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Title: A CRITIQUE OF THE INCIDENCE AND ORDER AXIOMS OF GEOMETRY

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A set of axioms of incidence and order for geometry was formulated by David Hilbert in 1898. In this paper these axioms are reformulated and particular care is taken with the two relations of order and incidence. Such phrases as " point P lies on line ℓ" are defined in terms of the incidence relation. A set of models is developed which illustrates and clarifies the axioms and establishes their independence.

A major portion of the paper is devoted to the implications of these axioms. Hilbert gave a list of theorems in his book Foundations of Geometry which he believed were provable on the basis of only these axioms. Some of the proofs were sketched and some were not given. The three major theorems which are a consequence of these axioms are

(1) The ordering of a finite number of points on a line.

(2) The ordering of angles with a common side.

(3) The Jordan Theorem for polygons.

A full proof is given of each of these theorems based upon elementary and not metric concepts. Some consequences of the theorems are investigated. Special attention is payed to the significance of Pasch's Axiom in the proof of these theorems.

A Critique of the Incidence and Order Axioms of Geometry

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A CRITIQUE OF THE INCIDENCE AND ORDER AXIOMS OF GEOMETRY

I. INTRODUCTION

In the logical development of an area of mathematics there must be a starting point; a basis of presupposed information. In most instances the development is founded upon the structure and theorems of other areas which in themselves may encompass a vast number of concepts and many intricate proofs. For example, calculus, as taught to freshmen in college is based upon the real number system with the implied ideas of a limit and algebraic structure, set theory, some intuitive topology and the concept of a function.

In investigating the foundations of geometry or other areas an attempt is made to minimize the amount of material which is presumed, limiting it to those concepts which would logically precede the material to be developed but which would not be classified as part of that body of knowledge. In the following discussion of the first two axiom groups of geometry we will presuppose only the necessary concepts and theorems from set theory, foundations of the number system and logic.

A second aspect of finding an initial basis for our investigations is that we will not be able to define all our terms; this is logically impossible. An attempt to do so will necessarily result in circular definition, that is a finite sequence of definitions each term defined in terms of the previous term with our initial term defined by using the last term. For example, we might try to define a line's being incident upon a pair of points in terms of each point's being on the line. This in turn might be defined in terms of the line passing through each of the two points. Finally, we might say that a line will pass through a point if there is another point such that the line is incident upon these two points. We end up back where we started without actually being able to define all of the terms.

In the following discussion we will take point, line and plane as undefined terms. All we will know about these terms is that they are names for the elements of three sets. To start with, we will not even know that these sets are nonempty. We will also leave several relationships between these entities undefined, such as incidence between lines and pairs of points. Through a set of axioms and definitions we will impose a structure upon these sets and the relationships between them.

One hint seems in order. Our intuition as to how points, lines and planes should be related to each other can be a great help, but it can also create no end of difficulties if we let it lead us to believe that our points, lines, and planes have properties or relationships which we have neither proved from our axioms nor assumed in our axioms. For example, how do we know if points A, B, and C lie on some line that exactly one of the points must lie "between" the other two? This must either be assumed or derived logically from our axioms even though our intuition tells us that this is "obviously" the case. Or, even more basic than this, we are not assuming that either lines or planes are sets of points. In order to make it clear that we cannot trust our intuition and at the same time to clarify the meaning of the various axioms and show their independence we will present models of our axiom system which will satisfy some of our axioms and definitions but not necessarily all of them. Most of these models will be developed in either set theory or in coordinate geometry. In coordinate geometry our geometric entities can be defined in terms of set theory and the real line so that we can speak of points, lines and planes independently of our axiomatic development. Hence points are ordered pairs or triples of real numbers, etc.

The purpose of this paper then can be categorized under three headings:

(1) To develop the axiom groups of incidence and betweenness for geometry including proofs which are usually omitted in most presentations. This will include the Jordan Curve Theorem for Polygons.

(2) To develop models which will illustrate the axioms and their independence.

(3) To investigate what other mathematical ideas can be

introduced on this limited axiomatic basis, that is, without the ideas of metric, congruence or parallelism.

We should also comment here that we will pay particular attention to the powerful impact that Pasch's Axiom has had upon the study of geometry. This will be seen in the number of lemmas and theorems which are based upon it and in the models.

Historically the quest for a consistent and independent set of axioms has been one of the major thrusts of mathematics. It is well known that Euclid's works contain an axiom system for geometry and many of its theorems. At the turn of the century David Hilbert wrote his Foundations of Geometry (Hilbert, 1899). This book has been revised many times since it was first published including the addition of several supplements. We will use the first two groups of axioms from this work as a guide for our work. We will include in our presentation the theorems from Foundations of Geometry and many of the details which Hilbert omitted.

I. AXIOMS OF INCIDENCE

1.1. In thinking about points lines and planes one of the first things which should occur to us is that there should be interrelationships among the elements of these three sets. We would expect points to lie on lines and lines in planes, etc. How to describe this axiomatically is the problem! In high school geometry texts one of the postulates is often "two points determine a line." We will rephrase this more precisely and use it as our point of departure.

1.2. Undefined Relations. We will utilize the word incidence in two different undefined relations. First, we assume an incidence relation between lines and pairs of points and secondly an incidence relation between planes and triples of points. Utilizing these two basic relations we will be able to define such phrases as "line ℓ lies on plane α," "line ℓ passes through point P," etc. One should note that the term relation here is being used in its usual set theoretic sense as a subset of a Cartesian product.

1.3.1. Axiom I-1. Given 2 distinct points there exists one and only line incident upon them. If the points are P and Q then we will denote the line incident upon P and Q by $\overleftrightarrow{PQ}$.

1.3.2. Note. The axiom actually gives us two bits of information.

First, if we have two different points we will always have a line which will be said to be incident upon them. Secondly, it tells us that there will be only one such line.

1.4. Models

1.4.1. M.1. Let $\Phi = \{(x, y) \mid x \geq 0\}$ be our set of points. (i.e., we will take the points in the closed right half plane of 2 dimensional coordinate geometry.) For our set of lines, denoted by Λ, we will take the union of two sets, Λ_R and Λ_C such that Λ_R is the set of rays with initial point on the Y axis and parallel to the positive X axis Λ_C is the set of "half" circles with centers on the Y axis (see Figure 1 with point P, and $\ell_1 \in \Lambda_R$, $\ell_2 \in \Lambda_C$.) An element of Λ_R is given by $\ell_1 = \{(x, a) \mid x \geq 0, a \in R\}$ an element of Λ_C, by $\{(x, y) \mid x^2 + (y-a)^2 = b, b > 0, x \geq 0\}$.

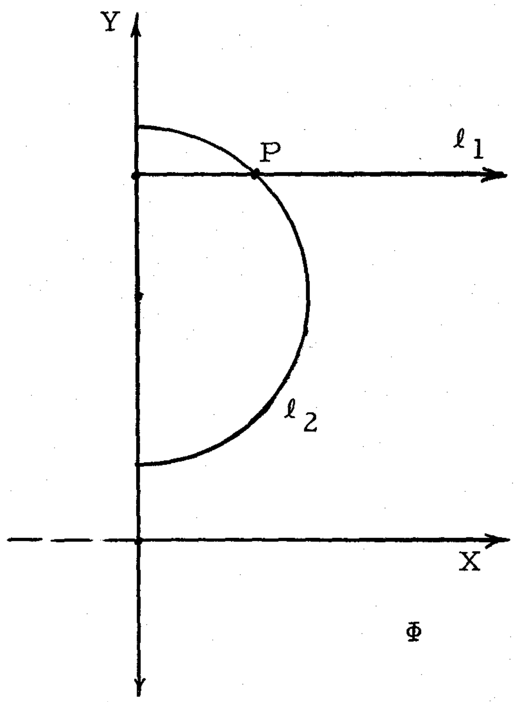


Figure 1.

To show that M-1 satisfies I-1 let P and Q be any two elements of Φ given by $P = (x_1, y_1)$, $Q = (x_2, y_2)$. Here incidence will be defined to be both points satisfying the equation of a "line." Now either

$y_1 = y_2$ or $y_1 \neq y_2$. In the former case let $a = y_1 = y_2$ then P and Q both lie on exactly one line of Λ_R. If $y_1 \neq y_2$ then there exists one and only one circle with center $(0,k)$ and passing through (x_1, y_1) and (x_2, y_2). To see this we can solve simultaneously the system of equations

$$x_1^2 + (y_1 - k)^2 = r^2$$

$$x_2^2 + (y_2 - k)^2 = r^2$$

for k and r, thus yielding the center $(0,k)$ and the radius r of the "line" passing through P and Q. We get the value of k to be

$$\frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(y_2 - y_1)}$$

and substituting this value in either of the equations we can solve for r. Thus in either case we have exactly one element of Λ which contains P and Q and therefore M-1 satisfies I-1. But, the elements of Λ are not what one would normally call lines.

<u>1.4.2.</u> It will be convenient to use the above notation in the future. We will designate the set of lines in our discussion by Λ and the set of points by Φ.

<u>1.4.3. M-2.</u> Let Φ = set of lines through the origin in 3 dimensional space and Λ be the set of planes through the origin. Then it is obvious that any two distinct elements of Φ will determine a unique element of Λ which contains them, hence we define incidence in this manner. Here we have a model in which our set of "points" is a set of lines and our set of "lines" is a set of planes. This, again, is not what our intuition would like to call points and lines.

<u>1.4.4. M-3.</u> Now, we are prepared to look for models of our axiom system which have less obvious intuitive content. Given the set $S = \{A, B, C, \}$. Let $\Phi = S$ and Λ be the subsets of S with 2 elements, i.e., $\Lambda = \{\{A, B\}, \{A, C\}, \{B, C\}\}$. Let $\ell_1 = \{A, B\}$, $\ell_2 = \{A, C\}$ and $\ell_3 = \{B, C\}$, so $\Lambda = \{\ell_1, \ell_2, \ell_3\}$. We give the meaning to the incidence of a line and a pair of points so that the line will be incident upon a pair of points if it is the subset of S consisting of that pair of points. Thus $\{A, B\}$ is the line incident upon the pair A and B. It is clear that each pair has exactly one line incident upon it, so this finite model satisfies I-1.

<u>1.4.5. M-4.</u> Now the reader should be ready for a more obscure model, but one which will illuminate our discussion later.

Let $S = \{A, B, C, D\}$. We will take our points to be the elements of the set S as in M-3. We will take our lines to be the following pairs of pairs of points.

$$\ell_1 = \{\{A, B\}, \{C, D\}\}$$

$$\ell_2 = \{\{A, C\}, \{B, D\}\}$$

$$\ell_3 = \{\{A, D\}, \{B, C\}\}$$

Thus $\Lambda = \{\ell_1, \ell_2, \ell_3\}$.

The subtlety of the model comes in the interpretation of the incidence relation and its consequences. We shall say in our model that a line is incident on a pair of points iff the subset consisting of those two elements is one of the two subsets making up the line. For example, the line incident upon B and D is the line ℓ_2 while the line incident upon A and D is ℓ_3. Again we see that for each pair there is exactly one line incident upon it. So we have a model in which I-1 holds but which is divorced from any intuitive interpretation of points or lines.

1.4.6. It will be a good idea to keep these and later models in mind as examples of the axiom system as it develops. We will return to them whenever they will serve as illustrations of the concepts being investigated.

1.5. Definition. The reader may have noticed that we have not used the terminology, "point P lies on line ℓ," or "P is a point of line ℓ" etc. We are able to define these terms with reference to the incidence relation above. We will say that P lies on line ℓ, or is a point <u>of</u> line ℓ, or ℓ <u>contains</u> the point P iff there exists a point Q, distinct from P, such that ℓ coincides with $\overleftrightarrow{PQ}$. (Two lines <u>coincide</u> when they are the same element of Λ, that is when they are the same line.)

<u>1.6.1. Axiom I-2.</u> If P and Q are distinct points of line ℓ then $\overleftrightarrow{PQ}$ and ℓ coincide.

<u>1.6.2.</u> At first, Axiom I-2 doesn't seem to add anything to the structure of geometry and in fact it seems redundant. But, we should look deeper. We have no guarantee in I-1 or Definition 1.5 that if two points lie on a line ℓ the line which is incident upon them is ℓ. In the sketch at the right we find that P and Q lie on ℓ since there exist points R and S such that $\ell = \overleftrightarrow{PR}$ and $\ell = \overleftrightarrow{QS}$ but the line incident upon P and Q might be m.

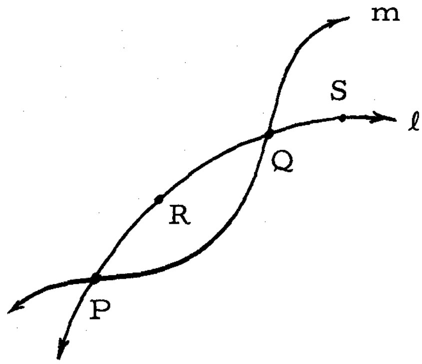


Figure 2.

<u>1.7.</u> Now let us return to some of the models of Section 1.4 and examine whether I-2 holds for them.

<u>1.7.1.</u> In Example M-1 we can see that if P and Q lie on ℓ, an element of either Λ_P or Λ_C then ℓ will be the only line through these two points. Recall that in this example "incidence" was

defined in terms of the coordinate of pair of point satisfying the equation of an element of Λ. Hence "P lies on ℓ" if its coordinates satisfy the equation of ℓ. (The details of the computations will be left to the reader.)

1.7.2. Let us take a closer look at Examples M-2, 3 and 4. In M-3 the points A and C both lie on line ℓ_2 since in order for A to lie on ℓ_2 there must be another point different from A, in this case C, such that $\ell_2 = \overleftrightarrow{AC}$. But $\ell_2 = \{A, C\}$ and thus we see, if we check out the other possibilities M-3 also satisfies I-2. (Notice that since we are dealing with sets $\overleftrightarrow{AC} = \{A, C\} = \{C, A\}$.)

Now consider Model 4. Each point lies on every line in this model. For instance A lies on line ℓ_1 since point B is different from A and $\overleftrightarrow{AB} = \ell_1$. (That is the line which contains the set $\{A, B\}$ as one of its elements is ℓ_1.) Again D lies on line ℓ_1 also since $\overleftrightarrow{CD} = \{\{A, B\}, \{C, D\}\} = \ell_1$. But now the line incident on A and D is ℓ_3 and not ℓ_1. Thus we have both A and D on ℓ_1 but ℓ_1 is not incident on A and D. This establishes that M-4 satisfies I-1 and not I-2. Granted, this model may seem artificial, but with it we have shown that there is nothing in the statement of Axiom I-1 which implies I-2.

It is clear that Axiom I-2 also holds in model M-2.

1.7.3. A set of axioms is said to be independent if each axiom is not a consequence of the others. If an axiom could be deduced from the remaining axioms in the set then in any model in which the other axioms held it would necessarily hold. Therefore, to show that a set of axioms is independent we demonstrate a model for each axiom such that the other axioms hold but it doesn't. In the preceding paragraph we have shown that M-4 is a model showing the independence of I-2 from I-1. This is not to say that the concepts formulated in Axiom I-2 do not require that I-1 be stated first. This is obviously the case since if two points are on some line then we must know there is a line incident upon these two points before we can finish the statement of the axiom.

1.8.1. We still do not know if our sets of points, lines and planes are empty or not. In order to clarify this situation we have:

1.8.2. Axiom I-3. On each line there are at least two points.

1.8.3. Notice this doesn't say that we have any lines or planes, but if we do have any lines then each must contain a pair of distinct points.

1.8.4. The reader should observe that in Examples M-1 through M-4, Axiom I-3 is satisfied. To see that this isn't necessarily the case consider the following model:

M-5. Let $\Phi = \{A, B\}$ and $\Lambda = \{\{A, B\}\{A\}\}$ and let $\ell_1 = \{A, B\}$ and $\ell_2 = \{A\}$. A line is said to be incident upon X and Y iff the set which is the line contains X and Y. Axioms I-1 and I-2 are satisfied because given the two points A and B there exists the unique line ℓ_1 which is incident upon them. On the other hand there are no points on ℓ_2 not even A according to Definition 1.5. Consequently in this example we have a line, ℓ_2, without any points on it and Axiom I-3 does not hold.

1.9.1. Now it seems appropriate to delineate the relationships among planes, points and lines.

1.9.2. Definition. A set of points will be called collinear iff they all lie on the same line. They will be called noncollinear iff there does not exist a line such that they are all points of it.

1.9.3. Axiom I-4. If 3 points are noncollinear then there exists one and only one plane incident upon them.

1.10.1. Our examples up to now have dealt with points and lines. We have not defined planes in any of them and they are not necessarily well suited to any extension. In M-3 we could let $S = \{A, B, C\}$ be a plane and call it α. If we designate ψ as our set of planes then $\psi = \{\alpha\}$. We must define incidence of planes and triples of points. We will take the obvious definition that a plane β will be incident upon P, Q, and R iff $\beta = \{P, Q, R\}$. From this definition we see that α is incident upon A, B and C. The uniqueness of the incidence relations follows since there is only one plane. Thus we have I-4 satisfied. Note that there is no confusion between line and planes since every line has only two points on it.

1.10.2. Let us investigate the extension of one of the other models. In M-1 we have only one possibility for a plane, $[0, \infty) \times (-\infty, \infty)$. But we might try to extend the model to 3 dimensional space. If we take $\Phi = \{(x, y, z) \mid x \geq 0\}$, and define Λ, ψ and our incidence relations in an analogous fashion to M-1 we can extend the model to 3-space. For instance, Λ could be the set of rays perpendicular to the Y-Z plane union with the set of semicircles with centers on the Y-Z plane and whose planes (in the sense of analytic geometry) are perpendicular to it, both sets lying in Φ. With a similar definition for ψ and the definition of incidence in terms of the coordinates of the points we could extend the model and check the axioms. But, it should be obvious to the reader, the algebraic analysis of the geometry becomes quite involved. To pursue it further here, would be to deviate from our development of axiomatic geometry farther than seems appropriate.

1.11.1. Just as in the development of the axioms about lines we will need to consider a relationship between single points and planes.

<u>1.11.2. Definition.</u> A point P will be called a point <u>of</u> plane α iff there exist two other distinct points Q and R such that P, R and S are noncollinear and the unique plane incident upon them is α. (Its existence follows from Axiom I-4.) We will call a set of points <u>coplanar</u> if there exists a plane such that all the points of the set are points of the plane. A set of points is said to be <u>non-coplanar</u> if they do not form a coplanar set.

<u>1.12. Axiom I-5.</u> The plane which is incident upon any three noncollinear points of a plane α, is α.

<u>1.13. M-6.</u> In all our previous models if I-4 held I-5 would also. So let us try to construct an example in which I-1 through I-4 hold but I-5 doesn't. We can use M-4 as an example and try to do the same thing for planes that it does for lines.

Let $\Phi = \{A, B, C, D, E, F\}$ and ψ, the set of planes, be the set of all pairs of disjoint subsets of Φ consisting of three elements each. For example $\alpha = \{\{A, B, C\}, \{D, E, F\}\}$ and $\beta = \{\{A, C, F\}, \{B, D, E\}\}$ are typical elements of ψ. We will define our incidence relation to be: γ is incident upon P, Q and R iff $\{P, Q, R\}$ is one of the two sets forming γ. Thus α is incident upon A, B, and C and D, E, and F while β is incident upon A, C, and F and B, D, and E. But, by our Definition 1.11.2 each point is a point of every plane. Therefore A, B and C are points of plane β, but β is not incident upon the triple A, B, C and I-5 should be violated.

Note in this model we have not defined our set of lines Λ. In order to discuss whether I-4 and I-5 hold in this model we must be able to say whether a set of points is collinear or noncollinear. Hence we define Λ to be the set of all subsets of Φ consisting of two elements, so a typical line would be $\{A, B\}$. Thus $\Lambda = \{\{A, B\}, \{A, C\}, \ldots, \{E, F\}\}$. It is obvious that I-1, I-2 and I-3 hold with this definition of Λ and incidence defined in terms of set equivalence. Now notice that only pairs of points can be called collinear. Thus, any triple of distinct points is automatically a noncollinear set and by our definition of ψ we then have a unique plane incident upon any such triple. Therefore I-4 is satisfied but we see from the previous paragraph that I-5 is not. Thus we have a model in which I-1 through I-4 are satisfied, but I-5 is not, in other words we have established the independence of Axiom I-5.

1.14. The question of the existence of lines and planes is still open. It seems appropriate now to hypothesize their existence.

Axiom I-6. There exists at least one plane and on any plane there exist at least 3 noncollinear points.

The reader can see that up to this point none of our definitions or axioms required the existence of a plane. Even in Axiom I-4 we do not require this since we may not even have 3 noncollinear points in a model. (Recall M-5 in Section 1.8.)

1.15. We would expect lines to lie in planes so it seems reasonable to include in our system the following definition.

Definition. A line ℓ is said to be a line of plane α or lie in plane α iff every point of ℓ lies in α.

1.16.1. Recall our attempt to generalize M-1 in Section 1.10.2. We take our planes to be restrictions of the usual planes of coordinate geometry to the set Φ. In this case we could have two points of a line ℓ in a plane α but not all of the points of ℓ in α according to Definition 1.15. Our axiom system should be restricted in some way so that this cannot occur.

1.16.2. Axiom I-7. If two distinct points of a line are points of some plane then the line lies in the plane. (Note: This tells us that if two points of a line ℓ lie in plane α then all the points of ℓ lie in α according to Definition 1.15.)

1.16.3. In M-6, I-7 automatically holds since each line contains exactly 2 points. Let us construct a finite model of the Axioms I-1 through I-6 which violates I-7.

M-7. Let $\Phi = \{A, B, C, D\}$, $\Lambda = \{\{A, B, C\}, \{A, D\}, \{B, D\}, \{C, D\}\}$ and $\psi = \{\{A, B, D\}, \{A, C, D\}, \{B, C, D\}\}$.

Incidence both for lines and planes will be defined in terms of set inclusion. Thus the line $\{A, B, C\}$ is incident upon A and C since $\{A, C\} \subset \{A, B, C\}$. Also from our Definition 1.5 A lies on line $\{A, B, C\}$ since $\{A, B, C\}$ is incident upon A and C. Now since each line contains at least two points Axiom I-3 holds. Each pair of points has exactly one line incident upon them so I-1 is satisfied and I-2 also holds due to our definitions above. There are 3 sets of triples of noncollinear points and these are the 3 planes of ψ so by checking the various possibilities we see that I-4 and I-5 are satisfied. Also I-6 is obviously satisfied. But I-7 is not since plane $\{A, B, D\}$, for example, contains points A and B of line $\{A, B, C\}$ but it doesn't contain point C.

1.17.1. Two distinct planes should intersect in a line if they intersect at all, but it is sufficient to state:

Axiom I-8. If two distinct planes have a point P in common then there exists a point Q, distinct from P, such that Q also lies in both planes.

1.17.2. The above axiom guarantees that if two distinct planes α and β have a point P in common they have a line in common and the line contains P. This follows since there exists a point Q distinct from P on both α and β by I-8 and then by I-7 $\overleftrightarrow{PQ}$

lies in both α and β. There can be no other point R such that R is on α and β and R is not on $\overleftrightarrow{PQ}$, since if there were then P, Q, and R would be noncollinear and by I-5 α and β would coincide contradicting the fact that α and β are distinct.

1.18. M-8. We now construct a model satisfying I-1 through I-7 but not satisfying I-8. Let Φ be a set of five points, say $\Phi = \{A, B, C, D, E\}$, Λ the set of all subsets of Φ consisting of 2 elements each and ψ the set of all subsets of Φ consisting of 3 elements each. (i.e., $\Lambda = \{\{AB\}\{AC\}, \ldots, \{DE\}\}$ and $\psi = \{\{A, B, C\}, \{A, B, D\}, \ldots, \{C, D, E\}\}$.) We will define incidence in terms of set inclusion again. Thus line $\{A, B\}$ is incident upon A and B, plane $\{A, B, C\}$ is incident upon A, B, and C. From this it follows that A is a point of line $\{A, B\}$ and a point of plane $\{A, B, C\}$ etc.

Now I-1 through I-7 hold trivially, but if we check planes $\{A, B, E\}$ and $\{C, D, E\}$ we see that they have point E in common but no other point so we have I-8 violated. Thus I-8 is independent of the previous axioms.

1.19.1. We still have not forced the existence of any points other than those contained in the plane required by I-6. Since we wish our system to exclude this situation we introduce the following axiom.

Axiom I-9. There exist at least 4 noncoplanar points.

1.19.2. Recall Model-3 (1.4.4). If we define $\psi = \{\{A, B, C\}\}$ then in this model all the axioms except I-9 are satisfied, so as before, I-9 is independent of I-1 through I-8.

1.19.3. M-9. If we extend M-3 by letting $\Phi = \{A, B, C, D\}$, $\Lambda = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$, $\psi = \{\{A, B, C\}, \{A, B, D\}, \{A, C, D\}, \{B, C, D\}\}$ and incidence by set inclusion then our entire axiom system is satisfied. This system is suggested by the sketch at the right (Moise, 1963).

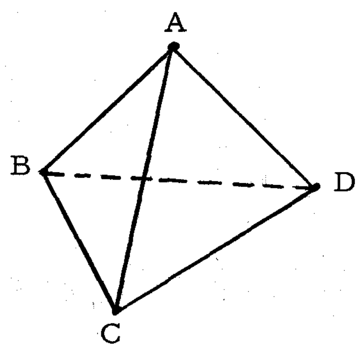


Figure 3.

1.20. Theorems of Incidence. There are several theorems which we can prove from the axioms of incidence. Such theorems as "every line is in some plane" or "two distinct lines have at most one point in common" are relatively easy to prove and will not be proved here. We will prove, below, several theorems which seem to be of more interest. Some of them will be useful later (Borsuk, Szmielew, 1960; Hilbert, 1963; Moise, 1962).

1.20.1. Theorem I-1. Exactly one plane contains a given line and a point not on the line.

Proof. Let ℓ be a line and P be a point not on ℓ. By I-3 there exist at least 2 points Q and R on ℓ and neither of them is the point P by our assumption that P is not on ℓ. By the definition of noncollinear we see that the points are noncollinear and therefore there exists a unique plane α incident upon them by I-4. If any other plane β contained ℓ and P then it would contain Q and R and by I-5 β and α coincide.

1.20.2. Definition. Two lines, a line and a plane or two planes will be called intersecting iff they have at least one point in common. That is, if lines ℓ and m intersect then there exists at least one point P such that P is a point of both lines. (Note: This is not set intersection since we have not stated that lines or planes are sets of points.)

1.20.3. Theorem I-2. Two distinct lines intersect in at most one point.

Proof. Suppose P lies on lines ℓ and m. Now if a second point Q, distinct from P, was also a point of ℓ and m then by I-2 ℓ and m would coincide contradicting our hypothesis.

Q.E.D.

1.20.4. Theorem I-3. Exactly one plane contains two distinct intersecting lines.

Proof. Let ℓ and m be a pair of intersecting lines and P their point of intersection. By our Definitions 1.20.2 and 1.5 and I-3 there is a point R on ℓ, $R \neq P$ and a point Q on m, $Q \neq P$. Now $R \neq Q$, since if R and Q were the same point then ℓ and m would coincide by I-2 contradicting the fact that ℓ and m are distinct lines. Now by I-4 there exists a unique plane α incident upon $P, Q,$ and R and by I-7 we have ℓ and m lying in α. No other plane could contain ℓ and m since this would imply that it contained P, Q, and R and by I-5 the plane would be the same as α.

1.20.5. Theorem I-4. If α and β are two distinct planes which both contain a point P then there exists a line ℓ containing P and lying in both α and β.

Proof. (See 1.17.2.)

1.20.6. Theorem I-5. There exist at least 6 distinct lines and 4 distinct planes.

1.20.7. Lemma I-1. If ℓ is a line then there exists a plane containing ℓ.

Proof. By I-6 there exists a plane α. Either ℓ lies in α or it doesn't. If ℓ lies in α we are done, so, suppose not. Again by I-6 there exist 3 noncollinear points on α so at least one of them which we call P doesn't lie on ℓ. Now by Theorem I-6 there exists a plane β containing both ℓ and P. Q.E.D.

1.20.8. Lemma I-2. If A, B, C, and D are four distinct noncoplanar points then any subset consisting of three elements is noncollinear.

Proof. Assume the contrary, say A, B, and C lie on line ℓ. Now either D lies on ℓ or it doesn't. If D lies on ℓ then by Lemma I-1 ℓ lies in some plane and A, B, C, and D are coplanar, (by definition of ℓ lying in the plane). If D is not on ℓ then by Theorem I-1 there exists a plane containing ℓ and D and again A, B, C, and D are coplanar. Therefore, our assumption that A, B, and C are collinear leads to a contradiction. Q.E.D.

Proof of Theorem I-5. By I-8 there exist at least 4 noncoplanar points. Call them A, B, C, and D. Now by Lemma I-2 each subset of $\{A, B, C, D\}$ consisting of three elements each is noncollinear and therefore determines a unique plane by I-4. There are 4 such subsets $\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$, and $\{B, C, D\}$ and therefore

4 distinct planes incident upon them.

There are 6 distinct subsets of $\{A, B, C, D\}$ consisting of two elements each: $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, $\{C, D\}$. Each of these pairs of points has a line incident upon them and these lines are distinct by Lemma I-2. Q.E.D.

II. AXIOMS OF ORDER

2.1. In this chapter we will further develop the properties of the undefined entities, points, lines, and planes. We will have an undefined relation between single points and pairs of distinct points. Our relation will be that of betweenness and we will say point B, for example, lies between A and C, only when A, B, and C are distinct collinear points. We will expend most of our effort in investigating the properties which our axioms will imply. We will integrate the theorems with our axioms and models where appropriate.

2.2. Axiom 0-1. If point B is between A and C then B is also between C and A.

2.3. In order to see that 0-1 is independent of the incidence axioms we need another model.

M-11. Consider the set $\Phi = \{A, B, C, D, E\}$ in 3 space given by A: $(0, -1, 0)$, B: $(0, 0, 0)$, C: $(0, 1, 0)$, D: $(1, 0, 0)$, and E: $(0, 0, 1)$. See the sketch at the right. Notice that each point lies on a coordinate axis. Using this we will define Q to be

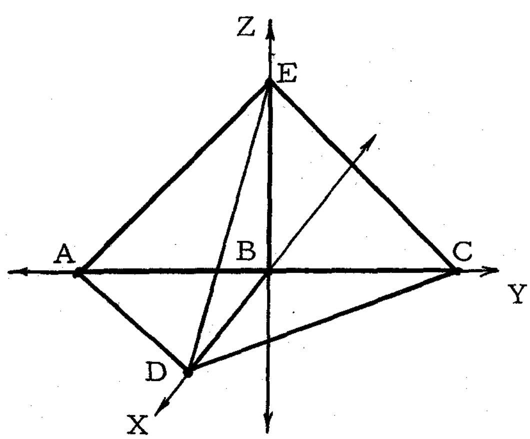


Figure 4.

between P and R iff the appropriate coordinate of Q is greater than that of P and less than that of R. Thus the y coordinate of B is 0 while that of A is -1 and C is 1, so our definition yields B between A and C. Notice that B is *not* between C and A. Now to complete the rest of the model. Let $\Lambda = \{\{A, B, C\}, \{A, D\}, \{C, D\}, \{A, E\}, \{B, E\}, \{C, E\}, \{D, E\}\}$ and $\psi = \{\{A, B, C, D\}, \{A, B, C, E\}, \{A, D, E\}, \{B, D, E\}, \{C, D, E\}\}$. Our set of lines and planes is *suggested* by the figure, but the lines are pairs of points and *not* the lines they determine in 3 space. We will use set inclusion again as our definition of incidence. It is relatively easy to see that all our incidence axioms are satisfied and therefore 0-1 is independent of the incidence axioms.

2.4. We need to clarify the relation of betweenness further:

Axiom 0-2. If A, B, and C are distinct collinear points then at most one of them is between the other two.

2.5. Notation. It will be convenient to indicate that point Q lies between P and R by PQR and that point Q does not lie between P and R by $\widetilde{PQR}$. Using this notation we can rephrase Axiom 0-1 as "ABC implies CBA," and Axiom 0-2 as ABC implies $\widetilde{ACB}$ and $\widetilde{BAC}$.

2.6. M-12. Consider the same set of points as in M-11. If we could take the Y-axis and make a circle out of it we would get a figure like the one at the right. The sketch suggests that we can have the same sets for our lines and planes, but a new betweenness relationship for the line $\{A, B, C\}$. The circle suggests that we could define all possible betweenness relations as holding. Thus we will define ABC, CBA, ACB, BCA, BAC, and CAB. Again it is easy to see that I-1 through I-8 and 0-1 hold with the same incidence relations as in M-12, but that 0-2 doesn't hold. So 0-2 is independent of the preceding system.

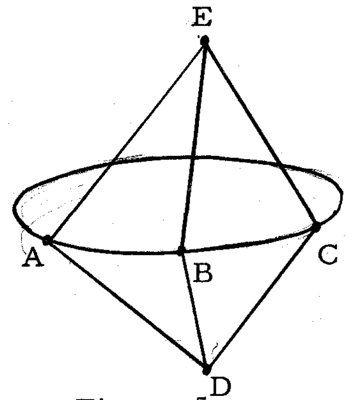


Figure 5.

2.7.1. In the development of our axiom system we would expect to find more than two points on each line and we would also expect to be able to order these points.

2.7.2. Axiom 0-3. If A and C are distinct points of a line ℓ then there exists a point D on ℓ such that ACD.

2.7.3. Notice that 0-3 does not tell us that there exists a point B such that ABC.

2.8.1. M-13. We can easily revise M-11 in order to illustrate the independence of 0-3 by taking the usual betweenness for points A, B, and C, that is ABC and CBA. Note there does not exist a point D such that ACD.

2.8.2. In order to develop models which will illustrate the remaining axioms and their independence we will restrict ourselves to planar models. Many of them can be extended to the nonplanar case but become too complex for the brief expositions which suit our purposes.

2.8.3. M-14. We should develop a model for which 0-3 holds but in which A and C on ℓ does not imply the existence of a point B such that ABC. Let Π be the Cartesian Plane and $\Phi = \Pi - \{(x,y) \mid x^2+y^2 < 1\} = \{(x,y) \mid x^2 + y^2 \geq 1\}$ (see Figure 6.). (Note: We could have defined Φ to be 3 space minus $\{(x,y,z) \mid x^2 + y^2 + z^2 < 1\}$ but this just complicates the discussion.) Now we will define lines, as usual, in terms of the sets of points which satisfy $ax + by + c = 0$, with a and b not both zero. A line will be incident upon a pair of points iff

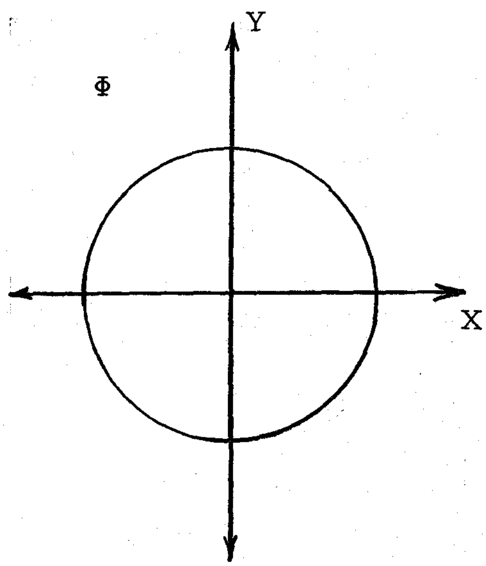


Figure 6.

the coordinates of the points satisfy the equation of the line. From the usual development of coordinate geometry we know the incidence axioms for the plane are satisfied. For example, given two different points (x_1, y_1) and (x_2, y_2) there is exactly one linear equation, and therefore one line, which both points will satisfy.

Betweenness will also be defined in the usual way in terms of the order relations on the X and Y axes. It follows that 0-1 and 0-2 are satisfied. In order to see that 0-3 holds let $P:(x_1, y_1)$ and $Q:(x_2, y_2)$ be two distinct points. Now either $x_1 \neq x_2$ or $y_1 \neq y_2$ so we have the following possibilities

$$x_1 = x_2 \quad y_1 < y_2$$
$$x_1 = x_2 \quad y_1 > y_2$$
$$x_1 < x_2 \quad y_1 < y_2$$
$$x_1 < x_2 \quad y_1 = y_2$$
$$x_1 < x_2 \quad y_1 > y_2$$
$$x_1 > x_2 \quad y_1 < y_2$$
$$x_1 > x_2 \quad y_1 = y_2$$
$$x_1 > x_2 \quad y_1 > y_2.$$

We investigate one of the possibilities, say $x_1 < x_2$ and $y_1 > y_2$. Let $ax + by + c = 0$ be the equation of the line ℓ incident upon P and Q. Since $x_1 < x_2$ let $x_3 = x_2 + 1$ and we have $x_1 < x_2 < x_3$. Now solving $ax + by + c = 0$ for y we have $y = -\frac{a}{b}x - \frac{c}{b}$. Also we know that

$$-\frac{a}{b} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Now $x_1 < x_2$ implies $x_2 - x_1 > 0$, $y_1 > y_2$ implies $y_2 = y_1 < 0$ and therefore $-\frac{a}{b} < 0$. Substituting x_3 we have

$$y_3 = -\frac{a}{b}x_3 - \frac{c}{b} = -\frac{a}{b}(x_2+1) - \frac{c}{b} < -\frac{a}{b}x_2 - \frac{c}{b} = y_2$$

Thus $x_1 < x_2 < x_3$ and $y_1 > y_2 > y_3$ and we have a point $R:(x_3, y_3)$ such that PQR but, R may not be in Φ. (i.e., $x_3^2 + y_3^2 < 1$.) If we choose $x_4 = 1$ then $y_4 = -\frac{a}{b} - \frac{c}{b}$. $x_4^2 + y_4^2 \geq x_4^2 = 1$. Call this point S, then we have S on ℓ and S in Φ. If R is not in Φ then $x_4 = 1 > x_3$ since $x_3^2 < x_3^2 + y_3^2 < 1$. Thus by an argument similar to the one above we will have $x_4 > x_3 > x_2 > x_1$ and $y_4 < y_3 < y_2 < y_1$. Therefore PQS and in either case a point with the desired characteristics.

<u>2.9.1.</u> Now we should begin to define some of the familiar concepts of plane geometry such as line segments, triangles, etc.

<u>2.9.2. Definition.</u> An <u>open line segment</u> is the set of points on a line lying between two points of the line.

<u>Notation.</u> If A and C are points of the line ℓ then the oepn segment determined by A and C will be denoted by AC.

Notice that from comment 2.7.3 we do not know whether AC is empty or not.

<u>2.9.3. Definition.</u> A <u>closed line segment</u> or simply a <u>line segment</u> will be the set of points on a line between two given points along with the two given points. We will denote the line segment determined by A and C on line ℓ by $\overline{AC}$. From the above definition we have $\overline{AC} = AC \cup \{A, C\}$.

<u>2.9.4.</u> Now we are able to define a triangle. <u>Definition:</u> Let A, B, and C be three noncollinear points then the <u>triangle</u> determined by A, B, and C, denoted by ΔABC, is $\overline{AB} \cup \overline{BC} \cup \overline{AC} = AB \cup BC \cup AC \cup \{A, B, C\}$. The open segments AB, BC, and AC are called the sides of ΔABC and the points A, B, and C are called its vertices. Since the three points are noncollinear there exists a unique plane α incident upon them and each of the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$ lies in α. Therefore all the points of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ lie in α. We will say a geometric figure <u>lies in</u> a plane when all its points lie in the plane. Thus we have just shown that the ΔABC will lie in the unique plane incident upon its three vertices.

<u>2.10.1.</u> We could go further with our defining process but first let us add one more axiom to our axioms of order. This axiom, although

somewhat innocent in appearance, has a very significant role to play in all our further development.

2.10.2. Axiom 0-4. If ΔABC lies in plane α and if ℓ is a line of α which intersects one of the sides of the triangle than ℓ must have another point in common with ΔABC. (The axiom is called Pasch's axiom.)

2.10.3. A sketch will help us interpret the meaning of the axiom. Suppose ℓ intersects AB at a point D, then there exists at least a second point on ℓ, call it E, such that E is a point of ΔABC. The sketch suggests some of the various possibilities for the "position" of E.

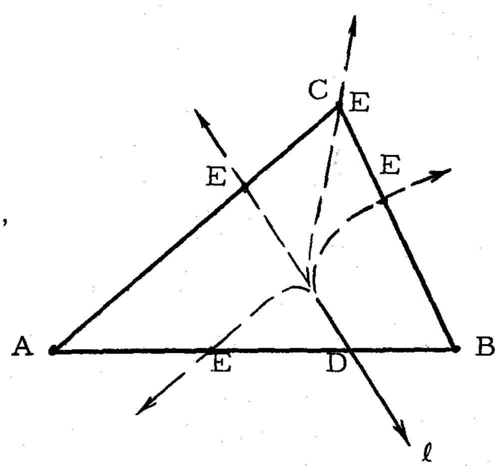


Figure 7.

2.11.1. Now we will explore the implications and relationships implied by our axiom system. In the remaining work of this chapter we will assume that our lines, etc. lie on some plane α unless otherwise stated. We can justify this by applying the lemmas and theorems in Chapter I.

2.11.2. Lemma 0-1. If a line ℓ intersects a side PR of ΔPQR and $\overleftrightarrow{PQ}$ in a point N such that PQN or NPQ then

ℓ meets $\overline{RQ}$.

Comment. Here we are being a little more specific than in Pasch's Axiom in order to get a more useful formulation.

Notation. It will be useful for us to use $\frown$ as a symbol for the intersection of two planar figures, even though this, in the sense we are using it, will not always represent the intersection of sets (see 1.20.2).

Proof. Let ΔPQR and line ℓ in plane α be given such that

(1) $\Phi(\ell) \frown PR = M$

(i.e., PMR and M on ℓ.)

(2) $\Phi(\ell) \frown \overleftrightarrow{PQ} = N$ such that PQN.

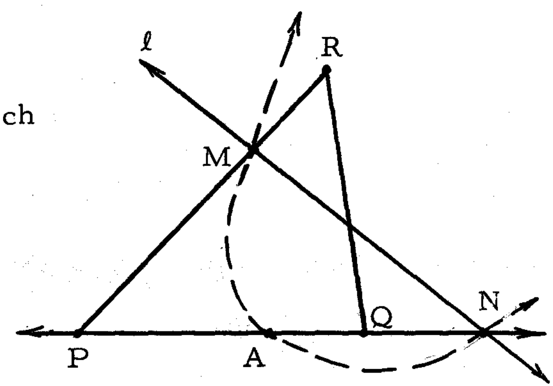


Figure 8.

(Note: If we take NPQ the proof is identical). By Pasch's Axiom ℓ intersects either $\overline{PQ}$ or $\overline{RQ}$. Our proof in a nutshell is to show that ℓ cannot intersect $\overline{PQ}$ because this would force, P, Q, and R to be collinear.

Assume ℓ intersects $\overline{PQ}$ at A. $A \neq N$ since PQN while A on $\overline{PQ}$ implies $A = P$, $A = Q$, or PAQ by definition, any of which would contradict PQN by 0-2 or the distinctness of P, Q, and N. Then $\ell = \overleftrightarrow{AN} = \overleftrightarrow{PQ}$ by I-2. But since PMR we know that P, M, and R are distinct points and therefore R lies on $\overleftrightarrow{PM}$. We already know that P and M are points of ℓ so R is on ℓ, again by I-2. Thus P, Q, and R are collinear points, contradicting our hypotheses. Thus ℓ cannot be intersect $\overline{PQ}$, but by 0-4 it must intersect $\overline{RQ}$ or $\overline{PQ}$, therefore ℓ intersects $\overline{RQ}$. We already know ℓ cannot intersect $\overline{RQ}$ at Q since this would mean it also intersected $\overline{PQ}$. ℓ cannot intersect $\overline{RQ}$ at R either, since if it did we would either have M = R contradicting PMR or ℓ intersecting $\overleftrightarrow{PR}$ at two distinct points. This forces $\overleftrightarrow{RP} = \ell$ and this cannot be since P and N on ℓ implies that $\overleftrightarrow{PQ}$ also equals ℓ giving P, Q, and R collinear again. There ℓ must intersect RQ at some point S. (That is S on ℓ and RSQ.)

<u>2.11.3. Theorem 0-1.</u> If A and C are any two points of line ℓ then there exists a point B on ℓ such that ABC (Hilbert, 1963).

<u>Proof.</u> By I-6 there exists a point E not on ℓ and by 0-3 there exists a point F such that AEF. F is not on ℓ since this would imply $\overleftrightarrow{AF} = \ell$ and E on ℓ. Therefore there exists $\overleftrightarrow{FC}$ distinct from ℓ by I-1 and a point G on $\overleftrightarrow{FC}$ such that FCG, according to 0-3. Now applying Lemma 0-1 to ΔACF with FCG and AEF we get a point B on AC, in other words there exists a

point B such that ABC. (Notice we needed Pasch's Axiom before we were able to assert that between any two points on a line there is a third point.)

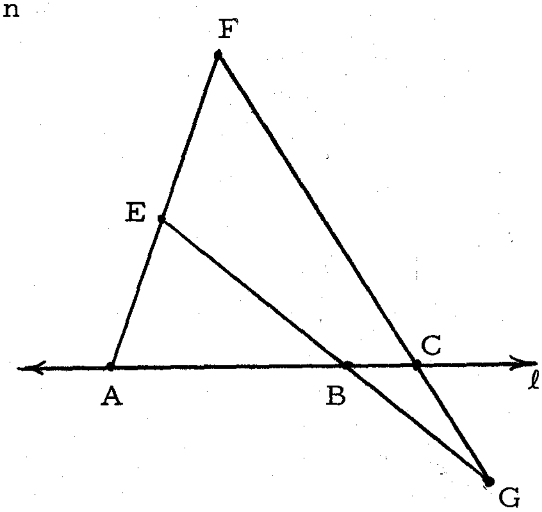


Figure 9.

2.11.4. Theorem 0-2. If A, B, C are points of line ℓ then at least one of the relations ABC, ACB, or BAC holds (Hilbert, 1962). (Note: This theorem combined with Axiom 0-2 says that "given any three points on a line exactly one of the points lies between the other two." Here again we first needed Pasch's Axiom, before this intuitively clear fact could be established from our axiom system.)

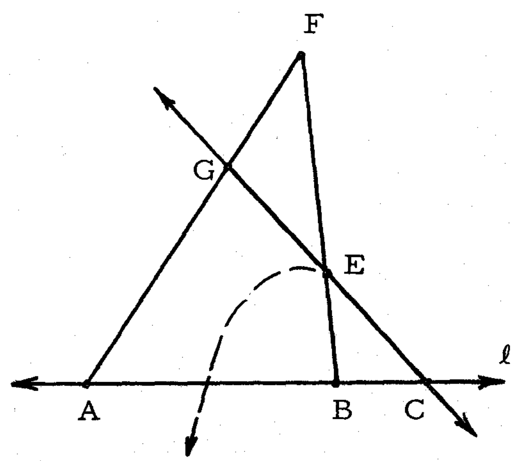


Figure 10.

Proof. Assume $\widetilde{ACB}$ and $\widetilde{BAC}$. There exists a point E not on line ℓ by I-6 and a point F such that BEF. F is not on ℓ since this would imply E on ℓ. Consider lines $\overleftrightarrow{AF}$ and $\overleftrightarrow{EC}$. A, B, and F determine $\triangle ABF$

since F is not on ℓ. Applying Pasch's Axiom to this triangle we see that $\overleftrightarrow{CE}$ intersects either $\overline{AF}$ or $\overline{AB}$. If $\overleftrightarrow{EC}$ intersects $\overline{AB}$, then since $\widetilde{ACB}$ we must have $\overleftrightarrow{EC}$ meeting ℓ in two distinct points and $\ell = \overleftrightarrow{EC}$, contradicting E not on ℓ. Therefore $\overleftrightarrow{EC}$ meets $\overline{AF}$ at some point G. G cannot be A since E would lie on ℓ again, as above. Also G cannot be F since E and F are distinct and $\overleftrightarrow{BE}$ and $\overleftrightarrow{CE}$ are distinct lines (i.e., $B \neq C$). Thus we must have G on AF that is AGF. G cannot be on ℓ since this would mean that either $G = A$, a possibility already eliminated, or G and A two distinct points of $\overleftrightarrow{AF}$ on ℓ forcing $\overleftrightarrow{AF} = \ell$ contradicting F not on ℓ. In a similar fashion, considering lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{CF}$, we get a point H on $\overleftrightarrow{FC}$ such that CHF. Now, A, H, and F are noncollinear since A cannot be on $\overleftrightarrow{FC}$ that is A, C, and F are noncollinear. (If A were on $\overleftrightarrow{FC}$ then we would have $\overleftrightarrow{FC} = \overleftrightarrow{AC} = \ell$ and F on ℓ, a contradiction.) By Lemma 0-1 we get AEH, since FGA and FHC. H is not on ℓ and therefore we have ΔAHC defined. Using AEH, CHF, and

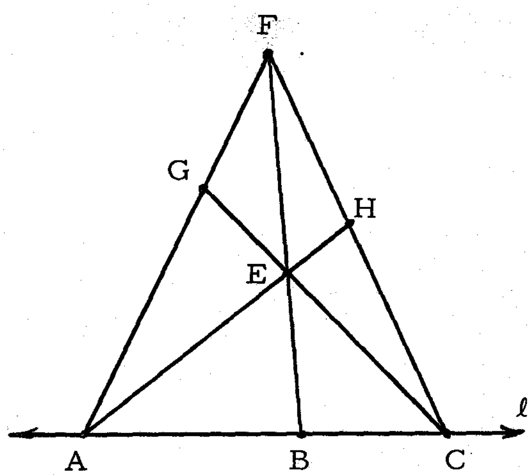


Figure 11.

Lemma 0-1 on ΔAHC we have ABC. Q.E.D.

2.11.5. Theorem 0-3. Given any three points on a line exactly one of the points lies between the other two.

Proof. See note after the statement of Theorem 0-2.

2.12.1. One of our major interests is in the number and ordering of points on a line. The previous lemma plays a key role in proving the following theorem which is our first step in this direction.

2.12.2. Theorem 0-4. Given any 4 distinct, collinear points they can be named A, B, C, and D in such a way that ABC, ABD, ACD, and BCD. Furthermore, this can be done in exactly two ways (Hilbert, 1962).

2.12.3. In the lemma below it will be shown that the relations ABC and ACD or ABC and BCD are sufficient to imply the other two. Because of this we can give the theorem in two other forms:

Theorem 0-4 (second form). Given any 4 distinct collinear points they can be named in such a way that ABC and ACD.

Theorem 0-4 (third form). Given any 4 distinct collinear points they can be named in such a way that ABC and BCD. (It will be convenient later in the proof to call the four points P, Q, R, and

S and the line ℓ.) First let us show that the second and third forms imply the first.

2.12.4. Lemma 0-2. If A, B, C, and D are points of line ℓ such that ABC and BCD then ABD and ACD. (Note: we need only show ACD since the proof is symmetric.)

Proof. ℓ lies in some plane α by Lemma I-1. There exists a point E on α such that E is not on ℓ according to I-6. There exists a point F such that DEF by 0-3. Now since E is not on ℓ {A, B, E}, {A, D, E}, and {B, D, E} are noncollinear sets and determine ΔABE, ΔADE, and ΔBDE according to the definition of a triangle. From FED and BCD we have a point G on BE by applying Lemma 0-1 to ΔBDE. Thus BGE. By Lemma 0-1, this time applied to ΔABE, with ABC and BGE we have a point H such that AHE. Finally, again applying Lemma 0-1, but this time to ΔADE with AHE and DEF we get ACD. A completely symmetric argument yields ABD. Q.E.D.

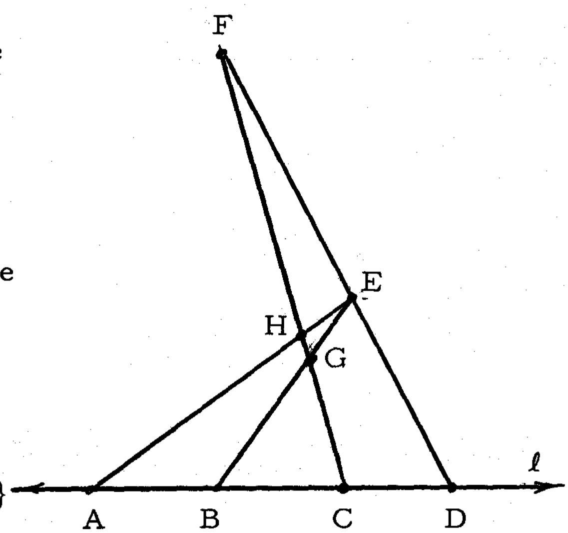


Figure 12.

2.12.5. Lemma 0-3. If A, B, C, and D are points of line ℓ such that ABC and ACD then ABD and BCD. (Our strategy here will be to show that BCD is a consequence of our hypothesis and then note that ABD follows from Lemma 0-2.)

Consider the proof and figure for Lemma 0-2. We repeat the first 3 steps of the proof getting E not on ℓ and F such that DEF. We will have the following triangles formed, ΔAED, ΔABE, and ΔBDE. Now applying Lemma 0-1 to ΔADE we have a point H such that AHE since ACD and DEF. With AHE and ABC Lemma 0-1 applied to ΔABE yields G on BE, or BGE. Finally, again with Lemma 0-1, we get BCD from DEF and BGE applied to ΔBDE. Now ABD follows by applying Lemma 0-2 with ABC and BCD. Q.E.D.

2.12.6. We are ready to return to Theorem 0-4. We see by Lemma 0-3 that the second form of the theorem implies the first, also Lemma 0-2 tells us that the third form implies the first.

The conclusion of the theorem requires that we establish two things:

2.12.7. (1) P, Q, R, and S on ℓ can be renamed with A, B, C, and D in such a way that ABC and ACD, using the second form, or ABC and BCD using the third form.

2.12.8. (2) That this renaming can be done in exactly two ways. If (1) can be shown then any other renaming of P, Q, R, and S would be a permutation of A, B, C, and D. So we must consider all permutation of the letters A, B, C, and D and show that there is exactly one permutation besides the identity which preserves the ordering given in the first form of the theorem.

2.12.9. Proof of (1). We are given four points P, Q, R, and S on line ℓ. Consider only the three points P, Q, and R for the moment. We know by Theorem 0-3 that exactly one of them lies between the other two, rename it Q' and the other two P' and R' so we have the following mutually exclusive possibilities: (i) P'R'S, (ii) SP'R', (iii) P'SR' by Theorem 0-3.

Case (i). We rename P' as A, Q' as B, R' as C, and S as D. Thus we have ABC and ACD and the second form of Theorem 0-4.

Case (ii). We rename S as A, P' and B, Q' as C, and R' as D. Then we have ABD and BCD and the second form again. (Note BCD and ABD is symmetric to ABC and ACD.)

Case (iii). In this case we must consider three subcases: (a) P'SQ', (b) P'Q'S, and (c) SP'Q'. (Recall we have P'Q'R' and P'SR'.)

Case (iii)(a). Here we have three betweenness relations: P'Q'R'*, P'SR', and P'SQ'**. By naming P' as A, S as B, Q' as C, and R' as D we have ACD and ABC from * and ** above. This yields the second form of the theorem. We must check the third relation, P'SR', to see if it agrees with the ordering implied by Lemma 0-3. P'SR' goes into ABD, which is consistent with the ordering derived in the lemma.

Case (iii)(b). For this case we have the relations: P'Q'R', P'Q'S*, and P'SR'**. By naming P' as A, Q' as B, S as C, and R' as D from * and ** we have ABC and ACD, and again the second form. Notice that P'Q'R' goes into ABD which is consistent according to Lemma 0-3.

Case (iii)(c). Finally, in this case we have: P'Q'R', P'SR', and SP'Q'. By 0-1 SP'Q' is equivalent to Q'P'S. From Lemma 0-1 P'SR' and Q'P'S yield Q'P'R'. But this and P'Q'R' are impossible by Axiom 0-2. That is, we cannot have P', Q', R', and S arranged on ℓ in such a way that Case (iii)(c) occurs.

Therefore, we have shown that in every case possible we can rename the points P, Q, R, and S in such a way that we get the four relations listed in the first form of the theorem.

2.12.10. It still remains to be shown that this renaming can be done in exactly 2 ways, that is part (2). We just established that there exists at least one way. We must show that there exists exactly one other permutation of (A, B, C, D), besides the identity, such that the four betweenness relations ABC, ABD, ACD, and BCD hold. Here we may let a picture and our intuition suggest the path to follow. Consider the four points as shown on the top line at the right. It would seem reasonable that renaming the points in the "opposite" order as indicated would also yield the same betweenness relations. That is, renaming by the permutation (AD)(BC). (Two comments on notation seem in order, we will use (A, B, C, D) as the ordered 4-tuple and ABCD to indicate the betweenness relations ABC, ACD, ABD, and BCD. Thus the permutation (AD) operating on (A, B, C, D) yields (D, B, C, A).)

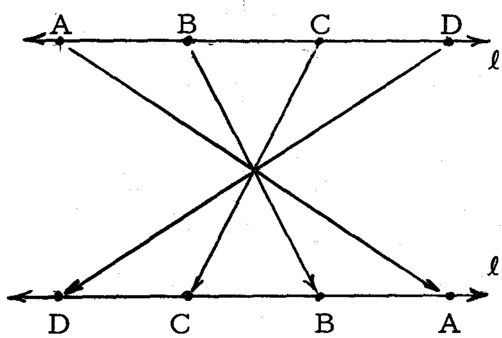


Figure 13.

If we apply (AD)(CB) to the betweenness relation ABCD,

$$\begin{array}{lcl} ABC & & DCB = BCD \\ ABD & & DCA = ACD \\ ACD & \xrightarrow{(AD)(BC)} & DBA = ABD \\ BCD & & CBA = ABC \end{array}$$

the results are shown above. (That is ABCD goes into DCBA.) We see that our betweenness relations are preserved, that is each of the relations goes into one of the others under the permutation.

We now know that there are at least two permutations of A, B, C, and D which yield ABCD. (Note: DCBA = ABCD according to 0-2.) We shall use the theory of the permutation group on four letters to show that there are exactly two.

<u>2.12.11.</u> The group of permutations on 4 letters is usually called S_4. Our modus operandi will be to show first that the set of permutations which preserves these relations form a subgroup of S_4 and then to show that the only subgroup of S_4 with this property is $\{e, (AD)(BC)\}$. For convenience we will designate $\{e, (AD)(BC)\}$ as $\mathcal{A}$ and the subset which preserves the betweenness relations as $\mathcal{B}$.

<u>2.12.12.</u> The proof that $\mathcal{B}$ is a subgroup of S_4 is trivial since if two permutations preserve the betweenness relations then their product will also. Thus we have $\mathcal{B}$ closed under our group operation and therefore a subgroup of S_4.

<u>2.12.13.</u> Now we want to establish that $\mathcal{B} = \mathcal{A}$. By our definition of $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ is a subgroup of $\mathcal{B}$ since $\mathcal{A}$ is closed under our operation. Now by Lagrange's Theorem, the order of $\mathcal{A}$ divides the order of $\mathcal{B}$. Thus the order of $\mathcal{B}$ is a multiple of 2.

Again, by Lagrange's Theorem $o(\mathcal{B})$ must be a divisor of $o(S_4) = 24$. So the possible values for $o(\mathcal{B})$ are 2, 4, 6, 8, 12, and 24.

The remainder of the proof will consist of showing that any subgroup, containing $\mathcal{a}$, of order greater than two must contain an element which does not preserve the betweenness relations. This will be done by first finding the subgroups of order 4 which contain $\mathcal{a}$ and demonstrating the existence of such an element in each. Then we will use this result to demonstrate that the subgroup of order 8 containing $\mathcal{a}$ also contains the ones of order 4 and thus cannot preserve the relation. Then, finally we will show that the subgroups of order 6 or 12 must have an element which does not preserve our betweenness relations by appealing to the order of the subgroups and the order of their elements.

From group theory we know that any group of order 4 must be isomorphic to either the Cyclic group of order 4 or the Klein 4's group. Now if $\mathcal{B}$ is isomorphic to the Cyclic group of order 4 then $\mathcal{B}$ if of the form $\{e, a, a^2, a^3\}$. What does a represent? If we use functional notation to represent our permutation denoting (AD)(BC) by f, (that is $f(A) = D$, $f(D) = A$, $f(B) = C$, and $f(C) = B$), then e, a, a^2, and a^3 are functions. In order for $\mathcal{B}$ to be cyclic $a \neq e$, $a^2 \neq e$ and $a^3 \neq e$ but $a^4 = (a^2)^2 = e$. Since $[(AD)(BC)]^2 = f^2 = e$ we must have $a^2 = f$. We have then that $a^2 = (a^3)^2 = f$, so to solve for a we are seeking the images of A, B, C, and D under a such that $a^2(A) = D$, $a^2(B) = C$, $a^2(C) = B$, and $a^2(D) = A$.

Let

$$a(A) = X \quad (1)$$

$$a(B) = Y \quad (2)$$

$$a(C) = Z \quad (3)$$

$$a(D) = W \quad (4)$$

where $\{A, B, C, D\} = \{X, Y, Z, W\}$. (That is $X \in \{A, B, C, D\}$ etc. Also X, Y, Z, and W are distinct since a is a permutation.)

If we solve for the values of X, Y, Z, and W then we will have the form a and a^3. Since $a^2(A) = a(a(A)) = a(X) = D$, we also have the equation $a(X) = D$ (5) and in a similar fashion

$$a(Y) = C \quad (6)$$

$$a(Z) = B \quad (7)$$

$$a(W) = A \quad (8)$$

Now X cannot be A or D, for suppose $X = A$, then we would have $a(A) = A$ from (1) and $a(A) = D$ from (5) contradicting our definition of a. Similarly W cannot be A or D and Y or Z cannot be B or C. Now suppose $\underline{X = B}$. Then $a(A) = B$ from (1) and $a(B) = D$ from (5) but this tells us that

$\underline{Y = D}$ from (2). In (6) we get $a(D) = C$ and this tells us from (4) that $\underline{W = C}$. Finally in (8) we have $a(C) = A$ and this yields $\underline{Z = A}$ from (3). This checks out in (7). So $a = (ABDC)$. To check consider $a^2 = (ABDC)^2 = (AD)(BC) = f$. Now $a^3 = (ACDB)$. Therefore $\underline{\mathcal{B} = \{e, (ABDC), (AD)(BC), (ACDB)\}}$. The other possible choice for X, $X = C$ yields $(ACDB)$ after a similar analysis. Thus we have found the only possible candidate for $\mathcal{B}$ if it is to be Cyclic of order 4.

We still have to investigate if there are subgroups of S_4 isomorphic to the Klein 4's group which contain α. This group would be of the form $\{e, a, b, ab\}$ where $a^2 = b^2 = (ab)^2 = e$. If we let $ab = (AD)(BC)$ then two obvious candidates for a and b should occur to us. Let $a = (AD)$ and $b = (BC)$ then it follows immediately that $\{e, (AD), (BC), (AD)(BC)\}$ is a Klein 4's subgroup containing a. Are there any others? If there are then we must have $a^2(X) = b^2(X) = X$ for $X \in \{A, B, C, D\}$ and

$$ba(A) = D = ab(A) \quad (1)^{1/}$$

$$ba(B) = C = ab(B) \quad (2)$$

$$ba(C) = B = ab(C) \quad (3)$$

$$ba(D) = A = ab(D) \quad (4)$$

1/ $ab = (AD)(CB)$, which can be chosen arbitrarily by the nature of the Klein 4's group.

since any group of order 4 is abelian. Again we will denote

$$(5) \quad a(A) = X$$

$$(6) \quad a(B) = Y$$

$$(7) \quad a(C) = Z$$

$$(8) \quad a(D) = W$$

with X, Y, Z, and W distinct. From column one in (1-4) above we have

$$(9) \quad ba(A) = b(X) = D$$

$$(10) \quad ba(B) = b(Y) = C$$

$$(11) \quad ba(C) = b(Z) = B$$

$$(12) \quad ba(D) = b(W) - A.$$

Now if $X = A$ then $a(A) = A$ and $b(A) = D$ from (5) and (9) since $\mathcal{B}$ is abelian $ba(A) = ab(A) = a(D) = D$. Thus a acts as the identity on A and D. If we would let $Z = C$ by a similar argument we see that $a(C) = C$ and $a(B) = B$. Thus $a = e$ and $b = (AD)(BC)$. So $\mathcal{B}$ has two elements instead of 4. On the other hand if Z were chosen to be B then from (11) $b(Z) = b(B) = B$ and b acts as the identity on B and C. In this case since $ab = (AD)(BC)$ we have $a = (BC)$ and $b = (AD)$ and the first possibility investigated.

So we might try either B or C for X, say B. That is $\underline{a(A) = B}$ from (5). Immediately we have $\underline{b(B) = D}$ from (9) and from (2) $ba(B) = ab(B) = \underline{a(D) = C}$. Now $ba(D) = A$ from (4) yields $\underline{b(C) = A}$. Thus from (12) we have $\underline{W = C}$, (i.e., we have $\underline{\underline{X = B}}$ and $\underline{\underline{W = C}}$.) We then have Y as either A or D. But, we also know $a^2 = b^2 = e$ so $a^2(A) = \underline{a(B) = A}$ and in the same way $a^2(D) = \underline{a(C) = D}$, $b^2(B) = \underline{b(D) = B}$, and $b^2(C) = \underline{b(A) = C}$. ($\underline{\underline{Y = A}}$ and $\underline{\underline{Z = D}}$.)

Summarizing

$$
\begin{array}{ll}
a(A) = B & b(A) = C \\
a(B) = A & b(B) = D \\
a(C) = D & b(C) = A \\
a(D) = C & b(D) = B
\end{array}
$$

or $a = (AB)(CD)$ and $b = (AC)(BD)$.

Checking we see that $a^2 = b^2 = e$ and $ab = (AD)(BC)$. If we had chosen X as C this would have interchanged a and b. We have investigated all possibilities for the values of X, Y, Z, and W if $\mathcal{B}$ is a Klein 4's group. So in summary there exist three subgroups of S_4 of order 4 containing $\mathcal{A}$, they are:

one cyclic subgroup $\mathcal{C} = \{e, (ABDC), (AD)(BC), (ACDB)\}$

and Klein 4's subgroups $\mathcal{K}_1 = \{e, (AD), (BC), (AD)(BC)\}$

and $\mathcal{K}_2 = \{e, (AB)(CD), (AC)(BD), (AD)(BC)\}$.

Now we must demonstrate that each of these subgroups contains a permutation which does not preserve the betweenness relations.

In $\mathcal{C}$ consider (ABDC)'s action on our betweenness relations

$$\begin{array}{ccc} ABC & & BDA \\ ABD & & BDC \\ ACD & \xrightarrow{(ABDC)} & BAC \\ BCD & & DAC \end{array}$$

Now our betweenness relations are not preserved since ABC is contradicted by BAC due to 0-2. So $\mathcal{B} \neq \mathcal{C}$.

In $\mathcal{K}_1$ consider (AD)'s action on these relations.

$$\begin{array}{ccc} ABC & & DBC \\ ABD & & DBA \\ ACD & \xrightarrow{(AD)} & DCA \\ BCD & & BCA \end{array}$$

DBC contradicts BCD for example, so $\mathcal{B} \neq \mathcal{K}_1$.

Finally in $\mathcal{K}_2$, consider (AB)(CD)'s action with BAD contradicting ABD according to 0-2.

$$\begin{array}{ccc} ABC & & BAD \\ ABD & & BAC \\ ACD & \xrightarrow{(AB)(CD)} & BDC \\ BCD & & ADC \end{array}$$

Therefore we have shown that $\mathcal{B}$ cannot be $\mathcal{C}$, $\mathcal{K}_1$, or $\mathcal{K}_2$, that is the order of $\mathcal{B}$ is not 4.

Could $\mathcal{B}$ have order 8? By Corollary IV to the 2nd Sylow Theorem (Carmichael, 1937) there exists in any group of order 8 containing $\mathcal{a}$ a subgroup of order 4 which also contains $\mathcal{a}$. But the only subgroups of order 4 containing $\mathcal{a}$ do not preserve the betweenness relations, so any group of order 8 containing $\mathcal{a}$ would not either. (See the adjoining sheet for the 2-Sylow subgroups of S_4 containing $\mathcal{a}$. The two conjugates of this group are given below it in roster form.)

So far we have been able to eliminate 8 and 4 as possibilities for the order of $\mathcal{B}$. If $\mathcal{B}$ has order 6 or 12 it must, according to Sylow's first theorem contain a subgroup of order 3, but any subgroup of order 3 must be cyclic. But any cyclic subgroup of order 3 must be of the form $\{e, (XYZ), (XZY)\}$ where X, Y, and Z are distinct elements of $\{A, B, C, D\}$. This follows since a cyclic subgroup of order 3 is of the form $\{e, a, a^2\}$ where $a^3 = e$. If we check the possibilities for a, (for example if $a = (XYZW)$ then

The 2-Sylow Subgroup of S_4 Containing

$\mathcal{A}$ indicated by

*	e	(AD)	(BC)	(AB)(CD)	(AC)(BD)	(AD)(BC)	(ABDC)	(ACDB)
e	e	(AD)	(BC)	(AB)(CD)	(AC)(BD)	(AD)(BC)	(ABDC)	(ACDB)
(AD)	(AD)	e	(AD)(BC)	(ACDB)	(ABDC)	(BC)	(AC)(BD)	(AB)(CD)
(BC)	(BC)	(AD)(BC)	e	(ABDC)	(ACBD)	(AD)	(AB)(CD)	(AC)(BD)
(AB)(CD)	(AB)(CD)	(ABDC)	(ACDB)	e	(AD)(BC)	(AC)(CD)	(AD)	(BC)
(AC)(BD)	(AC)(BD)	(ACDB)	(ABDC)	(AD)(BC)	e	(AB)(CD)	(BC)	(AD)
(AD)(BC)	(AD)(BD)	(BC)	(AD)	(AC)(BD)	(AB)(CD)	e	(ACDB)	(ABDC)
(ABDC)	(ABDC)	(AB)(CD)	(AC)(BD)	(BC)	(AD)	(ACDB)	(AD)(BC)	e
(ACDB)	(ACDB)	(AC)(BD)	(AB)(CD)	(AD)	(BC)	(ABDC)	e	(AD)(BC)

The two other 2-Sylow subgroups.

{e, (AB), (CD), (AB)(CD), (AC)(BD), (AD)(BC), (ACBD), (ADBC)}

{3, (AC), (BD), (AB)(CD), (AC)(BD), (AD)(BC), (ABCD), (ADCB)}

$a^2 = (XZ)(YW)$ and $a^3 = (XWZY) \neq e$), we find that this is the only possible form. For whatever values of X, Y, and Z are chosen there exists a betweenness relation involving these values. For example X, Y, and Z might be A, B, and C respectively. (Note: We may assume this since we can write our cycles with any first letter and since we have the second and third letters in both possible orders in (XYZ) or (XZY) it would not make any difference whether Y = B or C.) One of our betweenness relations is ABC. But (ABC) takes ABC into BCA (or ACB), and we know by Axiom 0-2 this cannot occur. The same argument can be carried out for any other possible values of X, Y, and Z. Therefore the order of $\mathcal{B}$ is neither 6 nor 12. It follows that the order of $\mathcal{B}$ cannot be 24 either since this is the order of S_4 and we have established that S_4 contains elements which do not preserve the betweenness relations. Thus the only possible value for the order of $\mathcal{B}$ is 2 and we have $\mathcal{B} = \mathcal{A}$. (This completes the proof of Theorem 0-4.)

2.12.14. Several comments should be made about the previous proof: First, the two lemmas, which were the keys to the proof, and a sketch of the existence of at least one such ordering can be found in Hilbert's later works. He did not establish that there exist exactly two such orderings. In his first edition the above theorem was taken as an axiom. This was revised later after it was established to be a consequence of the previous axioms by E.H. Moore in Trans. Am. Math. Soc. (1902).

Secondly both in this proof and in the proof of Theorem 0-5, which follows, more detail is given than would be necessary for a reader who is well versed in the theory of permutation groups. This was done because I hope to be able to utilize this material in an undergraduate geometry course where such detail would be necessary.

2.13.1. The obvious extension of the previous theorem is the ordering of n points of a line for any n.

Theorem 0-5. Given n points $P_1, P_2, \ldots, P_n$ on line ℓ they can be renamed $A_1, A_2, \ldots, A_n$ in exactly two ways such that $1 \leq i < j < k \leq n$ or $1 \leq k < j < i \leq n$ iff $A_iA_jA_k$ (Hilbert, 1962).

Proof. Again we have two parts to the proof: (1) There exists a renaming such that the indicated betweenness relations hold, and (2) That this can be done in exactly two ways. We will prove both (1) and (2) by induction.

2.13.2. Let $\mathcal{B}_n$ be the subgroup of S_n preserving the between relation $A_iA_jA_k$ iff $1 \leq i < j < k \leq n$ or $1 \leq k < j < i \leq n$. The proof consists of showing that $o(\mathcal{B}_n) = 2$ for $n \geq 2$ by showing (1), that is $\mathcal{B}_n \neq \phi$, by induction, and then showing (2) by first establishing $o(\mathcal{B}_n) \geq 2$ and then that $o(\mathcal{B}_n) > 2$ results in a contradiction.

(This is essentially the same procedure that we used in Theorem 0-4.)

2.13.3. First we see that the theorem is true for $n = 2, 3,$ or 4. It holds for $n = 2$ trivially; for $n = 3$ by Theorem 0-3 and an appropriate renaming. For $n = 4$ we have Theorem 0-4 and renaming $A = A_1$, $B = A_2$, $C = A_3$, and $D = A_4$. So we have more than enough to start our induction.

2.13.4. Assume $B_1, B_2, \ldots, B_m$ are m distinct points of a line ℓ ordered such that if $1 \leq i < j < k \leq m$ then $B_iB_jB_k$ and let P be any other point of ℓ. Now consider the points P, B_1, and B_m, we must have exactly one of the three cases (i) B_1B_mP, (ii) B_1PB_m, or (iii) PB_1B_m.

In Case (i) we rename $B_i = A_i$ and $P = A_{m+1}$ and we must show $A_iA_jA_k$ iff $1 \leq i < j < k \leq m+1$. If $k \leq m$ then $A_iA_jA_k$ follows from our induction assumption for the B's. So consider $k = m+1$. Thus we have three points $A_i = B_i$, $A_j = B_j$, and $A_k = A_{m+1} = P$. We know that B_1B_mP so if $i = 1$, and $j = m$ the betweenness relation is immediate. If $i = 1$ and $j \neq m$ then $B_1B_jB_m$ by our induction assumption and B_1B_mP implies B_1B_jP or $A_1A_jA_{m+1}$ by Lemma 0-3. On the other hand if $i \neq 1$ but $j = m$ then we have $B_1B_jB_m$ and B_1B_mP implying

B_iB_mP or $A_iA_mA_{m+1}$ again by Lemma 0-3. Finally if $1 < i < j < m$ then B_1B_mP and $B_1B_iB_m$ yields B_iB_mP by Lemma 0-3 which together with $B_iB_jB_m$ by the induction assumption, gives us B_iB_jP or $A_iA_jA_{m+1}$ again by Lemma 0-3. Thus in all possible cases $1 \leq i < j < k \leq m+1$ implies $A_iA_jA_k$. So we have established the existence of the desired ordering for any n in Case (i).

Case (iii) follows in a similar manner to Case (i) by renaming P as A_1 and B_i as A_{i+1}.

Case (ii) We have B_1PB_m. Now, either B_1B_2P or B_1PB_2. (Note: We cannot have PB_1B_2 since PB_1B_2 and $B_1B_2B_m$ implies PB_1B_m contradicting B_1PB_m by 0-3.) If B_1PB_2 then we can name B_1 as A_1, P as A_2, and B_i as A_{i+1} for $2 \leq i \leq m$. Then applying Lemma 0-3 as in Case (i) we have $A_iA_jA_k$ when $1 \leq i < j < k \leq m+1$. So we will assume B_1B_2P. Let q be the first integer greater than 1 such that B_1B_qP and B_1PB_{q+1}. Does such an integer q exist? Let $T = \{t \mid B_1PB_t\}$. T is nonempty since $m \in T$ by hypothesis. We also have $t \geq 2$ from the assumption B_1B_2P. T is a finite set of integers and therefore contains a least element, call it $q+1$. It follows from the definition of T that B_1PB_{q+1} but since $q+1$ is the least element of T we must have B_1B_qP. (Note: PB_1B_q is impossible as it would contradict B_1PB_m by Lemma 0-3.)

We now rename B_i as A_i for $1 \leq i \leq q$, P as A_{q+1} and B_i as A_{i+1} for $q+1 \leq i \leq m$. We must establish the desired betweenness relations. There are four subcases possible

(a) $P \notin \{A_i, A_j, A_k\}$, (b) $P = A_k$, (c) $P = A_j$, or (d) $P = A_i$.

Case (ii)(a). We will have $A_iA_jA_k$ since these points are ordered according to the subscripts of the corresponding B's. It will be left to the reader to check the various possible relationships between i, j, k and q.

Notice. From B_1B_qP and B_1PB_{q+1} we have B_qPB_{q+1} by Lemma 0-3. We will use this several times in the remainder of the proof.

Case (ii)(b). Since $P = A_k$ we will have $B_i = A_i$ and $B_j = A_j$. A still finer breakdown is needed: $A_j = A_q$ or $A_j \neq A_q$.

First if $A_j = A_q$ then we have $A_i = B_i$ and $A_i = A_q = B_q$. We know $B_iB_qB_{q+1}$ and applying Lemma 0-3 to this and B_qPB_{q+1} above we have B_iB_qP. So we have $A_iA_qA_{q+1}$ or $A_iA_jA_k$ as desired.

If $A_j \neq A_q$ then, in terms of the B's and from the definition of i, j, and k we get $B_iB_jB_q$. By applying the proof of B_iB_qP we know B_jB_qP and applying Lemma 0-2 to this situation we get B_iB_jP or $A_iA_jA_k$. Thus Case (ii)(b) is completed.

Case (ii)(d). If $A_i = P$ then an argument quite similar to the one for Case (ii)(b) can be constructed and will not be pursued further.

Case (ii)(c). Here, we recall, we have $A_j = P = A_{q+1}$. Thus we want to show $A_i A_{q+1} A_k$ when $1 \leq i < q+1 < k \leq m+1$. Because $i \leq q$ we know $A_i = B_i$ and since $k > q+1$ we have $A_k = B_{k-1}$ with $k-1 > q$. If $i = q$ and $k = q+2$ we have $B_q P B_{q+1}$ and thus $A_q A_{q+1} A_{q+2}$. If $k > q+2$ and $i = q$ we have $B_q P B_{q+1}$ as above and $B_q B_{q+1} B_{k-1}$ from the induction assumption and combining with Lemmas 0-3 and 0-2 we get $B_q P B_{k-1}$ or $A_q A_{q+1} A_k$. In a similar fashion if $i < q$ and $k = q+2$ we get $A_i A_{q+1} A_{q+2}$. Finally if $i < q$ and $k > q+2$ we first get $A_i A_{q+1} A_{q+2}$ as above. Then, also from Lemma 0-3 applied to $B_q P B_{q+1}$ and $B_q B_{q+1} B_{k-1}$ we have $P B_{q+1} B_{k-1}$ which yields on renaming $A_{q+1} A_{q+2} A_k$. Now combining these two results by Lemma 0-2 we have $A_i A_{q+1} A_k$.

This completes the proof of the final case of part (1) of the proof and therefore in all possible cases we have $1 \leq i < j < k \leq m+1$ implying $A_i A_j A_k$. Conversely, for the namings above, if $A_i A_j A_k$ then $i < j < k$ or $i > j > k$. Suppose, for instance, that $i < k < j$, then by the proof above, $A_i A_k A_j$ and we have a contradiction to Axiom 0-2. Similarly in the other possible cases there are contradictions if at least one of $i < j < k$ or $i > j > k$ does not hold. Consequently $A_iA_jA_k$ implies $i < j < k$ or $i > j > k$. Thus we have shown by induction the existence of a naming of the points so that the desired ordering of the points occurs. For convenience we will denote these betweenness relations on $\{A_1, A_2, \ldots, A_n\}$ as $A_1A_2A_3\ldots A_{n+1}$, just as we did with A, B, C, and D in Theorem 0-4. We still must show that this naming can be done in exactly two ways; but first we will prove the following lemma which will expedite our work.

<u>2.13.5. Lemma 0-4.</u> If $A_1, A_2, \ldots, A_n$ are any n collinear points such that $A_iA_jA_k$ iff $i < j < k$ or $k < j < i$ then any cycle of length $m \geq 3$ applied to the A_i's cannot preserve the betweenness relations.

<u>Proof.</u> Without loss of generality we can assume $m = n$, this saves us notational difficulties. We recall that a permutation is a one-to-one function from a finite set onto itself. We will denote our permutation as α and note by hypothesis that it is a cycle of length $m \geq 3$. Now there exists an A_r and an A_s such that $\alpha(A_1) = A_r$ and $\alpha(A_s) = A_1$ since α is an onto function. $s \neq r$ since if it were we could write α as a disjoint product of a cycle of length $m-2$ with (A_1A_r) contradicting α a cycle of length m. Now since $r \neq s$ we must have $r < s$ or $r > s$, see Figure 14(a) and (b) below.

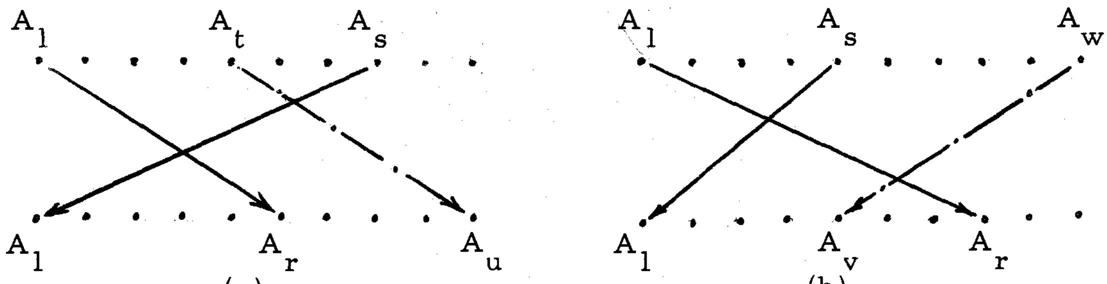


Figure 14.

<u>Case (i).</u> $r < s$, see Figure 14(a). Now there are $s-2$ points A_i such that $1 < i < s$ and $r-2$ points A_j such that $1 < j < r$ and since $r < s$ we have $r-2 < s-2$. Therefore there exists a point A_t such that $\alpha(A_t) = A_u$ and $1 < t < s$ but $u > r > 1$. Therefore $A_1 A_t A_s$ but $\wideparen{\alpha(A_1)\alpha(A_t)\alpha(A_s)}$ since $A_1 A_r A_u = \alpha(A_s)\alpha(A_1)\alpha(A_t)$.

<u>Case (ii).</u> See Figure 14(b). In this case we have $r-2 > s-2$ and therefore there exists an A_v with $1 < v < r$ which is the image of A_w under α and with $1 < s < w$. Thus we have $A_1 A_s A_w$ but $A_1 A_v A_r = \alpha(A_s)\alpha(A_w)\alpha(A_1)$ and therefore $\wideparen{\alpha(A_1)\alpha(A_s)\alpha(A_n)}$ by 0-2.

<u>2.13.6. Proof of Part (2).</u> The method we will use was outlined in 2.13.2. Essentially it will be to show that $o(\mathcal{B}_n) = 2$. From part (1) we have $o(\mathcal{B}_n) \geq 1$. We will use the form of induction which assumes that $o(\mathcal{B}_m) = 2$ for all $m \leq n$ and use this to show

$o(\mathcal{B}_{n+1}) = 2$. First we should note that, as with 4 points, the involution permutation

$$\rho_n = (A_1A_n)(A_2A_{n-1}) \ldots = \prod_{i=1}^{[n/2]} (A_iA_{n-i})$$

also preserves the betweenness relation since if $A_iA_jA_k$ then we must have $i < j < k$ or $k < j < i$ which in turn implies $n-i > n-j > n-k$ or $n-k > n-j > n-i$ and this leads us to $A_{n-i}A_{n-j}A_{n-k}$. Thus we have a renaming which will preserve our betweenness relations, in other words we have shown $o(\mathcal{B}_n) \geq 2$ for all n.

Now assume that $o(\mathcal{B}_m) = 2$ for all $m \leq n$, that is that the only permutation which preserve the ordering of $A_1, A_2, \ldots, A_m$ is the identity or the involution. Consider $\alpha \in \mathcal{B}_{n+1}$ and let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ be the unique disjoint cyclic decomposition of α. We may assume for convenience that A_1 is in the orbit of $\mathcal{C}_1$ and in fact $\mathcal{C}_1$ starts with A_1. (Recall that the orbits of the cycles are the equivalence classes of the set $\{A_1, A_2, \ldots, A_{n+1}\}$ under the equivalence relation $A_i \equiv A_j$ iff there exist an ℓ such that $\alpha^{\ell}(A_i) = A_j$.) Thus since these orbits are disjoint and subsets of $\{A_1, A_2, \ldots, A_{n+1}\}$ the action of α on each of them is to preserve the betweenness relation within each orbit since α is an element of $\mathcal{B}_{n+1}$. But applying Lemma 0-4 to the orbits of each of the cycles we see that each $\mathcal{C}_i$ must be of length one or two. (To apply the lemma we would first rename the elements of the orbit of $\mathcal{C}_i$ as B_1, B_2 to B_{n_i}, where n_i is the length of the orbit of $\mathcal{C}_i$. We would assign the subscripts in agreement with those of the B_i's.) Thus we can write C_1 as (A_1) or $(A_1, \alpha(A_1))$.

Assume $\mathcal{C}_1 = (A_1)$, that is α takes A_1 into A_1. Now $\mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_r$ by our induction assumption must be either the identity or the involution permutation acting on $A_2, A_3, \ldots, A_{n+1}$ since α is assumed to preserve the betweenness relations. If $\mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_n$ is the involution permutation then $\mathcal{C}_2 = (A_2A_{n+1})$ but then α takes $A_1A_2A_{n+1}$ into $A_1A_{n+1}A_2$ contradicting $A_1A_2A_{n+1}$ by 0-2. Therefore $\mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_r$ is the identity on $A_2, A_3, \ldots, A_{n+1}$. That is if α takes A_1 into A_1, α must act as the identity on all the A_i's.

The other possibility is that $\mathcal{C}_1 = (A_1A_k)$ for $1 < k \leq n+1$. Assume $\mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_r$ acts as the identity on the remaining A_i's. If $k \neq n+1$ then α acting on $A_1A_kA_{n+1}$ yields $A_kA_1A_{n+1}$ contradicting $A_1A_kA_{n+1}$ by 0-2. If $k = n+1$ and since $n \geq 3$ we know that A_2, A_3, and A_{n+1} are distinct points. Consider the effect of α upon $A_1A_2A_3$, A_1 goes into A_{n+1}, A_2 into A_2, and A_3 into A_3 so we have $A_2A_3A_{n+1}$ which by 0-1 is equivalent to $A_{n+1}A_3A_2$ but this is $\alpha(A_1)\alpha(A_3)\alpha(A_2)$ that is

$\overgroup{\alpha(A_1)\alpha(A_2)\alpha(A_3)}$. This does not preserve the betweenness relation $A_1A_2A_3$. Therefore $\mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_r$ must be the involution permutation on $A_2, A_3, \ldots, A_n$. We have established then that for any $n \geq 3$ the only permutations which preserve the betweenness relations on $A_1, A_2, \ldots, A_n$ are the identity and involution permutation. Thus we have completed the proof of Theorem 0-5.

<u>2.14.</u> The remainder of our work in this chapter will be devoted to investigating the further consequences of our axiom system. We will want to see how far we can develop geometry, what concepts can be introduced, and what theorems can be proved solely on the basis of the axioms of incidence and betweenness. Two rather immediate consequences of our preceding theorem are the following two theorems.

<u>2.15.1. Theorem 0-6.</u> There exists an infinite number of points on any line. (Notice: A set S is infinite iff there exists a one-to-one map of the natural numbers into S. So if a set is not infinite we can presume it contains only a finite number of elements.)

<u>Proof</u>. Assume the contrary, that is assume there exists a line ℓ with only a finite number of points on it. Denote the number of points by n. By Theorem 0-5 the points can be named $A_1, A_2, \ldots, A_n$ such that $A_iA_jA_k$ iff $i < j < k$ or $i > j > k$. Now by Axiom 0-3 there is a point P on ℓ such that A_1A_nP. This implies

immediately A_1 and A_n are distinct from P. P cannot be A_i for any $1 < i < n$ since this would imply both A_1PA_n and A_1A_nP contradicting 0-2. Thus P is a point distinct from the previous points and we have $n+1$ points on ℓ contradicting the assumption that there are n points on ℓ. Therefore there exists an infinite number of points on each line.

2.15.2. Theorem 0-7. Between any two points of a line there are infinitely many points (Hilbert, 1962). (In the following proof we will exhibit a one-to-one function from N into the points of a line between two given points. See the note after Theorem 0-6.)

Proof. Let A_0 and P be any arbitrary points on any line ℓ. By Theorem 0-1 there exists a point A_1 such that A_0A_1P. Similarly there exist points A_2, A_3, and so on such that $A_iA_{i+1}P$ for $i \geq 0$. Now we wish to prove by induction that if $A_1, A_2, \ldots, A_n$ are distinct points on ℓ between A_0 and P then there exists a point A_{n+1} distinct from the others such that A_{n+1} is between A_0 and P. We may assume $A_0, A_1, \ldots, A_n$ ordered such that $A_iA_jA_k$ iff $i < j < k$ or $i > j > k$, by Theorem 0-5. We may also assume $i < j$ implies A_iA_jP by Theorem 0-1, and Lemmas 0-2 and 0-3. Now there exists a point A_{n+1} such that $A_nA_{n+1}P$ by Theorem 0-1. A_{n+1} is not A_n nor P by definition. If $A_{n+1} = A_j$ for $0 \leq j < n$ then A_jA_nP holds by our induction assumption but

A_nA_jP must hold by the equivalence $A_j = A_{n+1}$. So A_{n+1} must be distinct from the previous points according to 0-2. Now A_0A_nP holds according to our induction assumption, and this, along with $A_nA_{n+1}P$, implies $A_0A_{n+1}P$ by Lemma 0-2. So we have the existence of n points between A_0 and P implies the existence of n+1 points there also. Now if we define $F(n) = A_n$ we have a one-to-one function mapping N into the points between A_0 and P on ℓ and therefore this is an infinite set.

2.15.3. The implications of the above theorems are that each line and line segment contains an infinite number of points. It should be noted that Theorem 0-7 actually implies 0-6 but the ordering of the points is somewhat different in the two theorems. We could have established Theorem 0-7 first and taken 0-6 as a corollary, but the restriction that the infinity of points lying between two fixed points seems unnecessary so the above presentation was chosen.

2.16.1. We have been careful in our discussion to distinguish our undefined terms of point and line from the sets of points on them. That is we have not assumed that lines or planes are sets of points. It will be convenient to designate the set of points on line ℓ by $\Phi(\ell)$ and those on plane α by $\Phi(\alpha)$.

2.16.2. Each point of a line ℓ separates $\Phi(\ell)$ into 3 disjoint sets. Intuitively we have the set consisting of the point P, the points on ℓ "to the left of P" and the points on ℓ "to the right of P." To make this more rigorous we know there exists another point Q on ℓ by I-3. Then we establish the three subsets of $\Phi(\ell)$ by utilizing Q to determine them. Let

$\mathcal{S}_1 = \{R \mid PRQ, P = Q \text{ or } PQR\}$, $\mathcal{S}_2 = \{R \mid RPQ\}$, and $\mathcal{S}_3 = \{P\}$. From Theorem 0-5 we see this exausts the possible "positions" for R on ℓ, so we have $\Phi(\ell) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. The definitions of $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ along with Axiom 0-2 give us the disjointness of the sets.

Now if $R_1, R_2 \in \mathcal{S}_1$ then $\overline{R_1R_2} \subset \mathcal{S}_1$. Assume not. Then there exists a points $S \in R_1R_2$ such that $S \in \mathcal{S}_2 \cup \mathcal{S}_3$. S cannot be R_1 or R_2 because they cannot be in $\mathcal{S}_2 \cup \mathcal{S}_3$ from their definition. If $S \in \mathcal{S}_3$ then $S = P$ and this implies R_1PR_2 by definition of a segment. But R_1PR_2 and PR_2Q implies R_1PQ by Lemma 0-2, contradicting $R_1 \in \mathcal{S}_1$ by 0-2. If $S \in \mathcal{S}_2$ then SPQ. Now from the above we have either R_1R_2P or R_2R_1P and these combined with R_1SR_2 yield SR_2P or SR_1P by Lemma 0-3. But the former combined with SPQ yields R_2PQ while the latter yields R_1PQ by Lemma 0-3, which in the first case contradicts $R_2 \in \mathcal{S}_1$ and in the second $R_1 \in \mathcal{S}_1$. Thus $\overline{R_1R_2} \subset \mathcal{S}_1$.

We also have if R_1 and R_2 are points of $\mathcal{S}_2$ then

$\overline{R_1R_2} \subset \mathcal{S}_2$. If not then there is an $S \in \overline{R_1R_2}$ such that $S \in \mathcal{S}_1 \cup \mathcal{S}_3$.

First, we claim that $R_1, R_2 \in \mathcal{S}_2$ implies R_1R_2PQ or R_2R_1PQ. Consider the points R_1, R_2, and P. We have 3 mutually exclusive possibilities according to Theorem 0-5: R_1R_2P, R_2R_1P, or R_1PR_2. R_1R_2P and R_2PQ implies R_1R_2PQ by Lemma 0-2. R_2R_1P and R_1PQ implies R_2R_1PQ by Lemma 0-2. Thus, we must show R_1PR_2 cannot occur. Suppose it does. We also have the 3 cases R_1R_2Q, R_2R_1Q, and R_1QR_2. If R_1R_2Q and R_1PR_2 then PR_2Q, contradicting $R_2 \in \mathcal{S}_2$. Similarly if R_2R_1Q and R_1PR_2 then PR_1Q contradicting R_1PQ. Finally if R_1QR_2 then R_1PQ implies PQR_2 by Lemma 0-3, and this contradicts R_1QP. R_1 and R_2 cannot be Q since this would contradict either R_1PQ or R_2PQ. Thus in all possible cases R_1PR_2 leads to a contradiction, so we must have either R_1R_2PQ or R_2R_1PQ. Now $S \in R_1R_2$ yields R_1SR_2 and this combined with either R_1R_2PQ or R_2R_1PQ yields SPQ by use of Lemmas 0-2 and 0-3. Therefore $S \in \mathcal{S}_2$.

Finally, if $R_1 \in \mathcal{S}_1$ and $R_2 \in \mathcal{S}_2$ then $P \in R_1R_2$. If not we would have R_1R_2P or R_2R_1P by Theorem 0-5. $R_2 \in \mathcal{S}_2$ implies R_2PQ which along with the two betweenness relations above yields R_1PQ, in the first case by Lemma 0-2 and in the second by Lemma 0-3. But R_1PQ contradicts R_1 in $\mathcal{S}_1$

according to 0-2. So the only possibility that remains is R_1PR_2 or $P \in R_1R_2$, which is what was to be proven.

We still must show that $\mathcal{S}_1$ and $\mathcal{S}_2$ are well defined. That is, if T were some other point on ℓ other than P and Q and if we define sets $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ as:

$\mathcal{T}_1 = \{R \mid PRT \text{ or } R = T \text{ or } PTR\}$, $\mathcal{T}_2 = \{R \mid RPT\}$, and $\mathcal{T}_3 = \{P\}$ then the three sets would agree with $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. It is obvious that $\mathcal{T}_3 = \mathcal{S}_3$. Now T must be an element of $\mathcal{S}_1$ or $\mathcal{S}_2$, say $\mathcal{S}_1$. If $T = Q$ we are done, so assume $T \neq Q$, so we have PTQ or PQT. These two betweenness relations imply Q is an element of $\mathcal{T}_1$. Now let R be any element of $\mathcal{T}_1$ and assume $R \notin \mathcal{S}_1$. $R \in \mathcal{T}_1$ implies $R \neq P$ so R must be in $\mathcal{S}_2$, that is RPQ. RPQ and PTQ imply RPT while RPQ and PQT imply RPT also, but this says $R \in \mathcal{T}_2$ and not $\mathcal{T}_1$. Therefore we have $\mathcal{T}_1 \subset \mathcal{S}_1$, but by symmetry and $Q \in \mathcal{T}_1$ we get $\mathcal{S}_1 \subset \mathcal{T}_1$ which yields $\mathcal{T}_1 = \mathcal{S}_1$. Finally since $\Phi(\ell)$ is the disjoint union of $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ as well as $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ we must have $\mathcal{T}_2 = \mathcal{S}_2$.

Now if $T \in \mathcal{S}_2$ then we get $\mathcal{T}_1 = \mathcal{S}_2$ and $\mathcal{T}_2 = \mathcal{S}_1$ by arguments similar to the above. Thus our sets are well defined.

2.16.3. Definition. We will define the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ as the two open rays on the line $\overleftrightarrow{PQ}$ determined by P. If we take the union of each of the open rays with $\{P\}$ we get what we will call the two closed rays or simply rays on $\overleftrightarrow{PQ}$ determined by P. We will designate the open ray $\mathcal{S}_1$ as $\overset{\circ\rightarrow}{PQ}$ and $\overrightarrow{PQ}$ will designate $\{P\} \cup \mathcal{S}_1$.

Summarizing we get the following theorem:

2.16.4. Theorem 0-8. Any point P on a line divides the remaining points of the line into two disjoint sets called open rays, in such a way that if R_1 and R_2 are points of the same ray then $\overline{R_1R_2}$ is a subset of that ray while if R_1 and R_2 are in different rays then R_1PR_2.

2.16.5. Corollary 1. If $P \in \overset{\circ\rightarrow}{AB}$ then $\overset{\circ\rightarrow}{AP} = \overset{\circ\rightarrow}{AB}$ and $\overrightarrow{AP} = \overrightarrow{AB}$.

This is a consequence of the definitions and our proof that $\mathcal{S}_1$ is well defined.

2.17.1. We would expect a line to separate a plane into two "half planes" just as a point separates a line into two open rays, so we have the following Plane Separation Theorem:

2.17.2. Theorem 0-9. Any line ℓ of a plane α separates the remaining points of α into two disjoint nonempty sets H_1 and H_2, called half planes, such that if A and B are two points of the same half plane then $\overline{AB} \cap \Phi(\ell) = \emptyset$ while if they are points of different half planes then $\overline{AB} \cap \Phi(\ell) \neq \emptyset$. (Hilbert, 1962).

2.17.3. Definition. Given ℓ a line of plane α and P a point of α not on ℓ then define $H_1 = \{Q \text{ on } \alpha \mid \overline{PQ} \cap \Phi(\ell) = \emptyset\}$ and $H_2 = \{Q \text{ on } \alpha \mid \overline{PQ} \cap \Phi(\ell) \neq \emptyset\}$. It will be convenient to denote H_1 by $H(\ell, P)$ and H_2 by $H(\ell, \widetilde{P})$ in some of our later work.

2.17.4. Proof. Notice we have immediately that H_1 and H_2 are disjoint sets. We must show that they are well defined and have the above properties. In order to show that H_1 and H_2 are well defined let P' by any other point of α not on ℓ and let H_1' and H_2' be defined in a similar way

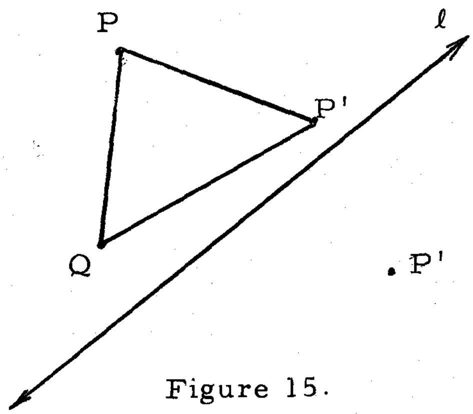


Figure 15.

to H_1 and H_2. (i.e., $H_1' = \{Q \text{ on } \alpha \mid \overline{QP'} \cap \Phi(\ell) = \emptyset\}$ and $H_2' = \{Q \text{ on } \alpha \mid \overline{QP'} \cap \Phi(\ell) \neq \emptyset\}$. If P' is in H_1 then $\overline{PP'} \cap \Phi(\ell) = \emptyset$. We must show that $H_1 = H_1'$ and $H_2 = H_2'$. Let Q be any other point of H_1', then $QP' \cap \Phi(\ell) = \emptyset$. Now assuming Q, P, and P' are noncollinear, by Pasch's Axiom ℓ cannot intersect $\overline{PQ}$. Therefore Q is a point of H_1. If P, P', and Q are collinear we need the following observation:

2.17.5. Lemma 0-6. If A, B, and C are 3 collinear points such that ABC then $AB \subset AC$ and $BC \subset AC$. This follows from Lemma 0-3 since if $D \in AB$ for example, then ADB. But ADB

and ABC imply ADC giving us $D \in AC$. From this observation it follows that a line can intersect either two or none of the open segments AB, BC, and AC. (More specifically it cannot intersect AB and BC without being the line $\overleftrightarrow{AC}$ by I-2. That is it cannot intersect all three open segments AB, BC, and AC.)

2.17.6. Since P, P', and Q are assumed to be collinear, exactly one of them must be between the other two. Then if $Q \in H_1'$, ℓ does not intersect P'Q or PP' and hence it cannot intersect PQ. Consequently in all cases $Q \in H_1'$ implies $Q \in H_1$ and so $H_1' \subset H_1$. In exactly the same way, taking $Q \in H_1$ we could show $H_1 \subset H_1'$. Therefore $H_1' = H_1$.

Since $\Phi(\ell)$ and the two half planes are disjoint sets whose union equals $\Phi(\alpha)$ we see that if $Q \in H_2$ it is not on ℓ or in $H_1 = H_1'$ and it must be in H_2'. Following through on this reasoning we see that $H_2' = H_2$. Therefore our sets agree if $P' \in H_1$, so we must investigate what occurs if $P' \in H_2$.

If $P' \in H_2$ then $P'P \cap \Phi(\ell) \neq \emptyset$, suppose $P'P$ meets ℓ in point R. If Q is a point of H_1 then $\overline{PQ} \cap \Phi(\ell) = \emptyset$ and so by Pasch's Axiom, ℓ must have a point in common with P'Q if the points are noncollinear. If they are collinear the result follows from Lemma 0-6. So we have $H_1 \subset H_2'$.

If $Q \in H_2'$ then we have $P'Q \cap \Phi(\ell) \neq \emptyset$ and

$PP' \cap \Phi(\ell) \neq \emptyset$. If the points are collinear $QP \cap \Phi(\ell) = \emptyset$ follows from the remark at the end of the proof of Lemma 0-6. If the points are not collinear then we need to prove that ℓ cannot intersect PQ. Therefore we need the following Lemma:

2.17.7. Lemma 0-7. A line cannot intersect all three sides of any triangle.

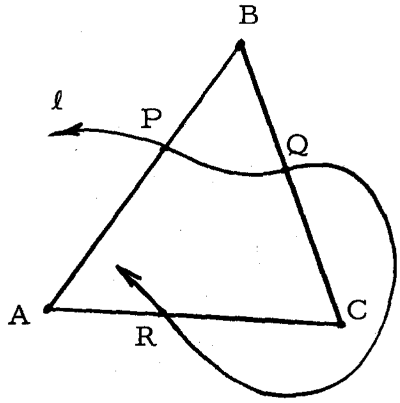


Figure 16.

Let ΔABC be given and assume ℓ meets AB at P, BC at Q, and AC at R. Now we must have one of the three cases PQR, PRQ, or QPR. Assume PQR, (the other cases follow in a similar manner). Now consider ΔAPR. We have BPA and PQR and according to Lemma 0-1 this implies $\overleftrightarrow{BC}$ meets AR at some point S such that ASR. But $\overleftrightarrow{AC}$ meets $\overleftrightarrow{BC}$ at C and so we have $C = S$ or else $\overleftrightarrow{BC} = \overleftrightarrow{AC}$ contradicting the definition of a triangle. But ARC and ASR do not allow C to be the same point as S by 0-2. Therefore we have a contradiction to ℓ meeting all three sides of ΔABC. If PRQ or QPR similar contradictions occur. Therefore a line can not intersect all three sides of a trangle.

2.17.8. Corollary 1. A line can meet either two or no sides of a given triangle.

The proof follows immediately from Pasch's Axiom and the preceding lemma.

2.17.9. Corollary 2. If $Q \in H(\overleftrightarrow{AB}, P)$ then $H(\overleftrightarrow{AB}, Q) = H(\overleftrightarrow{AB}, P)$ and $H(\overleftrightarrow{AB}, \widetilde{Q}) = H(\overleftrightarrow{AB}, \widetilde{P})$.

Let R be any point in $H(\overleftrightarrow{AB}, P)$ other than P or Q. Then if P, Q, and R are noncollinear we have, according to Corollary 1 above, that either both PR and QR each contain a point of $\overleftrightarrow{AB}$ or that neither does. If P, Q, and R are collinear the same result follows from Lemma 0-6. Hence $H(\overleftrightarrow{AB}, P) = H(\overleftrightarrow{AB}, Q)$ and $H(\overleftrightarrow{AB}, \widetilde{P}) = H(\overleftrightarrow{AB}, \widetilde{Q})$ by definition.

2.17.10. From the lemma we have $PQ \cap \Phi(\ell) = \emptyset$ since $P'Q \cap \Phi(\ell)$ and $PP' \cap \Phi(\ell)$ are both nonempty. Thus $H'_2 \subset H_1$ and combining this with $H_1 \subset H'_2$ we get $H_1 = H'_2$. Now it follows as in the previous case $H_2 = H'_1$ by the definitions of the sets. Therefore in all possible cases the half planes are well defined.

To see that the two half planes are nonempty we note that the existence of our point $P \notin \Phi(\ell)$ is assured by I-6. By I-3 we know that ℓ contains some point Q and by 0-3 there exists a point R on $\overleftrightarrow{PQ}$ such PQR. Then R must be in $H(\ell, \widetilde{P})$ and both half planes are nonempty.

2.17.11. Now we must show that if A and B are two points of the same half plane then $AB \cap \Phi(\ell) = \emptyset$ and if A and B are in different half planes then $AB \cap \Phi(\ell) \neq \emptyset$.

The steps to show that if A and B are in H_1 the $AB \cap \Phi(\ell) = \emptyset$ are identical to the ones we used to show $P'Q \cap \Phi(\ell) = \emptyset$ when P' and Q were points of H_1. We will not repeat the steps here. If A and B are in H_2 then the steps agree with P' and Q in H_2. Now if A is in H_1 and B in H_2 we must show $AB \cap \Phi(\ell) \neq \emptyset$. Since $A \in H_1$. $AP \cap \Phi(\ell) = \emptyset$ and since $B \in H_2$, $BP \cap \Phi(\ell) \neq \emptyset$. Therefore we can apply Pasch's Axiom and get $AB \cap \Phi(\ell) \neq \emptyset$. Q.E.D.

2.17.12. Corollary 1. If A and B are two points of plane α such that they both lie in the same half plane determined by line ℓ then $\overline{AB}$ is a subset of that half plane.

Assume A and B are two points of half plane H_1 defined as in Theorem 0-9. Now $AB \cap \Phi(\ell) = \emptyset$. Assume AB contains a point C of H_2 then $AC \cap \Phi(\ell) \neq \emptyset$, say AC meets ℓ at D. Thus ADC. But ADC and ACB imply ADB or in other words $\{D\} \subset AB \cap \Phi(\ell)$ contradicting $AB \cap \Phi(\ell) = \emptyset$. Therefore $\overline{AB} \subset H_1$. If A and B were points of H_2 the proof is identical. Q.E.D.

2.17.13. Corollary 2. If $P \in H(\ell, Q)$ then $H(\ell, Q) = H(\ell, P)$ and

$H(\ell, \widetilde{Q}) = H(\ell, \widetilde{P})$.

This is just a restatement of the fact that the half planes are well defined which we established in 2.17.4 through 2.17.10.

<u>2.18.1. Definition.</u> If H is a half plane of α determined by ℓ we define the <u>closed half plane</u> $\overline{H}$ as $H \cup \Phi(\ell)$.

<u>2.18.2. Definition.</u> If β is a set of points such that for every $P, Q \in \beta$, $\overline{PQ} \subset \beta$ then β is called a <u>convex set.</u>

<u>2.18.3. Corollary 3.</u> The set of points on a line, the set of points in a plane, line segments, rays and half planes are all convex sets.

The proofs are immediate consequences of our definitions, axioms, Theorem 0-8 and Corollary 1 of Section 2.17.12.

<u>2.18.4. Corollary 4.</u> Closed half planes are convex sets.

Let P and Q be elements of $\overline{H}_1$. If P and Q are in H_1 then we are done by the Corollary 3 above. Similarly if P and Q are on ℓ we are done, so assume $P \in H_1$ and Q on ℓ. We must show $\overline{PQ} \subset \overline{H}_1$. Now $\Phi(\alpha) = H_1 \cup H_2 \cup \Phi(\ell)$, and therefore $\Phi(\alpha) - \overline{H}_1 = H_2$. Also $\overline{PQ} \subset \Phi(\alpha)$ so if the theorem were false there would exist a point of PQ in H_2, say R. Now by Theorem 0-9 there is a point S of ℓ on PR, that is PSR and $S \in \Phi(\ell)$. But PSR and PRQ imply PSQ by Lemma 0-3.

Since $S \neq Q$ we have $\overleftrightarrow{QS} = \ell$ by I-2 and therefore $P \in \Phi(\ell)$ contradicting $P \in H_1$ and the disjointness of the sets H_1, H_2, and $\Phi(\ell)$. Therefore we must have $\overline{PQ} \subset \overline{H}_1$. Q.E.D.

2.18.5. The following theorem is called the Space Separation Theorem. The proof, which is analogous to the proof of Theorem 0-9 and will not be given here.

Theorem 0-10. A plane α divides the remaining points of space into two disjoint sets called half spaces such that if A and B are two points of the same half space then $\overline{AB} \cap \Phi(\alpha) = \emptyset$ while if A and B are points of different half spaces then $AB \cap \Phi(\alpha) \neq \emptyset$.

2.18.6. Definition. A closed half space is defined as the union of the half space with the points of its defining plane. We will designate a closed half space by $\overline{S}$.

2.18.7 Corollary 1. A closed half space is a convex set.

The proof is almost identical to the proof of Corollary 4, section 2.18.4 and will not be pursued further.

2.19.1. The following lemma could be classed as a theorem if we were to judge it by its importance and if our objective were to investigate convexity. For us it will be a handy tool to apply to the development of the more classical concepts of geometry.

2.19.2. Lemma 0-8. The intersection of any finite number of convex sets is convex.

Let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be a finite collection of convex sets and let P and Q be two points of $\beta = \bigcap_{i=1}^{n} \beta_i$. By the definition of intersection P and Q are elements of each β_i. Since each β_i is convex $\overline{PQ} \subset \beta_i$, that is if $R \in \overline{PQ}$ then $R \in \beta_i$ for all i. But then $R \in \beta$ which implies $\overline{PQ} \subset \beta$. Notice that if β does not contain two distinct points our lemma holds vacuously. (i.e., if $\beta = \emptyset$ or if β contains only one point.) Q.E.D.

2.20.1. In order to complete our development of geometry we still have two things to define, angles and polygons. We will first consider the concept of an angle:

2.20.2. Definition. An angle is a pair of distinct rays with a common end point. The rays will be called the sides of the angle, the common end point the vertex. If the rays are collinear (i.e., their union is the set of points of a line) then the angle will be called a straight angle. If B is the vertex of some angle and A and C are points on different sides of the angle, neither of which is B then we will designate the angle by $\angle ABC$ or $\angle CBA$. We define the interior of $\angle ABC$ as $H(\overleftrightarrow{BA}, C) \cap H(\overleftrightarrow{BC}, A)$ if A, B, and C are noncollinear. (Note that the definition holds for angles which are not straight angles.) We will designate the interior of $\angle ABC$

by Int $\angle ABC$. Notice that our definition of an angle does not include angles which are usually called reflex angles.

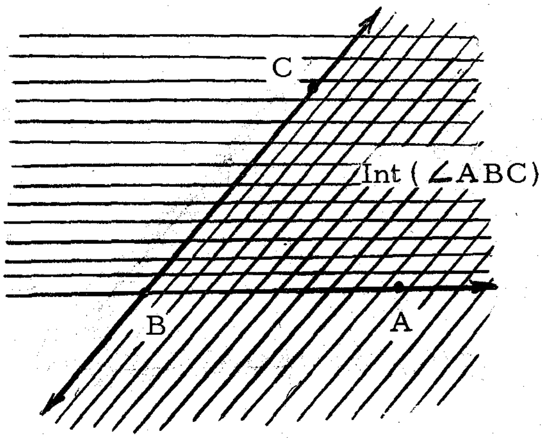


Figure 17.

2.20.3. Lemma. The interior of a non-straight angle is nonempty.

Let $\angle ABC$ be given. Then $C \notin \Phi(\overleftrightarrow{AB})$. Consider AC. There exists a point D such that ADC by Theorem 0-1. $CD \cap \Phi(\overleftrightarrow{AB}) = \emptyset$, since if not we would have $\overleftrightarrow{AC} = \overleftrightarrow{AB}$ contrary to hypothesis. Hence $D \in H(\overleftrightarrow{AB}, C)$ by definition. Similarly $D \in H(\overleftrightarrow{BC}, A)$. Thus $D \in \text{Int } \angle ABC$ by definition and hence $\text{Int } \angle ABC \neq \emptyset$.

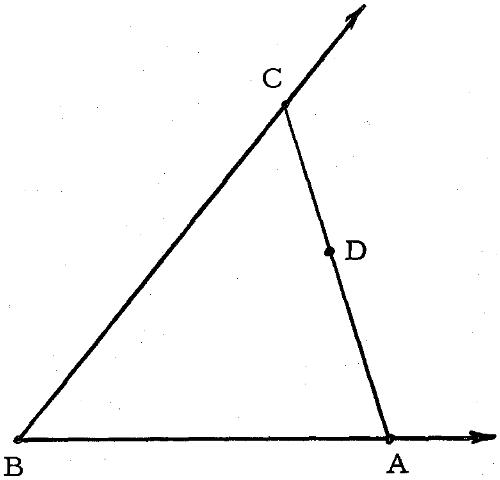


Figure 18.

2.20.4. One thing which we would like to be able to do is define an order relation on angles with a common vertex. But first we need to develop some intermediate theory, including a closer look at triangles.

2.20.5. Definition. The interior of $\triangle ABC$ will be designated as $\text{Int } \triangle ABC$ and is defined as $H(\overleftrightarrow{AB}, C) \cap H(\overleftrightarrow{AC}, B) \cap H(\overleftrightarrow{BC}, A)$.

2.20.6. Lemma. The interior of any triangle is nonempty.

Consider $\triangle ABC$. There exists a point D such that BDC by Theorem 0-1. By the proof of Lemma Section 2.20.3, $D \in H(\overleftrightarrow{AB}, C) \cap H(\overleftrightarrow{AC}, B)$. Hence by Corollary 2, Section 2.17.9 we have $H(\overleftrightarrow{AB}, C) = H(\overleftrightarrow{AB}, D)$ and $H(\overleftrightarrow{AC}, B) = H(\overleftrightarrow{AC}, D)$. There exists a point E such that AED. ED cannot meet $\overleftrightarrow{AC}$ or $\overleftrightarrow{AB}$ and

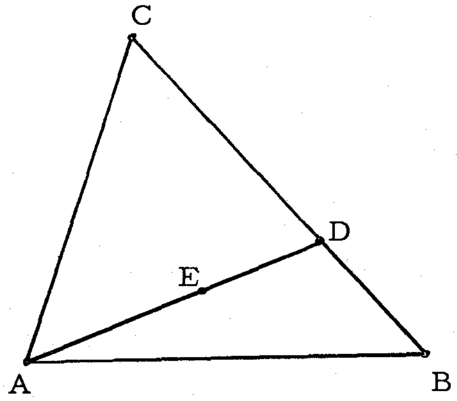


Figure 19.

AE cannot meet DC by collinearity arguments. Consequently $E \in H(\overleftrightarrow{AC}, D)$, $E \in H(\overleftrightarrow{AB}, C)$, and $E \in H(\overleftrightarrow{BC}, A)$ and therefore E and hence every point of AD are in the intersection of these half planes.

Q.E.D.

2.20.7. Theorem 0-11. The interior of a triangle and of a non-straight angle are convex sets.

Proof. The proof is immediate if we apply Lemma 0-8 to our definitions. That is we know that half planes are convex from Corollary 0-2 and from Lemma 0-8 the intersection of a finite number of convex sets is convex. When these results are applied to the definitions of the interior of an angle or a triangle we get our desired results.

2.21.1. The following sequence of theorems builds the theoretical structure we need to define the order relation on angles mentioned in 2.20.3. The theorems also are of interest in their own right.

<u>2.21.2. Lemma 0-9.</u> If A, B, and C are noncollinear points of plane α then $\overset{\circ\!\!\rightarrow}{AC} \subset H(\overleftrightarrow{AB}, C)$.

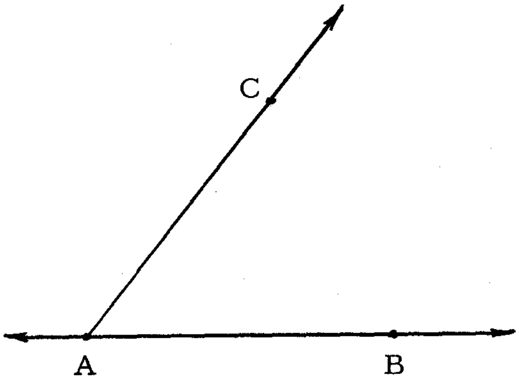


Figure 20.

Let P be any other point of $\overset{\circ\!\!\rightarrow}{AC}$. By definition of $\overset{\circ\!\!\rightarrow}{AC}$ we must have APC or ACP. But this implies $A \notin \overline{PC}$. Since A, B, and C are noncollinear $\overline{PC} \cap \Phi(\overleftrightarrow{AB}) = \emptyset$, otherwise, $\overleftrightarrow{PC}$ and $\overleftrightarrow{AB}$ would have two distinct points in common and we would have $\overleftrightarrow{PC} = \overleftrightarrow{AB}$ by I-2. Therefore by definition $P \in H(\overleftrightarrow{AB}, C)$ and we have $\overset{\circ\!\!\rightarrow}{AC} \subset H(\overleftrightarrow{AB}, C)$.

<u>2.21.3.</u> The following theorem is often referred to as the Cross Bar Theorem. We will find it particular useful in establishing some of our other results (Moise, 1963).

<u>2.21.4. Theorem 0-12.</u> Given any triangle and a point in its interior. A ray from one of its vertices through the point must intersect the side opposite the vertex. (The following proof was first shown to me by two of my students at Oregon College of Education, Bruce Tiedeman and Harold Murray.)

Proof. Let $\triangle ABC$ be the given triangle and D the point in its interior. There exists a point E such that EAB by 0-3. B, C, and E determine $\triangle BCE$ by definition since $C \notin \Phi(\overleftrightarrow{AB})$. By Pasch's Axiom $\overleftrightarrow{AD}$ must intersect

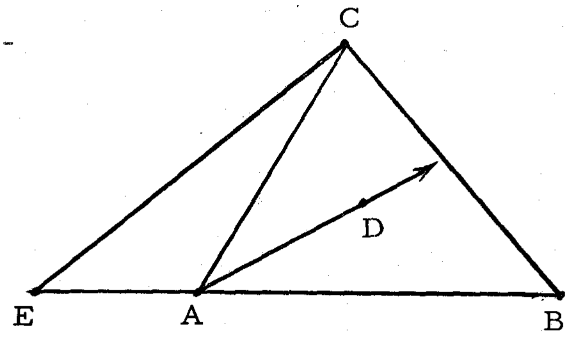


Figure 21.

either $\overline{CE}$ or $\overline{CB}$ since $\overleftrightarrow{AD}$ has the point A between E and B. We know that $D \in H(\overleftrightarrow{AC}, B)$ and so $\overrightarrow{AD} \subset H(\overleftrightarrow{AC}, B)$ by Lemma 0-9 and the fact that our half planes are well defined. Similarly $\overrightarrow{CE} \subset H(\overleftrightarrow{AC}, E)$. But E and B are on opposite sides of $\overleftrightarrow{AC}$ since $EB \cap \Phi(\overleftrightarrow{AC}) = \{A\}$. Therefore $\overrightarrow{AD}$ and $\overrightarrow{CE}$ lie in opposite half planes with respect to $\overleftrightarrow{AC}$. But $CE \subset \overrightarrow{CE}$ by definition. Therefore since $H(\overleftrightarrow{AC}, B)$ and $H(\overleftrightarrow{AC}, E)$ are disjoint sets $\overrightarrow{AD}$ cannot meet CE. But $\overrightarrow{AD}$ cannot contain the points B or C either since this would imply D on $\overleftrightarrow{AC}$ or $\overleftrightarrow{AB}$ contradicting $D \in \text{Int } \triangle ABC$. Then the only possibility is that $\overrightarrow{AD} \cap CB \neq \emptyset$.

Q.E.D.

2.21.5. Theorem 0-13. If a line ℓ contains a point of the interior of a triangle then $\Phi(\ell)$ intersects the triangle in two points.

Proof. Let $\triangle ABC$ be given triangle and D the point of $\text{Int } \triangle ABC$. Now either $\overleftrightarrow{AD} = \ell$ or $\overleftrightarrow{AD} \neq \ell$. If $AD = \ell$ then

$\overrightarrow{AD}$ intersects DC at some point E by the Cross Bar Theorem and hence ℓ meets $\triangle ABC$ at the points A and E. So assume $\overleftrightarrow{AD} \neq \ell$. As above there exists a point E on $\overrightarrow{AD}$ such that $E \in BC$. Consider $\triangle ABE$ and $\triangle ACE$. Since $D \in \text{Int}\, \triangle ABC$, D is neither the point E nor A, that is ADE. Applying Pasch's Axiom to each of these triangles we get another point of each on ℓ. Since these points cannot be on AE, they are distinct, since the only points the two triangles have in common are those of AE. Consequently these two points must be on $\triangle ABC$.

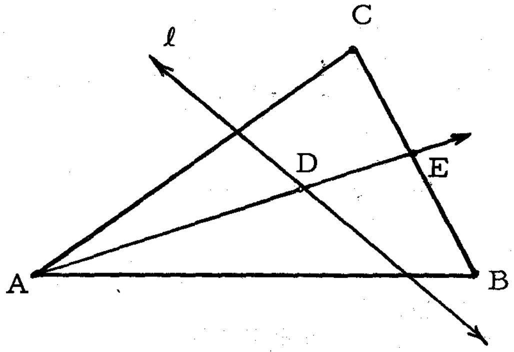


Figure 22.

Q.E.D.

<u>2.21.6. Corollary 1.</u> If ℓ contains a point P in the interior of $\triangle ABC$ then ℓ intersects $\triangle ABC$ in two points, say Q and R, such that QPR.

From Theorem 0-13 we know that points Q and R exist. If $\widetilde{QPR}$ then either PQR or PRQ. Assume PQR. Without loss of generality we can assume Q lies on $\overline{AB}$. Then $R \in \overline{AC}$ or $\overline{BC}$. In either case $R \in H(\overleftrightarrow{AB}, C)$. But by the Plane Separation Theorem this implies $P \in H(\overleftrightarrow{AB}, \tilde{C})$. But this contradicts the assumption that $P \in \text{Int}\, \triangle ABC$. In a similar way PRQ leads to a contradiction. Hence we must have QPR. Q.E.D

<u>2.22.1.</u> Theorem 0-13 is a departure from our goal of ordering angles with a common vertex. We return to this objective with the following lemmas.

<u>2.22.2. Lemma 0-10.</u> If two open rays $\overset{\circ\!\!\rightarrow}{AC}$ and $\overset{\circ\!\!\rightarrow}{AD}$ both lie on the same side of $\overleftrightarrow{AB}$ then one of the open rays lies in the interior of the angle formed by $\overrightarrow{AB}$ and the other ray.

Suppose $\overset{\circ\!\!\rightarrow}{AC}$ does not lie in the interior of $\angle BAD$, then $C \notin H(\overleftrightarrow{AD}, B)$ but by hypothesis $C \in H(\overleftrightarrow{AB}, D)$. The hypothesis implies $\overset{\circ\!\!\rightarrow}{AC}$ and $\overset{\circ\!\!\rightarrow}{AD}$ are distinct rays, and consequently $C \notin \overrightarrow{AD}$ by Corollary 1 to Theorem 0-8.

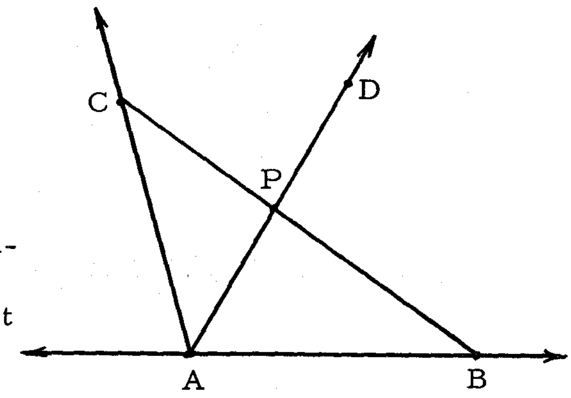


Figure 23.

$C \in H(\overleftrightarrow{AB}, D)$ implies $C \notin \Phi(\overleftrightarrow{AB})$ or $C \notin H(\overleftrightarrow{AB}, \tilde{D})$ and hence C cannot be in $\Phi(\overleftrightarrow{AD}) - \overset{\circ\!\!\rightarrow}{AD}$. We must have $C \in H(\overleftrightarrow{AD}, \tilde{B})$ and therefore there is a point P such that $\{P\} = BC \cap \Phi(\overleftrightarrow{AD})$, that is BPC and $P \in \overleftrightarrow{AD}$. Now $P \in \overset{\circ\!\!\rightarrow}{BC}$ by definition and by Lemma 0-9 $P \in H(\overleftrightarrow{AB}, C)$. Similarly $P \in \overset{\circ\!\!\rightarrow}{CB}$ and hence $P \in H(\overleftrightarrow{AC}, B)$. Consequently we have $P \in H(\overleftrightarrow{AB}, C) \cap H(\overleftrightarrow{AC}, D) = \text{Int}\ \angle ABC$. Now by Corollary 1 to Theorem 0-8 and Lemma 0-9 we have

$\overset{\circ\!\!\rightarrow}{AP} = \overset{\circ\!\!\rightarrow}{AD} \subset \text{Int}\ \angle BAC$. Q.E.D.

<u>2.22.3. Corollary 1.</u> If $\overrightarrow{AD}$ lies in the interior of $\angle BAC$ then $BC \cap \overrightarrow{AD} \neq \emptyset$.

This corollary is established as part of the proof above.

<u>2.22.4. Definition.</u> We will say $\angle BAD$ is less than $\angle BAC$ iff $\overrightarrow{BD}$ lies in the interior of $\angle BAC$. We will denote this by $\angle BAD < \angle BAC$.

<u>2.22.5. Lemma 0-11.</u> If $\angle BAD < \angle BAC$ then $BC \cap \overrightarrow{AD} \neq \emptyset$.

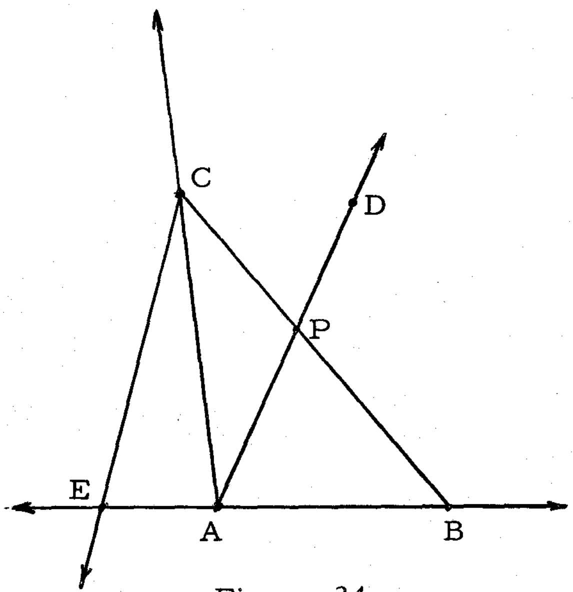


Figure 24.

Let E be a point of $\overleftrightarrow{AB}$ such that EAB. Since $C \notin \overleftrightarrow{AB}$ by hypothesis C, B, and E define a triangle. $D \in \text{Int} \angle BAC$ by definition and hence $D \in H(\overleftrightarrow{AC}, B)$. Thus, according to Lemma 0-9 $\overrightarrow{AD} \subset H(\overleftrightarrow{AC}, B)$ while $\overrightarrow{CE} \in H(\overleftrightarrow{AC}, \tilde{B})$ by our choice of the point E. From Pasch's Axiom $\overleftrightarrow{AD}$ must meet either $\overline{CE}$ or $\overline{CB}$ in a point P. P cannot be on the ray from A opposite to D since this ray is in $H(\overleftrightarrow{AB}, \tilde{D}) = H(\overleftrightarrow{AB}, \tilde{C})$. All the points of $\triangle CBE$ lie in $\bar{H}(\overleftrightarrow{AB}, C)$, hence $P \in \overrightarrow{AD}$. But since E and D lie in opposite sides of $\overleftrightarrow{AC}$, P must be on CB. Q.E.D.

2.22.6. Corollary 1. $\angle BAD < \angle BAC$ iff $BC \cap \overrightarrow{AD} \neq \emptyset$.

Lemma 0-11 gives that $\angle BAD < \angle BAC$ implies $BC \cap \overrightarrow{AD} \neq \emptyset$. On the other hand if $BC \cap \overrightarrow{AD} = \{P\}$ then by the proof of Lemma 10 we have $\overrightarrow{AD} \subset \text{Int} \angle BAC$ and so $\angle BAD < \angle BAC$ by the definition.

2.23.1. In Sections 2.21 and 2.22 we have been putting together the machinery we need to order any finite number of angles with common side $\overrightarrow{AB}$ and which lie on the same side of $\overleftrightarrow{AB}$. In order to do this we need one more lemma:

2.23.2. Lemma 0-12. If open rays $\overrightarrow{AC_1}, \overrightarrow{AC_2}, \ldots, \overrightarrow{AC_n}$ all lie on the same side of $\overleftrightarrow{AB}$ then the points $C_1, C_2, \ldots, C_n$ can be renamed in such a way that $\angle BAC_i < \angle BAC_n$ for all $i < n$.

The proof will be by induction. We have our first step established in Lemma 0-10. Hence we assume the theorem holds for $n = k$ and let $\overrightarrow{AD}$ be an open ray in $H(\overleftrightarrow{AB}, C_k)$, $\overrightarrow{AD} \neq \overrightarrow{AC_i}$ for $1 \leq i \leq k$. By Lemma 0-10 either $\overrightarrow{AD} \subset \text{Int} \angle BAC_k$ or $\overrightarrow{AC_k} \subset \text{Int} \angle BAD$. In the former case we are done since we have our conclusion, by the definition, for the $k+1$ open rays, (no renaming is necessary). So assume $\overrightarrow{AC_k} \subset \text{Int} \angle BAD$. In this case we rename D as C_{k+1}. We have immediately that

$\overrightarrow{AC_k} \subset \text{Int} \angle BAC_{k+1}$.

Consider $\overrightarrow{AC_i}$, $i < k$. Now $BC_{k+1} \cap \overrightarrow{AC_k}$ at a point P

by Lemma 0-11. By Corollary 1, Section 2.16.5 $\overset{\circ\!\!\rightarrow}{AP} = \overset{\circ\!\!\rightarrow}{AC_k}$ and hence by Lemma 0-11 $PB \cap \overset{\circ\!\!\rightarrow}{AC_i} \neq \phi$. Call the point of intersection Q. Then BPC_{k+1} and BQP which according to Lemma 0-3 yields BQC_{k+1}. Therefore by Corollary 1, Section 2.22.6 $\angle BAC_i < \angle BAC_{k+1}$. Thus we have, in both possible cases, our conclusion holding and hence by induction our lemma is valid for all n.

<u>2.23.3. Theorem 0-14.</u> If open rays $\overset{\circ\!\!\rightarrow}{AP_1}, \overset{\rightarrow}{AP_2}, \ldots, \overset{\circ\!\!\rightarrow}{AP_n}$ all lie on the same side of $\overleftrightarrow{AB}$ then the points $P_1, P_2, \ldots, P_n$ can be renamed $C_1, C_2, \ldots, C_n$ in such a way that $\angle BAC_i < \angle BAC_j$ iff $i < j \leq n$.

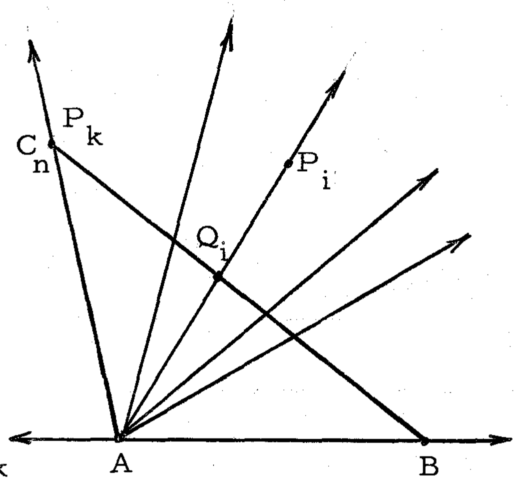


Figure 25.

<u>Proof.</u> By Lemma 0-12 there exists P_k such that $\angle BAP_i < \angle BAP_k$ for $i \neq k$. We rename P_k as C_n. Now by Lemma 0-11 BC_n meets each of the other $\overset{\circ\!\!\rightarrow}{AP_i}$'s at distinct points Q_i. Now the points B, C_n, and Q_i all lie on $\overleftrightarrow{BC_n}$. By Theorem 0-5 the points B, C_n, Q_i, $i \neq k$ can be renamed $D_1, D_2, \ldots, D_{n+1}$ in exactly two ways such that $1 \leq i < j < k \leq n+1$ implies $D_iD_jD_k$. Now we know BD_iC_n for all $D_i \neq B$ or C_n from Lemma 0-11. Consequently C_n must be named D_{n+1} or D_1. We are free to

choose either naming, so we will choose D_{n+1}. Then we have $D_1 = B$. Thus if $i < j \leq n+1$ we have BD_iD_j giving us $\angle BAD_i < \angle BAD_j$ for $2 \leq i < j \leq n+1$ by Corollary 1, Section 2.22.6. Now each D_i will be on some $\overrightarrow{AP_k}$. We rename each P_k as C_{i-1} according to the subscript of the D_i. By Corollary 1, Section 2.16.5 $\overrightarrow{AD_i} = \overrightarrow{AC_{i-1}}$ and therefore $\angle BAC_i < \angle BAC_j$ iff $1 \leq i < j \leq n$. Q.E.D.

2.23.4. We have just established an order relation for angles with a common side whose other sides all lie in the same half plane determined by the common side. We will not try to push the relation any further since Theorem 0-14 is actually more than is needed for our later development.

2.24.1. Our final objective is to establish the properties of polygons which are a consequence of the first two sets of axioms. We first derive some miscellaneous results which fit in with this development.

2.24.2. Definitions. A polygonal path, $\langle P_1, P_2, \ldots, P_n \rangle$, is defined to be $\bigcup_{i=1}^{n-1} \overline{P_iP_{i+1}}$. We will also denote this path as $\langle P_i \rangle_{i=1}^{n}$. Notice that we have not said that the P_i's are distinct nor have we said that $\overline{P_iP_{i+1}} \cap \overline{P_jP_{j+1}}$ is empty for all $i \neq j$. Thus we could have a polygonal path which "looks" like the figure below. Many other such illustrations could be devised. P_1 will be called the <u>initial</u> point and P_n the <u>terminal</u> point of the path. A <u>simple</u> polygonal path is a polygonal path in which $P_iP_{i+1} \cap P_jP_{j+1} = \emptyset$ for all $1 \leq i < j < n$ and all the P_i's are distinct except possibly P_1 and P_n. If $P_1 = P_n$ then $<P_1, P_2, \ldots, P_n>$ will be called

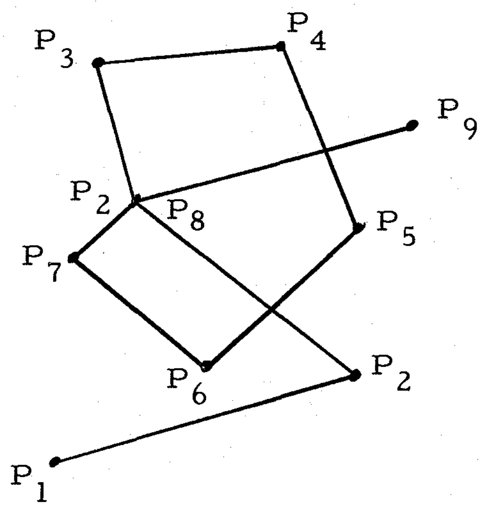


Figure 26.

a <u>polygon</u> for $n \geq 4$ and P_i, P_{i+1}, P_{i+2} noncollinear for $1 \leq i \leq n-2$.

The points P_1 to P_n will be called the <u>vertices</u> of the polygonal path or polygon while the segments P_1P_2 to $P_{n-1}P_n$ will be called its <u>sides</u> or <u>edges</u>. Notice, from our definition we have the number of edges of a polygon equals the number of vertices. A set $\mathcal{S}$ will be called <u>polygonally</u> connected or simply, <u>connected</u> if for every pair of points $P, Q \in \mathcal{S}$ there exists a polygonal path $\Pi \subset \mathcal{S}$ such that P is the initial point of Π and Q the terminal point.

<u>2.24.3. Lemma.</u> Every convex set is connected.

The proof is immediate since if $\mathcal{S}$ is convex and if $P, Q \in \mathcal{S}$ then $\overline{PQ} \subset \mathcal{S}$ by the definition of a convex set. So we take $\Pi = <P, Q> = \overline{PQ}$. Q.E.D.

<u>2.24.4.</u> As a consequence of this lemma we have that the interior of any triangle or angle, any closed or open half plane, or any line, line segment or ray is a connected set.

<u>2.24.5. Definition.</u> We define the <u>exterior</u> of $\angle ABC$, which is not straight as $H(\overleftrightarrow{AB}, \tilde{C}) \cup H(\overleftrightarrow{BC}, \tilde{A})$. We designate the exterior of $\angle ABC$ as $\text{Ext}\angle ABC$. Similarly we define the exterior of ΔABC, denoted by $\text{Ext}(\Delta ABC)$, as $H(\overleftrightarrow{AB}, \tilde{C}) \cup H(\overleftrightarrow{AC}, \tilde{B}) \cup H(\overleftrightarrow{BC}, \tilde{A})$.

<u>2.24.6. Lemma 0-13.</u> If Σ and T are connected sets and $\Sigma \cap T \neq \emptyset$ then $\Sigma \cup T$ is connected.

Let P and Q be any elements of $\Sigma \cup T$ and let R be an element of $\Sigma \cup T$. Since P is in one of the sets Σ or T there exists a polygonal path Π_P from P to R lying in one of the sets. Similarly there exists a polygonal path Π_Q from R to Q also lying in one of the sets. We take $\Pi = \Pi_R \cup \Pi_Q$. From our definition of polygonal paths it is clear that Π is a polygonal path from P to Q and that $\Pi \subset \Sigma \cup T$. Hence $\Sigma \cup T$ is connected.

<u>2.24.7. Lemma 0-14.</u> The exterior of an angle which is not straight is a connected set which is not convex.

Let $\angle ABC$ be given such that A, B, and C are noncollinear by an argument quite similar to the one which was used in the

Lemma of Section 2.20.3 we have $H(\overleftrightarrow{AB}, \tilde{C}) \cap H(\overleftrightarrow{BC}, \tilde{A}) \neq \emptyset$. Therefore by Lemma 0-13 $H(\overleftrightarrow{AB}, \tilde{C}) \cup H(\overleftrightarrow{BC}, \tilde{A})$ is connected but this set is $\mathrm{Ext}\angle ABC$. Hence the exterior of an angle is a connected set.

To see that it is not convex consider $\overleftrightarrow{AC}$. There exist points P and Q on $\overleftrightarrow{AC}$ such that PAC and ACQ by 0-3. These are points of $\mathrm{Ext}\angle ABC$. By Lemma 0-2 we have PAQ. But $A \notin \mathrm{Ext}\angle ABC$ and therefore $\overline{PQ} \not\subset \mathrm{Ext}\angle ABC$. Thus there are points of $\mathrm{Ext}\angle ABC$ such that the closed segment, with them as endpoints, is not in $\mathrm{Ext}\angle ABC$.

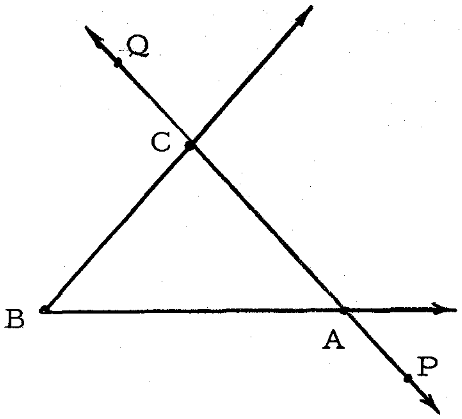


Figure 27.

2.25.1. We will now further explore some of the properties of triangles. We already know that the interior of a triangle is a convex and therefore connected set. What about the exterior?

2.25.2. Lemma 0-15. The exterior of a triangle is a connected nonconvex set.

Let ΔABC be given. By exactly the same steps as in Lemma 0-14 we have that $H(\overleftrightarrow{AB}, \tilde{C}) \cup H(\overleftrightarrow{BC}, \tilde{A})$ is a connected set. But again, as in Lemma 0-14, we know that $H(\overleftrightarrow{AC}, \tilde{B}) \cap H(\overleftrightarrow{AB}, \tilde{C}) \neq \emptyset$ and therefore $H(\overleftrightarrow{AC}, \tilde{B}) \cap [H(\overleftrightarrow{AB}, \tilde{C}) \cup H(\overleftrightarrow{BC}, \tilde{A})] \neq \emptyset$. Hence, applying Lemma 0-13 we have Ext $\triangle ABC$ is connected. To establish that Ext $\triangle ABC$ is not convex we can use a construction similar to that used in Lemma 0-14, by choosing two points on two sides of the triangle and proceeding in the same way.

<u>2.25.3. Lemma 0-16.</u> If $P \in$ Int $\triangle ABC$ and $Q \in$ Ext $\triangle ABC$ then $PQ \cap \triangle ABC \neq \emptyset$.

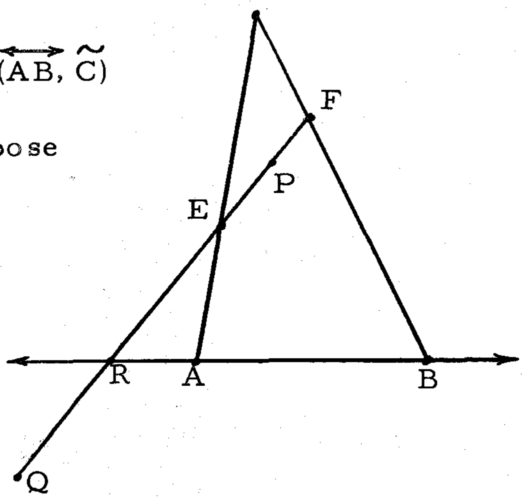


Figure 28.

If $Q \in$ Ext $\triangle ABC$ then $Q \in H(\overleftrightarrow{AB}, \tilde{C})$ or $H(\overleftrightarrow{AC}, \tilde{B})$ or $H(\overleftrightarrow{BC}, \tilde{A})$. Suppose $Q \in H(\overleftrightarrow{AB}, \tilde{C})$. But $P \in H(\overleftrightarrow{AB}, C)$ and hence $PQ \cap \Phi(\overleftrightarrow{AB}) \neq \emptyset$. Call the point of intersection R. Now if $R \in \overline{AB}$ we are done, so suppose $R \notin \overline{AB}$ then RAB or ABR by Theorem 0-2. Presume RAB. By Theorem 0-13 $\overleftrightarrow{PQ}$ intersects $\triangle ABC$ at two points, say E and F. Neither of these can be on AB since this would imply $\overleftrightarrow{AB} = \overleftrightarrow{PQ}$, contradicting $P \in H(\overleftrightarrow{AB}, C)$. Let $E \in AC$ and $F \in BE$. Now since RAB we have $R \in H(\overleftrightarrow{AC}, \tilde{B})$ and since $P \in H(\overleftrightarrow{AC}, B)$ we have REP by the Plane Separation Theorem. But QRP and REP imply QEP by Lemma 0-3. Thus $\{E\} = PQ \cap \triangle ABC$ in this case. The other cases lead in the same way to the same result.

<u>2.25.4. Definition.</u> A set theoretic definition which we will find

useful is that of the complement of a set. If $\beta \subset \alpha$ we define the complement of β, denoted by β', as $\beta' = \{x \in \alpha \mid x \notin \beta\}$. For example if ℓ is a line of plane α and P a point of α not on ℓ then the complement of $H(\ell, P)$ in α is $\overline{H}(\ell, \widetilde{P})$ by the Plane Separation Theorem. We will not point out in our discussions what the subset relation is if it is clear from the context. That is in $H(\ell, P)' = \overline{H}(\ell, \widetilde{P})$ there is no need to mention that $H(\ell, P) \subset \Phi(\alpha)$.

2.25.5. Lemma 0-17. Any $\triangle ABC$ in a plane α, divides $\Phi(\alpha)$ into three disjoint subsets, $\triangle ABC$, Int $\triangle ABC$, and Ext $\triangle ABC$ such that $\Phi(\alpha) = \triangle ABC \cup \text{Int } \triangle ABC \cup \text{Ext } \triangle ABC$.

The disjointness of the three sets is a trivial consequence of their definitions. To see that their union is $\Phi(\alpha)$, let P be any point of α not in Int $\triangle ABC$. Then

$P \notin H(\overleftrightarrow{AB}, C) \cap H(\overleftrightarrow{AC}, B) \cap H(\overleftrightarrow{BC}, A)$. This implies

$P \in (H(\overleftrightarrow{AB}, C) \cap H(\overleftrightarrow{AC}, B) \cap H(\overleftrightarrow{BC}, A))'$, (the complement of Int ($\triangle ABC$) in $\Phi(\alpha)$). But this set is

$H(\overleftrightarrow{AB}, C)' \cup H(\overleftrightarrow{AC}, B)' \cup H(\overleftrightarrow{BC}, A)'$. Consider $H(\overleftrightarrow{AB}, C)'$. From the Plane Separation Theorem this must be the closed half plane

$\overline{H}(\overleftrightarrow{AB}, \widetilde{C}) = \Phi(\overleftrightarrow{AB}) \cup H(\overleftrightarrow{AB}, \widetilde{C})$. Thus

$P \in H(\overleftrightarrow{AB}, \widetilde{C}) \cup H(\overleftrightarrow{AC}, \widetilde{B}) \cup H(\overleftrightarrow{BC}, \widetilde{A}) \cup \Phi(\overleftrightarrow{AB}) \cup \Phi(\overleftrightarrow{BC}) \cup \Phi(\overleftrightarrow{AC})$.

If P is in any of the first three sets we are done, so assume P is on one of the lines say $P \in \Phi(\overleftrightarrow{AB})$. Now we have the 4 mutually exclusive cases PAB, APB, ABP or $P \in \{A, B\}$. If $P \in \{A, B\}$ or APB we have $P \in \overline{AB}$ and we are done since this implies $P \in \triangle ABC$. So assume PAB. This implies by the Plane Separation Theorem that $P \in H(\overleftrightarrow{AC}, \widetilde{B})$. (See the adjoining sketch.) Therefore $P \in \text{Ext}\ \triangle ABC$. If ABP then $P \in H(\overleftrightarrow{BC}, \widetilde{A})$ and again $P \in \text{Ext}\ \triangle ABC$. If P were on $\overleftrightarrow{AC}$, or $\overleftrightarrow{BC}$ similar steps would lead to P either on $\triangle ABC$ or $P \in \text{Ext}\ \triangle ABC$. Thus $P \in \Phi(\alpha)$ implies P in one of our three sets and the proof is complete.

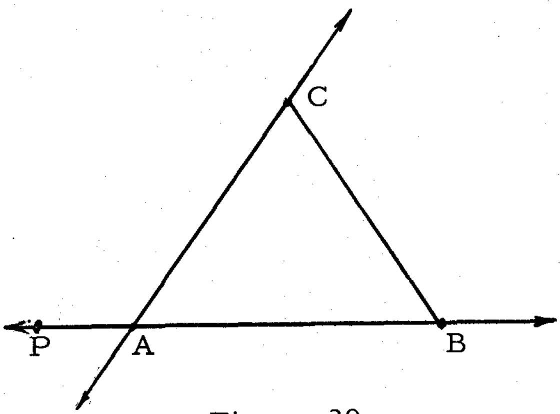


Figure 29.

2.25.6. Definition. We define the closure of the interior of a triangle as the union of the triangle with its interior. Given $\triangle ABC$ we denote the closure of $\text{Int}\ \triangle ABC$ by $\overline{\text{Int}}\ \triangle ABC$. That is, $\overline{\text{Int}}\ \triangle ABC = \text{Int}\ \triangle ABC \cup \triangle ABC$. (The closure of $\text{Ext}\ \triangle ABC$, denoted by $\overline{\text{Ext}}\ \triangle ABC$ is defined in a similar manner.)

2.25.7. Lemma. Given $\triangle ABC$, $\overline{\text{Int}}\ \triangle ABC = \overline{H}(\overleftrightarrow{AB}, C) \cap \overline{H}(\overleftrightarrow{AC}, B) \cap \overline{H}(\overleftrightarrow{BC}, A)$.

From our definition each of the open half planes is a subset of the corresponding closed half plane. Hence $\text{Int}\ \triangle ABC$

$$= H(\overleftrightarrow{AB}, C) \cap H(\overleftrightarrow{AC}, B) \cap H(\overleftrightarrow{AC}, A) \subset \overline{H}(\overleftrightarrow{AB}, C) \cap \overline{H}(\overleftrightarrow{AC}, B) \cap \overline{H}(\overleftrightarrow{BC}, A).$$

By definition $\overline{AB} \subset \overline{H}(\overleftrightarrow{AB}, C)$ and by the Plane Separation Theorem $\overline{AB} \subset \overline{H}(\overleftrightarrow{AC}, B)$ and $\overline{AB} \subset \overline{H}(\overleftrightarrow{BC}, A)$. Then $\overline{AB} \subset \overline{H}(\overleftrightarrow{AB}, C) \cap \overline{H}(\overleftrightarrow{AC}, B) \cap \overline{H}(\overleftrightarrow{BC}, A)$. Similarly $\overline{BC}$ and $\overline{AC}$ both lie in the intersection of the closed half planes and hence so does $\overline{\text{Int}}\ \triangle ABC$. Now by Lemma 0-17 since the points of the plane of the triangle are separated into the three disjoint sets Int $\triangle ABC$, Ext $\triangle ABC$, and $\triangle ABC$ by $\triangle ABC$, if $P \notin \overline{\text{Int}}\ \triangle ABC$, $P \in$ Ext $\triangle ABC$. But this implies $P \in H(\overleftrightarrow{AB}, \tilde{C}) \cup H(\overleftrightarrow{AC}, \tilde{B}) \cup H(\overleftrightarrow{BC}, \tilde{A})$. Presume $P \in H(\overleftrightarrow{AB}, \tilde{C})$ then by the Plane Separation Theorem and the definition of $\overline{H}(\overleftrightarrow{AB}, C)$ we see that $P \notin \overline{H}(\overleftrightarrow{AB}, C)$. In a like manner we see that P is in either $H(\overleftrightarrow{AC}, \tilde{B})$ or $H(\overleftrightarrow{BC}, \tilde{A})$ leads to $P \notin \overline{H}(\overleftrightarrow{AC}, B)$ or $\overline{H}(\overleftrightarrow{BC}, A)$, respectively, and therefore $P \notin \overline{H}(\overleftrightarrow{AB}, C) \cap \overline{H}(\overleftrightarrow{AC}, B) \cap \overline{H}(\overleftrightarrow{BC}, A)$. Consequently we have the equivalent definition of $\overline{\text{Int}}\ \triangle ABC = \overline{H}(\overleftrightarrow{AB}, C) \cap \overline{H}(\overleftrightarrow{AC}, B) \cap \overline{H}(\overleftrightarrow{BC}, A)$.

<u>2.25.8. Corollary 2.</u> $\overline{\text{Int}}\ \triangle ABC$ is a convex set.

The proof is a direct consequence of Lemma 0-8, and Corollary 4, Section 2.18.4. That is, each of the closed half planes are convex sets and the intersection of three convex sets is convex.

<u>2.25.9. Corollary 3.</u> Given $\triangle ABC$ and points P and Q such that $P \in AB$ and $Q \in BC$ then $\overline{PQ} \subset \overline{\text{Int}}\ \triangle ABC$ and $PQ \subset$ Int $\triangle ABC$.

The statement that $\overline{PQ} \subset \overline{\text{Int}}\ \triangle ABC$ is a direct result of the convexity of $\overline{\text{Int}}\ \triangle ABC$.

To show $PQ \subset \text{Int}\ \triangle ABC$, suppose there is a point $R \in PQ$ such that $R \notin \text{Int}\ \triangle ABC$. By Lemma 0-17 $R \in \text{Ext}\ \triangle ABC$ or $R \in \triangle ABC$. In the first case we contradict $R \in \overline{\text{Int}}\ \triangle ABC$ which follows from $R \in PQ \subset \overline{PQ}$. If $R \in \triangle ABC$ then R on $\overline{AB}$, $\overline{BC}$, or $\overline{AC}$. R cannot be on $\overline{AB}$ or $\overline{BC}$ since this would yield two points of $\overleftrightarrow{PQ}$ on either $\overline{AB}$ or $\overline{BC}$ giving us P and Q collinear with either A, B, or C, B; and in either case a contradiction. If $R \in AC$ then we have a contradiction by Lemma 0-7. Consequently $R \in \text{Int}\ \triangle ABC$. Q.E.D.

<u>2.25.10. Corollary 4.</u> For any $\triangle ABC$, $\overline{\text{Int}}\ \triangle ABC$ is a connected set.

The proof follows from the fact that $\overline{\text{Int}}\ \triangle ABC$ is a convex set and that every convex set is connected by the lemma of Section 2.24.3.

<u>2.25.11.</u> A similar result, that the union of a triangle with any of its open sides is convex and therefore connected, follows by an analogous set of steps.

<u>2.25.12. Corollary 5.</u> For any $\triangle ABC$, $\overline{\text{Ext}}\ \triangle ABC$ is a connected set.

Let $P, Q \in \overline{\text{Ext}}\ \triangle ABC$. If $P, Q \in \text{Ext}\ \triangle ABC$ there exists a path connecting P and Q by Lemma 0-15. If $P, Q \in \triangle ABC$, we can connect P and Q by using the edges of $\triangle ABC$ as our path. Hence in both these cases we have our desired result. So assume $P \in \text{Ext } \triangle ABC$ and $Q \in \triangle ABC$. Without loss of generality we may assume $Q \in \overline{AB}$. There exists a point R on $\overleftrightarrow{CQ}$ such that CQR. Hence $R \in H(\overleftrightarrow{AB}, \widetilde{C})$ and thus $R \in \text{Ext } \triangle ABC$. Since $\text{Ext } \triangle ABC$ is connected there exists a path Ω which connects P and R. Let $\Omega^* = \Omega \cup \overline{QR}$. It follows that Ω^* is a path in $\overline{\text{Ext}}\ \triangle ABC$ connecting Q and P. Q.E.D.

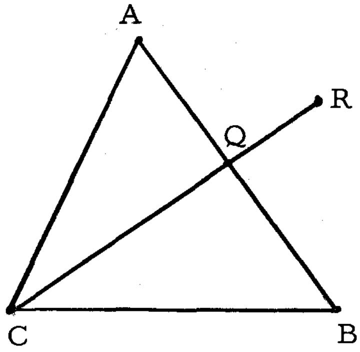


Figure 30.

<u>2.25.13. Lemma 0-18.</u> Given any $\triangle ABC$ there exists a line ℓ such that $\Phi(\ell) \subset \text{Ext } \triangle ABC$.

There exist points P and Q such that PAB and QAC by 0-3. $P \in H(\overleftrightarrow{AC}, \widetilde{B})$ and $Q \in H(\overleftrightarrow{BC}, \widetilde{C})$ by the Plane Separation Theorem. The line $\overleftrightarrow{PQ}$ seems to be a reasonable candidate for our line ℓ. $\overleftrightarrow{PQ}$ cannot meet $\overline{AB}$ or $\overline{AC}$ since

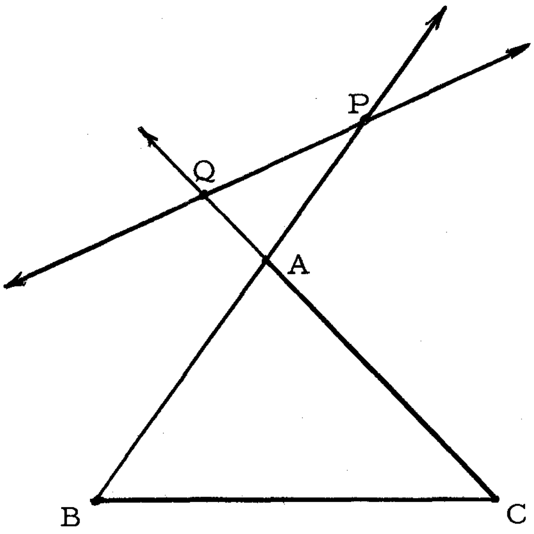


Figure 31.

this would imply $P \in \overline{AB}$ or $Q \in \overline{AC}$. If $\overleftrightarrow{PQ}$ meets BC then it also must have a point in common with $\overline{AC}$ or $\overline{AB}$ by Pasch's Axiom again leading to the same contradiction. Hence $\overleftrightarrow{PQ}$ has no points of ΔABC on it and by Theorem 0-13 it therefore cannot have a point in common with $\text{Int}\,(\Delta ABC)$. Consequently by Lemma 0-17 $\overleftrightarrow{PQ} \subset \text{Ext}\,(\Delta ABC)$.

<u>2.25.14.</u> The following lemma will be a helpful tool when we tackle the Jordan Theorem for polygons.

<u>2.25.15. Lemma 0-19.</u> If $\Delta ABC \subset \overline{\text{Int}}\,\Delta DEF$ then $\text{Int}\,\Delta ABC \subset \text{Int}\,\Delta DEF$.

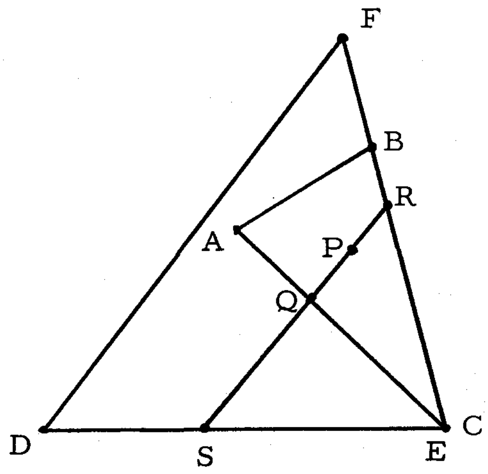


Figure 32.

We have two cases to consider: (1) $\Delta DEF = \Delta ABC$, (that is the vertices correspond) and (2) There exists at least one point of ΔABC in $\text{Int}\,\Delta DEF$.

In Case (1) the proof is immediate since $\text{Int}\,\Delta ABC = \text{Int}\,\Delta DEF$ by definition.

Case 2: Let $P \in \text{Int}\,\Delta ABC$ and Q be a point of ΔABC in $\text{Int}\,\Delta DEF$. By Theorem 0-13 $\overleftrightarrow{PQ}$ meets ΔABC in two points one of which will be Q. Let the other be R. By Corollary 1, Section 2.21.6 we have QPR. If $R \in \text{Int}\,\Delta DEF$ we have

$\overline{QR} \subset \text{Int } \Delta DEF$ by the convexity of $\text{Int } \Delta DEF$, and therefore $P \in \text{Int } \Delta DEF$. We also must investigate the possibility that $R \in \Delta DEF$. If so the $\overleftrightarrow{PQ}$ must meet ΔDEF at another point, say S, such that RQS by Theorem 0-13. But RQS and QPR implies $R \in RS$ by Lemma 0-3. Finally by Corollary 3, Section 2.25.9 $P \in \text{Int } \Delta DEF$. Q.E.D.

<u>2.25.16. Lemma 0-20.</u> If Q is a point of ΔABC then there exist points $P \in \text{Int } \Delta ABC$ and $R \in \text{Ext } \Delta ABC$ such that PQR.

We must consider two cases Q a vertex of the triangle or Q and element of one of the sides. First assume Q is a vertex of ΔABC. Without loss of generality we can assume $Q = A$. There exists a point D such that BDC by Theorem 0-1. Consider the line $\overleftrightarrow{DA}$. There exist points P and R such that DPA and DAR. (see Figure 33.) By Lemma 0-3 we have PQR.

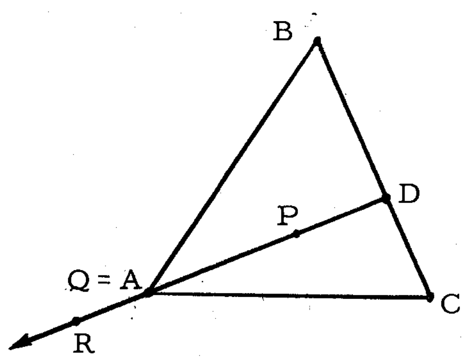


Figure 33.

We claim that $P \in \text{Int } \Delta ABC$ and $R \in \text{Ext } \Delta ABC$. Since QPD we have $P \in H(\overleftrightarrow{BC}, A)$. Since $D \in BC$, $H(\overleftrightarrow{AB}, C) = H(\overleftrightarrow{AB}, D)$ and $H(\overleftrightarrow{AC}, B) = H(\overleftrightarrow{AC}, D)$. But $PD \cap \Phi(\overleftrightarrow{AB}) = \emptyset$ and $PD \cap \Phi(\overleftrightarrow{AC}) = \emptyset$, hence $P \in H(\overleftrightarrow{AB}, C) \cap H(\overleftrightarrow{AC}, B)$. Therefore $P \in \text{Int } \Delta ABC$. Since PQR, we have $R \in H(\overleftrightarrow{AB}, \tilde{C})$ and therefore $P \in \text{Ext } \triangle ABC$ by definition.

Next, consider $Q \in AB$. (The case that Q is on either of the other sides follows in the same manner.) By steps quite similar to the ones above we get PQR on $\overleftrightarrow{QC}$ with $R \in \text{Ext } \triangle ABC$ and $P \in \text{Int } \triangle ABC$. Q.E.D.

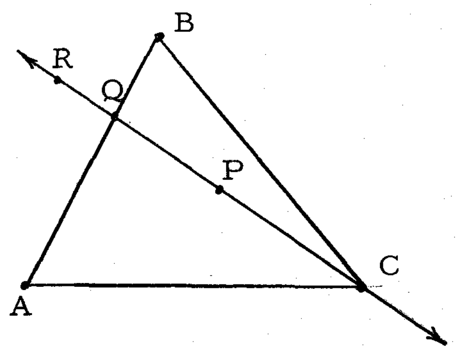


Figure 34.

2.25.17. Corollary 1. If Q is a point of $\triangle ABC$ then there exist points $P \in \text{Int } \triangle ABC$ and $R \in \text{Ext } \triangle ABC$ such that PQR, $PQ \subset \text{Int } \triangle ABC$ and $QR \subset \text{Ext } \triangle ABC$.

Any point on PQ will be in the same half planes determined by the sides and vertices of the triangle as P since $PQ \cap \triangle ABC = \emptyset$. (Recall QPC or CPQ and in either case PQ could not meet the triangle.) Hence $PQ \subset \text{Int } \triangle ABC$. Similarly we have that any point of RQ must be in the opposite half plane from P and consequently in the exterior of the triangle.

2.25.18. Corollary 2. If Q is any point on AB in $\triangle ABC$ then there exist points P and R on any line ℓ containing Q and noncollinear with A and B such that $P \in \text{Int } \triangle ABC$, $R \in \text{Ext } \triangle ABC$ and PQR.

By Pasch's Axiom ℓ will intersect $\triangle ABC$ in some point S such that $S \in AC \cup BC \cup \{C\}$. Let $P \in QS$ and R be such that RQS.

By the same argument we used in Lemma 20 it follows that

$R \in \text{Ext}\,\triangle ABC$ and $P \in \text{Int}\,\triangle ABC$. Q.E.D.

2.25.19. Definition. Simplices: a point will be called a 0-simplex, an open segment a 1-simplex, the interior of a triangle a 2-simplex. In order to define a 3-simplex we first must define a tetrahedron. Given any set of four noncoplanar points, each triple will determine a triangle by Theorem I-5. A tetrahedron will be defined as the union of the triangles determined by any four such points. The four points will be called the vertices of the tetrahedron. The plane of any of the triangles together with the fourth point will define a half space (see Theorem 0-10). Hence for any tetrahedron we will have four such half spaces defined. The interior of the tetrahedron will be defined to be the intersection of these half spaces. A 3-simplex will be the interior of any tetrahedron.

The faces of a 1-simplex are its end points, the 0-faces of a 2-simplex are its vertices, the 1-faces of a 2-simplex are its edges. For a 3-simplex we define it 0-faces as its vertices; the 1-faces, the open segments between the vertices; and the 2-simplices determined by each triple of vertices, as its 2-faces. The closure of a simplex σ is the union of σ with its faces and will be denoted by $\overline{\sigma}$. A complex K is a finite set of simplices with the following properties: (1) If σ is a simplex in K then all the faces of σ are in K, (2) If σ_1 and σ_2 are distinct simplices in K then $\sigma_1 \cap \sigma_2 = \emptyset$. We define $|K|$ to be $\bigcup_{\sigma \in K} \sigma$ and call $|K|$ the <u>geometric complex generated</u> by K. A <u>2-complex</u> will be a complex such that every 0 or 1 simplex of K is the face of some 2-simplex of K. A <u>2-complex</u> will be called <u>edge-connected</u> iff given any two 2-simplices of K, α, and β, there exists a sequence of 2-simplices of K $\{\sigma_1, \ldots, \sigma_n\}$ such that $\sigma_1 = \alpha$, $\sigma_n = \beta$ and $\overline{\sigma}_i \cap \overline{\sigma}_{i+1}$ is a common edge and its vertices for $1 \leq i < n$. A set of points Z is <u>triangulated</u> if there exists an edge-connected 2-complex K such that $|K| = Z$, K is called a <u>simplicial decomposition</u> of Z. We define $\beta(\sigma)$ as the set of faces of σ when σ is an n-simplex for $1 \leq n \leq 3$. β is called the Boundary Function (Aleksandrov, 1947; Lefschetz, 1949).

<u>2.25.20. Lemma.</u> Given ΔABC, $\overline{\text{Int}}\ \Delta ABC$ is triangulated.

The proof is a direct consequence of the definition of triangulated.

<u>2.25.21. Lemma.</u> If K is an edge-connected 2-complex then $|K|$ is connected.

By induction, utilizing Lemma 0-13 and the comments in Section 2.24.6 we have our results.

<u>2.25.22. Lemma.</u> Any set Z which has a triangulation is connected.

This follows from the definitions and the above Lemma.

<u>2.25.23. Lemma.</u> If Z is a planar set, $Z = |K|$ for some complex K and e a 1-simplex of K then e is a face of at most two 2-simplices of K.

Assume there exists three 2-simplices σ_1, σ_2, and σ_3 with but not collinear with e. ℓ will meet an edge or vertex of each of the simplices σ_1, σ_2, and σ_3 in points P_1, P_2, and P_3 respectively. Let P_i and P_j stand for any two of the points above. If QP_iP_j then consider

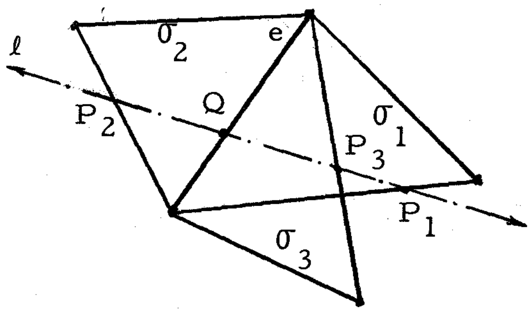


Figure 35.

a point R such that QRP_iP_j. By convexity of $\overline{\sigma}_i$ and $\overline{\sigma}_j$ we have $R \in \sigma_i \cap \sigma_j$, but this contradicts $\sigma_i \cap \sigma_j = \emptyset$ by the definition of a complex. Hence P_i and P_j must lie such that P_iQP_j. Thus P_1QP_2 and P_2QP_3, but these two imply either P_1P_3Q or P_3P_1Q and a contradiction. Therefore K cannot contain three simplices with a common edge. Q. E. D.

2. 26. 1. Now we need to consider whether a polygon has the same plane separation characteristics as a triangle. Recall that a triangle separates the points of the plane in which it lies into three disjoint subsets, its interior, its exterior and the triangle itself. In the preceding Theorems, Lemmas and Corollaries we have developed some of the characteristics of these sets. For example there exists a line lying in the exterior of any triangle according to Lemma 0-18, or the interior of a triangle is a convex and hence a connected set by the Lemma of Section 2. 24. 3. When we consider polygons we would

expect a polygon, also, to separate the points of the plane in which it lies into three disjoint sets, an interior, an exterior and the polygon itself. Intuitively it would be too much to expect the interior of a polygon to be convex but we would expect it to be connected. Before we get too far with our conjectures we should first recall our definition of a polygon Π: A polygon $\Pi = \langle P_1, P_2, \ldots, P_n \rangle$ is a polygonal path with $P_iP_{i+1} \cap P_jP_{j+1} = \emptyset$, $P_i \neq P_j$ for all i and j such that $1 \leq i < j < n$, $P_1 = P_n$, and P_i, P_{i+1}, P_{i+2} are noncollinear for $1 \leq i \leq n-2$. (See Section 2.24.2.)

We are ready to tackle the more difficult task of establishing the existence and properties of the interior and exterior of a polygon. Intuitively, taking our clues from the properties of a triangle and its interior and exterior, we would expect the following properties of a polygon Π in a plane α:

Π divides $\Phi(\alpha) - \Pi$ into two disjoint regions called the Interior and Exterior of Π denoted by Int Π and Ext Π such that:

(1) Int Π and $\overline{\text{Ext}}\ \Pi$ are connected sets.

(2) If a line contains a point $Q \in$ Int Π then $\Phi(\ell) \cap \Pi$ contains at least 2 points P and R such that PQR.

(3) There exists a line ℓ such that $\Phi(\ell) \subset$ Ext Π.

(4) If $Q \in \Pi$ then there exist point $P \in$ Int Π and $R \in$ Ext Π such that PQR and $PQ \subset$ Int Π, $QR \subset$ Ext Π. If Q is not a vertex of Π then Q is on some edge

P_iP_{i+1} of Π and any line $\ell \neq \overleftrightarrow{P_iP_{i+1}}$ containing Q contains two other points P and R with the above properties.

(5) If $S \in \text{Ext}\,\Pi$ and $T \in \text{Int}\,\Pi$ then $ST \cap \Pi \neq \emptyset$.

(6) If Ω is another polygon such that $\Omega \subset \overline{\text{Int}}\,\Pi$ and Ω has these six properties then $\text{Int}\,\Omega \subset \text{Int}\,\Pi$.

($\overline{\text{Int}}\,\Pi = \text{Int}\,\Pi \cup \Pi$.)

<u>2.26.2.</u> In the remainder of our discussion we will be dealing with the Jordan Theorem and its consequences. That is, we will be only interested in planar figures and hence will not mention the plane in which the figure lies unless it is necessary for clarification.

<u>2.26.3. Theorem 0-16. Jordan Theorem for Polygons.</u> If Π is a polygon of plane α then Π divides $\Phi(\alpha) - \Pi$ into two disjoint regions denoted by $\text{Int}\,\Pi$ and $\text{Ext}\,\Pi$ having properties (1) through (6) above.

The proof will be by induction on the number of vertices of Π. We have already established these properties if Π and Ω are triangles. See the Lemma of Section 2.24.3, Theorem 0-13, Lemma 0-15, Lemma 17 and its Corollary 5, Lemma 0-18, Lemma 0-16, Lemma 0-19 and Lemma 0-20 and its corollaries. Hence we will presume the validity of the theorem for polygons with m vertices such that $3 \leq m \leq k-1$. That is we are assuming properties (1) through

(6) hold for all polygons with a number of vertices less than or equal to $k-1$. From this induction assumption we will show the validity of the theorem for polygons with k vertices. (We choose $k-1$ as the number of vertices because if $\Pi = \langle P_1, P_2, \ldots, P_k \rangle$, Π has $k-1$ distinct vertices since $P_1 = P_k$.)

<u>2.26.4.</u> Initially we must establish a decomposition lemma for polygons with 4 or more vertices.

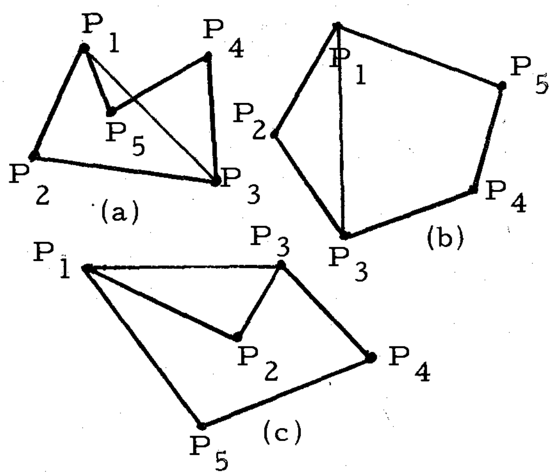


Figure 36.

<u>Lemma 0-21.</u> If Π is any polygon $\langle P_i \rangle_{i=1}^{n}$ with $n \geq 4$ then either $P_1P_3 \cap \Pi = \emptyset$ or there exists an i, $4 \leq i < n$, such that $P_2P_i \cap \Pi = \emptyset$.

If $P_1P_3 = \emptyset$ we are through so we assume $P_1P_3 \cap \Pi \neq \emptyset$. (See Figure 36 for examples for $n = 5$.) Our assumption implies there exists a subset of $\{P_1, P_2, \ldots, P_{n-1}\}$ lying in $P_1P_3 \cup \text{Int}\, \Delta P_1P_2P_3$. (See Figure 36(a).) This holds since $P_1P_3 \cap \Pi \neq \emptyset$, which means there exists P_i such that $\overline{P_iP_{i+1}} \cap P_1P_3 \neq \emptyset$ with $i \geq 3$. (We can exclude $i = 1$ or 2 since this would imply collinearity of P_1, P_2, and P_3 contrary to hypothesis.) Hence either P_i or P_{i+1} must lie in $P_1P_3 \cup \text{Int}\, \Delta P_1P_2P_3$. (If not then $\overrightarrow{P_iP_{i+1}} \cap (P_1P_2 \cup P_2P_3 \cup \{P_2\}) \neq \emptyset$ by Pasch's Axiom contradicting the definition of a polygon.) Thus we have established that the subset of $\langle P_1 \rangle_{i=1}^{n-1}$ which lies in $P_1P_3 \cup \text{Int } \Delta P_1P_2P_3$ is nonempty. Let this set be $\{Q_1, Q_2, \ldots, Q_m\}$.

We have two cases to consider: All the Q_i's lie on P_1P_3 or there exists a Q_i such that $Q_i \in \text{Int } \Delta P_1P_2P_3$. In the first case we can choose any of the Q_i's as our P_i and $P_2P_i \cap \Pi$ will be empty, since if not Pasch's Axiom would imply one of the Q_i's in $\text{Int } \Delta P_1P_2P_3$. (The details of this part of the discussion are trivial and will be omitted.)

The more difficult case is the case that the set of Q_i's in $\text{Int } \Delta P_1P_2P_3$ is nonempty. We order the angles formed by the rays $\overrightarrow{P_1Q_i}$ and $\overrightarrow{P_1P_2}$ by Theorem 0-14. (Notice that since all the points Q_i lie in $P_1P_3 \cup \text{Int } \Delta P_1P_2P_3$ the rays all lie in $H(\overleftrightarrow{P_1P_2}, P_3)$ hence Theorem 0-14 is applicable.) Since the set of Q_i's is finite there exists one of the Q_i's say Q_j such that $\angle P_2P_1Q_j < \angle P_2P_1Q_i$ for all $i \neq j$. Thus Q_j will be some P_i, say P_{i_0}. (In Figure 37 it is P_8.) We claim that $P_2P_i \cap \Pi = \phi$. For if it is not, let

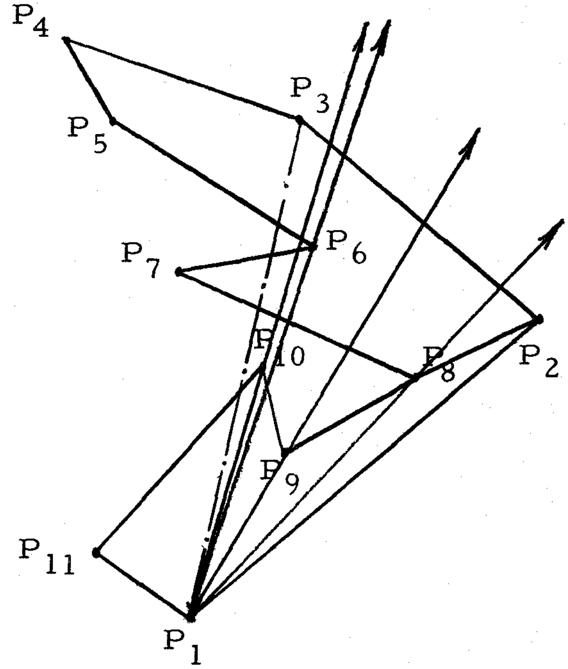


Figure 37.

$T \in P_2P_{i_0} \cap \Pi$.

Since all the Q_i's lie in $P_1P_3 \cup \text{Int}\, \Delta P_1P_2P_3$ we must have $\angle P_2P_1Q_i \leq \angle P_2P_1P_3$ for all i by the definition. (See Section 2.22.4.) By Lemma 0-11 we have that each $\overrightarrow{P_1Q_i}$ must intersect $P_2P_3 \cup \{P_3\}$ in some point, in particular $\overrightarrow{P_1P_{i_0}} \cap P_2P_3$ in a point R_{i_0}. Now if $T \in \Pi$ lies on $P_2P_{i_0}$ then $T \in \text{Int}\, \angle P_2P_1P_3$ by definition. Assume that $\overline{P_jP_{j+1}}$ is the segment of Π containing T. By Theorem 0-13 $\overleftrightarrow{P_jP_{j+1}}$ must meet $\Delta P_1P_2R_{i_0}$ in two points. Neither P_j or P_{j+1} can lie in $\text{Int}\, \Delta P_1P_2R_{i_0}$ since then it would lie in $\text{Int}\, \angle R_{i_0}P_1P_2$ and hence either $\angle P_2P_1P_j$ or $\angle P_2P_1P_{j+1}$ would be less than $\angle P_2P_1P_{i_0}$. By the definition of a polygon $\overline{P_jP_{j+1}}$ cannot intersect either $\overline{P_1P_2}$ or $\overline{P_2R_{i_0}} \subset \overline{P_2P_3}$. But we have established that $\overline{P_jP_{j+1}}$ meets $\Delta P_1P_2R_{i_0}$ in two points. Hence these points must both lie on $P_1R_{i_0}$; for, if not, $\overline{P_jP_{j+1}}$ would meet $\overline{P_1P_2}$ or $\overline{P_2R_{i_0}}$. But then we must have $\overleftrightarrow{P_jP_{j+1}} = \overleftrightarrow{P_1R_{i_0}}$ and consequently $T \in \overrightarrow{P_1R_{i_0}}$ contradicting the fact that $T \in P_2P_{i_0}$. Thus $P_2P_{i_0} \cap \Pi = \emptyset$.

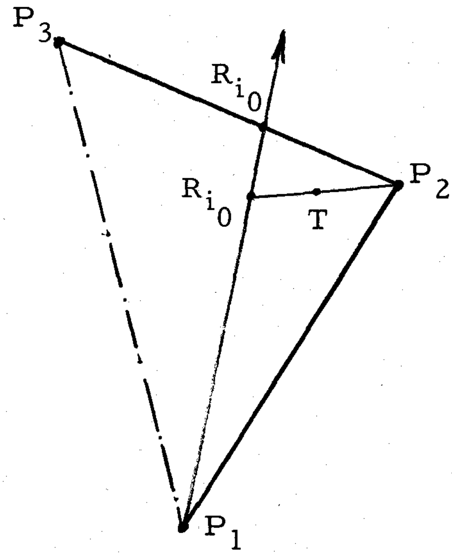


Figure 38.

Q.E.D.

The preceding can be summarized as a lemma and will provide a useful tool in our later development.

2.26.5. Lemma 0-22. Given a triangle $P_1P_2P_3$ and a polygonal path Ω such that $(\text{Int}\ \Delta P_1P_2P_3 \cup P_1P_3) \cap \Omega \neq \emptyset$ but $\Omega \cap (\overline{P_1P_2} \cup \overline{P_2P_3}) = \emptyset$, except possibly for P_1 or P_3, then there exists a vertex Q of Ω such that $QP_2 \cap \Omega = \emptyset$.

2.26.6. Now we will establish a set theoretic result which we can apply to our considerations:

Lemma 0-23. If A and B are nonempty proper subsets of U then exactly one of the following five cases must hold:

(1) $A \subset B$ and $B' \subset A'$

(2) $A \subset B'$ and $B \subset A'$

(3) $A' \subset B$ and $B' \subset A$

(4) $A' \subset B'$ and $B \subset A$

or (5) $A \cap B \neq \emptyset$, $A \cap B' \neq \emptyset$, $A' \cap B \neq \emptyset$, and $A' \cap B' \neq \emptyset$.

If we assume that (5) does not hold then at least one of the intersections in (5) must be empty. If $A \cap B = \emptyset$ then this implies $A \subset B'$, and also $B \subset A'$ and so we have case (2). If $A \cap B' = \emptyset$ then we have $A \subset B$ and $B' \subset A'$, or (1). If $A' \cap B = \emptyset$ then $A' \subset B'$ and $B \subset A$, or 4. Finally if $A' \cap B' = \emptyset$ then $A' \subset B$ and $B' \subset A$ which is case (3). Two or more of the intersections, in (5), empty, implies two or more of the subset relations (1) through (4) holding. But this is impossible since each possibility leads to a contradiction. For example, if we had (1) and (2) holding simultaneously, transitivity of the subset relation yields $A \subset A'$ and the only way this can occur is if $A = \emptyset$ in opposition to our hypotheses. Q.E.D.

2.26.7. Definition. We define the mod 2 union of sets $A_1, \ldots, A_n$ as $\{x \in \bigcup_{i=1}^{n} A_i \mid x$ is an element of an odd number of A_i's$\}$, we will denote this as $A_1 \oplus A_2 \oplus \ldots \oplus A_n = \bigoplus_{i=1}^{n} A_i$.

2.26.8. Now to complete the proof of the theorem. Let $\Pi = \langle P_1, P_2, \ldots, P_k, P_{k+1} \rangle$ be any polygon with $k \geq 4$ vertices. Recall that $P_{k+1} = P_1$ by definition. By Lemma 0-20 either P_1P_3 or $P_2P_{i_0}$ does not meet Π for some $i_0 \geq 4$. In the former case we can define two polygons $\Pi_1 = \langle P_1, P_2, P_3, P_1 \rangle$ and $\Pi_2 = \langle P_1, P_3, P_4, \ldots, P_{k+1} \rangle$ such that $\Pi_1 \oplus \Pi_2 \oplus \{P_1, P_3\} = \Pi$. In the second case we let $\Pi_1 = \langle P_1, P_2, P_{i_0}, P_{i_0+1}, \ldots, P_{k+1} \rangle$ and $\Pi_2 = \langle P_2, P_3, \ldots, P_{i_0} P_2 \rangle$ and again we have $\Pi_1 \oplus \Pi_2 \oplus \{P_2, P_{i_0}\} = \Pi$. (See Figure 36.) The fact that Π_1 and Π_2 in both cases are polygons follows from the definition and Π being a polygon. The equivalences for the mod 2 unions follow because the only points which both Π_1 and Π_2 have in common are those on $\overline{P_1P_3}$ in the first case and $\overline{P_2P_{i_0}}$ in the second.

By the induction assumption Π_1 and Π_2 have properties (1) through (6) since each has fewer than k vertices. Now $(\text{Int } \Pi_1)' = \text{Ext } \Pi_1 \cup \Pi_1 = \overline{\text{Ext}}\, \Pi_1$ and similarly $(\text{Int } \Pi_2)' = \overline{\text{Ext}}\, \Pi_2$. Therefore we can apply Lemma 0-22 to this situation and we have the following 5 disjoint cases:

(1) $\text{Int } \Pi_1 \subset \text{Int } \Pi_2$ and $\overline{\text{Ext}}\, \Pi_2 \subset \overline{\text{Ext}}\, \Pi_1$

(2) $\text{Int } \Pi_1 \subset \overline{\text{Ext}}\, \Pi_2$ and $\text{Int } \Pi_2 \subset \overline{\text{Ext}}\, \Pi_1$

(3) $\overline{\text{Ext}}\, \Pi_1 \subset \text{Int } \Pi_2$ and $\overline{\text{Ext}}\, \Pi_2 \subset \text{Int } \Pi_1$

(4) $\overline{\text{Ext}}\, \Pi_1 \subset \overline{\text{Ext}}\, \Pi_2$ and $\text{Int } \Pi_2 \subset \text{Int } \Pi_1$

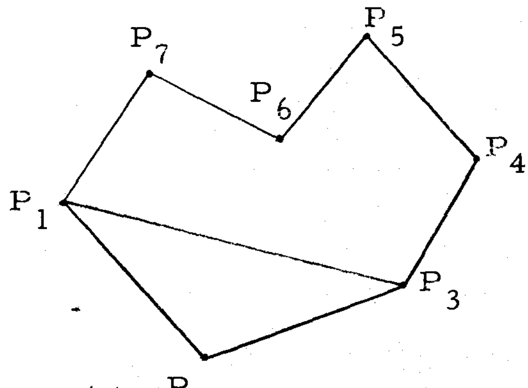
(5) $\text{Int } \Pi_1 \cap \text{Int } \Pi_2 \neq \emptyset$, $\text{Int } \Pi_1 \cap \overline{\text{Ext}}\, \Pi_2 \neq \emptyset$,

$\overline{\text{Ext}}\, \Pi_1 \cap \text{Int } \Pi_2 \neq \emptyset$, and $\overline{\text{Ext}}\, \Pi_1 \cap \overline{\text{Ext}}\, \Pi_2 \neq \emptyset$.

We will show that cases (3) and (5) contradict the definition of Π or the properties of Π_1 and Π_2. In the remaining cases we will inductively define $\text{Int } \Pi$ and $\text{Ext } \Pi$ and show they have properties (1) through (6).

<u>Case (3).</u> $\overline{\text{Ext}}\, \Pi_1 \subset \text{Int } \Pi_2$ and $\overline{\text{Ext}}\, \Pi_2 \subset \text{Int } \Pi_1$.

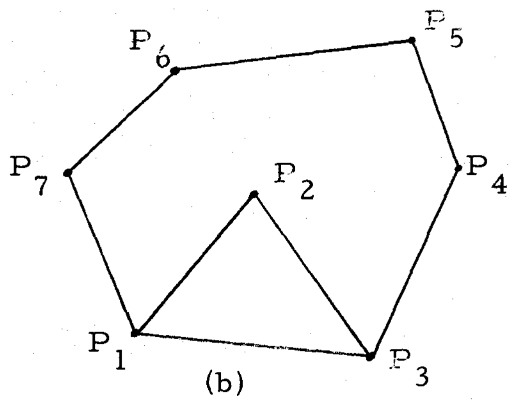
Since $\text{Ext } \Pi_1$ contains the points of some line ℓ and since $\text{Ext } \Pi_1 \subset \overline{\text{Ext}}\, \Pi_1$ we have $\Phi(\ell) \subset \overline{\text{Ext}}\, \Pi_1$. Hence $\Phi(\ell) \subset \text{Int } \Pi_2$. But by property (3) $\Phi(\ell)$ cannot be a subset of $\text{Int } \Pi_2$ since $\Phi(\ell) \cap \Pi_2$ contains at least two points. Therefore case (3) leads to a



(a)

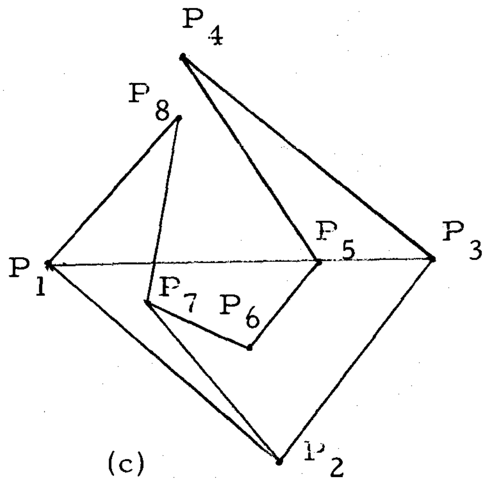
$P_1P_3 \cap \Pi = \emptyset$

$\Pi_1 = \langle P_1, P_2, P_3, P_1 \rangle$



(b)

$\Pi_2 = \langle P_1, P_3, P_4, P_5, P_6, P_7, P_1 \rangle$

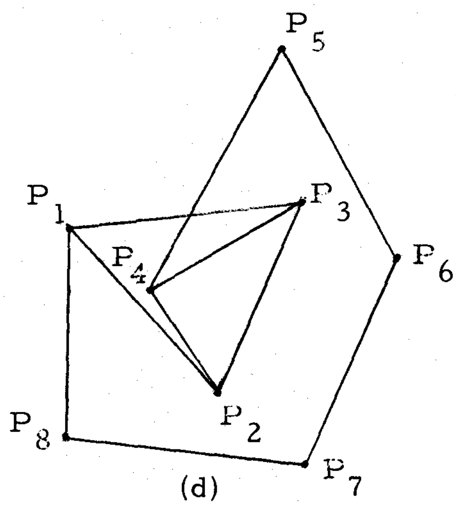


(c)

$P_1P_3 \cap \Pi \neq \emptyset$

$\Pi_1 = \langle P_1, P_2, P_7, P_8, P_1 \rangle$

$\Pi_2 = \langle P_2, P_3, P_4, P_5, P_6, P_7, P_2 \rangle$

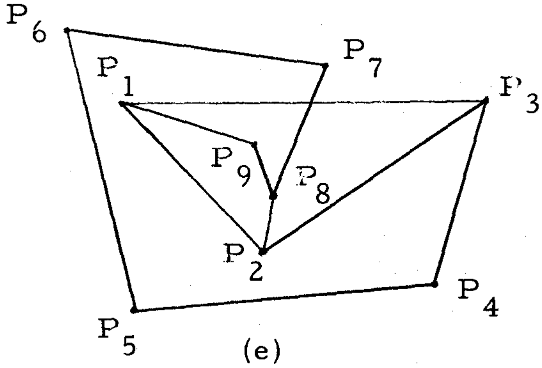


(d)

$P_1P_3 \cap \Pi \neq \emptyset$

$\Pi_1 = \langle P_1, P_2, P_4, P_5, P_6, P_7, P_8, P_1 \rangle$

$\Pi_2 = \langle P_2, P_3, P_4, P_2 \rangle$



(e)

$P_1P_3 \cap \Pi \neq \emptyset$

$\Pi_1 = \langle P_1, P_2, P_8, P_9, P_1 \rangle$

$\Pi_2 = \langle P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_2 \rangle$

Figure 39

contradiction and hence cannot occur.

Case (5). $\mathrm{Int}\,\Pi_1 \cap \mathrm{Int}\,\Pi_2 \neq \emptyset$, $\mathrm{Int}\,\Pi_1 \cap \overline{\mathrm{Ext}}\,\Pi_2 \neq \emptyset$, $\overline{\mathrm{Ext}}\,\Pi_1 \cap \mathrm{Int}\,\Pi_2 \neq \emptyset$, and $\overline{\mathrm{Ext}}\,\Pi_1 \cap \overline{\mathrm{Ext}}\,\Pi_2 \neq \emptyset$.

Let e be the common edge of Π_1 and Π_2. (i.e., $e = P_1P_3$ or $P_2P_{i_0}$.) In order to eliminate this case we first note that according to the definition of Π_1 and Π_2, $\Pi_1 - e$ and $\Pi_2 - e$ can meet only at the endpoints of e since they are disjoint subsets of Π which cannot cross itself. We proceed by establishing the following lemmas:

Lemma 0-23. If Q is a point of Π_1 which is also in $\mathrm{Int}\,\Pi_2$ then $\Pi_1 - e \subset \mathrm{Int}\,\Pi_2$.

Assume the conclusion is false, that is assume there exists $R \in \Pi_1$ such that $R \in \overline{\mathrm{Ext}}\,\Pi_2$. We can determine a path Ω using $\Pi_1 - e$ such that $\Omega \subset \Pi_1 - e$ and $\Omega = \langle Q_1, Q_2, \ldots, Q_m \rangle$ with $Q = Q_1$ and $R = Q_m$. Q and R may not be vertices of $\Pi_1 - e$, but we can use them and the edges of $\Pi_1 - e$ to determine edges of Ω.

Hence all but the first and last edges of Ω will be edges of $\Pi_1 - e$ and these two will be subsets of edges of $\Pi_1 - e$. There exists a last vertex Q_i of Ω such that $Q_i \in \mathrm{Int}\,\Pi_2$, consequently $Q_{i+1} \in \overline{\mathrm{Ext}}\,\Pi_2$. But either Q_{i+1} is on Π_2 or there exists a point $T \in Q_iQ_{i+1}$ such that $T \in \Pi_2$ by property (5) for

Π_2. In either case we have $\Pi_1 - e \cap \Pi_2 \neq \emptyset$. Hence $\Pi_1 - e$ and $\Pi_2 - e$ have a point in common or $\Pi_1 - e \cap e \neq \emptyset$. The first possibility contradicts the fact that $\Pi_1 - e$ and $\Pi_2 - e$ have no points in common except the vertices of e and neither of these vertices can be vertices of Ω. (If either of the vertices of e were vertices of Ω then two edges of Ω, and hence $\Pi_1 - e$, would also have this vertex as an end point. But this is impossible since this would imply that Π_1 would have these two edges and e with this vertex as endpoints contradicting the fact that Π_1 is a polygon.) If $\Pi_1 - e \cap e \neq \emptyset$ then again we have a contradiction to the fact that Π_1 is a polygon since the edges of a polygon cannot meet except at the vertices. Consequently if $\Pi_1 - e$ contains a point of $\text{Int}\,\Pi_2$ then $\Pi_1 - e$ lies in $\text{Int}\,\Pi_2$. Q.E.D.

Lemma 0-24. If Π_1 contains a point of $\text{Ext}\,\Pi_2$ then $\Pi_1 - e \subset \text{Ext}\,\Pi_2$.

This follows by an analogous argument, and will be left to the reader.

2.26.9. By the two previous lemmas we have two mutually exclusive cases, $\Pi_1 - e \subset \text{Int}\,\Pi_2$ or $\Pi_1 - e \subset \text{Ext}\,\Pi_2$. By a symmetric argument we also have the two possibilities $\Pi_2 - e \subset \text{Int}\,\Pi_1$ or $\Pi_2 - e \subset \text{Ext}\,\Pi_1$. Hence either $\Pi_1 \subset \overline{\text{Int}}\,\Pi_2$ or $\Pi_1 \subset \overline{\text{Ext}}\,\Pi_2$ and $\Pi_2 \subset \overline{\text{Int}}\,\Pi_1$ or $\Pi_2 \subset \overline{\text{Ext}}\,\Pi_1$.

Now if $\Pi_1 \subset \overline{\text{Int}}\ \Pi_2$ then $\text{Int}\ \Pi_1 \subset \text{Int}\ \Pi_2$ by property (6), but then $\text{Int}\ \Pi_1 \cap \overline{\text{Ext}}\ \Pi_2 \neq \emptyset$. Thus if case (5) is to occur we must have $\Pi_1 \subset \overline{\text{Ext}}\ \Pi_2$. Similarly $\Pi_2 \subset \overline{\text{Ext}}\ \Pi_1$.

Let P_1 be any point in $\text{Int}\ \Pi_1$ and P_2 any point in $\text{Int}\ \Pi_2$. Since by hypothesis, the intersection of these two sets is nonempty there exists a point Q in $\text{Int}\ \Pi_1 \cap \text{Int}\ \Pi_2$. Property (1) of our induction assumption tells us that there exists a path Ω_1 connecting P_1 and Q and lying entirely in $\text{Int}\ \Pi_1$ and a path Ω_2 connecting Q and P_2 lying within $\text{Int}\ \Pi_2$. Ω_1 cannot intersect Π_2 since $\Pi_2 \subset \overline{\text{Ext}}\ \Pi_1$ and Ω_1 cannot meet Π_1 either since $\Omega_1 \subset \text{Int}\ \Pi_1$. Similarly Ω_2 cannot meet Π_1 or Π_2. Let $R \in \text{Int}\ \Pi_2 \cap \overline{\text{Ext}}\ \Pi_1$ which is nonempty by hypothesis. There exists a path Ω_3 connecting R and P_1 lying in $\text{Int}\ \Pi_2$ since $\text{Int}\ \Pi_2$ is connected. Now none of the points of Ω_3 including R can be on Π_1 since $\Pi_1 \subset \overline{\text{Ext}}\ \Pi_2$. Thus if we let $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ we have a path connecting P_1 with R which does not meet Π_1. But $P_1 \in \text{Int}\ \Pi_1$ and $R \in \text{Ext}\ \Pi_1$ since $R \notin \Pi_1$. That is we have a path connecting a point of $\text{Int}\ \Pi_1$ with a point of the $\text{Ext}\ \Pi_1$ but not intersecting Π_1. Let $\Omega = \langle S_1, S_2, \ldots, S_m \rangle$ where $S_1 = P_1$ and $S_m = R$. There exists i such that $S_i, \ldots, S_{i-1}$ are elements of $\text{Int}\ \Pi_1$ but $S_i \in \overline{\text{Ext}}\ \Pi_1$. Consider $\overline{S_{i-1}S_i}$. By our induction assumption property (5) $S_{i-1}S_i$ must meet Π_1 or else $S_i \in \Pi_1$. In either case we contradict the assumption that

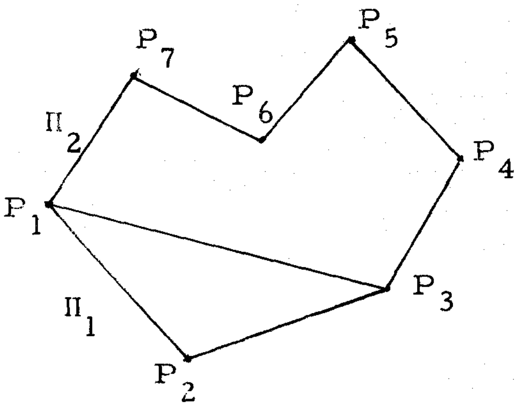
$\Omega \cap \Pi_1 = \emptyset$. Hence our case (5) leads us to a contradiction and we can eliminate it as one of our possibilities.

2.26.10. We establish next that cases (1) and (4) can both be reduced to case (2). We will assume case (1) and show that it can be reduced to (2). It then follows that (4) can also be reduced to (2) by just renaming Π_1 as Π_2 and Π_2 as Π_1 in the proof. See Figure 40 for illustrations of the various cases. Note that the possibilities shown are taken from Figure 39.

2.26.11. We proceed by first establishing the following lemma:

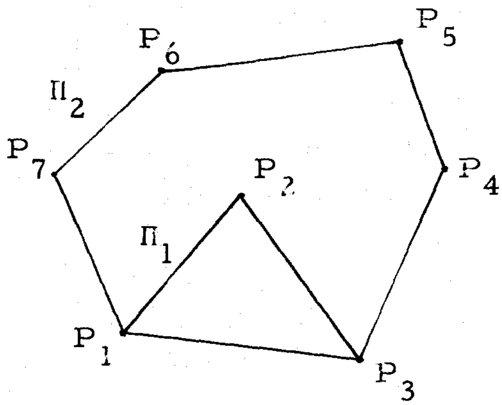
Lemma 0.25. Given a point P on a line ℓ and a finite set of nonintersecting open segments $\mathcal{T} = \{t_i\}_{i=1}^{m}$, none of which contain P such that at least one end point of each t_i lies in half plane H determined by ℓ, then there is a segment $t \in \mathcal{T}$ such that the segment between P and one of t's end point, T, does not meet any of the $\bar{t}_i$'s (see Figure 41(a)).

Assume the contrary and let $\{\rho_i\}_{i=1}^{n}$ be the set of the segments from P to the end points of the t_i's. Let T_i and $\hat{T}_i$ be the end points of t_i. By hypothesis we have that either T_i or $\hat{T}_i$ lies in H and by assumption we have that each of the segments PT_i and $P\hat{T}_i$ meet some $\bar{t}_j$ at a point other than an end point. We order these points on each PT_i and $P\hat{T}_i$.



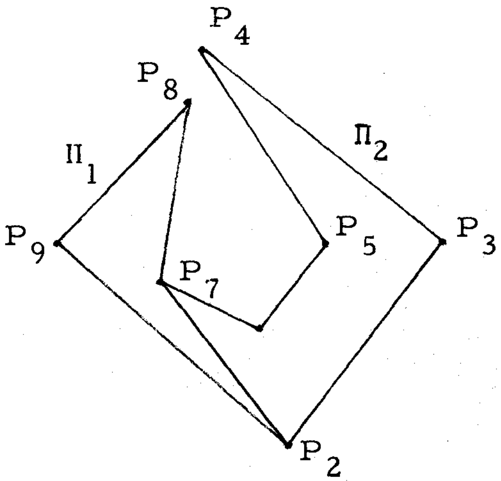
(a)

Case (2)



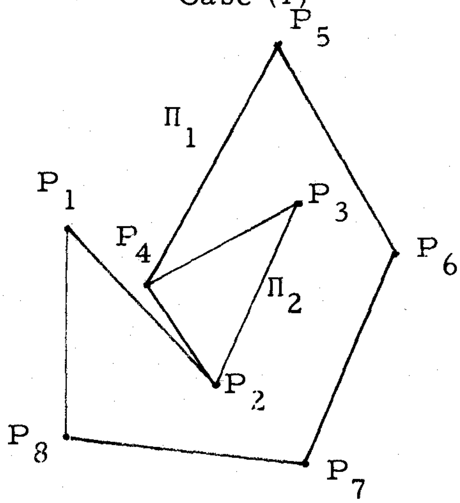
(b)

Case (1)



(c)

Case (2)



(d)

Case (4)

Figure 40..

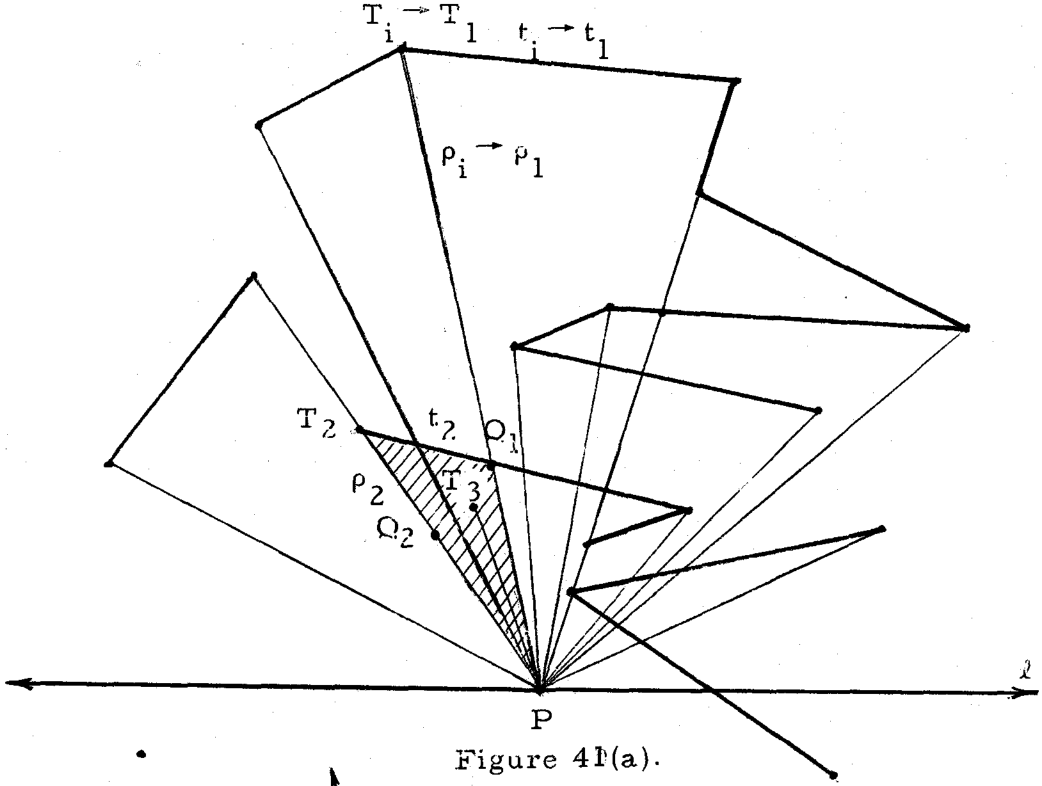


Figure 41(a).

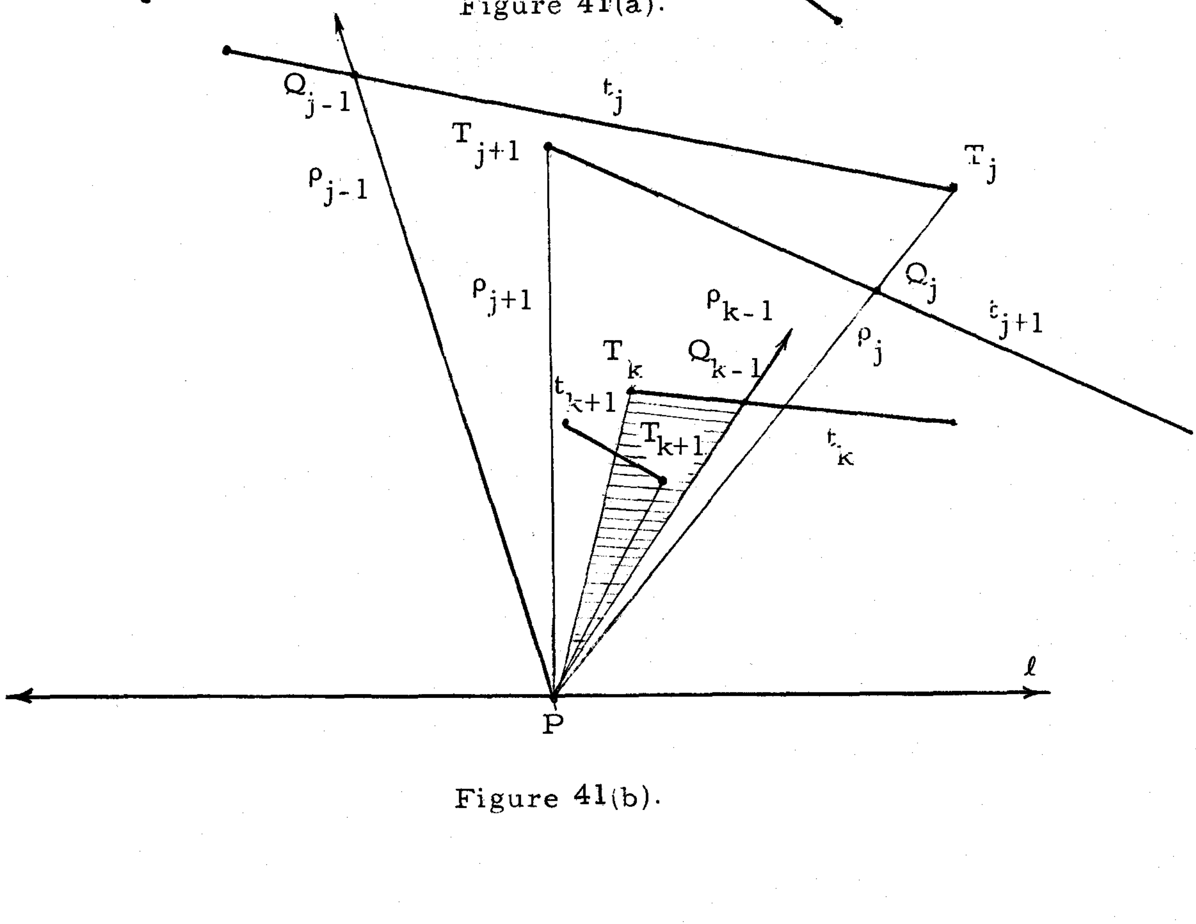


Figure 41(b).

Let Q_k be the point of intersection of ρ_k with the one of the t_i's which is "closest" to P, that is such that if R_k is any other point of ρ_k which is a point of some segment $\overline{s_i}$ then PQ_kR_k. By our assumption Q_k cannot be an end point of any t_i.

We single out any ρ_j and denote it as ρ_1, the segment in $\mathcal{T}$ whose end point determined ρ_1 we denote as t_1, the common end point as T_1 and the point on ρ_1 "closest" to P as Q_1. (Obviously, renaming the points and segments may be necessary.) (See Figure 41(a).)

Now the segment of $\mathcal{T}$ which contains Q_1 cannot be t_1 since Q_1 is not an end point. Call this segment t_2. At least one of the end points of t_2 will lie in H. We choose one that is in H and call it T_2. PT_2 will be one of our ρ_j's. Rename it ρ_2. We now have a triangle defined by PQ_1 and T_2. There will exist a point Q_2 on $\rho_2 = PT_2$ which is a point of one of the segments in $\mathcal{T}$. Rename this segment t_3. Since Q_2 is not an end point of t_3 one of its end points we claim is in Int ΔPQ_1T_2, for otherwise t_3 would meet $\overline{t_2}$, contrary to hypothesis, or it would meet PQ_1, contradicting the definition of Q_1. We choose this end point of t_3 as T_3 and define ρ_3 as PT_3. Notice that t_3 cannot be the same segment as t_2 by the definition of Q_2. We also have that t_3 cannot be t_1. Since if it were we would have it meeting PT_2 at Q_2 and PQ_1 at T_1 such that PQ_1T_1, hence by Lemma 0-1 t_3 must intersect t_2 at some point between T_2 and Q_1, contrary to hypothesis.

Hence by induction we may assume that we have defined k such segments t_j with end point T_j such that if $2 < j \leq k$ $T_j \in \text{Int}\, \Delta PQ_{j-2}T_{j-1}$ and each of the t_j's are distinct. Now there exists a point Q_k on PT_k which will be a point, other than an end point of some segment in $\mathcal{T}$ by assumption. We rename this segment t_{k+1}. By Lemma 0-19 each of the triangles lies in the interior of the one previously defined. Consequently $T_k \in \text{Int}\, \Delta PQ_{i-2}T_{i-1}$ for all $i < k$. But then t_{k+1} contains an end point in the interior of each of these triangles namely T_{k+1}. Thus t_{k+1} must be distinct from the previous t_j's. To see this we assume $t_{k+1} = t_j$ for some $2 < j < k$. Hence one of the end points of t_j is T_j. But then since $T_{k+1} \in \text{Int}\, \Delta PQ_jT_{j+1}$ and T_jQ_jP we have that t_{k+1} must meet t_{j+1} in some point between T_{j+1} and Q_j by Lemmas 0-16 and 0-1. But this contradicts that t_j and t_{j+1} are distinct nonintersecting segments. Hence t_{k+1} is distinct from the previous t_i's (see Figure 41(b)).

This proof implies that the number of elements in $\mathcal{T}$ is infinite, contradicting our hypothesis. In other words the procedure we developed above must terminate at some t_k and hence PT_k will not meet any element of $\mathcal{T}$ at some internal point.

2.26.12. Let Π be a polygon with k vertices and let Π_1 and Π_2 be defined as in Section 2.26.8, Case (1), that is $\text{Int}\,\Pi_1 \subset \text{Int}\,\Pi_2$ and $\overline{\text{Ext}}\,\Pi_2 \subset \overline{\text{Ext}}\,\Pi_i$. By general induction we may assume that Π_1 and Π_2 have properties (1) through (6). We will show that there exist a pair of vertices of Π, say P_i and P_j, such that $P_i \in \Pi_1$, $P_j \in \Pi_2$, and $P_iP_j \subset \text{Int}\,\Pi_2 \cap \text{Ext}\,\Pi_1$. Using these vertices we can define two new polygons Π_1' and Π_2' in which Case (2) will apply.

By use of property (4) and by an argument analogous to Lemma 0-23 we may assume $\Pi_1 \subset \overline{\text{Int}}\,\Pi_2$. (The details are left to the reader.)

By induction there exists a line ℓ such that $\Phi(\ell) \subset \overline{\text{Ext}}\,\Pi_1$ and hence $\Phi(\ell) \subset \overline{\text{Ext}}\,\Pi$. Let P and P^* be any two points of ℓ (see Figure 42). Let $\{r_i\}_{i=1}^m$ be the set of rays from P to the n vertices of Π (i.e., the vertices of Π_1 and Π_2). We should note that $m \leq n$ since some of the r_i's may contain more than one vertex of Π. Let H be the half plane determined by ℓ in which Π_2 lies and hence Π_1 also. Thus all the r_i's lie in $\overline{H}$. (Some of these rays are illustrated in Figure 42 for the examples shown.)

For all $i \leq m$ let Q_i be a point on r_i other than P. Now we can order the points Q_i such that $\angle Q_iPP^* < Q_jPP^*$ if $i < j$, according to Theorem 0-14. Let Q_m be the point such that

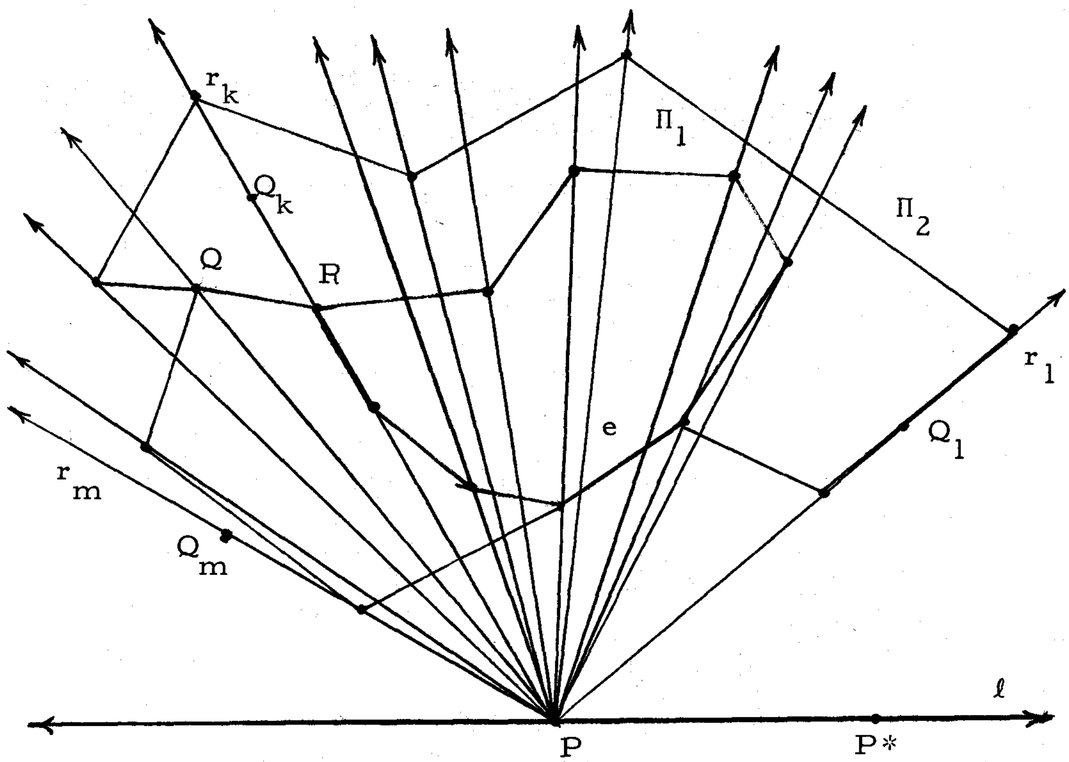


Figure 42(a).

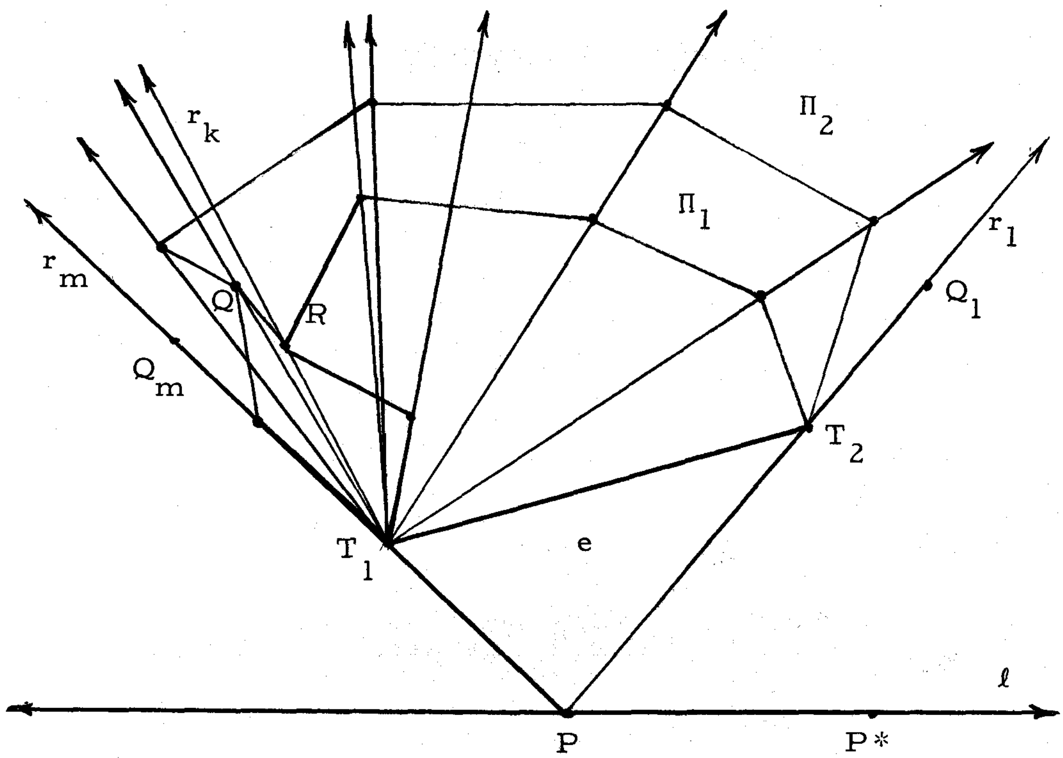


Figure 42(b).

$\angle Q_m PP^*$ is the "greatest" of these angles and Q_1 such that $\angle Q_1 PP^*$ is the "smallest". (Notice we will reassign subscripts to the Q_i's and also the r_i's.)

We claim there are two cases: (a) The vertex of Π which determines either r_1 or r_m will be a vertex of Π_2 and that ray will contain no points of Π_1 or (b) both r_1 and r_m contain the end points of e. Since $\Pi_1 \cap \Pi_2 = \overline{e}$ it follows that if Case (b) does not occur then either r_1 or r_m must not contain an end point of e. Say r_m does not contain an end point of e. Then r_m will be determined by some vertex of Π_1 or Π_2. If it is determined by a vertex of Π_1, other than an end point of e, then this vertex is in Int Π_1 and we must have vertices of Π_2 on both sides of the ray r_m. That is there must be vertices of Π_2 in Ext $\angle Q_m PP^*$, contradicting the ordering of the Q_i's. (To see that Π_2 must have vertices on both sides of r_m we can apply property (5) to the vertex of Π_1 and to a point in $H(\overleftrightarrow{PQ_m}, \widetilde{P}^*)$ which is in the exterior of Π_2. Such a point exists since $H(\ell, \widetilde{Q}_m) \cap H(\overleftrightarrow{PQ_m}, \widetilde{P}^*)$ will contain such points. There must be a point of Π_2 between these two points and hence a vertex of $\Pi_2 \in H(\overleftrightarrow{PQ}, \widetilde{P}^*)$.) Consequently we have that r_m can contain only vertices of Π_2 and Case (a) occurs.

Case (a). All the vertices of Π_1 lie in Int $\angle Q_m PP^*$. (We will suppose that r_m is the ray which does not contain any vertices of Π_1.) Let r_k be the ray such that it forms the largest angle with $\overrightarrow{PP^*}$ of any of the rays to the vertices of Π_1. We have immediately that $\angle Q_m PP^* > \angle Q_k PP^*$ (see Figure 42(a)). Now call a vertex of Π_1 on r_k, R. By our ordering of the angles there exist vertices of Π_2 in $H(\overleftrightarrow{PQ_k}, \tilde{P}^*)$. Now we can apply Lemma 0-25 to this situation and we see that we have a vertex Q of Π_2 in this half plane such that RQ does not meet Π_2. Since no vertex of Π_1 lies in this half plane we can infer that RQ does not intersect Π_1 either. Hence we have established the desired result for Case (a).

Case (b). (We only give a sketch of this portion of the proof since the steps are similar to those in Case (a).) Let T_1 be the end point of e on r_m. Order the rays to the vertices of Π from T_1. It follows that the largest (or smallest if $\Pi_1 \subset \overline{\text{Int}}\, \Delta PT_1T_2$) angle determined by these rays and $\overrightarrow{T_1P}$ must contain only points of Π_2. We let the last ray containing a vertex of Π_1 be called r_k (see Figure 42(b). Now we have exactly the same type of situation as we did in Case (a) and we can apply Lemma 0-25 to get points R on Π_1 and Q on Π_2 such that RQ does not intersect Π_1 or Π_2.

Now R will be some vertex P_i of Π and Q some vertex P_j of Π. Without loss of generality we may assume $i < j$.

We define Π_1' as $<P_1, P_2, \ldots, P_i, P_j, P_{j+1}, \ldots, P_k>$ and Π_2' as $<P_i, P_{i+1}, \ldots, P_j, P_i>$. Now we need to see that Case (2) applies to Π_1' and Π_2'. Since $P_j \in \Pi_1 \subset \overline{\text{Int}}\ \Pi_2$ and in both Cases (a) and (b) $P_j \notin \bar{e}$ we have $P_j \in \text{Int}\ \Pi_2$. Hence $P_jP_i \cap \Pi_2 = \emptyset$ implies that $P_jP_i \subset \text{Int}\ \Pi_2$ by property (4). Similarly $P_jP_i \subset \text{Ext}\ \Pi_1$.

Assume $\Pi_1' \subset \text{Int}\ \Pi_2'$ (i.e., Case (1)). Since all the vertices of Π_2' are in $\overline{\text{Int}}\ \Pi_2$ we have that $\Pi_2' \subset \overline{\text{Int}}\ \Pi_2$ and by property (6) $\text{Int}\ \Pi_1' \subset \text{Int}\ \Pi_2' \subset \text{Int}\ \Pi_2$. e is not an edge of Π_1' or Π_2' but its end points are vertices of Π_1' and Π_2' by definition. Hence since e cannot meet Π_1' or Π_2', e must lie entirely in the interior or exterior of Π_2'. But since $\Pi_1' \subset \text{Int}\ \Pi_2'$ we must have $e \subset \text{Int}\ \Pi_2'$ and hence $e \subset \text{Int}\ \Pi_2$ contradicting the original definition of e. Consequently Π_1' and Π_2' cannot satisfy Case (1) and in a similar fashion we can eliminate Case (4). Therefore Π_1' and Π_2' satisfy the conditions of Case (2).

<u>2.26.13.</u> In the preceding sections we have established that the only case we need to consider is Case (2). We define:

$$\text{Int}\ \Pi = \text{Int}\ \Pi_1 \cup \text{Int}\ \Pi_2 \cup e \quad \text{and} \quad \text{Ext}\ \Pi = \text{Ext}\ \Pi_1 \cap \text{Ext}\ \Pi_2.$$

<u>2.26.14.</u> First we show that $\text{Int}\ \Pi$ and $\overline{\text{Ext}}\ \Pi$ are connected. Let P_1 and Q_1 be points in $\text{Int}\ \Pi$. If P_1 and Q_1 are both

elements of the same set in $\text{Int}\,\Pi_1 \cup \text{Int}\,\Pi_2 \cup e$ then there exists a path completely within that set connecting P and Q by property (1) of our induction assumption or by the Lemma of Section 2.24.3. So, suppose P_1 is in $\text{Int}\,\Pi_1$ and Q_1 in $\text{Int}\,\Pi_2$ and let R be any element of e.

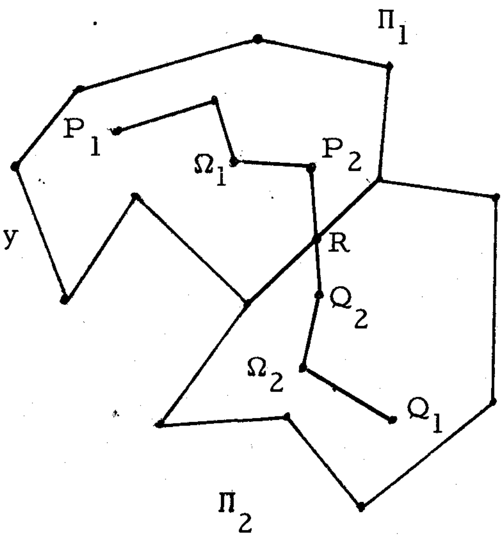


Figure 43.

By property (4) there exists a segment P_2Q_2 such that P_2RQ_2 with $P_2 \in \text{Int}\,\Pi_1$ and $Q_2 \in \text{Ext}\,\Pi_1$. Through the application of property (4) to Π_2 we may assume P_2 lies in either $\text{Ext}\,\Pi_2$ or $\text{Int}\,\Pi_2$. But P_2 cannot lie in $\text{Int}\,\Pi_2$ since this would imply $\text{Int}\,\Pi_1 \cap \text{Int}\,\Pi_2 \neq \emptyset$, contrary to hypothesis. Hence $P_2 \in \text{Ext}\,\Pi_2$ and $Q_2 \in \text{Int}\,\Pi_2$. Property (4) also tells us that any point of $\overline{P_2Q_2} - \{R\}$ must either be in $\text{Int}\,\Pi_1$ or $\text{Int}\,\Pi_2$ and hence $\overline{P_2Q_2} \subset \text{Int}\,\Pi$. Let Ω_1 be a path in $\text{Int}\,\Pi_1$ connecting P_1 and P_2 and Ω_2 be a path in $\text{Int}\,\Pi_2$ connecting Q_1 and Q_2. Consequently $\Omega = \overline{P_2Q_2} \cup \Omega_1 \cup \Omega_2$ is a path in $\text{Int}\,\Pi$ connecting P_1 and Q_1 and hence $\text{Int}\,\Pi$ is connected.

Let P_3 and Q_3 be any points in $\overline{\text{Ext}}\,\Pi$. Without loss of generality we may assume P_3 and Q_3 are in $\text{Ext}\,\Pi$. By definition P_3 and Q_3 are in both $\text{Ext}\,\Pi_1$ and $\text{Ext}\,\Pi_2$. Therefore there exists a path Ω_3 connecting P_3 and Q_3 lying entirely in $\overline{\text{Ext}}\,\Pi_1$. If $\Omega_3 \cap (\text{Int}\,\Pi_2 \cup e) = \emptyset$ we are done, since by definition then $\Omega_3 \subset \text{Ext}\,\Pi_2$ also. If Ω_3 intersects Π_2 there will be a first and last edge of Ω_3 which contains points of $\text{Int}\,\Pi_2$. Let $\Omega_3 = <T_1, \ldots, T_n>$. Then from above there exists a T_i and T_k such that if $j < i$ or $j > k+1$, $\overline{T_{j-1}T_j} \cap \text{Int}\,\Pi_2 = \emptyset$. Let T' and T'' be the first point on $\overline{T_iT_{i+1}}$ and the last point on $\overline{T_kT_{k+1}}$ contained on Π_2. (Notice if T_iT_{i+1} is the first edge containing points in $\text{Int}\,\Pi_2$ then $T_i \in \overline{\text{Ext}}\,\Pi_2$ by properties (4) and (5). Hence $\overline{T_iT_{i+1}}$ must meet Π in at least one point. We can order these points by Theorem 0-5 counting only the vertices if an edge of Π_3 or a portion of an edge of Π_3 is a subset of $\overline{T_iT_{i+1}}$. Consequently using this ordering we can select T'. T'' can be selected in a similar fashion on $\overline{T_kT_{k+1}}$.) Now let Ω_3' be the subpath of Ω_3 connecting P_3 and T' and Ω_3'' be the subpath of Ω_3 connecting T'' and Q_3. Now let Ω_4 be a path on Π connecting T' and T''. We define $\hat{\Omega} = \Omega_3' \cup \Omega_4 \cup \Omega_3''$. By definition $\hat{\Omega}$ is a path connecting P_3 and Q_3 and lying in $\overline{\text{Ext}}\,\Pi$. Hence $\text{Int}\,\Pi$ and $\overline{\text{Ext}}\,\Pi$ are connected sets and property (1) holds for Case (2).

2.26.15. To show that Π possesses property (2), let P be a point in $\text{Int}\,\Pi$. If $P \in e$ then by property (4) any line meeting e at P will contain a point in $\text{Int}\,\Pi_1$ and $\text{Int}\,\Pi_2$. (See the argument in the preceding paragraph.) If the line contains e then it meets Π at the end points of e and we are done. Therefore we may assume $P \in \text{Int}\,\Pi_1$ without loss of generality. By property (2) any line will meet Π_1 in two points R and S such that RPS. Now if R and $S \in \Pi_1 - e$ we have R and S on Π and the proof is complete. So, assume that S is an element of e. Then by property (4) there exists a point T such that $T \in \text{Int}\,\Pi_2$ and PST. But then by property (2) applied to Π_2 there exists a point Q on Π_2 such that STQ. Applying Lemma 0-2 twice we have RPQ with R and Q on Π. Q.E.D.

2.26.16. We next show that Π has property (3) in Case (2). By induction there exist lines ℓ_1 and ℓ_2 such that Π_1 lies in one of the half planes determined by ℓ_1 and Π_2 lies in one of the half planes determined by ℓ_2. If any of the half planes contain both Π_1 and Π_2 we are done. Thus we will assume that Π_2 meets both half planes determined by ℓ_1 (see Figure 44). Let the half plane determined by ℓ_1 and containing Π_1 be H_1 the other H_2. We are assuming $\Pi_2 \cap H_2 \neq \emptyset$ and $\Pi_2 \cap H_1 \neq \emptyset$. Now, let $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ be the set of all segments connecting the vertices of Π_2 lying in H_2 with those of Π_1 and Π_2 lying in H_1. (Some of these segments are indicated in Figure 44.) Since Π has only a finite number of vertices Σ will have only a finite

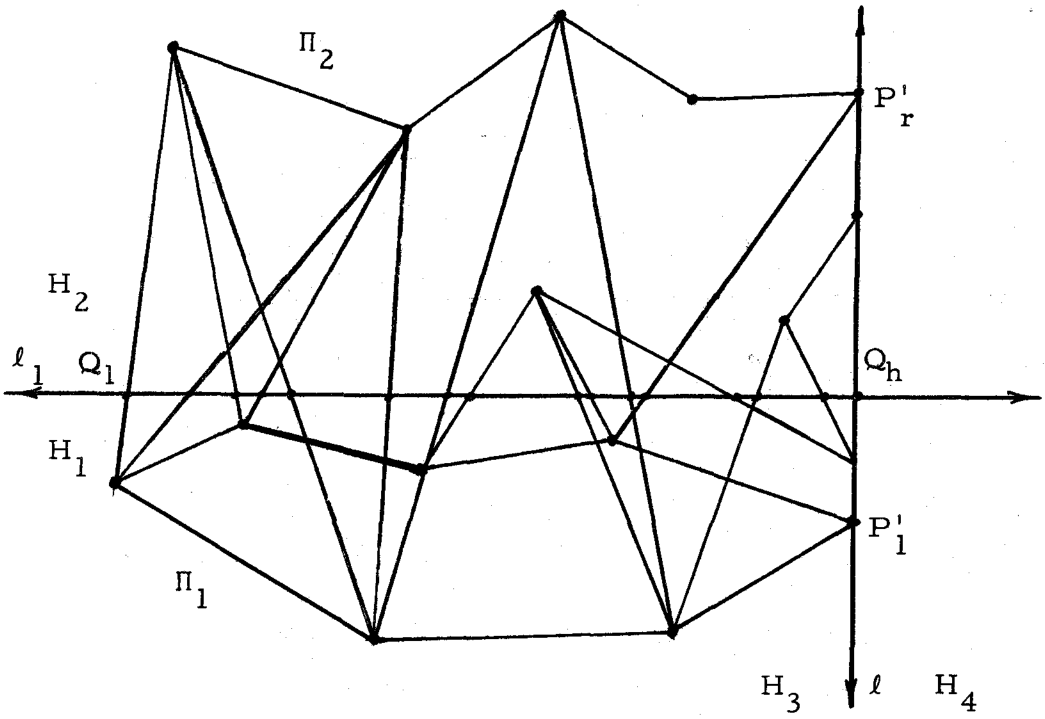


Figure 44.

number of elements. From our definition and the plane separation theorem each of the σ_i's will meet ℓ_1 in some point Q_j. We can order the set $\{Q_j\}_{j=1}^{h}$ so that if $1 \leq i < j < k \leq h$, $Q_iQ_jQ_k$ by Theorem 0-5. Notice that some of the σ_i's may intersect ℓ_1 in the same point so we have $h \leq m$. Let σ be the segment containing Q_h and let ℓ be the line determined by this segment (i.e., the line such that $\sigma \subset \Phi(\ell)$). We claim that all the points of Π lie in one of the closed half planes determined by ℓ. To see this, let $\Gamma = \{P'_1, P'_2, \ldots, P'_r\}$ be the set of vertices of Π lying on ℓ ordered by Theorem 0-5. P'_1 and P'_r lie in opposite sides of ℓ_1 since $\sigma \cap \Phi(\ell_1) \neq \emptyset$. Without loss of generality we may assume $P_1' \in H_1$ and $P_r' \in H_2$. The line ℓ separates H_1 and H_2 into four disjoint sets. If H_3 and H_4 are the half planes determined by ℓ, then these four disjoint sets are $H_1 \cap H_3$, $H_1 \cap H_4$, $H_2 \cap H_3$, and $H_2 \cap H_4$. If Π meets both $H_1 \cap H_3$ and $H_1 \cap H_4$ let P'' be a vertex of Π lying in $H_1 \cap H_3$, P''' a vertex of Π lying in $H_1 \cap H_4$. Then since $P''P'''$ meets ℓ in some point we must have $\overrightarrow{P_r'P_1'} \subset \operatorname{Int} \angle P''P_r'P'''$ and hence Q between the point of intersection of $P''P_r'$ with ℓ_1 and the point of intersection of $P'''P_r'$ with ℓ_1, contrary to the ordering of the Q_i's. Therefore we may assume that all the vertices of Π lying in $\overline{H}_1$ lie in $\overline{H}_3$. A similar argument establishes that all the vertices of Π lying in $\overline{H}_2$ lie either in $\overline{H}_3$ or $\overline{H}_4$ but not in both. If they were elements of H_4 then the segment connecting P_1' with one of these vertices would meet ℓ at a point in H_4 while the segment connecting P_r' with a vertex of Π lying in $H_3 \cap H_1$ would meet ℓ in H_3 and thus Q_h would lie between these two points, contrary to our ordering again. Thus $\Pi \subset \overline{H}_3$.

Next, from P_r' we order the angles of the form $\angle P_1'P_r'P$ where P is any vertex of Π not on ℓ. (See Figure 45.) There exists S, a vertex of Π, such that if P is any other vertex of Π not on ℓ, $\angle P_1'P_r'P \leq \angle P_1'P_r'S$. This is a

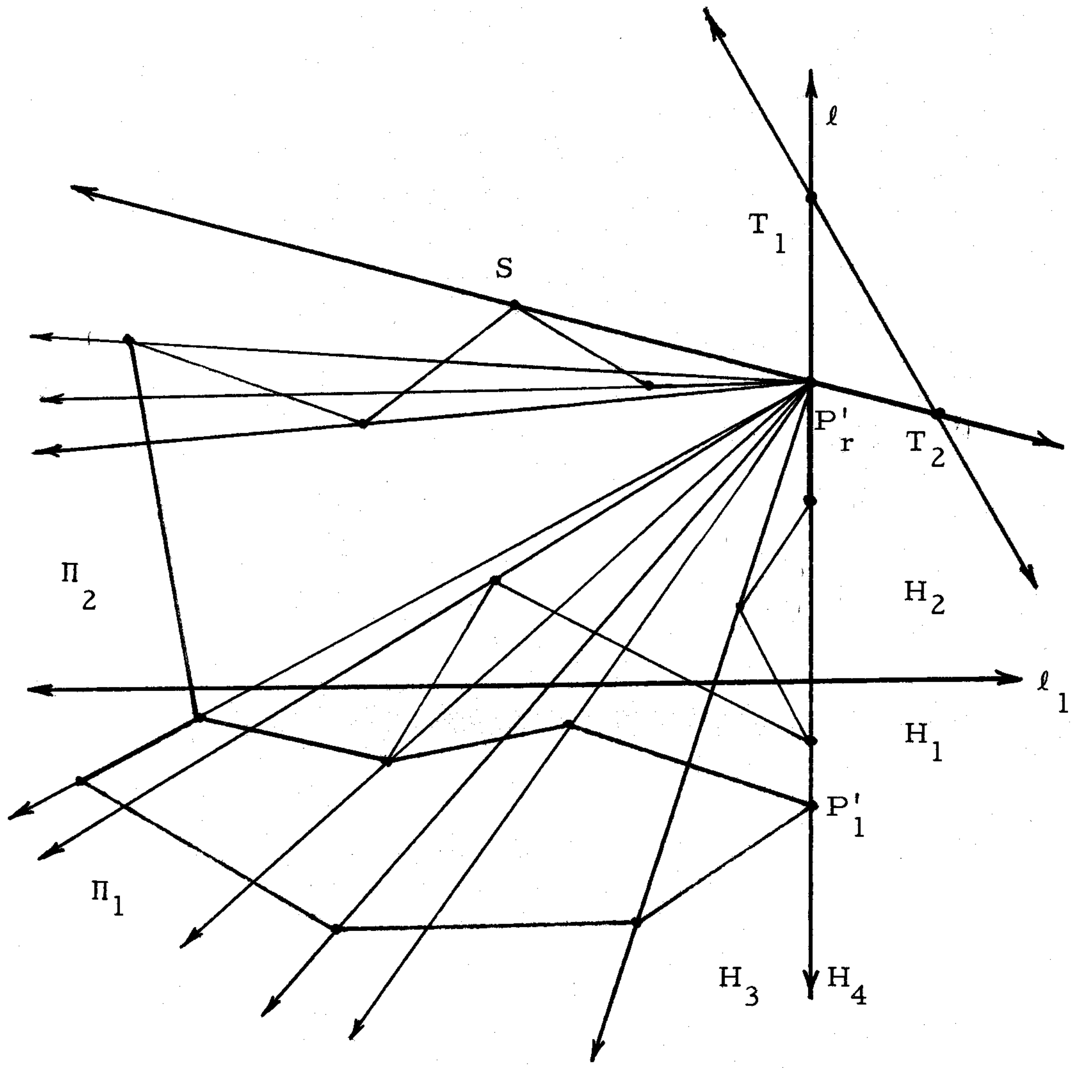


Figure 45.

consequence of our ordering and the fact that Π has only a finite number of vertices.

Consequently all vertices of Π lie in $\bar{H}(\overleftrightarrow{P_1'P_r'}, S) \cap \bar{H}(\overleftrightarrow{P_r'S}, P_1')$. Let $T_1 \in \overleftrightarrow{SP_r'}$ such that $P_1'P_r'T_1$ and $T_2 \in \overleftrightarrow{P_1'P_r'}$ such that $P_1'P_r'T_2$. Consider line $\overleftrightarrow{T_1T_2}$. By definition and the plane separation theorem $\Phi(\overleftrightarrow{T_1T_2}) \subset H(\overleftrightarrow{P_1'P_r'}, \tilde{S}) \cup H(\overleftrightarrow{SP_r'}, \tilde{P}_1')$. Since all the vertices of Π are in $\bar{H}(\overleftrightarrow{P_1'P_r'}, S) \cap \bar{H}(\overleftrightarrow{SP_r'}, P_1')$ it follows that $\Pi \subset \bar{H}(\overleftrightarrow{P_1'P_r'}, S) \cap \bar{H}(\overleftrightarrow{SP_r'}, P_1')$. Consequently $\overleftrightarrow{T_1T_2}$ has no points of Π on it. Q.E.D.

We should note that this proof could have been carried out independent of our induction but it fits into our development here and having $\Pi_1 \subset H_1$ simplifies our discussion somewhat.

<u>2.26.17.</u> In order to establish property (4) for Π let Q be a point of Π. If $Q \notin \bar{e}$ we have $Q \in \Pi_1 - \bar{e}$ or $Q \in \Pi_2 - \bar{e}$. Assume $Q \in \Pi_1 - \bar{e}$. By our induction assumption, we have a point $P \in \text{Int}\,\Pi_1$ and a point $R \in \text{Ext}\,\Pi_1$ such that $PQ \subset \text{Int}\,\Pi_1$ and $QR \subset \text{Ext}\,\Pi_1$. Now since $\text{Int}\,\Pi_1 \subset \text{Ext}\,\Pi_2$, $PQ \subset \text{Int}\,\Pi$ by definition. If $QR \cap \Pi_2$ is empty we can use R as the desired point for property (4) for Π.

If $QR \cap \Pi_2 \neq \emptyset$ then we can order the points of intersection, counting only vertices if some edge of $\Pi_2 \subset QR$. There will be a point S which is "closest" to Q, that is if $T \in \Pi_2 \cap QR$ then QST. Let R' be a point in QS. We claim that $R' \in \text{Ext}\,\Pi_2$. If not then, since $P \in \text{Int}\,\Pi_1 \subset \text{Ext}\,\Pi_2$ we must have $PR' \cap \Pi_2 \neq \emptyset$ and by our ordering it follows that Q must be on $\Pi_2 - \bar{e}$. But $Q \in \Pi_1 - \bar{e}$ and hence a contradiction since $\Pi_1 \cap \Pi_2 = \bar{e}$. Thus $R' \in \text{Ext}\,\Pi_2 \cap \text{Ext}\,\Pi_1 = \text{Ext}\,\Pi$. Therefore R' will serve as the point required in (4).

If Q is an end point of e then Q is a vertex of Π_1 and Π_2. Let the vertices of Π_1 and Π_2 adjacent to Q, but not end points of e be P_{i-1} and P_{i+1}. Consider $\angle P_{i-1}QP_{i+1}$ and let S be a point in interior of the angle.

We may assume that QS does not intersect $\Pi_1 - e$ or $\Pi_2 - e$ since we can order the points of intersection as above. Thus QS will lie entirely in $\text{Int}\,\Pi$, $\text{Ext}\,\Pi$, or e. Let T be a point such that TQS. Again we may assume TQ does not meet $\Pi_1 - e$ or $\Pi_2 - e$. It follows that either T or S lies in $\text{Int}\,\Pi$. Call that point P and the other R. Consequently the points P, Q, and R satisfy the conditions for property (4) for Π.

If Q is not a vertex of Π then it is on an edge of $\Pi_1 - e$ or $\Pi_2 - e$. If ℓ is any line containing Q and not collinear with the edge of Π containing Q then the existence of the points P on R follows in exactly the same way as in the first part of the proof above. Q.E.D.

<u>2.26.18.</u> To show that Π has property (5) in this case, let $T \in \operatorname{Int} \Pi$ and $S \in \operatorname{Ext} \Pi$. That is $T \in \operatorname{Int} \Pi_1 \cup \operatorname{Int} \Pi_2 \cup e$ and $S \in \operatorname{Ext} \Pi_1 \cap \operatorname{Ext} \Pi_2$. If $T \in \operatorname{Int} \Pi_1$ then there exists $R \in TS \cap \Pi_1$ such that TRS by property (5) of our induction assumption. If $R \in \Pi_1 - e$ we are done, so suppose $R \in e$.

By property (4) there exist P and Q such that $P \in \operatorname{Int} \Pi_1$, $Q \in \operatorname{Ext} \Pi_1$ and PRQ. It follows that $Q \in \operatorname{Int} \Pi_2$. (See the analogous argument, Section 2.26.11.) Then QS contains a point of Π_2, call it U such that QUS by property (5). Applying Lemma 0-2 twice we have SUT and we are done if $e \not\subset \Phi(\overleftrightarrow{ST})$. If this occurs then the appropriate end point of e will suffice. If $S \in e \cup \operatorname{Int} \Pi_2$ the proof is essentially the same and will not be pursued further.

<u>2.26.19.</u> The final property to be established for Π is: If Ω is a polygon such that $\Omega \subset \overline{\operatorname{Int}} \Pi$ and if Ω has properties (1) through (6) then $\operatorname{Int} \Omega \subset \operatorname{Int} \Pi$.

If $\Omega \subset \overline{\operatorname{Int}} \Pi_1$ or $\Omega \subset \overline{\operatorname{Int}} \Pi_2$ we are done since this implies $\operatorname{Int} \Omega \subset \operatorname{Int} \Pi_1$ or $\operatorname{Int} \Omega \subset \operatorname{Int} \Pi_2$ and hence $\operatorname{Int} \Omega \subset \operatorname{Int} \Pi$. If $\Omega = \Pi$ we are also done by definition of $\operatorname{Int} \Omega$. Hence we assume that $\operatorname{Int} \Pi_1$ and $\operatorname{Int} \Pi_2$ both contain points of Ω.

Assume there exists a point $P \in \operatorname{Int} \Omega$ such that $P \in \operatorname{Ext} \Pi = \operatorname{Ext} \Pi_1 \cap \operatorname{Ext} \Pi_2$. Let T be any other point in $\operatorname{Ext} \Pi$.

There exists a path α from P to T which does not contain any points in $\operatorname{Int}\Pi_1$ or $\operatorname{Int}\Pi_2$. (See the proof of (2).) Hence α cannot cross Ω since $\Omega \subset \overline{\operatorname{Int}\Pi}$. But this implies that every point of $\operatorname{Ext}\Pi$ is in $\operatorname{Int}\Omega$ and we know that this cannot be true since there exists a line ℓ such that $\Phi(\ell) \subset \operatorname{Ext}\Pi \subset \operatorname{Int}\Omega$ contradicting property (3) for Ω. Hence $\operatorname{Int}\Omega \subset \operatorname{Int}\Pi$. (Notice that property (5) was not used in the proof of properties (1) and (2) for Π and thus we do not have a circular argument.)

2.26.20. Our final theorem is called the Simplical Decomposition Theorem for Polygons:

Theorem 0-17. If Π is a polygon then there exists a 2-complex K such that:

(1) The 0-simplices of K are the vertices of Π

(2) $|K| = \overline{\operatorname{Int} K}$.

Proof. The proof will be by induction on n, the numbers of vertices of Π. The proof is immediate for $n = 3$, that is a triangle. So we assume that the theorem is true for all polygons with fewer than k vertices. Let Π be a polygon with k vertices. In the proof of the Jordan Theorem we established that Π can always be divided into two polygons Π_1 and Π_2 by adding an edge which lies in $\operatorname{Int}\Pi$ between two nonadjacent vertices of Π. By our induction assumption there exists a simplicial decomposition K_1 for $\overline{\operatorname{Int}\Pi_1}$

and a simplical decomposition K_2 for $\overline{\text{Int}}\ \Pi_2$. Let $K = K_1 \cup K_2$. Since the vertices of Π are vertices of either Π_1 or Π_2 we must have property (1) for K. By our definition

$\overline{\text{Int}}\ \Pi = \overline{\text{Int}}\ \Pi_1 \cup \overline{\text{Int}}\ \Pi_2$ hence $\overline{\text{Int}}\ \Pi = |K_1| \cup |K_2| = |K|$.

Q. E. D.

III. SUMMARY

3.1. Comments. In the previous two chapters we filled in many of the gaps in the development of the first two axiom groups of geometry. There are several paths still open for investigation, particularly the implications of the area of the Simplicial Decomposition Theorem and in space geometry. For example, it might be fruitful to investigate the relationships between the Jordan Theorem, simplicial decomposition and whether a particular set is planar or not.

It is impressive that Hilbert recognized that the Jordan Theorem is a consequence of the first two axiom groups. As the reader can see the proof that the Jordan Theorem holds for polygons without using topological or metric concepts is far from trivial. It also should be pointed out that Jordan's original proof for the general case of a simple closed curve was not complete. For a brief discussion of some of the history of the theorem see Newman's The World of Mathematics. For another discussion of the proof of the case for polygons see Arnold's Intuitive Concepts in Elementary Topology. For a complete discussion of the general case see Alexandrov's Combinational Topology, Volume 1.

3.2. Below you will find listed the axioms, terms and theorems of Chapters I and II. We list Chapter I under 3.2.1 and Chapter II under 3. 2. 2. Included at the end of each is the section in which the axiom, definition or theorem is to be found.

3. 2. 1. 1. Axioms of Incidence.

I-1. Given 2 distinct points there exists one and only one line incident upon them (1. 3. 1).

I-2. If P and Q are distinct points of line ℓ then $\overleftrightarrow{PQ}$ and ℓ coincide (1. 6. 1)

I-3. On each line there exist at least 2 points (1. 8. 2).

I-4. If 3 points are noncollinear then there exists one and only one plane incident upon them (1. 9. 3).

I-5. The plane which is incident upon any 3 noncollinear points of a plane α, is α (1. 12).

I-6. There exists at least one plane and on any plane there exist at least 3 noncollinear points (1. 14).

I-7. If two distinct points of a line are points of some plane then the line lies in the plane (1. 16. 2).

I-8. If two distinct planes have a point P in common then there exists a point Q, distinct from P such that Q also lies in both planes (1. 17. 1).

I-9. There exist at least 4 noncoplanar points (1.19.1).

3. 2. 1. 2. Definitions.

P is a point of line ℓ, ℓ contains the point P, etc. (1. 5).

Line ℓ and m coincide (1.5).

Collinear and noncollinear (1.9.2).

Coplanar and noncoplanar (1.11.2).

Line ℓ lies in plane α; ℓ is a line of plane α, etc. (1.15).

Intersection of lines, planes, etc. (1.20.2).

3.2.1.3. Theorems and Lemmas.

Th. I-1. Exactly one plane contains a given line and a point not on the line (1.20.1).

Th. I-2. Two distinct lines intersect in at most one point (1.20.3).

Th. I-3. Exactly one plane contains two distinct intersecting lines (1.20.4).

Th. I-4. If α and β are 2 distinct planes which both contain a point P then there exists a line ℓ containing P and lying in both α and β (1.20.5).

Th. I-5. There exist at least 6 distinct lines and 4 distinct planes (1.20.6).

Lemma I-1. If ℓ is a line then there exists a plane containing ℓ (1.20.7).

Lemma I-2. If A, B, C, and D are four distinct noncoplanar points then any subset consisting of three elements is noncollinear (1.20.8).

3.2.2.1. Axioms of Order.

0-1. If B is between A and C then B is between C and A (2.2).

0-2. If A, B, and C are distinct collinear points then at most one of them is between the other two (2.4).

0-3. If A and C are distinct points of a line ℓ then there exists a point D on ℓ such that ACD (2.7.2).

0-4. Pasch's Axiom. If ΔABC lies in plane α and if ℓ is a line of α which intersects one of the sides of the triangle, then ℓ must have another point in common with ΔABC (2.10.2).

3.2.2.2. Definitions.

Open line segment (2.9.2).

Closed line segment (2.9.3).

A triangle, and its sides and vertices, etc. (2.9.4).

Rays (2.16.3).

Half planes (2.17.2).

Closed half planes (2.18.1).

Convex sets (2.18.2).

Half space (2.18.5).

Closed half space (2.18.6).

Angles, the interior of an angle, etc. (2.20.2).

Interior of a triangle (2.20.5).

The <u>order</u> <u>relation</u> or angles with a common side (2.22.4).

<u>Polygonal</u> <u>paths</u>, <u>polygons</u>, <u>connected</u>, etc. (2.24.2).

<u>Exterior</u> of an angle and a triangle (2.24.5).

<u>Closure</u> of the interior and exterior of a triangle (2.25.6).

<u>Simplices</u>, <u>complexes</u> and <u>simplicial</u> <u>decomposition</u> (2.25.19).

<u>3.2.2.3. Theorems, Lemmas and Corollaries.</u>

(Some of the less important corollaries and lemmas are not included here.)

<u>Lemma 0-1.</u> If a line ℓ intersects a side PR of ΔPQR and $\overleftrightarrow{PQ}$ in a point N such that PQN or NPQ then ℓ meets RQ (2.11.2).

<u>Theorem 0-1.</u> If A and C are two points of line ℓ then there exists a point B on ℓ such that ABC (2.11.3).

<u>Theorem 0-2.</u> If A, B, and C are points of line ℓ then at least one of the relations ABC, ACB, or BAC holds (2.11.4).

<u>Theorem 0-3.</u> Given any three points on a line exactly one of the points lies between the other two (2.11.5).

<u>Theorem 0-4.</u> Given any four distinct, collinear points they can be named A, B, C, and D in such a way that ABC, ABD, ACD, and BCD. Furthermore this can be done in exactly two ways (2.12.2).

<u>Lemma 0-2.</u> If A, B, C, and D are points of line ℓ such that ABD and BCD then ABD and ACD (2.12.4).

<u>Lemma 0-3.</u> If A, B, C, and D are points of line ℓ such that ABC and ACD then ABD and BCD (2.12.5).

<u>Theorem 0-5.</u> Given n points $P_1, P_2, \ldots, P_n$ on line ℓ they can be renamed $A_1, A_2, \ldots, A_n$ in exactly two ways such that $1 \leq i < j < k \leq n$ iff $A_i A_j A_k$ (2.13.1).

<u>Lemma 0-4.</u> If $A_1, A_2, \ldots, A_n$ are any n collinear points such that $A_i A_j A_k$ iff $i < j < k$ or $k < j < i$ then any cycle of length $m \geq 3$ applied to A_i's cannot preserve the betweenness realtions (2.13.5).

<u>Theorem 0-6.</u> There exists an infinite number of points on any line (2.15.1).

<u>Theorem 0-7.</u> Between any two points of a line there are infinitely many points (2.15.2).

<u>Theorem 0-8.</u> Any point P on a line divides the remaining points of the line into two disjoint sets called open rays, in such a way that if R_1 and R_2 are points of the same ray then $\overline{R_1 R_2}$ is a subset of that ray while if R_1 and R_2 are in different rays than $R_1 P R_2$ (2.16.4).

<u>Theorem 0-9.</u> Any line ℓ of a plane α separates the remaining points of α into two disjoint nonempty sets H_1 and H_2 called <u>half planes</u> such that if A and B are two points of the same half plane then $\overline{AB} \cap \Phi(\ell) = \emptyset$ while if they are points of different half planes then $\overline{AB} \cap \Phi(\ell) \neq \emptyset$ (2.17.2).

Lemma 0-7. A line cannot intersect all three sides of any triangle (2.17.7).

Corollary 2. If $Q \in H(\overleftrightarrow{AB}, P)$ then $H(\overleftrightarrow{AB}, Q) = H(\overleftrightarrow{AB}, P)$ (2.17.9).

Corollary 3. The set of points on a line, the set of points in a plane, line segments, rays and half planes are all convex sets (2.18.3).

Corollary 4. Closed half planes are convex sets (2.18.4).

Theorem 0-10. A plane α divides the remaining points of space into two disjoint sets called half spaces such that if A and B are two points of the same half space then $\overline{AB} \cap \Phi(\alpha) = \emptyset$ while if A and B are in different half spaces then $AB \cap \Phi(\alpha) \neq \emptyset$ (2.18.5).

Lemma 0-8. The intersection of a finite number of convex sets is convex (2.19.2).

Lemma. The interior of a non-straight angle is nonempty (2.20.3).

Lemma. The interior of a triangle exists (2.20.6).

Theorem 0-11. The interior of a triangle and a non-straight angle are convex sets (2.20.7).

Lemma 0-9. If A, B, and C are noncollinear points of plane α then $\overrightarrow{AC} \subset H(\overleftrightarrow{AB}, C)$ (2.21.2).

Theorem 0-12. (Cross Bar Theorem) Given any triangle and a point in its interior. A ray from one of its vertices to the point must intersect the side opposite the vertex (2.21.4).

Theorem 0-13. If a line ℓ contains a point of the interior of a triangle then $\Phi(\ell)$ intersects the triangle in two points (2.21.5).

Corollary 1. If ℓ contains a point P in the interior of ΔABC then ℓ intersects ΔABC in two points, say Q and R such that QPR. (2.21.6).

Lemma 0-10. If two open rays $\overrightarrow{AC}$ and $\overrightarrow{AD}$ both lie on the same side of $\overleftrightarrow{AB}$ then one of the open rays lies in the interior of the angle formed by $\overrightarrow{AB}$ and the other ray (2.22.2).

Corollary 1. If $\overrightarrow{AD}$ lies in the interior of $\angle BAC$ then $BC \cap \overrightarrow{AD} \neq \emptyset$ (2.22.3).

Lemma 0-11. If $\angle BAD < \angle BAC$ then $BC \cap \overrightarrow{AD} \neq \emptyset$ (2.22.5).

Corollary 1. $\angle BAD < \angle BAC$ iff $BC \cap \overrightarrow{AD} \neq \emptyset$ (2.22.6).

Lemma 0-12. If open rays $\overrightarrow{AC_1}, \overrightarrow{AC_2}, \ldots, \overrightarrow{AC_n}$ all lie on the same side of $\overleftrightarrow{AB}$ then the points $C_1, C_2, \ldots, C_n$ can be renamed in such a way that $\angle BAC_i < \angle BAC_n$ for all $i < n$ (2.23.2).

Theorem 0-14. If open rays $\overrightarrow{AP_1}, \overrightarrow{AP_2}, \ldots, \overrightarrow{AP_n}$ all lie on the same side of $\overleftrightarrow{AB}$ then the points $P_1, P_2, \ldots, P_n$ can be renamed $C_1, C_2, \ldots, C_n$ in such a way that $\angle BAC_i < \angle BAC_j$ iff $i < j \leq n$ (2.23.3).

Lemma. Every convex set is connected (2.24.3).

Lemma 0-13. If Σ and T are connected sets and

$\Sigma \cup T \neq \emptyset$ then $\Sigma \cup T$ is connected (2.24.6).

Lemma 0-14. The interior of an angle is a connected set which is not convex (2.24.7).

Lemma 0-15. The exterior of a triangle is a connected non-convex set (2.25.2).

Lemma 0-16. If $P \in \text{Int}\,\triangle ABC$ and $Q \in \text{Ext}\,\triangle ABC$ then $PQ \cap \triangle ABC \neq \emptyset$ (2.25.3).

Lemma 0-17. Any $\triangle ABC$ in a plane α divides $\Phi(\alpha)$ into three disjoint subsets $\triangle ABC$, $\text{Int}\,\triangle ABC$, and $\text{Ext}\,\triangle ABC$ such that $\Phi(\alpha) = \triangle ABC \cup \text{Int}\,\triangle ABC \cup \text{Ext}\,\triangle ABC$ (2.25.5).

Lemma. Given $\triangle ABC$, $\overline{\text{Int}}\,\triangle ABC = \overline{H}(\overleftrightarrow{AB}, C) \cap \overline{H}(\overleftrightarrow{AC}, B) \cap \overline{H}(\overleftrightarrow{BC}, A)$ (2.25.7).

Corollary 2. $\overline{\text{Int}}\,\triangle ABC$ is a convex set (2.25.8).

Corollary 3. Given $\triangle ABC$ and points P and Q such that $P \in AB$ and $Q \in BC$ then $\overline{PQ} \subset \overline{\text{Int}}\,\triangle ABC$ and $PQ \subset \text{Int}\,\triangle ABC$ (2.25.9).

Corollary 4. For any $\triangle ABC$, $\overline{\text{Int}}\,\triangle ABC$ is a connected set (2.25.10).

Corollary 5. For any $\triangle ABC$, $\overline{\text{Ext}}\,\triangle ABC$ is a connected set (2.25.12).

Lemma 0-18. Given any $\triangle ABC$ there exists a line ℓ such that $\Phi(\ell) \subset \text{Ext}\,\triangle ABC$ (2.25.13).

Lemma 0-19. If $\triangle ABC \subset \overline{\text{Int}}\,\triangle DEF$ then

Int $\Delta ABC \subset$ Int ΔDEF (2.25.15).

Lemma 0-20. If Q is a point of ΔABC then there exist points $P \in$ Int ΔABC and $Q \in$ Ext ΔABC such that PQR (2.25.16).

Corollary 1. If Q is a point of ΔABC then there exist points $P \in$ Int ΔABC and $Q \in$ Ext ΔABC such that PQR, $PQ \subset$ Int ΔABC and $QR \subset$ Ext ΔABC (2.25.17).

Corollary 2. If Q is any point on AB in ΔABC then there exist points P and R on any line ℓ containing Q and noncollinear with A and B such that $P \in$ Int ΔABC, $R \in$ Ext ΔABC and PQR (2.25.18).

Lemma. If Z is a planar set, $Z = |K|$ for some complex K and e a 1-simplex of K then e is a face of at most two 2-simplices of K (2.25.23).

Theorem 0-16. (Jordan Theorem for Polygons) If Π is a polygon of plane α then Π divides $\Phi(\alpha) - \Pi$ into two disjoint regions denoted by Int Π and Ext Π having properties (1) through (6) listed below (2.26.3).

(1) Int Π and $\overline{\text{Ext}}\ \Pi$ are connected sets.

(2) If a line contains a point $Q \in$ Int Π then $\Phi(\ell) \cap \Pi$ contains at least 2 points P and R such that PQR.

(3) There exists a line ℓ such that $\Phi(\ell) \subset$ Ext Π.

(4) If $Q \in \Pi$ then there exist point $P \in$ Int Π and $R \in$ Ext Π

such that PQR and $PQ \subset \text{Int}\,\Pi$, $QR \subset \text{Ext}\,\Pi$. If Q is not a vertex of Π then Q is on some edge P_iP_{i+1} of Π and any line $\ell \neq \overleftrightarrow{P_iP_{i+1}}$ containing Q contains two other points P and R with the above properties.

(5) If $S \in \text{Ext}\,\Pi$ and $T \in \text{Int}\,\Pi$ then $ST \cap \Pi \neq \emptyset$.

(6) If Ω is another polygon such that $\Omega \subset \overline{\text{Int}}\,\Pi$ and Ω has these six properties then $\text{Int}\,\Omega \subset \text{Int}\,\Pi$.

$(\overline{\text{Int}}\,\Pi = \text{Int}\,\Pi \cup \Pi\,.)$

Lemma 0-21. If Π is any polygon $\langle P_i \rangle_{i=1}^{n}$ with $n \geq 4$ then either $P_1P_3 \cap \Pi = \emptyset$ or there exists an i, $4 \leq i < n$ such that $P_2P_i \cap \Pi = \emptyset$ (2.26.4).

Lemma 0-22. Given a triangle $P_1P_2P_3$ and a polygonal path Ω such that $(\text{Int}\,\Delta P_1P_2P_3 \cup P_1P_3) \cap \Omega \neq \emptyset$ but $\Omega \cap (\overline{P_1P_2} \cup \overline{P_2P_3}) = \emptyset$, except possibly for P_1 or P_3, then there exists a vertex Q of Ω such that $QP_2 \cap \Omega = \emptyset$ (2.26.5).

Simplicial Decomposition Theorem. If Π is a polygon then there exists a 2-complex K such that:

(1) The 0-simplices of K are vertices of Π

(2) $|K| = \overline{\text{Int}}\,K$ (2.26.20).

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