AN ABSTRACT OF THE THESIS OF

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Abstract approved: ______________________________

Edward Waymire

We consider some mathematical problems involving the asymptotic analysis of rooted tree structures. River channel networks, patterns of electric discharge, eletrochemical deposition and botanical trees themselves are examples of such naturally occuring structures. In this thesis we will study the width function aymptotics of some random trees as well as certain deterministic self-similar trees. The width function counts the number of branches as a function of the distance to the root. The main results are: (i) A proof of convergence of the width function to a Brownian excursion local time process; (ii) A probabilistic derivation of the expected width function asymptotic; (iii) Asymptotic computation of width functions for a class of deterministic self-similar trees. The result (i) solves a weak version of a conjecture of Aldous (1991) in the case of geometrically distributed offspring and corrects a scale factor there. This result provides the basis for the approach in (ii). The results in (iii) apply to the expected Galton-Watson branching tree under appropriate conditioning.

SOME TREE STRUCTURE FUNCTION ASYMPTOTICS

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Qing Zhang, Author

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SOME TREE STRUCTURE FUNCTION ASYMPTOTICS

1. INTRODUCTION

The abundance in nature of branching structures has led to extensive studies of their properties. Population growth, river channel networks, electrochemical deposition and botanical trees themselves are examples of such structures. The study of *branching processes* associated with branching structures has a long history and involves a large number of interesting mathematical and statistical problems. Branching processes are often modeled by various *tree networks* (or simply *trees*). In this thesis we focus on two rooted tree models, one random and the other deterministic. *A rooted tree* is defined as consisting of a set of *vertices* connected to one another by *edges* so as not to produce loops; see Definition 2.1.1. One such random rooted tree model is the *Galton-Watson tree* (also called *Galton-Watson branching process*). This model was originally introduced as population growth model by Galton in 1873 in the study of extinction of family names in the British peerage and first successfully attacked by Watson in that year (see Harris (1963)). The model can be thought of as representing an evolving population of particles. It starts at time 0 with 1 particle representing the root in the corresponding tree graph, which after one unit of time generates a random number of offspring according to some probability law. The total number of offspring produced by the particle constitutes the first generation. These go on to produce a second generation, and so on. The number of "offspring" produced by a single "parent" particle at any given time is independent of the history of the process, and of other particles existing at the present. This makes the evolution a *Markov Process.* Another rooted tree model is *river channel network* which can be modeled by *a finite binary random tree* obtained by conditioning the binary Galton-Watson tree on its *size* (i.e: *total progeny*). In this way all trees with a given size are equally likely to occur; see Shreve (1966, 1967) and Definition 2.1.4. One considers a rooted tree and thinks of the water as flowing from the *exterior vertices* of the tree (the sources of the river) to the root (the outlet of the river). The distance between successive *junctions* in such a river network are not constant, and therefore are modeled by random variables, generically referred to as *weight.*

In this thesis we will study the so called *width function asymptotics.* The *width function* of a tree network counts the number of branches as a function of the distance to the root of the tree. The main results are: (i) A probabilistic derivation of the expected width function asymptotics; (ii) A proof of convergence of the width function to a *Brownian excursion local time process*; (iii) Aymptotic computation of width functions for a class of deterministic *self-similar trees.*

In Chapter 2 we give the formal definition for a rooted labeled tree network. Some terms related to tree networks such as *height, width function and weighted trees* etc are also defined. Meanwhile some tree structure examples and previous results about width function asymptotics are described.

In Chapter 3 we state the results (i) and (ii). For (i) we give a new approach to previous results by Troutman and Karlinger (1984) under some special assumptions. They proved the results by using renewal structure via transform techniques. Our new proof is a probabilistic approach. First we establish a relationship between the width function of our special tree and *local time of a random walk excursion on the* *tree.* Then we use *local time of Brownian excursion*, which is the limiting process of corresponding local time of random walk excursion, to approximate the width function of the tree which gives the *Rayleigh density.* Meanwhile (ii) gives a proof of weak version of one of the several conjectures in Aldous (Conjecture 4, 1991). Our solution also corrects a scale factor there.

In Chapter 4 we define and study some deterministic self-similar trees. Result (iii) gives the computations of width functions of two classes of self-similar trees. One is the *Peano tree* whose width function was considered in Marani etc (1991). The other one is the so-called *b-ary tree* which describes the "expected Galton-Watson tree" when a vertex has 0 or b offspring with equal probability. *Cascade measures* and the special structures are used to show that the width function of a Peano tree *weakly converges* to a *singular measure* with a Cantor distribution function. On the other hand, for the b-ary trees the asymptotic width function is absolutely continuous and in fact uniformly distributed.

2. DEFINITIONS, EXAMPLES AND SOME KNOWN RESULTS

In this chapter we begin with the definition of tree networks (or simply trees). Then we will introduce some notation used in the sections that follow. We will define some functions associated with the tree structure such as *weight, height, and width function* etc. Some classical examples will be discussed. Finally we will state some known results and an open problem about the asymptotic width functions of networks.

2.1. SOME DEFINITIONS AND NOTATION OF TREE STRUCTURE

Definition 2.1.1 *A rooted labeled tree* T is a collection of finite sequences of non-negative integers $< 0, i_1, i_2, \ldots, i_n >$, $i_j \in \{1, 2, \ldots\}$, such that

(i) The root of T is represented by $< 0 >$.

(ii) If $< 0, i_1, i_2, \ldots, i_n > \in T$, then $< 0, i_1, i_2, \ldots, i_k > \in T$ for $0 \leq k \leq n$ where the case $k = 0$ means the sequence $< 0 >$.

(iii) If $< 0, i_1, i_2, \ldots, i_n > \in T$ and if $1 \leq j \leq i_n$, then $< 0, i_1, i_2, \ldots, i_{n-1}, j > \in T$.

Each $v = < 0, i_1, i_2, \ldots, i_n >$ is called *vertex* of T. A vertex $< 0, i_1, i_2, \ldots, i_k >$ with $k < n$ is a *prefix* to a vertex $< 0, i_1, i_2, \ldots, i_n >$. The special prefix $< 0, i_1, i_2, \ldots, i_{n-1} >$ to $< 0, i_1, i_2, \ldots, i_n >$ is called the *parent* of $< 0, i_1, i_2, \ldots, i_n >$. A pair of vertices are connected by an *edge* (adjacent) if and only if one of them is the parent of the other. An edge may be labeled by the *descendent* vertex, ie: let $< 0, i_1, i_2, \ldots, i_k >$ denote the corresponding edge e which connects vertex $< 0, i_1, i_2, \ldots, i_{k-1} >$ to vertex $< 0, i_1, i_2, \ldots, i_{k-1}, i_k >$. This specifies the graph structure of T, and according to Definition 2.1.1 this makes T a rooted connected graph without cycles.

Definition 2.1.2 The *height of a vertex* $v =< 0, i_1, i_2, \ldots, i_n >\in T$ is $h(v) = n$. The *height of a tree* T is

$$h(T) = max_v\{h(v) : v \in T\}. \tag{2.1}$$

Definition 2.1.3 If T is a tree and $n \in \mathbb{N}$, the *truncation of* T to its first n level is

$$T|n := \{v \in T : h(v) \leq n\}, \tag{2.2}$$

which is a subtree of height at most n.

A tree is called *locally finite* if its truncation to every finite level is finite. Let $\mathcal{T}$ be the space of locally finite trees. We define a metric on $\mathcal{T}$ by

$$d(T, T') = \frac{1}{sup\{n : T|n = T'|n\}}. \tag{2.3}$$

Then $\mathcal{T}$ is a complete and separable metric space. The collection of all finite trees is a countable dense subset of $\mathcal{T}$. For a finite tree $T \in \mathcal{T}$ and arbitrary n, $[T]_n = \{S \in \mathcal{T} : S|n = T|n\}$ defines a finite dimensional event. Let $\mathcal{B}$ denote the σ-field generated by the finite dimensional events. Then $\mathcal{B}$ is the Borel σ-field of $\mathcal{T}$. We now make the following definition.

Definition 2.1.4 A *random tree* is a probability distribution on $(\mathcal{T}, \mathcal{B}, P)$.

We shall also refer to a $\mathcal{T}$ valued random variable T with random tree distribution P as a random tree. Let $\nu = \nu(T)$ be the total number of vertices in T. Let $< e >$ be an edge (or link) of T. We associate with each edge $< e >$ of the tree T a *random weight* $W(e)$ where $\{W(e) : e =< 0, i_1, \ldots, i_n >, n \geq 1, i_n \geq 1\}$ is a denumerable collection of independent and identically distributed random variables (i.i.d), independent of T. We let $Z_0 = Z_0(T)$ be the number of vertices in T at level 0. Obviously, $Z_0 = 1$. Vertices at level 1 are labeled as $< 0, i_1 >$, $1 \leq i_1 \leq Z_1$ where $Z_1 = Z_1(T)$ is the total number of vertices in T at level 1. We arrange the order of these vertices from left to right, i.e: $< 0, 1 >$ is on the far left, next (if it exists) is $< 0, 2 >$, etc, and $< 0, Z_1 >$ is on the far right. Recursively, the label$< 0, i_1, i_2, \ldots, i_{k+1} >$ with $i_j \geq 1$ and $1 \leq j \leq k+1$ is assigned to one of the vertices adjacent to $< 0, i_1, i_2, \ldots, i_k >$. Let $Z_k = Z_k(T)$ be the total number of vertices in T at level k. In general, a label of the form $< 0, i_1, i_2, \ldots, i_k >$ is a vertex of T which is connected to the root $< 0 >$ by a self-avoiding path in T of exactly k edges. For convenience, one may add an edge (stem) to the root $< 0 >$, so that root can be regarded as connected to a ghost vertex by a stem. Figure 2.1 is an example of a labeled tree as described above.

Definition 2.1.5

(1) *The weighted height of vertex* $< 0, i_1, i_2, \ldots, i_k >$ is defined as

$$H_w(0, i_1, i_2, \ldots, i_k) = \sum_{j=1}^{k} W(0, i_1, i_2, \ldots, i_j). \tag{2.4}$$

(2) *The weighted tree height* is defined as

$$H_w(T) = max_{<e> \in T} H_w(e). \tag{2.5}$$

Definition 2.1.6 *The contour size* of the weighted random tree T at height h, also called the *width function* of T, is defined as

$$Z(h) = \#\{< e >\in T : H_w(e) \leq h < H_w(e')\}, \tag{2.6}$$

where $< e >$ is the parent of $< e' >$, and $\#$ denotes the cordinality of the set.

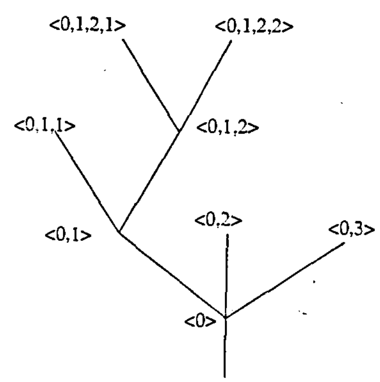


Figure 2.1. A rooted labeled tree

If we consider the link lengths as weight $\{W(e)\}$ for $< e >\in T$, in the special case where all $W(e) = W(0, i_1, i_2, \ldots, i_k) = 1$, the width function is given by

$$\begin{aligned} Z(h) &= \#\{< e >\in T : H_w(e) \leq h < H_w(e')\} \\ &= \#\{< 0, i_1, i_2, \ldots, i_k >\in T, k \leq h < (k+1)\} \\ &= Z_k \cdot 1(k \leq h < k+1), \end{aligned} \tag{2.7}$$

where Z_k is the total number of vertices at level k.

2.2. EXAMPLES OF TREE STRUCTURE

In this section, we will present two examples related to tree structure. Some notation and results from these examples will be used in the chapters that follow.

2.2.1. Example: Galton-Watson Branching Process

There are two ways of describing the *Galton-Watson branching process.* One is the usual way in which the Galton-Watson branching process is decribed in terms of generation random variables Z_n (defined in the previous section) as a Markov process. The other is a way of directly putting a measure on the space of trees. We start with the second one. Let p_k be the offspring distribution, i.e: p_k is the probability that a single particle generates k particles, $k = 0, 1, 2, \ldots$, and $\sum_k p_k = 1$. The Galton-Watson process can be defined by consistently specifying a probability distribution of the finite dimensional sets (defined in section 2.1) as follows: Let $T \in \mathcal{T}$.

At $n = 0$: Define $[T]_0 = \{S \in \mathcal{T} : S|0 = T|0\} = \{S \in \mathcal{T} : S|0 =< 0 >\} = \mathcal{T}$. Define $P([T]_0) = 1$.

At $n = 1$: Define $[T]_1 = \{S \in \mathcal{T} : S|1 = T|1\} = \{S \in \mathcal{T} : S|1 = \{< 0 >, < 0, 1 >, < 0, 2 >, \ldots, < 0, Z_1(T) >\}\}$. Define $P([T]_1) = p_i$ where $i = Z_1(T)$.

At $n = 2$: Define

$$[T]_2 = \{S \in \mathcal{T} : S|2 = T|2\} = \{S \in \mathcal{T} : S|2 = \{<0>, <0,1>, \ldots, <0, Z_1(T)>,$$
$$<0,1,1>, <0,1,2>, \ldots, <0,1,j_1>,$$
$$<0,2,1>, <0,2,2>, \ldots, <0,2,j_2>,$$
$$\ldots\ldots$$
$$<0, Z_1(T), 1>, <0, Z_1(T), 2>, \ldots, <0, Z_1(T), j_{Z_1(T)}>\}.$$

Define $P([T]_2) = p_i p_{j_1} p_{j_2} \ldots p_{j_i}$ where $Z_1(T) = i$, $Z_2(T) = j_1 + j_2 + \ldots + j_i$. Let $v \in T$ be a vertex with height $h(v) = n$. Let $v_n = v_n(T) = \{v \in T : h(v) = n\}$. Let $\#(v) = \#\{j \in T :< v, j >\in T\}$, i.e: number of particles at the $(n+1)$th level generated by $v \in v_n$. By induction, define

$$P([T]_{n+1}) = P([T]_n) \cdot \prod_{v \in v_n(T)} p_{\#(v)}.$$

Consistency holds since $\sum_j p_j = 1$ and can be checked by induction. So this defines a probability measure P on $(\mathcal{T}, \mathcal{B})$ by the Kolmogorov consistency theorem.

The Galton-Watson branching process is also usually defined as follows: Let Z_n be the random variable which denotes the number of vertices at the level n, the nth generation. A Galton-Watson process is a *Markov chain* $\{Z_n : n = 0, 1, 2, \ldots\}$ on the nonnegative integer $\mathbb{Z}^+$ with initial distribution $P(Z_0 = 1) = 1$ and transition function defined by $p(i,j) = P(Z_{n+1} = j | Z_n = i)$ where

$$P(Z_{n+1} = j | Z_n = i) = \begin{cases} p_j^{*i} & \text{if } i \geq 0,\ j \geq 0 \\ \delta_{0j} & \text{if } i = 0,\ j \geq 0 \end{cases}$$

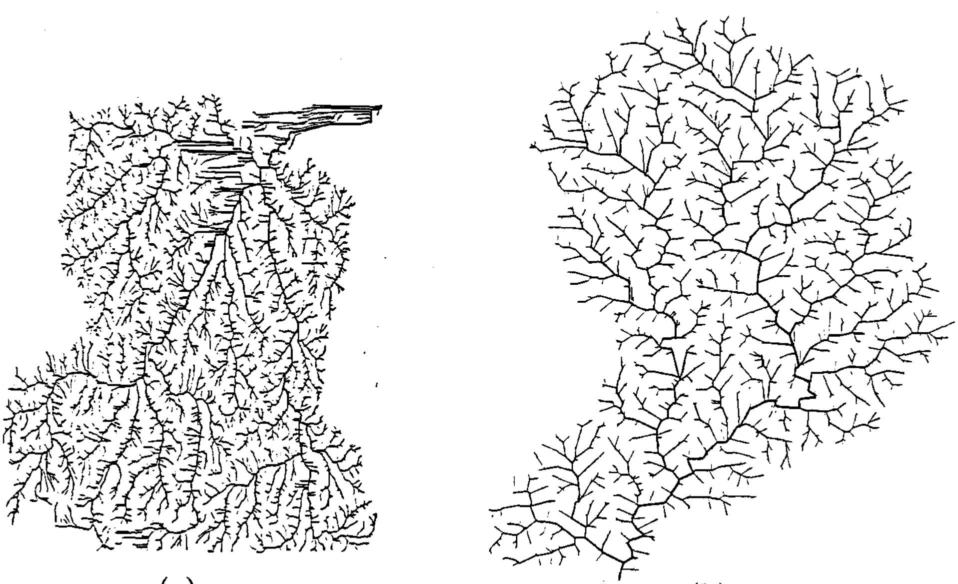
and p_j^{*i} is the i-fold convolution of p_j and δ_{0j} is the Kronecker delta. The consistency of these definitions is given in Harris (1963).

If the offspring distribution p_k is *geometric*, i.e: $p_k = pq^{k-1}$ with $p + q = 1$ for $k = 1, 2, \ldots$, then the associated tree is called a *geometric tree*. We will discuss the asymptotic width function of geometric tree in Chapter 3, where we see the advantage of this choice of distribution is its connection with *random walk excursion* given in Durrett, Kesten, Waymire (1991). If $p_0 = p$, $p_2 = q$, $p + q = 1$, then the branching tree is called the *binary tree*. The *extinction time* of the branching process is usually defined as $\tau = \tau(T) = min\{k \geq 1 : Z_k = 0\}$. So the usual extinction time coincides with the height defined in (2.1). The *total progeny* ν is the total number of individuals produced by the process. i.e: $\nu = \nu(T) = \sum_{k=0}^{\tau-1} Z_k$ and will be referred to as the *size* of the tree in the context of this paper.

2.2.2. Example: Random Model of River Channel Networks

In this section we will give an introduction to *Shreve's random tree model* for river channel networks. In section 2.3 we will state some known results about the asymptotic behavior of the width functions of these random trees. We will also describe some of the emperical structure of naturally occuring river networks. These are obtained using the River-Tools computer code written by Scott Peckham (1995a) using *Interactive Data Language*, and one-degree digitized elevation data produced by the defence Mopping Agency and distributed by the U.S. Geological Survey over the world wide web.

The pioneering work in the stochastic analysis of channel networks is Horton (1945) in which the *Horton Laws* were formulated. By restricting attention only to the graphical structure of channel networks, Shreve (1966, 1967) introduced a widely acknowledged random model in which Horton Laws and other morphometric relationships would be studied. In Shreve's construction, an idealized river channel network is represented by a binary branching tree graph. The root of the tree corresponds to the *outlet* or point farthest downstream of the network. The exterior vertices of the binary tree denote *sources* of the network which are points farthest upstream. The interior vertices of tree represent *junctions* which are points at which any two channels join. Exterior edges of the tree are referred to be the segments of the network connecting a junction to a source, called *exterior links.* The segments between two succesive junctions or between the outlet and the first junction upstream are called *interior links* which are represented by interior edges of the tree. The number of sources (or exterior vertices) is called the network *magnitude* (see Figure 2.2).



(a) (b)

Figure 2.2. (a) The South Willamette River (b) The Sub-Kentucky River

A binary tree of magnitude m has size $\nu = 2m - 1$ (total progeny). Let C_m be the number of different trees of magnitude m. Then

$$C_m = \frac{1}{2m-1}\begin{pmatrix} 2m-1 \\ m \end{pmatrix}. \tag{2.8}$$

This result goes back to Cayley (1859). It also can be derived by using the recursive character of the tree structure. Shreve (1966) proposed that, for a given magnitude m, in the absence of geologic controls the C_m distinct trees should be considered as equally likely. In the study of river channel networks hydrologists often supplement the Shreve's random tree model by the assumption that the link lengths are independently identically distributed (i.i.d) random variables. Under these assumptions Hack (1957) compared the length of *main channel* (the largest path from the outlet to a source) to basin area and found the relation $L_H \sim CA^\beta$ where L_H denotes the length of the main channel, the exponent β is a parameter, C is a coefficient that depends on the units of measurement, and A is the network size. This log-log linear relation between main channel and size has come to be known as *Hack's Law.*

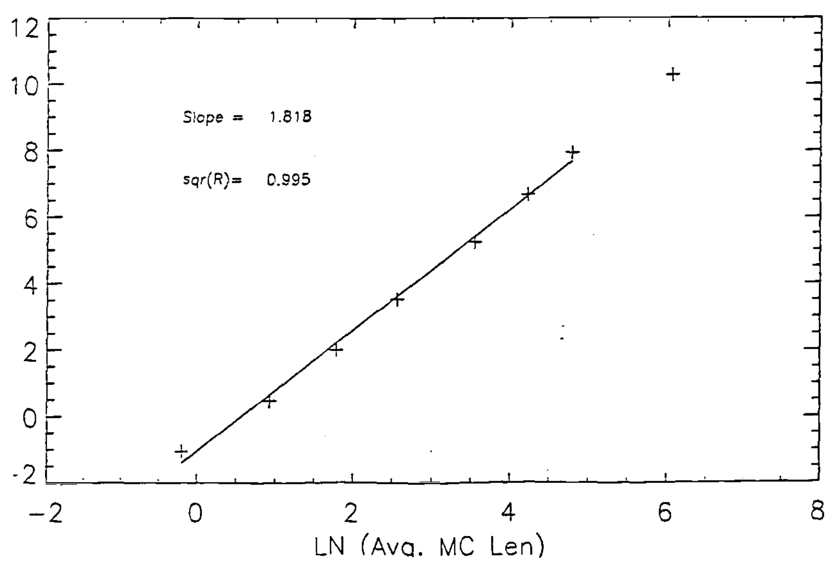


Figure 2.3. Hack's law curve for the Willamette River

There is an important connection between the Shreve's random tree for river network of magnitude m and the Galton-Watson branching process by conditioning on the event that the size of the branching tree equals $2m-1$. Conditioning on total progeny, while natural from the consideration here, does not seem to be very prominent in the classical branching process literature. However there is a large literature on asymptotics conditioned on time to extinction which were motivated by biological applications (see Harris (1968)). Using this connection and the assumption that the link lengths are i.i.d random variables, various authors have studied the asymptotic bahavior of the main channel length. Gupta, Mesa and Waymire (1990) showed that if the link lengths are i.i.d random variables according to an exponential distribution with rate parameter λ, then the conditioned mean main channel length has asymptotic form $\frac{2\sqrt{m\pi}}{\lambda}$. The same asymptotic formula is valid with $\lambda = 1$ in the case where the link lengths are nonrandom and equal, cf. Troutman and Karlinger (1984); Waymire (1989). Durrett, Kesten and Waymire (1991) extended the results to very general link length distributions.

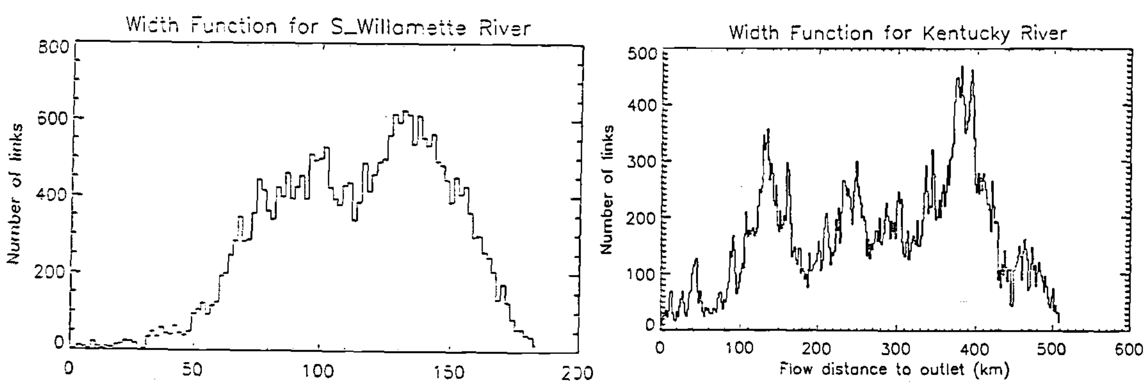


Figure 2.4. Examples of width function

The width function of a network is often considered by hydrologists as a means of estimating flow hydrographs from a map of the river basin in place of actual gauges. In the idealization of constant flow the width function provides the probability of randomly distributed input to reach the outlet by time t. The width function for the South Willamette River (see Figure 2.2) is depicted below for illustration. In section 2.3 we shall describe some corresponding theoretical results for the width function for Shreve's random model.

In the study of river networks, Horton (1945) devised a scheme for indexing the hierarchical structure of the streams. Streams starting from the sources of a river network are assigned the order 1 and, moving downstream, a confluene of streams raises the order of the resulting stream. Strahler (1964)) modified this ordering scheme to make it independent of metric or directional properties of the streams. Either scheme applies to all tree-like structures.

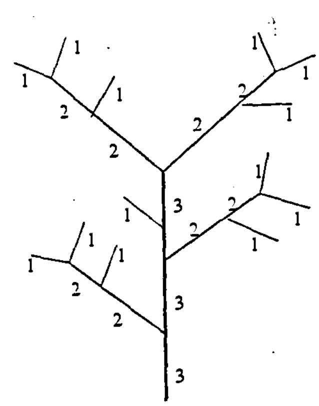


Figure 2.5. Example of Horton-Strahler ordering

Given a rooted tree structure, the Horton-Strahler orders are defined recursively by the following rules: All exterior edges are given the order 1, and an interior edge starting where two edges of order i and order j meet is given the order $max\{i,j\} + \delta_{i-j,0}$, where δ_{mn} is the Kronecker delta. A branch of order i is a maximal connected sequence of edge of order i. When the order of a branch is equal to that of one of its sub-branches, it is considered to be a continuation of this branch, otherwise it is considered to be a new branch. The order of a whole tree $O(T)$ is defined to be the order of the root, its lowest branch. When a branch has more than two sub-branches, only the two of highest orders are considered. Figure 2.5 indicates the Horton-Strahler ordering scheme on a tree of order $O(T) = 3$. The motivation for this ordinary algorithm will be made clear in section 4.1.

2.3. SOME KNOWN RESULTS

Here we present some results with regard to the asymptotic behavior of the width function of a binary random tree T.

Let $\{W(e) :< e >\in T\}$ be independent random weights such that the distribution of $W(e)$ depends on the location of $< e >$ only through the total progeny $\nu = \nu(T) = n$ of T. Let G_n denote the distribution function of a weight $W(e)$, then

$$G_n(x) = P[W(e) \leq x|\nu = n]. \tag{2.9}$$

We assume that $\{G_n(x)\}$ are of same type up to scale change, that is, there is a sequence β_n of positive scale parameters such that

$$G_n(x) = G_1(\beta_n x), n = 1, 2, \ldots. \tag{2.10}$$

Remark: There are qualitatively different geomorphologic cases of interests. The first one is furnished by planimetric maps where the $\{W(e)\}$ represent the link lengths of channel networks. The second case involves topographic maps where the $\{W(e)\}$ denote drops in elevation between channel junctions.

Let $\mu_n(x)$ be the mathematical expectation of the width function evaluated at x given total progeny $\nu = n$, i.e:

$$\mu_n(x) = E[Z(x)|\nu = n]. \tag{2.11}$$

Let K_n be the normalization constant defined by

$$K_n = \int_0^\infty \mu_n(x)\, dx. \tag{2.12}$$

We define a measure F_n such that

$$\frac{d}{dx}F_n(x) = a_n K_n^{-1} \mu_n(a_n x), h \geq 0 \tag{2.13}$$

where a_n is positive scale parameter. In the following we state some results about asymptotic behavior of F_n.

Theorem 2.1

Case 1: For $\beta_n = 1$, i.e: $\{W(e)\}$ are i.i.d random variables. If $G_1(x)$ has a moment generating function in a neighborhood of the origin, then taking $a_n = \mu\sqrt{n}$, where μ is the mean of G_1, F_n converges weakly to a measure F which is uniquely determined by

$$\int_0^\infty h^k F(dx) = 2\sqrt{\pi}\frac{k!}{\Gamma(\frac{k+1}{2})}, k = 1, 2, \ldots. \tag{2.14}$$

Moreover, F has a Rayleigh density $F'(x) = \frac{x}{2}e^{-\frac{x^2}{4}}$.

Case 2: For $\beta_n = e^{(n-1)\theta}$, $\theta > 0$, if in addition G_1 is exponentially distributed, then taking $a_n = \mu$, $F_n \overset{d}{\Longrightarrow} F$ and

$$\int_0^\infty x^k F(dx) = \frac{\mu^k e^{k\theta} \tilde{r}(\frac{1}{4}e^{-(k+1)\theta})k!}{\tilde{r}(\frac{1}{4}e^{-\theta})\sqrt{\prod_{j=1}^k (1-e^{-j\theta})}}, k = 1, 2, \ldots, \tag{2.15}$$

where

$$\tilde{r}(s) = \sum_n r_n s^n = \frac{1}{2}(1 - \sqrt{1-4s}), \tag{2.16}$$

is the generating function for the number r_n of trees of total progeny n, respectively.

Remarks:

(1) Case 1 was proved by Troutman and Karlinger (1984), and Case 2 is in Waymire (1992). Both results are approached by analytical methods using renewal structure via transform techniques. In Chapter 3, we will use probabilistic approach to prove Case 1 under some special assumptions. In particular we will give a solution to a special case of Conjecture 4 in Aldous (1991).

(2) For $\beta_n = n^\theta$, $\theta > 0$, the problem solved in Theorem 2.1 is still an open problem. Gupta and Waymire (1989) gave some numerical approach to the case $\beta_n = n^\theta$, which suggested numerically that a limit function exists. David Aldous in a private conversation with Waymire made the following conjectures about this case: If $\theta > \frac{1}{2}$, and $a_n = \sqrt{n}$, then the limit distribution has a expression in terms of Brownian excursion which involves θ not, F_1. If $\theta > \frac{1}{2}$ and $a_n = 1$, then limit distribution has an expression in terms of the discrete infinite tree which involves θ, not F_1.

In Troutman and Karlinger (1984), the expected width function of the random tree T given both total progeny $\nu(T) = n$ and order $O(T) = m$ was also studied. If we let

$$\xi_{m,n} = E[Z(x)|\nu(T) = n, O(T) = m] \tag{2.17}$$

and

$$K_{m,n} = \int_0^\infty \xi_{m,n}(x)\,dx, \tag{2.18}$$

then one has

Theorem 2.2 *If $H_{m,n}$ is the distribution function of $\xi_{m,n}(x)$, then*

$$\lim_{n\to\infty} H_{m,n}(\sqrt{n}\mu x) = H_m(x), \tag{2.19}$$

where $H_m(x)$ has density given by $h_m(x) = \beta_m^{-1} \cdot 1_{[0,\beta_m]}(x)$, and β_m is a scale parameter.

In Chapter 4 we shall explore a deterministic version of this result in which we replace the Galton-Watson random tree T conditioned on size (total progeny) and order by a deterministic tree which has an "expected" self-similarity property of a Galton-Watson tree conditioned on size and order.

3. A PROBABILISTIC APPROACH

3.1. INTRODUCTION AND STATEMENT OF RESULTS

In Chapter 2 we introduced Shreve's random model for networks. Based on this model, Troutman and Karlinger (1984) (see also Theorem 2.1 case 1) have studied the width functions of channel networks having arbitrary independent exterior and interior link length distributions. Here link length is considered as random weight $W(e)$. The width $Z(x)$ for a fixed $x \geq 0$ is defined to be the number of links with the property that the weighted distance to the downstream junction from the outlet is $< x$ and the distance to the upstream junction from the outlet is $\geq x$ (see also Definition 2.1.6). From the viewpoint of branching processes the width function $Z(x)$ can be interpreted as the number of individuals alive in the branching population at time x. In (2.11) and (2.12), $\mu_n(x)$ and K_n are respectively defined as the conditional expectation of $Z(x)$ and the normalization constant. If we take $a_n = \sqrt{n}$ in (2.13), then $F_n \Longrightarrow F$ where $F'(x) = \frac{x}{2}e^{-\frac{x^2}{4}}$, a *Rayleigh density* and "$\Longrightarrow$" denotes the *convergence in distribution*; i.e: pointwise convergence F_n to F at all points of continuity of F. This result was first obtained in Theorem 5.4 of Troutman and Karlinger (1984) by analytic techniques.

In this chapter we will provide a probabilistic approach to prove the above result under some special assumptions in Theorem 3.1. This new proof makes the special form of limiting Rayleigh density entirely transparent from a probabilistic point of view. Meanwhile we will give a proof of weak version of Conjecture 4 in

Aldous (1991) under the following assumptions in Theorem 3.2. We assume that the random tree model T has the following properties:

(1) The tree T is ordered as in section 2.1 and $\nu(T) = n$;

(2) The links of T are of equal length. Without loss of generality, let $W(e) = 1$, for $< e > \in T$;

(3) The offspring distribution of T is geometric. i.e: $p_k = P[Z_1 = k | Z_0 = 1] = r^k(1-r)$, $k = 0, 1, 2, \ldots$ $(0 < r < 1)$.

In section 3.2 some background for this approach will be provided. In section 3.3 we will set up a connection between random walk and this special tree model T. Finally we will use some known results summarized in section 3.4 to show the following theorems in section 3.5.

Theorem 3.1: *Let $Z(x)$ be the width function of such T. $\mu_n(x)$ and K_n are as defined in (2.11) and (2.12). If $f_n(x) = \frac{\mu_n(x)}{K_n}$ and $F_n(x) = \int_0^{\sqrt{n}x} f_n(y)\, dy$, then $F_n \Longrightarrow F$ and $F'(x) = 2xe^{-x^2}$ for $x \geq 0$.*

Theorem 3.2: *Let Z_n be the total number of individuals alive in the nth level of T. Let $W_0^+(t)$ be the Brownian excursion on $[0,1]$ defined in section 3.2.2. Then*

$$\int_a^b \frac{Z_{[\sqrt{n}t]}}{\sqrt{n}}\, dt \Longrightarrow \int_0^1 1\left(\frac{a}{\sqrt{2}} \leq W_0^+(t) < \frac{b}{\sqrt{2}}\right) dt. \tag{3.1}$$

for any $0 < a < b$.

Remark: The Rayleigh density in Theorem 3.1 differs from that computed in Troutman and Karlinger (1984) by a scale factor of $\frac{1}{2}$. The reason for this is that T in that paper is binary tree. While the scale factor which appears in Aldous (1991) is corrected by (3.1).

3.2. SOME BACKGROUND

3.2.1. Random walk and Random walk excursion

We may think of a particle moving randomly among the integers according to the following rules. At the time $n = 0$ the particle is at the origin. At the time $n = 1$ it moves either one unit forward to +1 or one unit backward to -1, with respective probability p and $1-p$. In the case $p = \frac{1}{2}$ this process may be accomplished by tossing a fair coin and making the particle move forward or backward corresponding to the occurence of a "head" or a "tail", respectively. Let $\{X_i : i = 1, 2, \ldots\}$ be a sequence of identically independently random variables (i.i.d) defined on a probability space $(\Omega, \mathcal{F}, P)$. We assume $P[X_i = +1] = P[X_i = -1] = \frac{1}{2}$. Let

$$S_n = X_1 + X_2 + \ldots + X_n, S_0 = 0 \tag{3.2}$$

be the position process. Then $\{S_n : n = 0, 1, 2, \ldots\}$ is a discrete-parameter *Markov chain* with transition probability $p_{ij} = \frac{1}{2}\delta_{i,i+1} + \frac{1}{2}\delta_{i,i-1}$ and intial state 0.

Definition 3.2.1 The stochastic process $\{S_n : n = 0, 1, 2, \ldots\}$ is called the *simple symmetric random walk* starting at the origin.

Sample realizations of a stochastic process are referred to as *sample paths*. Since the sample paths of $\{S_n\}$ are of the form $\{S_0, S_1, S_2, \ldots, S_n, \ldots\} \in \mathbb{Z}$, therefore, discontinous, it is convenient to *linearly interpolate* the random walk between one dot and the next. The *polygonal process* $\{S_n(t)\}$ is defined by

$$S_n(t) = S_k + (t-k)(S_{k+1} - S_k), \tag{3.3}$$

where $k \le t \le k+1$ and $k = 0, 1, \ldots, n-1$. $\{S_n(t)\}$ is referred to as the *interpolated random walk.* Figure 3.1(a) plots the sample paths of $\{S_n\}$ up to time $n = 8$ if $X_1 = -1, X_2 = +1, X_3 = +1, X_4 = +1, X_5 = +1, X_6 = -1, X_7 = +1$ and $X_8 = +1$.

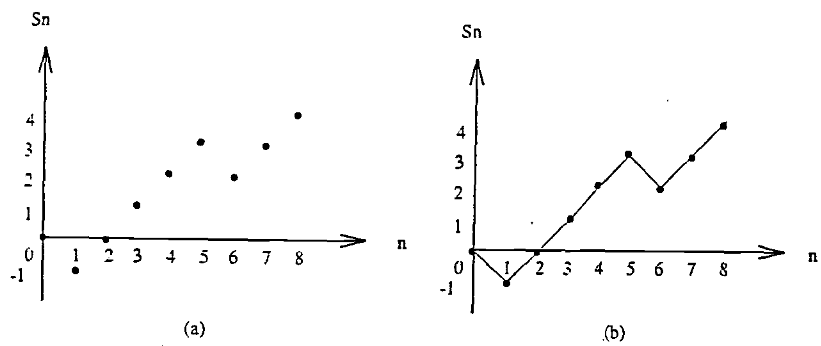


Figure 3.1. Random walk (a) and (b)

Figure 3.1(b) plots the corresponding sample paths of interpolated random walk $\{S_n(t)\}$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n, n \ge 1\}$ an increasing sequence of sub-σ-algebra of $\mathcal{F}$, that is, $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$. A measurable function $\tau = \tau(\omega)$ taking values $1, 2, \ldots, \infty$ is called a *stopping time* relative to $\{\mathcal{F}_n\}$ or simply $\{\mathcal{F}_n\}$*-stopping time* if $\{\tau = k\} \in \mathcal{F}_k$, $k = 1, 2, \ldots$. A stopping time is completely determined by the sets $\{\tau = n\}$, $1 \le n \le \infty$.

Definition 3.2.2 Let $\tau = \tau^1 = \inf\{k \ge 1 : S_k = 0\}$ and $\tau^{(j+1)} = \inf\{k \ge 1 : S_{\tau^j + k} = 0\}$. Then $\tau_j = \sum_{i=1}^{j} \tau^i$ are called the *return times to the origin.* Especially, τ is called the *first return time to 0.*

Note: Each τ_j is stopping time for $\{S_n\}$ in the sense that $\{\tau_j = n\} \in \sigma\{S_k : k \le n\}$.

In the following we will consider stochastic processes of the form $\{Y_k : k = 0, 1, 2, \ldots\}$, say, defined on a probability space $(\Omega, \mathcal{F}, P)$ and an event $A_n \in \sigma(Y_0, Y_1, \ldots, Y_n)$ with $P(A_n) > 0$. We use the notation by restricting the process $\{Y_k : 0 \le k \le n\}$ to $(\Omega, \mathcal{F} \cap A_n, P_n)$, where $P_n(E) = \frac{P(E)}{P(A_n)}$, $E \in \mathcal{F} \cap A_n$.

Definition 3.2.3 The conditioned process $\{S_k : 0 \le k \le n | \tau = n\}$ up to time n is called the *(simple symmetric) random walk excursion* on $[0, n]$.

Similar to random walk, we can connect the sample paths of $\{S_k : 0 \le k \le n | \tau = n\}$ to get the polygonal process $\{S_n(t) : 0 \le t \le n | \tau = n\}$ which we refer to as the sample paths of the *interpolated random walk excursion.* Figure 3.2 is an example of the sample paths of the interpolated random walk excursion $\{S_k : 0 \le k \le 8 | \tau = 8\}$ such that $X_1 = +1, X_2 = +1, X_3 = -1, X_4 = +1, X_5 = +1, X_6 = -1, X_7 = -1, X_8 = -1$.

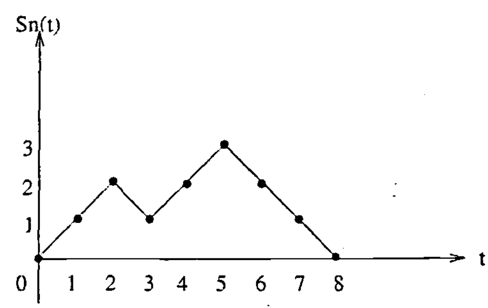


Figure 3.2. Random walk excursion

3.2.2. Brownian motion and Brownian excursion

In section 3.2.1 we defined two discrete-parameter stochastic processes, random walk and random walk excursion, and their continuous extension by interpolation. In this section we will look at some continuous-parameter stochastic processes, so called *Brownian motion* and *Brownian excursion* (see Bhattacharya and Waymire (1990), Billingsley (1986) and Ito and Mckean (1965)) which can be obtained as limits of the random walk and random walk excursion after suitable re-scaling of space and time.

Definition 3.2.4 A *Brownian motion with drift μ and diffusion coefficient σ^2* starting at 0 is a stochastic process $\{W_t : t \geq 0\}$ having continuous sample paths and the following three additional properties:

(1) $P[W_0 = 0] = 1$

(2) The increments are independent: if

$$0 \leq t_0 \leq t_1 \leq \ldots \leq t_k,$$

then for Borel sets $H_1, H_2, \ldots,$

$$P[W_{t_i} - W_{t_{i-1}} \in H_i, i \leq k] = \prod_{i \leq k} P[W_{t_i} - W_{t_{i-1}} \in H_i] \tag{3.4}$$

(3) For $0 \leq s < t$, the increment $W_{t+s} - W_t$ is normally distributed with mean $s\mu$ and variance $s\sigma^2$: for a Borel set H

$$P[W_{t+s} - W_t \in H] = \frac{1}{\sqrt{2\pi\sigma^2 s}} \int_H e^{-\frac{(x-\mu s)^2}{2\sigma^2 s}}\, dx. \tag{3.5}$$

A Brownian motion with zero drift and diffusion coefficent of 1 is called the *standard Brownian motion* denoted by $W(t)$ for $t \geq 0$.

Definition 3.2.5 Let $\{W(t) : t \geq 0\}$ be the standard Brownian motion and

$$\begin{aligned} \tau_1 &= sup\{s \leq 1 : W(s) = 0\} \\ \tau_2 &= inf\{s \geq 1 : W(s) = 0\}. \end{aligned} \tag{3.6}$$

We call

$$W_0^+(t) = \frac{|W(\tau_1 + t(\tau_2 - \tau_1))|}{\sqrt{\tau_2 - \tau_1}} \tag{3.7}$$

the *Brownian excursion* over $[0, 1]$ which is a continuous, non-homogenous Markov process with transition density (Ito and Mckean (1965)).

$$p(t, y) = \frac{y^2 e^{-\frac{y^2}{2t(1-t)}}}{\sqrt{2\pi t^3 (1-t)^3}}. \tag{3.8}$$

Remark: As will be discussed more precisely in section 3.4, a number of authors have studied the weak convergence of some conditioned families of Brownian processes. These results may be found in Billingsley (1968), Kaigh (1974, 1975, 1976) and Iglehart (1978) etc. The relation between Brownian motion and Brownian excursion was established in Durret, Iglehart and Miller (1976). They showed that Brownian excursion is the weak limit of *Brownian bridge* conditioned to be positive, where Brownian bridge B_t^* is Brownian motion "tied down at 0 at time 1"; more precisely $B_t^* = B_t - tB_1$, $0 \leq t \leq 1$.

3.2.3. Local time and Occupation time

Let $\{S_n : n = 0, 1, 2, \ldots\}$ be a simple symmetric random walk, i.e: $S_n = \sum_{i=1}^{n} X_i$ and $S_0 = 0$ where each X_i takes on the values $+1$ or -1 with probability $\frac{1}{2}$. Let $S_n(t)$ be the corresponding interpolated random walk in (3.3).

Definition 3.2.6 We define for any $x \in \mathbb{R}$,

$$L_n(S_n, x) = \frac{1}{2} \sum_{k=0}^{n} 1(|S_k| = [x]) \tag{3.9}$$

as the *local time of random walk* $\{S_n\}$ *at* x, and for a Borel set $A \in \mathcal{B}(\mathbb{R})$

$$O_n(S_n, A) = \frac{1}{2} \sum_{k=0}^{n} 1(|S_k| \in A) \tag{3.10}$$

as the *occupation time of random walk* $\{S_n\}$ *on* A.

Note: Observe that (3.9) is a special case of (3.10) for $A = \{x\}$ and

$$L_n(S_n, x) = \frac{1}{2} \int_0^n 1([x] \leq |S_n(t)| < [x] + 1)\, dt. \tag{3.11}$$

Let $\{W(t) : t \geq 0\}$ be a standard Brownian motion on $\mathbb{R}$. If $S_n^*(t)$ is defined by setting

$$S_n^*(\frac{k}{n}) = \sum_{i=1}^{k} \frac{X_i}{\sqrt{n}} \tag{3.12}$$

for $k = 0, 1, 2, \ldots$ and interpolating linearly, a theorem of Donsker states that $S_n^* \Longrightarrow W(t)$. Observe that $S_n^*(t)$ is the rescaled interpolated random walk corresponding to S_n. Knight (1963) studied the joint continuous process $L(t, x)$ which satisfies

$$L(t, x) = \frac{1}{2} \frac{d}{dx} \int_0^t 1(W(s) \leq x)\, ds \tag{3.13}$$

for all $t \geq 0$ and $x \in \mathbb{R}$ a.s. Let L_n^* on $[0,\infty)$x$[-\infty,\infty]$ by setting

$$L_n^*(\frac{j}{n}, \frac{k}{\sqrt{n}}) = \frac{1}{2}\sum_{i=0}^{j-1} \frac{1(S_i = k)}{\sqrt{n}} \tag{3.14}$$

for $j = 0, 1, 2, \ldots$ and $k \in \mathbb{Z}$. Note that L_n^* is the local time corresponding to S_n^*. In the spirit of Knight (1963), Perkins (1981) used techniques from nonstandard analysis to get $(S_n^*, L_n^*) \Longrightarrow (W, L)$ on $C(\mathbb{R})$x$C([0,\infty)$x$[-\infty,\infty],\mathbb{R})$.

Definition 3.2.7 $L(t, x)$ is called *the local time of Brownian motion* $W(t)$.

Let $\{W_0^+(t)\}$ be a Brownian excursion process defined in section 3.2.2. Chung (1976) defined the occupation time of $\{W_0^+(t)\}$ during an excursion in the following definition.

Definition 3.2.8 *The occupation time of* $W_0^+(t)$ is

$$S([a,b)) = \int_0^1 1(a \leq W_0^+(t) < b)\, dt \tag{3.15}$$

for $0 < a \leq b < \infty$.

The expectation of $S([a,b))$ was also calculated by Chung (1976); see section 3.4.

3.3. CONNECTION WITH RANDOM WALK EXCURSION

Let $\{S_n\}$ be simple symmetric random walk starting at 0. Let T be the tree satisfying the three properties as described in section 3.1. In this section we will encode the tree T as random walks by referring to Durrett, Kesten and Waymire

(1991). The description of such an encoding is as follow: Add an artificial vertex $< -1 >$ to T, which is connected by the edge to $< 0 >$, but not to any other vertex. Order the vertices of T *lexicographically* with $< -1 >$ at the beginning. For example, the tree of Figure 2.1 yields the list $< -1 >$, $< 0 >$, $< 0, 1 >$, $< 0, 1, 1 >$, $< 0, 1, 2 >$, $< 0, 1, 2, 1 >$, $< 0, 1, 2, 2 >$, $< 0, 2 >$ and $< 0, 3 >$. Now assign to T a random walk path by drawing a closed polygonal path going around T clockwise. Start at $< -1 >$ and go to $< 0 >$. If the path is at $< e >$ and $< e' >$ is its first child (in the lexicographical ordering) not yet "surrounded" by the path, then move to $< e' >$. If there are no such children, then the path moves back to the parent $< e >$. Continue this process until all vertices of T have been reached twice including $< -1 >$. This process will generate a sequence of +1's and -1's. If the polygonal path moves from a vertex to its child (respectively its parent), then the random walk corresponding to T will take a step of +1 (respectively -1). Since every edge of T corresponds to two steps of the random walk, +1 and -1, thus each T generates a unique random walk excursion path

$$S_{2n}^{+}(t) = \{S_{2n}(t) : S_{2n}(0) = S_{2n}(2n) = 0, S_{2n}(t) > 0, 0 < t < 2n\}. \tag{3.16}$$

Harris (1956, p.485) proved that the probability of a random particle starting at 0 reaches 0 for the first time after $2n$ steps equals to the probability that the family tree T became extinct after producing a total of n individuals.

One also can easily get the relation between the width function of T and the occupation time of the corresponding random walk excursion. For example, the width function of T in Figure 2.1 is

$$Z(x) = Z_k \cdot 1(k \leq x < k + 1) \tag{3.17}$$

for $k = 0, 1, 2$. Figure 3.3 show the paths of random walk excursion corresponding to the tree T in Figure 2.1.

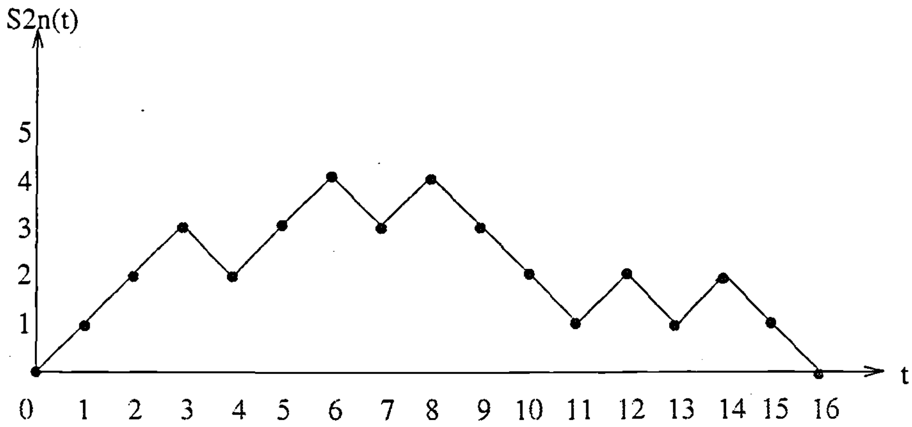


Figure 3.3. Random walk excursion corresponding to T

From Figure 3.3, we have

$$Z_k = O_{16}(S_{16}^+(t), [k, k+1)) = \frac{1}{2}\int_0^{16} 1(k \leq S_{16}^+(t) < k+1)\, dt. \tag{3.18}$$

More generally, a simple induction shows that

$$Z_k = O_n(S_{2n}^+(t), [k, k+1)) \tag{3.19}$$

for $k = 0, 1, 2, \ldots$. We will use this relation and some theorems in the next section to prove our theorems in section 3.6

3.4. DEFINITION AND LEMMAS

Let S be a metric space with Borel σ-field $\mathcal{B}$. Let $\{X_n\}$, X be random variables on $(\Omega, \mathcal{F}, P)$ with values in S. Let P_{X_n} and P_X be the distribution of X_n and X respectively. So $P_{X_n}(B) = P(X_n \in B)$ and $P_X(B) = P(X \in B)$ for $B \in \mathcal{B}$.

Definition 3.4.1 If such probability measures P_{X_n} and P_X satisfy

$$E[f(X_n)] = \int_S f(x)P_{X_n}(dx) \to \int_S f(x)P_X(dx) = E[f(X)]$$

as $n \to \infty$, for every bounded, continuous real valued function f on S, we say that P_{X_n} *conveges weakly to* P_X and write $P_{X_n} \Longrightarrow P_X$ or $X_n \Longrightarrow X$.

Let $h : S \to \mathbb{R}$ be a measurable map. The following result generalizes the Mann-Wald Theorem (1943) for Euclidean space to arbitrary metric space; see Billingsley (1968, p.30-31).

Lemma 3.1 *If* $X_n \Longrightarrow X$ *and* $P(X \in D_h) = 0$*, then* $h(X_n) \Longrightarrow h(X)$*, where* D_h *is the set of discontinuities of* h*.*

Note: In the following we will be interested in the special case $S = C[0,1]$ with the uniform metric.

Let $\{S_n : S_0 = 0, n = 1, 2, \ldots\}$ be a simple symmetric random walk starting at the origin. By linear interpolation between the integers $\{S_n : n = 0, 1, 2, \ldots\}$, we can extend $\{S_n\}$ to corresponding interpolated random walk $S_n(t)$ on $[0, \infty)$. In section 3.2.1, we defined the random walk excursion as the random walk conditioned by returning to zero after stopping time $T = n$. Kaigh (1976) studied the limiting process in $S = C[0,1]$

$$\frac{S_{2n}^{+}(2nt)}{\sqrt{2n}} = [\frac{S_{2n}(2nt)}{\sqrt{2n}}|S_0 = S_{2n} = 0, S_{2n}(2nt) > 0] \tag{3.20}$$

for $t \in [0,1]$, which is the rescaled paths of the random walk excursion $S_{2n}^{+}(t)$.

Lemma 3.2 (Kaigh) *In* $C([0,1])$,

$$\frac{S_{2n}^{+}(2nt)}{\sqrt{2n}} \Longrightarrow W_0^+, \tag{3.21}$$

where $W_0^+(t)$ *is a Brownian excursion process over* $[0,1]$.

Note: We will also use W_0^+ to denote the distribution of the process $W_0^+(t)$ when it is clear from the context what it means.

Chung (1976) calculated the expected occupation time during a Brownian excursion.

Lemma 3.3 (Chung) *Let* $S([a,b))$ *be the occupation time of Brownian excursion on* $[0,1]$ *defined in section 3.2.3,*

$$E[S([a,b))] = \int_a^b 4xe^{-2x^2}\,dx \tag{3.22}$$

for all $0 \le a < b < \infty$.

3.5. PROOFS OF THEOREMS

3.5.1. Proof of Theorem 3.1

In this section, we will prove Theorems 3.1 stated in section 3.1. From (3.19) in section 3.3, we have

$$\begin{aligned} Z_k &= O_n(S_{2n}^+(t), [k, k+1)) \\ &= \frac{1}{2}\int_0^{2n} 1(k \le S_{2n}^+(t) < k+1)\, dt, \end{aligned} \tag{3.23}$$

where $S_{2n}^+(t)$ is the interpolated random walk excursion defined in (3.16) corresponding to the T given the total progeny $\nu(T) = n$. Let $[\sqrt{n}x]$ replace k, thus

$$Z_{[\sqrt{n}x]} = \frac{1}{2}\int_0^{2n} 1([\sqrt{n}x] \le S_{2n}^+(t) < [\sqrt{n}x] + 1)\, dt, \tag{3.24}$$

where $[\sqrt{n}x]$ is the largest integer less than $\sqrt{n}x$. Let $t = 2nu$, then

$$\begin{aligned} Z_{[\sqrt{n}x]} &= n\int_0^1 1([\sqrt{n}x] \le S_{2n}^+(2nu) < [\sqrt{n}x] + 1)\, du \\ &= n\int_0^1 1(\frac{[\sqrt{n}x]}{\sqrt{2n}} \le \frac{S_{2n}^+(2nu)}{\sqrt{2n}} < \frac{[\sqrt{n}x]+1}{\sqrt{2n}})\, du. \end{aligned} \tag{3.25}$$

On the other hand, one has for the distribution function that

$$\begin{aligned} F_n(x) &= \int_0^{\sqrt{n}x} f_n(y)\, dy \\ &= \int_0^x \sqrt{n} f_n(\sqrt{n}y)\, dy \\ &= \int_0^x \sqrt{n}\frac{\mu_n(\sqrt{n}y)}{K_n}\, dy. \end{aligned} \tag{3.26}$$

Since $K_n = n$ and $\mu_n(\sqrt{n}x) = E[Z(\sqrt{n}x)|\nu = n]$, it follows that

$$\begin{aligned} F_n(x) &= \frac{1}{\sqrt{n}} \int_0^x E[Z(\sqrt{n}y)|\nu(T) = n]\, dy \\ &= \frac{1}{\sqrt{n}} \int_0^x E[Z_{[\sqrt{n}y]}|\nu(T) = n]\, dy. \end{aligned} \tag{3.27}$$

Combining (3.25) and (3.27), we get

$$F_n(x) = \sqrt{n} \int_0^x E[\int_0^1 1(\frac{[\sqrt{n}y]}{\sqrt{2n}} \leq \frac{S_{2n}^+(2nu)}{\sqrt{2n}} < \frac{[\sqrt{n}y]+1}{\sqrt{2n}})\, du]\, dy. \tag{3.28}$$

Let $\sqrt{n}y$ replace y, then

$$\begin{aligned} F_n(x) &= \int_0^{\sqrt{n}x} E[\int_0^1 1(\frac{[y]}{\sqrt{2n}} \leq \frac{S_{2n}^+(2nu)}{\sqrt{2n}} < \frac{[y]+1}{\sqrt{2n}})\, du]\, dy \\ &= \int_0^{\sqrt{n}x} \int_0^1 P[\frac{[y]}{\sqrt{2n}} \leq \frac{S_{2n}^+(2nu)}{\sqrt{2n}} < \frac{[y]+1}{\sqrt{2n}}]\, du\, dy \\ &= \int_0^1 P[0 \leq \frac{S_{2n}^+(2nu)}{\sqrt{2n}} < \frac{x}{\sqrt{2}} + \frac{1}{\sqrt{2n}}]\, du. \end{aligned} \tag{3.29}$$

By Lemma 3.2,

$$\begin{aligned} \lim_{n\to\infty} F_n(x) &= \int_0^1 P[0 \leq W_0^+(u) < \frac{x}{\sqrt{2}}]\, du \\ &= \int_0^1 E[1(0 \leq W_0^+(u) < \frac{x}{\sqrt{2}})]\, du \\ &= E[S([0, \frac{x}{\sqrt{2}})]. \end{aligned} \tag{3.30}$$

By Lemma 3.3, we have

$$\lim_{n\to\infty} F_n(x) = \int_0^{\frac{x}{\sqrt{2}}} 4ye^{-2y^2}\,dy$$
$$= \int_0^x 2ye^{-y^2}\,dx. \tag{3.31}$$

Therefore, this concludes the proof.

3.5.2. Proof of Theorem 3.2

From (3.25), we have for any $0 < a < b$

$$\int_a^b \frac{Z_{[\sqrt{n}t]}}{\sqrt{n}}\,dt = \sqrt{n}\int_a^b \int_0^1 1\left(\frac{[\sqrt{n}t]}{\sqrt{2n}} \le \frac{S_{2n}^+(2nu)}{\sqrt{2n}} < \frac{[\sqrt{n}t]+1}{\sqrt{2n}}\right) du\,dt. \tag{3.32}$$

Let $s = \sqrt{n}t$ on the right side, then

$$\begin{aligned}
\int_a^b \frac{Z_{[\sqrt{n}t]}}{\sqrt{n}}\,dt &= \int_{\sqrt{n}a}^{\sqrt{n}b} \int_0^1 1\left(\frac{[s]}{\sqrt{2n}} \le \frac{S_{2n}^+(2nu)}{\sqrt{2n}} < \frac{[s]+1}{\sqrt{2n}}\right) du\,ds \\
&= \int_0^1 \int_{\sqrt{n}a}^{\sqrt{n}b} 1\left(\frac{[s]}{\sqrt{2n}} \le \frac{S_{2n}^+(2nu)}{\sqrt{2n}} < \frac{[s]+1}{\sqrt{2n}}\right) ds\,du \\
&= \int_0^1 1\left(\frac{[\sqrt{n}a]}{\sqrt{2n}} \le \frac{S_{2n}^+(2nu)}{\sqrt{2n}} < \frac{[\sqrt{n}b]+1}{\sqrt{2n}}\right) du.
\end{aligned} \tag{3.33}$$

Let

$$R_n^+(u) = \frac{S_{2n}^+(2nu)}{\sqrt{2n}} \tag{3.34}$$

for $u \in [0,1]$.

For fixed $0 < a < b$, we define $h : C[0,1] \to \mathbb{R}$ by

$$\begin{aligned} h(\omega) &= \int_0^1 1(\omega(t) \in \Delta)\, dt \\ &= \lambda\{t : \omega(t) \in \Delta\} \end{aligned} \tag{3.35}$$

for $\omega(t) \in C[0,1]$ and $\Delta = [\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$. λ denotes Lebesgue measure. Then note that h is clearly not continuous on $C[0,1]$. Let D_h be the set of discontinuities of h. Therefore in view of Lemma 3.1, Lemma 3.2 and (3.33) we need only show $W_0^+(D_h) = 0$, where W_0^+ denotes the distribution of the process $\{W_0^+(t) : 0 \le t \le 1\}$. As we will see the key to this proof is the fact from Ito and Mckean (1963) that $\{W_0^+(t)\}$ has an absolutly continuous distribution with respect to Lebesgue measure for each t; see (3.8). Beyond this we apply a standard "Fubini argument" as given in Billingsley (1968, p.231), for example.

Let $1_\Delta(\omega(t)) = 1(\omega(t) \in \Delta)$. Then the indicator function 1_Δ is continuous Lebesgue a-e. Note that since the map $(\omega, t) \to \omega(t)$ is measurable on $C[0,1]\text{x}[0,1] \to \mathbb{R}$ and since I_Δ: $C[0,1]\text{x}[0,1] \to \mathbb{R}$ by $I_\Delta(\omega, t) = 1_\Delta(\omega(t))$, one has

$$h(\omega) = \int_0^1 I_\Delta(\omega(t))\, dt, \tag{3.36}$$

it follows from Fubini theorem that h is measurable. By (3.8), for $A = \{(\omega, t) : \omega(t) \in \partial\Delta\}$, $\partial\Delta = \{\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\}$,

$$W_0^+(\{\omega \in C[0,1] : (\omega, t) \in A\}) = \textstyle\int_{\partial\Delta} p(t,y)\, dy = 0. \tag{3.37}$$

Therefore by Fubini theorem,

$$\int_{C[0,1]} \lambda(\{t \in [0,1] : (\omega, t) \in A\}) W_0^+(d\omega)$$
$$= \int_{[0,1]} W_0^+(\{\omega \in C[0,1] : (\omega, t) \in A\}) \lambda(dt) = 0. \tag{3.38}$$

Therefore,

$$\lambda(\{t \in [0,1] : (\omega, t) \in A\}) = 0 \tag{3.39}$$

for W_0^+-a.a.ω.

Let

$$B = \{\omega \in C[0,1] : \lambda(\{t \in [0,1] : (\omega, t) \in A\}) = 0\} \tag{3.40}$$

So $W_0^+(B^c) = 0$. If $\omega \in B$, then for $\omega_n \in C[0,1]$,

$$1_\Delta(\omega_n(t)) \to 1_\Delta(\omega(t)) \; a.a.t \tag{3.41}$$

and therefore

$$h(\omega_n) = \int_0^1 1_\Delta(\omega_n(t))\, dt \to \int_0^1 1_\Delta(\omega(t))\, dt = h(\omega) \tag{3.42}$$

by Lebesgue Dominated Convergence Theorem. Thus h is continuous except on a set of W_0^+-measure 0, which concludes the proof.

4. SELF-SIMILAR TREES

In this chapter we begin with an alternative description of the Horton-Strahler ordering scheme of tree structure which is equivalent to that in section 2.2.2. Then we will give the formal definition of *self-similar trees* in terms of their *generators*. In section 4.2 and 4.3 we will define two classes of special deterministic self-similar trees, so called the *Peano trees* and *b-ary trees*. We will show that the width function asymptotics of the latter class of these deterministic trees are the same as those of random Galton-Watson trees conditioned on size and order in section 2.3.

4.1. INTRODUCTION OF SELF-SIMILAR TREES

4.1.1. Notation and Definitions

Horton (1945) introduced the stream ordering scheme described in section 2.2.2 as a way of subdividing river networks into major and minor streams. Strahler (1957) modified Horton's scheme. There is a more geometric and intuitive way of introducing the Horton-Strahler ordering scheme than recursive rules in section 2.2.2. The following interpretation of ordering is given in Peckham (1995b). For a finite tree graph we prune away all exterior edges, and classify them as order 1 branches. The resulting new tree is a more structurally "coarse" tree that has its own exterior edges. Some of these exterior edges correspond to a contiguous chain of interior edges in the former tree. Each edge of this chain of interior edges is designated as an order 2 edge. If we continue to prune the finite tree in this way, the process will terminate after a finite number of steps. Each edge of the chain of interior edges that gets pruned in the kth iteration (the exterior edges of the tree after $(k-1)$th prunings) is referred to as an order k edge. The highest order Ω of edges in the tree is called the *order of the tree.*

Let us look at a class of deterministic trees of order Ω where every edge of order ω has $b \geq 2$ branching edges (or child edges) of order $\omega - 1$. Let $T_{\omega,k}$ be the number of subtrees of order k in the chain of edges of order ω where $2 \leq \omega \leq \Omega$ and $1 \leq k \leq \omega - 1$. We can put the numbers $T_{\omega,k}$ in a $(\Omega-1)$x$(\Omega-1)$ lower triangular matrix as follow:

$$\{T_{\omega,k}\} = \begin{pmatrix} T_{2,1} & 0 & 0 & \dots & 0 \\ T_{3,1} & T_{3,2} & 0 & \dots & 0 \\ T_{4,1} & T_{4,2} & T_{4,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{\Omega,1} & T_{\Omega,2} & T_{\Omega,3} & \dots & T_{\Omega,\Omega-1} \end{pmatrix}$$

.

This matrix $\{T_{\omega,k}\}$ is referred to as *generator matrix* of these deterministic trees.

Definition 4.1.1 *Self-similar trees* are the trees with generators $\{T_{\omega,k}\}$ that satisfy the constraint that $T_{\omega,\omega-k} = T_k$, where T_k is the number of subtrees of order $\omega - k$ in the chain of edges of order ω.

Note that in terms of the above matrix, self-similar trees are defined by the condition that the *Toeplitz generator matrix* have constant values along diagonals.

Figure 4.1 shows several examples of self-similar trees of order 3. Note that two trees can have the same generators and still be different, owing to two freedoms in adding the edges. One is that edges can enter a tree from the left or right side. This is illustrated by comparing Figure 4.1(a) and 4.1(b). The second freedom arises from the many possible ways in which subtrees of different orders can be interspersed, which is illustrated in Figure 4.1(b) and 4.1(c).

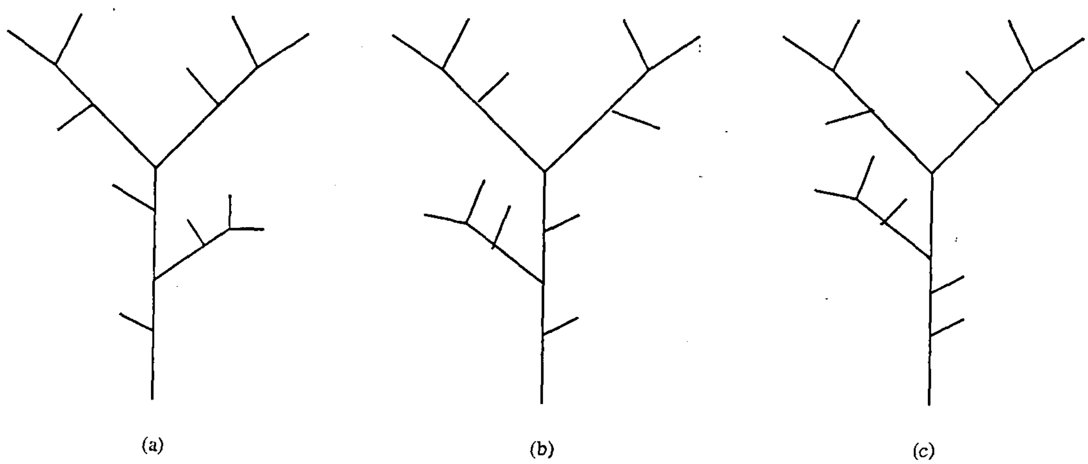


Figure 4.1. Self-similar trees with order 3 and $T_k = 2^{k-1}$

Let T be a tree of order $\Omega = n$. Let $Z_{n,k}$ be the number of vertices in the kth level, i.e: the total number of offspring in the kth generation from the branching point of view.

Definition 4.1.2 The width function of self-similar trees of order Ω is defined as

$$Z(x) = \sum_{k=0}^{H_n} Z_{n,k} \cdot 1(k \leq x \leq k+1) \tag{4.1}$$

where H_n is the height of the tree.

In section 4.2 and 4.3 we will compute the width function asymptotics of two special classes of self-similar trees.

4.1.2. Connection to Shreve's Random Tree Model

An example of self-similar trees is Shreve's random tree, also called Shreve's random model for river channel networks introduced in section 2.2.2. Shreve (1969) found the following results for his random tree model.

Theorem 4.1 $ET_{ij} = 2^{j-i-1}$ *for Shreve's random model of river channel networks, where* T_{ij} *is the number of subtrees of order* j *in the chain of edges of order* i.

Remark: It is in this sense that on average, Shreve's tree behaves like a deterministic self-similar tree with generator $T_k = 2^{k-1}$, i.e: $E[T_{\omega,\omega-k}] = E[T_k] = 2^{k-1}$.

Here we give two examples of the generator matrix of Shreve's random model (Peckham (1995b)). The average generator measured for the Kentucky River are

$$\begin{pmatrix} 1.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3.2 & 1.1 & 0 & 0 & 0 & 0 & 0 \\ 7.6 & 2.8 & 1.2 & 0 & 0 & 0 & 0 \\ 15.6 & 6.2 & 2.9 & 1.0 & 0 & 0 & 0 \\ 54.8 & 20.3 & 10.8 & 3.2 & 1.8 & 0 & 0 \\ 87.0 & 27.0 & 16.0 & 5.3 & 2.0 & 1.0 & 0 \\ 408.0 & 115.0 & 40.0 & 20.0 & 12.0 & 3.0 & 1.0 \end{pmatrix}.$$

The second one is a local river, the Willamette River.

$$\begin{pmatrix} 1.57 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3.85 & 1.09 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10.20 & 3.31 & 1.17 & 0 & 0 & 0 & 0 & 0 \\ 23.89 & 9.15 & 3.18 & 1.08 & 0 & 0 & 0 & 0 \\ 56.40 & 22.08 & 6.92 & 2.38 & 1.24 & 0 & 0 & 0 \\ 135.44 & 67.67 & 24.22 & 7.56 & 4.00 & 1.44 & 0 & 0 \\ 332.67 & 115.67 & 43.33 & 12.67 & 6.00 & 4.00 & 0.67 & 0 \\ 600.00 & 357.00 & 155.00 & 62.00 & 23.00 & 7.00 & 1.00 & 1.00 \end{pmatrix}$$

4.2. PEANO TREES

In this section we will introduce some background about Peano trees and their connections with *cascade* structures. We will show that the normalized width functions of Peano trees weakly converges to some singular measures. This result was first noted by Marani, Rigon, Rinaldo (1991) without complete proofs. We fill in the complete details in section 4.2.3.

4.2.1. Introductions of Peano trees

In this *Peano tree model* every link generates four links for every change of scale, two derived from the subdivision in the middle of the previous link, and two new links. The Peano tree models of order 2, 3 and 4 are given in Figure 4.2.

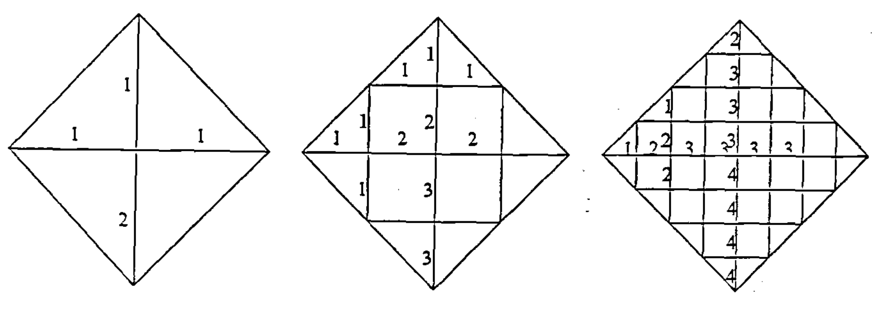


Figure 4.2. The Peano models

In fact, as noted by Peckham (1995b), the Peano tree can be representated by a class of self-similar trees with branching number 3 and generators $\{T_1 = 0, T_k = 2^{k-1} : k = 2, 3, \ldots\}$. The following figures are examples of the Peano trees with order 2, 3, and 4 corresponding to Figure 4.2, from the point of view of branching structure.

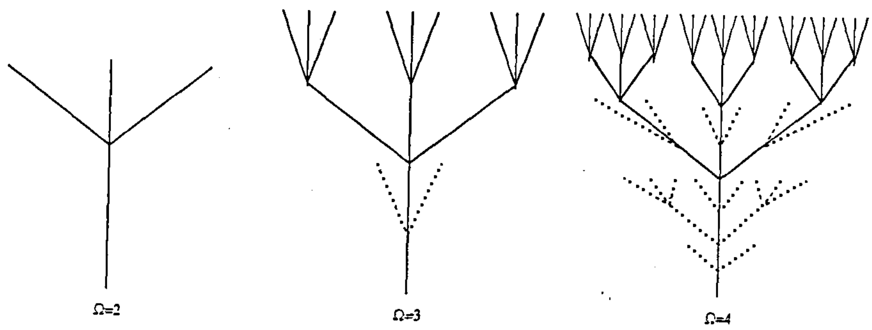


Figure 4.3. The Peano trees

The dashed lines are subtrees which are put in according to the generators.

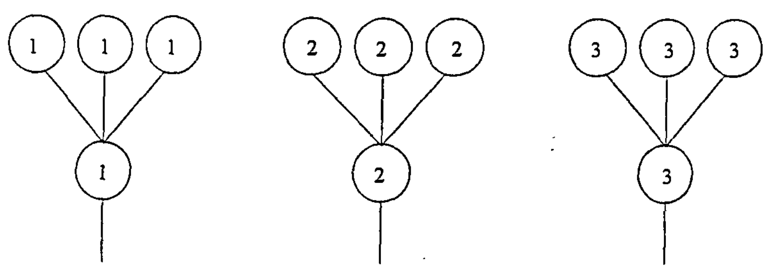


Figure 4.4. Cluster forms of the Peano trees

In order to compute the numbers of the kth generation of a Peano tree of order n, it is also convenient to draw cluster forms of the Peano trees corresponding to Figure 4.3 so that the recursive relations can be established.

For example, a Peano tree with order 3 consists of four Peano trees of order 2. Three of them are parallelly located on the same level, and the other one is down below. In general, a Peano tree of order $\Omega = n$ has four sub-Peano trees of order n-1.

Let $Z_{n,k}$ be the numbers of the kth generation of a Peano tree of order n, then we have the following recursive relations

$$Z_{n.k} = Z_{n-1,k}$$

$$Z_{n,k+2^{n-2}} = 3Z_{n-1,k}, \tag{4.2}$$

where $k = 1, 2, \ldots, 2^{n-2}$.

For example, in the case of $\Omega = 2$, we have

$$Z_{2,1} = Z_{1,1}$$

$$Z_{2,2} = 3Z_{1,1},$$

where $Z_{1,1} = 1$.

Similarly,

$$Z_{3,1} = Z_{2,1}$$

$$Z_{3,2} = Z_{2,2}$$

$$Z_{3,3} = 3Z_{2,1}$$

$$Z_{3,4} = 3Z_{2,2}.$$

According to the definition of the width function of self-similar trees in section 4.1, the width functions of a Peano tree of order n and generator $\{T_0 = 0, T_k = 2^{k-1}; k = 1, 2, \ldots\}$ are

$$Z_n(x) = \sum_{k=0}^{2^{n-1}-1} Z_{n,k+1} \cdot 1(k \le x < k+1), \tag{4.3}$$

where $n = 1, 2, \ldots$

We define a normalized width density f_n as

$$f_n(x) = \frac{Z_n(x)}{N_n}, \tag{4.4}$$

where $x \in [0, 2^{n-1})$ and $N_n = 4^{n-1}$ denotes the total progeny. Then f_n defines a measure μ_n with distribution function

$$\begin{aligned} F_n(x) &= \int_0^{2^{n-1}x} f_n(y)\, dy \\ &= \int_0^x 2^{n-1} f_n(2^{n-1}y)\, dy \end{aligned} \tag{4.5}$$

for $x \in [0, 1)$. Then we have the following theorem asserted without complete proof in Marani etal (1991).

Theorem 4.2 $F_n(x)$ *converges pointwise to a continuous singular function for* $x \in [0, 1)$. *Equivalently,* μ_n *converges weakly to a singular probability measure on* $[0, 1]$.

Proofs of the theorem will be given in 4.2.3.

4.2.2. Cascade Measures

We now consider some special measures on the unit interval $[0, 1]$. These measures have a natural multiplicative structure, referred to as a *cascade structure*. *Cascade measures* on the unit interval are a special class of mass distributions defined recursively in terms of a natural hierarchial sequence of scales of resolution of the interval. There is an extensive literature on random cascade measures (see Kahane (1987); Waymire and Williams (1994)), however in the present case the measures are deterministic and the *Martingale* arguments for existence are not applicable.

The simplest example is to consider the hierarchy of scales associated with the successive binary partitions of $X = [0,1]$. We let $\Delta(0) = [0, \frac{1}{2}]$ and $\Delta(1) = [\frac{1}{2}, 1]$, given $n = 1$. When $n = 2$, $\Delta(00) = [0, \frac{1}{4}]$, $\Delta(01) = [\frac{1}{4}, \frac{2}{4}]$, $\Delta(10) = [\frac{2}{4}, \frac{3}{4}]$ and $\Delta(11) = [\frac{3}{4}, 1]$. Recursively, at the nth level, we have

$$\Delta(t_1, t_2, \ldots, t_n) = [\sum_{i=1}^{n} \frac{t_i}{2^i}, \sum_{i=1}^{n} \frac{t_i}{2^i} + 2^{-n}), \tag{4.6}$$

where $t_i \in \{0, 1\}$.

We associate each subinterval with a non-negative weight respectively, namely W_0, W_1, W_{00}, W_{01}, W_{10}, $\ldots$, $W_{t_1 t_2 \ldots t_n}$, $\ldots$, called *cascade generators*. With these generators, the density functions are defined as follows:

$$d_1(x) = W_0, x \in \Delta(0) = [0, \frac{1}{2}]$$
$$d_1(x) = W_1, x \in \Delta(1) = [\frac{1}{2}, 1]$$

and

$$d_2(x) = W_{00}, x \in \Delta(00) = [0, \frac{1}{4}]$$
$$d_2(x) = W_{01}, x \in \Delta(00) = [\frac{1}{4}, \frac{1}{2}]$$
$$d_2(x) = W_{10}, x \in \Delta(00) = [\frac{1}{2}, \frac{3}{4}]$$
$$d_2(x) = W_{11}, x \in \Delta(00) = [\frac{3}{4}, 1].$$

In general,

$$d_n(x) = W_{t_1 t_2 \ldots t_n}, x \in \Delta(t_1 t_2 \ldots t_n) = [\sum_{i=1}^{n} t_i 2^{-i}, \sum_{i=1}^{n} t_i 2^{-i} + 2^{-(n)}), \tag{4.7}$$

where $t_i \in \{0, 1\}$.

The corresponding cascade mass distributions are defined by

$$\mu_n(A) = \int_A d_n(x)\, dx, for A \in X = [0, 1]. \tag{4.8}$$

If the limit exists in the sense of *vague convergence*, say $\mu_\infty = \lim_{n\to\infty} \mu_n$, then we call μ_∞ the cascade measure defined by the fine scale limit measure. In general, the limit measures may not have a density, i.e: the limit measures are *singular measures.*

4.2.3. Weak Convegences of Width Functions of Peano Trees

In section 4.2.1, we defined the width function of a Peano tree as

$$Z_n(x) = \sum_{k=0}^{2^{n-1}-1} Z_{n,k+1} \cdot 1(k \le x < k+1)$$

and corresponding width measure as

$$f_n(x) = \frac{Z_n(x)}{N_n}$$

We let $\{X_i\}$ be i.i.d random variables with $P(X_i = 0) = \frac{1}{4}$ and $P(X_i = 1) = \frac{3}{4}$. Let $W_{t_1 t_2 \ldots t_n} = 2^n P(X_1 = t_1, X_2 = t_2, \ldots, X_n = t_n)$, for $t_i \in \{0, 1\}$. Then the cascade generators can be expressed in terms of density function as follows:

$$d_1(x) = W_0 = 2^1 \frac{1}{4} = 2^1 \frac{Z_{2,1}}{N_2}, x \in \Delta(0) = [0, \frac{1}{2}]$$
$$d_1(x) = W_1 = 2^1 \frac{3}{4} = 2^1 \frac{Z_{2,2}}{N_2}, x \in \Delta(1) = [\frac{1}{2}, 1],$$

or

$$d_1(x) = 2^1 \frac{Z_2(2x)}{N_2} = 2^1 f_2(2x), x \in [0, 1). \tag{4.9}$$

Similarly, we have

$$d_2(x) = W_{00} = 2^2\frac{1}{4^2} = 2^2\frac{Z_{3,1}}{N_3}, x \in \Delta(00) = [0, \frac{1}{4}]$$
$$d_2(x) = W_{01} = 2^2\frac{3}{4^2} = 2^2\frac{Z_{3,2}}{N_3}, x \in \Delta(00) = [\frac{1}{4}, \frac{1}{2}]$$
$$d_2(x) = W_{10} = 2^2\frac{3}{4^2} = 2^2\frac{Z_{3,3}}{N_3}, x \in \Delta(00) = [\frac{1}{2}, \frac{3}{4}]$$
$$d_2(x) = W_{11} = 2^2\frac{9}{4^2} = 2^2\frac{Z_{3,4}}{N_3}, x \in \Delta(00) = [\frac{3}{4}, 1],$$

or

$$d_2(x) = 2^2 f_3(2^2 x), x \in [0, 1). \tag{4.10}$$

By induction,

$$\begin{aligned} d_{n-1}(x) &= W_{t_1 t_2 \dots t_{n-1}} \\ &= 2^{n-1}(\frac{1}{4})^{\sum_{i=1}^{n-1} t_i}(\frac{3}{4})^{n-1-\sum_{i=1}^{n-1} t_i} \\ &= 2^{n-1}\frac{3^{n-1-\sum_{i=1}^{n-1} t_i}}{4^{n-1}} \\ &= 2^{n-1}\frac{Z_{n,k}}{N_n} \\ &\quad x \in \Delta(t_1 t_2 \dots t_{n-1}) = [\sum_{i=1}^{n-1} t_i 2^{-i}, \sum_{i=1}^{n-1} t_i 2^{-i} + 2^{-(n-1)}), \end{aligned}$$

where $k = 1, 2, \dots, 2^{n-1}$. Thus we have

$$d_{n-1}(x) = 2^{n-1} f_n(2^{n-1} x) \tag{4.11}$$

for$x \in [0, 1)$.

By comparing (4.5) with (4.8), we notice that

$$F_n(x) = \mu_{n-1}([0, x]), \tag{4.12}$$

for $x \in [0,1)$. Therefore, it is sufficient to show that $\mu_n([0,x]) \longrightarrow \mu_\infty([0,x])$, as n goes to infinity for $x \in [0,1)$, where $\mu_\infty([0,x])$ is a continous singular function. Since $\{\mu_n\}$ are probability measures defined on a compact set and therefore by tightness, μ_n has a weakly convergent subsequence. Thus

$$\mu_n \stackrel{d}{\Longrightarrow} \mu_\infty,$$

where μ_∞ is a probability measure over [0, 1). For $\{X_i\}$ defined at the beginning of this section, let $X = \sum_{i=1}^\infty X_i 2^{-i}$ and $\mu_\infty([0,x]) = P[X \le x]$ be the distribution function of X for $x \in [0,1)$. Since x can have at most two dyadic expansions such that $x = \sum_i t_i 2^{-i}$, where $t_i \in \{0,1\}$. So

$$P[X = x] = \lim_{n\to\infty} P[X_i = t_i, i = 1,2,\ldots] = 0.$$

Therefore, $\mu_\infty([0,x])$ is continous over [0, 1).

For each n , we chose k_n so that $x \in I_n = [k_n 2^{-n}, (k_n+1)2^{-n})$, so

$$\lim_{n\to\infty} \frac{P[X \in I_n]}{2^{-n}} = \lim_{n\to\infty} \frac{\mu_\infty((k_n+1)2^{-n}) - \mu_\infty(k_n 2^{-n})}{2^{-n}} = \mu'_\infty([0,x]). \tag{4.13}$$

If $\mu'_\infty([0,x]) \ne 0$, then

$$1 = \lim_{n\to\infty} \frac{\frac{P[X\in I_{n+1}]}{2^{-(n+1)}}}{\frac{P[X\in I_n]}{2^{-n}}} = 2 \lim_{n\to\infty} \frac{P[X \in I_{n+1}]}{P[X \in I_n]},$$

i.e:

$$\lim_{n\to\infty} \frac{P[X \in I_{n+1}]}{P[X \in I_n]} = \frac{1}{2}. \tag{4.14}$$

On the other hand, since $x \in I_n$, so we can write x as $x = \sum_{i=1}^n t_i 2^{-i}$, thus

$$\begin{aligned} P[X \in I_{n+1}] &= P[X_i = t_i, i = 1,2,\ldots,n+1] \\ P[X \in I_n] &= P[X_i = t_i, i = 1,2,\ldots,n]. \end{aligned} \tag{4.15}$$

By independence, we have

$$\lim_{n\to\infty} \frac{P[X \in I_{n+1}]}{P[X \in I_n]} = P[X_{n+1} = t_{n+1}]$$
$$= \tfrac{1}{4} or \frac{3}{4} \tag{4.16}$$

where $t_i \in \{0,1\}$. Obviously, (4.16) contradicts (4.14), so $\mu'_\infty([0,x]) = 0$ everywhere except on a set of Lebesgue measure zero. Therefore $\mu_\infty([0,x])$ is a continuous singular function on [0, 1). This is called a *devil's staircase* by Marani etc (1991).

4.3. B-ARY TREES

In this section, we will study width functions of another class of self-similar trees, the b-ary trees (ie: branching number $b = 2, 3, \ldots$) with corresponding generators $\{T_k(b) = (b-1)2^{k-1}$, k=1, 2, ...$\}$. For example, a binary tree with generator $\{T_k(2) = 2^{k-1}, k = 1, 2, \ldots\}$ is one of these trees discussed in section 4.1. Based on the special construction of these self-similar trees, we will show that the width functions of a b-ary tree of given order, as a normalized probabilty measure, converge weakly to a uniform distribution function over [0, 1] given the orders of trees. In section 2.3, we have seen the similar result of Shreve's random model in Troutman and Karlinger (1984).

4.3.1. Construction of b-ary self-similar trees

One may note that trees can have the same generators and still be different due to the freedom in adding subtrees. Since we are only interested in the width functions, trees having same width functions are referred to as trees without distinction. We will construct those trees according to the following rules:

Rule (1). All edges are added to the trees from right sides.

Rule (2). Subtrees of order n-1, n-2, ..., 2, 1 will be added to the *"principal path"* (a chain of edges connecting root to the top of the tree) of order n, so that two parallel subtrees of order n-1 are constructed in the principal path, together with the other two parallel subtrees of order n-1 in the upper level.

First, we implement these rules to the special case $b = 2$ to construct binary self-similar trees with generator $T_k(2) = 2^{k-1}, k = 1, 2, \ldots$, then expand to the general case.

If order $\Omega = 2$, so $k = 1$ and the generator $T_{2,1} = 1$. According to our rules we have the following self-similar tree of order 2.

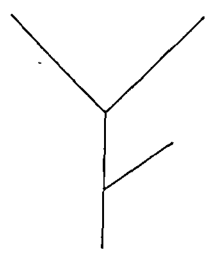


Figure 4.5. Binary self-similar tree of order $\Omega = 2$

For self-similar trees one has the corresponding cluster form shown in Figure 4.6 so that we could get nice recursive equations of the numbers of the kth generation $\{Z_{2,k}, k = 1, 2, 3\}$.

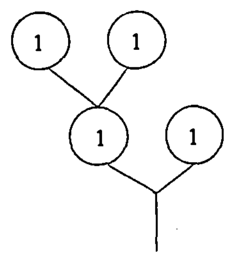


Figure 4.6. Corresponding cluster form

It is easy to see

$$Z_{2,1} = 1$$
$$Z_{2,2} = 2Z_{1,1}$$
$$Z_{2,3} = 2Z_{1,1} \tag{4.17}$$

and total progeny

$$N_2 = 4N_1 + 1 = 4 + 1, \tag{4.18}$$

where $Z_{1,1} = 1$ and $N_1 = 1$.

If $\Omega = 3$, the generator matrix is

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

So by our rules, we need to bring two subtrees of order 1 and one subtree of order 2 to the principal path of order 3. The one with higher order is put below the two of lower order so that the self-similar tree of order 3 has four subtrees of order 2, where two of these are parallelly located in the lower level and the other subtrees are in the upper level; see Figure 4.7.

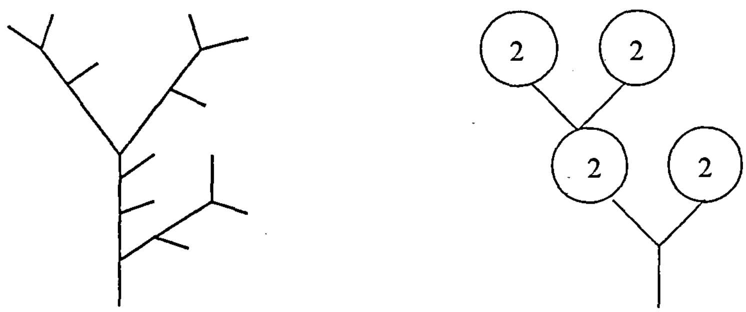


Figure 4.7. Binary self-similar tree of order 3

$\{Z_{3,k}, k = 1, 2, \ldots, 2^3 - 1\}$ satisfies the following recursive equations:

$$\begin{aligned} Z_{3,1} &= 1 \\ Z_{3,2} &= 2Z_{2,1} \\ Z_{3,3} &= 2Z_{2,2} \\ Z_{3,4} &= 2Z_{2,3} \\ Z_{3,5} &= 2Z_{2,1} \\ Z_{3,6} &= 2Z_{2,2} \\ Z_{3,7} &= 2Z_{2,3}. \end{aligned} \tag{4.19}$$

The total progeny is

$$N_3 = 4N_2 + 1 = 4^2 + 4 + 1. \tag{4.20}$$

Similarly, we can construct a self-similar tree of order 4 according to the generator matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

From last row of above matrix we know that we need to put one subtree of order 3, two subtrees of order 2 and four subtrees of order 1 on the principal path. According to our rules we will arrange these subtrees upward as followings: One of order 3 is on the bottom, next is one of order 2, then two of order 1, one of order 2, and two of order 1. In this way we can get the self-similar tree of order 4 consisting of four subtrees of order 3, two of them are parallelly located below the other parallel two. $\{Z_{4,k}, k = 1, 2, \ldots, 2^4 - 1\}$ satifies:

$$\begin{aligned} Z_{4,1} &= 1 \\ Z_{4,j+1} &= 2Z_{3,j} \\ Z_{4,j+2^3} &= 2Z_{3,j}. \end{aligned} \tag{4.21}$$

for $j = 1, 2, \ldots, 2^3 - 1$.

The total progeny is

$$N_4 = 4N_3 + 1 = 4^3 + 4^2 + 1. \tag{4.22}$$

In general, we can get a self-similar tree of order $\Omega = n$ $(n = 2, 3, \ldots)$ in this way. Thus the corresponding $\{Z_{n,k}, k = 1, 2, \ldots, 2^n - 1\}$ can be expressed in the following recursive equations

$$\begin{aligned} Z_{n,1} &= 1 \\ Z_{n,j+1} &= 2Z_{n-1,j} \\ Z_{n,j+2^{n-1}} &= 2Z_{n-1,j}, \end{aligned} \tag{4.23}$$

where $j = 1, 2, \ldots, 2^{n-1} - 1$.

The total progeny is

$$N_1 = 1, N_n = 4N_{n-1} + 1 = 4^{n-1} + 4^{n-2} + \ldots + 1 \tag{4.24}$$

for $n = 2, 3, \ldots$.

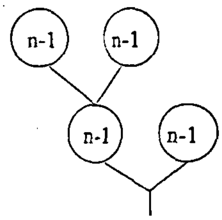


Figure 4.8. Cluster form of order $\Omega = n$

Now we will follow the same ideas as we did in the binary case to construct the b-ary self-similar trees with generator $\{T_k(b) = (b-1)2^{k-1}$, k=1, 2,...$\}$.
If $\Omega = 2$, the generator $T_{2,1} = b - 1$.

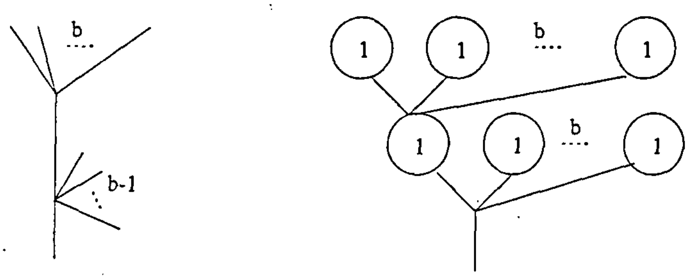


Figure 4.9. b-ary trees of $\Omega = 2$

From Figure 4.9 It is easy to see that $\{Z_{2,k}, k = 1, 2, 2^2 - 1\}$ satisfy the following relations:

$$Z_{2,1} = 1$$

$$Z_{2,2} = bZ_{1,1}$$

$$Z_{2,3} = bZ_{1,1}. \tag{4.25}$$

The total progeny is

$$N_2 = 2bN_1 + 1 = 2b + 1. \tag{4.26}$$

When $\Omega = 3$, the generator matrix is

$$\begin{pmatrix} b-1 & 0 \\ 2(b-1) & b-1 \end{pmatrix}$$

According to our rules, 2(b-1) subtrees of order 1 and b-1 subtrees of order 2 are needed to put in the principal path of order 3, so that we get the following figures:

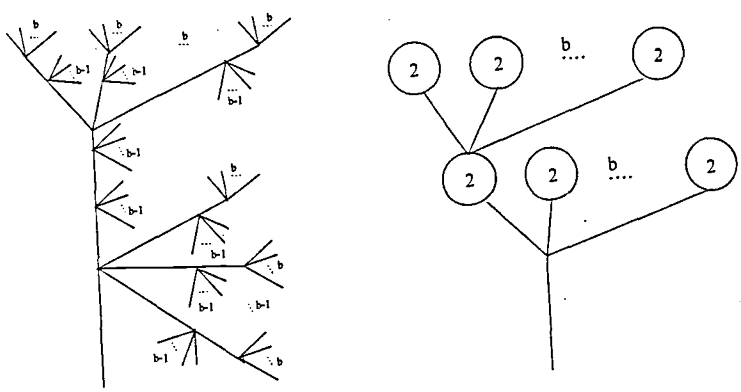


Figure 4.10. b-ary trees of $\Omega = 3$

Thus

$$Z_{3,1} = 1$$

$$Z_{3,j+1} = bZ_{2,j}$$

$$Z_{3,j+2^2} = bZ_{2,j}, \tag{4.27}$$

where $j = 1, 2, \ldots, 2^2 - 1$ and the total progeny is

$$N_3 = 2bN_2 + 1 = (2b)^2 + 2b + 1. \tag{4.28}$$

In general, if $\Omega = n$ we can get the following cluster form of b-ary tree.

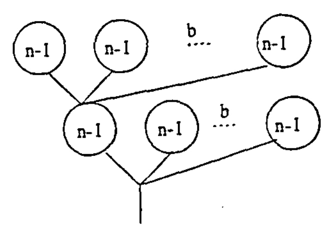


Figure 4.11. b-ary trees of $\Omega = n$

The recursive equations of $\{Z_{n,k}, k = 1, 2, \ldots, 2^n - 1\}$ are:

$$Z_{n,1} = 1$$

$$Z_{n,j+1} = bZ_{n-1,j}$$

$$Z_{n,j+2^{n-1}} = bZ_{n-1,j}, \tag{4.29}$$

for $j = 1, 2, \ldots, 2^{n-1} - 1$. The total progeny is

$$N_n = 2bN_{n-1} + 1 = (2b)^{n-1} + (2b)^{n-2} + \ldots + 1 = \frac{(2b)^n - 1}{2b - 1}, \tag{4.30}$$

where $b = 2, 3, \ldots$, $n = 2, 3, \ldots$, and $Z_{1,1} = N_1 = 1$.

4.3.2. Width Functions of b-ary Self-similar Trees

According to the definition in section 4.1, the width function of b-ary self-similar trees of order n with generator $\{T_k(b) = (b-1)2^{k-1}, \text{k=1,2,}\dots\}$ is

$$Z_n(x) = \sum_{k=0}^{2^n-2} Z_{n,k+1} \cdot 1(k \le x < k+1), \tag{4.31}$$

where $n = 2, 3, \dots$

We define the density function as

$$f_n(x) = \frac{Z_n(x)}{N_n}. \tag{4.32}$$

If we let H_n denote the height of b-ary tree of order $\Omega = n$, i.e: $H_n = 2^n - 1$, thus the distribution function is

$$\begin{aligned} F_n(x) &= \int_0^{H_n x} f_n(y)\, dy \\ &= \int_0^{H_n x} \frac{Z_n(y)}{N_n}\, dy \\ &= \int_0^{x} \frac{Z_n(H_n y)}{\frac{N_n}{H_n}}\, dy. \end{aligned} \tag{4.33}$$

If let X_n be the random variable with the density function

$$\frac{Z_n(H_n x)}{\frac{N_n}{H_n}}, \tag{4.34}$$

then we will have the following theorem.

Theorem 4.3 *$F_n(x)$ converges to a uniformly distributed function on $[0, 1]$ as $n \to \infty$, ie:*

$$X_n \Longrightarrow X,$$

where X is the uniformly distributed random variable and $\Longrightarrow$ denotes weak convergence.

The proofs of above theorem immediately follow the results of the following lemmas.

Lemma 4.1 *Let $\phi_n(s)$ be the moment generating function of X_n, ie:*

$$\phi_n(s) = \int_0^1 e^{sx} \frac{Z_n(H_n x)}{\frac{N_n}{H_n}}\, dx, \tag{4.35}$$

then $\phi(s) = \lim_{s\to\infty} \phi_n(s)$ exists, for $s \in [0,1]$.

Proof: Let

$$\psi_n(s) = \int_0^{H_n} e^{sx} Z_n(x)\, dx. \tag{4.36}$$

Since

$$\begin{aligned} Z_n(x) &= \sum_{k=0}^{2^n-2} Z_{n,k+1} \cdot 1(k \le x < k+1) \\ &= Z_{n,1} \cdot 1(0 \le x < 1) \\ &\quad + \sum_{k=1}^{2^{n-1}-1} Z_{n,k+1} \cdot 1(k \le x < k+1) \\ &\quad + \sum_{k=2^{n-1}}^{2^n-2} Z_{n,k+1} \cdot 1(k \le x < k+1) \\ &= Z_{n,1} \cdot 1(0 \le x < 1) \\ &\quad + \sum_{k=1}^{2^{n-1}-1} Z_{n,k+1} \cdot 1(k \le x < k+1) \\ &\quad + \sum_{k=1}^{2^{n-1}-1} Z_{n,k+2(n-1)} \cdot 1(k \le x - 2^{n+1} + 1 < k+1) \end{aligned} \tag{4.37}$$

and the recursive relation in (4.29), so we get

$$Z_n(x) = Z_{n-1,1} \cdot 1(0 \le x < 1)$$

$$
\begin{aligned}
&+b \cdot \sum_{k=1}^{2^{n-1}-1} Z_{n-1,k} \cdot 1(k \leq x < k+1) \\
&+b \cdot \sum_{k=1}^{2^{n-1}-1} Z_{n-1,k} \cdot 1(k \leq x - 2^{n-1} + 1 < k+1) \\
= {}& Z_{n-1,1} \cdot 1(0 \leq x < 1) \\
&+b \cdot \sum_{k=0}^{2^{n-1}-2} Z_{n-1,k+1} \cdot 1(k \leq x - 1 < k+1) \\
&+b \cdot \sum_{k=0}^{2^{n-1}-2} Z_{n-1,k+1} \cdot 1(k \leq x - 2^{n-1} < k+1) \\
= {}& Z_{n-1,1} \cdot 1(0 \leq x < 1) \\
&+bZ_{n-1}(x-1) \cdot 1(1 \leq x < 2^{n-1}) \\
&+bZ_{n-1}(x - 2^{(n-1)}) \cdot 1(2^{n-1} \leq x < 2^n - 1).
\end{aligned} \tag{4.38}
$$

Therefore,

$$
\begin{aligned}
\psi_n(s) = {}& \int_0^{H_n} e^{sx}[Z_{n-1,1} \cdot 1(0 \leq x < 1) \\
&+bZ_{n-1}(x-1) \cdot 1(1 \leq x < 2^{n-1}) \\
&+bZ_{n-1}(x - 2^{n-1}) \cdot 1(2^{n-1} \leq x < 2^n - 1)]\, dx \\
= {}& \int_0^1 e^{sx}\, dx + b \int_1^{2^{n-1}} e^{sx} Z_{n-1}(x-1)\, dx \\
&+b \int_{2^{n-1}}^{2^n-1} e^{sx} Z_{n-1}(x - 2^{n-1})\, dx \\
= {}& \frac{e^s - 1}{s} + b \int_0^{2^{n-1}-1} e^{s(x+1)} Z_{n-1}(x)\, dx \\
&+b \int_0^{2^{n-1}-1} e^{s(x+2^{n-1})} Z_{n-1}(x)\, dx
\end{aligned}
$$

Finally, we get the following recursive equations

$$\begin{aligned}
\psi_n(s) &= \frac{e^s - 1}{s} + b(e^s + e^{2^{n-1}s})\psi_{n-1}(s) \\
\psi_0(s) &= 0 \\
\psi_1(s) &= \frac{e^s - 1}{s},
\end{aligned} \tag{4.39}$$

where $n = 1, 2, \ldots$.

By comparing (4.35) with (4.36), we have

$$\phi_n(s) = \frac{1}{N_n}\psi_n(\frac{s}{H_n}). \tag{4.40}$$

So we can get the recursive equation of $\phi_n(s)$ from (4.39)

$$\begin{aligned}
\phi_n(s) &= \frac{1}{N_n}\frac{e^{\frac{s}{H_n}} - 1}{\frac{s}{H_n}} + b(e^{\frac{s}{H_n}} + e^{\frac{2^{n-1}s}{H_n}})\frac{N_{n-1}}{N_n}\phi_{n-1}(\frac{H_{n-1}}{H_n}s) \\
\phi_0(s) &= 0 \\
\phi_1(s) &= \frac{e^s - 1}{s},
\end{aligned} \tag{4.41}$$

where $H_n = 2^n - 1$ and

$$N_n = \frac{(2b)^n - 1}{2b - 1}$$

for $n = 1, 2, \ldots$.

If we let $P_n(\cdot)$ be probability measure corresponding to $\phi_n(s)$, $\delta_{\{0\}}(\cdot)$ be Dirac measure at 0 and $\lambda(\cdot)$ be uniform distribution function, then

$$\begin{aligned}
P_n(\cdot) = &\frac{1}{N_n}\delta_{\{0\}}(\cdot) * \lambda(\cdot, n[0, \frac{1}{H_n}]) \\
&+ \frac{N_{n-1}}{N_n}b(\delta_{\{\frac{1}{H_n}\}}(\cdot) + \delta_{\{\frac{2^{n-1}}{H_n}\}}(\cdot)) * P_{n-1}(\frac{H_{n-1}}{H_n}\cdot),
\end{aligned} \tag{4.42}$$

where $*$ denotes convolution. Since $\phi_1(s) = \frac{e^s-1}{s}$, so $P_1(\cdot)$ has uniform distribution on $[0, 1]$. Also $P_2(\cdot)$ has compact support by $[0, \frac{H_1}{H_n}]$ or $[0, \frac{1}{2^2-1}]$. By induction, $P_n(\cdot)$ has compact support by $[0, \frac{H_{n-1}}{H_n}]$. We also notice that

$$\lim_{n\to\infty}\frac{N_{n-1}}{N_n}b = \lim_{n\to\infty}\frac{(2b)^{n-1}-1}{(2b)^n-1}b = \frac{1}{2}. \tag{4.43}$$

Thus $P_n(\cdot)$ converges to a probability measure $P(\cdot)$. Therefore , $\lim_{n\to\infty}\phi_n(s)$ exists by tightness and *Helly Selection Theorem.*

Lemma 4.2 *Let* $\phi(s) = \lim_{n\to\infty}\phi_n(s)$. *Then*

$$\phi(s) = \frac{e^s-1}{s},$$

ie: $\phi(s)$ *is moment generating function of a uniformly distributed random variable on* $[0,1]$.

Proof: If we take limits on both sides of (4.41), then we have

$$\phi(s) = \frac{(1+e^{s/2})}{2}\phi(\frac{s}{2}). \tag{4.44}$$

Let X be the random variable distributed on $[0,1]$ and $\phi(s)$ denote the moment generating function of X. If let $\{U_i\}$ be i.i.d Bernouli 0-1, with probablity $\frac{1}{2}$, we know the moment generating function of U_i is

$$\frac{(1+e^s)}{2}.$$

From (4.44) we get for any $n \geq 1$

$$\begin{aligned} X &= \frac{U_i + X}{2} \\ &= \sum_{i=1}^{n}\frac{U_i}{2^i} + \frac{X}{2^n}, \end{aligned} \tag{4.45}$$

where = means both sides have the same distribution. Since $\{\frac{U_i}{2^i}\}$ are independent and $\sum_i E(\frac{U_i}{2^i}) < \infty$ and $\sum_i Var(\frac{U_i}{2^i}) < \infty$, it follows by Kolmogorov theorem that $U = \sum_{i=1}^{\infty}\frac{U_i}{2^i}$ converges a.s. and therefore in distribution. For arbitrary $\epsilon_i \in \{0,1\}$ and $k \geq 1$,

$$P(\sum_{i=1}^{k} \frac{\epsilon_i}{2^i} \leq U < \sum_{i=1}^{k} \frac{\epsilon_i}{2^i} + \frac{1}{2^k}) = P(U_1 = \epsilon_1, \ldots, U_k = \epsilon_k)$$
$$= \frac{1}{2^k} = |[\sum_{i=1}^{k} \frac{\epsilon_i}{2^i}, \sum_{i=1}^{k} \frac{\epsilon_i}{2^i} + \frac{1}{2^k})|. \tag{4.46}$$

Since the binary rationals are dense in $(0,1)$, it follows that $P(a \leq U < b) = b - a$, $0 \leq a < b \leq 1$. Therefore U is uniformly distributed over $[0,1]$. Since $X/2^n$ goes to 0 in distribution. Thus, X is uniformly distributed over $[0,1]$. Therefore, this concludes the proof of Theorem 4.3.

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