AN ABSTRACT OF THE THESIS OF

Kiwon Nam for the degree of Doctor of Philosophy in Mathematics presented on June 1, 1995. Title: Detecting Piecewise Linear Approximate Fibrations

Abstract approved: Redacted for Privacy

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In this thesis we study which manifolds N have the following property: if the fibers of a proper piecewise linear map p from a closed, orientable, connected, $(n+k)$-dimensional manifold M^{n+k} onto a k-manifold B all have the homotopy type of N then p is automatically an approximate fibration.

We call such a manifold a codimension-k PL h-fibrator and a manifold is a PL h-fibrator if it is a codimension-k PL h-fibrator for any positive integer k.

We prove that a closed, orientable, aspherical manifold with a hopfian fundamental group is a PL h-fibrator if it is a codimension-2 PL fibrator.

We also prove that for a closed, orientable, connected, hopfian n-dimensional manifold N^n with a hopfian fundamental group such that $H^j(N^n) = 0$ for $0 < j < m$ for some integer $m \geq 1$ and $H^n(N^n)$ is in the subring of $H^*(N^n)$, generated by $H^m(N)$, N^n is a PL h-fibrator if it is a codimension-$(m+1)$ PL h-fibator.

Using this result we show that the connected sum of two copies of complex projective space $\mathbb{C}P^2$ is a PL h-fibrator.

Detecting Piecewise Linear Approximate Fibrations

by

Kiwon Nam

A Thesis

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Completed June 1, 1995
Commencement June 1996

Doctor of Philosophy thesis of Kiwon Nam presented on June 1, 1995

APPROVED:

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Kiwon Nam, Author

ACKNOWLEDGEMENT

I would like to thank my advisor, Dennis Garity, who has taken me under his wing and helped me grow as a mathematician. He is always willing to share his insights in mathematics with me.

It was a great experience to learn from Robert Daverman at the University of Tennessee and I am very much indebted to him for my research resulted in this thesis. His guidance was inspirational.

I also thank to Bill Bogley, Bella Bose, Stuart Newberger, Bent Petersen, and Dick Schori to serve as members of my committee.

I wish to express gratitude to the Department of Mathematics at Oregon State University for making it possible for me to pursue a graduate degree. For this, I especially want to thank Don Solmon, Kenneth Ross, John Leahy, Bob Burton, and Frank Flaherty.

Finally I thank to many faculty members and fellow students, past and present, who have educated me on and off classrooms.

TABLE OF CONTENTS

DETECTING PIECEWISE LINEAR APPROXIMATE FIBRATIONS

1. INTRODUCTION

The main goal of topology is an understanding of topological spaces. One way to achieve this is by considering various continuous functions (maps) between spaces. Homeomorphisms, embeddings, covering maps, homotopy equivalences, fiber bundle projections, and fibrations are some of the continuous functions that play a major role in topology.

In this thesis we study a class of maps called approximate fibrations which may be considered as a generalization of fibrations. In particular we investigate which homotopy types of manifolds can appear as a homotopy type of point inverses of piecewise linear maps such that these maps are always approximate fibrations.

This thesis is organized as follows. In Chapter 1 we present some definitions, notation, and terminology relevant to our investigation. In particular, the definitions of an approximate fibration and of a codimension-k PL h-fibrator are given. We also include some properties of approximate fibrations for completeness even though we are mainly concerned with piecewise linear approximate fibrations. The last section of Chapter 1 distinguishes the setting that we work in from the setting worked in by R. J. Daverman and explains the main problem that we investigate.

In Chapter 2 we discuss some tools used in detecting PL approximate fibrations.

Two of the main tools in our investigation are the Fundamental Recognition Theorem of R. J. Daverman and the existence of Wang exact sequences for approximate fibrations. For completeness we present a detailed proof of Daverman's Fundamental Recognition Theorem which allows us to recognize piecewise linear approximate fibrations.

The existence of Wang exact sequences for PL approximate fibrations over k-sphere was observed by R. J. Daverman. We include a new proof of the existence of Wang exact sequence associated with an approximate fibrations over a k-sphere in a slightly more general setting.

In Chapter 3, we establish the Basic Lemma 3.2.2 which enables us to prove Theorem 3.3.1 stating that an aspherical n-manifold with a hopfian fundamental group is a PL h-fibrator if it is a codimension-2 PL h-fibrator. Also we prove Theorem 3.4.2 concerning a case of a closed, orientable, hopfian manifold with a special cohomology ring structure. As a corollary of this we show that complex projective spaces $\mathbb{C}P^n$,$n > 1$, are PL h-fibrators.

In Chapter 4 we study the connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ of two copies of $\mathbb{C}P^2$ and we prove that it is a PL h-fibrator (Theorem 4.2.1).

The reader interested in getting quickly to the main results in this thesis can just read the last section of Chapter 1, the statements of Theorems 2.1.13 and 2.2.4 in Chapter2 and then go directly to Chapters 3 and 4. Chapter 2 may be then be referred back to in more detail later if necessary.

1.1. DEFINITIONS AND NOTATION

In this section we review several relevant definitions and establish some standard notation used in the thesis.

Following usual practice, by spaces we mean topological spaces and maps are continuous functions. We will try to follow the terminology and definitions of some standard textbooks such as Spanier's [20] and Munkres' [17]. Spanier's book contains the standard material on algebraic topology and on CW complexes that we need. Munkres' book contains the standard material on general topology that we need. All homology and cohomology groups are with integer coefficients unless otherwise stated.

Throughout this thesis for any map $F : X \times I \to Y$, where I is the the closed interval $[0,1]$, we write $F_t(x)$ for $F(x,t)$ unless stated otherwise.

Since we mainly deal with piecewise linear maps between subsets of Euclidean space $\mathbb{R}^n$ we include the following definitions here. A standard reference for the material that we use on Piecewise Linear Topology is [18]. A subset V of $\mathbb{R}^n$ is a polyhedron if for each point v in V there exists a compact subset L of $\mathbb{R}^n$ such that v is contained in the subset N of V, where $N = v * L = \{\lambda v + \mu l : \lambda, \mu \geq 0, \lambda + \mu = 1, l \in L\}$. We call N a cone neighborhood of v in V and L a link of v in V.

Note that N is the collection of all line segments in V connecting v and points in L.

All finite simplicial complexes are polyhedra and many manifolds can be given a PL structure. A map $f : V \to Y$ between polyhedra is a *piecewise linear* (PL) map if each point v in V has a neighborhood $N = v * L$ such that $f(\lambda v + \mu l) = \lambda f(v) + \mu f(l)$ for all l in L, $\lambda, \mu \geq 0, \lambda + \mu = 1$.

Recall that a fiber bundle (E, B, F, p) consists of a total space E, a base space B, and a fiber bundle projection $p : E \to B$ with each fiber $p^{-1}(b)$, $b \in B$, being homeomorphic to some fixed space F such that the following conditions hold: for any $b \in B$ there exists an open neighborhood U of b in B and a homeomorphism $\Phi_U : U \times F \to p^{-1}(U)$ such that the composite map $p \circ \Phi_U : U \times F \to U$ is a projection map onto the first factor of the product space $U \times F$.

One of the useful properties of a fiber bundle with a paracompact base space B is that a bundle projection p has the homotopy lifting property with respect to any space. We say that a map $p : E \to B$ has the *homotopy lifting property with respect to a space* X if for any map $f : X \to E$ and a homotopy $H : X \times I \to B$ with $H_0 = p \circ f$ there exists a *lifted* homotopy $\tilde{H} : X \times I \to E$ such that $p \circ \tilde{H} = H$ and $\tilde{H}_0 = f$.

Sometimes we say that that $\tilde{H}$ is a solution of the homotopy lifting problem with the data f, and H.

A map is a (*Hurewicz*) *fibration* if it has a homotopy lifting property with respect to any space. A homeomorphism is trivially a fibration and any projection map of a product space onto one of its factor space is a fibration. For a fibration $p : E \to B$, we still call E a total space, B a base space, and point inverses fibers.

Occasionally we write F_b for a fiber $p^{-1}b$ for b in B.

One salient difference between a fiber bundle projection and a fibration is that the fibers of a fibration are not necessarily homeomorphic. However, they are always of the same homotopy type when the base space B is path connected.

See [20] for a proof of this. A simple example of a fibration with non-homeomorphic fibers is the projection map of a right triangle onto its base.

One of the useful consequences of a fibration $p : E \to B$ is the existence of the following homotopy exact sequence, relating the homotopy groups of the total space E, the base space B, its fibers F_b's [20]:

$$\to \pi_{j+1}(B,b) \to \pi_j(F_b,e) \to \pi_j(E,e) \to \pi_j(B,b) \to$$

where e is in F_b.

It is known that a uniform limit of a sequence of fibrations is not necessarily a fibration [1]. But its limit, called an approximate fibration exhibits several useful, analogous properties of a fibration. In 1977, D.S.Coram and P.F.Duvall [1] introduced the concept of approximate fibrations between metric spaces. This concept generalizes the concept of fibrations. We need some terminology and notation in order to define approximate fibrations. Let ϵ be a positive number. By an ϵ-cover of a metric space (Y,d) we mean a coverμ of Y where the diameters of members of the μ are less than ϵ. Two maps $f, g : X \to Y$ are said to be ϵ-close if for any x in$X$ both $f(x)$ and $g(x)$ are contained in some member of the ϵ-cover μ. Note that $d(f(x), g(x)) < \epsilon$ for any x in X.

Definition 1.1.1 *(Approximate Homotopy Lifting Property) [2] We say that a surjective map $p : E \to B$ between metric spaces has the approximate homotopy lifting property (AHLP) with respect to a space X if for any map $g : X \to E$ and a homotopy $H : X \times I \to B$ with $H_0 = p \circ g$ and, for any positive number ϵ, there exists a map $\tilde{H} : X \times I \to E$ such that $\tilde{H}_0 = g$, $p \circ \tilde{H}$ and H are ϵ-close. We call $\tilde{H}$ an ϵ-approximate lifted homotopy of H.*

Definition 1.1.2 *(Approximate Fibration) [1] A surjective map $p : E \to B$ is an approximate fibration if p has the AHLP with respect to every space X.*

It follows that fibrations are approximate fibrations. So the concept of approximate fibrations can be considered as a generalization of the concept of a fibration involving separable metric spaces.

There are a number of reasons why approximate fibrations are worth studying. Cell-like maps between finite dimensional locally contractible spaces are approximate fibrations [8]. The approximate homotopy lifting property of cell-like maps is an important ingredient of a proof of R. Edwards' cell-like approximation theorem [5] which furthered the understanding of manifolds. Another advantage of approximate fibrations over fibrations is the fact that in the space of maps between two compact ANR's with the sup-norm metric, the set of approximate fibrations is a closed subset while the subset of fibrations may not be closed [2]. It is known that the limit of a sequence of homeomorphisms between compact manifolds in the compact-open topology is necessarily cell-like. This may be considered as a simple example of the fact that a limit of sequence of fibrations is necessarily an approximate fibration. [1].

For completeness, we include the following definitions. A standard reference on Absolute Neighborhood Retracts is [13].

Definition 1.1.3 *(Absolute Neighborhood Retract: ANR) An absolute neighborhood retract (ANR) is a separable metric space X such that for any separable metric space Y and for any closed subset A of Y and for any map $f : A \to X$ there exists an open set U containing A and an extension map $F : U \to X$ such that $F : U \mid A = f$. If Y can be taken as U for all cases, X is called an absolute retract (AR).*

The closed interval $I = [0, 1]$ is an absolute retract by Tietze extension theorem. The n-sphere S^n is an ANR which is not an AR. All CW complexes and manifolds are ANR's.

The comb space $C = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, x = 0, \frac{1}{n},\ or\, y = 0, 0 \leq x \leq 1\}$ in $\mathbb{R}^2$ is an example of a space which is not an ANR.

1.2. SOME PROPERTIES OF APPROXIMATE FIBRATIONS

In the last section, we gave some reasons why the study of approximate fibrations is important. In this section we discuss some properties of approximate fibrations which are relevant to our investigation and we present Coram and Duvall's characterization of approximate fibrations in terms of completely movable maps. The reader interested in getting to the main results of the thesis can skip the material on completely movable maps.

The results of this section will be used in Chapter 2.

Coram and Duvall [1] showed that the approximate fibrations have the following properties similar to properties of fibrations: shape equivalence of fibers, fibers being fundamental absolute neighborhood retracts, and the existence of a homotopy exact sequence. The corresponding properties of fibrations are homotopy equivalence of fibers, ANR nature of fibers, and the existence of a homotopy exact sequence.

Let $p : E \to B$ be a surjective map between metric spaces.

Theorem 1.2.1 *(Corollary 2.5 of [1]) If $p : E \to B$ is an approximate fibration, then for any b in B its fiber $p^{-1}b$ is strongly movable.*

For the definition of strongly movable set see [14] and [1]. A strongly movable metric compactum is also called a fundamental absolute retract(FANR). Thus each fiber of an approximate fibration is a fundamental absolute neighborhood retract while fibers of a fibration are known to be ANR's.

The next theorem [1] is an analogue to the fact that all fibers of a fibration over a path connected base space have the same homotopy type.

Theorem 1.2.2 *Let $p : E \to B$ be an approximate fibration. If B is path connected, then any two fibers have the same shape.*

See [14] for the definition of the shape.

The last but most useful property for us is the existence of an exact sequence associated with an approximate fibration $p : E \to B$, involving the homotopy groups of E, B, and fibers F_b [1].

Theorem 1.2.3 *For each approximate fibration $p : E \to B$ and each b in B and e in the fiber $F_b = p^{-1}b$, there exists the following exact sequence:*

$$\to \check{\pi}_n(F_b, e) \to \pi_n(E, e) \to \pi_n(B, b) \to \check{\pi}_{n-1}(F_b, e) \to$$

Here $\check{\pi}_n(F, e)$ is defined as an inverse limit of $\pi_n(U_j, e)$ where (U_j, α_{ij}) is an inverse ANR neighborhood sequence with inclusions as bonding maps. That is,

$$\check{\pi}_n(F, e) = \varprojlim \pi_n(U_j, e).$$

See [1] for more detail.

Thus, as Coram and Duvall showed us, approximate fibrations share important properties with fibrations.

We have also seen why approximate fibrations arise naturally in studying fibrations. So it is desirable to have a criterion to determine whether a given map is an approximate fibration.

In order to characterize approximate fibrations, Coram and Duvall [2] introduced the following concept of a completely movable map.

Definition 1.2.4 *(Completely Movable Map) [2] A proper map $p : E \to B$ between ANR's is completely movable if for each b in B and each neighborhood U of the fiber F_b there exists a neighborhood V of F_b in U such that if F_c is any fiber in V and W is any neighborhood of F_c in V, then there is a homotopy $H : V \times I \to U$ such that $H(x, 0) = x$ and $H(x, 1) \in W$ for each x in V and $H(x, t) = x$ for all x in F_c and $t|inI$.*

Example. Here is an example of non-movable map [2]. Define a map $f : S^1 \times D^2 \to D^2$ by $f(x, y) = \|y\|x$, where $x \in S^1$, $y \in D^2$, 2-disc, and $\|\|$ is the Euclidean norm. Then fibers consist of all meridional circles of various radii and the center circle of the solid torus $S^1 \times D^2$, an exceptional fiber which is mapped to the center of 2-disc D^2. Note that f fails to be completely movable at the origin of the disc.fiber.

In terms of completely movable maps Coram and Duvall [2] proved the following characterization theorem.

Theorem 1.2.5 *[2] Let $p : E \to B$ be a proper map. Then p is an approximate fibration if and only if p is completely movable.*

Using the above example of a non-movable map we can construct the following proper map between closed manifolds S^3 and S^2 with the circle fibers. Consider S^3 as an adjunction space of two solid tori T_1, T_2 via a map $g : \partial T_1 \cong S^1 \times S^1 \to \partial T_2$ that interchanges meridional circles with longitudinal circles of the two boundary tori. So $S^3 \cong T_1 \cup_g T_2$. Decompose T_1 into the disjoint union of meridional circles and the center circle C and decompose T_2 into longitudinal circles. This induces a decomposition of S^3 and its associated quotient map f is a proper map from S^3 onto S^2. But f fails to be completely movable at the point $s = f(C)$ in S^2. So f is not an approximate fibration. But note that the restriction map $f \mid f^{-1}(S^2 - \{s\})$ is an approximate fibration.

More generally Coram and Duvall showed that for any surjection $p : E \to B$ between ANR's where F_b has the shape of S^1 there exists a dense open subset B_0 in B such that $p \mid p^{-1}(B_0) : p^{-1}(B_0) \to B_0$ is an approximate fibration [3].

1.3. CODIMENSION-K H-FIBRATORS AND H-FIBRATORS

Since the latter part of the 1980's R.J.Daverman has investigated which manifolds have the following property:

If $p : M \to B$ is a proper map and each $p^{-1}(b)$ has the homotopy type of the given manifold , then p is always an approximate fibration.

In the 1989 paper [4] Daverman called a closed n-manifold N a *codimension-k fibrator* if for any $(n+k)$-manifold M, any decomposition map $p : M \to B$, with $B = M/G$, where G is the upper semi–continuous decomposition associated with p, with each member g in G being shape equivalent to N, and with $\dim(M/G)$ finite, the map p is always an approximate fibration. Daverman proved the following three theorems among others [6].

Theorem 1.3.1 *S^n is a codimension-k fibrator for all $k \leq n$.*

But S^1 fails to be a codimension-2 fibrator as we have seen in the above example.

Theorem 1.3.2 *Every simply connected closed manifold is a codimension-2 fibrator.*

Theorem 1.3.3 *Any closed two dimensional manifold N with nonzero Euler characteristic is a codimension-2 fibrator.*

The torus $T^2 = S^1 \times S^1$ is the only closed, connected, orientable 2-manifold which is not a codimension-2 fibrator. Daverman also showed that the real projective space $\mathbb{R}P^3$ is a codimension-2 fibrator but $\mathbb{R}P^3 \# \mathbb{R}P^3$ fails to be a codimension-2 fibrator [4].

In the 1992 paper [8] Daverman defines an orientable n-manifold N^n to be a codimension-k PL fibrator if, for all orientable PL $(n+k)$-manifold M^{n+k} and PL maps $p : M^{n+k} \to B^k$ such that each fiber collapses to an n-complex homotopy equivalent to N^n, p is always an approximate fibration.

The hypothesis of fibers collapsing to complexes of homotopy type of the manifold N may be stringent but it allows one to show that the base space B is a nice space.

Specifically he proved the following theorem.

Theorem 1.3.4 *[8] Suppose that $p : M^{n+k} \to B$ is an approximate fibration such that each $p^{-1}b$ collapses to an n-complex homotopy equivalent to a closed n-manifold. Then B is a simplicial homotopy k-manifold.*

In this thesis we study similar problems assuming only that each fiber has the homotopy type of a fixed n-manifold N without assuming collapsibility.

Since one of the necessary conditions for a PL map to be an approximate fibrations over path connected spaces is that all point inverses have the same homotopy type, we impose this condition on maps of our interest and we introduce the following terminology. Let N be an orientable, connected, closed n-manifold and M be an orientable, connected, closed $(n+k)$-manifold.

Definition 1.3.5 *(N-type map) A proper surjective PL map $p : M^{n+k} \to B^k$ is called an N-type map if for any b in B, $p^{-1}b$ has the homotopy type of N. Furthermore if $p^{-1}b$ collapses to a complex of the homotopy type of N for all b in B, we say that p is an N-like map.*

In terms of this terminology we can rephrase the goal of our investigation: For which manifolds N^n is any N^n-type map $p : M^{n+k} \to B^k$ always an approximate fibration ?

As mentioned before Daverman called a manifold N^n a codimension-k PL fibrator if any N^n-like map $p : M^{n+k} \to B^k$ from (n+k)-manifold onto a k-dimensional polyhedron is automatically an approximate fibration. A manifold is a PL fibrator if it is a codimension-k fibrator for any positive integer k. In order to distinguish the case of N-type maps from N-like maps, we will introduce the following definitions.

Definition 1.3.6 *(Codimension-k PL h-fibrator) A closed, connected, orientable manifold N^n is called a codimension-k PL h-fibrator if any N^n-type map $p : M^{n+k} \to B^k$ from $(n+k)$-manifold onto a k-dimensional manifold is always an approximate fibration. We call N an PL h-fibrator if it is a codimension-k PL h-fibrator for any positive integer k.*

Example. It is straightforward to prove that a manifold is a codimension-j h-fibrator for all $j \geq k-1$ if it is a codimension-k h-fibrator. Note that we require the base space B to be a manifold in the definition of a codimension-k h-fibrator while we only assume B to be a polyhedron for a codimension-k fibrator. In general the collapsibility of fibers in the case of N^n-like maps $p : M^{n+k} \to B^k$ forces the base space to be a nicer space whereas the weaker condition of being an N^n-type map will not guarantee the base space to be as nice. There is an example of N^n-type map approximate fibration from a manifold onto a base space which is not a manifold. This example is due to R. J. Daverman.

Example. Here is an example of an approximate fibration whose base space is not a manifold. Let Σ^3 be a non-simply connected homology 3-sphere that bounds a contractible, but not a collapsible 4-manifold W^4. Construct a 7-manifold as follows.

$M^7 = (S^3 \times W^4) \bigcup_\partial (S^3 \times \partial W^4 \times [1, \infty))$ where the attaching is done via the identification $(x, y) \sim (x, y, 1)$ for $y \in \partial W^4$.

Consider a map $p : M^7 \to B^4$ with $p^{-1}(b_0) = (S^3 \times W^4) \bigcup_\partial (S^3 \times \partial W^4 \times \{1\})$ and $p^{-1}(b) = S^3 \times q \times r$ where $r \in [1, \infty)$, $q \in W^4 - \partial W^4$. Then $B - b_0 \cong \partial W^4 \times (1, \infty)$ and b_0 is a non-manifold point since its link is not simply connected. On the other hand the base space of an N^n-like approximate fibration is necessarily a manifold as mentioned above (See Theorem 1.3.4).

In conclusion, using the terminology we have introduced, we again restate the main question that this thesis investigates.

Which manifolds N^n are PL h-fibrators?

2. TOOLS TO DETECT PL APPROXIMATE FIBRATIONS

In this chapter we present necessary materials that facilitate our attempt to determine for which manifolds N any proper PL map p with each fiber having the homotopy type of N is always an approximate fibration.

Most of the material is technical and can be skipped on a first reading. Those interested in getting quickly to the main results in the thesis should just read the statements of Theorem 2.1.13 and Theorem 2.2.4 and then proceed directly to Chapters 3 and 4.

In the first section we discuss theorems used to detect approximate fibrations among PL maps. We show that for PL maps certain algebraic topological data provide a criterion to check whether a given PL map is completely movable (2.1.11). Then we discuss a detailed proof of the Fundamental Recognition Theorem (2.1.13) of R. J. Daverman which is a key tool used to recognize PL approximate fibrations.

In the second section we present Wang exact sequences associated with approximate fibrations over a k-sphere.

2.1. THE FUNDAMENTAL RECOGNITION THEOREM

In this section we present one of the main tools in our study. The results in this section are due to R.J.Daverman.

The following lemma of R. J. Daverman [8] concerning PL maps defined on PL manifolds will be indispensable in our investigation.

Here we do not distinguish simplicial complexes and their underlying topological spaces for the sake of brevity. Again, we refer the reader to Rourke and Sanderson [18] for the relevant terminology.

Lemma 2.1.1 *(Codimension Reduction Lemma) [8] Let* $p : M \to B$ *be a proper surjective PL map from an* $(n+k)$*-manifold* M *onto a polyhedron* B*. Then each* b *in* B *has a PL neighborhood* $S = b * L$ *in* B *such that* $p^{-1}S$ *is a regular neighborhood of* $p^{-1}b$ *in* M *and* $p^{-1}L = \partial(p^{-1}S)$ *is an* $(n+k-1)$*-manifold.*

Remark. Occasionally we write $L' = p^{-1}L$ and $S' = p^{-1}S$. If S' is a regular neighborhood of $p^{-1}b$ in M with the property stated in the above lemma, we can find a smaller regular neighborhood of $p^{-1}b$ which is contained in S' such that above lemma holds with respect to this smaller neighborhood.

The following is a useful property of simplicial maps mentioned in R. J. Daverman's paper [8] without its proof. This will be used in the proof of the Fundamental Recognition Theorem (Theorem 5.1 in [8]).

For completeness, we present it here as a proposition with a proof. Note the remarkable resemblance to the definition of a fiber bundle.

Proposition 2.1.2 *Let* $p : M \to B$ *be a simplicial map between finite simplicial complexes. Then for any simplex* σ *of* B *and* $x \in Int(\sigma)$ *there exists a PL homeomorphism*

$$\Psi : p^{-1}(\mathrm{Int}(\sigma)) \to \mathrm{Int}(\sigma) \times p^{-1}x$$

such that $(\mathrm{proj})_1 \circ \Psi = p$.

PROOF: Let $\sigma = < v_0, v_1, \cdots, v_k >$ and fix $x \in \mathrm{Int}(\sigma)$. Then

$$x = \sum_{i=0}^{k} x_i v_i$$

where $x_i > 0$ and

$$\sum_{i=0}^{k} x_i = 1.$$

Now $p^{-1}(\sigma)$ is a full subcomplex of M and $p^{-1}(x) \subset p^{-1}(\sigma)$. Order and label all vertices in $p^{-1}(\sigma)$ so that $\{w_0, w_1, \cdots, w_N\} = p^{-1}(\sigma) \cap M^{(0)}$ where $M^{(0)}$ is the set of vertices of M. Consider a function $f : \{0, 1, \cdots, N\} \to \{0, 1, \cdots, k\}$ defined by $f(j) = i$ if $p(w_j) = v_i$. Then

$$p^{-1}(x) = \{\sum_{j=0}^{N} a_i w_j \in M \mid \sum_{j \in f^{-1}(i)} a_j = x_i\}$$

and

$$p^{-1}(Int(\sigma)) = \{\sum_{j=0}^{N} b_j w_j \mid \sum_{j \in f^{-1}(i)} b_j > 0, \sum_{i=0}^{k}(\sum_{j \in f^{-1}(i)} b_j) = 1\}.$$

Define a function $\Phi : \text{Int}\sigma \times p^{-1}(x) \to p^{-1}(\text{Int}\sigma)$ by

$$\Phi(y, z) = \Phi(\sum_{i=0}^{k} y_i v_i, \sum_{j=0}^{N} a_j w_j) = \sum_{i=0}^{k} \sum_{j \in f^{-1}(i)} y_i \frac{a_j}{x_i} w_j.$$

Note that Φ is properly defined since $x_i > 0$ for all i and $p(\Phi(y, z)) \in \text{Int}\sigma$. We check that Φ is continuous.

Next we define a function $\Psi : p^{-1}(\text{Int}\sigma) \to \text{Int}\sigma \times p^{-1}(x)$ by

$$\Psi(r) = \Psi(\sum_{j=0}^{N} b_j w_j) = (\sum_{i=0}^{k}(\sum_{j \in f^{-1}(i)} b_j) v_i, \sum_{i=0}^{k} \sum_{j \in f^{-1}(i)} \frac{x_i b_j}{(\sum_{l \in f^{-1}(i)} b_l)} w_j).$$

Note that Ψ is properly defined since $\sum_{j \in f^{-1}(i)} b_j > 0$. Ψ is also continuous. We can check that Φ and Ψ are inverse functions of each other and we conclude that Ψ is a PL homeomorphism. $\square$

In order to state and to prove the next proposition which will be used in proving the Fundamental Recognition Theorem of an approximate fibration, we need several lemmas. Recall that a subset A of a space X is a deformation retract if there exists a map $H : X \times I \to X$ such that $H_0 = id_X$, $H_1(X) \subset A$, and $H_1(a) = a$

for all a in A. H is called a deformation retraction. Furthermore if $H_t(a) = a$ for all t in I, H is called a strong deformation retraction and A is called a strong deformation retract of X. The single point $A = \{(0,1)\}$ of the comb space C is a deformation retract of X but A is not a strong deformation retract. But with ANR spaces the following is true.

Lemma 2.1.3 *[11] Let A be a subset of an ANR space X. Then A is a strong deformation retract of X if A is a deformation retract.*

Lemma 2.1.4 *[13] Let X be a connected ANR and A be a connected closed ANR subset of X. Then A is a deformation retract of X if and only if the inclusion $i : A \to X$ induces isomorphisms for all homotopy groups $i_\# : \pi_n(A) \cong \pi_n(X)$, $n \geq 1$.*

That is, the inclusion map i is a weak homotopy equivalence. Recall that a map $f : X \to Y$ is called a weak homotopy equivalence if for any x in X, $f_\# : \pi_q(X, x) \to \pi_q(Y, f(x))$ is an isomorphism for all q. The following useful theorem of J. H. C. Whitehead is used throughout the thesis whenever it is applicable, without explicit reference.

Theorem 2.1.5 *[20] A map between CW complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

As a corollary of the above theorem and Lemma 2.1.4, we deduce that in the CW category a closed subset A in X is a strong deformation retract of X if the inclusion map is a homotopy equivalence.

Lemma 2.1.6 *Let $f : X \to Y$, $g : Y \to Z$, and $h : Y \to Z$ be maps. Suppose $g \simeq h$. Then $g \circ f \simeq h \circ f$.*

PROOF: One can directly construct the necessary homotopies using the given homotopies. □

Lemma 2.1.7 *Let $f : X \to Y$ and $g : Y \to Z$ be maps. Suppose both g and $g \circ f$ are homotopy equivalences. Then f is also a homotopy equivalence.*

PROOF: This is a straightforward application of above lemmas. □

Now we can state and prove the proposition which will be used in proving the Fundamental Recognition Theorem.

Proposition 2.1.8 *Let X be a connected ANR and G be an upper semi-continuous decomposition of X into connected ANR subsets. Suppose g in G is a strong deformation retract of X via a strong deformation retraction R and for some g' in G the restriction of R_1 to g' is a homotopy equivalence. Then g' is also a strong deformation retract of X.*

PROOF: Let $j' : g' \to X$ be the inclusion map. Note that $R_1 \mid g' : g' \to g$ is just the composite map $R_1 \circ j' : g' \to X \to g$. Since R is a strong deformation retraction, the retraction R_1 is a homotopy equivalence. Since $R_1 \circ j'$ is a homotopy equivalence, the inclusion $j' : g' \to X$ is a homotopy equivalence by Lemma 2.1.7. Then j' induces an isomorphism on homotopy groups so that g' is a deformation retract of X according to Lemma 2.1.4. Now Lemma 2.1.3 shows that g' is a strong deformation retract. □

We will present the Fundamental Recognition Theorem somewhat differently from the way it is presented in Daverman's paper in order to study its proof in more detail. We still need a few results before presenting the theorem.

Theorem 2.1.9 *(Regular Neighborhood Theorem) [18] Suppose N_1 and N_2 are regular neighborhoods of X in Y . Then there exists an isotopy H of Y fixed on X and of compact support carrying N_1 onto N_2, i.e., $H_1(N_1) = N_2$. Moreover if Y is a manifold and $X \subset \mathrm{Int}(Y)$ then we can assume further that H is fixed on any regular neighborhood $N \subset (\mathrm{Int}(N_1) \cap \mathrm{Int}(N_2))$ and outside any open neighborhood U of $N_1 \cup N_2$.*

The following fact can be checked as in Lemma 2.1.7

Lemma 2.1.10 *Let $f : X \to Y$ and $g : Y \to Z$ be a map. Suppose f and $g \circ f$ are homotopy equivalences. Then g is also a homotopy equivalence.*

The next theorem makes it possible to detect completely movable maps among PL maps using algebraic topological data.

Theorem 2.1.11 *(Recognition Theorem for Completely Movable PL Maps) A proper surjective PL map $p : M \to B$ defined on a manifold is a completely movable map if the following two conditions are met:*

(1) each point v in B has a stellar neighborhood $S = v * L$ in B whose preimage $p^{-1}S$ collapses onto $p^{-1}v$ via a strong deformation retraction $D : p^{-1}S \times I \to p^{-1}S$
(2) for any b in S the restriction map of retraction D_1 in the above condition (1) to the fiber $p^{-1}b$, $D_1 \mid p^{-1}b : p^{-1}b \to p^{-1}v$ is a homotopy equivalence.

PROOF: Consider any proper surjective PL map $p : M \to B$ defined on a manifold that satisfies the given hypothesis. We prove that p is a completely movable map. Let v be an arbitrary point in B. Consider any neighborhood U of $p^{-1}v$ in M. By the given hypothesis, we can find a stellar neighborhood $S = v * L$ in B whose preimage, a regular neighborhood $p^{-1}S$ of $p^{-1}v$ collapses onto $p^{-1}v$ via a strong deformation retraction $D : p^{-1}S \times I \to p^{-1}S$ such that for any b in S the restriction map of D_1 to the fiber $p^{-1}b$, $D_1 \mid p^{-1}b : p^{-1}b \to p^{-1}v$ is a homotopy equivalence.

We write $S' = p^{-1}S$ and $L' = p^{-1}L$. Since $D_1 \mid p^{-1}b$ is just the composite map $D_1 \circ i_{S'}$ where $i_{S'} : p^{-1}b \to S'$ is the inclusion and D is a strong deformation retraction, by Lemma 2.1.7, $i_{S'} : p^{-1}b \to S'$ is a homotopy equivalence for any b in S.

Find a smaller regular neighborhood V of $p^{-1}v$ in $S' \cap U$. There exists a strong deformation retraction $R : V \times I \to V$ that collapses V onto $p^{-1}v$. We claim that the inclusion $i_V : p^{-1}b \to V$ is a homotopy equivalence for any $p^{-1}b$ in V. To see this, we note that the inclusion $j : V \to S'$ is a homotopy equivalence since V is a strong deformation retract of S' according to Theorem 2.1.9, the Regular Neighborhood Theorem. Also we know that the inclusion $i_{S'} : p^{-1}b \to S'$ is a homotopy equivalence. Since $i_{S'}$ is a composite map $j \circ i_V$ and j is a homotopy equivalence, by the above Lemma 2.1.10, $i_V : p^{-1}b \to V$ is also a homotopy equivalence and we have established the claim.

Then $R \circ i_V = R \mid p^{-1}b : p^{-1}b \to p^{-1}v$ is a homotopy equivalence. Since $p^{-1}v$ is a strong deformation retract of V, by Proposition 2.1.8, we conclude that $p^{-1}b$ is a strong deformation retract of V. This implies that our PL map p is a completely movable map. $\square$

Remark. Note that for any proper surjective PL map $p : M \to B$ defined on a manifold and for each v in B we can find a stellar neighborhood $S = v * L$ in B by the Codimension Reduction Lemma. So we only need to check the homotopy condition with respect to this neighborhood to see whether p is a completely movable map.

The following proposition provides a link between the above theorem and the Fundamental Recognition Theorem.

Proposition 2.1.12 *We assume the hypothesis and notation of the above theorem 2.1.11. That is, let p be proper surjective PL map $p : M \to B$ defined on a manifold and suppose that each point v in B has a stellar neighborhood $S = v * L$ in B whose preimage $p^{-1}S$ collapses onto $p^{-1}v$ via a strong deformation retraction $D : p^{-1}S \times I \to p^{-1}S$.*

Then following conditions are equivalent:

(2) $D_1 \mid p^{-1}y : p^{-1}y \to p^{-1}v$ is a homotopy equivalence for any y in S.

(2)′ $D_1 \mid p^{-1}x : p^{-1}x \to p^{-1}v$ is a homotopy equivalence for any x in L.

PROOF: We need to show $(2)'$ implies (2). Consider any v in B. Find S, L, S', and L' as in the Codimension Reduction Lemma. Recall that in the Codimension Reduction Lemma S is a stellar neighborhood of v and L is a link with respect to a subdivision of the complex B . In the following proof, all simplicial structures refer to the original triangulation before the subdivision except L and S. Consider any $y \in S - v$. There exists a unique simplex σ in B such that $y \in \mathrm{I}nt(\sigma)$. By Proposition 2.1.2, we have a PL homeomorphism $\Psi : p^{-1}(\mathrm{I}nt(\sigma)) \to \mathrm{Int}(\sigma) \times p^{-1}y$. Choose $x \in L \cap \mathrm{I}nt(\sigma)$. (Note that this is possible because the simplex σ belongs to simplicial complex before the subdivision.)

Then there is a homeomorphism $\Psi \mid p^{-1}x : p^{-1}x \to \{x\} \times p^{-1}y$. So if $D_1 \mid p^{-1}x : p^{-1}x \to p^{-1}v$ is a homotopy equivalence, then $D_1 \mid p^{-1}y : p^{-1}y \to p^{-1}v$ is also a homotopy equivalence. $\square$

The above proposition says that we only need to check the homotopy equivalence condition for the link L rather than whole star neighborhood S to see whether a PL map p is a completely movable map.

Since by the Theorem 1.2.5 a proper map is an approximate fibration if and only if it is a completely movable map, we have the following theorem which we call the Fundamental Recognition Theorem. This will be a main tool to recognize approximate fibrations among proper PL maps.

Theorem 2.1.13 *(**Fundamental Recognition Theorem**) A proper surjective PL map $p : M \to B$ defined on a manifold M is an approximate fibration if each v in B has a stellar neighborhood $S = v * L$ whose preimage $p^{-1}S$ collapses to $p^{-1}v$ via a strong deformation retraction $D : p^{-1}S \times I \to p^{-1}S$ such that for all b in L the map $D_1 \mid p^{-1}b : p^{-1}b \to p^{-1}v$ is a homotopy equivalence.*

PROOF: Consider any proper surjective PL map $p : M \to B$ defined on a manifold M. Assume that we can find a stellar neighborhood $S = v * L$ whose preimage $p^{-1}S$ collapses to $p^{-1}v$ via a strong deformation retraction $D : p^{-1}S \times I \to p^{-1}S$ such that the map $D_1 \mid p^{-1}b : p^{-1}b \to p^{-1}v$ for any b in L is a homotopy equivalence. Then Proposition 2.1.12 implies that $D_1 \mid p^{-1}y : p^{-1}y \to p^{-1}v$ for any y in S and p is a completely movable map by Theorem 2.1.11 . $\square$

2.2. THE EXACT SEQUENCES

For PL approximate fibrations over the k-sphere S^k, R. J. Daverman shows that there exists an associated Wang homology exact sequence as in the case of fibrations [4]. We state here a somewhat more general version of this without restricting to a PL map. This will be one of our main tools.

We need the following results to prove the existence of the Wang exact sequence:

Lemma 2.2.1 *Let $p : M \to B$ be an approximate fibration onto a contractible space B. Then for any fiber F, the inclusion $i : F \to M$ is a weak homotopy equivalence.*

This lemma is a straightforward consequence of the homotopy exact sequence of the approximate fibration p. The above result may be considered as an analogue of the theorem concerning a fibration [20] which says that any fibration with a contractible space is equivalent to a trivial bundle.

Next we need the following result of Coram and Duvall [2].

Proposition 2.2.2 *[2] Let $p : E \to B$ be a proper surjective map between ANR spaces. Then p is an approximate fibration if p has the approximate lifting property with respect to any n-cells, $I^n, n \geq 0$.*

For a fibration $p : E \to B$, a restriction of p to $p^{-1}(A)$, where A is any subset of B , is also a fibration . This may not hold in the case of approximate fibrations but we have the following result.

Proposition 2.2.3 *Let $p : E \to B$ be an approximate fibration between ANR spaces E and B. Let A be an open subset of B. Then $p \mid p^{-1}(A) : p^{-1}(A) \to A$ is an approximate fibration.*

PROOF: First, note that both A and $p^{-1}(A)$ are ANRs as open subsets of an ANR. By the above Proposition 2.2.2, it suffices to show that $p \mid p^{-1}(A)$ has the

AHLP with respect to any cell I^n. Consider a homotopy lifting problem for a map $f : I^n \to p^{-1}(A) \subset E$ and a homotopy $H : I^n \times I \to A \subset B$ with $H_0 = p \circ f$. Note that $H(I^n \times I)$ is a compact subset of the open set A. Since $B \setminus A$ is a closed subset of B, there exists a positive number $\eta = \inf\{d(H(x,t), y) \mid (x,t) \in I^n \times I,\ y \in B \setminus A\}$. Consider any positive number ϵ strictly less than η. Now the above is also a homotopy lifting problem for an approximate fibration $p : E \to B$ so that there exists an ϵ-lift $\tilde{H}$ of H into E such that $\tilde{H}_0 = f$ and $p \circ \tilde{H}$ and H are ϵ-close.

Now we claim that $\tilde{H}$ actually maps $I^n \times I$ into $p^{-1}(A)$. Otherwise there is a point $(x,t) \in I^n \times I$ such that $p \circ \tilde{H}(x,t) \in B \setminus A$. But this implies that $d(H(x,t), p(\tilde{H}(x,t))) \geq \eta > \epsilon$ and we have a contradiction. Thus $\tilde{H}(I^n \times I) \subset p^{-1}(A)$ and we conclude that $p \mid p^{-1}(A) : p^{-1}(A) \to A$ is an approximate fibration. □

Theorem 2.2.4 (Wang Homology Exact Sequence) *Let $p : M^{n+k} \to S^k$ be an approximate fibration defined on a closed, connected $(n+k)$-manifold M onto the k-sphere for $k \geq 1$ such that each fiber has a homotopy type of a closed, connected, orientable n-manifold N. Then there exists a Wang exact sequence:*

$$\to H_j(F) \to H_j(M) \to H_{j-k}(F) \to H_{j-1}(F) \to$$

Here F is a typical fiber which has the homotopy type of the n-manifold N.

PROOF: Consider an exact sequence of the pair (M, F) for any fiber F:

$$\to H_j(F) \to H_j(M) \to H_j(M, F) \to H_{j-1}(F) \to$$

By Lefschetz duality, $H_j(M, F) \cong H^{n+k-j}(M - F)$.

Claim. $H^l(F) \cong H^l(M - F)$.

We can prove the above claim as follows. Choose any fiber $F' \subset (M - F)$. Now $p(M-F) = S^k - \{b_0\} \cong \mathbb{R}^k$. Then the restriction $p \mid (M-F)$ is an approximate fibration over a k-cell by Proposition 2.2.3 above.

Then by the above Lemma 2.2.1 the inclusion $F' \subset (M - F)$ is a weak homotopy equivalence so that $H^l(F') \cong H^l(M - F)$ by Theorem 7.7.25 of [20].

For these F, F', we can find another fiber F'' such that $F, F' \subset (M - F'')$. Then by the above Lemma 2.2.1 again, we have

$H^l(F') \cong H^l(M - F'') \cong H^l(F)$. This shows that $H^l(F) \cong H^l(M - F)$.

Now we have $H_j(M, F) \cong H^{n+k-j}(M-F) \cong H^{n+k-j}(F)$ by the above Claim. By Poincare duality $H^{n+k-j}(F) \cong H_{j-k}(F)$ and we have proved the theorem. $\square$

Similarly there exists the following Wang exact cohomology sequence associated with an approximate fibrations:

$$\to H^j(M) \to H^j(F) \to H^{j-k+1}(F) \to H^{j+1}(M) \to$$

Note that the first homomorphisms in both sequences are induced by an inclusion map.

Also for a PL approximate fibration $p : E \to B$ the exact sequence in Theorem 1.2.3 reduces to a usual homotopy exact sequence:

$$\to \pi_n(F_b, e) \to \pi_n(E, e) \to \pi_n(B, b) \to \pi_{n-1}(F_b, e) \to$$

where b in B and e in F_b.

3. SOME CLASSES OF MANIFOLDS WHICH ARE PL H-FIBRATORS

In this chapter we present new results identifying several classes of manifolds which are PL h-fibrators.

After discussing the concept of hopfian manifolds and reviewing some results of R. J. Daverman in section 1, we obtain the Basic Lemma in section 2. This Basic Lemma is useful in dealing with the non-simply connected case. In section 3, we prove that a closed, orientable, aspherical manifold with a hopfian fundamental group is a PL h-fibrator if it is a codimension-2 PL fibrator by using the Basic Lemma. In section 4, we examine certain manifolds with a special cohomology ring structure. We prove that for a closed, orientable, connected, hopfian n dimensional manifold N^n with a hopfian fundamental group such that $H^j(N^n) = 0$ for $0 < j < m$ for some integer $m \geq 1$ and such that $H^n(N^n)$ is in the subring of $H^*(N^n)$, generated by $H^m(N)$, N^n is a PL h-fibrator if it is a codimension-$(m+1)$ PL h-fibrator.

This result gives as a corollary that quarternionic projective spaces are PL h-fibrators . First we need some definitions.

3.1. HOPFIAN MANIFOLDS

Definition 3.1.1 *[4] An orientable, closed manifold N is hopfian if any degree one map $f : N \to N$ inducing an isomorphism on the fundamental group is a homotopy equivalence.*

We now exhibit examples of hopfian manifolds. Recall that a group G is hopfian if any epimorphism $\phi : G \to G$ is an isomorphism. Any finite group is hopfian and $(\mathbb{Z}, +)$ is also hopfian. More generally any finitely generated abelian group is hopfian.

The next three propositions are known to R. J. Daverman.

Proposition 3.1.2 *[10] A simply connected, orientable, closed n-manifold N^n is a hopfian manifold.*

PROOF: Let $f : N^n \to N^n$ be a degree 1 map. The map f induces an isomorphism on the fundamental group trivially. Since $f_* : H_n(N^n) \to H_n(N^n)$ is an isomorphism, $f_* : H_j(N^n) \to H_j(N^n)$ is an epimorphism for $1 \leq j \leq n$ by Theorem 67.2 in [16]. Since N^n is a compact manifold, each $H_j(N^n)$'s is a finitely generated abelian group and hence is a hopfian group. So $f_* : H_j(N^n) \to H_j(N^n)$ is an isomorphism for any j. Since N^n is simply connected, f is a homotopy equivalence by the Whitehead theorem. □

Proposition 3.1.3 *[10] An orientable, closed, connected n-manifold N^n with a finite fundamental group is a hopfian manifold.*

PROOF: Let $f : N^n \to N^n$ be a degree 1 map that induces an isomorphism on the fundamental group. We only need to check the case of non-trivial fundamental group. Consider a universal cover $p : \tilde{N} \to N$ and the composite map $f \circ p : \tilde{N} \to N$. Since $\tilde{N}$ is simply connected, there exists a lift $\tilde{f} : \tilde{N} \to \tilde{N}$ such that $p \circ \tilde{f} = f \circ p$. Since $\pi_1(N)$ is a finite non-trivial group and p is a finite cover, deg p is nonzero finite integer. Then deg $(p \circ \tilde{f})$ = deg p deg $\tilde{f}$ = deg f deg p so that deg $\tilde{f}$ = deg $f = 1$. Since $\tilde{N}$ is simply connected, by the previous proposition, $\tilde{f} : \tilde{N} \to \tilde{N}$ is a homotopy equivalence. Then $\tilde{f}_\# : \pi_i(\tilde{N}) \to \pi_i(\tilde{N})$ is an isomorphism for any i.

Now $\pi_i(\tilde{N}) \cong \pi_i(N)$ for all $i \geq 2$ since p is a universal covering map. Since $\tilde{f}_\#$, $p_\#$ are isomorphisms for $i \geq 2$, $f_\# : \pi_i(N) \rightarrow \pi_i(N)$ is also an isomorphism for any $i \geq 2$. Combining this with the hypotheses, we have that $f_\# : \pi_i(N) \rightarrow \pi_i(N)$ is an isomorphism for all i. It follows that f is a homotopy equivalence. □

Note that in the proof of the above propositions the hopfian nature of the fundamental group plays a crucial role. The following proposition shows that if the fundamental group of a manifold N is hopfian, then every degree one self map induces an isomorphism on its fundamental group.

Proposition 3.1.4 *[10] Let $f : N \rightarrow N$ be a degree one self map of a closed, connected, orientable manifold N with a finite fundamental group. Then $f_\# : \pi_1(N) \rightarrow \pi_1(N)$ is an epimorphism.*

Proof: Suppose not. Then $f_\#(\pi_1(N))$ is a proper subgroup of $\pi_1(N)$. Consider a covering map $\theta : \tilde{N} \rightarrow N$ corresponding to the subgroup $f_\#(\pi_1(N))$. Note that $\tilde{N}$ is a closed, orientable n-manifold. Since $f_\#(\pi_1(N)) \subset \theta_\#(\pi_1(\tilde{N})) = f_\#(\pi_1(N))$, there exists a lift $\tilde{f} : N \rightarrow \tilde{N}$ such that $f = \theta \circ \tilde{f}$. Then $1 = \deg(f) = \deg\ (\theta \circ \tilde{f}) = \deg(\theta)$ $\deg(\tilde{f})$. This implies $\deg(\theta)=\pm 1$ and $\deg(\tilde{f})=\pm 1$. Now $\theta_\#$ is a monomorphism and $\deg(\theta)=\pm 1$ implies that $\theta_\#$ is an isomorphism. Thus we have a contradiction. □

It is known that fundamental groups of 2-manifolds are hopfian groups [12]. Although there is a conjecture that the fundamental groups of any 3-manifold are hopfian, there exists a 4-manifold whose fundamental group is not hopfian. This is due to the existence of a finitely presented group $G =< a, b \mid a^{-1}b^2a = b^3 >$ which is not a hopfian group [12] and the fact that any such finitely presented group can be realized as a fundamental group of some 4-manifold [15]. Any 3 or 4-manifold with a hopfian fundamental group is a hopfian manifold [4]. The following theorem of Swarup implies that any aspherical manifold is hopfian.

Theorem 3.1.5 *[21] Let $f : (M_1, x) \to (M_2, f(x))$ be a map of closed, oriented n-manifold which induces an isomorphism of fundamental groups. Suppose that $\pi_i(M_1)$ and $\pi_1(M_2)$ are trivial for $1 < i < n-1$. Then f is a homotopy equivalence if and only if the degree of f is ± 1.*

Next we present a lemma which we call the Basic Lemma. This lemma is very useful in our pursuit of h-fibrators.

3.2. BASIC LEMMA

Recall that according to the Fundamental Recognition Theorem we have to check whether the following composite map $p^{-1}b \to L' \to S' \to p^{-1}v$ is a homotopy equivalence for all b in B where $L' = p^{-1}L$ and $S' = p^{-1}S$ as in the Codimension Reduction Lemma to conclude that p is an approximate fibration. Since we are dealing with the PL category, this can be achieved by studying the induced homomorphism on the homotopy groups of involved spaces.

$$\pi_l(p^{-1}b) \to \pi_l(L') \to \pi_l(S') \to \pi_l(p^{-1}v).$$

The first two homomorphisms are induced by inclusion maps and the last one is induced by a strong deformation retraction. In the case of an N^n-like map $p : M^{n+k} \to B^k$, due to the collapsibility of each fiber, there is a useful result [4] concerning the inclusion induced homomorphism $j_\# : \pi_l(L') \to \pi_l(S')$. This result says that $j_\#$ is

an isomorphisms for $l \leq k-2$ and is an epimorphism when $l = k-1$. But in the case of an N^n-type map we do not have the above results on hand.

The best we can do is show that the inclusion $j : L' \to S'$ induces an epimorphism on fundamental groups. We will prove that that $\pi_1(L') \to \pi_1(S')$ is an epimorphism using the following lemma which we call the Inclusion Lemma.

Lemma 3.2.1 *(Inclusion Lemma) Let M be a compact, orientable, connected PL $(n+k)$-manifold with a nonempty boundary. Let g be a compact subset of M which has the homotopy type of an n-dimensional, orientable, connected, closed manifold N. Suppose $k \geq 2$ and g is a strong deformation retract of M and g has an open neighborhood U such that $U \subset \overline{U} \subset \mathrm{Int}(M)$ and U collapses to g. (Note that $(M, M-U)$ is a relative homology $(n+k)$-manifold.) Then the inclusion $i : M-g \to M$ induces an epimorphism on the fundamental groups.*

Remark. The hypothesis $k \geq 2$ cannot be removed since the similar proposition for $k = 1$ is not true. Consider the following example: let M be a torus with an open disc removed and g be one point union of a longitudinal circle and a meridional circle of the torus. Then $\pi_1(M) \cong \mathbb{Z} * \mathbb{Z}$ but $\pi_1(M-g) \cong \mathbb{Z}$. The condition of having an open neighborhood described in the lemma is also crucial as we can see by the following example with k = 2: let M be a solid torus and g be the solid torus with a non-separating open ball removed. Then $\pi_1(M-g) \cong \mathbb{Z}$ but $\pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Before proving the Inclusion Lemma, we have the following as its corollary. We will refer this as the Basic Lemma which will be used throughout the remainder of this work.

Corollary 3.2.2 (Basic Lemma) *Let $p : M^{n+k} \to B^k$ be an N^n-type map where $k \geq 2$. Then for any v in B with a stellar neighborhood $S = v * L$ as in the Codimension Reduction Lemma, the inclusion $i : L' \to S'$ induces an epimorphism $i_{\#} : \pi_1(L') \to \pi_1(S')$.*

Proof: Note that S' collapses to $p^{-1}v$ and $S' - p^{-1}v$ collapses to L' so that $S' - p^{-1}v \simeq L'$. By hypothesis, $p^{-1}v \simeq N^n$. Choose a smaller regular neighborhood W of $p^{-1}v$ which is contained in S' such that $p^{-1}v \subset (W - \partial W) \subset W \subset (S' - L')$. Then the open subset $W - \partial W$ collapses to $p^{-1}v$. Take $p^{-1}v$ for g, S' for M, and $W - \partial W$ for U in the Inclusion Lemma to conclude that the inclusion induced homomorphism $\pi_1(S' - p^{-1}v) \to \pi_1(S')$ is an epimorphism. Since $S' - p^{-1}v$ collapses to L', we have an epimorphism $i_\# : \pi_1(L') \to \pi_1(S')$ $\square$

Now we prove the Inclusion Lemma. Our proof is somewhat long and we need several lemmas. Rather than obscuring the main proof by presenting necessary results in advance, we give a main proof first and discuss the needed lemmas after the proof.

Proof: (Proof of the Inclusion Lemma) Choose a base point x_0 in M. Since M is path connected, without loss of generality, we may choose x_0 to be in $M - g$. Consider any non-identity element $[\alpha] \in \pi_1(M, x_0)$ and a universal covering $\theta : \tilde{M} \to M$. The loop α in M based at x_0 has a lifted path $\tilde{\alpha}$ in $\tilde{M}$ such that $\tilde{\alpha}(0) = \tilde{x_0}$ and $\tilde{\alpha}(1) = \tilde{x_0}'$ where $\tilde{x_0}, \tilde{x_0}' \in \theta^{-1}(x_0)$.

<u>Claim 1.</u> $M - g$ is path connected.

This claim can be established by considering the following homology exact sequence of the pair $(M, M - g)$:

$$\to H_1(M) \to H_1(M, M - g) \to H_0(M - g) \to H_0(M) \to H_0(M, M - g) \to$$

Find a neighborhood U of g as in the hypothesis so that $H_1(M, M - g) \cong H_1(M, M - U)$. We may choose U to be the interior of a regular neighborhood of g that does not intersect the boundary of M. By Lefschetz duality $H_1(M, M - U) \cong H^{n+k-1}(U) \cong H(g)$ which is trivial since g has a homotopy type of N and $k \geq 2$.

Similarly the last group in the above sequence is also trivial. Then by the exactness of the sequence, $H_0(M-g) \cong H_0(M) \cong \mathbb{Z}$ so that $M-g$ is path connected. Next we consider $\theta^{-1}(M-g) = \tilde{M} - \theta^{-1}(g)$. Let $\tilde{g} = \theta^{-1}(g)$. We note that $\theta \mid \tilde{g} : \tilde{g} \to g$ is a universal covering by the Proposition 3.2.3 below.

Claim 2. $\tilde{M} - \tilde{g}$ is path connected.

Since the universal cover $\tilde{M}$ is not necessarily compact, we will use Alexander duality [[20] p.342] and we need some justification to use it in establishing this claim. Note that $\tilde{M} - \partial\tilde{M}$ is an open manifold. Since $\tilde{g} \cap \partial\tilde{M} = \emptyset$, $\tilde{M} - \tilde{g}$ is path connected if $(\tilde{M} - \partial\tilde{M}) - \tilde{g}$ is path connected. Let $\tilde{M}' = \tilde{M} - \partial\tilde{M}$. Consider the following long exact homology sequence of the pair $(\tilde{M}', \tilde{M}' - \tilde{g})$.

$$\to H_1(\tilde{M}') \to H_1(\tilde{M}', \tilde{M}' - \tilde{g}) \to H_0(\tilde{M}' - \tilde{g}) \to H_0(\tilde{M}') \to H_0(\tilde{M}', \tilde{M}' - \tilde{g})$$

By Alexander duality $H_1(\tilde{M}', \tilde{M}' - \tilde{g}) \cong H_c^{n+k-1}(\tilde{g})$. Since $\tilde{g}$ and $\tilde{N}$ belong to the same proper homotopy equivalence class by Proposition 3.2.9 below , by the homotopy axiom of Alexander cohomology, we have $H_c^{n+k-1}(\tilde{g}) \cong H_c^{n+k-1}(\tilde{N})$. But $H_c^{n+k-1}(\tilde{N}) \cong 0$ since dim $\tilde{N} = n$ and $k \geq 2$. Similarly the last group of the above exact sequence is also trivial. Then the exactness of the sequence shows that $H_0(\tilde{M}' - \tilde{g}) \cong H_0(\tilde{N}) \cong \mathbb{Z}$ and we have proved that $\tilde{M}' - \tilde{g}$ is path connected. Then as mentioned above, since $\tilde{g} \cap \partial\tilde{M} = \emptyset$, this implies the claim, i.e. the path connectedness of $\tilde{M} - \tilde{g}$.

So by the above two claims both $M-g$ and $\tilde{M} - \tilde{g}$ are path connected.

Since $\{x_0\} \cap g = \emptyset$ by our choice, we have $\theta^{-1}(x_0) \cap \theta^{-1}(g) = \emptyset$ so that $\theta^{-1}(x_0) \subset (\tilde{M} - \theta^{-1}(g))$ and $\tilde{x_0}, \tilde{x_0}' \in (\tilde{M} - \theta^{-1}(g))$.

Since $\tilde{M} - \tilde{g}$ is path connected, there exists a path $\tilde{\beta} \subset \tilde{M} - \tilde{g}$ such that $\tilde{\beta}(x_0) = \tilde{x_0}$ and $\tilde{\beta}(x_1) = \tilde{x_0}'$. [Here we do not distinguish a path as a map from its image.] Since $\tilde{M}$ is simply connected as a universal cover of M, $\tilde{\beta}$ is homotopic to $\tilde{\alpha}$ in $\tilde{M}$. Then $\theta \circ \tilde{\beta}$ is a loop homotopic to $\alpha = \theta \circ \tilde{\alpha}$ in M.

Note $\theta \circ \tilde{\beta} \subset (M - g)$, i.e. $[\theta \circ \tilde{\beta}] \in \pi_1(M - g)$ and $\theta \circ \tilde{\beta} \simeq \alpha$. We conclude that $i_\# : \pi_1(M - g) \to \pi_1(M)$ is an epimorphism. $\square$

Now we present a series of necessary results to prove the Inclusion Lemma.

Proposition 3.2.3 *Let $g \subset M$ be a strong deformation retract of M. Suppose $\theta : \tilde{M} \to M$ is a universal covering. Then $\theta \mid \tilde{g} : \tilde{g} \to g$ is an universal covering.*

PROOF: We have to show that $\tilde{g}$ is simply connected and $\theta \mid \tilde{g}$ is a covering map.

Let $D : M \times I \to M$ be a strong deformation retraction such that $D_0 = id_M$ and R=$D_1 : M \to g$ is a retraction. Define a map $(\theta \times id_I) : \tilde{M} \times I \to M \times I$ by $\theta \times id_I(x,t) = (\theta(x), t)$. Since $\tilde{M} \times I$ is simply connected, the composite map $D \circ (\theta \times id_I) : \tilde{M} \times I \to M$ can be lifted to a map $\tilde{D} : \tilde{M} \times I \to \tilde{M}$ such that $\theta \circ \tilde{M} = D \circ (\theta \times id_I)$. Note that $(\theta \circ \tilde{D})(z,0) = D(\theta(z),0) = \theta(z) = \theta \circ id_{\tilde{M}}(z)$. Then we may consider $\tilde{D}$ as a lifted homotopy,i.e. as a solution for the homotopy lifting problem with the data, $id_{\tilde{M}} : \tilde{M} \to \tilde{M}$ and the homotopy $D \circ (\theta \times id_I) : \tilde{M} \times I \to M$. Then $\tilde{D}_0 = id_{\tilde{M}}$.

We claim that $\tilde{D} : \tilde{M} \times I \to \tilde{M}$ is a strong deformation retraction and $\tilde{D}_1 : \tilde{M} \to \tilde{g}$ is a retraction. To see this, consider any $z \in \tilde{g}$. Then $\theta(z) \in g$ so that $\theta \circ \tilde{D}(z,t) = D \circ (\theta \times id_I)(z,t) = D(\theta(z),t) = \theta(z)$ for all t in I. So $\tilde{D}(z,t) \in \theta^{-1}(\theta(z))$ for all t in I. Since $\tilde{D}(z,I)$ is the continuous image of the connected set I, which lies in the discrete space $\theta^{-1}(\theta(z))$, we conclude that $\tilde{D}(z,I)$ is a point. But $\tilde{D}(z,0) = z$. So $\tilde{D}(z,I) = \{z\}$, i.e. $\tilde{D}$ fixes all z in $\theta^{-1}(g)$. Next for any z in $\tilde{D}$, $\theta \circ \tilde{D}(z,1) = D(\theta(z),1) = R(\theta(z)) \in g$ so that $\tilde{D}(z,1) \in \theta^{-1}(g) = \tilde{g}$ and $\tilde{D}_1(M) \subset \tilde{g}$. Since $\tilde{D}(z,1) = z$ for z in $\tilde{g}$, we have $\tilde{D}_1(M) = \tilde{g}$. We conclude that $\tilde{D} : \tilde{M} \times I \to \tilde{M}$ is a strong deformation retraction and $\tilde{D}_1 : \tilde{M} \to \tilde{g}$ is a retraction. Thus $\tilde{g}$ is path connected. Since g is also path connected, by Lemma 2.1 of [15], $\theta \mid \tilde{g} : \tilde{g} \to g$ is a covering map. Since $\tilde{g}$ has the homotopy type of a simply connected space $\tilde{M}$, $\tilde{g}$ is also simply connected. So $\theta \mid \tilde{g} : \tilde{g} \to g$ is a universal covering. $\square$

For the next proposition we need the following two definitions.

Definition 3.2.4 *(Properly Discontinuous Group Action) [20] A group G acts properly discontinuously on a space Y if for each y in Y there exists an open neighborhood U of y in Y such that if $g(U) \cap g'(U) \neq \emptyset$ for some g, g' in G, then $g = g'$.*

Remark. Note that above definition is equivalent to saying that $g \neq id$ implies that $U \cap g(U) = \emptyset$.

Definition 3.2.5 *(Discrete Group Action) [19] We say that a group G acting on a space X acts discretely if for any compact subset C of X, $\{g \in G \mid g(C) \cap C \neq \emptyset\}$ is a finite set.*

Remark. Note that any finite group action is discrete. So the group $\mathbb{Z}_2$ acting trivially on a space X by $gx = x$ is a discrete group action but it is not a properly discontinuous group action. But we have the following proposition:

Proposition 3.2.6 *Let G be a group acting properly discontinuously on a locally compact Hausdorff space. Then G acts on X discretely.*

PROOF: We only need to consider the case of an infinite group G. Consider any compact subset K in X. For each z in K we can find an open neighborhood W_z of z in X such that $W_z \cap h(W_z) = \emptyset$ for all nonidentity elements h in G and such that $\overline{W_z}$ is compact. Then $\{W_z \mid z \in K\}$ is an open cover of the compact set K, so there exists a finite subcover $\{W_{z_j} \mid j = 1, \cdots, n\}$ such that $K \subset \bigcup_{j=1}^n W_{z_j}$. Write W_j for W_{z_j}. For any h in G, we have $h(K) \subset \bigcup_{j=1}^n h(W_{z_j})$ and

$$K \cap h(K) \subset (\bigcup_{i=1}^n W_i) \cap (\bigcup_{j=1}^n h(W_j)) = \bigcup_{i=1}^n \bigcup_{j=1}^n (W_i \cap h(W_j)).$$

Suppose $h \neq id$. Then if $i = j$, $W_i \cap h(W_j) = \emptyset$ by our construction. So if $h \neq id$, we have $K \cap h(K) \subset \bigcup_{i \neq j} (W_i \cap h(W_j))$.

Claim. Given i and j, $1 \leq i, j \leq n$ as above, $W_i \cap h(W_j) \neq \emptyset$ for only finitely many h in G.

Note that if this claim is true, the conclusion of the proposition follows since the set $A = \{h \in G \mid W_i \cap h(W_j) \neq \emptyset, 1 \leq i, j \leq n\}$ is finite and for any $h \in G - A$, $K \cap h(K) = \emptyset$.

Proof of the Claim. Fix i and j. Suppose the claim is not true. Then there exists a subset $\{h_k \mid k \in \mathbb{N}\} \subset G$ such that $W_i \cap h_k(W_j) \neq \emptyset$. Note that $h_k(W_j) \cap h_l(W_j) = \emptyset$ if $k \neq l$. So $\{h_k(W_j) \mid k \in \mathbb{N}\}$ is a collection of pairwise disjoint open sets. Now $\overline{W_i}$ is compact and we have $\overline{W_i} \cap h_k(W_j) \neq \emptyset$, $k \in \mathbb{N}$. Since X is locally compact Hausdorff, X is a regular space and its subspaces are also regular. For each $k \in \mathbb{N}$, choose a point $x_k \in W_i \cap h_k(W_j) \neq \emptyset$. Using regularity we can find an open subset V_k such that $x_k \in V_k \subset \overline{V_k} \subset W_i \cap h_k(W_j)$.

Define $U_l = \bigcup_{k=1}^l (W_i \cap h_k(W_j))$, $l = 1, 2, \cdots$ so that $U_1 \subset U_2 \subset \cdots$. Note that $F = \bigcup_{k=1}^\infty (\overline{W_i} - \overline{V_k}) = \overline{W_i} - (\bigcap_{k=1}^\infty \overline{V_k})$ is a relatively open subset of the closed set $\overline{W_i}$. Then $\{F, U_l \mid l \in \mathbb{N}\}$ is an open cover of the compact set $\overline{W_i}$ which does not admit a finite subcover. This is a contradiction and we have established the claim. Now the claim implies the proposition. $\square$

As a corollary of the above proposition we get the following result needed for the proof of Proposition 3.2.9 which we used to prove the Inclusion Lemma.

Lemma 3.2.7 *Let $\theta : \tilde{g} \to g$ be a covering map. Let $G = G(\tilde{g} \mid g)$ be a group of covering transformations. Then for any compact subsets A, B in $\tilde{g}$, the set $\{h \in G(\tilde{g} \mid g) \mid A \cap h(B) \neq \emptyset\}$ is finite.*

PROOF: The covering transformation group G acts on the covering space $\tilde{g}$ properly discontinuously. Then by the above Proposition 3.2.6 applied to the compact set $K = A \cup B$, we have since $A \cap h(B) \subset K \cap h(K)$, $\{h \in G \mid A \cap h(B) \neq \emptyset\} \subset \{h \in G \mid K \cap h(K) \neq \emptyset\}$ is finite. $\square$

The above Lemma 3.2.7 and the following result will be used to prove Proposition 3.2.9.

Lemma 3.2.8 *Let $\theta : \tilde{g} \to g$ be a universal covering. Let $F : g \times I \to g$ be a homotopy. Define a map $(\theta \times id_I) : \tilde{g} \times I \to g \times I$ by $(\theta \times id_I)(x,t) = (\theta(x), t)$. Then the homotopy $F \circ (\theta \times id_I) : \tilde{g} \times I \to g$ has a lift $H : \tilde{g} \times I \to \tilde{g}$ such that $H_0 = id_{\tilde{g}}$ and $\theta \circ H = F \circ (\theta \times id_I)$. Let $h \in G(\tilde{g} \mid g)$ be a covering transformation. So $\theta \circ h = \theta$. Then $H \circ (h \times id_I) : \tilde{g} \times I \to \tilde{g} \times I \to \tilde{g}$ is equal to $h \circ H : \tilde{g} \times I \to \tilde{g} \to \tilde{g}$, i.e. $H(h(z), t) = h(H(z,t))$ for all z in $\tilde{g}$ and t in I.*

PROOF: The first part of the existence of the lift is just a consequence of the hypothesis $\theta : \tilde{g} \to g$ is a covering map. For the second part, we show that $\theta \circ h^{-1} \circ H \circ (h \times id_I) = \theta \circ H$. Since $h^{-1} \in G(\tilde{g} \mid g)$, we have $\theta \circ h^{-1} \circ H \circ (h \times id_I) = \theta \circ H \circ (h \times id_I) = (F \circ \theta \times id_I) \circ (h \times id_I) = F \circ [(\theta \circ h) \times id_I] = F \circ \theta \times id_I = \theta \circ H$.

Thus $\theta \circ [h^{-1} \circ H \circ (h \times id_I)] = \theta \circ H$ so that both $h^{-1} \circ H \circ (h \times id_I)$ and H are lifts of the map $F \circ \theta \times id_I$. Now $H_0 = id_{\tilde{g}}$, i.e. $H(z, 0) = id_{\tilde{g}}(z) = z$. But $h^{-1} \circ H \circ (h \times id_I)(z, 0) = h^{-1}(H(h(z), 0)) = h^{-1}(h(z)) = z$ so that $[h^{-1} \circ H \circ (h \times id_I)]_0 = id_{\tilde{g}}$, too. So by the uniqueness of lifts, we have $h^{-1} \circ H \circ (h \times id_I) = H$ and $h^{-1}(H(h(z), t)) = H(z, t)$.

Now we have $H(h(z), t) = h(H(z, t))$. $\square$

Proposition 3.2.9 *Let g, N be compact spaces of the same homotopy type, which have universal covers $\tilde{g}, \tilde{N}$ respectively. Then $\tilde{g}, \tilde{N}$ belong to the same proper homotopy equivalence class. [See p.320 of [20] for the definition of proper homotopy.]*

PROOF: We prove this in several steps.

Claim 1. $g \simeq N$ implies $\tilde{g} \simeq \tilde{N}$.

Proof of the claim. Let $\mu : g \to N$ be a homotopy equivalence and $\nu : N \to g$ be its homotopy inverse. Let $\theta_g : \tilde{g} \to g$ and $\theta_N : \tilde{N} \to N$ be universal coverings. Since $\tilde{g}$ is simply connected, the map $\mu \circ \theta_g : \tilde{g} \to N$ has a unique lift $\widetilde{\mu \circ \theta_g} : \tilde{g} \to \tilde{N}$ such that $\theta_N \circ \widetilde{\mu \circ \theta_g} = \mu \circ \theta_g$. Similarly, the map $\nu \circ \theta_N : \tilde{N} \to g$ has the unique lift $\widetilde{\nu \circ \theta_N} : \tilde{N} \to \tilde{g}$ such that $\theta_g \circ \widetilde{\nu \circ \theta_N} = \nu \circ \theta_N$. We write $\tilde{\mu} = \mu \circ \theta_g$ and $\tilde{\nu} = \nu \circ \theta_N$. We assert that $\tilde{\mu} \circ \tilde{\nu} \simeq id_{\tilde{N}}$ and $\tilde{\nu} \circ \tilde{\mu} \simeq id_{\tilde{g}}$. Let $H : g \times I \to g$ be a homotopy between id_g and $\nu \circ \mu$. Define a map $(\theta_g \times id_I) : \tilde{g} \times I \to g \times I$ and a construct a homotopy $H \circ (\theta_g \times id_I) : \tilde{g} \times I \to g \times I \to g$. Note that $(H \circ (\theta_g \times id_I))_0 = \theta_g = \theta_g \circ id_{\tilde{g}}$. Now we can view this as a homotopy lifting problem for the covering map $\theta_g : \tilde{g} \to g$ and we have a lifted homotopy $\tilde{H} = H \circ \widetilde{(\theta_g \times id_I)} : \tilde{g} \times I \to \tilde{g}$ such that $\tilde{H}_0 = id_I : \tilde{g}$ and $\theta_g \circ \tilde{H} = H \circ (\theta_g \times id_I)$.

We check that $\tilde{H}_1 = \tilde{\nu} \circ \tilde{\mu}$ as follows. Since $\theta_g \circ \tilde{H} = H \circ (\theta_g \times id_I)$, we have $\theta_g \circ \tilde{H}(z, 1) = H \circ (\theta_g \times id_I)(z, 1) = H(\theta_g(z), 1) = (\nu \circ \mu)(\theta_g(z)) = \nu \circ (\mu \circ \theta_g)(z) = \nu \circ (\theta_N \circ \tilde{\mu})(z) = (\nu \circ \theta_N) \circ \tilde{\mu}(z) = \theta_g \circ (\tilde{\nu} \circ \tilde{\mu})(z)$. This shows that both $\tilde{\nu} \circ \tilde{\mu}$ and $\tilde{H}_1$ are the lifts of the map $[H \circ (\theta_g \times id_I)]_1$. Because of the uniqueness of lifts, we conclude that $\tilde{H}_1 = \tilde{\nu} \circ \tilde{\mu}$. Thus $\tilde{H}$ is a homotopy between $id_{\tilde{g}}$ and $\tilde{\nu} \circ \tilde{\mu}$.

Similarly, the lift $\tilde{G} = G \circ (\theta_N \tilde{\times} id_I)$ of $G : N \times I \to N$ where $G : N \times I \to N$ is a homotopy between id_N and $\mu \circ \nu$ can be checked to be a homotopy between $id_{\tilde{N}}$ and $\tilde{\mu} \circ \tilde{\nu}$. Thus $\tilde{\nu} \circ \tilde{\mu} \simeq id_{\tilde{g}}$ and $\tilde{\mu} \circ \tilde{\nu} \simeq id_{\tilde{N}}$ so that $\tilde{\mu} : \tilde{g} \to \tilde{N}$ is a homotopy equivalence, i.e. $\tilde{g} \simeq \tilde{N}$. We have proven claim 1.

Claim 2. $\tilde{g} \simeq \tilde{N}$ is a proper homotopy equivalence.

<u>Proof of the claim.</u> We will show that the lifted homotopy $\tilde{H} : \tilde{g} \times I \to \tilde{g}$ is a proper map. Consider any compact subset $C \subset \tilde{g}$. We want to show that $\tilde{H}^{-1}(C) = \{(z,t) \in \tilde{g} \times I \mid \tilde{H}(z,t) \in C\}$ is compact in $\tilde{g} \times I$. For any z in $\tilde{g}$, find a neighborhood N_z of z such that $\theta^{-1}(\theta(N_z))$ is a disjoint union of open subsets in $\tilde{g}$, each of which is homeomorphic to N_z, i.e. $\theta(N_z)$ is an evenly covered open set in g and $\overline{N_z}$ is compact in $\tilde{g}$. (Here we assume that $\tilde{g}$ is locally compact Hausdorff.) Note $\tilde{H}(\overline{N_z} \times I)$ is compact in $\tilde{g}$. By applying Lemma 3.2.7 with two compact sets $\tilde{H}(\overline{N_z} \times I)$ and C, we find the subset of the covering transformation group, $G_z = \{f \in G(\tilde{g} \mid g) \mid f \circ \tilde{H}(\overline{N_z} \times I) \cap C \neq \emptyset\}$ is finite for each z. We write $G_z = \{f_{z,1}, f_{z,2}, \cdots, f_{z,n(z)}\}$ where $n(z)$ is a positive integer. Now the collection $\{\theta(N_z) \mid z \in \tilde{g}\}$ is an open cover of the compact space g. Choose a finite subcover $\{\theta(N_{z_1}), \theta(N_{z_2}), \cdots \theta(N_{z_m})\}$ of g. For these z_i's, we have corresponding finite sets $G_{z_1}, \cdots G_{z_m}$.

Define sets

$$L_j = \bigcup_{i=1}^{n(z_j)} \{f_{z_j,i}(\overline{N_{z_j}}) \mid f_{z_j,i} \in G_{z_j}\}.$$

Let $L = \bigcup_{j=1}^{m} L_j$. Note that L_j's and L are compact subsets of $\tilde{g}$.

<u>Subclaim.</u> $\tilde{H}^{-1}(C) \subset L \times I$

We check the subclaim as follows. Consider any $(z,t) \in \tilde{H}^{-1}(C)$. Then $\tilde{H}(z,t) \in C$. Since $\theta(z)$ is in g, there exists N_{z_l} such that $\theta(z) \in \theta(N_{z_l})$. Then $z \in \theta^{-1}[\theta(N_{z_l})]$ and there exists $\hat{f} \in G(\tilde{g} \mid g)$ such that $z \in \hat{f}(N_{z_l})$. There exists a point $w \in N_{z_l}$ such that $\hat{f}(w) = z$.

Recall $G_{z_l} = \{f \in G(\tilde{g} \mid g) \mid f \circ \tilde{H}(\overline{N_z} \times I) \cap C \neq \emptyset\}$. Since $\tilde{H}(z,t) \in C$, we have $\tilde{H}(\{z\} \times I) \cap C \neq \emptyset$.

Now $\tilde{H}(z, \ ) : I \to \tilde{g}$ is a path for each z beginning at $\tilde{H}(z,0) = z$. Also $\tilde{H}(w, \ ) : I \to \tilde{g}$ is a path starting at $\tilde{H}(w,0) = w = \hat{f}(z)$.

By Lemma 3.2.8 since $\hat{f} \in G(\tilde{g} \mid g)$ we have $\tilde{H}(z,t) = \tilde{H}(\hat{f}(w),t) = \hat{f} \circ \tilde{H}(w,t)$. Since $\tilde{H}(z,t) \in C$, $\hat{f} \circ \tilde{H}(w,t) \in C$. But $\hat{f} \circ \tilde{H}(w,t) \in \hat{f} \circ \tilde{H}(N_{z_l},t)$ so that $\hat{f} \circ \tilde{H}(N_{z_l} \times I) \cap C \neq \emptyset$. This implies that $\hat{f} \in G_{z_l}$. Also we have $z \in \hat{f}(N_{z_l})$. Then $\hat{f} \in G_{z_l}$ and $z \in \hat{f}(N_{z_l})$ imply that $z \in L_l = \bigcup_{i=1}^{n(z_l)} \{f_{z_l,i}(\overline{N_{z_l}}) \mid f_{z_l,i} \in G_{z_l}\}$. Thus $z \in L$ and $(z,t) \in L \times I$. Since $(z,t) \in \tilde{H}^{-1}(C)$ is arbitrary, we have proven that $\tilde{H}^{-1}(C) \subset L \times I$ and the subclaim is established.

Now $\tilde{H}^{-1}(C)$ is a compact subset as a closed subset of the compact set $L \times I$. Thus $\tilde{H} : \tilde{g} \times I \to \tilde{g}$ is a proper homotopy. □

With these preliminaries, we can obtain some results.

3.3. THE CASE OF ASPHERICAL MANIFOLDS

In this section we present several classes of manifolds which turn out to be PL h-fibrators.

Theorem 3.3.1 *Let N^n be a closed, orientable, connected, aspherical n-manifold with a hopfian fundamental group. Then N^n is a PL h-fibrator if it is a codimension-2 PL h-fibrator.*

PROOF: We have to show that any N^n-type map $p : M^{n+k} \to B^k$ is an approximate fibration for all $k \geq 3$. We use induction on k. By the hypothesis the statement is true for $k = 2$. Assuming the statement is true for all $j \leq k-1$, we will prove the statement for $j = k$ where $k \geq 3$. Let $p : M^{n+k} \to B^k$ be any N^n-type map between manifolds. Consider any v in B and get L, S, L', and S' be as in the Codimension Reduction Lemma. Let $D : S' \times I \to S'$ be a strong deformation retraction and $r = D_1 : S' \to p^{-1}v$ be obtained by the Codimension Lemma. The restriction map $p \mid L' : L' \to L$ is an N^n-type map of codimension-$(k-1)$. By the induction hypothesis, $p \mid L'$ is an approximate fibration. Then we have the following homotopy exact sequence associated with the approximate fibration $p \mid L'$ for any b in L:

$$\to \pi_2(L) \to \pi_1(p^{-1}b) \to \pi_1(L') \to \pi_1(L) \to$$

Here B is a k-manifold and the link L of v in B is homeomorphic to S^{k-1}. Since $k \geq 3$, the last group $\pi_1(L)$ in the above exact sequence is trivial. Then by exactness the inclusion induced middle homomorphism $\pi_1(p^{-1}b) \to \pi_1(L')$ is an epimorphism. Now consider the composite homomorphism

$$\pi_1(p^{-1}b) \xrightarrow{i_\#} \pi_1(L') \xrightarrow{j_\#} \pi_1(S') \xrightarrow{r_\#} \pi_1(p^{-1}v).$$

The inclusion induced middle homomorphism in the above is an epimorphism by the Basic Lemma and the last one is an isomorphism because it is induced by a collapsing map. Since the first one is an epimorphism by the above homotopy exact sequence, the composite homomorphism is also an epimorphism between a hopfian group $\pi_1(p^{-1}b) \cong \pi_1(p^{-1}v) \cong \pi_1(N)$. So $(r \mid p^{-1}b)_\# : \pi_1(p^{-1}b) \to \pi_1(p^{-1}v)$ is an isomorphism. Since N^n is an aspherical manifold, we conclude that the map $r \mid p^{-1}b : p^{-1}b \to p^{-1}v$ is a homotopy equivalence. This holds for any b in L and thus p is an approximate fibration by the Fundamental Recognition Theorem. Thus N^n is a codimension-k PL h-fibrator and hence it is PL h-fibrator by induction. $\square$

The following proposition shows that we only need to check the case of one dimension higher than the dimension of a manifold to determine whether it is a PL h-fibrator if it is a hopfian manifold with a hopfian fundamental group.

Proposition 3.3.2 *Let N^n be a hopfian n-manifold with a hopfian fundamental group. Suppose N^n is a codimension-$(n+1)$ PL h-fibrator. Then it is a PL h-fibrator.*

PROOF: We need to show that any N^n-type map $p : M^{n+k} \to B^k$ is an approximate fibration for any k greater than $n+1$. We prove this by induction on k. It is true for $k = n+1$ by the hypothesis. Suppose N^n is a codimension-j h-fibrator for all $n+1 \leq j \leq k$. We prove the case of codimension $k+1$. Consider any N^n-type map $p : M^{n+(k+1)} \to B^{k+1}$. The restriction of p to L', $p \mid L' : L' \to L \cong S^k$ is an approximate fibration and the associated homotopy exact sequence gives:

$$\to \pi_2(L) \to \pi_1(p^{-1}b) \to \pi_1(L') \to \pi_1(L) \to$$

Since $L \cong S^k$ and $k \geq 2$ the rightmost group of the above exact sequence is trivial, and the middles homomorphism $\pi_1(p^{-1}b) \to \pi_1(L')$ is an epimorphism. Then as in the proof of the previous proposition, by using the Basic Lemma, we can see that the induced composite homomorphism $(r \mid p^{-1}b)_{\#} : \pi_1(p^{-1}b) \to \pi_1(p^{-1}v)$ between a hopfian group is an isomorphism. Next we show that the degree of the map $r \mid p^{-1}b$ is one by proving that the composite homomorphism,

$$H_n(p^{-1}b) \to H_n(L') \to H_n(S') \to H_n(p^{-1}v)$$

is an isomorphism. Note that this induced by $r \mid p^{-1}b$. Consider the long exact homology sequence of a pair (S', L'):

$$\to H_{n+1}(S', L') \to H_n(L') \to H_n(S') \to H_n(S', L') \to$$

Note that $H_{n+1}(S', L') \cong H_{n+1}(S', S' - p^{-1}v) \cong H^k(p^{-1}v) \cong H^k(N^n) \cong 0$. The first isomorphism is due to the fact that $L' \simeq S' - p^{-1}v$ and the second one is due to Lefschetz duality. The last one is the result of our assumption that $k \geq n+2$. Similarly $H_n(S', L')$ is trivial so that the middle homomorphism $H_n(L') \to H_n(S')$ is an isomorphism. Next, from the Wang homology exact sequence of the approximate fibration $p \mid L' : L' \to L \cong S^k$, we have, for any b in L, the exact sequence,

$$\to H_{n-k-1}(p^{-1}b) \to H_n(p^{-1}b) \to H_n(L') \to H_{n-k} \to.$$

The middle homomorphism is an isomorphism since the two outside groups are trivial. Then the composite homomorphism $H_n(p^{-1}b) \to H_n(L') \to H_n(S') \to H_n(p^{-1}v)$ is an isomorphism. We conclude that the map $r \mid p^{-1}b$ is a degree one map. Since this map also induces an isomorphism on its fundamental group and N^n is a hopfian manifold, $r \mid p^{-1}b$ is a homotopy equivalence. By the Fundamental Recognition Theorem, p is an approximate fibration and N is a PL h-fibrator. $\square$

The following theorem was obtained by R. J. Daverman for the more general case of a non- PL setting but we present it in the PL case with a PL proof.

Theorem 3.3.3 *Let N^n be a closed, orientable, $(k-1)$-connected n-manifold where $n > k \geq 2$. Then N^n is a codimension-k PL h-fibrator.*

PROOF: Since N^n is simply connected, N^n is a codimension-2 h-fibrator by Theorem 1.3.2. Assuming N^n is a codimension-$(j-1)$ PL h-fibrator for all $j \leq k$, we prove that any N^n-type map $p : M^{n+j} \to B^j$ is an approximate fibration, i.e. N^n is a codimension-j PL h-fibrator. Fix v in B. Since $p \mid L' : L' \to L \cong S^{j-1}$ is a map with $(k-1)$-connected fibers, $H_{j-1}(L') \cong H_{j-1} \cong \mathbb{Z}$, by the Vietoris-Begle mapping theorem. Consider the Cohomology exact sequence of the pair (S', L'):

$$\to H^n(S', L') \to H^n(S') \to H^n(L') \to H^{n+1}(S', L') \to$$

Now $H^{n+1}(S', L') \cong H^{n+1}(S', S' - p^{-1}v) \cong H_{j-1}(p^{-1}v) \cong H_{j-1}(N^n) \cong 0$ since $dim(S') = n + j$, $j \leq k$, and N^n is $(k-1)$-connected. So the last group in the above exact sequence is trivial. Also $H^n(S') \cong H^n(N^n) \cong \mathbb{Z}$. By Poincare duality $H^n(L') \cong H_{j-1}(L')$ since dim $(L') = n + j - 1$. Since $H_{j-1}(L') \cong \mathbb{Z}$ as obtained above as a consequence of Vietoris-Begle mapping theorem, the exactness of the above sequence forces that $H^n(S') \to H^n(L')$ is an epimorphism from to itself, $\mathbb{Z}$. So the inclusion induced homomorphism $H^n(S') \to H^n(L')$ is an isomorphism. Next, the Wang cohomology exact sequence of the approximate fibration $p \mid L' : L' \to L \cong S^{j-1}$ gives, for any b in L,

$$\to H^{n-j+1}(p^{-1}b) \to H^n(L') \to H^n(p^{-1}b) \to H^{n-(j-1)+1}(p^{-1}b) \to .$$

Now $H^{n-j+1}(p^{-1}b) \cong H^{n-j+1}(N^n) \cong H_{j-1}(N^n) \cong 0$. Similarly the right-most group of the above exact sequence is also trivial so that $H^n(L') \to H^n(p^{-1}b)$ is an isomorphism. Then the composite homomorphism $H^n(p^{-1}v) \to H^n(S') \to H^n(L') \to H^n(p^{-1}b)$ is an isomorphism and $r \mid p^{-1}b$ is a degree one map. Since N is a hopfian manifold, p is an approximate fibration. $\square$

3.4. THE CASE OF MANIFOLDS WITH SPECIAL COHOMOLOGY RING STRUCTURE

In this section we study a class of manifolds with some special cohomology ring structure.

First we need a lemma:

Lemma 3.4.1 *Let N^n be a closed, orientable, connected n-manifold such that $H^n(N^n)$ is in the subring of $H^*(N^n)$, generated by $H^m(N^n)$, for some integer $m \geq 1$. Suppose a map $f : N \to N$ induces an epimorphism $f^* : H^m(N) \to H^m(N)$. Then f is a degree one map.*

PROOF: Since N is a compact manifold, $H^m(N)$ is finitely generated. There exist $\alpha_1, \alpha_2, \cdots, \alpha_j$ in $H^m(N)$ such that $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_j$ generates $H^n(N) = \mathbb{Z}$ by the given hypothesis. Since $f^* : H^m(N) \to H^m(N)$ is an epimorphism, for each α_i in $H^m(N)$ there exists β_i in $H^m(N)$ such that $f^*(\beta_i) = \alpha_i$. Then $f^*(\beta_1 \cup \cdots \cup \beta_j) = f^*(\beta) \cup \cdots \cup f^*(\beta) = \alpha_1 \cup \cdots \cup \alpha_j$. That is, there exists an element $\beta_1 \cup \cdots \cup \beta_j$ in H^n(N) which is mapped into a generator $\alpha_1 \cup \cdots \cup \alpha_j$ of $H^n(N)$. Thus $f^* : H^n(N) \to H^n(N)$ is an epimorphism between copies of a hopfian group $\mathbb{Z}$. We conclude that f is a degree one map. □

Now we present the main theorem in this section.

Theorem 3.4.2 *Let N^n be a closed, orientable, connected, hopfian manifold with a hopfian fundamental group. Suppose $H^j(N^n) = 0$ for $0 < j < m$ for some integer $m \geq 1$ and $H^n(N^n)$ is in the subring of $H^*(N^n)$, generated by $H^m(N)$. Then N^n is a PL h-fibrator if it is a codimension-$(m+1)$ PL h-fibrator.*

PROOF: We will show that under the hypothesis any N^n-type map $p : M^{n+k} \to B^k$ is an approximate fibration. By the hypothesis, p is an approximate fibration for $k = m + 1$. In order to use an induction on the codimension k, assume that p is an approximate fibration for all $k \leq j - 1$, $j \geq m + 2$. To prove the case when $k = j$, consider any N^n-type map $p : M^{n+j} \to B^j$. Let v in B be arbitrary. Using the Codimension Reduction Lemma, we get S, L, S', L' for v. Since $p \mid L' : L' \to L \cong S^{j-1}$ is an approximate fibration, we have the associated homotopy exact sequence:

$$\to \pi_2(L) \to \pi_1(p^{-1}b) \to \pi_1(L') \to \pi_1(L) \to .$$

Since $L \cong S^{j-1}$ and $j \geq 3$ by our hypothesis, the rightmost group of the above exact sequence is trivial so that $\pi_1(p^{-1}b) \to \pi_1(L')$ is an epimorphism. Then the composite homomorphism induced by the map $r \mid p^{-1}b$, $\pi_1(p^{-1}b) \to \pi_1(L') \to \pi_1(S') \to \pi_1(p^{-1}v)$ is an epimorphism on a hopfian group so that it is an isomorphism. Now if we can show that $r \mid p^{-1}b$ is a degree one map, then we are done since N is a hopfian manifold. By Lemma 3.4.1, we only need to show that

$$H^m(p^{-1}v) \to H^m(S') \to H^m(L') \to H^m(p^{-1}b)$$

is an epimorphism. The first homomorphism in this exact sequence is an isomorphism since S' collapses to $p^{-1}v$.

For the middle homomorphism, consider a cohomology exact sequence of the pair (S', L'):

$$\to H^m(S', L') \to H^m(S') \to H^m(L') \to H^{m+1}(S', L') \to .$$

Then the leftmost group is trivial since $H^m(S', L') \cong H^m(S', S' - p^{-1}v) \cong H_{n+j-m}(p^{-1}v) \cong H_{n+j-m}(N^n)$, $j \geq m+2$, and N is an n-manifold. Similarly, the last group of the above exact sequence is also trivial. So the middle homomorphism $H^m(S') \to H^m(L')$ is an isomorphism.

It remain to check the last homomorphism.

<u>Claim.</u> $H^m(L') \to H^m(p^{-1}b)$ is an isomorphism.

To see this consider the Wang cohomology exact sequence of the fibration $p \mid L' : L' \to L \cong S^{j-1}$:

$$\to H^{m-1-(j-1)+1}(p^{-1}b) \to H^m(L') \to H^m(p^{-1}b) \to H^{m-(j-1)+1}(p^{-1}b) \to .$$

The first group $H^{m-j+1}(p^{-1}b) \cong H^{m-j+1}(N^n)$ is trivial since $j \geq m+2$. For the last group, we have $H^{m-j+2}(p^{-1}b) \cong H^{m-j+2}(N^n)$ and we have two cases. If $j \geq m+3$, then we have an isomorphism. If $j = m+2$, then this last group is isomorphic to $\mathbb{Z}$ and we analyze this case as follows. In this case the above exact sequence becomes:

$$0 \to H^m(L') \to H^m(p^{-1}b) \to \mathbb{Z} \to$$

Now from the above $H^m(L') \cong H^m(S') \cong H^m(N^n)$ so that the second and third groups in the exact sequence are isomorphic and they are finitely generated abelian groups. Since the middle homomorphism is injective and all subgroups of $\mathbb{Z}$ are of the form $q\mathbb{Z}$ for some integer q, we conclude that q has to be zero and $H^m(L') \to H^m(p^{-1}b)$ is also an isomorphism.

Now with this claim we have an epimorphism $H^m(p^{-1}v) \to H^m(S') \to H^m(L') \to H^m(p^{-1}b)$. Hence $r \mid p^{-1}b$ is a degree one map by the Lemma 3.4.1 and the conclusion of the theorem follows. □

Proposition 3.4.3 *$\mathbb{Q}P^n$ is a codimension-5 PL h-fibrator.*

PROOF: Let N denote the 4n-manifold $\mathbb{Q}P^n$. Since N is 3-connected, it is a codimension-4 PL h-fibrator. Consider any N-type map $P : M^{4n+5} \to B^5$. Choose any v in B and get S, L, S', and L' as in the Codimension Reduction Lemma and as in the hypothesis of the Fundamental Recognition Theorem. Since N is simply connected and it is a hopfian manifold, it suffices to show that $r \mid p^{-1}b$ is a degree one map for any b in L. Then by Lemma 3.4.1 we only need to check that the composite homomorphism

$$H^4(p^{-1}v) \to H^4(S') \to H^4(L') \to H^4(P^{-1}b)$$

is an epimorphism since the cohomology ring of N, H^* is generated by $H^4(N)$. Since $H^4(p^{-1}v) \to H^4(S')$ is an isomorphism, we only need the following claim:

<u>Claim.</u> $H^4(S') \to H^4(L') \to H^4(P^{-1}b)$ is an epimorphism.

We prove the claim as follows. Since $p \mid L' : L' \to L \cong S^4$ is an approximate fibration, we have the Serre cohomology exact sequence [20]: $H^i(L) = 0$ for $1 \leq i < 4$ and $H^j(N) = 0$ for $1 \leq j < 4$ so that

$$\to H^3(p^{-1}b) \to H^4(L) \to H^4(L') \to H^4(p^{-1}b) \to H^5(L) \to$$

The two outside groups are trivial so that

$$H^4(L') \cong (p \mid L')^*((H^4(L)) \oplus \tilde{i}^*(H^4(p^{-1}b)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Let $\tau \in H^4(L) \cong \mathbb{Z}$ be its generator and $\tau' = (p \mid L')^*(\tau)$. And let $\gamma \in H^4(p^{-1}b) \cong \mathbb{Z}$ so that $\gamma' = \tilde{i}^*(\gamma)$. Then γ' and τ' generates $H^4(L')$. Note that $\tau \cup \tau = 0$ since it belongs to the trivial group $H^8(L)$. So $\tau' \cup \tau' = 0$. Let $\alpha \in H^4(S') \cong H^4(p^{-1}v) \cong H(N) \cong \mathbb{Z}$ be its generator. Then $\alpha' = j^*(\alpha) = a\gamma' + b\tau'$. Note that $H^8(S') \cong \mathbb{Z}$ and $\alpha \cup \alpha$ is its generator. From the cohomology exact sequence of the pair (S', L'), we have

$$\to H^8(S', L') \to H^8(S') \to H^8(L') \to H^9(S', L') \to H^9(S') \to$$

Here, since dim $S' = 4n + 5$, $H^8(S', L') \cong H_{4n-3}(p^{-1}v) = 0$ and $H^9(S', L') \cong H_{4n-4}(p^{-1}v) \cong H_4(p^{-1}v) \cong \mathbb{Z}$ by duality. So $H^8(L') \cong j^*(H^8(S')) \oplus \mathbb{Z}$ and $\alpha' \cup \alpha'$ is one of the generator of $H^8(L')$. And $\alpha' \cup \alpha' = j^*(\alpha) \cup j^*(\alpha) = (a\gamma' + b\tau') \cup (a\gamma' + b\tau') = a[a(\gamma' \cup \gamma') + 2b(\gamma' \cup \tau')]$ is one of the basis elements of $H^8(L')$. We conclude that $a = \pm 1$ so that $\alpha \mapsto \pm\gamma' + b\tau'$ and $\gamma' \mapsto \gamma$. Thus $H^4(S') \to H^4(p^{-1}b)$ is an epimorphism and the claim is proven.

By combining above results we conclude that the composite homomorphism $H^4(p^{-1}v) \to H^4(S') \to H^4(L') \to H^4(P^{-1}b)$ is an epimorphism to finish the proof. $\square$

Similarly we can prove that $\mathbb{C}P^n, n > 1$ is a codimension-3 PL h-fibrator.

Then as a corollary of Theorem 3.4.2 and above proposition we have

Theorem 3.4.4 *$\mathbb{Q}P^n$ and $\mathbb{C}P^n, n > 1$ are PL h-fibrators.*

Note that $\mathbb{C}P^1 \cong S^2$ fails to be a codimension-2 fibrator but a four dimensional manifold $\mathbb{C}P^2$ is a PL h-fibrator. In the next chapter we will see that $\mathbb{C}P^2 \# \mathbb{C}P^2$ is also a PL h-fibrator.

4. THE CASE OF $\mathbb{C}P^2 \# \mathbb{C}P^2$

R. J. Daverman has investigated extensively which 2-manifolds and 3-manifolds are codimension k-fibrators [6,7]. Among the simply connected 4-manifolds, S^4 is a codimension-4 fibrator but it fails to be a PL h-fibrator. $S^2 \times S^2$ is not a codimension-3 fibrator and $S^1 \times S^3$ is not even a codimension-2 fibrator.

In this chapter we study cases of connected sums of a finite number of copies of $\mathbb{C}P^2$ as a first step in determining which simply connected four manifolds are PL h-fibrators and we prove that a connected sum of two copies of $\mathbb{C}P^2$ is a PL h-fibrator.

It is interesting to observe that the connected sum does not preserve the property of being a fibrator, i.e. $M\#M$ is not necessarily a fibrator even though M is. For example, $\mathbb{R}P^3\#\mathbb{R}P^3$ is not a codimension-3 fibrator though $\mathbb{R}P^3$ is [4]. On the other hand the connected sum of two tori $T^2\#T^2$ is a PL h-fibrator while T^2 itself is not a codimension-2 fibrator.

4.1. THE COHOMOLOGY RING STRUCTURE

In this section we include some material that we need in order to investigate the case of a connected sum of finitely many copies of $\mathbb{C}P^2$.

Denote a connected sum of k copies of $\mathbb{C}P^2$ by ${N_k}^4$. That is, ${N_k}^4 = \mathbb{C}P^2\#\cdots\#\mathbb{C}P^2$.

Let M_1 and M_2 be connected, orientable, closed n-dimensional manifolds and let $\tilde{M}_i$ be M_i with an open n-ball removed. Note that $\tilde{M}_i$'s are n-dimensional manifolds with their boundaries homeomorphic to S^{n-1}. Recall that the connected sum $M_1\#M_2$ is obtained by gluing $\tilde{M}_1$ and $\tilde{M}_2$ together by an orientation preserving homeomorphism between the spheres S^{n-1} which are the boundaries of $\tilde{M}_i$'s.

Using a Mayer-Vietoris sequence we have that

$$H^j(M_1\#M_2) \cong \begin{cases} H^j(M_i) \cong \mathbb{Z} & \text{for j=n, } 0 \\ H^j(M_1) \oplus H^j(M_2) & \text{for } 0 < j < n \end{cases}$$

Also the above isomorphism preserves cup products and

if $u \in H^i(M_1), v \in H^j(M_2), i, j > 0$, then $u \cup v = 0$ in $H^{i+j}(M_1\#M_2)$.

Now for the case of two copies of $\mathbb{C}P^2$ we let ${N_2}^4 = \mathbb{C}P^2\#\mathbb{C}P^2$. Then

$$H^j({N_2}^4) = \begin{cases} \mathbb{Z} & \text{for j=4, } 0 \\ \mathbb{Z}\oplus\mathbb{Z} & \text{for j=2} \\ 0 & \text{otherwise} \end{cases}$$

Let ξ,η be two generators of $H^2(N^4) = \mathbb{Z}\oplus\mathbb{Z}$. Then $\xi \cup \xi = \eta \cup \eta$ is a generator of $H^4(N^4) = \mathbb{Z}$ and $\xi \cup \eta = 0$.

More generally

$$H^j({N_k}^4) = \begin{cases} \mathbb{Z} & \text{for j=4, } 0 \\ \mathbb{Z}\oplus\cdots\oplus\mathbb{Z} & \text{for j=2} \\ 0 & \text{otherwise} \end{cases}$$

The cohomology ring $H^*({N_k}^4)$ of ${N_k}^4$ is generated by a free abelian group of rank k, $H^2({N_k}^4)$. In particular $H^4({N_k}^4)$ is generated by elements of $H^2({N_k}^4)$ and the 4-manifold ${N_k}^4$ satisfies the hypothesis of special cohomology ring structure of Theorem 3.4.2.

Even though we do not have a result for ${N_k}^4, k > 2$, it is worth discussing the problem in this general setting.

We have to prove that any ${N_k}^4$-type map $p : M^{4+3} \to B^3$ is an approximate fibration to conclude that ${N_k}^4$ is a PL h-fibrator.

Then according to Theorem 3.4.2 it suffices to show that the composite homomorphism

$$H^2(p^{-1}v) \xrightarrow{\cong} H^2(S') \xrightarrow{j^*} H^2(L') \xrightarrow{i^*} H^2(p^{-1}b)$$

is an epimorphism. (Here we use notation as in the Codimension Reduction Lemma.)

Since the first homomorphism in the above exact sequence is an isomorphism, we only need to determine whether $H^2(S') \to H^2(L') \to H^2(p^{-1}b)$ is an epimorphism.

For this we study the approximate fibration $p \mid L' : L' \to L \cong S^2$. From the Serre exact sequence [20] associated with $p \mid L'$, we have a split exact sequence

$$0 \to H^2(L) \cong \mathbb{Z} \xrightarrow{(p \mid L')^*} H^2(L') \xrightarrow{i^*} H^2(p^{-1}b) \cong (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \to 0.$$

Then $H^2(L') \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})$ is generated by $\tau', (\alpha'_1, \cdots \alpha'_k)$ which are images of generators under a suitable homomorphism [See the next section for more detail].

Also from the exact sequence of the pair (S', L') we have

$$0 \cong H^2(S', L') \to H^2(S') \cong (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \xrightarrow{j^*} H^2(L') \to H^3(S', L') \cong \mathbb{Z} \to 0 \cong H^3(S')$$

Since above exact sequence splits, $H^2(L') \cong (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \oplus \mathbb{Z}$ is generated by $(\rho'_1, \cdots \rho'_k), \tilde{\Lambda}'$.

Thus we have two sets of bases of the free abelian group $H^2(L')$ of rank $(k+1)$ to work with. Using the fact that $\rho'_i \cup \rho'_i$ is a basis element of $H^4(L')$, we can investigate whether the composite homomorphism of interest is an epimorphism.

The fundamental tools for this analysis are the dual pairing bases of $H^2(L') \cong H^4(L')$ [Theorem 68.1 in [16]] and a property of the Wang exact sequence [Theorem VII 3.1 [22]].

These enable us to find cup products of basis elements of $H^2(L')$ in terms of a dual pairing basis of $H^4(L')$ and we can get information about the homomorphisms i^* and j^*. We have not been completely successful in this endeavor except in the case of $k = 2$ and we will present the analysis for ${N_2}^4$ in the next section.

4.2. THE CASE OF $\mathbb{C}P^2\#\mathbb{C}P^2$

As in the previous section we let $N^4 = \mathbb{C}P^2\#\mathbb{C}P^2$. Then $\pi_1(N^4) = 1$.

Note that N^4 is a hopfian manifold as it is simply connected. And $N^4 = \mathbb{C}P^2\#\mathbb{C}P^2$ is a codimension-2 PL h-fibrator since it is simply connected.

The main theorem in this chapter is the following.

Theorem 4.2.1 $N^4 = \mathbb{C}P^2\#\mathbb{C}P^2$ *is a PL h-fibrator.*

PROOF: Since N^4 satisfies the hypothesis of Theorem 3.4.2, this is a corollary of Theorem 3.4.2 and the next Theorem 4.2.2. □

Theorem 4.2.2 $N^4 = \mathbb{C}P^2\#\mathbb{C}P^2$ *is a codimension-3 PL h-fibrator.*

PROOF: Consider any N^4-type PL map $p : M^{4+3} \to B^3$ from a connected, orientable 7-manifold M onto a 3-manifold B. Let $v \in B$ be arbitrary. Using the Codimension Reduction Lemma, we get L, S, L', and S' such that $S' \searrow p^{-1}v$. Note that N^4 is a simply connected hopfian manifold and hence it is a codimension-2 PL fibrator by Theorem 3.3.3. Also $H^4(N^4)$ is in the subring of $H^*(N^4)$, generated by $H^2(N^4)$.

Then by the Fundamental Recognition Theorem and by Lemma 3.4.1 we only need to check whether the following composite homomorphism

$$H^2(p^{-1}v) \xrightarrow{\cong} H^2(S') \xrightarrow{j^*} H^2(L') \xrightarrow{i^*} H^2(p^{-1}b)$$

is an epimorphism for any b in L.

Since $p \mid L' : L' \to L \cong S^2$ is an approximate fibration (N^4 is a codimension-2 fibrator), we have a Serre exact sequence [9]

$$0 \cong H^1(p^{-1}b) \to H^2(L) \xrightarrow{(p \mid L')^*} H^2(L') \xrightarrow{i^*} H^2(p^{-1}b) \to H^3(L) \cong 0$$

The first and last groups are trivial. Let τ be a generator of $H^2(L) \cong H^2(S^2) \cong \mathbb{Z}$ and α and β be generators of $H^2(p^{-1}b) \cong \mathbb{Z} \oplus \mathbb{Z}$. Since the above exact sequence splits, there exists a homomorphism $s^* : H^2(p^{-1}b) \to H^2(L')$ such that $i^* \circ s^* = \mathbb{1}_{H^2(p^{-1}b)}$.

Define $\tau' = (p \mid L')^*(\tau)$ and $\alpha' = s^*(\alpha)$, $\beta' = s^*(\beta)$. Then $H^2(L') \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z})$ is generated by τ', α', and β'.

Note $\tau \cup \tau = 0$ so that $\tau' \cup \tau' = 0$. Although $\alpha \cup \beta = 0$ and $\alpha \cup \alpha = \beta \cup \beta$, we note that $\alpha' \cup \beta' = 0$, and $\alpha' \cup \alpha' = \beta' \cup \beta'$ may not be true since the section homomorphism s^* is not necessarily induced by a map. That is, cup products are not necessarily preserved under s^*. We also note that $\alpha \cup \alpha = \beta \cup \beta$ generates $H^4(p^{-1}b) \cong \mathbb{Z}$ as discussed in the previous section.

We have the exact sequence of the pair (S', L'):

$$0 \cong H^2(S', L') \to H^2(S') \xrightarrow{j^*} H^2(L') \to H^3(S', L') \to H^3(S') \cong 0$$

Now $H^2(S', L') \cong H^2(S', S' - p^{-1}v) \cong H_{7-2}(p^{-1}v) \cong 0$ and the last group is trivial. Since $H^3(S', L') \cong H_4(p^{-1}v) \cong \mathbb{Z}$, the above exact sequence splits. Let (ρ, ω) be generators of $H^2(S') \cong H^2(p^{-1}v) \cong (\mathbb{Z} \oplus \mathbb{Z})$ and let $\tilde{\Lambda}$ be a generator of $H^3(S', L') \cong H_4(p^{-1}v) \cong \mathbb{Z}$.

Then $H^2(L') \cong (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z}$ is generated by the corresponding images $(\rho', \omega'), \tilde{\Lambda}'$.

Also

$$0 \cong H^4(S', L') \to H^4(S') \xrightarrow{j^*} H^4(L') \to H^5(S', L') \to H^5(S') \cong 0$$

Note $H^4(S') \cong H^4(p^{-1}v) \cong \mathbb{Z}$ is generated by $\rho \cup \rho = \omega \cup \omega = \Lambda$. Since $H^5(S', L') \cong H_2(p^{-1}v) \cong (\mathbb{Z} \oplus \mathbb{Z})$, the above is a split exact sequence so that $H^4(L') \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z})$ is generated by $\Lambda', (\tilde{\rho}', \tilde{\omega}')$. Here $(\tilde{\rho}, \tilde{\omega})$ is a generating pair for $H^5(S', L') \cong H_2(p^{-1}v) \cong (\mathbb{Z} \oplus \mathbb{Z})$.

Now $\rho \cup \rho = \omega \cup \omega$ generates $H^4(S') \cong \mathbb{Z}$.

Since $H^4(L')$ has $j^*(H^4(S'))$ as one of its direct summands, $\rho' \cup \rho' = \omega' \cup \omega'$ is an element of some basis for $H^4(L')$. Also $\rho' \cup \omega' = 0$ since j^* is inclusion induced.

Since $H^4(L') \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \cong H^2(L')$, we have a dual pairing basis $\{\tau'', \alpha'', \beta''\}$ of $H^4(L')$ associated with $\{\tau', \alpha', \beta'\}$ of $H^2(L')$ such that $\tau' \cup \tau'' = \Gamma$, $\alpha' \cup \alpha'' = \Gamma$, $\beta' \cup \beta'' = \Gamma$, $\tau' \cup \alpha'' = 0 = \tau' \cup \beta''$, and $\alpha' \cup \beta'' = 0$ where Γ is a generator of $H^6(L')$ [[16], Theorem 68.1].

Next we need some facts about the Wang exact sequence of the approximate fibration $p \mid L' : L' \to L \cong S^2$. [22]

The Wang exact sequence gives

$$0 \cong H^1(p^{-1}b) \to H^{1-2+1}(p^{-1}b) \xrightarrow{h^*} H^2(L') \xrightarrow{i^*} H^2(p^{-1}b) \to H^{2-2+1}(p^{-1}b) \cong 0$$

The two outside groups are trivial. Let λ be a generator of $H^{1-2+1}(p^{-1}b) = H^0(p^{-1}b) \cong \mathbb{Z}$. Now α, β generates $H^2(p^{-1}b) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the above exact sequence splits so that $\lambda', (\alpha', \beta')$ generates $H^2(L') \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z})$.

Here $\lambda' = h^*(\lambda)$.

Note that h^* is not induced by a map [22] so that it does not preserve cup products. But for $x \in H^{q-n}(p^{-1}b), y \in H^r(L')$ we have $h^*(x \cup i^*(y)) = h^*(x) \cup y$ [Theorem VII, p.336 [22]].

By comparing the above split exact sequence with the Serre exact sequence, we conclude that $h^*(\lambda) = \lambda' = (p \mid L')^*(\tau) = \tau'$, since $i^*(\lambda') = 0$ and $i^*(\tau') = 0$. [The other direct summands $\mathbb{Z} \oplus \mathbb{Z}$ are generated by the same α', β' in both exact sequences.]

Our plan is to obtain necessary information about the ring structure of $H^*(L')$. In particular, we need $\tau' \cup \alpha'$, $\tau' \cup \beta'$, $\alpha' \cup \alpha'$, $\beta' \cup \beta'$, and $\alpha' \cup \beta'$ in $H^4(L')$. Since the generator $\lambda \in H^0(p^{-1}b)$ is a multiplicative unit element in the cohomological ring $H^*(p^{-1}b)$, according to G. Whitehead's Theorem VII 3.1 [22], with $\lambda \in H^{2-2}(p^{-1}b)$ and $\alpha' \cup \beta' \in H^4(L')$, we have

$$\begin{aligned} h^*(\lambda \cup i^*(\alpha' \cup \beta')) &= h^*(\lambda) \cup (\alpha' \cup \beta') = \tau' \cup \alpha' \cup \beta' \\ &= h^*(\lambda \cup \alpha \cup \beta) \\ &= h^*(\alpha \cup \beta) = h^*(0) = 0. \end{aligned}$$

So

$$\tau' \cup \alpha' \cup \beta' = 0.$$

Also by the same theorem, with $\alpha', \beta' \in H^2(L')$ [i.e. $q = 2$, $n = 2$, $r = 2$ in the statement of the Theorem]

$$h^*(\lambda \cup i^*(\alpha')) = h^*(\lambda) \cup \alpha' = \tau' \cup \alpha'$$

$$= h^*(\lambda \cup \alpha) = h^*(\alpha).$$

So $h^*(\alpha) = \tau' \cup \alpha'$. Similarly $h^*(\beta) = \tau' \cup \beta'$. That is, in the Wang exact sequence we have:

$$0 \to H^2(p^{-1}b) \xrightarrow{h^*} H^4(L') \to H^4(p^{-1}b) \to 0$$

Let Θ be a generator of $H^4(p^{-1}b) \cong \mathbb{Z}$. Then $H^4(L') \cong (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z}$ is generated by $(\tilde{\alpha}, \tilde{\beta}), \tilde{\Theta}$.

we have $\tilde{\alpha} = \tau' \cup \alpha', \tilde{\beta} = \tau' \cup \beta'$. [Here $\tilde{\alpha} = h^*(\alpha), \tilde{\beta} = h^*(\beta)$]

Now using the dual pairing basis $\{\tau'', \alpha'', \beta''\}$ of $H^4(L')$, we let

$$\alpha' \cup \beta' = A_0\alpha'' + B_0\beta'' + C_0\tau'' \quad (1)$$

$$\tilde{\alpha} = \tau' \cup \alpha' = A_1\alpha'' + B_1\beta'' + C_1\tau'' \quad (2)$$

$$\tilde{\beta} = \tau' \cup \beta' = A_2\alpha'' + B_2\beta'' + C_2\tau'' \quad (3)$$

$$\alpha' \cup \alpha' = A_3\alpha'' + B_3\beta'' + C_3\tau'' \quad (4)$$

$$\beta' \cup \beta' = A_4\alpha'' + B_4\beta'' + C_4\tau'' \quad (5)$$

and we determine the expansion coefficients A_i, B_i, and C_i's as follows. From (1), since $\tau' \cup \alpha' \cup \beta' = 0$, we have $0 = \tau' \cup (\alpha' \cup \beta') = C_0\tau' \cup \tau'' = C_0\Gamma$ so that $C_0 = 0$.

From (2), we obtain $0 = \beta' \cup (\tau' \cup \alpha') = B_1\beta' \cup \beta'' = B_1\Gamma$ so that $B_1 = 0$. Using (3), we get $0 = \alpha' \cup (\tau' \cup \beta') = A_2\Gamma$ so that $A_2 = 0$.

From (2) again, $\tau' \cup (\tau' \cup \alpha') = (\tau' \cup \tau') \cup \alpha' = 0$ so that $C_1 = 0$. From (3), $\tau' \cup (\tau' \cup \beta') = 0$ so that $C_2 = 0$.

Thus

$$\alpha' \cup \beta' = A_0\alpha'' + B_0\beta''$$

$$\tau' \cup \alpha' = A_1\alpha''$$

$$\tau' \cup \beta' = B_2\beta''$$

In order to determine $\alpha' \cup \alpha'$ and $\beta' \cup \beta'$, let

$$\alpha' \cup \alpha' = A_3\alpha'' + B_3\beta'' + C_3\tau'' \qquad (4')$$

$$\beta' \cup \beta' = A_4\alpha'' + B_4\beta'' + C_4\tau'' \qquad (5')$$

We use G. Whitehead's theorem again: with $\alpha', \beta' \in H^2(L')$, $i^*(\alpha') = \alpha \in H^2(p^{-1}b)$, we get

$$\begin{aligned} h^*(\alpha \cup i^*(\alpha')) &= h^*(\alpha) \cup \alpha' = \tilde{\alpha} \cup \alpha' = (\tau' \cup \alpha') \cup \alpha' \\ &= h^*(\alpha \cup \alpha) \end{aligned}$$

So $\tau' \cup \alpha' \cup \alpha' = h^*(\alpha \cup \alpha)$. Similarly we get $\tau' \cup \beta' \cup \beta' = h^*(\beta \cup \beta)$.

Now from the Wang exact sequence, we have

$$\to H^5(p^{-1}b) \to H^{5-2+1}(p^{-1}b) \xrightarrow{h^*} H^6(L') \xrightarrow{i^*} H^6(p^{-1}b) \to$$

The two outside groups are trivial so that $\mathbb{Z} \cong H^6(L') \cong H^{5-2+1}(p^{-1}b) = H^4(p^{-1}b)$. Note that $H^4(p^{-1}b) \cong \mathbb{Z}$ is generated by $\Theta = \alpha' \cup \alpha' = \beta' \cup \beta'$.

Thus $h^*(\alpha \cup \alpha) = \pm\Gamma = h^*(\beta \cup \beta)$ is a generator Γ of $H^6(L')$. That is,

$$h^*(\alpha \cup \alpha) = \tau' \cup \alpha' \cup \alpha' = \pm\Gamma$$

$$h^*(\beta \cup \beta) = \tau' \cup \beta' \cup \beta' = \pm\Gamma$$

Then from $(4')$ and $(5')$

$$\tau' \cup \alpha' \cup \alpha' = C_3\tau' \cup \tau'' = C_3\Gamma$$

$$\tau' \cup \beta' \cup \beta' = C_4\Gamma$$

so, $C_3, C_4 = \pm 1$.

Using $\alpha' \cup \beta' = A_0\alpha'' + B_0\beta''$, we have $\beta' \cup (\alpha' \cup \beta') = B_0\beta' \cup \beta'' = B_o\Gamma$ and $\beta' \cup (\alpha' \cup \beta') = \alpha' \cup (\beta' \cup \beta') = \alpha' \cup (A_4\alpha'' + B_4\beta'' + C_4\tau'') = A_4\Gamma$. So $A_4 = B_0$.

Similarly $\alpha' \cup (\alpha' \cup \beta') = A_0\Gamma = (\alpha' \cup \alpha') \cup \beta' = B_3\Gamma$ so that $B_3 = A_0$. Also, from $\tau' \cup \alpha'' = A_1\alpha''$, we get $A_1\Gamma = \alpha' \cup (\tau' \cup \alpha') = \tau' \cup (\alpha' \cup \alpha') = C_3\Gamma$ so that $A_1 = C_3$.

Using $\tau' \cup \beta' = B_2\beta''$, one gets $B_2\Gamma = \beta' \cup (\tau' \cup \beta') = \tau' \cup (\beta' \cup \beta') = C_4\Gamma$ so that $B_2 = C_4$.

Thus

$$\begin{array}{lcccccccc}
\alpha' \cup \alpha' & = & & & A_0\alpha'' & + & B_0\beta'' \\
\tau' \cup \alpha' & = & & & C_3\alpha'' & & \\
\tau' \cup \beta' & = & & & & & C_4\beta'' \\
\alpha' \cup \alpha' & = & C_3\tau'' & + & A_3\alpha'' & + & A_0\beta'' \\
\beta' \cup \beta' & = & C_4\tau'' & + & B_0\alpha'' & + & B_4\beta''
\end{array}$$

Here $C_3, C_4 = \pm 1$. Let $C_3 = m, C_4 = n$, where $m, n = \pm 1$ and let $A_0 = d, B_0 = e, A_3 = f$, and $B - 4 = g$.

With this notation we summarize our results in the following table:

$\cup$	τ'	α'	β'
τ'	0	$m\alpha''$	$n\beta''$
α'	$m\alpha''$	$m\tau'' + f\alpha'' + d\beta''$	$d\alpha'' + e\beta''$
β'	$n\beta''$	$d\alpha'' + e\beta''$	$n\tau'' + e\alpha'' + g\beta''$

Recall that we are checking whether the composite homomorphism

$$H^2(S') \xrightarrow{j^*} H^2(L') \xrightarrow{i^*} H^2(p^{-1}b)$$

is an epimorphism.

For j^* we need to analyze the portion of the exact sequence of the pair (S', L').

$$H^2(S',L') \to H^2(S') \xrightarrow{j^*} H^2(L') \to H^3(S',L') \to H^3(S')$$

Note $H^2(S',L') \cong H_{7-2}(p^{-1}b) = 0$ and $H^3(S') = 0$. Since $H^3(S',L') \cong H_4(p^{-1}b) \cong \mathbb{Z}$, the above is a split exact sequence so that $H^2(L') \cong (\mathbb{Z}\oplus\mathbb{Z})\oplus\mathbb{Z}$ is generated by $(\rho',\omega'), \tilde{\Lambda}'$.

Here $\rho' = j^*(\rho)$, $\omega' = j^*(\omega)$.

Note that ρ', ω' are basis elements for some basis of $H^2(L')$. Now we write ρ', ω' in $H^2(L')$ in terms of the basis $\{\tau', \alpha', \beta'\}$ of $H^2(L')$:

$$\rho' = a_1\tau' + b_1\alpha' + c_1\beta',$$

$$\omega' = a_2\ \tau' + b_2\alpha' + c_2\beta'.$$

We need expressions $\rho' \cup \rho', \omega' \cup \omega'$, and $\rho' \cup \omega'$. Using $\tau' \cup \tau' = 0$, one gets:

$$\rho' \cup \rho' = 2a_1b_1\tau' \cup \alpha' + 2a_1c_1\tau' \cup \beta' + 2b_1c_1\alpha' \cup \beta' + {b_1}^2\alpha' \cup \alpha' + {c_1}^2\beta' \cup \beta'.$$

$$\omega' \cup \omega' = 2a_2b_2\tau' \cup \alpha' + 2a_2c_2\tau' \cup \beta' + 2b_2c_2\alpha' \cup \beta' + {b_2}^2\alpha' \cup \alpha' + {c_2}^2\beta' \cup \beta'$$

and

$\rho' \cup \omega' = (a_1b_2 + a_2b_1)\tau' \cup \alpha' + (a_1c_2 + a_2c_1)\tau' \cup \beta' + (b_1c_2 + b_2c_1)\alpha' \cup \beta' + b_1b_2\alpha' \cup \alpha' + c_1c_2\beta' \cup \beta'$.

Note that $\rho \cup \omega = 0$ hence $\rho' \cup \omega' = 0$ and $\rho \cup \rho, \omega \cup \omega$ are generators of $H^4(S') \cong H^4(p^{-1}v) \cong \mathbb{Z}$.

Also $\rho' \cup \rho', \omega' \cup \omega'$ are basis elements of $H^4(L')$ because of the following exact sequence of the pair (S', L'):

$$H^4(S',L') \to H^4(S') \xrightarrow{j^*} H^4(L') \to H^5(S',L') \to H^5(S')$$

We can check that the two outside groups are trivial and $H^5(S', L') \cong H_2(p^{-1}v) \cong \mathbb{Z}\oplus\mathbb{Z}$. As before we let $\tilde{\rho}, \tilde{\omega}$ be generators of $H^2(p^{-1}v) \cong \mathbb{Z}\oplus\mathbb{Z}$ and $\rho\cup\rho = \omega\cup\omega = \Lambda$ be a generator of $H^4(S') \cong H^4(p^{-1}v) \cong \mathbb{Z} \oplus \mathbb{Z}$. Then $H^4(L') \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z})$ is generated by $\Lambda', (\tilde{\rho}', \tilde{\omega}')$ so that $\rho' \cup \rho'$ and $\omega' \cup \omega'$ are basis elements of some basis of $H^4(L')$.

Using the cup product table, we rewrite the above equations as follows:

$$\rho' \cup \rho' = (2ma_1b_1 + 2b_1c_1d + {b_1}^2 f + {c_1}^2 e)\alpha'' \tag{6}$$
$$+ (2na_1c_1 + 2b_1c_1e + {b_1}^2 d + {c_1}^2 g)\beta'' \tag{7}$$
$$+ (m{b_1}^2 + n{c_1}^2)\tau'' \tag{8}$$
$$\omega' \cup \omega' = (2ma_2b_2 + 2b_2c_2d + {b_2}^2 f + {c_2}^2 e)\alpha'' \tag{9}$$
$$+ (2na_2c_2 + 2b_2c_2e + {b_2}^2 d + {c_2}^2 g)\beta'' \tag{10}$$
$$+ (m{b_2}^2 + n{c_2}^2)\tau'' \tag{11}$$
$$0 = \rho' \cup \omega' = [m(a_1b_2 + a_2b_1) + d(b_1c_2 + b_2c_1) + b_1b_2f + c_1c_2e]\alpha'' \tag{12}$$
$$+ [n(a_1c_2 + a_2c_1) + e(b_1c_2 + b_2c_1) + b_1b_2d + c_1c_2g]\beta'' \tag{13}$$
$$+ (mb_1b_2 + nc_1c_2)\tau'' \tag{14}$$

Claim 1 $(b_1, c_1) = 1$ and $(b_2, c_2) = 1$, i.e. b_i, c_i are relatively prime. Otherwise, we can see that $\rho' \cup \rho'$ is a multiple of some element, contradicting the fact that $\rho' \cup \rho'$ is a basis element. Thus $(b_1, c_1) = 1$. Similarly, $\omega' \cup \omega'$ is a basis element so that $(b_2, c_2) = 1$. Now we can compare coefficients in above expressions since $\rho' \cup \rho' = \omega' \cup \omega'$, $\rho' \cup \omega' = 0$.

Then, equating (8) and (11) gives $mb_1^2 + nc_2^2 = mb_2^2 + nc_2^2$ and equation (14) gives $mb_1b_2 + nc_1c_2 = 0$.

Claim 2 At least one of $\{b_1, c_1, b_2, c_2\}$ is zero.

Proof Suppose not, i.e. all b_i, c_i are non-zero.

$$mb_1b_2 + nc_1c_2 = 0 \Rightarrow c_2 = -\frac{mb_1}{nc_1}b_2$$

then

$$\begin{aligned} mb_1^2 + nc_1^2 = mb_2^2 + n\tfrac{m^2b_1^2}{n^2c_1^2}b_2^2 &= mb_2^2\left(1 + \tfrac{mb_1^2}{nc_1^2}\right) \\ &= \tfrac{mb_2^2}{nc_1^2}\left(nc_1^2 + mb_1^2\right) \\ &\Rightarrow b_2^2 = \tfrac{n}{m}c_1^2. \end{aligned}$$

Note that if n and m have different signs, $b_2^2 = -c_1^2$ so that $b_2 = 0 = c_1$. This is a contradiction. So $nm = 1$, i.e. n, m have the same signs. Then $b_2^2 = c_1^2$ so that $b_2 = c_1$ or $b_2 = -c_1$. We check each case separately.

Case 1 $b_2 = c_1$ and $c_2 = -b_1$ From (12),

$$\begin{aligned} 0 &= m(a_1c_1 + a_2b_1) + d(-b_1^2 + c_1^2) + b_1c_1f - b_1c_1e \\ &\Rightarrow m(a_1c_1 + a_2b_1) = (e - f)b_1c_1 + d(b_1^2 - c_1^2) \qquad (15) \end{aligned}$$

From (13)

$$\begin{aligned} 0 &= n(-a_1b_1 + a_2c_1) + e(-b_1^2 + c_1^2) + b_1c_1d - b_1c_1g \\ &\Rightarrow n(a_1b_1 - a_2c_1) = (d - g)b_1c_1 - e(b_1^2 - c_1^2) \qquad (16) \end{aligned}$$

Equating (11) and (14), we get

$$\begin{aligned} & 2ma_1b_1 + 2b_1c_1d + b_1^2f + c_1^2e = 2ma_2b_2 + 2b_2c_2d + b_2^2f + c_2^2e \\ \Rightarrow\quad & 2ma_1b_1 + 2b_1c_1d + b_1^2f + c_1^2e = 2ma_2c_1 - 2b_1c_1d + c_1^2f + b_1^2e \\ \Rightarrow\quad & 2m(a_1b_1 - a_2c_1) + 4b_1c_1d + (b_1^2 - c_1^2)f + (c_1^2 - b_1^2)e = 0 \end{aligned}$$

Using (16) above, we can rewrite this as, (Recall that $m = n$)

$$2(d-g)b_1c_1 - 2e(b_1^2 - c_1^2) + 4b_1c_1d + (b_1^2 - c_1^2)(f-e) = 0$$

$$\Rightarrow \qquad (f-3e)b_1^2 + 2(3d-g)b_1c_1 - (f-3e)c_1^2 = 0 \qquad (17)$$

$$\Rightarrow \qquad b_1\left[(f-3e)b_1 + 2(3d-g)c_1\right] = (f-3e)c_2^2$$

Since $(b_1, c_1) = 1$ hence $(b_1, c_1^2) = 1$, $b_1 \mid f-3e$. Similarly, we have $c_1 \mid f-3e$. Then $f-3e = Mb_1c_1$ for some integer M.

Next equate (7) and (10). Then since $b_2 = c_1$, $c_2 = -b_1$

$$2na_1c_1 + 2b_1c_1e + b_1^2d + c_1^2g = 2na_2c_2 + 2b_2c_2e + b_2^2d + c_2^2g$$

$$\Rightarrow \quad 2na_1c_1 + 2b_1c_1e + b_1^2d + c_1^2g = -2na_2b_1 - 2b_1c_1e + c_1^2d + b_1^2g$$

$$\Rightarrow \quad 2n(a_1c_1 + a_2b_1) + 4b_1c_1e + (b_1^2 - c_1^2)d + (c_1^2 - b_1^2)g = 0$$

Using (15) and recalling that $m = n$, one gets

$$2(e-f)b_1c_1 + 2d(b_1^2 - c_1^2) + 4b_1c_1e + (b_1^2 - c_1^2)(d-g) = 0$$

Simplifying the above expression, we get

$$(3d-g)b_1^2 + 2(3e-f)b_1c_1 - (3d-g)c_1^2 = 0 \quad (18)$$

Using a similar argument as above, since $(b_1, c_1) = 1$, one concludes that

$$3d - g = Nb_1c_1$$

for some integer N.

Then one can rewrite (17) and (18) as follows, using $f-3e = Mb_1c_1$ and $3d-g = Nb_1c_1$,

$$(Mb_1c_1)b_1^2 + 2(Nb_1c_1)b_1c_1 - (Mb_1c_1)c_1^2 = 0$$

$$(Nb_1c_1)b_1^2 - 2(Mb_1c_1)b_1c_1 - (Nb_1c_1)c_1^2 = 0$$

Since $b_1c_1 \neq 0$, we get

$$M(b_1^2 - c_1^2) + 2Nb_1c_1 = 0 \quad (19)$$

$$N(b_1^2 - c_1^2) + 2Mb_1c_1 = 0 \quad (20)$$

Eliminating the second terms in the above two equations, we get $(M^2 + N^2)({b_1}^2 - {c_1}^2) = 0$. So either $b_1^2 - c_1^2 = 0$ or $M^2 + N^2 = 0$ and we check each case.

<u>subcase 1</u> ${b_1}^2 - {c_1}^2 = 0$.

Then from (19), $2\mathrm{N}b_1c_1 = 0$ and from (20), $2\mathrm{M}b_1c_1 = 0$. Since $b_1c_1 \neq 0$, we have $M = 0 = N$. This is just the next subcase since $M^2 + N^2 = 0$ implies $M = 0 = N$ and we only need to consider:

<u>subcase 2</u> $M^2 + N^2 = 0$, i.e.$M = 0 = N$.

In this case $f = 3e$ and $g = 3d$. From (12), since in our case $b_2 = c_1$, $c_2 = -b_1$, and $m = n$, we have

$0 = m(a_1c_1 + a_2b_1) + d(-{b_1}^2 + {c_1}^2) + 3eb_1c_1 - eb_1c_1$. This becomes

$$0 = m(a_1c_1 + a_2b_1) - d({b_1}^2 - {c_1}^2) + 2eb_1c_1 \quad (21)$$

So

$$0 = ma_1c_1 + d{c_1}^2 + 2eb_1c_1 + ma_2b_1 - d{b_1}^2 \quad (22)$$

From this we get $c_1(ma_1 + dc_1) = b_1(db_1 - ma_2 - 2ec_1)$

Since $(c_1, b_1) = 1$, $b_1 \mid (ma_1 + dc_1)$ so that $ma_1 + dc_1 = Jb_1$ for some integer J.

Similarly, we write (22) as $b_1(ma_2 - db_1) = -c_1(ma_1 + dc_1 + 2eb_1)$.

So $c_1 \mid (ma_2 - db_1)$ and $ma_2 - db_1 = Kc_1$ for some integer K.

Substituting $ma_1 = Jb_1 - dc_1$ and $ma_2 = db_1 + Kc_1$ into (21), we get

$0 = c_1(Jb_1 - dc_1) + b_1(db_1 + Kc_1) - d({b_1}^2 - {c_1}^2) + 2eb_1c_1$. This gives $0 = (J + K + 2e)b_1c_1$ so that $J + K + 2e = 0$.

Since (6)=(9) and m=n, $b_2 = c_1$,$c_2 = -b_1$, we get

$2ma_1b_1 + 2b_1c_1d + 3e{b_1}^2 + e{c_1}^2 = 2na_2b_2 + 2b_2c_2d + 3e{b_2}^2 + e{c_2}^2 = 2ma_2c_1 - 2b_1c_1d + 3e{c_1}^2 + e{b_1}^2$.

From this we get

$$2ma_1b_1 - 2ma_2c_1 + 4b_1c_1d + 2e({b_1}^2 - {c_1}^2) = 0 \quad (23)$$

Substituting $ma_1 = Jb_1 - dc_1$ and $ma_2 = db_1 + Kc_1$ into the above equation, we get

$2b_1(Jb_1 - dc_1) - 2c_1(db_1 + Kc_1) + 4b_1c_1d + 2e({b_1}^2 - {c_1}^2) = 0$. This gives

$$(J + e){b_1}^2 = (K + e){c_1}^2 \quad (24)$$

Since ${b_1}^2$ and ${c_1}^2$ are relatively prime, we conclude that ${c_1}^2 \mid (J + e)$ and ${b_1}^2 \mid (K + e)$ so that $J + e = P{c_1}^2$ and $K + e = Q{b_1}^2$ for some integers P and Q.

From (24), using $J + e = P{c_1}^2$ and $K + e = Q{b_1}^2$, we have $P{c_1}^2{b_1}^2 = Q{b_1}^2{c_1}^2$ and $P = Q$. So $J + e = P{c_1}^2$ and $K + e = P{b_1}^2$. Then $J + K + 2e = P({b_1}^2 + {c_1}^2)$. But $J + K + 2e = 0$ and ${b_1}^2 + {c_1}^2 \neq 0$ by assumption. So $P = 0$ and $J = -e, K = -e$. Hence we have $ma_1 = -eb_1 - dc_1$ and $na_2 = db_1 - ec_1$ where $m = n$.

From (1), $\rho' \cup \rho' = [2(-eb_1 - dc_1)b_1 + 2b_1c_1d + 3eb_1{}^2 + ec_1{}^2]\alpha'' + [2(-eb_1 - dc_1)c_1 + 2b_1c_1e + db_1{}^2 + 3dc_1{}^2)]\beta'' + (mb_1{}^2 + mc_1{}^2)\tau''$.

This becomes

$\rho' \cup \rho' = e(b_1{}^2 + c_1{}^2)\alpha'' + d(b_1{}^2 + c_1{}^2)\beta'' + m(b_1{}^2 + c_1{}^2)\tau'' = (b_1{}^2 + c_1{}^2)(e\alpha'' + d\beta'' + m\tau'')$.

Since $\rho' \cup \rho'$ is a basis element for some basis of $H^4(L')$, $b_1{}^2 + c_1{}^2{=}1$. Since b_1 and c_1 are integers, one of them is zero and this contradicts our assumption. Thus we establish the Claim 2 and at least one of b_i, c_i is zero.

Next if one of b_1, c_1, b_2, c_2 is zero, say $b_1 = 0$, we need either $c_1 = 0$ or $c_2 = 0$ in order to have $b_1b_2 = -c_1c_2$. But if $b_1{=}0{=}\ c_1$, then $\rho' \cup \rho'$ $=0$ contradicting that $\rho' \cup \rho'$ is a basis element in $H^4(L')$. Thus $b_1 = 0$, $c_2 = 0$, $c_1 \neq 0$, and $b_2 \neq 0$. So suppose that $b_1 = 0$, $c_2 = 0$, $c_1 \neq 0$, and $b_2 \neq 0$. Then $\rho' \cup \rho' = c_1{}^2e\alpha'' + (2na_1c_1 + c_1{}^2g)\beta'' + nc_1{}^2\tau''$ $= c_1[c_1e\alpha'' + (2na_1 + c_1g)\beta'' + nc_1\tau'']$. Since $\rho' \cup \rho'$ is a basis element, $c_1 = \pm 1$ so that $\rho' = a_1\tau' \pm \beta'$.

Also $\omega \cup \omega = (2ma_2b_2 + b_2{}^2f)\alpha'' + b_2{}^2d\beta'' + mb_2{}^2\tau'' = b_2[(2ma_2 + b_2f)\alpha'' + b_2d\beta'' + mb_2\tau'']$ so that $b^2 = \pm 1$ and $\omega' = a_2\tau' \pm \tau'$.

Thus in this case, under the composite homomorphism $H^2(S') \to H^2(L') \to H^2(p^{-1}b)$,

$$\rho \mapsto a_1\tau' \pm \beta' \mapsto \pm\beta \quad \text{and } \omega \mapsto a_2\tau' \pm \alpha' \mapsto \pm\alpha$$

and the composite homomorphism is an epimorphism.

Because of the cohomology ring structure of $N^4 = CP^2\#CP^2$ and by Lemma 2.7 this implies that the map of our interest is a degree one map and we are done.

For the other case of $c_1 = 0, b_2 = 0, c_2 \neq 0$, and $b_1 \neq 0$, we have $\rho' \cup \rho'$ $=(2ma_1b_1 + b_1{}^2f)\alpha'' + b_1{}^2d\beta'' + mb_1{}^2\tau'' = b_1[(2ma_1 + b_1f)\alpha'' + b_1d\beta'' + mb_1\tau'']$ so that $b_1 = \pm 1$.

Also $\omega' \cup \omega' = c_2{}^2e\alpha'' + (2na_2c_2 + c_2{}^2g)\beta'' + nc_2{}^2\tau'' = c_2[c_2e\alpha'' + (2na_2 + c_2g)\beta'' + nc_2^2\tau'']$ so that $c_2 = \pm 1$.

Thus $\rho \mapsto a_1\tau' \pm \alpha' \mapsto \pm\alpha$ and $\omega \mapsto a_2\tau' \pm \beta' \mapsto \pm\beta$ under the composite homomorphism $H^2(S') \to H^2(L') \to H^2(p^{-1}b)$ and we are done as before.

It remains to check

<u>Case 2</u> $b_2 = -c_1$ and $c_2 = b_1$

We may proceed as in case 1 to show that the map of our interest is a degree one map. But simpler thing to do is to observe that expressions of $\rho' \cup \rho'$,$\omega' \cup \omega'$, and $\rho' \cup \omega'$ are symmetric under the following changes: $m \leftrightarrow n, b_1 \leftrightarrow c_1, b_2 \leftrightarrow c-2, \alpha'' \leftrightarrow \beta'', d \leftrightarrow e, f \leftrightarrow g$, and case 1 $\leftrightarrow$ case 2. So case 2 is essentially the same as case 1.

Thus we have proved that the composite homomorphism

$H^2(p^{-1}v) \to H^2(S') \to H^2(L') \to H^2(p^{-1}b)$ is an epimorphism in all cases. Then by Theorem 3.4.2 the associated map is a degree one map.

Now $N^4 = \mathbb{C}P^2\#\mathbb{C}P^2$ is a simply connected manifold hence hopfian so that the map $r \mid p^{-1}b$ is a homotopy equivalence for any b in L' and we conclude that given N^4-type PL map $p : M^7 \to B^3$ between manifolds is an approximate fibration and $N^4 = \mathbb{C}P^2\#\mathbb{C}P^2$ is a codimension-3 PL h-fibrator. $\square$

As mentioned before the above results implies that $\mathbb{C}P^2\#\mathbb{C}P^2$ is a PL h-fibrator.

5. CONCLUSION

R. J. Daverman has studied which manifolds N^n have the following property. If $p : M^{n+k} \to B^k$ be any proper PL map from a closed, orientable, connected, PL $(n+k)$-manifold M^{n+k} onto a polyhedron B such that each fiber $p^{-1}b$ collapses to an n-complex homotopy equivalent to N, then p is an approximate fibration.

In this thesis we have investigated similar problem without assuming that each fiber $p^{-1}b$ collapses to an n-complex homotopy equivalent to N. We only assume that each fiber $p^{-1}b$ has the homotopy type of the given manifold N^n. Since this problem is not easily tractable, we require the codomain space B to be a PL k-manifold.

We have shown that a closed, orientable, aspherical manifold with a hopfian fundamental group is a PL h-fibrator if it is a codimension-2 PL fibrator.

Also we have proved that for a closed, orientable, connected, hopfian n-dimensional manifold N^n with a hopfian fundamental group such that $H^j(N^n) = 0$ for $0 < j < m$ for some integer $m \geq 1$ and $H^n(N^n)$ is in the subring of $H^*(N^n)$, generated by $H^m(N)$, N^n is a PL h-fibrator if it is a codimension-$(m+1)$ PL h-fibator.

Finally, using the fact that the cohomology ring of $\mathbb{C}P^2\#\mathbb{C}P^2$ satisfies the above condition, we have established that $\mathbb{C}P^2\#\mathbb{C}P^2$ is a PL h-fibrator.

One of the immediate open questions is whether $\mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2$ is a PL h-fibrator. More generally which simply connected, smooth 4-manifolds are PL h-fibrators?

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