

## AN ABSTRACT OF THE THESIS OF

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Robert Burton

In 1989, Burton and Keane showed under the conditions of stationarity and ergodicity, percolation in  $\mathcal{Z}^2$  results (a.s.) in topological strips. That is, for  $X = \{x|x : \mathcal{Z}^2 \rightarrow \{0,1\}\}$ , if  $Q$  is the event that  $x \in X$  contains a ribbon whose complement in  $\mathcal{Z}^2$  has at least three components, then  $P(Q) = 0$ . Furthermore, with the added constraint of finite energy, there can be at most two infinite ribbons.

Following up those results, we construct a  $\mathcal{Z}^2$  model in which two colors do indeed percolate under the conditions of stationarity, ergodicity and uniform finite energy (implying finite energy). This is done by constructing “yin-yang” building blocks and iteratively building up the space. The required measure is obtained by a *cut and stack* method.

We continue to show that the limitation of only two colors percolating does not hold in  $\mathcal{Z}^3$  by constructing a model which has infinitely many colors percolate under the conditions of stationarity, ergodicity and uniform finite energy. Unlike the two dimensional model, we build this model from one to two to three dimensions. After extending our space to three dimensions, we induce uniform finite energy and show percolation occurs.

Dependent Site Percolation Models

by

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Paul R. Krouss, Author

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# DEPENDENT SITE PERCOLATION MODELS

## 1. INTRODUCTION

### 1.1. WHAT IS PERCOLATION?

The theory of percolation comes in many flavors, all of which have different uses. Percolation theory was born out of a physical model. In 1957, Broadbent and Hammersley [3] were attempting to develop a model for a gas or liquid passing through a random medium. Unlike a diffusion process, the randomness of the system lies in the medium, not in the gas or liquid. A solid material submerged in water is an example for which a percolation model would be useful. Are there a series of channels through which the water may pass so as to wet the center of the solid?

When describing site percolation to those not familiar with the field, it is often helpful to give a simple example. Suppose one has an infinite checkerboard. Instead of coloring the squares in their usual, predictable fashion, one colors them randomly by flipping a coin (possibly biased) at each square to decide how it should be colored. After the coloring is done, can an ant walk from one square, or *site*, to some other site (moving only in the north, east, south and west directions) without leaving the color? If he can, then we say these two sites are connected. If the ant can walk an infinite distance on the same color without ever returning to the same square twice, we say that percolation occurs.

A site percolation model can be roughly described as some random process of coloring sites and the rules by which we define sites as neighbors. Neighbors are a path distance of one apart. This path distance need not be Euclidean. Another

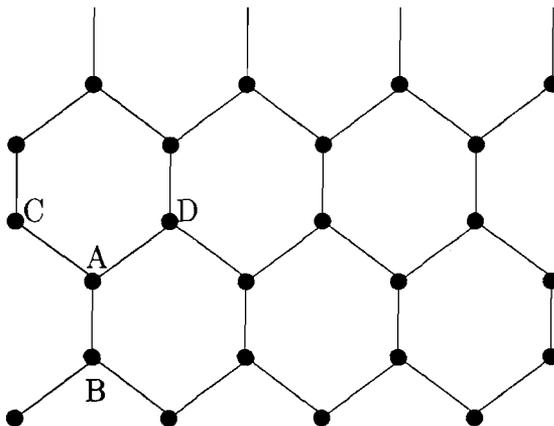


Figure 1.1. Honeycomb Lattice

very popular percolation model is bond percolation. In this model, bonds (edges of a graph) in a structure are randomly colored. Bonds are neighbors if they share a common endpoint (vertex). The field of percolation is not limited to square lattices. For instance, one may examine the honeycomb lattice as depicted in Figure 1.1.

It is a fact that the set of bond percolation models is a proper subset of the set of site percolation models [16]. That is, any bond percolation model can be represented as a site percolation model. One simply maps each bond to a site and defines the connections between sites so as to preserve the relations as in the bond model. If one were to translate a site percolation model to a bond percolation model, then each site would become a bond and have the same neighbor relation as the vertex (site) did in the site percolation model.

This translation of a site model to a bond model is not possible in every case. We can see that there is no translation to a bond model from the honeycomb lattice. As a site model, site  $A$  is a neighbor to each of the sites  $B$ ,  $C$  and  $D$ . As well,  $B$ ,  $C$  and  $D$  are not neighbors with one another. As a bond model, we cannot preserve

this relationship. If bond  $A$  is to be a neighbor to bonds  $B$ ,  $C$  and  $D$ , at least two of the bonds  $B$ ,  $C$  and  $D$  share a common vertex and the neighbor relationship is not preserved.

Since this paper deals with site percolation on the integer lattice, we will describe  $\mathcal{Z}^d$  site percolation in more detail and introduce some definitions and ideas basic to the field.

**Definition 1** For any  $w, z \in \mathcal{Z}^d$ ,  $w = (w_1, w_2, \dots, w_d)$ ,  $z = (z_1, z_2, \dots, z_d)$ , we define the **path distance**  $\delta(w, z)$  from  $w$  to  $z$  to be

$$\delta(w, z) = \sum_{i=1}^d |z_i - w_i|.$$

This number represents the length of the shortest path from  $w$  to  $z$  along integral points, travelling a Euclidean distance of one for each step. We also define the distance from the origin

$$|z| = \delta(z, 0)$$

where 0 is defined to be the origin for  $\mathcal{Z}^d$ .

For a two color percolation model, each site  $z \in \mathcal{Z}^d$  is given one of two colors. Often, we refer to the colors as being open and closed, or 1 and 0 (respectively). Associated with each site is a probability of being open or closed. For  $z \in \mathcal{Z}^d$ , we have  $p_z = P(z = 1)$  and  $q_z = P(z = 0)$  with  $p_z + q_z = 1$ .

Elements  $w, z \in \mathcal{Z}^d$  are said to be **neighbors** if the path distance  $\delta(w, z) = 1$ . Consider the set  $S \subseteq \mathcal{Z}^d$  where for a fixed color assignment  $x : \mathcal{Z}^d \rightarrow \{0, 1\}$ ,

$$S = \{z \in \mathcal{Z}^d | x(z) = 1\}.$$

The saturation of the relation “ $z_1, z_2 \in S$  are neighbors” forms an equivalence relation. The equivalence classes are called **open clusters**. Similarly, we can define

closed clusters.  $C(z)$  will denote the open cluster to which  $z$  belongs, with  $C(z) = \emptyset$  if  $z \notin S$ .

## 1.2. I.I.D. BERNOULLI SITE PERCOLATION

Often, percolation models exhibit phase transitions. The critical point for transition is of great interest. For example, suppose the sites of a percolation model are colored independently and have the same probability of being open. That is,  $p_z = p$  for all  $z \in \mathcal{Z}^d$ . We can think of the sites as independent, identically distributed (i.i.d.) Bernoulli random variables.

The function  $\theta(p) = P(\#(C(0)) = \infty)$  is called the percolation probability. Broadbent and Hammersly [3] showed the existence of a critical value  $p_c = \sup\{p | \theta(p) = 0\}$  with  $0 < p_c < 1$  for  $d \geq 2$ . An equivalent way in which to think of  $p_c$  is in the context of  $\theta(p)$ .

$$\theta(p) = 0 \text{ if } p < p_c$$

and

$$\theta(p) > 0 \text{ if } p > p_c.$$

The value  $p_c$  has been elusive thus far for many popular percolation models. For site percolation on  $\mathcal{Z}^d$ ,  $p_c$  has yet to be calculated for  $d \geq 2$  with  $d = 1$  a trivial case. Estimates for  $d = 2$  are approximately 0.59 with rigorous bounds of  $0.5059 \leq p_c \leq 0.6819$  [20].

## 1.3. EXAMPLES OF PHYSICAL PERCOLATION MODELS.

Percolation theory covers a wide range of applications and is a popular topic in many physics and math journals. It can be used to model natural phenomenon

such as spontaneous magnetization, forest fires [20] and galactic evolution [19]. Though these studies are not the focus of this dissertation, it is worthwhile to give a few examples in order to illustrate some of these applications.

### 1.3.1. Galactic Evolution

Our first example deals with galactic evolution [19]. The evolution of galaxies may be modeled in terms of directed bond percolation. In a directed bond percolation model, one or more of the dimensions' bonds have a flow. In this case, the time dimension is directed. Because of the thinness of galaxies relative to their diameter, the bulk of work in the area has been done on disk galaxy models. Hence, the model of primary interest is the (2+1) dimensional model, where 2 refers to the spatial dimensions and 1 refers to the directed temporal dimension.

Massive stars have violent and short (in star years) lives. Approximately ten million years after their birth, their lives end as supernovae. By the shock wave of a supernova, or via high energy densities which are radiated over a star's lifetime, interstellar gas can be compressed into high enough densities whereby gravity takes over and allows the gas to collapse into a star. It has been proposed by some that it is by this self-propagation that the majority of stars are formed [13].

The model is defined as follows. For  $\alpha = (i_1, i_2)$ , a point in the 2-dimensional lattice, we assign a random variable  $\sigma_\alpha$  which takes on the values 0 or 1. For each  $\alpha$ , define  $R_\alpha$  to be the set of sites neighboring to  $\alpha$ .  $R_\alpha$  may contain more sites than the nearest neighbors one would find in a  $\mathcal{Z}^2$  model. Each active neighbor  $\beta \in R_\alpha$  (that is  $\sigma_\beta(t) = 1$ ) at time  $t$  has a probability  $p$  of activating site  $\alpha$  at time  $t + 1$ . This is given by

$$\sigma_\alpha(t + 1) = 1 - \prod_{\beta \in R_\alpha} (1 - A_{\alpha\beta t} \sigma_\beta(t))$$

where  $A_{\alpha\beta t}$  is an independent random variable indicating activation of  $\alpha$  at time  $t$  by  $\beta$ .  $A_{\alpha\beta t}$  takes the value 1 with probability  $p$  and 0 with probability  $1 - p$ .

In this model, one can examine such things as the density of stars and under what conditions clusters arise to form the spiraling arms of our galaxies. It treats the formation of stars in a probabilistic manner rather than in the conventional way.

### 1.3.2. Magnetization and the Ising Model

Ferromagnetic materials can be described as transition materials, examples of which include iron, nickel and cobalt. These elements have non-paired electrons in their outer shells. For most materials, there is a cancellation of electron spin magnetic moments between electron pairs. For ferromagnetic materials, this cancellation does not occur and the atoms have large magnetic moments.

Thermal energy randomizes these magnetic moments at higher temperatures and the material is not magnetized. At cooler temperatures (in the case of iron, cooler than 770 degrees centigrade) these materials possess a spontaneous magnetization, i.e. magnetization in the absence of an applied magnetic field. The temperature below which spontaneous magnetization can occur is called the Curie temperature  $T_c$ .

A model for ferromagnetism was introduced by Ising in 1925 [15]. Today, the Ising model is one in a family of random cluster models [9] to which percolation also belongs. The resemblance of percolation models to the Ising model was recognized shortly after percolation was introduced in 1957 [3]. It was not until 1968 that a precise relation was established [8].

The finite spin- $\frac{1}{2}$  Ising system is represented as a finite graph  $G$  where each vertex represents an electron spin (either +1 or -1). The exchange energy for

neighboring electrons is  $-J$  if the spins are parallel and  $+J$  if antiparallel. Assuming no external magnetic field on this system, we can form the Hamiltonian

$$H = \sum_{(s_i, s_j)} -J(\sigma_{s_i} \sigma_{s_j})$$

where  $(s_i, s_j)$  are neighboring sites and  $\sigma_{s_k}$  are  $\{-1, +1\}$  valued for all integers  $k$ . In mathematical physics, the Hamiltonian represents the total energy of a system.

With the Hamiltonian in hand, we are able to provide the canonical partition function

$$Z = \sum_{\sigma} \exp\left\{-\frac{1}{kT} H(\sigma)\right\}$$

where  $\sigma$  is a particular configuration of spins,  $k$  is the Boltzmann constant and  $T$  is the absolute temperature. This partition function is central to statistical mechanics. From the partition function one is able to derive many important thermodynamic properties such as internal energy, specific heat and magnetism. Fortuin and Kasteleyn [9] showed this partition function can be rewritten in the language of percolation and then continued to tie the two fields together.

Many concepts for the Ising model have analogies in percolation theory. The most obvious analogy is that of a critical probability (at which percolation may occur) and critical temperature (at which spontaneous magnetization may occur). It is worth making special note that the Ising model is a dependent model. That is, the sites do not act independently from one another. The focus of this thesis is on dependent site percolation models.

## 1.4. SUMMARY

One of the attractive features of percolation theory is that many of its difficult and interesting problems are easily stated. This attribute is likely due to the many relations one can make between the theory and the physical world.

The remainder of this dissertation is outlined as follows. In Chapter 2, basics to probability theory and other relevant topics will be provided. With this basis in hand, in Chapter 3 we present a two dimensional site percolation model which is an extension of the work done by Burton and Keane (1989).

In Chapter 4, we construct a three dimensional model with many of the same properties as the two dimensional model, but resulting in a different conclusion. This will also show that certain results of Burton and Keane do not extend beyond two dimensions. Chapter 5 will summarize the work done here and discuss some of the possible extensions of this work.

## 2. SELECT BASICS

### 2.1. INTRODUCTION

This chapter will provide readers with the definitions of terms and some basic ideas which are used in this paper. We are not attempting to give a comprehensive review of measure theory or probability theory. At the conclusion of this chapter, we will review work relevant to the topic of the research presented as well as a more detailed description of the research.

### 2.2. PROBABILITY THEORY

We begin by defining a probability space.

**Definition 2** *A probability space is a triple  $(\Omega, \mathcal{F}, P)$  where  $\Omega$  is a set,  $\mathcal{F}$  is a  $\sigma$ -field on that set, and  $P : \mathcal{F} \rightarrow [0, 1]$  is a measure such that  $P(\Omega) = 1$  and  $P(\emptyset) = 0$ .  $\Omega$  is often referred to as the sample space or the set of outcomes, and  $\mathcal{F}$  is comprised of measurable sets and denotes the set of events which may occur.*

From our description of a two color  $\mathbb{Z}^d$  site percolation model in chapter one, the sample space is

$$\Omega = \prod_{z \in \mathbb{Z}^d} \{0, 1\}$$

with a  $\sigma$ -field  $\mathcal{F} = \mathcal{P}(\Omega)$ , the power set of  $\Omega$ . Unless otherwise specified, we take as a probability measure for this space the product measure

$$P = \prod_{z \in \mathbb{Z}^d} \mu_z$$

where  $\mu_z(\omega(z) = 1) = p_z$  and  $\mu_z(\omega(z) = 0) = q_z$ , and  $p_z + q_z = 1$ . An equivalent way to look at  $\Omega$  is as the space of maps

$$\mathcal{X} = \{x|x : \mathcal{Z}^d \rightarrow \{0, 1\}\}.$$

A central concept of probability theory is that of a **random variable**. In the following definition,  $\mathcal{R}$  is the set of real numbers and  $\mathcal{B}$  is the Borel  $\sigma$ -field on  $\mathcal{R}$ .

**Definition 3** *A function  $X : (\Omega, \mathcal{F}) \rightarrow (\mathcal{R}, \mathcal{B})$  is called a random variable if  $X$  is a measurable map; that is for  $B \in \mathcal{B}$ ,  $X^{-1}(B) \in \mathcal{F}$ . Hence the notation  $P(X = k)$  is taken to mean  $P(\{\omega \in \Omega | X(\omega) = k\})$ .*

The **distribution function**  $F(x)$  for a random variable  $X$  is defined by

$$F(x) = P(X \leq x)$$

where  $P(X \leq x) = P(\{\omega \in \Omega | X(\omega) \leq x\})$ . Two random variables which have the same distribution function are said to have the same distribution.

When  $\int_{\Omega} |X| dP < \infty$ , we can define

$$EX = \int_{\Omega} X dP.$$

This is called the **expected value** of  $X$ . A useful random variable is the indicator function  $1_A(\omega)$  where  $A \in \mathcal{F}$ .  $1_A(\omega) = 1$  when  $\omega \in A$  and  $1_A(\omega) = 0$  otherwise. One should note that for the indicator function  $1_A(\omega) : \Omega \rightarrow \{0, 1\}$ , we have

$$P(A) = E(1_A).$$

Since the expectation is an integral, it has all the properties which apply to integrals. A consequence of this is the Borel-Cantelli Lemma, which we prove. The

Borel-Cantelli Lemma is concerned with the probability that a sequence of events occurs infinitely often (i.o.). Given a sequence of events  $\{A_n\} \subseteq \mathcal{F}$ ,

$$\{A_n \text{ occurs infinitely often}\} = \{\omega \in \Omega \mid \omega \in A_n \text{ for infinitely many } n\}.$$

**Lemma 1 (Borel-Cantelli Lemma)** [?] If  $\sum_{i=1}^{\infty} P(A_i) < \infty$ , then  $P(A_i \text{ i.o.}) = 0$ .

*proof:* Suppose  $(\Omega, \mathcal{F}, P)$  is a probability space and  $\{A_n\} \subseteq \mathcal{F}$ . Assume  $\sum_{i=1}^{\infty} P(A_i) < \infty$ . We define a random variable  $N = \sum_{i=1}^{\infty} 1_{A_i}$ .  $N$  counts the number of times  $A_i$  occurs. Since

$$\sum_{i=1}^{\infty} P(A_i) < \infty,$$

we can rewrite this as

$$\sum_{i=1}^{\infty} E(1_{A_i}) < \infty.$$

Since  $1_{A_i} \geq 0$ , by Fubini's theorem we can exchange the summation and expectation. Thus

$$E\left(\sum_{i=1}^{\infty} 1_{A_i}\right) < \infty,$$

or, stated otherwise,

$$EN < \infty.$$

Thus the measure of  $\{\omega \mid N(\omega) = \infty\}$  is zero, and we have our result.

Another concept dealing with expected value is the notion of **conditional expectation**. In this case, one wonders about the average value of a random variable, or expectation, when some extra information about the outcome is given.

**Definition 4** Given a probability space  $(\Omega, \mathcal{F}, P)$ , a  $\sigma$ -field  $\mathcal{G} \subseteq \mathcal{F}$ , and an  $\mathcal{F}$ -measurable random variable  $X$  where  $E|X| < \infty$ , we define  $E(X|\mathcal{G})$ , the **expected value of  $X$  given  $\mathcal{G}$** , to be a random variable  $Y$  such that a)  $Y$  is  $\mathcal{G}$ -measurable and b) for any event  $G \in \mathcal{G}$ ,  $\int_G X dP = \int_G Y dP$ .

As before, there is a connection from conditional expectation to conditional probability. Namely,

$$P(A|\mathcal{G}) = E(1_A|\mathcal{G}).$$

Probabilities can also be conditioned on events. Bayes formula describes this in terms of the previous definition. For  $G \in \mathcal{G}$ , we can define

$$P(G|A) = \frac{\int_G P(A|\mathcal{G})dP}{\int_{\Omega} P(A|\mathcal{G})dP}$$

**Example 1** *We give now a simple example to illustrate conditional probability. Suppose you flip a fair coin twice. There are four equally likely possible outcomes*

$$\Omega = \{(h, h), (h, t), (t, h), (t, t)\}.$$

*The  $\sigma$ -field is  $\mathcal{P}(\Omega)$ , the power set of  $\Omega$ . Each outcome is equally likely, so if  $A$  is the event that the coin comes up with two heads then  $P(A) = \frac{1}{4}$ . For the event  $B$  “there is at least one head,”  $P(A|B) = \frac{1}{3}$ . In this case, the added information that there was at least one observed head decreased the number of possible equally likely outcomes and consequently increased the probability that the event  $(h, h)$  occurred.*

Another concept central to probability theory is that of independence.

**Events  $A$  and  $B$  are independent if**

$$P(A \cap B) = P(A)P(B).$$

Independence is used to describe more than events. **Random variables  $X$  and  $Y$  are independent if for all Borel sets  $C$  and  $D$ ,**

$$P(X \in C, Y \in D) = P(X \in C)P(Y \in D).$$

This can be restated by saying that random variables  $X$  and  $Y$  are independent if the events  $A = \{X \in C\}$  and  $B = \{Y \in D\}$  are independent for all Borel sets  $C$  and  $D$ . More generally, two  $\sigma$ -fields  $\mathcal{F}$  and  $\mathcal{G}$  are **independent** if for all  $A \in \mathcal{F}$  and  $B \in \mathcal{G}$ ,  $A$  and  $B$  are independent.

A sequence of distribution functions  $\{F_n\}$  **converges weakly** to a limit  $F$  ( $F_n \Rightarrow F$ ) if  $F_n(y) \rightarrow F(y)$  for all  $y$  that are continuity points of  $F$ . A sequence of random variables  $\{X_n\}$  **converges in distribution** to a random variable  $X$  if their distribution functions converge weakly. This is written  $X_n \Rightarrow X$ . We can now state the Central Limit Theorem [7] for independent, identically distributed (i.i.d.) random variables.

**Theorem 1 (The Central Limit Theorem)** *Let  $X_1, X_2, \dots$  be i.i.d. with  $EX_1 = \mu$  and  $\text{var}(X_1) = E(X_1^2) - [EX_1]^2 \in (0, \infty)$ . We write  $\text{var}(X_1) = \sigma^2$ . If  $S_n = X_1 + \dots + X_n$ , then*

$$(S_n - n\mu)/\sigma n^{1/2} \Rightarrow \mathcal{N}(0, 1)$$

where  $\mathcal{N}(0, 1)$  has the standard normal distribution.

Independence and conditional expectation are not unrelated notions. When one considers

$$E(X|\mathcal{F}),$$

one is asking the question how much information is added to the random variable  $X$  by  $\mathcal{F}$ . In the case when  $X$  is independent of  $\mathcal{F}$ , no information is added and

$$E(X|\mathcal{F}) = E(X).$$

As well, events  $A$  and  $B$  are **conditionally independent given  $\mathcal{F}$**  if

$$P(AB|\mathcal{F}) = P(A|\mathcal{F})P(B|\mathcal{F}).$$

That is, knowing  $\mathcal{F}$  has occurred,  $A$  and  $B$  add no information to one another.

In this paper, we will be considering collections of random variables which are indexed by  $\mathcal{Z}^d$ . Given a process

$$\{X_z | z \in \mathcal{Z}^d\}$$

we say that this process is **stationary** if for  $\mathcal{C} \subset \mathcal{Z}^d$

$$\{X_z | z \in \mathcal{C}\}$$

has the same distribution as

$$\{X_{z+z_0} | z \in \mathcal{C}\}$$

for any  $z_0 \in \mathcal{Z}^d$ .

Given a probability space  $(\Omega, \mathcal{F}, P)$ , a measurable map  $T : \Omega \rightarrow \Omega$  is called **measure preserving** if  $P(T^{-1}(A)) = P(A)$  whenever  $A \in \mathcal{F}$ . The concept of measure preserving and stationarity are closely related. Given a measure preserving transformation  $T$  on a probability space  $(\Omega, \mathcal{F}, P)$  and a random variable  $X$  on  $(\Omega, \mathcal{F})$ , we can construct a stationary sequence of random variables  $X_n$  by defining  $X_n(\omega) = X(T^{n-1}\omega)$ . In fact, in some sense every stationary sequence can be written in this way.

### 2.3. ERGODICITY

We now move towards making definitions concerning ergodic theory, a deep and wonderful field. We will need only the barest of essential definitions in this paper, so much will be omitted. Loosely said, a process is ergodic if it is not a mixture of more than one stationary sequence.

**Definition 5** *A measure preserving transformation  $T$  on a probability space  $(\Omega, \mathcal{F}, P)$  is called **ergodic** if  $T^{-1}(A) = A$  (a.s.) implies that  $P(A) = 0$  or 1.*

Since we will be considering measure preserving transformations which are  $\mathcal{Z}^d$  actions, we can give an alternate view of ergodic transformations with respect to  $\mathcal{Z}^d$ . Suppose we are given a collection  $\{T_1, \dots, T_d\}$  of commuting measure preserving transformations. Then

$$\{T | T = T_1^{i_1} \circ \dots \circ T_d^{i_d}\}$$

forms a  $\mathcal{Z}^d$  action where  $(i_1, \dots, i_d) \in \mathcal{Z}^d$ . If each measure preserving transformation  $S$  in this  $\mathcal{Z}^d$  action is ergodic, we say the  $\mathcal{Z}^d$  action

$$\{T | T = T_1^{i_1} \circ \dots \circ T_d^{i_d}\}$$

is totally ergodic. We next look at a one dimensional example to help illustrate ergodic transformations.

**Example 2** *Let us consider the space  $[0, 1)$  with Borel  $\sigma$ -field and Lebesgue measure as a probability space. The measure preserving transformation that we will examine is*

$$T_\theta(x) = x + \theta \pmod{1}.$$

*For rational  $\theta$ ,  $T_\theta$  is not ergodic. To see this, write  $\theta = a/b$  where  $a, b \in \mathcal{Z}$ . If  $\theta = 0$  then any set is invariant under  $T_\theta$ . Without loss of generality we assume  $\theta \in (0, 1)$ . Let*

$$A = [0, \frac{1}{2b}) \cup [\frac{2}{2b}, \frac{3}{2b}) \cup \dots \cup [\frac{2b-2}{2b}, \frac{2b-1}{2b}).$$

*In this case,  $T_\theta^{-1}(A) = A$  and  $P(A) = 1/2$ .*

*For irrational  $\theta$ ,  $T_\theta$  is ergodic. We can see this by first noting that for any  $x \in (0, 1)$ ,*

$$\{T_\theta(x), T_\theta^2(x), T_\theta^3(x), \dots\}$$

is dense in  $[0, 1]$  [2]. Suppose that  $A$  is a measurable set with measure greater than zero and  $A$  is  $T$  invariant. For any  $\delta > 0$ , there exists an interval  $J$  such that

$$P(A \cap J) > (1 - \delta)P(J).$$

That is to say,  $A \cap J$  is “almost” an interval. See [18] for more details. Since  $A$  is  $T$  invariant,

$$A = T_\theta^{-1}(A) \cup T_\theta^{-2}(A) \cup \dots$$

and thus

$$A \supseteq T_\theta^{-1}(A \cap J) \cup T_\theta^{-2}(A \cap J) \cup \dots$$

This tells us that our “almost” interval  $A \cap J$  can be moved anywhere by  $T_\theta$  and still be contained in  $A$ . Since  $P(T_\theta^{-n}(A \cap J)) > (1 - \delta)P(J)$  and  $\{T_\theta^{-n}(x)\}_{n=0}^\infty$  is dense, then

$$P(A) > 1 - \delta.$$

Letting  $\delta \rightarrow 0$ , we have  $P(A) = 1$  and thus  $T$  is an ergodic transformation.

We can even say a bit more about this. If we consider a collection of transformations,  $(T_{\theta_1}, \dots, T_{\theta_n})$ , this will be ergodic if and only if at least one  $\theta_i$  is irrational and will be totally ergodic if and only if  $\theta_1, \dots, \theta_n$  are linearly independent over the rationals. Totally ergodic implies that any finite composition of transformations in the collection is also ergodic.

The following strong limit theorem is often referred to as the ergodic theorem. In this theorem,  $\Upsilon$  is the class of invariant sets with respect to the measure preserving transformation  $T$ .

**Theorem 2 (The Ergodic Theorem)** *Let  $T$  be a measure preserving transformation on  $(\Omega, \mathcal{F}, P)$ . Then for  $X$  any random variable such that  $E|X| < \infty$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} X(T^k \omega) = E(X|\Upsilon) \text{ a.s.}$$

As a corollary to this, if  $T$  is also ergodic, then  $\Upsilon = \mathcal{F}$  and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} X(T^k \omega) = EX \text{ a.s.}$$

**Definition 6** *An idea more general than ergodicity is the notion of mixing. A measure preserving transformation  $T$  on a probability space  $(\Omega, \mathcal{F}, P)$  is called mixing if*

$$\lim_{n \rightarrow \infty} P(A \cap T^{-n}B) = P(A)P(B)$$

where  $A, B \in \mathcal{F}$ .

**Proposition 1** *Mixing implies ergodicity.*

*Proof:* Suppose we let  $A$  be a  $T$ -invariant set ( $T^{-1}(A) = A$  a.s.). If  $T$  is mixing, then

$$\lim_{n \rightarrow \infty} P(A \cap T^{-n}A) = P(A)P(A).$$

As well,

$$\lim_{n \rightarrow \infty} P(A \cap T^{-n}A) = P(A),$$

so  $P(A) = P(A)^2$ . This implies that  $P(A) = 0$  or  $1$  and thus shows that if  $T$  is mixing then  $T$  is ergodic as well.

Many of our proofs will take advantage of the fact that in order to prove something for the entire space, it is enough to prove it on a generating set. A group of generating sets we will use are called **cylinder sets**. Cylinder sets are defined to be subsets of  $\Omega$  which are specified by fixing a finite number of components of the space.

Since the work presented here is concerned with  $\mathcal{Z}^d$  percolation, we take as our cylinder sets events which have a finite number of  $\mathcal{Z}^d$  coordinates specified. Call the collection of these events  $\mathcal{E}$ . We show now that if a measure preserving transformation is mixing with respect to  $\mathcal{E}$ , it is mixing with respect to  $\sigma(\mathcal{E})$ .

**Proposition 2** *Let  $\mathcal{E}$  be the set defined above and let  $m$  be a probability measure on  $\sigma(\mathcal{E})$ . If a measure preserving transformation  $T$  has the property that for  $A_0, B_0 \in \mathcal{E}$*

$$\lim_{n \rightarrow \infty} m(T^{-n}A_0 \cap B_0) = m(A_0)m(B_0)$$

*then  $T$  is mixing with respect to  $\sigma(\mathcal{E})$ .*

*proof: Suppose*

$$\lim_{n \rightarrow \infty} m(T^{-n}A_0 \cap B_0) = m(A_0)m(B_0)$$

*for all  $A_0, B_0 \in \mathcal{E}$ .*

*Let  $A, B \in \sigma(\mathcal{E})$ . We can write*

$$A = \cup_{i=1}^{\infty} E_i \text{ where } E_i \in \mathcal{E}$$

*and*

$$B = \cup_{j=1}^{\infty} E_j \text{ where } E_j \in \mathcal{E}.$$

*Since  $m$  is a finite measure, given any  $\epsilon > 0$ , there exist  $A_0, B_0 \in \mathcal{E}$  such that*

$$m(A \Delta A_0) < \epsilon \text{ and } m(B \Delta B_0) < \epsilon$$

*where  $A_0 \Delta A = (A_0 \cap A)^c \cap (A_0 \cup A)$ . For any  $i \geq 0$ ,*

$$(T^{-i}A \cap B) \Delta (T^{-i}A_0 \cap B_0) \subset (T^{-i}A \Delta T^{-i}A_0) \cup (B \Delta B_0).$$

*This implies  $m((T^{-i}A \cap B) \Delta (T^{-i}A_0 \cap B_0)) < 2\epsilon$ . In turn, this implies*

$$|m(T^{-i}A \cap B) - m(T^{-i}A_0 \cap B_0)| < 2\epsilon.$$

*We find from this*

$$\begin{aligned} |m(T^{-i}A \cap B) - m(A)m(B)| &\leq |m(T^{-i}A \cap B) - m(T^{-i}A_0 \cap B_0)| \\ &\quad + |m(T^{-i}A_0 \cap B_0) - m(A_0)m(B_0)| \\ &\quad + |m(A_0)m(B_0) - m(A)m(B)| \\ &\quad + |m(A)m(B_0) - m(A)m(B)| \end{aligned}$$

$$\begin{aligned} &\leq 2\epsilon + |m(T^{-i}A_0 \cap B_0) - m(A_0)m(B_0)| + \epsilon + \epsilon \\ &= 4\epsilon + |m(T^{-i}A_0 \cap B_0) - m(A_0)m(B_0)|. \end{aligned}$$

Letting  $i \rightarrow \infty$  and  $\epsilon \rightarrow 0$ , we get

$$\lim_{n \rightarrow \infty} m(T^{-n}A \cap B) = m(A)m(B).$$

Thus we have shown that  $T$  is mixing with respect to  $\sigma(\mathcal{E})$ .

## 2.4. CONCEPTS RELEVANT TO RESEARCH

For a two color percolation model, we can define the concept of finite energy.

**Definition 7** *A measure  $\mu$  is said to have finite energy if*

$$\mu(\{x_{0,0} = 0 | \mathcal{F}\}) > 0$$

$$\mu(\{x_{0,0} = 1 | \mathcal{F}\}) > 0$$

where  $\mathcal{F}$  is the  $\sigma$ -field generated by  $\{x_{i,j} | (i,j) \neq (0,0)\}$ .

We can define this more generally for spaces with countable colors.

**Definition 8** *A measure  $\mu$  is said to have finite energy if*

$$\mu(\{x_{0,0} = \alpha | \mathcal{F}\}) > 0$$

for all colors  $\alpha$  where  $\mathcal{F}$  is the  $\sigma$ -field generated by  $\{x_{i,j} | (i,j) \neq (0,0)\}$ .

The topics in which we are most interested deal with the number and nature of infinite clusters in various percolation models. The concept of finite energy plays a key role in the research presented here. In 1981, Newman and Schulman [17]

introduced this definition and published results concerning the number of infinite clusters under certain conditions. Specifically, they considered the  $\mathcal{Z}^d$  lattice. Given a probability measure  $P$  on the space of maps

$$X = \{x|x : \mathcal{Z}^2 \rightarrow \{0,1\}\}$$

where  $P$  is translation invariant as well as translation ergodic and  $P$  has finite energy, then the number of infinite open clusters is exactly one of the following:

1. 0
2. 1
3.  $\infty$ .

Note that independence is not required of this percolation process.

A topic which has played a role in the research we consider is the Fortuin, Kasteleyn, Ginibre (FKG) inequality [14].

We can define a partial order on  $\Omega$ , namely

$$\omega_1 \leq \omega_2 \iff \omega_1(z) \leq \omega_2(z)$$

for all  $z \in \mathcal{Z}^d$ . A random variable  $X$  on  $(\Omega, \mathcal{F}, P)$  is said to be *increasing* if

$$\omega_1 \leq \omega_2 \Rightarrow X(\omega_1) \leq X(\omega_2).$$

Likewise, event  $A \in \mathcal{F}$  is increasing if

$$\omega_1 \leq \omega_2 \Rightarrow I_A(\omega_1) \leq I_A(\omega_2).$$

A example of increasing events and increasing random variables can easily be put in the context of percolation theory. Consider two sites  $a$  and  $b$ . The event

that there is a path joining  $a$  to  $b$  is an increasing event and the number of different open paths from  $a$  to  $b$  is an increasing random variable.

For an i.i.d. Bernoulli percolation model with the natural probability measure  $P_p$  and expectation  $E_p$ , we can state the FKG Inequality.

**Theorem 3 (FKG Inequality)** *If  $X$  and  $Y$  are increasing random variables such that  $E_p(X^2) < \infty$  and  $E_p(Y^2) < \infty$ , then*

$$E_p(XY) \geq E_p(X)E_p(Y).$$

*If  $A$  and  $B$  are increasing events, then*

$$P_p(A \cap B) \geq P_p(A)P_p(B).$$

In 1989, Gandolfi, Keane and Russo [12] showed that for a  $\mathcal{Z}^2$  dependent site percolation process, under certain conditions for the probability measure  $P$ , there was (a.s.) a unique infinite cluster. The conditions for the probability measure  $P$  are

1.  $P$  is invariant under horizontal and vertical translation as well as horizontal and vertical axis reflection.
2.  $P$  is ergodic with respect to horizontal and vertical translation (separately).
3. Increasing events are positively correlated under  $P$  (also known as the FKG condition).
4. The number of infinite clusters is non-zero.

In 1989, Burton and Keane [4] proved quite a bit more along these lines. They showed that under the conditions of translation invariance (stationarity), ergodicity and finite energy, for all  $\mathcal{Z}^d$  models and some other lattice structures, there

was at most one infinite cluster. In a later publication [5], under the conditions of stationarity and ergodicity, the topological nature of the infinite clusters was classified. More detail on this problem will be given in the next chapter.

In 1991, Gandolfi, Keane and Newman [11] extended the technique used in the previously mentioned result to obtain results for long range percolation models. Long range percolation models define neighbor relations on a lattice beyond the nearest Euclidean sites.

Finally, we mention a result due to Benjamini and Schramm [1]. This result is not on the integer lattice to which most of the previous results refer. Instead, Benjamini examined site percolation in 2-dimensional hyperbolic space with compact quotient. For percolation on the ( $n$ -dimensional) integer lattice, as probability  $p$  increases, the number of infinite clusters (a.s) transitions from none to one. In the case of percolation in hyperbolic space, Benjamini showed that there are  $0 < p_1 < p_2 < 1$  such that, for the probability  $p_1$  that sites are open, there are infinitely many clusters and, for probability  $p_2$  that sites are open, there is exactly one cluster. In essence, one might say that there is enough room to grow in hyperbolic space to allow the phenomenon of infinitely many infinite clusters on a plane.

## 2.5. SUMMARY

At first glance, percolation theory seems to be concerned with the critical values of phase transitions in two and three dimensions. Closer inspection shows us there is much more. Describing and classifying the nature of infinite components in both Euclidean and non-Euclidean spaces is just one more dimension to this topic.

In Chapter 3, we will more closely examine the work of Burton and Keane [5]. Then, we will provide an example of a two dimensional integer lattice percolation

model in which the open and closed sites percolate (a.s.). Under the conditions imposed on this model, this will be the maximum number of infinite clusters. The measure on the space is constructed by a *cut and stack* method. For clarity, we will demonstrate this technique in one dimension before providing the measure for the two dimensional model.

Chapter 4 will examine this situation in three dimensions. It will be seen that it is indeed possible to construct a model in which infinitely many colors percolate. The measure in this model will come directly from the construction.

### 3. TWO DIMENSIONAL PERCOLATION MODEL

#### 3.1. INTRODUCTION

In 1989, Burton and Keane [4] showed that under the conditions of stationarity, ergodicity and finite energy, percolation in  $\mathcal{Z}^2$  results in a maximum of two infinite components. After sketching this result, we will construct a two dimensional percolation model and a stationary measure on the space of maps which has uniform finite energy and such that shift transformations are ergodic with respect to that measure. The space of maps will, under this measure, result in the percolation of both the open and closed sites (a.s.) thus demonstrating the existence of such a space. We begin with some introductory definitions and concepts, repeating some definitions for the reader's convenience.

#### 3.2. DEFINITIONS AND BACKGROUND

**Definition 9** We say that  $z_1, z_2 \in \mathcal{Z}^2$  are **neighbors** if their Euclidean distance is 1. Furthermore,  $z_1, z_2 \in \mathcal{Z}^2$  are **\*-neighbors** if their Euclidean distance is either 1 or  $\sqrt{2}$ .

**Definition 10** Suppose  $S \subseteq \mathcal{Z}^2$ . The saturation of the relation " $z_1, z_2 \in S$ ,  $z_1$  and  $z_2$  are neighbors" is an equivalence relation, and the equivalence classes are called the **components** of  $S$ .

We continue with three intuitive definitions of **interior**, **exterior** and **closure** in the context of percolation theory. Given  $S \subseteq \mathcal{Z}^2$ , define:

$$\text{int}(S) \equiv \bigcup \{K : K \text{ is a finite component of } \mathcal{Z}^2 - S\}$$

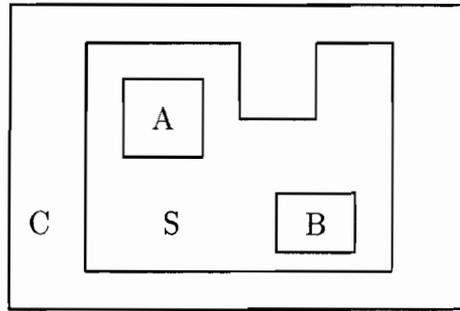


Figure 3.1. Interior, Exterior and Closure

$$\text{ext}(S) \equiv \bigcup \{K : K \text{ is an infinite component of } \mathcal{Z}^2 - S \}$$

$$\text{cl}(S) \equiv S \cup \text{int}(S)$$

We identify these sets in Figure 3.1.

$$\text{int}(S) = A \cup B$$

$$\text{ext}(S) = C$$

$$\text{cl}(S) = S \cup A \cup B.$$

Now suppose  $x : \mathcal{Z}^2 \rightarrow \{0, 1\}$  (i.e.,  $x$  is a realization of a  $\mathcal{Z}^2$  two color site percolation model). Two sets with which we will concern ourselves are

$$G_0 \equiv \{z \in \mathcal{Z}^2 | x(z) = 0\}$$

$$G_1 \equiv \{z \in \mathcal{Z}^2 | x(z) = 1\}.$$

**Definition 11** *The components of  $G_i$  are called  $i$ -clusters. Note that if a component of  $G_i$  is infinite, we say that  $i$  percolates.*

We classify clusters in the following way. Define a partial ordering “ $\prec$ ” on clusters  $C, C'$  via

$$C \prec C' \iff C \subseteq \text{cl}(C').$$

The maximal elements under this partial ordering are called **essential** clusters. All others are called **inessential**. Notice that inessential clusters are always finite, whereas essential clusters can be finite or infinite. We extend this definition to a map  $x : \mathcal{Z}^2 \rightarrow \{0,1\}$  by calling map  $x$  **essential** if every cluster is enclosed by an essential cluster. A map  $x : \mathcal{Z}^2 \rightarrow \{0,1\}$  is called **infinite cascade** if each cluster is inessential. It is a relatively simple to show that a map  $x : \mathcal{Z}^2 \rightarrow \{0,1\}$  is either essential or infinite cascade.

Define the set  $X = \{x|x : \mathcal{Z}^2 \rightarrow \{0,1\}\}$ . Let  $\mu$  be a probability measure on  $X$  which is stationary and ergodic under the action of  $\mathcal{Z}^2$  on  $X$  by translation, (i.e.  $(X, \mu, \mathcal{A}, T)$  is ergodic where  $\mathcal{A} = \mathcal{P}(X)$  and  $T$  is a  $\mathcal{Z}^2$  shift). We will see that for such a space, either  $x \in X$  is essential or infinite cascade ( $\mu - \text{a.s.}$ )

Let the event  $A \in \mathcal{A}$  be defined by

$$A = \{x \in X : x \text{ is essential}\}.$$

For a measure preserving shift  $T$ ,  $T^{-1}(A) = A$ . Because  $T$  is ergodic with respect to  $(X, \mu, A)$ , then  $\mu(A) = 0$  or  $\mu(A) = 1$ . Thus  $x \in X$  is essential (a.s.) or  $x \in X$  is infinite cascade (a.s.). From this point on, we'll concern ourselves with the case  $x \in X$  is essential.

This brings us finally to the structures which we will ultimately consider.

Define:

**0-ribbon**  $\equiv \cup\{C : C \text{ is a cluster and } C \text{ is enclosed by a fixed infinite essential cluster of color } 0 \}$

and

**1-ribbon**  $\equiv \cup\{C : C \text{ is a cluster and } C \text{ is enclosed by a fixed infinite essential cluster of color } 1 \}$

Let  $H \equiv \bigcup \{ \text{cl}(C) : C \text{ is an essential finite cluster} \}$ . The saturation of the relation “ $z_1, z_2 \in H$ ,  $z_1$  and  $z_2$  are  $*$ -neighbors” is an equivalence relation. **Rocks** are defined to be the finite equivalence classes of  $H$ , and **quilts** or **q-ribbons** are defined to be the infinite equivalence classes of  $H$ .

In classifying the structure of ribbons, from results by Burton and Keane we know that the topological structure of  $x$  (a.s.) is that of topological strips (i.e., infinite ribbons with at most two boundaries). The arguments first show that (a.s.) rocks play no role in classifying the structure of ribbons. In a similar argument, Burton and Keane show that a ribbon has at most two infinite components in its exterior (a.s.). To give an idea of these similar arguments, we provide the proof showing that rocks do not play a role in classifying the structure of ribbons.

**Proposition 3** *Let  $(X, \mu, \mathcal{A}, T)$  be a stationary, ergodic process where  $X$  is defined as above. If  $R$  is the event that there exists a rock with three or more ribbons as  $*$ -neighbors, then  $\mu(R) = 0$ .*

*proof: Suppose  $\mu(R) > 0$ . Then, by ergodicity,  $\mu(R) = 1$ . There must exist  $\epsilon > 0$  such that for (a.e.)  $x \in X$  there exist sufficiently large boxes  $[0, N] \times [0, N]$  which contain  $\epsilon N^2$  rocks with at least three ribbons as  $*$ -neighbors. This can be seen by application of the ergodic theorem.*

*For each rock, choose three infinite ribbons. Because the ribbons are infinite, these ribbons cross the boundaries of the  $[0, N] \times [0, N]$  box. For each ribbon, there exists a path from the rock to the border entirely contained in the ribbon. As a convention, if the path is contained in a ribbon which contains the origin, then the path will cross at the origin. Starting at the origin, traverse the boundary of the box in a clockwise manner.*

*For each rock, we call the second of the three points which cross the boundary the **central point** for a rock. From an elementary application of the Jordan curve*

theorem we see that the central points of the rocks are unique. Since the central points are unique, the border of  $[0, N] \times [0, N]$  contains at least  $\epsilon N^2$  points. This is a contradiction for large  $N$  and we conclude that (a.s.) a rock does not have three or more ribbons as  $*$ -neighbors.

The proof is similar for showing that a ribbon has at most two infinite components in its exterior.

**Theorem 4** *Let  $(X, \mu, \mathcal{A}, T)$  be as above. If  $Q$  be the event that  $x \in X$  contains a ribbon whose complement in  $\mathcal{Z}^2$  has at least three components. Then  $\mu(Q) = 0$ .*

The conclusion is that topologically, these ribbons are strips. We now come to another idea which has been previously discussed; finite energy.

**Definition 12** *A measure  $\mu$  is said to have **finite energy** if*

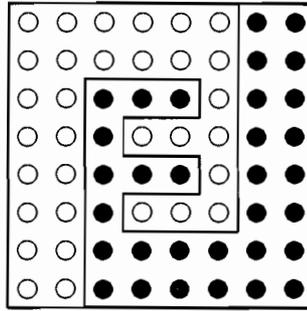
$$\mu(\{x_{0,0} = 0 | \mathcal{F}\}) > 0$$

$$\mu(\{x_{0,0} = 1 | \mathcal{F}\}) > 0$$

where  $\mathcal{F}$  is the  $\sigma$ -field generated by  $\{x_{i,j} | (i,j) \neq (0,0)\}$ .

**Proposition 4** *If  $\mu$  is a stationary ergodic measure with finite energy, then  $x \in X$  contains a.s. at most two ribbons.*

The argument is as follows. Suppose, with positive probability,  $x$  contains a ribbon with two ribbons as neighbors. It is then possible to change the configuration of a large enough box so that, with positive probability, a rock is produced with three ribbons as neighbors or a ribbon is produced with with three ribbons in its complement. This is a contradiction to the previous theorems.

Figure 3.2.  $\mathcal{E}_1$ -box

The idea of finite energy can be extended to uniform finite energy. In the subsequent work, we will show the probability measures to have this property. Finite energy is implied by uniform finite energy.

**Definition 13** : A measure  $\mu$  is said to have **uniform finite energy** if for some  $\epsilon > 0$ ,

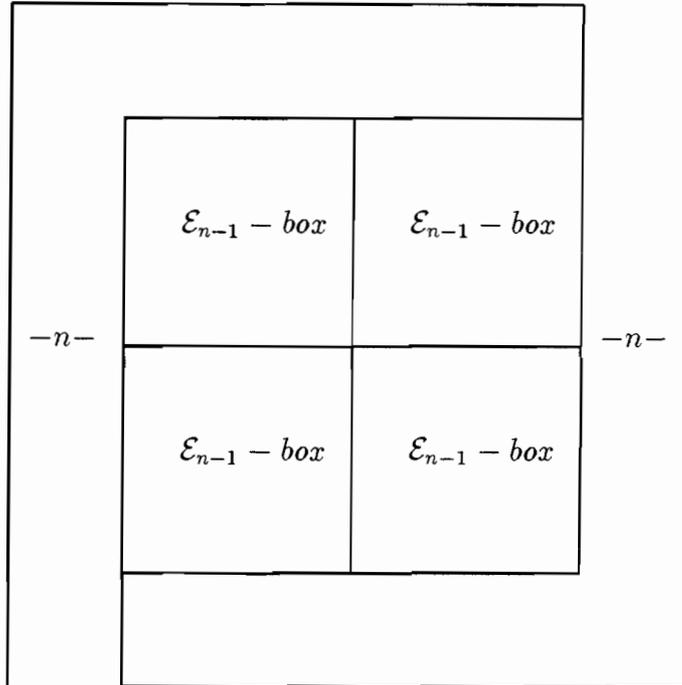
$$\mu(\{x_{0,0} = 0|\mathcal{F}\}) > \epsilon$$

$$\mu(\{x_{0,0} = 1|\mathcal{F}\}) > \epsilon$$

where  $\mathcal{F}$  is the  $\sigma$ -field generated by  $\{x_{i,j} | (i,j) \neq (0,0)\}$ .

### 3.3. CONSTRUCTION OF MODEL

We now move to the construction of a stochastic, ergodic measure  $\mu$  on  $X$  with uniform finite energy such that the ribbons of a realization of any  $x \in X$  (a.s.) share exactly one border with the other ribbon. From the above results, these maps will contain the maximum number of ribbons under these conditions. First, we will give a physical description of the model, and then we will proceed with a construction of the measure.

Figure 3.3.  $\mathcal{E}_n$ -box

Define an  $\mathcal{E}_1$ -*box* to have the configuration as in Figure 3.2. The solid circles are closed with probability  $p$  and open with probability  $1 - p$ . In a similar fashion, the open circles are open with probability  $p$  and closed with probability  $1 - p$ . For  $n \geq 2$ , we can define an  $\mathcal{E}_n$ -*box* as in Figure 3.3.

For  $n$  even, the sites in the upper left border are closed with probability  $p$  and open with probability  $1 - p$ . The sites in the lower right border are open with probability  $p$  and closed with probability  $1 - p$ . For  $n$  odd, this is reversed. It is fairly straight forward to verify that an  $\mathcal{E}_i$ -*box* has length and width  $a_i = 7 \cdot 2^i - 2i - 4$ .

If  $p = 1$ , it is clear that our “yin-yang” pattern results in both the open and closed sites percolating. Unfortunately, our model would not possess the other properties we desire. With  $p < 1$ , it is possible that a  $0$ -*cluster* is prevented from percolating by a group of sites colored 1 traversing the path of 0’s. We will show

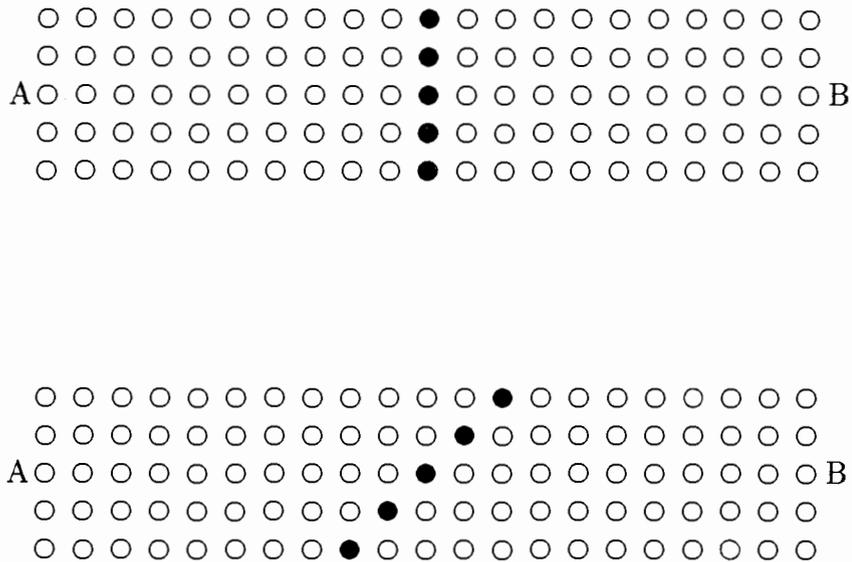


Figure 3.4. Cut Sets

that for certain values of  $p$ , the probability that a path is blocked infinitely often is zero.

A set of points which prevent a path from connecting one site to another is called a **cut set**. The configurations in Figure 3.4 are examples of cut sets which prevent site  $A$  from being connected to site  $B$  via a path of open circles. Notice that in the first case, the cut set is a path. The cut set is not a path in the second case.

In order for a path to escape the  $n - 1^{\text{st}}$  stage of construction, it must travel both vertically and horizontally a distance bounded by  $a_n - n + 1$ . Let us now consider the horizontal path. The number of cut sets of length  $l$  which prevent a path from transversing the block horizontally is bounded by  $a_n \cdot 7^l$ . Call the event that there is a cut set  $A_n$ .

$$\begin{aligned}
P(A_n) &\leq \sum_{l \geq n} a_n 7^l (1-p)^l \\
&= a_n 7^n (1-p)^n \sum_{l \geq 0} 7^l (1-p)^l
\end{aligned}$$

This geometric series converges for  $p > \frac{6}{7}$ . Letting  $p > \frac{6}{7}$ ,

$$a_n 7^n (1-p)^n \sum_{l \geq 0} 7^l (1-p)^l = a_n 7^n (1-p)^n \frac{1}{7p-6}$$

The probability that a cut set blocks a path vertically is bounded in the same way. Define the event  $B_n$  as a path being blocked at the  $n-1^{\text{st}}$  stage and not making it to the  $n^{\text{th}}$  stage. For  $p > \frac{6}{7}$ ,

$$P(B_n) < a_n 7^n (1-p)^n \frac{2}{7p-6}$$

To check if  $\{B_n\}$  occurs infinitely often, we examine the sum  $\sum_{n \geq 1} P(B_n)$ .

$$\begin{aligned}
\sum_{n \geq 1} P(B_n) &< \sum_{n \geq 1} \frac{2}{7p-6} a_n 7^n (1-p)^n \\
&< \sum_{n \geq 1} \frac{2}{7p-6} (14-14p)^n.
\end{aligned}$$

For  $p > 13/14$ , this geometric series converges and

$$\sum_{n \geq 1} P(B_n) < \infty.$$

Thus by Borel-Cantelli, the probability of being blocked infinitely often is zero, and hence a path escapes (a.s.).

### 3.4. CONSTRUCTION OF MEASURE

What we have at this point is a model in which both open and closed sites percolate (a.s.). We now need to provide the stationary measure with uniform finite

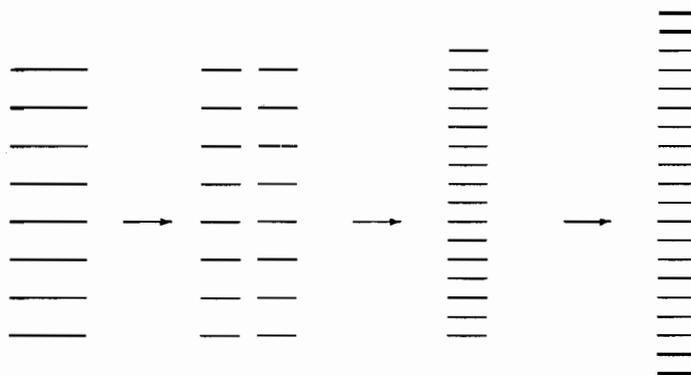


Figure 3.5. One Dimensional Cut and Stack Method

energy such that measure preserving shift transformations are ergodic with respect to the measure. We do this by a standard "cut and stack" method. For further treatment of this construction, see Friedman [10]. In order to demonstrate the construction of the asserted stationary, ergodic measure with uniform finite energy, we shall first do a similar construction in one dimension.

### 3.4.1. One Dimensional Construction

Consider the half open interval  $[0, 4/7)$  with Lebesgue measure. Subdivide this interval into eight half open subintervals of equal measure and stack the left stack upon the right stack. Call this set of points  $\mathcal{L}_1$ . We then "cut" these eight intervals in half and "stack" them, making sixteen half open intervals. In addition, on both ends we add two intervals of equal length. This set shall be denoted by  $\mathcal{L}_2$ . Our cut and stack method is illustrated in Figure 3.5.

In general,  $\mathcal{L}_n$  is defined in similar way. We "cut" an  $\mathcal{L}_{n-1}$  in half, "stack" the intervals, and add  $n$  intervals of equal measure on both ends.

It is a fairly straight forward calculation to show that

$$\mu(\lim_{n \rightarrow \infty} \mathcal{L}_n) = 1.$$

At any level of our construction, the transformation  $T$  on our space can be described as moving a point vertically upwards one tier. Consequently,  $T^{-1}$  moves a point down one tier.  $T$  is undefined at the top level.

**Proposition 5**  $T$  is  $\mu$  measure preserving and  $(\mathcal{L}, \mu, T)$  is ergodic.

Let  $A \subseteq \mathcal{L}$  be any measurable set and  $\epsilon > 0$ . Suppose  $\mu(A) = c$  where  $c \in [0, 1]$ . Choose  $n$  large enough so that

$$\mu(A - \mathcal{L}_n) < \frac{\epsilon}{2}$$

and that the length of one interval of  $\mathcal{L}_n$  is small enough. Specifically,

$$\left(\frac{4}{7}\right)\left(\frac{1}{2^{n+2}}\right) < \frac{\epsilon}{2}.$$

Since Lebesgue measure is invariant under translation,

$$c - \epsilon < \mu(T^{-1}(A \cap \mathcal{L}_n) \cap \mathcal{L}_n) \leq c = \mu(A).$$

Let  $\epsilon \rightarrow 0$ . Thus we have  $\mu(T^{-1}(A)) = \mu(A)$ , so  $T$  is  $\mu$  measure preserving.

Next we will show ergodicity. Suppose  $A$  is a measurable set and  $T^{-1}(A) = A$  (a.s.). Suppose  $\mu(A) > 0$ . There exists  $N$  large enough so that for all  $n \geq N$

$$\mu(A \cap \mathcal{L}_n) > 0.$$

For any  $\lambda > 0$ , there exists  $x \in (A \cap \mathcal{L}_n)$  and  $\delta > 0$  such that

$$\mu(B_\delta(x) \cap (A \cap \mathcal{L}_n)) > (1 - \lambda)2\delta$$

where  $\mathcal{B}_\delta(x)$  is the ball of radius  $\delta$  centered at  $x$ . In our case, we also choose  $\delta$  small enough so that  $\mathcal{B}_\delta(x)$  remains on a single connected interval of  $\mathcal{L}_n$ . For some later stage  $m > n$ , there is an interval  $I$  of length  $\rho$  such that

$$\mu(A \cap I) > (1 - \lambda)\rho.$$

$T$  is  $\mu$  measure preserving (and in the same way, so is  $T^{-1}$ ). For any interval  $J$  in the  $\mathcal{L}_m$  configuration,

$$\mu(A \cap J) > (1 - \lambda)\rho.$$

Furthermore, this is true for every subsequent stage. Letting  $\lambda \rightarrow 0$ , we have  $\mu(A) = 1$ . Therefore  $(\mathcal{L}, \mu, T)$  is ergodic.

### 3.4.2. Measure for Constructed Model

In this section we now construct a measure on the two dimensional space previously described which is stationary, ergodic and has uniform finite energy. We do this via the same *cut and stack* method used in the previous one dimension example. Recall that our  $\mathcal{E}_1$  – box was defined to be an eight by eight array colored according to a certain prescription. In an analogous way, we divide the interval  $[0, 16/49)$  into 64 half open subintervals of equal Lebesgue measure and arrange these subintervals into an eight by eight array. This configuration we shall call an  $\mathcal{E}_1$  set.

An  $\mathcal{E}_2$  set is constructed by further subdividing each subinterval in  $\mathcal{E}_1$  into four half open subintervals, moving the second of these intervals to the right of the original eight by eight configuration, the third moves directly below, and the fourth is moved the remaining lower right quadrant. In addition, we add a border two levels deep of half open intervals of the same measure. This cut and stack is depicted in Figure 3.6.

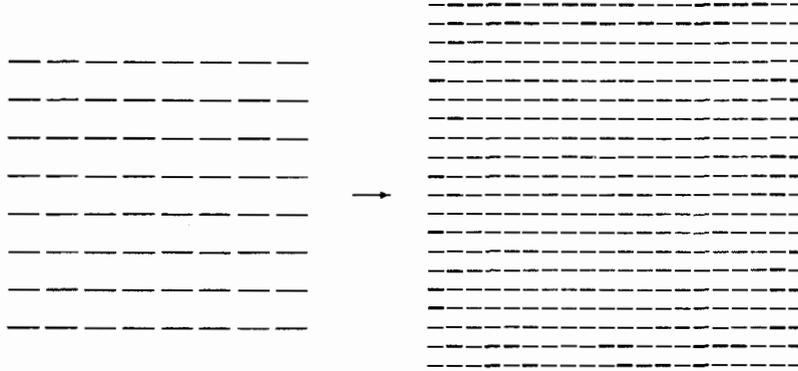


Figure 3.6. Two Dimensional Cut and Stack Method

Inductively, an  $\mathcal{E}_n$  set is constructed by equally subdividing each interval in an  $\mathcal{E}_{n-1}$  set into four half open subintervals, moving them as above, and adding a layer  $n$  deep. It is a fairly straight forward calculation to show that  $\mu(\lim_{n \rightarrow \infty} \mathcal{E}_n) = 1$ . It is worth noting that for an  $\mathcal{E}_n$  configuration, the  $\mu$ -measure of any segment is

$$\frac{16}{49} \cdot \frac{1}{2^{2n+4}}$$

and there are  $7(2^n) - 2(n + 2)$  such segments in each row. The transformations  $T$  and  $S$  on our space can be described as moving a point vertically upwards one level (respectively horizontally to the right one level).  $T$  is undefined on the top column and  $S$  is undefined on the right column. We now wish to know that these are measure preserving transformations and the system is ergodic. The proof will be very similar to that of the one dimensional case.

**Proposition 6**  $T$  and  $S$  are  $\mu$  measure preserving and  $(\mathcal{E}, \{T, S\}, \mu)$  is ergodic.

*Again we begin by showing measure preserving. It is enough to show that  $T$  and  $S$  are individually measure preserving. Let  $A$  be any measurable set and let  $\mu(A) = c$  where  $c \in [0, 1]$ . Let  $\epsilon > 0$ . Choose  $n$  large enough so that*

$$\mu(A - \mathcal{E}_n) < \frac{\epsilon}{2}$$

and so that the length of one interval of  $\mathcal{E}_n$  is small enough. That is

$$\left(\frac{1}{2^{2n+4}}\right) \cdot (7 \cdot 2^n - 2(n+2)) < \frac{\epsilon}{2}.$$

Since  $T^{-1}$  is a simple translation and Lebesgue measure is invariant under translation,

$$c - \epsilon < \mu(T^{-1}(A \cap \mathcal{E}_n) \cap \mathcal{E}_n) \leq c.$$

Letting  $\epsilon \rightarrow 0$ , we have that  $\mu(T^{-1}(A)) = \mu(A)$ . As well, the same reasoning applies to  $S$ . Thus  $T$  and  $S$  are  $\mu$  invariant.

To see ergodicity, we want to show that for any measurable set  $A$ , if  $T^{-1}(A) = A$  (a.s.) and  $S^{-1}(A) = A$  (a.s.) then  $\mu(A) = 0$  or  $\mu(A) = 1$ . Suppose for a measurable set  $A$  that  $T^{-1}(A) = A$  (a.s.),  $S^{-1}(A) = A$  (a.s.) and  $\mu(A) = c$  where  $c > 0$ . Choose  $n$  large enough so that  $\mu(A \cap \mathcal{E}_n) > 0$ . For any  $\lambda > 0$ , there exists  $x \in (A \cap \mathcal{E}_n)$  and  $\delta > 0$  such that

$$\mu(\mathcal{B}_\delta(x) \cap A \cap \mathcal{E}_n) > (1 - \lambda)2\delta.$$

For some stage  $m > n$ , there is an interval  $I$  of length  $\rho$  such that

$$\mu(A \cap I) > (1 - \lambda)\rho.$$

Since  $T, T^{-1}, S$  and  $S^{-1}$  are measure preserving, for any interval  $J$  in  $\mathcal{E}_m$ ,

$$\mu(A \cap J) > (1 - \lambda)\rho.$$

This is true for all subsequent stages. Letting  $\lambda \rightarrow 0$ , we have  $\mu(A) = 1$ .

Thus, the measure  $\mu$  we have described here is stationary with uniform finite energy and the measure preserving transformations  $T$  and  $S$  are ergodic with respect to this system. Since the transformations  $T$  and  $S$  generate all other shift transformations, we can conclude that any shift transformation is ergodic with respect to this system.

## 4. THREE DIMENSIONAL PERCOLATION MODEL

### 4.1. OUTLINE OF MODEL

In the previous chapter, we constructed a stationary measure with uniform finite energy on a space of maps such that every non-trivial measure preserving shift transformation is ergodic with respect to the resulting probability space. In that model, two colors percolate (a.s.). This is the maximum number of colors which can percolate under these conditions.

In this chapter, we consider the space of maps  $X = \{x|x : \mathcal{Z}^3 \rightarrow \mathcal{N}\}$  and construct a stationary measure with uniform finite energy such that “most” shift transformations are ergodic with respect to the resulting probability space and that (a.s.) every color percolates. It should be noted that only one measure preserving shift transformation is required for ergodicity. Unlike the previous model, the measure for this space of maps will not require any special construction.

We will build this model beginning with a one dimensional process and making appropriate extensions to two and three dimensions. A final modification will be made to add uniform finite energy. The last step will be to conclude that each color percolates (a.s.). We begin with the one dimensional process.

### 4.2. ONE DIMENSIONAL BASE PROCESS

Let  $X_1 = \{x|x : \mathcal{Z} \rightarrow \mathcal{N}\}$ , and define a probability measure  $P^1$  such that  $P^1(x(j) = i) = 1/2^i$  for all  $j \in \mathcal{Z}$ . Note that the color of any one point is independent from the coloring of any other point. As a matter of interest, we will observe

that the measure  $P^1$  is stationary and that measure preserving shifts are mixing with respect to  $(X_1, P^1)$ .

**Proposition 7**  $P^1$  is stationary and any measure preserving shift transformation  $T$  is mixing with respect to  $(X_1, P^1)$ .

*proof:* It is a direct consequence of independence that  $P^1$  is stationary. To see that  $P^1$  is stationary, let  $T$  be a shift transformation of  $a$ . For  $x = (\dots, x(-1), x(0), x(1), \dots)$ ,

$$T(x) = y$$

where  $y(i) = x(i + a)$  for all  $i \in \mathcal{Z}$ . For some cylinder set

$$A = \{x(j) = i_j, \dots, x(j + k) = i_{j+k}\}$$

we see the following:

$$\begin{aligned} & P(T^{-1}(A)) \\ &= P(\{x(j - a) = i_j, \dots, x(j + k - a) = i_{j+k}\}) \\ &= \frac{1}{2^{i_j}} \cdots \frac{1}{2^{i_{j+k}}} \\ &= P(A). \end{aligned}$$

Since this is true for all cylinder sets, it is indeed true for the entire probability space. Thus  $P^1$  is stationary.

To see that any shift transformation is mixing with respect to  $P^1$ , we note that for any two cylinder sets  $A, B$

$$P(A \cap T^{-l}B) = P(A)P(T^{-l}B)$$

given  $l$  is greater than some constant depending on  $A, B$  and  $T$ . Since  $T$  is measure preserving,

$$\lim_{l \rightarrow \infty} P(A)P(T^{-l}B) = \lim_{l \rightarrow \infty} P(A)P(B)$$

$$= P(A)P(B).$$

As shown in chapter 2, this is a property which found true for a family of cylinder sets implies the property follows for the space generated by those sets. Thus we conclude that  $P^1$  is stationary and  $T$  is mixing with respect to  $(X_1, P^1)$ .

### 4.3. TWO DIMENSIONAL PROCESS

We are now ready to extend our model to two dimensions. To do this extension, we shall use a two sided random walk. Let

$$\{\mathcal{X}_i\}_{i \in \mathbb{Z} - \{0\}}$$

be a family of i.i.d. random variables with

$$P(\mathcal{X}_1 = 1) = P(\mathcal{X}_1 = -1) = 1/2.$$

Given this, we define our random walk  $\{S_n\}$  by

$$S_n = \mathcal{X}_1 + \cdots + \mathcal{X}_n, n > 0$$

$$S_n = \mathcal{X}_{-1} + \cdots + \mathcal{X}_n, n < 0$$

$$S_0 = 0.$$

**Observation 1** *Something which we will want to note about the random walk process is that for a fixed interval  $[-a, a]$ ,  $P(S_n \in [-a, a]) \rightarrow 0$  as  $n \rightarrow \infty$ . This can be seen by invoking the Central Limit Theorem as given in chapter 2.*

$$\begin{aligned} &P(S_n \in [-a, a]) \\ &= P(S_n/\sqrt{n} \in [-a/\sqrt{n}, a/\sqrt{n}]) \end{aligned}$$

$$= F_n(a/\sqrt{n}) - F_n(-a/\sqrt{n}) \text{ (a.s.)}$$

where  $F_n$  is the distribution function for  $S_n/\sqrt{n}$ . By the central limit theorem,

$$= F_n(a/\sqrt{n}) - F_n(-a/\sqrt{n}) \text{ (a.s.)} \rightarrow 0$$

as  $n \rightarrow \infty$ .

We are now ready to extend our one dimensional process to two dimensions.

Let  $X_2 = \{x|x : \mathcal{Z}^2 \rightarrow \mathcal{N}\}$ . We define the initial space as before,

$$P^2(x(i, 0) = j) = 1/2^j$$

for all  $i \in \mathcal{Z}$ . For the remainder of the points, we define

$$x(i, k) = x(i - S_k, 0).$$

This makes a copy of our initial process and shifts it  $S_k$  units according to the random walk. In order illustrate this process, we shall present a few sample events.

**Example 3** Suppose that in our initial coloring, we have  $x(0) = 2$ ,  $x(1) = 2$ ,  $x(2) = 1$ ,  $x(3) = 8$ ,  $x(4) = 4$  and for our random walk,  $S_1 = -1$ ,  $S_2 = 0$ . A cylinder set is a finite array of coordinates which have been specified. From what we have described above, we get the set

$$A = \left[ \begin{array}{cccc} x(0, 2) = 2 & x(1, 2) = 2 & x(2, 2) = 1 & x(3, 2) = 8 \\ x(0, 1) = 2 & x(1, 1) = 1 & x(2, 1) = 8 & x(3, 1) = 4 \\ x(0, 0) = 2 & x(1, 0) = 2 & x(2, 0) = 1 & x(3, 0) = 8 \end{array} \right].$$

Likewise, if we are given the cylinder set  $A$ , we would be able to deduce  $x(4, 0) = 4$ ,  $x(-1, 1) = 2$ ,  $x(4, 2) = 4$ ,  $S_1 = -1$  and  $S_2 = 0$  as well as the points specified by  $A$ . We see from this that a cylinder set such as  $A$  reveals more about our process than just the points specified.

**Example 4** *Not every cylinder set is very revealing. Suppose*

$$B = \begin{bmatrix} x(0, 3) = 2 & x(1, 3) = 2 & x(2, 3) = 2 & x(3, 3) = 1 \\ x(0, 2) = 2 & x(1, 2) = 2 & x(2, 2) = 2 & x(3, 2) = 2 \\ x(0, 1) = 2 & x(1, 1) = 2 & x(2, 1) = 2 & x(3, 1) = 2 \\ x(0, 0) = 2 & x(1, 0) = 2 & x(2, 0) = 2 & x(3, 0) = 2 \end{bmatrix}.$$

From  $B$ , we are not able to determine  $S_1, S_2$ , and  $S_3$ , though we do know that  $S_3 = S_2 - 1$  and  $S_2 = S_1 - 1$ . Also, we cannot be certain what color points outside the cylinder set are, but that does not mean all points are independent from  $B$ . For instance, from how our process is defined

$$P^2(x(-1, 0) = 2) = 1/4.$$

Now let us consider  $P^2(x(-1, 0) = 2|B)$ . We will calculate this by

$$P^2(x(-1, 0) = 2|B) = \frac{P^2(x(-1, 0) \cap B)}{P^2(B)}.$$

Of the eight possibilities for  $S_1, S_2$  and  $S_3$ , only two are possible for event  $B$ .  $S_1 = 1, S_2 = 0$ , and  $S_3 = -1$  or  $S_1 = -1, S_2 = -2$ , and  $S_3 = -3$ . In the case of  $S_1 = 1$ , this requires six coordinates to be colored 2 followed by a coordinate colored 1. For  $S_1 = -1$ , five coordinates are colored 2 followed by a coordinate colored 1. Hence

$$P^2(B) = \frac{1}{2^3} \frac{1}{4^6} \frac{1}{2} + \frac{1}{2^3} \frac{1}{4^5} \frac{1}{2}$$

In the same way, we calculate

$$P^2(x(-1, 0) \cap B) = \frac{1}{2^3} \frac{1}{4^7} \frac{1}{2} + \frac{1}{2^3} \frac{1}{4^5} \frac{1}{2}$$

Finally we have

$$P^2(x(-1, 0) = 2|B) = 17/20.$$

Eventhough  $x(-1, 0)$  is not specified by  $B$ , it is certainly not independent from  $B$ .

#### 4.4. DEFINITION OF $\mathcal{G}_{A,B}$ AND $\mathcal{B}_{A,B}$

Before we show that certain measure preserving transformations are mixing with respect to  $(X_2, P^2)$ , we will identify a subset of all possible random walks such that, for fixed cylinder sets  $A$  and  $B$ ,  $A$  and  $B$  are conditionally independent from one another.

Let  $A$  and  $B$  be cylinder sets. We will denote the height of  $A$  ( $B$ ) by  $h(A)$  ( $h(B)$ ) and the width of  $A$  ( $B$ ) by  $w(A)$  ( $w(B)$ ). If the coordinates specified in the sets  $A$  and  $B$  do not share a common  $y$ -coordinate, then we can define the set

$$\mathcal{V}_{A,B} = \{\alpha | \alpha \text{ is a random walk between}$$

the  $y$ -coordinates of the nearer edges of  $A$  and  $B\}$

Depending on the dimensions of  $A$  and  $B$  and the distance between the specified coordinates of these two sets, there can be a dependence between  $A$  and  $B$ .

As a simple example, suppose

$$A_1 = [x(0,0) = 2, x(1,0) = 2, x(2,0) = 2]$$

and

$$B_1 = [x(0,3) = 1, x(1,3) = 1, x(2,3) = 1].$$

We know  $P^2(A_1) = 1/64$  and  $P^2(B_1) = 1/8$ . In order for  $A_1$  and  $B_1$  to occur simultaneously, it must be that  $S_3 = 3$  or  $S_3 = -3$ . These events are not independent since

$$P^2(A_1 \cap B_1) = \frac{1}{4} \cdot \frac{1}{64} \cdot \frac{1}{8}.$$

These events, though not independent, may be **conditionally independent** with respect to certain paths  $\alpha \in \mathcal{V}$ . Using our previous example as an illustration,

$$\begin{aligned} P^2(A_1 \cap B_1 | S_3 = -3) &= P^2(A \cap B \cap S_3 = -3) / P^2(S_3 = -3) \\ &= \frac{(1/64)(1/8)(1/8)}{1/8} = \frac{1}{64} \cdot \frac{1}{8} \end{aligned}$$

since the events are independent of the random walks between them.

To answer the question of how to be certain which paths  $A$  and  $B$  will be conditionally independent with respect to, we will want to consider the possible influences  $A$  and  $B$  have on the general process. We now find a collection of paths from the random walk for which two fixed cylinder sets are conditionally independent.

Consider the cylinder set  $A$  with height  $h(A)$  and width  $w(A)$ . Without loss of generality, suppose the base point of  $A$  is at  $(0, 0)$ . What points in the base process is it possible for  $A$  to affect? If the random walk were to always shift to the left ( $\mathcal{X}_1 = -1, \mathcal{X}_2 = -1, \dots, \mathcal{X}_{h(A)-1} = -1$ ), then the points  $\{x(0), \dots, x(w(A) + h(A) - 2)\}$  might be possible to be determined. In general, the only points in the base process which could possibly have some dependence on  $A$  are  $\{x(-h(A) + 1), \dots, x(w(A) + h(A) - 2)\}$ . The width of the influence of  $A$  is  $w(A) + 2h(A) - 3$ . Likewise, the width of the influence of  $B$  on the base process is  $w(B) + 2h(B) - 3$ .

Consider two cylinder sets  $A$  and  $B$  with no overlap in the  $y$  direction. For  $\alpha \in \mathcal{V}_{A,B}$ , let  $d(\alpha)$  be the  $x$  offset of the path  $\alpha$  between the nearer edges of  $A$  and  $B$ .

We are now in a position to define two disjoint sets of random paths  $\mathcal{G}_{A,B}$  and  $\mathcal{B}_{A,B}$ . For cylinder sets  $A, B$  with base point of  $A$  at  $(a_x, a_y)$  and base point of  $B$  at  $(b_x, b_y)$ ,

$$\mathcal{G}_{A,B} = \{\alpha \in \mathcal{V}_{A,B} \mid \{a_x - h(A) + 1, \dots, a_x + w(A) + h(A) - 2\}$$

$$\cap \{b_x - d(\alpha) - h(B) + 1 - \max\{h(A), h(B)\}, \dots,$$

$$b_x - d(\alpha) + w(B) + h(B) - 2 + \max\{h(A), h(B)\}\} = \emptyset\}$$

and

$$\mathcal{B}_{A,B} = \mathcal{V} - \mathcal{G}_{A,B}.$$

This guarantees that the sets  $A$  and  $B$  are conditionally independent with respect to any path in  $\mathcal{G}_{A,B}$ . For paths in  $\mathcal{B}_{A,B}$ , conditional independence is possible but not guaranteed.

Notice that for all  $\alpha \in \mathcal{V}_{A,B}$ , the intervals

$$[a_x - h(A) + 1, \dots, a_x + w(A) + h(A) - 2]$$

and

$$[b_x - d(\alpha) - h(B) + 1 - \max\{h(A), h(B)\}, \dots,$$

$$b_x - d(\alpha) + w(B) + h(B) - 2 + \max\{h(A), h(B)\}]$$

have a fixed width dependent only on the dimensions of  $A$  and  $B$ . It is not dependent on the base point nor on the random walk path  $\alpha$ .

**Proposition 8** *The space  $(X_2, P^2)$  is stationary and for any measure preserving transformation  $T = \langle a, b \rangle$  where  $b \neq 0$ ,  $T$  is mixing.*

*proof:* First, let us see why we exclude the case  $b = 0$ . Let  $A$  be the event

$$A = \left[ \begin{array}{ccc} x(0,1) = 1 & x(1,1) = 2 & x(2,1) = 2 \\ x(0,0) = 2 & x(1,0) = 2 & x(2,0) = 3 \end{array} \right].$$



also contained in  $T^{-1}A$  for  $\{S_{j-b+1} - S_{j-b}, \dots, S_{j+n-b} - S_{j+n-b+1}\}$ . In addition, the coloring in the one dimensional process is i.i.d. as well. Information contained in  $A$  concerning the coloring in the one dimensional process is simply transposed by  $T^{-1}A$ . Hence  $P^2(A) = P^2(T^{-1}A)$ , and  $P^2$  is stationary.

Let  $A, B$  be cylinder sets and let  $T = T_{\langle a, b \rangle}$  be a shift transformation with  $b \neq 0$ . We want to show that

$$\lim_{l \rightarrow \infty} P^2(A \cap T^{-l}B) = P^2(A)P^2(B).$$

We can see that

$$\begin{aligned} & \lim_{l \rightarrow \infty} P^2(A \cap T^{-l}B) \\ &= \lim_{l \rightarrow \infty} \left( \sum_{\alpha \in \mathcal{B}_{A, T^{-l}B}} P^2(A \cap T^{-l}B \cap \alpha) + \sum_{\alpha \in \mathcal{G}_{A, T^{-l}B}} P^2(A \cap T^{-l}B \cap \alpha) \right). \end{aligned}$$

Examining the first part, we see

$$\begin{aligned} & \sum_{\alpha \in \mathcal{B}_{A, T^{-l}B}} P^2(A \cap T^{-l}B \cap \alpha) \\ & \leq \sum_{\alpha \in \mathcal{B}_{A, T^{-l}B}} P^2(\alpha) \end{aligned}$$

By Observation 1, we know that

$$S_n/\sqrt{n} \Rightarrow \mathcal{N}(0, 1).$$

implying that for our random walk  $\{S_n\}$ ,  $P(S_n \in [-a, a]) \rightarrow 0$  for fixed  $a$ . Recall that the window of  $\mathcal{B}_{A, T^{-l}B}$  is fixed and not dependent on  $l$ . This implies that as  $l \rightarrow \infty$ ,

$$\sum_{\alpha \in \mathcal{B}_{A, T^{-l}B}} P^2(\alpha) \rightarrow 0.$$

As a corollary to this,

$$\sum_{\alpha \in \mathcal{G}_{A, T^{-l}B}} P^2(\alpha) \rightarrow 1.$$

Looking at the second part,

$$\sum_{\alpha \in \mathcal{G}_{A, T^{-l}B}} P^2(A \cap T^{-l}B \cap \alpha) = \sum_{\alpha \in \mathcal{G}_{A, T^{-l}B}} P^2(A \cap T^{-l}B | \alpha) P^2(\alpha).$$

The events  $A$  and  $T^{-l}B$  are conditionally independent with respect to the paths in  $\mathcal{G}_{A, T^{-l}B}$ . Thus

$$\sum_{\alpha \in \mathcal{G}_{A, T^{-l}B}} P^2(A \cap T^{-l}B | \alpha) P^2(\alpha) = \sum_{\alpha \in \mathcal{G}_{A, T^{-l}B}} P^2(A | \alpha) P^2(T^{-l}B | \alpha) P^2(\alpha).$$

Both  $A$  and  $T^{-l}B$  are independent from  $\alpha$ , so

$$\begin{aligned} \sum_{\alpha \in \mathcal{G}_{A, T^{-l}B}} P^2(A | \alpha) P^2(T^{-l}B | \alpha) P^2(\alpha) &= P^2(A) P^2(T^{-l}B) \sum_{\alpha \in \mathcal{G}_{A, T^{-l}B}} P^2(\alpha) \\ &\rightarrow P^2(A) P^2(B) \text{ as } l \rightarrow \infty. \end{aligned}$$

Thus we have shown that

$$\lim_{l \rightarrow \infty} P^2(A \cap T^{-l}B) = P^2(A) P^2(B).$$

Since  $T$  is mixing on these cylinder sets, it is mixing on the whole space.

#### 4.5. THREE DIMENSIONAL PROCESS

Our extension from two dimensions to three dimensions is similar to the extension from one dimension to two dimensions. As well, the proof to conclude stationarity and mixing will be similar as well. We begin with another two sided random walk. Let

$$\{W_i\}_{i \in \mathbb{Z} - \{0\}}$$

be a family of i.i.d. random variables with

$$P(W_1 = 1) = P(W_1 = -1) = 1/2.$$

Given this, we define our second two sided random walk  $\{R_n\}$  by

$$R_n = W_1 + \dots + W_n, n > 0$$

$$R_n = W_{-1} + \dots + W_n, n < 0$$

$$R_0 = 0.$$

We define the space  $X_3 = \{x|x : \mathcal{Z}^3 \rightarrow \mathcal{N}\}$  similarly as before with

$$x(i, j, k) = x(i - R_k, j).$$

Tracing this back to our base one dimensional process,

$$P^3(x(i, j, k) = c_m) = P^2(x(i - R_k, j) = c_m) = P^1(x(i - S_j - R_k) = c_m) = 1/2^m$$

**Proposition 9** *The space  $(X_3, P^3)$  is stationary and for any measure preserving transformation  $T = \langle a, b, c \rangle$  with  $b \neq 0, c \neq 0$ ,  $T$  is mixing.*

*proof: As in the two dimensional case, stationarity of  $P^3$  is a direct consequence of the construction of our space. We will not repeat the explanation from the two dimensional case.*

*We want to show that given a transformation  $T = T_{\langle a, b, c \rangle}$  where  $b \neq 0, c \neq 0$ ,  $T$  is mixing with respect to  $(X_3, P^3)$ .*

*Let  $A$  and  $B$  be cylinder sets. Again, we want to show*

$$P(A \cap T^{-l}B) \rightarrow P(A)P(B)$$

*as  $l \rightarrow \infty$ .*

*We will utilize the idea of the sets  $\mathcal{G}_{A,B}$  and  $\mathcal{B}_{A,B}$ . Rather than give the definition of these two sets of paths in an analogous way to the two dimensional model, let us simplify this. Let*

$$\mathcal{V}_{A,B} = \{(\alpha, \beta)\}$$

where

$\alpha$  is a random walk between the  $y$ -coordinates of the nearer edges of  $A$  and  $B$  and  $\beta$  is a random walk between the  $x$ -coordinates of the nearer edges of  $A$  and  $B$ .

Now we define

$$\mathcal{G}_{A,B} = \{(\alpha, \beta) \in \mathcal{V}_{A,B} \mid A \text{ and } B \text{ are cond. ind. with respect to } (\alpha, \beta)\}$$

and

$$B_{A,B} = \mathcal{V}_{A,B} - \mathcal{G}_{A,B}.$$

We proceed in a similar fashion as before.

$$\begin{aligned} & P(A \cap T^{-1}B) \\ &= \sum_{(\alpha, \beta)} P(A \cap T^{-1}B \cap (\alpha, \beta)) \\ = & \sum_{(\alpha, \beta) \in \mathcal{B}_{A, T^{-1}B}} P(A \cap T^{-1}B \cap (\alpha, \beta)) + \sum_{(\alpha, \beta) \in \mathcal{G}_{A, T^{-1}B}} P(A \cap T^{-1}B \cap (\alpha, \beta)). \end{aligned}$$

Again, we note that the first part will go to zero in the limit.

$$\begin{aligned} & \sum_{(\alpha, \beta) \in \mathcal{B}_{A, T^{-1}B}} P(A \cap T^{-1}B \cap (\alpha, \beta)) \\ & \leq \sum_{(\alpha, \beta) \in \mathcal{B}_{A, T^{-1}B}} P((\alpha, \beta)) \rightarrow 0. \end{aligned}$$

As well, our second part is much like the two dimensional case.

$$\begin{aligned} & \sum_{(\alpha, \beta) \in \mathcal{G}_{A, T^{-1}B}} P(A \cap T^{-1}B \cap (\alpha, \beta)) \\ &= \sum_{(\alpha, \beta) \in \mathcal{G}_{A, T^{-1}B}} P(A \cap T^{-1}B \mid (\alpha, \beta)) P((\alpha, \beta)) \\ = & \sum_{(\alpha, \beta) \in \mathcal{G}_{A, T^{-1}B}} P(A \mid (\alpha, \beta)) P(T^{-1}B \mid (\alpha, \beta)) P((\alpha, \beta)) \\ &= \sum_{(\alpha, \beta) \in \mathcal{G}_{A, T^{-1}B}} P(A) P(T^{-1}B) P((\alpha, \beta)) \end{aligned}$$

$$\begin{aligned}
&= P(A)P(B) \sum_{(\alpha, \beta) \in \mathcal{G}_{A, T^{-l}B}} P((\alpha, \beta)) \\
&\rightarrow P(A)P(B)
\end{aligned}$$

as  $l \rightarrow \infty$

Since

$$\lim_{l \rightarrow \infty} P(A \cap T^{-l}B) = P(A)P(B).$$

we have shown  $T$  is mixing.

#### 4.6. UNIFORM FINITE ENERGY

Still lacking from this model is the property of uniform finite energy. Certainly it is not present in the model as it exists. Given the following event

$$A = \left[ \begin{array}{cccc} x(0, 2, 0) = 4 & x(1, 2, 0) = 2 & x(2, 2, 0) = 3 & x(3, 2, 0) = 1 \\ x(0, 1, 0) = 2 & & x(2, 1, 0) = 1 & x(3, 1, 0) = 2 \\ x(0, 0, 0) = 4 & x(1, 0, 0) = 2 & x(2, 0, 0) = 3 & x(3, 0, 0) = 1 \end{array} \right].$$

it is easy to determine that  $x(1, 1, 0) = 3$ . To induce uniform finite energy, we, in effect, flip a biased coin at each location in order to decide if the color should be changed.

For any map  $x \in X_3$ , if  $x(i, j, k) = m$ , then

$$P(x(i, j, k) = m) = p$$

and for  $l \neq m$

$$P(x(i, j, k) = l) = (1 - p) \frac{1}{2^l} \frac{2^m}{2^m - 1}$$

where  $p \in (0.8258, 1)$ . We chose these values for  $p$  to ensure percolation which we prove in the next section. The model now meets the condition of uniform finite energy. Call this new space  $X$ .

## 4.7. PERCOLATION

It should be clear that prior to introducing the condition of uniform finite energy that every color percolates. Pick a color,  $l$ . Given any two adjacent sites in the one dimensional base model, the probability that the sites are colored  $l$  is  $1/2^{2l}$ . With probability one there exists a pair of two consecutive sites colored  $l$ . The subsequent shifts of one into two and three dimensions will result in an infinite connected component. That is to say,  $l$  percolates. This is true of any color.

What one asks now is if the arbitrary change of colors with probability  $p$  for the sites is enough to disrupt the percolation properties. It is not. The proof we give to show that every color percolates can be explained roughly in the following way. With probability one every color will occur two times in a row in our process of coloring the base space. After being extended to three dimensions, that color has been propagated into a “fat, squiggly plane.” As long as the probability of getting from a  $yz$  coordinate to a neighboring  $yz$  coordinate without leaving the color is larger than the critical percolation probability for two dimensions, that color will (a.s.) percolate. Though the exact value of the critical percolation probability for the square two dimensional lattice is not yet known, we know that it is less than one. Bounds are given in Chapter 1.

**Proposition 10** *Each color percolates (a.s.) in the space  $X$ .*

*proof:* Let  $m \in \mathcal{N}$  be an arbitrary color. With probability one, for each map  $x \in X_3$ , there exists  $i$  such that

$$x(i, 0, 0) = m \text{ and } x(i + 1, 0, 0) = m.$$

*In the map  $x$ , this manifests itself as a layer of the color  $m$  of depth two with respect to the  $yz$ -plane.*

Let us shift our thinking to the  $yz$ -plane. Given a point  $(0, j, k)$  on the  $yz$ -plane, we know that in the preliminary space  $X_3$

$$x(i + S_j + R_k, j, k) = x(i + S_j + R_k + 1, j, k) = m.$$

In our final space  $X$ ,

$$P(x(i + S_j + R_k, j, k) = m) = P(x(i + S_j + R_k + 1, j, k) = m) = p.$$

Due to independence,

$$P(x(i + S_j + R_k, j, k) = m) \cap (x(i + S_j + R_k + 1, j, k) = m) = p^2.$$

Now we can look at our three dimensional percolation model as a two dimensional percolation process (on the  $yz$ -plane). Projecting our “fat, squiggly plane” onto the  $yz$ -plane, we will say a point of the  $yz$ -plane is colored  $m$  if both points being projected onto the  $yz$ -plane are colored  $m$ . For a  $yz$  point  $(j, k)$ ,

$$P((j, k) = m) = p^2.$$

For  $p^2 > 0.6819$ , or  $p > 0.8258$ , we know (a.s.) percolation occurs for the color  $m$ .

Since this is true for arbitrary colors, every color percolates. We now have the space we set out to construct.

## 5. CONCLUSIONS AND CONJECTURES

### 5.1. INTRODUCTION

Chapters 1 and 2 give an introduction to percolation theory as well as a few tangential topics which are important to this dissertation. At first glance, percolation theory may seem to be a narrow field. Nothing could be further from the truth. This topic has threads through many different disciplines.

In chapter 3, we explored some research of Burton and Keane [5]. They operated in the context of a two dimensional integer lattice site percolation model. Under the conditions where a measure  $\mu$  on the space of maps

$$X = \{x|x : \mathcal{Z}^2 \rightarrow \{0,1\}\}$$

is stationary and a shift transformation  $T$  is ergodic, it was concluded that ribbons were like topological strips (infinite ribbons with at most two boundaries).

With the additional restriction of  $\mu$  having finite energy, it was concluded  $x \in X$  contains (a.s.) at most two ribbons. We then provided a construction of such a model containing two ribbons (a.s.). To do this, we showed both ribbons did percolate and we constructed, via a “cut and stack” method, the appropriate measure for this model.

Chapter 4 examined how these restrictions effected a three dimensional model. We conclude the two dimensional results do not extend to the dimensions by constructing a model with the appropriate conditions in which infinitely many colors percolate (a.s.). In this model, the challenge lies less in constructing the measure and more in the actual formulation of the model.

The rest of this chapter will address possible extensions of this work or observations made in doing this work. We will look at both the two and three dimensional models.

## **5.2. RELATION TO THE ISING MODEL**

Something to note about the two dimensional model presented in Chapter 3 is how it is different from an Ising model. Because of the construction of the model, we were able to cause both the open and closed sites to percolate. In the language of the Ising model, this would be analogous to spontaneous magnetization occurring for both the up and down spins. We could adjust the model to have a higher percentage of opens, thus appearing to have caused magnetism to occur. Even so, we would still not have a model with direct analogies to the Ising model since spontaneous magnetization would still have occurred in both directions.

## **5.3. EVOLUTION OF THE THREE DIMENSIONAL MODEL**

The three dimensional model presented in this dissertation is built up from one dimension to two dimensions and then, finally, to three dimensions. In the course of research, an entirely different approach was considered for this model. This work would have been somewhat similar to work by Burton [6]. In this approach, one could iteratively embed cubes into a cube of a larger size. At each level, one could ensure that a new color percolates and still allow enough randomness to permit the necessary conditions to hold.

One might think of this as a complex babushka doll. In this case, each time one level of the doll is opened, one would find multiple dolls in the space. As well, the space inside the doll would be conducive to having a particular color percolate.

Instead of this, we decided build this process from one dimension to two dimensions and into three dimensions. We considered a Markov process for the one dimensional layer. Though this would have been successful, we opted for the simplicity of the i.i.d. process. The Markov process did not add to the model and seemed unnecessarily complicated.

#### 5.4. REMOVING THE CONDITIONS ON $T, S$

It seems unfortunate to exclude some shift transformations in the three dimensional model as not being ergodic. Although this does not diminish the ultimate result, it somehow offends one's sense of completeness. A remedy for this might be to add one more perturbation to the system. Notice in the three dimensional model of chapter 4 that the original one dimensional process is undisturbed until the point when we induce uniform finite energy. The idea is to induce a two-dimensional random walk on the model before inducing uniform finite energy. This is said in more detail below.

After extending the model to three dimensions, we impose yet another random process on the model. Let  $\{V_i\}_{i \in \mathcal{Z} - \{0\}}$  be a family of i.i.d. random variables with

$$P(V_i = (1, 0)) = P(V_i = (0, 1)) = P(V_i = (-1, 0)) = P(V_i = (0, -1)) = 1/4.$$

Define  $\{Q_n\}_{n \in \mathcal{Z}}$  via

$$Q_n = V_1 + \dots + V_n \text{ for } n > 0$$

$$Q_n = V_{-1} + \dots + V_n \text{ for } n < 0$$

$$Q_0 = (0, 0).$$

For  $Q_i = (a, b)$ , let  $Q_i^1 = a$  and  $Q_i^2 = b$ . Now we can apply this to our model by letting  $x(i, j, k) = x(i, j + Q_i^1, k + Q_i^2)$ . This can be thought of as a further perturbation of our space running in the  $x$ -axis. The conjecture is that the resulting three dimensional model would allow for all shift transformations to be ergodic and retain the same properties as presented in Chapter 4.

## 5.5. UNIQUENESS AND CLASSIFICATION OF CLUSTERS

Burton and Keane [5] were able to do more than bound the number of colors which percolate in two dimensions. They were able to topologically classify the structure of ribbons and discuss some of their metric properties. One might ask if the same is possible for three and higher dimensions. As well, if it possible to classify the infinite structures, one might be able to say when three dimensional models are equivalent.

## 5.6. COMMENTS

One does not have to look far to find more questions to ask in this field. This research considered only integer lattices. These same questions can be asked on non-lattice percolation models, Euclidean and non-Euclidean.

The lure of percolation theory lies in that some of its most difficult problems are easily posed. The reader may wish to review the wind tree problem [14] so as to have something to do on sleepless nights.

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