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Title: Characterization of the Best Approximations by Classic Cubic Splines

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Joel Davis

This study deals specifically with classical cubic splines. Based on a lemma of John Rice, best approximation in the uniform norm by cubic splines is explored. The purpose of this study is to characterize the best approximation to a given continuous function $f(x)$ by a cubic spline with fixed knots by counting alternating extreme points of its error function $E(t)$.

Characterization of the Best Approximations by Classic Cubic Splines

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Tuen Tuen

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Redacted for Privacy

Head of Department of Mathematics

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Introduction

The study of cubic splines has been a large and interesting subject in mathematics for many years. Much research has been done with different kinds of cubic spline functions. This study will concentrate on the case of the classic cubic spline. *A classical cubic spline is a function that is a piecewise cubic polynomial and is twice continuously differentiable.* Cubic splines are noted for their accuracy of approximation and ease of calculation. In comparison with linear and quadratic splines the cubic spline provides a higher degree of accuracy. At the same time, it is more easily computed than either fourth or fifth degree splines.

The main purpose of this study is to present a theorem which characterizes the best approximation to a given continuous function $f(x)$ by a spline, with fixed knots by counting alternating extreme points of its error function $E(t)$.

Notation

Here is a list of the notation used in this paper:

1. := is the sign indicating " by definition".
2. N:= { 1, 2, 3, ... }
3. f:= is a continuous real valued function on the interval [a, b].
4. S:= is a cubic spline function.
5. I:= interval [a, b].
6. Error function $E := S - f$, where f is a continuous function defined on I and S is some spline approximation of f.
7. m:= is a counting number; $m \in N$.
8. P:= is a polynomial.
9. $\| \|$:= maximun norm on I.
10. **R**:= the real numbers.
11. Ξ:= is the set of <u>knots</u> and $\Xi=\{\xi_i \mid a= \xi_0 < \xi_1< \xi_2 <...< \xi_m< \xi_{m+1}=b\}$.
12. $$(x-\xi)_+^n = \begin{cases} (x-\xi)^n, & \text{for } x> \xi \\ 0, & \text{for } x \le \xi \end{cases}$$
13. A:= is a <u>set of coefficients</u> for a cublic spline; $A = (a_1, a_2, a_3, \dots, a_{m+4})$.

14. Φ:= is a cubic spline basis with $\Phi = \{\phi_j\}$.

$$\Phi = \{\phi_j\}_1^{m+4} = \begin{cases} \phi_j(x) = x^{j-1}, & j=1,2,3,4 \\ \phi_j(x) = (x-\xi_{j-4})_+^3, & j=5,6,\ldots,m+4 \end{cases}$$

15. $S(A,\Xi,X)$:= is a spline function of degree 3 with m interior knots and $S = S(A,\Xi,X) = \Sigma a_i\phi_i(x)$, for $i=1, 2, 3, \ldots, m+4$.

16. ***S*** := set of all cubic splines.

Definitions

1. Basic interval

A basic interval is a closed interval bounded by adjacent knots.

2. Uniform Norm, N

$$N = N(f) = \| f(t) \| = \max_{t\in[a,b]} | f(t) |$$

3. Alternating extreme points

<u>extreme points</u>:

Suppose a function f is defined on an interval [a,b] and c is a point in [a,b]. Then c is an extreme point if $| f(x) | \leq | f(c) |$ for every x in [a,b].

<u>Alternating extreme point</u>

Let $E : [a, b] \rightarrow \mathbf{R}$: A set of alternating extreme points of E is a subset $\{x_j\}$ of extreme points of E with (for each j)

1. $x_j < x_{j+1}$
2. $|E(x_j)| = ||E||$
3. $E(x_j)E(x_{j+1}) < 0$

Splines

In the introduction we defined a classical cubic spline to be *a function that is a piecewise cubic polynomial and is twice continuously differentiable.* A classic cubic spline can be represented using a linear combinations of $\{ \phi_j \}_1^{m+4}$.

In this section we are going to concentrate on root counting. In early college mathematics courses, students learn how to find roots of a polynomial and what single and double roots look like on a graph. Since root counting is an important fact which leads to the main theorem, we are going to define simple and double roots as follows:

Definition. A double root of a function will be an interior root of the function where the function does not change sign. Other roots will be called simple roots.

Remark. For a differentiable function, the derivative is zero at a double root.

Remark. In counting roots, a double root will count as two roots.

Definition. A classical cubic spline has κ suitably placed roots if it has at least κ roots, counting no more than $j+3$ roots in any union of j adjacent basic intervals for each positive integer $j < m+1$. The relationship between cubic splines and suitably placed roots is significant. John R. Rice proved a Lemma in 1967 about interpolation. (Numerical Analysis, Vol 4, No. 4, page 557.). We restate the lemma for the cubic spline case below:

Rice's Lemma. The system

$$(*) \qquad \Sigma\, a_i\phi_i(x_j)=y_j$$

has a solution A for arbitrary values y_j, j=1, 2, ..., m+4,

if and only if

$$x_i < \xi_i < x_{i+4}\ , \qquad \text{where } i=1, 2, ..., m.$$

in this case the solution of (*) is unique.

Following John R. Rice's Lemma, we discovered some interesting results which are specific to a classical cubic spline.

Theorem 1. A classical cubic spline with m interior knots which is not identically zero has no more than m+3 suitably placed roots.

Remark. This means that on any union of j adjacent basic intervals if the spline has at least j+3 roots it is identically zero on that union.

Proof. The proof depends on m, the number of interval knots using Rice's Lemma for a basic interval.

1. $m=0$

In this one interval case, the splines are cubic polynomials. For each root, there is a linear factor of $(x - r_j)$. There can not be more than three roots for a cubic polynomial. This works even for double roots.

2. For m interior knots in $m+1$ intervals

a). Distinct roots

If the spline has at least $m+4$ suitably placed and distinct roots and no more than 4 roots in any basic interval with spline basis $\{\phi_i\}_1^{m+4}$. Then by Rice's Lemma

$$\det(\phi_i(r_j))\neq 0 \qquad i, j=1, 2, \ldots, m+4.$$

and

$$S(r_j)=\Sigma a_i\phi_i(r_j)=0 \qquad i, j=1, 2, 3, \ldots, m+4$$

$$\Rightarrow \quad a_i=0$$

$$\Rightarrow \quad S\equiv 0$$

Therefore, this concludes the proof of Theorem 1 for simple roots.

b). Double roots

If some pairs of r_j are equal; i.e., we have double roots then we argue this way: Using Rice's lemma and positive integer n, we can interpolate a spline T that has the following properties:

$T(r_j) = 0$, for a simple root r_j of S

$= 1/n$, for $r_j = r_{j+1}$ (a double root) $S<0$ near root

$= -1/n$, for $r_j = r_{j+1}$ (a double root) $S>0$ near root

$= 0$, for $r_j = r_{j+1}$ (a double root) $S=0$ near root

(In this last case we also make $T(x)=0$ for some x near r_j where $S(x)$ is zero. If the $S>0$ and $S<0$ cases do not happen then T is identically zero.)

For large enough n, S+T will be a cubic spline with m+4 roots, counting just distinct roots. The spline S+T will have all the simple roots of S and two distinct roots for each double root of S. Thus S+T is identically zero by the previous argument. This is a contradiction since S is not identically zero and, for some j, $T(r_j)$ is not zero.

Thus, this concludes the proof of the Theorem.

In the opposite direction we have:

Theorem 2. Given $k \le m+3$ suitably placed potential roots, $\{r_j\}_1{}^k$, then there exists a cubic spline S which has only these $\{r_j\}_1{}^k$ roots and no others.

Proof.

1. m=0

   Suppose we are given 3 potential roots, $\{r_j\}_1{}^3$. So in one interval [a, b] we can find a cubic polynomial P which has these $\{r_j\}_1{}^3$ as its only roots. We can write P as $P=(x-r_1)(x-r_2)(x-r_3)$. These $\{r_j\}_1{}^3$ are all simple roots. If there is a double root, i.e.: $r_2=r_3$, then we can write P as: $P=(x-r_1)(x-r_2)^2$. Fewer than 3 roots is handled in a similar way. Since a cubic spline S in one interval is just a cubic polynomial, therefore, S exists and S is not identically zero.

2. m=1

   a). Distinct roots

   Suppose we are given 4 potential simple roots $\{r_j\}_1{}^4$ which are suitably placed. Then, by Rice's Lemma we can construct a cubic spline S which has only these 4 simple roots $\{r_j\}_1{}^4$.

   $$\text{Such } S=\Sigma a_i\phi_i(r_j)=0, \qquad \text{for } i=1, 2, \ldots, 5 \quad j=1, 2, 3, 4$$

   and let's choose a $t\in$ [a,b] such that

$S(t)=\Sigma a_i \phi_i(t)=1$, for $i=1, 2, ..., 5$.

Thus, S exists, is not identically zero, and has these $\{r_j\}_1{}^4$ roots. Also it has no other roots by Theorem 1.

b). Double root(s)

i). Given 4 potential roots $\{r_j\}_1{}^4$

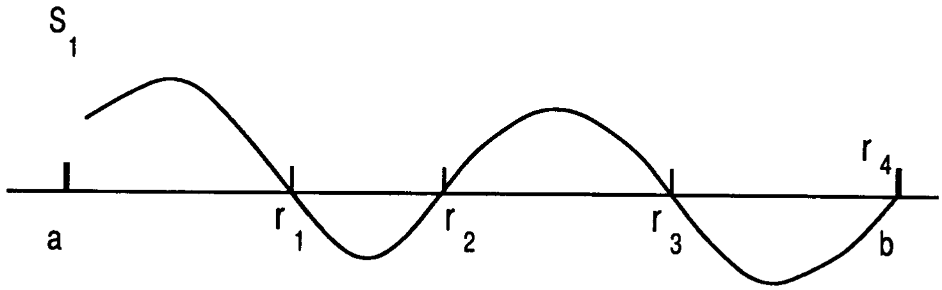
Suppose we are given 4 potential roots $\{rj\}_1{}^4$ which are suitably placed, with a double root, i.e.: $r_2=r_3$

Let $r_3 = r_2 + 1/n$ where $n=1, 2, 3,...$

Construct a spline S_n with roots r_1, r_2, r_3, and r_4 as in part 1) and norm which is defined in Part I. Let n go to ∞. Then, r_3 moves to r_2 as n goes to ∞. A subsequence of S_n converges to a spline S by compactness of bounded set of cubics. S has the appropriate roots. By previous theorem S has no additional suitably placed roots.

ii). Given 3 potential roots

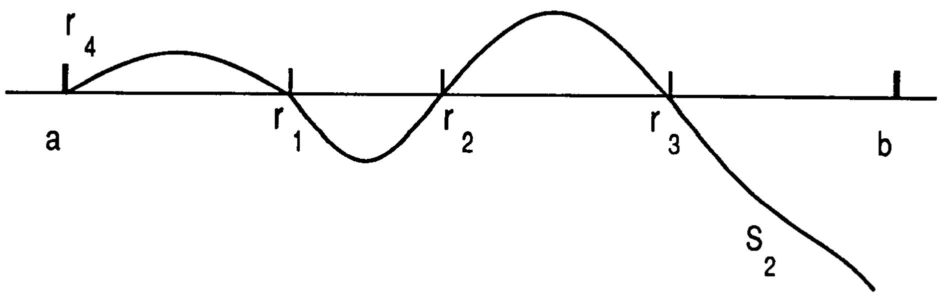
We know that we can make a cubic spline with 4 roots. Now, suppose we are given 3 potential roots. If a function S_1 has 4 roots and one root is at one of the end-point. (see Figire 1.)



Figire 1.

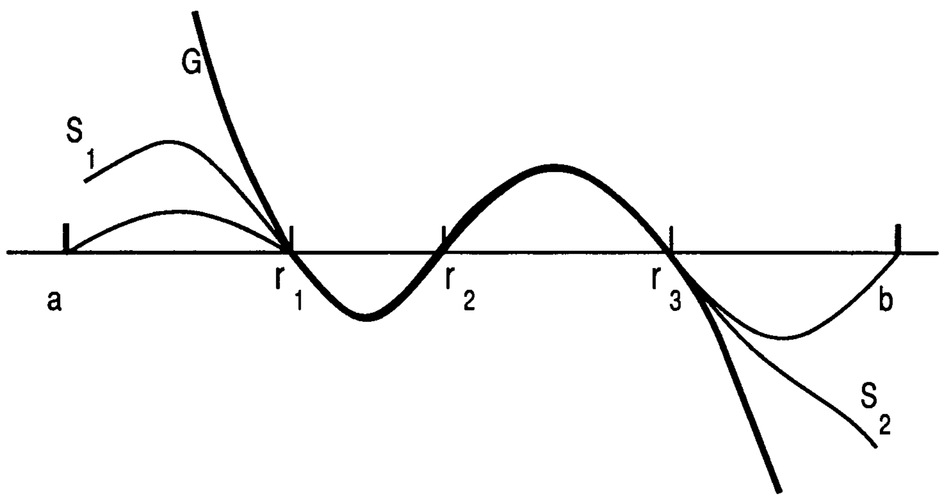
We can construct another function S_2 with 4 roots and one root is at the other end-point.

The other 3 roots are the same as S_1. (see Figire 2.)



Figire 2.

By the same argument as in the previous proof, each S has an unique solution with its four simple roots (counting double roots twice). So that if $S_1 + S_2 = G$, G would have only 3 roots at r_1, r_2, r_3 positions. (see Figire 3.)

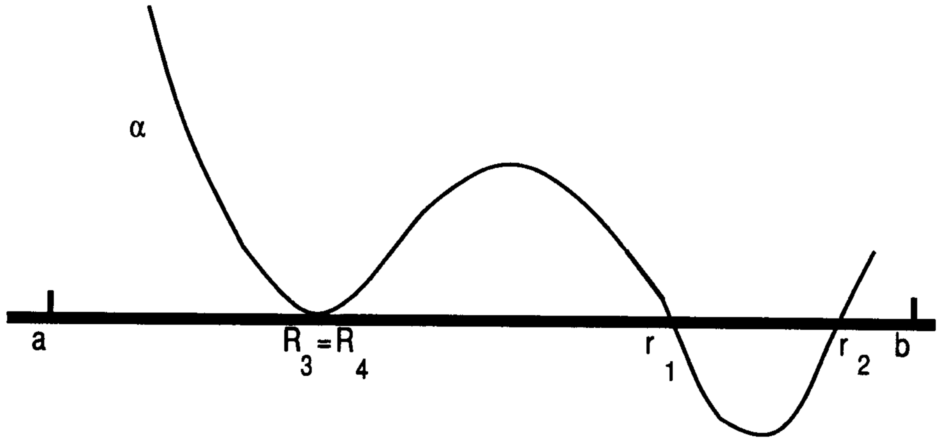


Figire 3.

If r_1=a, or r_3 = b, or both, we can extend the interval [a, b]. Then we are able to construct splines by previous method. Thus, we can contruct a cubic spline with 3 roots.

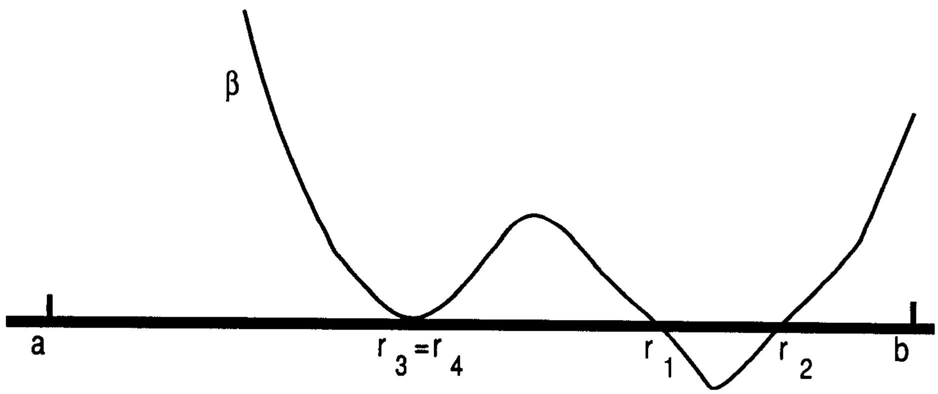
iii). Given 2 potential roots.

Since we know that we can construct a cubic spline with 4 roots, we construct a spline, α ,with 4 roots. Two are simple roots, and one is a double root. (see Figire 4.)



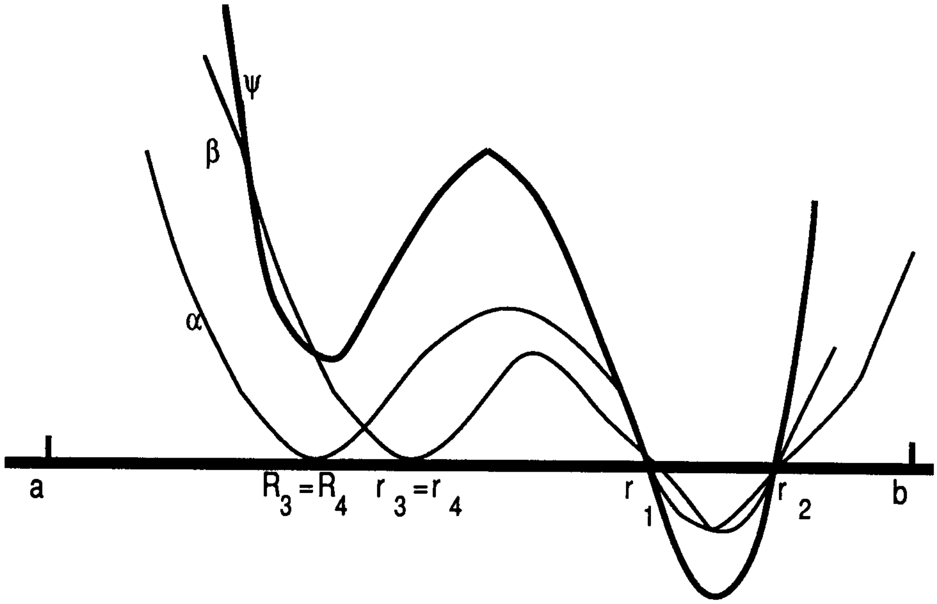
Figire 4.

Then, add a similar spline, β , with r_1 and r_2 as two simple roots, and one double root which is nearby α's double root. (see Figire 5.)



Figire 5.

Then let $\psi = \alpha + \beta$, and ψ has only two simple roots. (see Figire 6.)



Figire 6.

iv). Given one potential root

If there is only one potential root, then we can use the two end-points to construct 2 roots. Also by extending the interval, using previous 2, we can construct a cubic spline with one single root.

Thus, the result of m=1 case is cubic spline S having no more than 4 suitable placed roots.

3. m interior knots.

Suppose we are given m+3 potential simple roots $\{r_j\}_1^{m+3}$ which are suitably placed throughout [a,b]. i.e.: $r_i < \xi_i < r_{i+4}$,

$i=1,2,\ldots,m$, then we can construct a cubic spline S which has $\{r_j\}_1^{m+3}$ as it's roots. Such that

$$S(r_j)=\Sigma a_i \phi_i(r_j)=0 \qquad \text{for} \quad \begin{array}{l} i=1, 2, \ldots, m+4 \\ j=1, 2, \ldots, m+3 \end{array}$$

let's choose a $t \in [a,b]$ such that $S(t)=\Sigma a_i \phi_i(t)=1$. By Rice's Lemma, S exists, has a unique solution, and is not identical zero.

If there are some double roots amoung the m+3 potential roots $\{r_j\}_1^{m+3}$, then we use the same method as for the case m=1 part 2) to construct simple roots from all double roots. If there are less than m+3 potential roots, then we can use the end-point method, and/or the combine-splines-with-double-roots method. These methods were used on pages 10, 11,12,13, and 14.

This concludes the proof of the theorem.

The Best Approximation Theorem

The root-counting theorem in the previous section is a necessary tool for establishing a characterization theorem for best approximation. The error function, E, and alternating axtrema points, which are defined in the beginning of this paper, are the basis for the characterization theorem for best approximation.

Now, we restate **the Best Approximation Theorem** by I.M. Singer. Let f be a continous function. Then S is a best approximation in uniform norm if and only if the error function, $E=S-f$, satisfies the condition below:

$$\mathrm{Max}\ E(t)\ h(t) \geq 0 \quad \text{for } \forall\ h \in \boldsymbol{S}, \quad \text{where } t \in \text{ extreme points of } E.$$

Remark. Besides Singer's long and complex proof, one way to prove this is to use the directional derivative of $\| S-f \|$ with respect to S. At a minimum, this derivative must be nonnegative for each direction h.

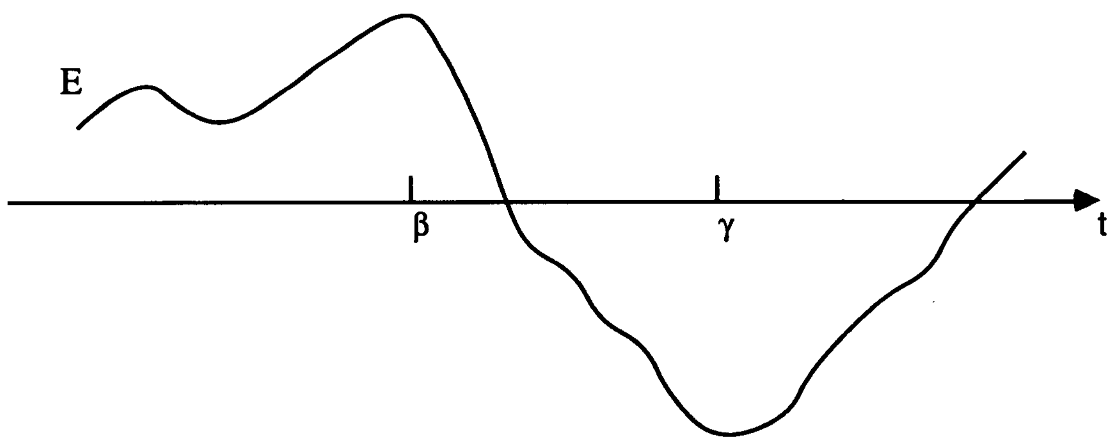
Theorem. Let f be a continous function on [a,b] and let $S(A,\Xi,X)$ be a cubic spline with m fixed interior knots. S is a best approximation to f if and only if for some nonnegative integer j, the error function, $E=S-f$, has j+5 alternating extreme points in the union of j+1 adjacent basic intervals.

One implication of the theorem states:

Let f be a continous function on [a,b] and let $S(A,\Xi,X)$ be a cubic spline with m fixed interior knots. Assume that for some nonnegative integer j, the error function, $E=S-f$, has j+5 alternating extreme points in the union of j+1 basic adjacent intervals. Then S is a best approximation to f.

Proof. Let j be the least j for which E has j+5 alternating extreme points in the union of j+1 adjacent basic intervals. without loss of generality, j = m; for if j < m, then we could apply the following proof to the j+1 basic intervals instead. Let E have m+5 alternating extreme points. So that E has at least m+4 roots in m+1 intervals. We use the method of contradiction to prove this implication of the theorem.

Assume $E(t)h(t)<0$ for all extreme points t .



Graph 7.

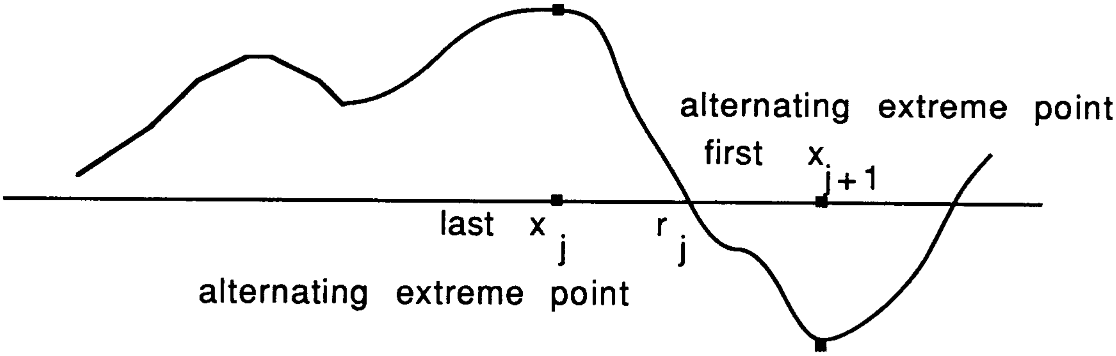
Let's look at one part of the error function E from the Figire 7 above. Let β and γ be adjacent alternating extreme points of E. Since h and E must have opposite sign at extreme points, h must have one root between β and γ, and must also change sign. That is true for every adjacent pair of alternating extreme points. So h has at least m+4 simple roots. Because of the choice of the least j at the beginning of the proof, these roots can be chosen to be suitably placed. However, by the root-counting theorem from the last section, a spline with m+4 suitably placed roots is identically zero. Thus

$$\Rightarrow \qquad h \equiv 0 \quad \text{where } h \in \boldsymbol{S}$$

$$\Rightarrow \qquad \max E(t)\ h(t) \geq 0$$

which is a contradiction with the assumption made above.

Before we prove converse of the last theorem, we are going to carefully describe a particular way to choose a root r_j of E between two adjacent alternating extreme points. To do this, we pick the very last point x_j of one group of alternating extreme points, and the very first point x_{j+1} of the next group of alternating extreme points with opposite sign. (see Figire 8.)



Figire 8.

Between each two opposite alternating extreme points, there is at least one root, R of E. So we choose r_j=R.

Now, we prove the converse of the theorem:
If S is the best approximation to f, then E has at least m+5 alternating extreme points in [a,b].

Proof. Let S be the best approximation to f in [a, b]; then by Singer's Theorem, the error function E satisfies the condition below:

$$\text{Max } E(t)h(t) \geq 0 \quad \text{for } \forall\ h \in \boldsymbol{S}, \text{ and } t \in \text{ extreme points of } E.$$

Assume that the error function E has m+4 or fewer alternating extreme points in [a,b]. From the previous section, we can find an $h \in \boldsymbol{S}$ so that h has only these m+3 simple roots $\{r_j\}$. We can choose the sign of h so that

$$E(t)\ h(t) < 0 \quad \text{for } t \in \text{ extreme points of } E$$

This contradicts our assumption.

Consequently, if S is the best approximation, then error function E must have m+5 or more alternating extreme points. If the above roots are not suitably placed then we can apply the above argument to some subinterval which is the union of several basic intervals. Thus, the proof of this theorem is completed.

We note that if E alternates too few times, then we have constructed a direction h to search for a better approximation. This is the basis for the Remes algorithm (also called exchange algorithm), which has been inducled in many approximation theory books.

Example

Here is an example using a spline with 3 fixed interior knots (m=3) on the interval [0,1] to approximate the function f = exp(x). The cubic spline of best approximation is

$$S(x) = a + bx + cx^2 + dx^3 + A(x-0.25)_+^{\,3} + B(x-0.50)_+^{\,3} + C(x-0.75)_+^{\,3}$$

with

a = 0.9999904756
b = 1.0005464294
c = 0.4940500173
d = 0.1937944127
A = 0.046738283
B = 0.070834890
C = 0.085522924

Figure 9 below is the graph of error function E.

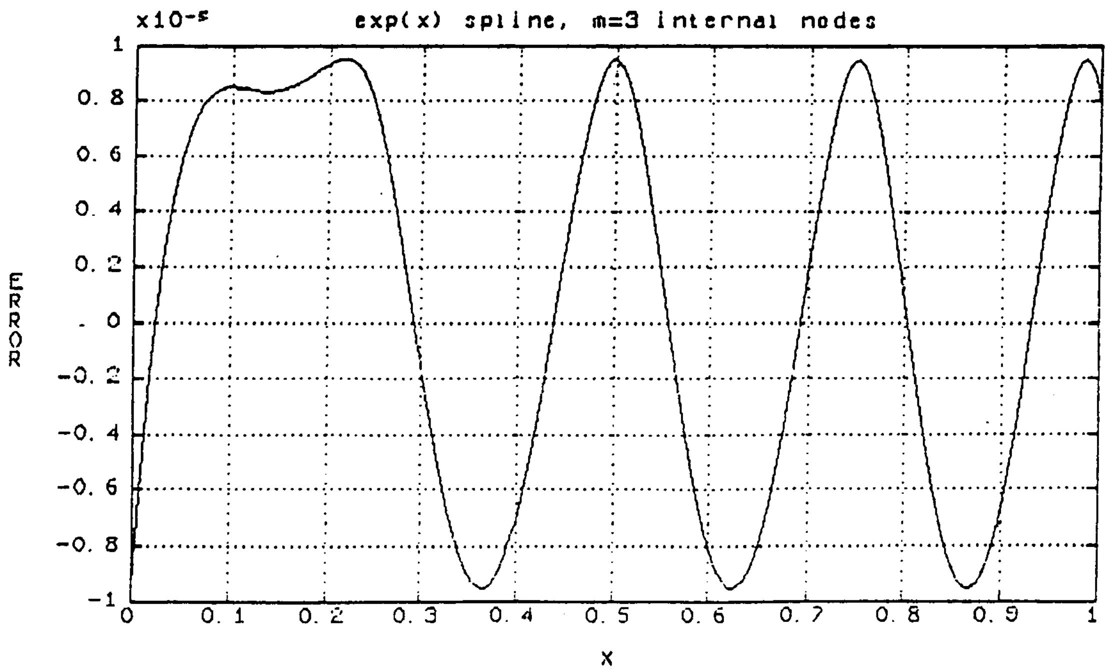


Figure 9

The maximum error = 9.524423E-0006 at x = 0 and seven other places. From the graph above, we can count that there are eight alternating extreme points. These alternating extreme points $\{q_j\}_0{}^7$ satisfy:

$$q_0 = 0$$

$$0.20 < q_1 < 0.25$$

$$0.35 < q_2 < 0.40$$

$$0.48 < q_3 < 0.52$$

$$0.61 < q_4 < 0.65$$

$$0.73 < q_5 < 0.77$$

$$0.85 < q_6 < 0.88$$

$$0.96 < q_7 < 1.00$$

We can apply the theorem for the specific case of three fixed knots. By the theorem we know that the cubic spline is the best approximation to the function f = exp(x).

Conclusion

The Theorem proved here describes the relationship between alternating extreme points and the best approximation. We characterized the best approximation to a given continuous function f by a spline using alternating extreme points of the error function E. This study deals specifically with classical cubic splines. In addition, this study leads to more open questions about splines in general. For example, how does the Hermite spline behave? Do Hermite splines behave in the same way as the classical cubic spline? What kinds of conditions do Hermite splines need? These questions require more study.

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