proof is exactly the proof of Case 3 with the A's and B's interchanged.

Case 6: $A_n > A_m$ and $A_n - A_m < E/2M$, $B_n > B_m$ and $B_n - B_m < E/2M$.

By multiplying inequalities, $A_nB_n > A_mB_m$.

Since $A_n \leq M$, $B_n \leq M$ for all n, by Lemmas 1 and 2 we find

$$A_n(B_n - B_m) + B_m(A_n - A_m) = (A_nB_n - A_nB_m) + (A_nB_m - A_mB_m) =$$

$$= A_nB_n - A_mB_m < ME/2M + ME/2M = E.$$

Therefore, $A_nB_n > A_mB_m$ and $A_nB_n - A_mB_m < E$, satisfying Definition 28, (2).

Case 7: $A_n < A_m$ and $A_m - A_n < E/2M$, $B_n < B_m$ and $B_m - B_n < E/2M$. The proof will be omitted since it is analogous to the proof of Case 6.

Case 8: $A_n > A_m$ and $A_n - A_m < E/2M$, $B_n < B_m$ and $B_m - B_n < E/2M$. From the Tricotomy Law it is true that

(1) $A_nB_n = A_mB_m$; or

(2) $A_nB_n > A_mB_m$; or

(3) $A_nB_n < A_mB_m$.

We must show that the inequalities of Definition 28 are satisfied in the subcases 2 and 3 above.

Subcase 1: $A_nB_n = A_mB_m$. Definition 28, (1) is satisfied.

Subcase 2: $A_nB_n > A_mB_m$.

Since $B_n \leq M$ for all n, we find by Lemma 2 and adding and simplifying;

$$A_nB_n < A_nB_m = B_m(A_n - A_m)+(A_mB_m) < M(E/2M)+(A_mB_m)<E+(A_mB_m).$$

Therefore, $A_nB_n - A_mB_m < E$, satisfying Definition 28, (2).

Subcase 3: $A_nB_n < A_mB_m$.

Since $A_n \leq M$ for all n, we find by Lemma 2 and adding and simplifying;

$A_mB_m < A_nB_m = A_n(B_m - B_n)+(A_nB_n) < M(E/2M)+(A_nB_n) < E +(A_nB_n)$.

Therefore, $A_mB_m - A_nB_n < E$, satisfying Definition 28, (3).

Case 9: $A_n < A_m$ and $A_m - A_n < E/2M$, $B_n < B_m$ and $B_m - B_n < E/2M$. The Tricotomy Law again gives three subcases for which the proof will be omitted since it is analogous to the proof of Case 8 and its three subcases.

In every case, $\xi\,\eta$ is a Cauchy sequence; thus the theorem is proved.

<u>Definition 37</u>: The Cauchy sequence constructed in Theorem 144 is denoted by $\xi \cdot \eta$ (. to be read "times"; however, the dot is usually omitted) and is called the <u>product</u> of ξ and η or the Cauchy sequence obtained from multiplication of ξ and η.

<u>Theorem 145</u>: If $\xi \sim \eta$, $\zeta \sim \upsilon$, then $\xi\,\zeta \sim \eta\,\upsilon$.

<u>Remark</u>: The class of the product thus depends only on the classes to which the "factors" belong.

<u>Proof</u>: Suppose $\xi \sim \eta$, $\zeta \sim \upsilon$; Theorem 144 and Definition 37 show that each of $\xi\,\zeta$ and $\eta\,\upsilon$ is a Cauchy sequence. We must show that given any E, there exists a $p(E)$ such that for all $m,n > p(E)$ it is true that

(1) $A_nB_n = A_mB_m$; or

(2) $A_nB_n > A_mB_m$ and $A_nB_n - A_mB_m < E$; or

(3) $A_nB_n < A_mB_m$ and $A_mB_m - A_nB_n < E$.

Given any E, there exists a $p_1(E/2M)$ [M is an upper bound for $\xi, \eta, \zeta,$.] such that for all $n>p_1(E/2M)$ it is true that

(1) $A_n = B_n$; or

(2) $A_n > B_n$ and $A_n - B_n < E/2M$; or

(3) $A_n < B_n$ and $B_n - A_n < E/2M$;

and there exists a $p_2(E/2M)$ such that for all $n>p_2(E/2M)$ it is true that

(1) $C_n = D_n$; or

(2) $C_n > D_n$ and $C_n - D_n < E/2M$; or

(3) $C_n < D_n$ and $D_n - C_n < E/2M$.

Let $p(E)$ be the maximum of $p_1(E/2M)$ and $p_2(E/2M)$, then for all $n>p(E)$ one of the following cases arises;

Case 1: $A_n = B_n$, $C_n = D_n$.

Case 2: $A_n = B_n$, $C_n > D_n$ and $C_n - D_n < E/2M$.

Case 3: $A_n = B_n$, $C_n < D_n$ and $D_n - C_n < E/2M$.

Case 4: $A_n > B_n$ and $A_n - B_n < E/2M$, $C_n = D_n$.

Case 5: $A_n < B_n$ and $B_n - A_n < E/2M$, $C_n = D_n$.

Case 6: $A_n > B_n$ and $A_n - B_n < E/2M$, $C_n > D_n$ and

$C_n - D_n < E/2M$.

Case 7: $A_n < B_n$ and $B_n - A_n < E/2M$, $C_n < D_n$ and $D_n - C_n < E/2M$.

Case 8: $A_n > B_n$ and $A_n - B_n < E/2M$, $C_n < D_n$ and $D_n - C_n < E/2M$.

Case 9: $A_n < B_n$ and $B_n - A_n < E/2M$, $C_n > D_n$ and $C_n - D_n < E/2M$.

By a proof analogous to the proof of Theorem 144, we could show that one of the three conditions of Definition 29 for tantamount is satisfied, hence the theorem is proved.

<u>Theorem</u> <u>146</u>: (Commutative Law of Multiplication):

$$\xi\,\eta \sim \eta\,\xi.$$

<u>Proof</u>: Given any E, take $p(E)=1$, then for all $n>p(E)$ it is true that

(1) $A_n B_n = B_n A_n$.

Hence, $\xi\,\eta \sim \eta\,\xi$.

<u>Theorem</u> <u>147</u>: (Associative Law of Multiplication):

$$(\xi\,\eta)\,\zeta \sim \xi\,(\eta\,\zeta).$$

<u>Proof</u>: Given any E, take $p(E)=1$, then for all $n>p(E)$ it is true that

(1) $(A_n B_n)\,C_n = A_n(B_n C_n)$.

Hence, $(\xi\ \eta)\ \zeta \sim \xi\ (\ \eta\ \zeta)$.

Theorem 148: (Distributive Law):

$$\xi(\eta + \zeta) \sim \xi\ \eta + \xi\ \zeta.$$

Proof: Given any E, take $p(E)=1$, then for all $n>p(E)$ it is true that

$$(1)\quad A_n(B_n + C_n) = A_nB_n + A_nC_n.$$

Hence, $\xi(\eta + \zeta) \sim \xi\ \eta + \xi\ \zeta$.

Theorem 149: If $\xi > \eta$, $\xi \sim \eta$, or $\xi < \eta$, then $\xi\ \zeta > \eta\ \zeta$, or $\xi\ \zeta \sim \eta\ \zeta$, or $\xi\ \zeta < \eta\ \zeta$, respectively.

Proof: If $\xi > \eta$, by Theorem 143, with a suitable υ, $\xi \sim \eta + \upsilon$, hence

$$\xi\ \zeta \sim (\eta + \upsilon)\ \zeta \sim (\eta\ \zeta + \upsilon\ \zeta) > \upsilon\ \zeta.$$

The second part is contained in Theorem 145; and the third is a consequence of the first, since if

$$\xi < \eta \text{ then } \eta > \xi,$$

so that by the first part, $\eta\ \zeta > \xi\ \zeta$, $\xi\ \zeta < \eta\ \zeta$.

Theorem 150: If $\xi\ \zeta > \eta\ \zeta$, or $\xi\ \zeta \sim \eta\ \zeta$, or $\xi\ \zeta < \eta\ \zeta$, then $\xi > \eta$, or $\xi \sim \eta$, or $\xi < \eta$, respectively.

Proof: Follows from Theorem 149, since the three cases are in both instances mutually exclusive and exhaust all possibilities.

Theorem 151: If $\xi > \eta$, $\zeta > \upsilon$, then $\xi\ \zeta > \eta\ \upsilon$.

Proof: By Theorem 149: $\xi \zeta > \eta \zeta$ and $\eta \zeta \sim \zeta \eta > \upsilon \eta \sim \eta \upsilon$ so that $\xi \zeta > \eta \upsilon$.

Theorem 152: If $\xi \gtrsim \eta$, $\zeta > \upsilon$, or $\xi > \eta$, $\zeta \gtrsim \upsilon$, then $\xi \zeta \gtrsim \eta \upsilon$.

Proof: Follows from Theorem 145 and Theorem 149 if the tantamount sign holds in the hypothesis; otherwise from Theorem 151.

Theorem 153: If $\xi \gtrsim \eta$, $\zeta \gtrsim \upsilon$, then $\xi \zeta \gtrsim \eta \upsilon$.

Proof: Follows from Theorem 145 if two tantamount signs hold in the hypothesis; otherwise from Theorem 152.

Definition 38: For any given rational number R, a sequence formed by having each term identically R is called a constant sequence. It will be denoted by R^0.

Theorem 154: Every constant sequence is a Cauchy sequence.

Proof: Let R^0 be a given constant sequence. For a lower bound, $R_n = R_1 = N$ for all n; and given any E, take p(E)=1, then for all n>p(E) it is true that

$$(1) \quad R_n = R_m.$$

Therefore, R^0 is a Cauchy sequence.

Theorem 155: $\xi \cdot 1^0 \sim \xi$.

Proof: Given any E, take $p(E)=1$, then for all $n>p(E)$ it is true that

$$(1) \quad A_n \cdot 1 = A_n.$$

<u>Theorem 156</u>: For any given ξ, the relation $\xi\,\nu \sim 1^o$ has a solution ν.

<u>Proof</u>: Let ξ be the given Cauchy sequence; that is, $A_n \geq H$ for all n, and given any E_1 there exists a $p(HHE_1)$ such that for all $m,n>p(HHE_1)$ it is true that

(1) $A_n = A_m$; or

(2) $A_n > A_m$ and $A_n - A_m < HHE_1$; or

(3) $A_n < A_m$ and $A_m - A_n < HHE_1$.

Let us consider as a solution the sequence ν defined by

$$D_n = 1 \quad \text{for all } n \leq p(HHE_1),$$

$$D_n = 1/A_n \quad \text{for all } n>p(HHE_1).$$

We must show that ν is a Cauchy sequence; that is, there is an N such that $D_n \geq N$ for all n, and given any E, there exists a $p(E)$ such that for all $m,n>p(E)$ it is true that

(1) $D_n = D_m$; or

(2) $D_n > D_m$ and $D_n - D_m < E$; or

$$(3)\quad D_n < D_m \quad \text{and} \quad D_m - D_n < E.$$

By Theorem 119, ξ is bounded, hence $A_n \leq M$ for all n, and we find

$$1/M = (A_n)1/A_nM \leq (M)1/A_nM = 1/A_n,$$

therefore, $1/A_n \geq 1/M$ for all n.

Pick $p(E) = p(HHE_1)$. Thus with $D_n = 1/A_n$, $D_m = 1/A_m$, the following cases arise.

Case 1: $A_n = A_m$. Then $1/A_n = 1/A_m$, satisfying Definition 28, (1).

Case 2: $A_n > A_m$ and $A_n - A_m < HHE$. Since by Lemma 2 $A_n - A_m = A_mA_n(1/A_m - 1/A_n) < HHE$,

then since $A_n \geq H$, $A_m \geq H$; $A_nA_m \geq H$ for all n and

$$1/A_nA_m = (HH)1/A_nA_mHH \leq (A_nA_m)1/A_nA_mHH = 1/HH.$$

Multiplying by $1/A_nA_m \leq 1/HH$, we have $1/A_m - 1/A_n < E$, satisfying Definition 28, (3).

Case 3: $A_n < A_m$ and $A_m - A_n < HHE$. Since by Lemma 2 $A_m - A_n = A_mA_n(1/A_n - 1/A_m) < HHE$,

then from above, $A_nA_m \geq HH$ and $1/A_nA_m \leq 1/HH$,

hence, multiplying by $1/A_nA_m \leq 1/HH$, we have $1/A_n - 1/A_m < E$, satisfying Definition 28, (2).

Therefore, ν is a Cauchy sequence.

We must show that ν satisfies $\xi \cdot \nu \sim 1^0$. That is, given any E, there exists a $p(E)$ such that for all $n>p(E)$ it is true that

$$(1)\quad A_nD_n = 1; \quad \text{or}$$

$$(2)\quad A_nD_n > 1 \quad \text{and} \quad A_nD_n - 1 < E; \quad \text{or}$$

$$(3)\quad A_nD_n < 1 \quad \text{and} \quad 1 - A_nD_n < E.$$

Given any E, take $p(E) = p(HHE_1)$, then for all $n>p(E)$ it is true that

$$(1)\quad A_nD_n = A_n(1/A_n) = 1.$$

Therefore, ν is a solution and the theorem is proved.

<u>Theorem 157</u>: The relation

$$\eta\,\nu \sim \xi$$

where ξ and η are given, has a solution. If ν_1 and ν_2 are solutions, then $\nu_1 \sim \nu_2$.

<u>Proof</u>: The second assertion of Theorem 157 is an immediate consequence of Theorem 150; for if

$$\eta\,\nu_1 \sim \eta\,\nu_2,$$

then, by Theorem 150, $\nu_1 \sim \nu_2$.

The existence of a ν (the first assertion of Theorem 157) is proved as follows. If ζ is the solution of

$$\eta\,\zeta \sim 1^0,$$

whose existence was proved by Theorem 156, then the sequence

$$\nu = \zeta\,\xi$$

satisfies the relation in Theorem 157; for we have by Theorem 155,

$$\eta\, \mathcal{V} \sim \eta(\zeta\ \xi) \sim (\eta\ \zeta)\ \xi \sim 1^{o}\ (\xi) \sim \xi.$$

CHAPTER II

SECS

1

Definition

Definition 39: By a sec (to be read "seek"), we mean the set of all Cauchy sequences which are tantamount to some fixed Cauchy sequence. (Such a set is therefore a class in the sense of §1 of Landau.)

The small Greek letters α, β, γ, and δ will be used to denote secs.

Definition 40: $\alpha = \beta$
(= to be read "equals") if the two sets consist of the same Cauchy sequences. Otherwise,

$$\alpha \neq \beta$$

($\neq$ to be read "is not equal to").
The following three theorems are trivial:

Theorem 158: $\alpha = \alpha$.

Theorem 159: If $\alpha = \beta$, then $\beta = \alpha$.

Theorem 160: If $\alpha = \beta$, $\beta = \gamma$, then $\alpha = \gamma$.

2

Ordering

<u>Definition 41</u>: $\alpha > \beta$

($>$ to be read "is greater than") if for a Cauchy sequence ξ of the set α, and for a Cauchy sequence η of the set β (hence for any such pair of Cauchy sequences, by Theorem 123) we have that

$$\xi > \eta.$$

<u>Definition 42</u>: $\alpha < \beta$

($<$ to be read "is less than") if for a Cauchy sequence ξ of the set α, and for a Cauchy sequence η of the set β (hence for any such pair of Cauchy sequences, by Theorem 124) we have that

$$\eta > \xi.$$

<u>Theorem 161</u>: For any given α, β, exactly one of $\alpha > \beta$, $\alpha = \beta$, $\alpha < \beta$ must be the case.

<u>Proof</u>: Theorem 122.

<u>Theorem 162</u>: If $\alpha > \beta$, then $\beta < \alpha$.

<u>Proof</u>: Theorem 120.

<u>Theorem 163</u>: If $\alpha < \beta$, then $\beta > \alpha$.

<u>Proof</u>: Theorem 121.

<u>Definition 43</u>: $\alpha \geq \beta$ means $\alpha > \beta$ or $\alpha = \beta$.

($\geq$ to be read "is greater than or equal to".)

<u>Definition</u> <u>44</u>: $\alpha \leq \beta$ means $\alpha < \beta$ or $\alpha = \beta$.
($\leq$ to be read "is less than or equal to".)

<u>Theorem</u> <u>164</u>: If $\alpha \geq \beta$, then $\beta \leq \alpha$.

<u>Proof</u>: Theorem 127.

<u>Theorem</u> <u>165</u>: If $\alpha \leq \beta$, then $\beta \geq \alpha$.

<u>Proof</u>: Theorem 128.

<u>Theorem</u> <u>166</u>: (Transitivity of Ordering): If

$$\alpha > \beta, \quad \beta > \gamma,$$

then $\alpha > \gamma$.

<u>Proof</u>: Theorem 129.

<u>Theorem</u> <u>167</u>: If $\alpha \leq \beta$, $\beta < \gamma$, or $\alpha < \beta$, $\beta \leq \gamma$ then $\alpha < \gamma$.

<u>Proof</u>: Theorem 130.

<u>Theorem</u> <u>168</u>: If $\alpha \leq \beta$, $\beta \leq \gamma$, then $\alpha \leq \gamma$.

<u>Proof</u>: Theorem 131.

3

Addition

Definition 45: By $\alpha + \beta$ ($+$ to be read "plus") we mean the class which contains a sum (hence, by Theorem 133, every such sum) of a Cauchy sequence from α and a Cauchy sequence from β.

This sec is called a sum of α and β, or the sec obtained from the addition of β to α.

Theorem 169: (Commutative Law of Addition):

$$\alpha + \beta = \beta + \alpha.$$

Proof: Theorem 134.

Theorem 170: (Associative Law of Addition):

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

Proof: Theorem 135.

Theorem 171: $\alpha + \beta > \alpha$.

Proof: Theorem 136.

Theorem 172: If $\alpha > \beta$, then $\alpha + \gamma > \beta + \gamma$.

Proof: Theorem 137.

Theorem 173: If $\alpha > \beta$, or $\alpha = \beta$, or $\alpha < \beta$, then $\alpha + \gamma > \beta + \gamma$, or $\alpha + \gamma = \beta + \gamma$, or $\alpha + \gamma < \beta + \gamma$, respectively.

Proof: Theorem 138.

<u>Theorem 174</u>: If $\alpha + \gamma > \beta + \gamma$, or $\alpha + \gamma = \beta + \gamma$, or $\alpha + \gamma < \beta + \gamma$, then $\alpha > \beta$, or $\alpha = \beta$, or $\alpha < \beta$, respectively.

<u>Proof</u>: Theorem 139.

<u>Theorem 175</u>: If $\alpha > \beta$, $\gamma > \delta$, then $\alpha + \gamma > \beta + \delta$.

<u>Proof</u>: Theorem 140.

<u>Theorem 176</u>: If $\alpha \geq \beta$, $\gamma > \delta$, or $\alpha > \beta$, $\gamma \geq \delta$, then $\alpha + \gamma \geq \beta + \delta$.

<u>Proof</u>: Theorem 141.

<u>Theorem 177</u>: If $\alpha \geq \beta$, $r \geq \delta$, then $\alpha + \gamma \geq \beta + \delta$.

<u>Proof</u>: Theorem 142.

<u>Theorem 178</u>: If $\alpha > \beta$, then $\beta + \delta = \alpha$ has exactly one solution δ.

<u>Remark</u>: If $\alpha \leq \beta$, there does not exist a solution, by Theorem 136.

<u>Proof</u>: Theorem 143.

<u>Definition 46</u>: The δ of Theorem 178 is denoted by $\alpha - \beta$ (- to be read "minus") and is called the <u>difference</u> α minus β, or the sec obtained by subtraction of the sec β from the sec α.

4

Multiplication

Definition 47: By $\alpha \cdot \beta$ (. to be read "times"; however, the dot is usually omitted) we mean the class which contains a product (hence, by Theorem 145, every such product) of a Cauchy sequence from α by a Cauchy sequence from β.

This sec is called the product of α by β, or the sec obtained from multiplication of the sec α by the sec β.

Theorem 179: (Commutative Law of Multiplication):

$$\alpha \beta = \beta \alpha,$$

Proof: Theorem 146.

Theorem 180: (Associative Law of Multiplication):

$$(\alpha \beta) \gamma = \alpha(\beta \gamma).$$

Proof: Theorem 147.

Theorem 181: (Distributive Law):

$$\alpha(\beta + \gamma) = \alpha \beta + \alpha \gamma.$$

Proof: Theorem 148.

Theorem 182: If $\alpha > \beta$, or $\alpha = \beta$, or $\alpha < \beta$, then $\alpha \gamma > \beta \gamma$, or $\alpha \gamma = \beta \gamma$, or $\alpha \gamma < \beta \gamma$, respectively.

Proof: Theorem 149.

Theorem 183: If $\alpha\gamma > \beta\gamma$, or $\alpha\gamma = \beta\gamma$, or $\alpha\gamma < \beta\gamma$, then $\alpha > \beta$, or $\alpha = \beta$, or $\alpha < \beta$, respectively.

Proof: Theorem 150.

Theorem 184: If $\alpha > \beta$, $\gamma > \delta$, then $\alpha\gamma > \beta\delta$.

Proof: Theorem 151.

Theorem 185: If $\alpha \geq \beta$, $\gamma > \delta$, or $\alpha > \beta$, $\gamma \geq \delta$, then $\alpha\gamma > \beta\delta$.

Proof: Theorem 152.

Theorem 186: If $\alpha \geq \beta$, $\gamma \geq \delta$, then $\alpha\gamma \geq \beta\delta$.

Proof: Theorem 153.

Theorem 187: The equation $\beta\delta = \alpha$ in which β and α are given, has exactly one solution δ.

Proof: Theorem 157.

Theorem 188: Every sec contains a sequence $\{B_n\}$ such that $B_n < B_{n+1}$ for $n=1, 2, \ldots$ and a sequence $\{C_n\}$ such that $C_n > C_{n+1}$ for $n=1, 2, \ldots$. The sequence $\{B_n\}$ is called strictly increasing and $\{C_n\}$ is called strictly decreasing.

Proof: Given any sec α, let $\{A_n\}$ be a Cauchy sequence contained in α. To find strictly increasing and strictly decreasing sequences, let us proceed in the following manner:

I) Choose $E_1 = N/2$ (N is a lower bound for $\{A_n\}$) and find $p(E_1)$ so that, for all $m,n \geq p(E_1)$ it is true that

(1) $A_n = A_m$; or

(2) $A_n > A_m$ and $A_n - A_m < E_1$; or

(3) $A_n < A_m$ and $A_m - A_n < E_1$.

Then, for all $n > p(E_1)$ one of the following is true

(4) $A_n = A_{p(E_1)}$; or

(5) $A_n > A_{p(E_1)}$ and $A_n - A_{p(E_1)} < E_1$; or

(6) $A_n < A_{p(E_1)}$ and $A_{p(E_1)} - A_n < E_1$.

But for all $n > p(E_1)$, conditions (4), (5), and (6) imply

(7) $A_{p(E_1)} - E_1 < A_n < A_{p(E_1)} + E_1$.

The proof for (7) is as follows:

The subtraction $A_{p(E_1)} - E_1$ is possible since $A_{p(E_1)} \geq N$ for all values of $p(E_1)$. Hence $A_{p(E_1)} > N/2$. Similarly, $A_{p(E_n)} \geq N > E_n$ for all values of n. The rest of the proof is by cases;

Case 1: If $A_n = A_{p(E_1)}$, then certainly $A_{p(E_1)} - E_1 < A_n < A_{p(E_1)} + E_1$.

Case 2: If $A_n > A_{p(E_1)}$ and $A_n - A_{p(E_1)} < E_1$, then $A_{p(E_1)} - E_1 < A_{p(E_1)} < A_n < A_{p(E_1)} + E_1$.

Case 3: If $A_n < A_{p(E_1)}$ and $A_{p(E_1)} - A_n < E_1$, then $A_{p(E_1)} - E_1 < A_n < A_{p(E_1)} < A_{p(E_1)} + E_1$.

Set $B_1 = A_{p(E_1)} - 2E_1$, $C_1 = A_{p(E_1)} + 2E_1$; then for all $n > p(E_1)$, $B_1 < A_n - E_1$, $A_n + E_1 < C_1$. These are true since by subtracting E_1 from condition (7) we obtain $A_{p(E_1)} - 2E_1 < A_n - E_1 < A_{p(E_1)}$, and adding E_1 to condition (7) we obtain $A_{p(E_1)} < A_n + E_1 < A_{p(E_1)} + 2E_n$.

II) Set $E_2 = E_1/2$ and find $p(E_2)$ so that $p(E_2) > p(E_1)$ and, for all $m,n \geq p(E_2)$ one of the following is true

(1) $A_n = A_m$; or

(2) $A_n > A_m$ and $A_n - A_m < E_2$; or

(3) $A_n < A_m$ and $A_m - A_n < E_2$.

Then for all $n > p(E_2)$ one of the following is true

(4) $A_n = A_{p(E_2)}$; or

(5) $A_n > A_{p(E_2)}$ and $A_n - A_{p(E_2)}$; or

(6) $A_n < A_{p(E_2)}$ and $A_{p(E_2)} - A_n < E_2$.

But as in part I), for all $n > p(E_2)$, conditions (4), (5), and (6) imply

(7) $A_{p(E_2)} - E_2 < A_n < A_{p(E_2)} + E_2$.

Set $B_2 = A_{p(E_2)} - 2E_2$, $C_2 = A_{p(E_2)} + 2E_2$. Then $B_1 < B_2$, $C_1 > C_2$. These are true since for $p(E_2) > p(E_1)$ the following inequalities are true

$$B_1 < A_{p(E_2)} - E_1 = A_{p(E_2)} - 2E_2 = B_2,$$

$$C_1 > A_{p(E_2)} + E_1 = A_{p(E_2)} + 2E_2 = C_2.$$ And for all $n > p(E_2)$ it is true that $B_2 < A_n - E_2$, $A_n + E_2 < C_2$. These are true since for all $n > p(E_2)$ it is true that $A_{p(E_2)} - E_2 < A_n < A_{p(E_2)} + E_2$.

By subtracting E_2 we obtain $A_{p(E_2)} - 2E_2 < A_n - E_2 < A_{p(E_2)}$ and adding E_2 we obtain $A_{p(E_2)} < A_n + E_2 < A_{p(E_2)} + 2E_2$.

III) Continuing by induction for the kth terms suppose we have E_{k-1}, $p(E_{k-1})$, $B_1, \ldots, B_{k-1}$, and $C_1, \ldots, C_{k-1}$, such that $B_1 < B_2 < \ldots < B_{k-1}$, $C_1 > C_2 > \ldots > C_{k-1}$ and for all $n > p(E_{k-1})$

$$B_{k-1} < A_n - E_{k-1}, \quad A_n + E_{k-1} < C_{k-1}.$$

We set $E_k = E_{k-1}/2$ and find $p(E_k)$ so that $p(E_k) > p(E_{k-1})$ and for all $m, n \geq p(E_k)$ one of the following is true

(1) $A_n = A_m$; or

(2) $A_n > A_m$; and $A_n - A_m < E_k$; or

(3) $A_n < A_m$ and $A_m - A_n < E_k$.

Then for all $n > p(E_k)$ one of the following is true

(4) $A_n = A_{p(E_k)}$; or

(5) $A_n > A_{p(E_k)}$ and $A_n - A_{p(E_k)} < E_k$; or

(6) $A_n < A_{p(E_k)}$ and $A_{p(E_k)} - A_n < E_k$.

But as in parts I) and II), for all $n \geq p(E_k)$, conditions (4), (5), and (6) imply

(7) $A_{p(E_k)} - E_k < A_n < A_{p(E_k)} + E_k$.

Set $B_k = A_{p(E_k)} - 2E_k$, $C_k = A_{p(E_k)} + 2E_k$. Then $B_k > B_{k-1}$, $C_k > C_{k-1}$. These are true since for $p(E_k) > p(E_{k-1})$ the following inequalities are true

$$B_{k-1} < A_{p(E_k)} - E_{k-1} = A_{p(E_k)} - 2E_k = B_k,$$

$$C_{k-1} > A_{p(E_k)} + E_{k-1} = A_{p(E_k)} + 2E_k = C_k.$$ And for all $n > p(E_k)$ it is true that $B_k < A_n - E_k$, $A_n + E_k < C_k$. These are true since for all $n > p(E_k)$ it is true that $A_{p(E_k)} - E_k < A_n < A_{p(E_k)} + E_k$.

By subtracting E_k we obtain $A_{p(E_k)} - 2E_k < A_n - E_k < A_{p(E_k)}$ and adding E_k we obtain $A_{p(E_k)} < A_n + E_k < A_{p(E_k)} + 2E_k$.

By construction $B_k < B_{k+1}$ and $C_k > C_{k+1}$ for all

$k=1, 2, 3, \ldots$. Hence $\{B_k\}$ is such that $B_1 < B_2 < \ldots < B_k < \ldots$ and that $\{C_k\}$ is such that $C_1 > C_2 > \ldots > C_k > \ldots$. Thus $\{B_k\}$ is strictly increasing and $\{C_k\}$ is strictly decreasing.

We must show that $\{B_n\}$ and $\{C_n\}$ are Cauchy sequences and that $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ are contained in the same sec.

First, to show that $\{B_n\}$ and $\{C_n\}$ are Cauchy sequences, we may take B_1 as a lower bound for $\{B_n\}$ and $\{C_n\}$ since $B_1 \leq B_n < C_n$ for all n and prove below that given any E, there exists a $p(E)$ such that for all $n>p(E)$ both the following are true

(1) $B_n < B_{n+1}$ and $B_{n+1} - B_n < E$; and

(2) $C_n > C_{n+1}$ and $C_n - C_{n+1} < E$.

From the construction of $\{B_n\}$ and $\{C_n\}$, the following are true for all n: $B_n < B_{n+1} < C_{n+1} < C_n$ and $C_n - B_n = 4E_n$. Then, for all n, we arrive at the following conclusions:

(1) If $B_n < B_{n+1} < C_n$ and $C_n - B_n = 4E_n$, then

$B_n < B_{n+1}$ and $B_{n+1} - B_n < 4E_n$.

From the inequalities, $B_n < B_{n+1}$. Adding and simplifying,

$$(C_n - B_n) + B_n + B_{n+1} < 4E_n + B_n + C_n,$$

$$C_n + B_{n+1} < 4E_n + B_n + C_n.$$

Cancelling C_n and simplifying, $B_{n+1} - B_n < 4E_n$.

(2) If $B_n < C_{n+1} < C_n$ and $C_n - B_n = 4E_n$, then

$$C_n > C_{n+1} \text{ and } C_n - C_{n+1} < 4E_n.$$

From the inequalities, $C_n > C_{n+1}$. Adding and simplifying,

$$(C_n - B_n) + B_n < 4E_n + C_{n+1},$$

$$C_n - C_{n+1} < 4E_n.$$

Given any E, find $p(E)$ so that $4E_{p(E)} < E$, then for all $n > p(E)$ both of the following are true

(1) $B_n < B_{n+1}$ and $B_{n+1} - B_n < 4E_n < 4E_{p(E)} < E$; and

(2) $C_n > C_{n+1}$ and $C_n - C_{n+1} < 4E_n < 4E_{p(E)} < E$.

Thus we have shown that $\{B_n\}$ and $\{C_n\}$ have lower bounds and that given any E, there exists a $p(E)$ such that for all $n > p(E)$, conditions (1) and (2) are true, hence $\{B_n\}$ and $\{C_n\}$ are Cauchy sequences.

Next, we must show that $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ are tantamount. Let us first show that $\{B_n\}$ and $\{C_n\}$ are tantamount. That is, given any E, there exists a $p(E)$ such that for all $n > p(E)$ the following is true

(3) $C_n > B_n$ and $C_n - B_n < E$.

Since $C_n > B_n$ and $C_n - B_n = 4E_n$ for all n, given any E, find $p(E)$ so that $4E_{p(E)} < E$, then for all $n > p(E)$ the following is true

(4) $C_n > B_n$ and $C_n - B_n = 4E_n < 4E_{p(E)} < E$.

Therefore, $\{B_n\} \sim \{C_n\}$.

Next, we will show that $\{A_n\} \sim \{B_n\}$. That is, given any E, there exists a $p(E)$ such that for all $n > p(E)$ one of the following is true

(A) $A_n = B_n$; or

(B) $A_n > B_n$ and $A_n - B_n < E$; or

(C) $A_n < B_n$ and $B_n - A_n < E$.

Since $\{A_n\}$ is a Cauchy sequence, given any E, find $p_1(E)$ so that $E_{p_1(E)} < E/4$. With this $E_{p_1(E)}$ there exists a $p(E_{p_1(E)})$ such that for all $n > p(E_{p_1(E)})$ one of the following is true

(1) $A_n = A_m$; or

(2) $A_n > A_m$ and $A_n - A_m < E_{p_1(E)}$; or

(3) $A_n < A_m$ and $A_m - A_n < E_{p_1(E)}$.

From the construction of $\{B_n\}$ and $\{C_n\}$ we know that for any n we may choose an E_n and find $p(E_n)$

such that for all $m > p(E_n)$ the following are true

(1) $B_n < A_m < C_n$ and

(2) $A_m - B_n < 3E_n$.

The proof for (2) will be given below.

Since $\{A_n\}$ is a Cauchy sequence, given any E_n, there exists a $p(E_n)$ such that for all $m > p(E_n)$, one of the following is true

(1) $A_m = A_{p(E_n)}$; or

(2) $A_m > A_{p(E_n)}$ and $A_m - A_{p(E_n)} < E_n$; or

(3) $A_m < A_{p(E_n)}$ and $A_{p(E_n)} - A_m < E_n$.

Thus the proof is by cases.

Case 1: $A_m = A_{p(E_n)}$, $A_m > B_n$. Then

$A_m = A_{p(E_n)} > A_{p(E_n)} - 2E_n = B_n$ and $A_m - B_n = 2E_n < 3E_n$.

Case 2: $A_m > A_{p(E_n)}$ and $A_m - A_{p(E_n)} < E_n$,

$A_m > B_n$.

Thus the following are true,

$$A_{p(E_n)} + E_n > A_m > B_n = A_{p(E_n)} - 2E_n.$$

Hence, $(A_{p(E_n)} + E_n) - B_n = (A_{p(E_n)} + E_n) - (A_{p(E_n)} - 2E_n) = 3E_n$.

Adding and simplifying,

$$(A_{p(E_n)} + E_n) - B_n + B_n + A_m < 3E_n + (A_{p(E_m)} + E_n) + B_n,$$

$$(A_{p(E_n)} + E_n) + A_m < 3E_n + (A_{p(E_n)} + E_n) + B_n.$$

Cancelling $(A_{p(E_n)} + E_n)$ and simplifying,

$$A_m - B_n < 3E_n.$$

Case 3: $A_m < A_{p(E_n)}$ and $A_m - A_{p(E_n)} < E_n$. Thus the following are true,

$$A_{p(E_n)} > A_m > A_{p(E_n)} - E_n > B_n.$$

Since $A_{p(E_n)} - B_n = 2E_n$, by adding and simplifying,

$$A_{p(E_n)} - B_n + A_m + B_n < 2E_n + A_{p(E_n)} + B_n,$$

then cancelling $A_{p(E_n)}$ and simplifying,

$A_m - B_n < 2E_n < 3E_n$. Thus for all $m > p(E_n)$, $A_m - B_n < 3E_n$.

For any $n > p_1(E)$ we may choose an E_n and pick a $p(E_n)$ such that for all $m > p(E_n)$ it is true that $A_m - B_n < 3E_n$. But for all $n > p_1(E)$ the following inequalities are true

$$E_n < E_{p_1(E)} < E/4.$$

Hence, for all $n > p_1(E_n)$ we may choose an E_n and pick a $p(E_n)$ such that for all $n > p_1(E)$, $A_m - B_n < 3E_n < 4E_{p_1(E)} < E$.

Thus with $p(E)$ the maximum of $p_1(E)$ and $p(E_{p_1(E)})$, for each $n>p(E)$ choose an $m>p(E_n)$ and the following cases arise.

Case 1: $A_n = A_m$. Since $A_m > B_n$ and $A_m - B_n < 3E_n$, by substituting equals, $A_n > B_n$ and $A_n - B_n < 3E_n < 3E_{p_1(E)} < E$, satisfying condition (A).

Case 2: $A_n > A_m$ and $A_n - A_m < E_{p_1(E)}$. Since $A_m > B_n$ and $A_m - B_n < 3E_n$, then $A_n > A_m > B_n$ and $(A_n - B_n) = (A_m - B_n) + (A_n - A_m) < 3E_n < 3E_{p_1(E)} < E$, satisfying condition (A).

Case 3: $A_n < A_m$ and $A_m - A_n < E_{p_1(E)}$. Since $A_m > B_n$ and $A_m - B_n < 3E_n$, then from the Tricotomy Law the following are true.

Subcase 1: $A_n = B_n$. Condition (A) is satisfied.

Subcase 2: $A_n > B_n$. By adding and simplifying,

$$(A_m - B_n) + (A_m - A_n) + 2A_n + B_n < 4_{p_1(E)} + 2A_n + B_n,$$

$$2A_m + A_n < 4E_{p_1(E)} + 2A_m + B_n.$$

Cancelling $2A_m$ and simplifying, $A_n - B_n < 4E_{p_1(E)} < E$, satisfying condition (A).

Subcase 3: $A_n < B_n$. The proof will be omitted since it is similar to Subcase 2.

Thus we have shown that given any E, there exists a $p(E)$ such that for all $n>p(E)$ one of (A), (B), or (C) is satisfied. Therefore, $\{A_n\} \sim \{B_n\}$. By Theorem 118; if $\{A_n\} \sim \{B_n\}$, $\{B_n\} \sim \{C_n\}$, then $\{A_n\} \sim \{C_n\}$ and the theorem is proved.

5

Rational Secs and Integral Secs

Definition 48: A sec is called a rational sec if it contains a sequence of the form R^0. The rational sec containing R^0 will be denoted by R^*.

Definition 49: A sec is called an integral sec if it contains a sequence of the form x^0. The integral sec containing x^0 will be denoted by x^*.

Theorem 189: If $X^0 > Y^0$, or $X^0 \sim Y^0$, or $X^0 < Y^0$, then $X^* > Y^*$, or $X^* = Y^*$, or $X^* < Y^*$, respectively and conversely.

Proof: I) 1) If $X^0 > Y^0$, then by Definition 41, $X^* > Y^*$.

2) If $X^0 \sim Y^0$, then clearly $X^* = Y^*$.

3) If $X^0 < Y^0$, then by Definition 42, $X^* < Y^*$.

II) The converse is obvious, since the three cases are, in both instances, mutually exclusive and exhaust all possibilities.

Theorem 190: $X^* + Y^* = (X + Y)^*$,

$X^* Y^* = (XY)^*$.

Remark: Thus, the sum and the product of two rational secs are themselves rational secs.

Proof: 1) By Theorem 132, $X^0 + Y^0 = (X+Y)^0$.

2) By Theorem 144, $X^0 Y^0 = (XY)^0$.

Theorem 191: The integral secs satisfy the five axioms of the natural numbers, provided that the role of 1 is assigned to the sec 1^* and that the role of successor to the sec x^* is assigned to the sec $(x')^*$; that is, $(x^*)' = (x')^*$.

Proof: Let $\mathcal{Z}^*$ be the set of all integral secs.

1) Certainly 1^* belongs to $\mathcal{Z}^*$.

2) For every x^* in $\mathcal{Z}^*$, the sec $(x^*)'$ is also in $\mathcal{Z}^*$.

3) This successor is always different from the sec 1^*, since $x' \neq 1$,

$$x' \neq 1,$$
$$(x')^0 \quad 1^0,$$
$$(x')^* \neq 1^*.$$

4) If $(x^*)' = (y^*)'$, then $x^* = y^*$.

$$(x')^* = (x^*)' = (y^*)' = (y')^*$$
$$(x')^0 \sim (y')^0,$$
$$x' = y',$$
$$x = y,$$
$$x^0 \sim y^0,$$
$$x^* = y^*.$$

5) Let a set $\mathfrak{M}^*$ of integral secs have the following properties:

I) 1^* belongs to $\mathfrak{M}^*$.

II) If x^* belongs to $\mathfrak{M}^*$, then so does $(x^*)'$.

Furthermore, denote by $\mathfrak{M}$ the set of all x for which x^* belongs to $\mathfrak{M}^*$. Then 1 belongs to $\mathfrak{M}$, and for any x belonging to $\mathfrak{M}$, its successor also belongs to $\mathfrak{M}$. Therefore, every integer belongs to $\mathfrak{M}$, so that every integral sec belongs to $\mathfrak{M}^*$.

CHAPTER III

SECS AND CUTS

In order to establish the existence of the irrational numbers, let us establish a one-to-one correspondence between our secs and Landau's cuts. By doing this, we will prove that our system of secs includes irrational numbers as well as rational numbers.

<u>Theorem 192</u>: The following rule sets up a one-to-one correspondence between the collection of all secs and the collection of all cuts as defined by Definition 28 of Landau.

<u>Rule A</u>: For any sec α, a number Z is in the cut corresponding to α if and only if for every one of the sequences in α, there are at most a finite number of terms of the sequence less than Z.

Before proceeding to the proof of Theorem 192, let us show that the following rules are equivalent to Rule A; that is, every number which satisfies Rule B or Rule B^1 also satisfies Rule A; if Rule B or Rule B^1 is not satisfied, then Rule A is not satisfied.

<u>Rule B</u>: For any sec α, a number Z is in the cut corresponding to α if and only if there is a strictly increasing sequence $\{A_n\}$ in α and an integer n_0 such

that $Z \leq A_{n_0}$.

Rule B^1: For any sec α, a number Z is in the cut corresponding to α if and only if for every strictly increasing sequence $\{A_n\}$ in α there is an integer n_0 such that $Z \leq A_{n_0}$.

Lemma 3: Every number satisfying Rule B satisfies Rule A and every number not satisfying Rule B does not satisfy Rule A.

Proof: Let us do the second part first; that is, any number which does not satisfy Rule B does not satisfy Rule A. Let Y be any number which does not satisfy Rule B, then Rule A is also not satisfied; for a strictly increasing sequence $\{A_n\}$ in α, whose existence is guaranteed by Theorem 188, has more than a finite number of terms less than Y; in fact, every term of $\{A_n\}$ is less than Y.

For the first part of Lemma 3, let Z be a number which satisfies Rule B; we must show that Z also satisfies Rule A. If Z satisfies Rule B, then $Z \leq A_{n_0}$ for some n_0, where $\{A_n\}$ is a strictly increasing sequence in α. Let $\{B_n\}$ be any sequence contained in α. Take $E = 1/2(A_{n_0+1} - A_{n_0})$; with this E, there exists a $p(E)$ such that if $m>p(E)$, it is true that

$$(1) \quad A_m = B_m; \qquad \text{or}$$

$$(2) \quad A_m > B_m \qquad \text{and} \qquad A_m - B_m < E; \quad \text{or}$$

$$(3) \quad A_m < B_m \qquad \text{and} \qquad B_m - A_m < E.$$

Then for all $m > p(E)$, $m > n_0$, the following inequality is true;

$$Z \leq A_{n_0} < A_m - E < B_m.$$

Hence we have shown that there are at most a finite number of terms of any sequence of α less than Z. Thus the Lemma is proved.

The proof of Lemma 3 also proves any number satisfying Rule B^1 also satisfies Rule A and any number not satisfying Rule B^1 does not satisfy Rule A. Therefore, Rule A and Rule B^1 are also equivalent. Also, since Rule B is equivalent to Rule A, Rules B and B^1 must be equivalent, so that any number satisfying one also satisfies the other. Hence we may use either of Rules B or B^1 interchangeably.

Let us now return to the proof of Theorem 192.

Proof: We must show that a set of rational numbers satisfying Rule A constitutes a cut; to this end, we must show that

1) It contains a rational number, but does not contain all rational numbers;

2) Every rational number of the set is smaller than every rational number not belonging to the

set;

3) It does not contain a largest rational number.

For the proof of 1), the set does contain a number; for by Lemma 3, we are able to determine a lower number for the cut, for instance A_1, the smallest number in any strictly increasing sequence contained in α (existence of strictly increasing sequence in α is by Theorem 188). And by Lemma 3, this number must also satisfy Rule A. For an upper number for the cut, consider $\{C_n\}$ a strictly decreasing sequence contained in α; then C_1 is an upper number for the cut since C_1 is greater than an infinite number of terms of $\{C_n\}$ and hence does not satisfy Rule A.

2) Every number not belonging to the set is larger than every one belonging to the set, for every X, a lower number, and Y, an upper number, one of the following must hold; $X \geq Y$, or $X < Y$. If $X \geq Y$, and X satisfies Rule A, then Y must satisfy Rule A, which it does not. Therefore, $X < Y$, as was to be proved.

3) To show that our set does not contain a largest rational number, let X be in the set and consider a strictly increasing sequence η contained in α. Then, for some n_0, the following inequality is true:

$$B_{n_0} \leq X < B_{n_0+1},$$

But by Landau's Theorem 91 we may find a number Z which lies between X and B_{n_0+1}. Hence $Z > X$ and Z satisfies Rule B. But by Lemma 3, Z must also satisfy Rule A. Hence we have shown that Rule A gives us a cut corresponding to a given sec.

Obviously any cut corresponding to a given sec is identical with any other cut corresponding to that same sec, hence the corresponding cut is unique.

Conversely, to show that there exists a sec corresponding to a given cut, let us find sequences which will be in the corresponding sec. We observe that two rational numbers can always be found, one of which is a lower number for the cut and the other an upper number for the cut; and such that their difference is numerically less than a given arbitrarily small rational number E, (Landau's Theorem 132). Let A be a lower number and D be such that $D < E$. Then of the numbers $A + D$, $A + 2D$, ... , $A + rD$, ... there must be a last one $A + rD$ which is a lower number, for $A + nD$ may be made as large as we please by taking n large enough; the next number $A + (r + 1)D$ is then an upper number; and these numbers $A + rD$, $A + (r + 1)D$ whose difference is $D < E$, are the two numbers required. Moreover, if B is an upper number, the two numbers may be so determined that both lie between A

and B; for we need only take D to be of the form $(1/s)(B-A)$, where s is an integer so chosen that $(1/s)(B-A) < E$.

Given $E=1$, determine A_1 a lower number, and A_2 an upper number for the given cut, so that $A_2-A_1 < E_1$; next take A_3 a lower number, and A_4 an upper number so that $A_4 - A_3 < E_2 = 1/2$; and such that A_3, A_4 both lie between A_1 and A_2. Proceeding in this way, we can determine A_{2n-1}, A_{2n} rational numbers of different classes, so that

$$A_{2n} - A_{2n-1} < E_n = 1/n,$$

then either of the sequences $\{A_1, A_3, ...\}$, $\{A_2, A_4, ...\}$ belong to the sec corresponding to the given cut. To prove this, we must show that $\{A_{2n-1}\}$, $\{A_{2n}\}$ are Cauchy sequences. To this end, we will show that $\{A_{2n}\}$, $\{A_{2n-1}\}$ have lower bounds and that given any E there exists a $p(E)$ such that for all $n>p(E)$ the following are true

$$\text{(I)} \quad A_{2n+1} > A_{2n-1} \qquad \text{and} \quad A_{2n+1} - A_{2n-1} < E; \text{ and}$$

$$\text{(II)} \quad A_{2n} > A_{2n+2} \qquad \text{and} \quad A_{2n} - A_{2n+2} < E.$$

For a lower bound, by the construction of $\{A_{2n}\}$ and $\{A_{2n-1}\}$ the term A_1 of the sequence $\{A_{2n-1}\}$ is less than all other terms of both sequences. Given any E, to find a $p(E)$, find an $E_{n_0} = 1/n_0$ such that $E_{n_0} < E$ and take $p(E)= n_0$. Then for all $n>n_0$ it is true that

$$A_{2n+1} - A_{2n-1} < A_{2n} - A_{2n-1} < 1/n < 1/n_0 < E; \text{ and}$$

$$A_{2n} - A_{2n+2} < A_{2n} - A_{2n-1} < 1/n < 1/n_0 < E.$$

But these are just conditions (I), (II); hence $\{A_{2n}\}$, $\{A_{2n-1}\}$ are Cauchy sequences.

Also given any E, find $E_{n_1} = 1/n_1$ such that $E_{n_1} < E$ and find $p(E)$ as above. Then with this $p(E)$ for all $n>p(E)$ it is true that

$A_{2n} - A_{2n-1} < E$ for any choices of A_{2n}, A_{2n-1} as prescribed. Hence the sequences are tantamount and are contained in the same sec.

We must show that the sec constructed above will, when Rule B is applied, correspond to the given cut. Let us examine X, a lower number, and Y, an upper number for the cut with regard to Rule B.

If $\{A_{2n-1}\}$ is a strictly increasing sequence of lower numbers contained in our sec, we must show that $X \leq A_{2n_0-1}$ for some n_0, and that Y does not satisfy Rule B.

1) If X is a lower number, we may find X_1, also a lower number, where $X_1 > X$ and $1/n_0 < X_1 - X$. Then $X + 1/n_0$ is a lower number, and for the same n_0, $A_{2n_0-1} + 1/n_0$ is an upper number, since $A_{2n_0} - A_{2n_0-1} < 1/n_0$ for all n_0 (A_{2n_0} is an upper number). Hence the

following inequality is true;

$$X + 1/n_0 < A_{2n_0-1} + 1/n_0 \quad \text{for some} \quad n_0.$$

Cancelling $1/n_0$, $X < A_{2n_0-1}$ for some n_0; hence every X is a lower number for the cut corresponding to the sec constructed from a given cut.

2) If Y is an upper number for the given cut, then $Y > A_{2n-1}$ for all n, and Rule B is not satisfied, hence every upper number for the given cut is an upper number for the cut corresponding to the sec constructed from this given cut.

We have thus shown that given a sec, there is a corresponding cut and conversely. We must now show that the correspondence is one-to-one; that is, given $\alpha \neq \beta$, the corresponding cuts, σ, τ, respectively, are such that $\sigma \neq \tau$. To this end, if $\alpha \neq \beta$, this means $\alpha > \beta$, or $\alpha < \beta$. Let us examine the case $\alpha > \beta$. If $\alpha > \beta$, by Definition 41, every pair of Cauchy sequences, ξ strictly increasing in α; η, strictly decreasing in β and such that $\xi > \eta$. If $\xi > \eta$, there exist E and $p(E)$ such that for all $n>p(E)$, $A_n - B_n \geq E$. Pick an $n_0>p(E)$, then $A_{n_0} > B_{n_0}$. By Theorem 91 of Landau, we may find a Z such that $A_{n_0} > Z > B_{n_0}$. Now Z is a lower number for σ since there is a strictly increasing sequence $\{A_n\}$ in α

and an n_0 such that $Z < A_{n_0}$. And for all $n > n_0 > p(E)$; $Z > B_{n_0}$, hence $\{B_n\}$ has an infinite number of terms less than Z, therefore Z does not satisfy Rule A and is an upper number for τ. Thus there exist lower numbers for which are upper numbers for τ; therefore, $\sigma > \tau$ and $\sigma \neq \tau$ as was to be proved. The case $\alpha < \beta$ is proved similarly.

<u>Theorem 193</u>: The ordering of secs and cuts is preserved by the one-to-one correspondence of Rule A; that is, if $\alpha > \beta$, or $\alpha = \beta$, or $\alpha < \beta$, then the corresponding cuts σ, τ are such that $\sigma < \tau$, or $\sigma = \tau$, or $\sigma < \tau$, respectively, and conversely.

<u>Proof</u>: 1) If $\alpha > \beta$, then by the proof in Theorem 192 that the correspondence of Rule A is one-to-one we may find lower numbers for σ which are upper numbers for τ; therefore $\sigma > \tau$.

2) If $\alpha = \beta$, every lower number for is a lower number for τ and every lower number for τ is a lower number for σ; therefore, $\sigma = \tau$.

3) If $\alpha < \beta$, then $\beta > \alpha$, hence, by 1) $\tau > \sigma$,

$$\sigma < \tau.$$

Conversely, the three cases are, in both instances, mutually exclusive and exhaust all possibilities.

Theorem 194: The operations of addition and multiplication are preserved by the one-to-one correspondence of Rule A; that is, given α, β and their corresponding cuts σ, τ, respectively; $\alpha + \beta$ corresponds to $\sigma + \tau$, and $\alpha\beta$ corresponds to $\sigma\tau$.

Proof: Given α, β and their corresponding cuts σ, τ, we must show that the cut corresponding to $\alpha + \beta$ consists of numbers of the form $X + Y$, where X is a lower number for σ and Y is a lower number for τ.

1) Let Z be in the cut corresponding to $\alpha + \beta$. Then, Z satisfies Rule B^1 for all strictly increasing sequences in $\alpha + \beta$. Let us examine the following sequences; $\{A_n\}$, strictly increasing, in α; $\{B_n\}$, strictly increasing, in β, both constructed by Theorem 188. Then $\{A_n + B_n\}$ is in $\alpha + \beta$. Since Z satisfies Rule B^1 for the strictly increasing sequence $\{A_n + B_n\}$ of $\alpha + \beta$, then for the sequence

$A_1 + B_1, A_2 + B_2, \ldots, A_n + B_n, \ldots$ satisfying

$A_1 + B_1 < A_2 + B_2 < \ldots < A_n + B_n < \ldots$, there

exists some n_0 such that

$$Z \leq A_{n_0} + B_{n_0}.$$

Since $Z \leq A_{n_0} + B_{n_0}$, certainly one of the following cases is true;

1) $A_{n_0} < Z \leq A_{n_0} + B_{n_0}$; or

2) $Z \leq A_{n_0}$.

In either case we must show that $Z = X + Y$, where X is a lower number for σ and Y is a lower number for τ.

Case 1: $A_{n_0} < Z \leq A_{n_0} + B_{n_0}$. Certainly A_{n_0} is a lower number for σ, since $A_{n_0} \leq A_{n_0}$ for all n_0, satisfying Rule B. And $Z - A_{n_0}$ is a lower number for τ since $Z - A_{n_0} \leq B_{n_0}$ for some n_0. Thus, $Z = A_{n_0}+(Z-A_{n_0})$ where Z satisfies Rule B and $Z=(X + Y)$, where $X= A_{n_0}$ and $Y=(Z - A_{n_0})$.

Case 2: $Z \leq A_{n_0}$. If L is a lower number for τ, $Z > L$, $Z - L$ is a lower number for σ since $Z - L \leq A_{n_0}$ for some n_0. Hence $Z=(Z-L) + L$ where Z satisfies Rule B and $Z=(X + Y)$, where $Y= L$ and $X=(Z - L)$.

Thus every number Z in the cut corresponding to $\alpha + \beta$ is an $X + Y$, where X is a lower number for and Y is a lower number for τ.

2) If Z is in $\sigma + \tau$, then $Z=X+Y$, where X is a lower number for σ and Y is a lower number for τ. We must show that Z is in the cut corresponding to $\alpha + \beta$. Let $\{A_n\}$ be a strictly increasing sequence in α,

$\{B_n\}$ a strictly increasing sequence in β, then if X is a lower number for σ, there must be some n_1 such that $X \leq A_{n_1}$ and if Y is a lower number for τ, there must be some n_2 such that $Y \leq B_{n_2}$.

Certainly the sequence $\{A_n + B_n\}$ is contained in $\alpha + \beta$ and

$A_1 + B_1 < A_2 + B_2 < \ldots < A_n + B_n < \ldots$. If we choose n_3 to be the maximum of n_1 and n_2, then

$Z = X + Y \leq A_{n_3} + B_{n_3}$, and Z satisfies our Rule B and is a lower number for the cut corresponding to $\alpha+\beta$.

Hence the cut $\sigma + \tau$ and the cut corresponding to $\alpha + \beta$ are the same cut, which is what we wished to prove. Therefore, $\alpha + \beta$ corresponds to $\sigma + \tau$.

The proof that $\alpha\beta$ corresponds to $\sigma\tau$ is quite similar to that given above and will be omitted.

<u>Theorem 195</u>: The one-to-one correspondence of Theorem 192 preserves the correspondence between the sec 1 and the cut 1 and also preserves the operation of successor between integral cuts and integral secs.

<u>Proof</u>: For the first part, we must show that the sec 1 corresponds to the cut 1. Given sec 1, apply Rule B to determine the corresponding cut. That is, if we have a strictly increasing sequence contained in 1^* and constructed by Theorem 188, Z is a lower number for the

cut if there exists an n_0 such that $Z \leq A_{n_0}$. Thus the cut determined has as lower numbers all rational numbers < 1, since for all n it is true that $Z \leq A_n < 1$. And the upper numbers will be all rational numbers ≥ 1, since certainly the strictly increasing sequence is such that there does not exist an n_0 such that $1 \leq A_{n_0}$. And this cut is clearly the cut 1.

Similarly, given any integral sec α and its corresponding integral cut x, we will show that the cut corresponding to the successor of α is the successor of x. With α', apply Rule B to determine the following cut; all rational numbers $< x'$ are lower numbers and all rational numbers $\geq x'$ are upper numbers. This cut is clearly the cut which corresponds to the number x'. But by Landau's Theorem 156, for a given integral cut x we may determine its successor, (cut x)', which is the cut x', hence the cut x' is the successor of cut x. Thus the theorem is proved.

In order that the one-to-one correspondence between secs and cuts gives a correspondence between every sec and every cut, let us make the following definition.

<u>Definition 50</u>: For a given cut which has no smallest upper number, define the corresponding sec to be an irrational sec. The irrational secs will be called

<u>irrational</u> <u>numbers</u> just as the rational secs have the name rational number attached to them.

With Definition 51 the following correspondences hold;

1) The integral secs correspond to the integral cuts and vice versa (by Theorem 195);

2) The rational secs correspond to the rational cuts and vice versa (rational secs are defined in terms of rational numbers);

3) The irrational secs correspond to the irrational cuts and vice versa (Definition 51).

Thus our system of secs includes not only the rational numbers, but the irrationals as well.

CHAPTER IV

CONVERGENCE AND LIMITS

From the system of rational numbers we were able to construct the system of secs which enabled us to extend our number system to include the irrational numbers as well as the rationals. Now let us examine the system of secs to see if it can also be extended by the same procedure. To do this, let us consider a sequence of secs.

Definition 51: A sequence of secs $\{\alpha_n\} = \{\alpha_1, \ldots, \alpha_n, \ldots\}$ is called a Cauchy sequence, if there is a rational sec N such that $\alpha_n \geq N$, $(n=1, 2, \ldots)$ and for every rational sec E (E arbitrarily small), there is an integral sec $p(E)$ such that for every pair of integers $m,n>p(E)$ one of the following is true.

(1) $\alpha_n = \alpha_m$; or

(2) $\alpha_n > \alpha_m$ and $\alpha_n - \alpha_m < E$; or

(3) $\alpha_n < \alpha_m$ and $\alpha_m - \alpha_n < E$.

Definition 52: Two Cauchy sequences of secs, $\{\alpha_n\}$ and $\{\beta_n\}$, are tantamount (in symbols, $\{\alpha_n\} \sim \{\beta_n\}$) if for every E, there is a $p(E)$ such that for all $n>p(E)$ one of the following is true.

(1) $\alpha_n = \beta_n$; or

(2) $\alpha_n > \beta_n$ and $\alpha_n - \beta_n < E$; or

(3) $\alpha_n < \beta_n$ and $\beta_n - \alpha_n < E$.

Otherwise, $\{\alpha_n\} \nsim \{\beta_n\}$.

Theorem 196: $\{\alpha_n\} \sim \{\alpha_n\}$.

Proof: Given any E, take $p(E)=1$, then for all $n>p(E)$

(1) $\alpha_n = \alpha_n$.

Theorem 197: If $\{\alpha_n\}\sim\{\beta_n\}$, then $\{\beta_n\}\sim\{\alpha_n\}$.

Proof: Suppose $\{\alpha_n\}\sim\{\beta_n\}$; given any E, there exists a $p(E)$ such that, for all $n>p(E)$

(1) $\alpha_n = \beta_n$; or

(2) $\alpha_n > \beta_n$ and $\alpha_n - \beta_n < E$; or

(3) $\alpha_n < \beta_n$ and $\beta_n - \alpha_n < E$.

But these cases are equivalent, respectively, to

(1) $\beta_n = \alpha_n$; or

(2) $\beta_n < \alpha_n$ and $\alpha_n - \beta_n < E$; or

(3) $\beta_n > \alpha_n$ and $\beta_n - \alpha_n < E$.

Therefore, $\{\beta_n\} \sim \{\alpha_n\}$.

Theorem 198: If $\{\alpha_n\}\sim\{\beta_n\}$, $\{\beta_n\}\sim\{\gamma_n\}$, then

$\{\alpha_n\} \sim \{\gamma_n\}$.

<u>Proof</u>: The proof is analogous to the proof of Theorem 118.

By Theorems 196 through 198, all Cauchy sequences of secs fall into classes, in such a way that

$$\{\alpha_n\} \sim \{\beta_n\}$$

if and only if $\{\alpha_n\}$ and $\{\beta_n\}$ belong to the same class.

<u>Definition 53</u>: A sequence $\{\alpha_n\}=\{\alpha_1, \alpha_2, \ldots\}$ of secs will be said to <u>converge</u> to a limit sec α_0 (in symbols, $\alpha_n \to \alpha_0$) if and only if given any E, there is a $p(E)$ such that for all $n>p(E)$ it is true that

(1) $\alpha_n = \alpha_0$; or

(2) $\alpha_n > \alpha_0$ and $\alpha_n - \alpha_0 < E$; or

(3) $\alpha_n < \alpha_0$ and $\alpha_0 - \alpha_n < E$.

<u>Theorem 199</u>: Every convergent sequence of secs is a Cauchy sequence; that is, if $\alpha_n \to \alpha_0$, then there exists an N such that $\alpha_n \geq N$ for all n and given E, there is a $p(E)$ such that for all $m,n>p(E)$ it is true that

(1) $\alpha_n = \alpha_m$; or

(2) $\alpha_n > \alpha_m$ and $\alpha_n - \alpha_m < E$: or

(3) $\alpha_n < \alpha_m$ and $\alpha_m - \alpha_n < E$.

<u>Proof</u>: If $\alpha_n \to \alpha_0$, to find an N such that $\alpha_n \geq N$ for all n, proceed as follows; given $E=1/2\ \alpha_0$, there exists a $p(E)$ such that for all $n>p(E)$ one of the following is true

(1) $\alpha_n = \alpha_0$. Then $\alpha_0 - E < \alpha_0 < \alpha_n$; or

(2) $\alpha_n > \alpha_0$ and $\alpha_n - \alpha_0 < E$. Then $\alpha_0 - E < \alpha_0 < \alpha_n$; or

(3) $\alpha_n < \alpha_0$ and $\alpha_0 - \alpha_n < E$. Then $\alpha_0 - E < \alpha_n < \alpha_0$. Hence all terms for $n>p(E)$ have $\alpha_0 - E = 1/2\alpha_0$ as a lower bound. For a lower bound for all n, pick N to be the minimum of $\alpha_1, \ldots, \alpha_{p(E)}$ and $1/2\alpha_0$. And given any E, there exists a $p(E/2)$ such that for all $n>p(E/2)$ it is true that

(1) $\alpha_n = \alpha_0$ or

(2) $\alpha_n > \alpha_0$ and $\alpha_n - \alpha_0 < E/2 < E$; or

(3) $\alpha_n < \alpha_0$ and $\alpha_0 - \alpha_n < E/2 < E$.

Then, with $m,n>p(E/2)$, the following cases arise:

Case 1: $\alpha_n = \alpha_0$, $\alpha_m = \alpha_0$. By logical equality, $\alpha_n = \alpha_m$, satisfying Definition 28, (1).

Case 2: $\alpha_n = \alpha_0$, $\alpha_m > \alpha_0$ and $\alpha_m - \alpha_0 < E/2 < E$.

$\alpha_n = \alpha_0$ gives $\alpha_m > \alpha_n$ and $\alpha_m - \alpha_n < E$, satisfying Definition 28, (3).

Case 3: $\alpha_n = \alpha_0$, $\alpha_m < \alpha_0$ and $\alpha_0 - \alpha_m < E/2 < E$.

$\alpha_n = \alpha_0$ gives $\alpha_n > \alpha_m$ and $\alpha_n - \alpha_m < E$, satisfying Definition 28, (2).

Case 4: $\alpha_n > \alpha_0$ and $\alpha_n - \alpha_0 < E$, $\alpha_m = \alpha_0$.

$\alpha_m = \alpha_0$ gives $\alpha_n > \alpha_m$ and $\alpha_n - \alpha_m < E$, satisfying Definition 28, (2).

Case 5: $\alpha_n < \alpha_0$ and $\alpha_0 - \alpha_n < E$, $\alpha_m = \alpha_0$.

$\alpha_m = \alpha_0$ gives $\alpha_m > \alpha_n$ and $\alpha_m - \alpha_n < E$, satisfying Definition 28, (3).

Case 6: $\alpha_n > \alpha_0$ and $\alpha_n - \alpha_0 < E$, $\alpha_m > \alpha_0$ and $\alpha_m - \alpha_0 < E$. The proof is by subcases;

1) Suppose $\alpha_n > \alpha_m > \alpha_0$.

Adding and simplifying

$$\alpha_n = (\alpha_n - \alpha_0) + \alpha_0 < E + \alpha_m,$$

therefore, $\alpha_n - \alpha_m < E$, satisfying Definition 28, (2).

2) Suppose $\alpha_n = \alpha_m > \alpha_0$. Satisfies Definition 28, (1).

3) Suppose $\alpha_m > \alpha_n > \alpha_0$.

Adding and simplifying

$$\alpha_m = (\alpha_m - \alpha_0) + \alpha_0 < E + \alpha_n,$$

therefore, $\alpha_m - \alpha_n < E$, satisfying Definition 28, (3).

Case 7: $\alpha_n < \alpha_0$ and $\alpha_0 - A_n < E$. $\alpha_m < \alpha_0$ and $\alpha_0 - \alpha_m < E$. The proof is Case 6 with the inequality signs of the subcases reversed.

Case 8: $\alpha_n > \alpha_0$ and $\alpha_n - \alpha_0 < E/2$. $\alpha_m < \alpha_0$ and $\alpha_0 - \alpha_m < E/2$. From the inequalities; $\alpha_n > \alpha_0 > \alpha_m$, therefore $\alpha_n > \alpha_m$;

Adding and simplifying; $(\alpha_n - \alpha_0) + (\alpha_0 - \alpha_m) + \alpha_m < E/2 + E/2 + \alpha_m$.

Therefore, $\alpha_n > \alpha_m$ and $\alpha_n - \alpha_m < E$, satisfying Definition 28, (2).

Case 9: $\alpha_n > \alpha_0$ and $\alpha_0 - \alpha_m < E/2$. $\alpha_m < \alpha_0$ and $\alpha_m - \alpha_0 < E/2$.

From the inequalities; $\alpha_m > \alpha_0 > \alpha_n$, therefore $\alpha_m > \alpha_n$;

Adding and simplifying; $(\alpha_m - \alpha_0) + (\alpha_0 - \alpha_n) + \alpha_n < E/2 + E/2 + \alpha_n$.

Therefore, $\alpha_m > \alpha_n$ and $\alpha_m - \alpha_n < E$, satisfying Definition 28, (3).

Hence in every case the sequence $\{\alpha_n\}$ is a Cauchy sequence and the theorem is true.

<u>Theorem 200</u>: Every Cauchy sequence of secs has a limit.

Before going on to the proof of Theorem 200, let us discuss ideas and notation which will be useful in proving Theorem 200. First of all, since each sec is composed of all sequences which are tantamount to a given Cauchy sequence, we may choose any one of these Cauchy sequences to represent our given sec. But each of these Cauchy sequences may in turn be approxamated by its mth term. Since this is the case, with a small E, we expect p(E) to be large and all $n>p(E)$ to give a better and better approxamation for the sec as p(E) gets large.

Also to be used in Theorem 200 is the following notation: A_{nm}. A rational number which is the nth term of a Cauchy sequence contained in the mth sec of a Cauchy sequence of secs; that is, the first subscript is the number of the term of the sequence and the second is the number of the sec that the sequence is contained in.

<u>Proof</u>: Let $\{\alpha_n\}$ be a given Cauchy sequence of secs. We must show that there is some sec α_0 such that given any E, there exists a p(E) such that for all $n>p(E)$ one of the following is true

(1) $\alpha_n = \alpha_0$; or

(2) $\alpha_n > \alpha_0$ and $\alpha_n - \alpha_0 < E$; or

(3) $\alpha_n < \alpha_0$ and $\alpha_0 - \alpha_n < E$.

Let us examine $\{\alpha_n\}$ term by term. That is, since each term is a sec, by Theorem 188 we are able to construct strictly increasing sequences contained in each sec. Let us proceed as follows; since $\{\alpha_n\}$ is a Cauchy sequence of secs, there is a rational sec N such that $\alpha_n \geq N$ for all n. Hence $\alpha_n > N/2$ for all n. But $N/2$ contains the constant sequence $(N/2)^0 = \{N/2, N/2, \ldots\}$ (here N is a rational number) and this sequence will be chosen to represent the sec $N/2$. As our first representations of the α_n's, choose any strictly increasing sequence as follows:

$$\nu_1 = \{B_{11}, B_{21}, \ldots, B_{n1}, \ldots\} \text{ in } \alpha_1,$$

$$\nu_2 = \{B_{12}, B_{22}, \ldots, B_{n2}, \ldots\} \text{ in } \alpha_2, \text{ and}$$

continuing for each n,

$$\nu_n = \{B_{1n}, B_{2n}, \ldots, B_{nn}, \ldots\} \text{ in } \alpha_n.$$

Since for all n it is true that $\alpha_n > N/2$, then with the ν_n's as representatives of the α_n's and with $(N/2)^0$ as the representative of $N/2$ it is true that $\nu_n > (N/2)^0$. But this means there exist E_n and $p_1(E_n)$ such that for all $m \geq p_1(E_n)$, $B_{mn} > N/2$.

Let us define as second representatives for the α_n's the following sequences:

$$\zeta_1 = \{C_{11}, C_{21}, \ldots, C_{n+1,1}, \ldots\} \text{ where}$$

$$C_{11} = B_{p_1(E_1),1},$$

$$C_{21} = B_{p_1(E_1)+1,1},$$

$$\vdots$$

$$C_{n+1,1} = B_{p_1(E_1)+n,1},$$

$$\vdots$$

Similarly, with $C_{m+1,n} = B_{p_1(E_m)+m,n}$ define the following sequence for a representative of α_n:

$$\zeta_n = \{C_{1n}, C_{2n}, \ldots, C_{m+1,n}, \ldots\} \text{ where}$$

$$C_{1n} = B_{p_1(E_n),n}$$

$$\vdots$$

$$C_{m+1,n} = B_{p_1(E_m)+m,n},$$

$$\vdots$$

Hence we have the following strictly increasing sequences:

$$\zeta_1 = \{C_{11}, C_{21}, \ldots, C_{n1}, \ldots\} \text{ in } \alpha_1,$$

$$\zeta_2 = \{C_{12}, C_{22}, \ldots, C_{n2}, \ldots\} \text{ in } \alpha_2, \text{ and}$$

continuing for each n,

$$\zeta_n = \{C_{1n}, C_{2n}, \ldots , C_{nn}, \ldots\} \text{ in } \alpha_n.$$

And since the α_n's are Cauchy sequences, the following are true: with $E_1=1$, there exists a $p_2(E_1)$ such that for all $m,n \geq p_2(E_1)$ one of the following is true

(1) $C_{n1} = C_{m1}$; or

(2) $C_{n1} > C_{m1}$ and $C_{n1} - C_{m1} < 1$; or

(3) $C_{n1} < C_{m1}$ and $C_{n1} - C_{m1} < 1$. And

continuing this process for each k, with $E_k = 1/k$ there exists a $p_2(E_k)$ such that for all $m,n \geq p_2(E_k)$ one of the following is true

(1) $C_{nk} = C_{mk}$; or

(2) $C_{nk} > C_{mk}$ and $C_{nk} - C_{mk} < 1/k$; or

(3) $C_{nk} < C_{mk}$ and $C_{mk} - C_{nk} < 1/k$.

Hence as our final representatives of the α_n's we define the following sequences:

$$\xi_1 = \{A_{11}, A_{21}, \ldots , A_{n1}, \ldots\} \text{ where}$$

$$A_{11} = C_{p_2(E_1),1},$$

$$A_{21} = C_{p_2(E_1)+1,1},$$

$$\vdots$$

$$A_{n1} = C_{p_2(E_1)+j,1},$$

$$\vdots$$

Similarly, with $A_{kk} = C_{p_2(E_k)+j,k}$ define the following sequence for a representative of α_k:

$$\xi_k = \{A_{1k}, A_{2k}, \ldots , A_{kk}, \ldots\} \text{ where}$$

$$A_{1n} = C_{p_2(E_k),k},$$

$$A_{2n} = C_{p_2(E_k)+1,k},$$

$$\vdots$$

$$A_{kk} = C_{p_2(E_k)+j,k},$$

$$\vdots$$

Hence we have the following strictly increasing sequences:

$$\xi_1 = \{A_{11}, A_{21}, \ldots , A_{n1}, \ldots\} \text{ in } \alpha_1,$$

$$\xi_2 = \{A_{12}, A_{22}, \ldots , A_{n2}, \ldots\} \text{ in } \alpha_2, \text{ and}$$

continuing for each n,

$$\xi_n = \{A_{1n}, A_{2n}, \ldots , A_{nn}, \ldots\} \text{ in } \alpha_n.$$

From this set of representatives of the α_n's choose the following sequence; $\xi_0 = \{A_{11}, A_{22}, \ldots , A_{nn}, \ldots\}$.

We shall show that ξ_0 is a Cauchy sequence and hence it is contained in some sec α_0. Also we shall show that

α_0 is the limit of our given Cauchy sequence of secs.

To show that $\{A_{nn}\}$ is a Cauchy sequence we must show that $\{A_{nn}\}$ has a lower bound and that given any E, there exists a $p(E)$ such that for all $m,n>p(E)$ one of the following is true

(1) $A_{nn} = A_{mm}$; or

(2) $A_{nn} > A_{mm}$ and $A_{nn} - A_{mm} < E$; or

(3) $A_{nn} < A_{mm}$ and $A_{mm} - A_{nn} < E$.

First, we have a lower bound by construction since $A_{nn} > N/2$ for all n. Next, we must show that given any E, there exists a $p(E)$ such that for all $m,n>p(E)$ conditions (1), (2), or (3) are true. By the hypothesis, given any E, there exists a $p_1(E/3)$ such that for all $m,n>p_1(E/3)$ one of the following is true

(1) $\alpha_n = \alpha_m$; or

(2) $\alpha_n > \alpha_m$ and $\alpha_n - \alpha_m < E/3$; or

(3) $\alpha_n < \alpha_m$ and $\alpha_m - \alpha_n < E/3$.

But these three conditions for secs translate into the following conditions since we may choose any sequence as a representation for our secs:

(1) $\xi_n \sim \xi_m$; or

(2) $\xi_n > \xi_m$ and $\xi_n - \xi_m < (E/3)^0$; or

$$(3) \quad \xi_n < \xi_m \quad \text{and} \quad \xi_m - \xi_n < (E/3)^o.$$

With a particular m,n the following conditions are a consequence of the above conditions since we may approximate a sequence by its kth term. That is, there exists a $p_2(E/3)$ such that for all $k>p_2(E/3)$ one of the following is true

$$\begin{array}{llll} (A) & A_{kn} = A_{km}; & \text{or} & \\ & A_{kn} > A_{km} & \text{and} & A_{kn} - A_{km} < E/3; \quad \text{or} \\ & A_{kn} < A_{km} & \text{and} & A_{km} - A_{kn} < E/3; \quad \text{or} \\ (B) & A_{kn} > A_{km} & \text{and} & A_{kn} - A_{km} < E/3; \quad \text{or} \\ (C) & A_{kn} < A_{km} & \text{and} & A_{km} - A_{kn} < E/3. \end{array}$$

But conditions (B) and (C) imply condition (A), so for all $k>p_2(E/3)$ condition (A) is a consequence of the original conditions on the given Cauchy sequence of secs.

With our given E, find k so that $1/k < E/3$, choose $p(E)=k$, then for $m,n>p(E)$ there are 27 cases which arise in showing that ξ_0 is a Cauchy sequence. Following will be given a proof for one of the cases with the remainder of the cases omitted since the proofs are similar.

$$\begin{array}{llll} \text{Case 1:} & A_{nn} > A_{kn} & \text{and} & A_{nn} - A_{kn} < 1/k < E/3, \\ & A_{kn} < A_{km} & \text{and} & A_{kn} - A_{km} < E/3, \\ & A_{km} < A_{mm} & \text{and} & A_{mm} - A_{km} < 1/k < E/3. \end{array}$$

From the Tricotomy Law it follows that

(1) $A_{nn} = A_{mm}$; or

(2) $A_{nn} > A_{mm}$; or

(3) $A_{nn} < A_{mm}$.

We must show that the inequalities of the Cauchy condition are satisfied in subcases 2 and 3 above.

Subcase 1: $A_{nn} = A_{mm}$. Definition 28, (1) is satisfied.

Subcase 2: $A_{nn} > A_{mm}$.

By adding, simplifying, and Lemma 1,

$$A_{nn} + A_{mm} = [(A_{nn} - A_{kn}) + A_{kn}] + [(A_{km} - A_{kn}) + A_{kn}] + (A_{mm} - A_{km})$$
$$< E + A_{mm} + A_{mm},$$

Cancelling A_{mm} and simplifying, $A_{nn} - A_{mm} < E$, satisfying Definition 28, (2).

Subcase 3: $A_{nn} < A_{mm}$. The proof is similar to that of subcase 2 hence will be omitted.

With the proof of all cases the sequence ξ_0 is a Cauchy sequence.

Certainly ξ_0 is contained in some sec, say α_0, for we may take all sequences tantamount to ξ_0 for our sec. We wish to show that α_0 is a limit to our given Cauchy sequence of secs. Below we will show that given any E, there exists a $p(E)$ such that for all $m,n > p(E)$ one of the following is true

(A) $A_{mn} = A_{mm}$; or

(B) $A_{mn} > A_{mm}$ and $A_{mn} - A_{mm} < E$: or

(C) $A_{mn} < A_{mm}$ and $A_{mm} - A_{mn} < E$.

From these conditions we may deduce that given any E, there exists a $p(E)$ such that for all $n>p(E)$ one of the following is true

(1) $\alpha_n = \alpha_0$; or

(2) $\alpha_n > \alpha_0$ and $\alpha_n - \alpha_0 < E$; or

(3) $\alpha_n < \alpha_0$ and $\alpha_0 - \alpha_n < E$.

If one of (1), (2), or (3) happens for all $n>p(E)$, then our given Cauchy sequence of secs has α_0 as a limit.

To show that one of (A), (B), or (C) happens, let us begin as follows; since ξ_0 is a Cauchy sequence, given any E, there exists a $p_1(E/2)$ such that for all $m,n>p_1(E/2)$ one of the following is true

(1) $A_{nn} = A_{mm}$; or

(2) $A_{nn} > A_{mm}$ and $A_{nn} - A_{mm} < E/2$; or

(3) $A_{nn} < A_{mm}$ and $A_{mm} - A_{nn} < E/2$.

And from the proof that ξ_0 is a Cauchy sequence we have that ξ_n is such that for all i,j one of the following is true

(1) $A_{in} = A_{jn}$; or

(2) $A_{in} > A_{jn}$ and $A_{in} - A_{jn} < 1/n$; or

(3) $A_{in} < A_{jn}$ and $A_{jn} - A_{in} < 1/n$.

Take $p_2(E/2) > 2/E$, then for each $n > p_2(E/2)$, choose $i=n$, $j=m$, then one of the following is true

(1) $A_{nn} = A_{mn}$; or

(2) $A_{nn} > A_{mn}$ and $A_{nn} - A_{mn} < 1/n < 1/p_2(E/2) < E/2$;

or

(3) $A_{nn} < A_{mn}$ and $A_{mn} - A_{nn} < 1/n < 1/p_2(E/2) < E/2$.

Take $p(E)$ to be the larger of $p_1(E/2)$ and $p_2(E/2)$. With this $p(E)$, for all $m,n > p(E)$, one of the following cases arises.

Case 1: $A_{nn} = A_{mm}$, $A_{nn} = A_{mn}$. Certainly $A_{mm} = A_{mn}$, satisfying condition (A).

Case 2: $A_{nn} = A_{mm}$, $A_{nn} > A_{mn}$ and $A_{nn} - A_{mn} < E/2 < E$. Substituting equals, $A_{mm} > A_{mn}$ and $A_{mm} - A_{mn} < E$, satisfying condition (C).

Case 3: $A_{nn} = A_{mm}$, $A_{nn} < A_{mn}$ and $A_{mn} - A_{nn} < E/2 < E$. Substituting equals, $A_{mn} > A_{mm}$ and $A_{mn} - A_{mm} < E$, satisfying condition (B).

Case 4: $A_{nn} > A_{mm}$ and $A_{nn} - A_{mm} < E/2 < E$, $A_{nn} = A_{mn}$. Substituting equals, $A_{mn} > A_{mm}$ and $A_{mn} - A_{mm} < E$, satisfying condition (B).

Case 5: $A_{nn} < A_{mm}$ and $A_{mm} - A_{nn} < E/2 < E$, $A_{nn} = A_{mn}$. Substituting equals, $A_{mm} > A_{mn}$ and $A_{mm} - A_{mn} < E$, satisfying condition (C).

Case 6: $A_{nn} > A_{mm}$ and $A_{nn} - A_{mm} < E/2$, $A_{nn} > A_{mn}$ and $A_{nn} - A_{mn} < E/2$. From the Tricotomy Law the proof is by subcases.

Subcase 1: $A_{mn} = A_{mm}$. Condition (A) is satisfied.

Subcase 2: $A_{mn} > A_{mm}$. Adding and simplifying,

$$(A_{nn} - A_{mm})+(A_{nn} - A_{mn}) + 2A_{mn} < E/2 + E/2 + 2A_{nn},$$

hence, $A_{mn} - A_{mm} < E$, and condition (B) is satisfied.

Subcase 3: $A_{mn} < A_{mm}$. The proof will be omitted since it is similar to Subcase 3.

Case 7: $A_{nn} < A_{mm}$ and $A_{mm} - A_{nn} < E/2$, $A_{nn} < A_{mn}$ and $A_{mn} - A_{mm} < E/2$. The proof will be omitted since it is similar to Case 6.

Case 8: $A_{nn} > A_{mm}$ and $A_{nn} - A_{mm} < E/2$. $A_{nn} < A_{mn}$ and $A_{mn} - A_{nn} < E/2$. From the inequalities, $A_{mn} > A_{mm}$. Adding and simplifying, $(A_{nn} - A_{mm})+(A_{mn} - A_{nn}) < E/2+E/2 = E$, hence, $A_{mn} > A_{mm}$ and $A_{mn} - A_{mm} < E$, satisfying condition (B).

Case 9: $A_{nn} < A_{mm}$ and $A_{mm} - A_{nn} < E/2$. $A_{nn} > A_{mn}$ and $A_{nn} - A_{mn} < E/2$. The proof will be omitted since it is similar to Case 8.

Hence for all $m,n > p(E)$, one of conditions (A), (B), or (C) is true and from these conditions we have deduced that α_0 is a limit for our given Cauchy sequence of secs. Thus the theorem is proved.

Theorem 201: Every equivalence class of Cauchy sequences of secs contains a constant Cauchy sequence of secs.

Proof: Given any equivalence class of Cauchy sequences of secs, we must show that any sequence of this class is tantamount to a constant Cauchy sequence of secs. Let $\{\alpha_n\}$ be any element of the equivalence class. By Theorem 200 we know that there is some sec α_0 such that $\alpha_n \to \alpha_0$; that is, given any E, there exists a $p_1(E)$ such that for all $n > p_1(E)$ it is true that

(1) $\alpha_n = \alpha_0$; or

(2) $\alpha_n > \alpha_0$ and $\alpha_n - \alpha_0 < E$; or

(3) $\alpha_n < \alpha_0$ and $\alpha_0 - \alpha_n < E$.

Consider a sequence $\{\beta_n\}$ in which $\beta_n = \alpha_0$ for all n. It is a Cauchy sequence of secs, for given any E, we may take $p_2(E)=1$ and, for all $m,n > p_2(E)$, it is true that

(4) $\beta_m = \alpha_0 = \beta_n$.

We must show that $\{\alpha_n\} \sim \{\beta_n\}$; that is, given any E, there exists a p(E) such that for all $n > p(E)$ it is true that

(5) $\alpha_n = \beta_n$; or

(6) $\alpha_n > \beta_n$ and $\alpha_n - \beta_n < E$; or

(7) $\alpha_n < \beta_n$ and $\beta_n - \alpha_n < E$.

But conditions (5), (6), and (7) are immediate consequences of conditions (1)-(4). Thus p(E) may be taken as the maximum of $p_1(E)$ and $p_2(E)$.

From the system of rational numbers we constructed the system of secs which extended the rational number system to include irrational numbers. In trying to extend our system of secs by the same procedure (Cauchy sequences of secs), Theorems 200 and 201 show that the system of secs cannot be extended in this way. Hence, let us make the following definition.

<u>Definition 54</u>: The secs will henceforth be called "positive numbers"; similarly, what we have been calling "rational numbers" and "integers" will henceforth be called "positive rational numbers" and "positive integers", respectively.

These positive numbers will also be called the positive real numbers. These real numbers are then equivalence classes of Cauchy sequences of rational numbers.

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