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A geometric condition on differentiable maps is given which is equivalent to the set of critical values being nowhere dense. In particular, the geometric condition is satisfied for radially ductile maps. On the other hand it is proved that the induced map on the de Rham complex will be a monomorphism, provided the set of regular values is dense. These results, together with Bredon's theorem for ductile maps, yields a generalization of the classical de Rham Theorem.

On the Theorems of Sard and de Rham

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# ON THE THEOREMS OF SARD AND de RHAM

## I. SARD'S THEOREM

Let  $C^k$  denote the class of  $k$ -times continuously differentiable maps. By manifold we shall always understand finite dimensional  $C^\infty$  manifold.

Suppose that  $f : X \rightarrow Y$  is a  $C^k$  ( $k \geq 1$ ) map between manifolds. A point  $p \in X$  is called a regular point of  $f$  if the differential at  $p$

$$Df_p : T_p X \rightarrow T_{f(p)} Y$$

is surjective. If  $p \in X$  is not a regular point it is called a singular point of the map  $f$ . The images of the singular points under the map  $f$  are called critical values or singular values. If  $q \in Y$  is not a critical value then it is called a regular value.

The following theorem dealing with the set of critical values is due to Sard [9].

Theorem 1.1 (Sard): Let  $U$  be an open set of  $\mathbb{R}^m$  and  $f : U \rightarrow \mathbb{R}^n$  a  $C^k$  map, where  $k > \max(m-n, 0)$ . Then the set of critical values in  $\mathbb{R}^n$  has outer measure zero.

This theorem admits a corollary first proved for a special

case by Brown [2] previous to the work of Sard.

Corollary 1.2: Let  $f : X \rightarrow Y$  be a map between manifolds of dimension  $m$  and  $n$ , respectively, which is of class  $C^k$ ,  $k > \max(m-n, 0)$ . Then the set of regular values is dense in  $Y$ .

Now we ask whether these theorems are in some sense best possible. If we use the class of differentiability of the map as the main hypothesis, then the answer is affirmative. This follows from the fact that Whitney [12] has constructed, for  $k > 1$ , maps of class  $C^{k-1}$  from the unit cube in  $R^k$  into  $R$  which are nonconstant on a connected set of their singular points.

If we are to extend these theorems, we must therefore find a condition which can replace the restriction on the differentiability class of the map. In this chapter we will seek such a condition and obtain a theorem analogous to Corollary 1.2.

Suppose  $Y$  is a  $n$ -dimensional manifold, by a chart  $(V, \phi)$  we shall mean that  $V$  is an open set in  $Y$  and  $\phi$  is a diffeomorphism of  $V$  into the model space  $R^n$  such that  $\phi(V)$  contains the origin and is convex. Let  $\bar{V} \subset R^n$  denote the image of  $V$  under the map  $\phi$ . If  $\xi$  is a vector field defined on  $V$ , define  $\bar{\xi} = D\phi(\xi)$ . Then  $\bar{\xi}$  is a vector field defined on  $\bar{V}$  contained in the model space  $R^n$ .

Let  $U$  be a neighborhood of a point  $p$  in the  $n$ -dimensional

manifold  $Y$ . Suppose we are given a vector field  $\xi$  defined on  $U - \{p\}$ . The vector field  $\xi$  will be called a radial vector field if there exists a chart  $(V, \phi)$  such that  $V \subset U$ ,  $\phi(p) = \text{origin}$ , and  $D\phi(\xi) = \bar{\xi} = -\frac{\partial}{\partial r}$ , where  $r$  is the radial distance of a point to the origin in  $\mathbb{R}^n$ . With this definition we can now state the condition which will replace the differentiability class.

Condition A: Given a surjective map of manifolds,  $f : X \rightarrow Y$  we demand that for every open set  $U \subset Y$  there exists a point  $p \in U$  and a neighborhood  $V \subset U$  of  $p$ , which admits a radial vector field  $\xi$  which lifts to a vector field  $\eta$  defined on  $f^{-1}V - f^{-1}p$ .

Thus we have the following relationship

$$(1) \quad Df(\eta) = \xi$$

between these vector fields. Condition A may be weakened to Condition B by requiring that  $\eta$  be any cross-section of the tangent bundle over  $f^{-1}V - f^{-1}p$ , possibly discontinuous. We note that (1) still holds, even though  $\eta$  is no longer a vector field.

The following theorem constitutes one of the main results.

Theorem 1.3: If  $f : X \rightarrow Y$  is a surjective map between manifolds then the set of critical values is nowhere dense in  $Y$  if and only if  $f$  satisfies Condition B.

Proof: Suppose  $f$  satisfies Condition B.

Let  $Q \subset Y$  be the set of critical values,  $CQ$  the closure of  $Q$  and  $iCQ$  the interior of the closure of  $Q$ . Lemma: Under the hypothesis of the theorem, for any open set  $U \subset Y$  there is a closed ball  $S$  with radius  $r > 0$  and center  $c$  such that  $S \subset U$  and  $S$  contains only regular values.

If  $iCQ$  is nonempty, it then follows that there exists a closed ball  $S \subset iCQ$  which contains only regular values. But  $c \notin CQ$ , since the open ball centered at  $c$  of radius  $r/2$  contains only regular values. Hence  $S \not\subset iCQ$ . Thus either the lemma is false or  $iCQ$  is empty or both. Since the proof of the lemma is given below we see that  $iCQ$  must be empty, which implies that  $Q$  is nowhere dense in  $Y$ .

Proof of the Lemma: Let the manifold  $Y$  have dimension  $n$ . The dimension of  $X$  is not specifically used in the proof, in fact it could even be infinite. Let the  $i^{\text{th}}$  unit co-ordinate vector field on the model space  $R^n$  of  $Y$  be defined by  $e_i$ .

By induction on the index  $i$  of the co-ordinate axes, we will construct a nested family of closed balls  $\{S_i\}$ , a family of vector fields  $\{\bar{\xi}_i\}$  defined on these balls, and a family of cones  $\{E_i\}$  where the principal part of the vector fields lie.  $\{S_i\}$ ,  $\{\bar{\xi}_i\}$  and  $\{E_i\}$  will all lie in the model space  $R^n$ .

For two given vector fields  $\xi$  and  $\eta$  defined on a subset  $W$  of  $\mathbb{R}^n$ , we define  $(\xi, \eta; W)$  by

$$\cos(\xi, \eta; W) = \sup \left\{ \frac{|\langle \xi(p), \eta(p) \rangle|}{\|\xi(p)\| \|\eta(p)\|} \text{ for } p \in W \right\}$$

where  $0 \leq (\xi, \eta; W) \leq \frac{\pi}{2}$ ,  $\langle, \rangle$  is the inner product in  $\mathbb{R}^n$ , and  $\|\cdot\|$  is the euclidean norm.

Since  $f$  satisfies Condition B we see that there is a point  $p \in U$  and a chart  $(V, \phi)$  about  $p$  which has defined on it a vector field  $\xi_1$  which is radial with respect to  $p$ .

Construction for  $i = 1$ : Choose a point  $c_1 \in \bar{V}$  on the positive co-ordinate axis corresponding to  $i = 1$ . Because  $\xi_1$  is represented in  $\bar{V}$  as  $\bar{\xi} = \frac{-\partial}{\partial r}$ , the radial vector field, there exists a closed ball  $S_1$  about  $c_1$  of radius  $r_1 > 0$  such that

$$\tan(e_1, \bar{\xi}_1; S_1) \leq \left(\frac{1}{2}\right)^2.$$

Let  $\{V_\alpha\}$  denote the cone in  $\mathbb{R}^n$  generated by the vectors  $V_\alpha$ . Now we shall define the cone  $E_1$  by

$$E_1 = \left\{ (1, a_2^1, a_3^1, \dots, a_n^1) : \left( \sum_{i \neq 1} (a_i^1)^2 \right)^{1/2} \leq \left(\frac{1}{2}\right)^2 \right\}$$

Because of the assumed bound on  $\tan(e_1, \bar{\xi}_1; S_1)$ , we see that the

principal part of the vector field  $\bar{\xi}_1$  restricted to  $S_1$  must lie in the cone  $E_1$ .

Suppose that the dimension of  $Y$  is greater than some integer  $m$ , and assume that we have constructed the balls  $\{S_i\}$ , the vector fields  $\{\bar{\xi}_i\}$ , and the cones  $\{E_i\}$  for  $1 \leq i \leq m$ .

Construction for  $i = m + 1$ : By applying Condition B with  $U$  equal to  $\phi^{-1}$  (interior of  $S_m$ ), there exist a point  $p_m \in U$ , a chart  $(W, \psi)$  with  $W \subset U$ , and a radial vector field  $\xi_{m+1}$  with respect to  $p_m$ . That is  $D\psi(\xi_{m+1}) = \frac{-\partial}{\partial r}$  in  $\bar{W}$ .

Let  $\theta$  be the diffeomorphism between subsets of  $R^n$  given by  $\theta = \phi \circ \psi^{-1}$ . Define  $E$  to be a sphere about the origin in  $R^n$  contained in  $\bar{W}$ . Then  $F = \theta(E)$  is a compact orientable hypersurface in  $R^n$  contained in  $\bar{V}$ . Define  $\eta$  and  $\lambda$  to be the outward directed unit normal vector field to the orientable hypersurfaces  $E$  and  $F$ , respectively. For a given point  $p \in F$  we can consider the tangent space to  $F$  at  $p$ ,  $TF_p$ , as a linear subspace of  $\bar{TV}_p$ . Since  $\theta$  is a diffeomorphism, we see that  $D\theta(\eta)$  lies in a complementary subspace to  $TF$  at every point of  $F$ . We may assume without loss of generality that the Jacobian of  $\theta$  is positive. Hence  $D\theta(\eta)$  also gives us an outward directed vector field defined on  $F$ .

Now we can define a homotopy  $H$  between the vector fields

$D\theta(\eta)$  and  $\lambda$  defined on  $F$  as follows:

$$H : F \times I \rightarrow \overline{TV}$$

$$H(q, t) = (1-t)D\theta(\eta(\theta^{-1}q)) + t\lambda(q)$$

for  $q \in F$  and  $t \in [0, 1]$ . Since both vector fields are directed outward, we know that  $H(q, t) \neq 0$  for all  $q \in F$  and  $t \in [0, 1]$ . Thus  $H$  maps into the nonzero portion of the tangent bundle and hence can be composed with the map  $g$  from Appendix I to give a homotopy

$$G = g \circ H : F \times I \rightarrow S^{n-1}$$

Since  $G|_{F \times 1} = g \circ \lambda$ , which by Theorem B of Appendix I is surjective, we know that  $\deg(G|_{F \times 1}) \neq 0$ . Because the degree is invariant under homotopy, we have that  $\deg(G|_{F \times 0}) \neq 0$ , which implies that  $G|_{F \times 0} = g \circ D\theta(\eta)$  is surjective. Now consider the constant vector field  $e_{m+1}$  restricted to  $F$  in  $\overline{V}$ . From the definition of  $g$  we see that  $g \circ e_{m+1}(F) = r$  where  $r$  is a point of  $S^{n-1}$ . Since  $G|_{F \times 0}$  is surjective there exists a point  $c_{m+1} \in F$  such that  $G|_{F \times 0}(c_{m+1}) = r$ . Thus

$$g \circ e_{m+1}(c_{m+1}) = g \circ D\theta(\eta)(c_{m+1})$$

Hence for  $q = \theta^{-1}(c_{m+1})$  the following is true by Lemma A of

Appendix I:

$$D\theta\left(\frac{-\partial}{\partial r}(q)\right) = aD\theta(\eta(q)) = be_{m+1}(c_{m+1})$$

for some constants  $a, b > 0$ . Thus the vector field  $\bar{\xi}_{m+1}$  at the point  $c_{m+1}$  is parallel to  $e_{m+1}$ . Because  $\bar{\xi}_{m+1}$  is a continuous vector field there exists a number  $r_{m+1} > 0$  such that (i) the ball  $S_{m+1}$  centered at  $c_{m+1}$  of radius  $r_{m+1}$  is contained in the ball  $S_m$ , and (ii)  $\bar{\xi}_{m+1}$  approximates the co-ordinate vector field  $e_{m+1}$  in the ball  $S_{m+1}$ , that is

$$\tan(e_{m+1}, \bar{\xi}_{m+1}; S_{m+1}) \leq \left(\frac{1}{2}\right)^{m+2}.$$

Because of this bound, the principal part of the vectors  $\bar{\xi}_{m+1}(q)$  for  $q \in S_{m+1}$  must lie in the cone  $E_{m+1}$  given by

$$E_{m+1} = \{(a_1^{m+1}, \dots, a_m^{m+1}, 1, a_{m+2}^{m+1}, \dots, a_n^{m+1}) : \left(\sum_{i \neq m+1} (a_i^{m+1})^2\right)^{1/2} \leq \left(\frac{1}{2}\right)^{m+2}\}$$

Now continue this construction until all the axes have been used, and define  $S = \bigcap_{1 \leq i \leq n} S_i$ .  $S$  will be equal to  $S_n$ , and hence have radius  $r = r_n$  and center  $c = c_n$ . For each point  $q \in S$  we have a set of vectors  $\{\bar{\xi}_i(q) : 1 \leq i \leq n\}$ . Because each vector field  $\xi_i$  was given by Condition B, we know by (1) that for each  $x \in f^{-1}p$  where  $p \in \phi^{-1}(S)$  and each index  $i$

$$(2) \quad \bar{\xi}_i(p) \in \text{ImDf}_x.$$

Since the chart map  $\phi$  is a diffeomorphism (2) becomes

$$\bar{\xi}_i(q) \in \text{ImD}(\phi \circ f)_x$$

for  $q \in S$ ,  $x \in (\phi \circ f)^{-1}q$  and for each index  $i$ . Hence if we can show that  $\{\bar{\xi}_i(q) : 1 \leq i \leq n\}$  spans the tangent space  $T_q \bar{V}$  for every  $q \in S$ , then  $S$  is the desired closed ball of regular values. But this is equivalent to the question, whether picking a vector from each cone  $E_i$  constitutes a basis for the model space  $\mathbb{R}^n$ . Let

$$\{(a_1^i, \dots, a_{i-1}^i, 1, a_{i+1}^i, \dots, a_n^i) : 1 \leq i \leq n = \dim \text{ of } Y\}$$

be such a selection of vectors, and consider these vectors as the rows of a matrix  $P$ . Thus  $P$  is a square matrix of dimension  $n$ . Now we observe that  $P$  is diagonally dominant,<sup>1</sup> since

$$|a_i^i| \geq \left( \sum_{\substack{j=1 \\ j \neq i}}^n (a_i^j)^2 \right)^{1/2} \leq \left( \frac{1}{2} \right)^{j+1}$$

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<sup>1</sup>A square matrix  $P$  is diagonally dominant if

$$|P_{ii}| > \sum_{\substack{j \\ j \neq i}} |P_{ij}| \quad \text{for all } i.$$

so that

$$\sum_{\substack{j=1 \\ j \neq i}}^n |a_i^j| < \frac{1}{2}$$

But it is a well known fact that diagonally dominant matrices are non-singular [9, 14]. Hence  $\{\bar{\xi}_i(q) : 1 \leq i \leq n\}$  are linearly independent for all  $q \in S$ . Thus the lemma is proved.

Now suppose that  $Q$  is nowhere dense. Then the complement of the closure of  $Q$ ,  $\sim(CQ)$ , is an open, dense set containing only regular values. Thus for any open set  $U \subset Y$ ,  $W = U \cap \sim(CQ)$  is an open, nonempty set containing only regular values. That is for any  $x \in f^{-1}q$  where  $q \in W$ ,  $Df_x$  is surjective. Thus any radial vector field about a point in  $W$  lifts to a (possibly discontinuous) section of the tangent bundle over  $f^{-1}W$ . |

For proper<sup>2</sup> surjective maps satisfying Condition A we have the following analog of Theorem 1.3.

Theorem 1.4: If  $f : X \rightarrow Y$  is a proper surjection between manifolds then the set of critical values is nowhere dense if and only if  $f$  satisfies Condition A.

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<sup>2</sup>A map is proper if for any compact set  $C$ ,  $f^{-1}C$  is compact.

Proof: If  $f$  satisfies Condition A then Theorem 1.3 states that the set of critical values  $Q$  is nowhere dense, then the complement of the closure of  $Q$ ,  $\sim(CQ)$ , is an open, dense set containing only regular values. Thus for any open set  $U \subset Y$ ,  $W = U \cap \sim(CQ)$  is an open, nonempty set containing only regular values.

Let  $(V, \phi)$  be a chart containing  $W$ . Choose a point  $p \in W$  and a ball  $S \subset \phi(W)$  of radius  $r > 0$  centered about  $\phi(p)$ . Consider the closed ball  $B$  of radius  $\frac{r}{2}$  centered at  $\phi(p)$ . Since  $f$  is a proper map and  $\phi$  is a diffeomorphism  $(\phi \circ f)^{-1}B$  is compact. For every point  $x \in (\phi \circ f)^{-1}B$  we know that  $Df_x$  is of maximal rank. Thus, by a result of Ehresmann [3],  $f$  induces a differentiable fibre bundle structure on  $(\phi \circ f)^{-1}B$ . Using this structure there exists a small open ball  $R$  about  $\phi(p)$  such that  $(\phi \circ f)^{-1}R$  is diffeomorphic to a product. Thus the radial vector field  $\frac{-\partial}{\partial r}$  defined on  $R$  lifts to a vector field defined on  $(\phi \circ f)^{-1}R$ . Hence the map  $f$  satisfies Condition A. |

In the remainder of this chapter we will restrict ourselves to a class of maps of interest in Chapter III. Suppose that  $f : X \rightarrow Y$  is a surjection between manifolds, then  $f$  will be called differentiably ductile if for each  $y \in Y$  and each neighborhood  $V$  of  $y$  there is a neighborhood  $U \subset V$  of  $y$  and a contraction of  $U$  to  $y$

of class  $C^\infty$  which lifts to a homotopy of  $f^{-1}U$  of class  $C^\infty$ .

Now if a surjective map  $f$  has a radial vector field satisfying Condition B for every point in the range, then we say that  $f$  satisfies Condition C. Clearly, Condition C implies Condition B. A contraction  $c$  of a neighborhood  $U$  of  $y$  to the point  $y$  is called a radial contraction if there exists a chart  $(V, \phi)$  such that (i)  $\phi(U)$  is a sphere about the origin in the model space, (ii)  $\phi(y) = 0$ , and (iii) the contraction is given by  $c(t, p) = (1-t)p$  for  $p \in \phi(U)$ . Now we can state the precise condition of interest to us. A surjective map  $f: X \rightarrow Y$  is said to be radially ductile if it is differentiably ductile with all the contractions being radial.

Theorem 1.5: If  $f: X \rightarrow Y$  is a radially ductile map between manifolds then  $f$  satisfies Condition C.

Proof: Let  $y \in Y$ . Since  $f$  is radially ductile there exists a neighborhood  $U$  of  $y$ , a contraction  $c$  of  $U$  to  $y$  which lifts and a chart  $(V, \phi)$  such that  $\phi(U)$  is a sphere centered at the origin,  $\phi(y) = 0$ , and  $c(t, p) = (1-t)p$  for  $p \in \phi(U)$ . Thus we see that the vector field  $\frac{-\partial}{\partial r}$  defined on  $\phi(U)$  is nothing but the vector field associated with the flow given by  $c$ . If we choose any point  $q \in \phi(U) - \{0\}$  it is on a unique flow curve of the contraction. Let  $x \in (\phi \circ f)^{-1}q$ . From the ductility condition we have the following commutative diagram:

$$\begin{array}{ccc}
 I \times X & \xrightarrow{\text{id} \times \phi \circ f} & I \times Q \\
 \downarrow h(t) & & \downarrow c(t, q) \\
 f^{-1}U & \xrightarrow{\phi \circ f} & \phi(U)
 \end{array}$$

Now define a tangent vector in  $T_x X$  by  $v_x = \lim_{t \rightarrow 1} v_{h(t)}$  where  $v_{h(t)}$  are the tangent vectors to the curve  $h(t)$  in  $f^{-1}U$ . Because the curve  $h(t)$  covers the curve  $c(t, q)$ , we see that

$$(3) \quad Df(v_{h(t)}) = \frac{-\partial}{\partial r} (\phi \circ f \circ h(t))$$

Since  $\frac{-\partial}{\partial r}$  is the tangent vector field to the curve  $c(t, q)$ . Because

$Df$  is a continuous map we have:

$$Df(v_x) = Df\left(\lim_{t \rightarrow 1} v_{h(t)}\right) = \lim_{t \rightarrow 1} Df(v_{h(t)})$$

But by Equation (3) we obtain:

$$\lim_{t \rightarrow 1} Df(v_{h(t)}) = \lim_{t \rightarrow 1} \frac{-\partial}{\partial r} (\phi \circ f \circ h(t)) = \frac{-\partial}{\partial r} (q)$$

Hence  $Df(v_x) = \frac{-\partial}{\partial r} (q)$ . Thus, we see that the radial vector field

lifts to a section in the tangent bundle of  $f^{-1}U - f^{-1}y$ .  $\square$

Now by combining this result with Theorem 1.3 and the observation that Condition C implies Condition B we have the following

theorem.

Theorem 1.6: If  $f : X \rightarrow Y$  is a radially ductile map between manifolds then the set of critical values is nowhere dense in  $Y$ .

## II. ISOMORPHISMS OF THE de RHAM COHOMOLOGY MODULES

In the following paragraph we will construct the de Rham complexes of a manifold. All the assertions about differential forms are justified in Lang's book [6]. If  $Y$  is a manifold let  $F^p(Y)$  denote the real vector space of differential  $p$ -forms over  $Y$  (cross-sections of the  $p^{\text{th}}$  alternating power of the contratangent bundle over  $Y$ ). The exterior differential  $d : F^p(Y) \rightarrow F^{p+1}(Y)$  makes  $\{F^p Y : p \geq 0\}$  into a non-negative cochain complex called the de Rham complex of the manifold  $Y$ . If  $f : X \rightarrow Y$  is a differentiable map between manifolds,  $f$  induces a linear map  $f^\# : F(Y) \rightarrow F(X)$  by  $\langle f^\# \omega; v_1, \dots, v_n \rangle(p) = \langle \omega; Df v_1, \dots, Df v_n \rangle f(p)$  where  $v_i \in T_p X$  for  $1 \leq i \leq n$  and  $\omega \in F^n(Y)$ . Thus  $F$  is a contravariant functor from the category of manifolds and smooth maps into the category of cochain complexes. The wedge product of differential forms gives us a multiplicative structure on the de Rham complex which commutes with the induced maps.

The homology functor applied to this complex gives the graded de Rham cohomology module  $H_{d.R}^*(Y)$  for the manifold  $Y$ . The wedge product induces a product on the cohomology module making  $H_{d.R}^*(Y)$  into a graded algebra.

Lemma 2.1: If two differential  $p$ -forms  $\omega$  and  $\omega'$  differ at point  $y \in Y$ , then they differ in an open neighborhood of  $y$ .

Proof: Since  $\omega$  and  $\omega'$  are cross-sections of a bundle which is a Hausdorff space the result follows. |

Let  $f : X \rightarrow Y$  be a differentiable map between manifolds. We shall call such a map a Sard map if the set of regular values of  $f$  is dense in  $Y$ . From the first chapter we have three criteria for a Sard map:

- (i)  $X$  and  $Y$  of finite dimensions  $m$  and  $n$  respectively with  $f \in C^k$  for  $k > \max(m-n, 0)$
- (ii)  $f$  satisfies Condition B
- (iii)  $f$  is radially ductile.

The following theorem is basic to the remainder of this chapter.

Theorem 2. 2: If  $f : X \rightarrow Y$  is a surjective Sard map between manifolds, then  $f^\#$  the induced map between the de Rham complexes is a monomorphism.

Proof: Suppose there exists a  $k$ -form  $\omega$  on  $Y$  such that  $f^\# \omega = 0$ . Also suppose there is a point  $p \in Y$  such that  $\omega(p) \neq 0$ . Then Lemma 2. 1 implies that  $\omega \neq 0$  in a neighborhood  $U$  of  $p$ . From the definition of the induced map we know,

$$(4) \quad \langle f^\# \omega; v_1, \dots, v_k \rangle(x) = \langle \omega; Df v_1, \dots, Df v_k \rangle(f(x)) = 0$$

for any point  $x \in X$  and any set of vectors  $\{v_i \in T_x X : 1 \leq i \leq k\}$ .

Since  $f$  is a surjective Sard map, there is a point  $q \in U$  which is a regular value. Because  $q \in U$ , there exist vectors

$\{u_i \in T_q Y : 1 \leq i \leq k\}$  such that  $\langle \omega; u_1, \dots, u_k \rangle(q) \neq 0$ . But  $q$  is a regular value, hence for all  $x \in f^{-1}q$  there exists a set of vectors

$\{w_i \in T_x X : 1 \leq i \leq k\}$  such that  $Df_x(w_i) = u_i$ . By (4) we obtain:

$$\langle f^\# \omega; w_1, \dots, w_k \rangle(x) = \langle \omega; u_1, \dots, u_k \rangle(q) = 0$$

Hence a contradiction to  $\omega \neq 0$  on  $U$ . Thus  $\omega(p) = 0$  and  $f^\#$  is a monomorphism.  $\blacksquare$

Since  $f^\#$  commutes with the exterior differential, we see that  $f^\# F(Y)$  is a subcomplex of the de Rham complex of  $X$ . This subcomplex also inherits the wedge product structure, because it too commutes with the induced map  $f^\#$ .

Theorem 2.3: If  $f : X \rightarrow Y$  is a surjective Sard map, then  $f$  induces an isomorphism of graded algebras

$$H_{d.R}^*(Y) \simeq H^*(f^\# F(Y))$$

Proof: By Theorem 2.2  $f^\#$  is a monomorphism, hence  $F(Y)$  and  $f^\# F(Y)$  are isomorphic complexes. Because the wedge product commutes with  $f^\#$  we see that the isomorphism preserves

products. By applying the cohomology functor  $H^*$  the result follows. |

Suppose  $f: X \rightarrow Y$  is a surjective Sard map, then by Theorem 2.2 we know that the induced map  $f^\#$  is a monomorphism. Hence the following sequence is exact.

$$(5) \quad 0 \rightarrow F(Y) \xrightarrow{f^\#} F(X) \xrightarrow{\pi} \frac{F(X)}{f^\# F(Y)} \rightarrow 0$$

By the standard argument, this short exact sequence of cochain complexes induces a long exact sequence of cohomology modules [10, p. 182].

$$(6) \quad \rightarrow H_{d.R}^p(Y) \xrightarrow{f^*} H_{d.R}^p(X) \xrightarrow{\pi^*} H_{d.R}^p\left(\frac{F(X)}{f^\# F(Y)}\right) \xrightarrow{\delta} H_{d.R}^{p+1}(Y) \rightarrow$$

There is another long exact sequence associated with a map, which we will now derive. If  $f: X \rightarrow Y$  is a differentiable map between manifolds, define the mapping cone cochain complex of differential forms as follows:  $F^q(f) = F^q(X) \oplus F^{q+1}(Y)$  with coboundary operator given by  $d_f = (-d_X + f^\#, d_Y)$ . If we let the cochain complex  $\overline{F}(X)$  denote  $F(X)$  with the negative coboundary operator, then the following sequence is exact.

$$(7) \quad 0 \rightarrow \overline{F}(X) \xrightarrow{i} F(f) \xrightarrow{j} F(Y) \rightarrow 0$$

Where  $i$  and  $j$  are the natural injection and projection maps of the direct sum.

Theorem 2. 4: If  $f : X \rightarrow Y$  is a differentiable map between manifolds and if  $F(f)$  is the mapping cone complex of forms associated with  $f$ , then the following sequence is exact.

$$(8) \quad \rightarrow H_{d.R}^p(Y) \xrightarrow{f^*} H_{d.R}^p(X) \xrightarrow{i^*} H^p(F(f)) \xrightarrow{j^*} H_{d.R}^{p+1}(Y) \rightarrow$$

Proof: Apply the homology functor to the short exact sequence (7) and note that the connecting homomorphism is just  $f^*$  and that  $H^*(\overline{F(X)})$  is canonically isomorphic with  $H_{d.R}^*(X)$  (Spanier has these calculations in the dual case of homology [10, p. 191]). |

Now we will define a map  $\rho^* : H^p(F(f)) \rightarrow H^p\left(\frac{F(X)}{f^{\#}F(Y)}\right)$  by the following construction. There is a map  $\rho : F(f) \rightarrow \frac{F(X)}{f^{\#}F(Y)}$  given by  $\rho(\alpha \oplus \beta) = \pi(\alpha)$  for  $\alpha \in F^p(X)$ ,  $\beta \in F^{p+1}(Y)$  and  $\pi$  the map from the sequence (5). The map  $\rho$  is not a cochain map since  $\rho \circ d_f = -d \circ \rho$ . However because  $\rho$  preserves the kernels and images of  $d_f$ , it descends to the homology level giving us the desired map  $\rho^*$ .

Lemma 2. 5: The following diagram composed of sequences (6) and (8) is commutative for  $f$  a surjective Sard map.

$$\begin{array}{ccccccc}
\rightarrow & H_{d.R}^p(Y) & \xrightarrow{f^*} & H_{d.R}^p(X) & \xrightarrow{\pi^*} & H^p\left(\frac{F(X)}{f^\# F(Y)}\right) & \xrightarrow{\delta} & H_{d.R}^{p+1}(Y) \rightarrow \\
(9) & \uparrow \text{id} & & \uparrow \text{id} & & \uparrow \rho^* & & \uparrow \text{id} \\
\rightarrow & H_{d.R}^p(Y) & \xrightarrow{f^*} & H_{d.R}^p(X) & \xrightarrow{i^*} & H^p(F(f)) & \xrightarrow{j^*} & H_{d.R}^{p+1}(Y) \rightarrow
\end{array}$$

Proof: Let  $a \in F^p(X)$  then  $\rho \circ i(a) = \pi(a)$  by the definition of  $\rho$ . Passing to cohomology we have  $\rho^* \circ i^* = \pi^*$ , hence the second square commutes. Let  $\gamma \in H^p(F(f))$  and let  $\gamma$  be represented by the cocycle  $a \oplus \beta$  where  $a \in F^p(X)$  and  $\beta \in F^{p+1}(Y)$ . Now calculate

$$\delta \circ \rho^*(\gamma) = \delta[\pi(a)] = [f^{\#-1} da] = [c]$$

where  $[b]$  is the cohomology class generated by the cocycle  $b$  and  $c$  is such that  $f^\# c = da$ . The other half of the third square yields  $j^*(\gamma) = [\beta]$ . Since  $a \oplus \beta$  is a cocycle we know that  $d_f(a \oplus \beta) = 0$  or  $d_f(a \oplus \beta) = (-da + f^\# \beta, d\beta) = 0$ . Hence  $da = f^\# \beta$ . But by Theorem 2.2  $f^\#$  is injective, which implies that  $c = \beta$ . Thus  $\delta \circ \rho^* = j^*$ . Hence the third square commutes. |

Theorem 2.6: If  $f : X \rightarrow Y$  is a surjective Sard map between manifolds, then  $\rho^*$  is an isomorphism of graded modules.

$$\rho^* : H^*(F(f)) \rightarrow H^*\left(\frac{F(X)}{f^\# F(Y)}\right)$$

Proof: Apply the 5-Lemma to Lemma 2.5. |

### III. THE GENERALIZED de RHAM THEOREM

Before we can state or prove the generalized de Rham theorem, some preliminary results are needed. We have already seen that  $F$  is a cochain functor. There are two other cochain functors which we shall consider, i. e.,  ${}_s C(*; \mathbb{R})$  the singular cochains with real coefficients and  ${}_d C(*; \mathbb{R})$  the singular cochains based on smooth singular simplexes with real coefficients.

For a given manifold  $X$ , the integral of differential forms over smooth chains induces a homomorphism

$$I_X : F(X) \rightarrow {}_d C(X; \mathbb{R})$$

by

$$[I_X(\omega)](\sigma) = \int_{\sigma} \omega$$

where  $\omega \in F^p(X)$  and  $\sigma : \Delta^p \rightarrow X$  is a smooth singular  $p$ -simplex.

Suppose  $f : X \rightarrow Y$  is a differentiable map between manifolds, then the following calculation shows that  $I$  is, in fact, a natural transformation between the functors  $F$  and  ${}_d C(*; \mathbb{R})$ . Let  $\omega \in F^p(Y)$  and  $\sigma : \Delta^p \rightarrow X$  be a smooth singular  $p$ -simplex, then we have the following:

$$\begin{aligned}
f_d^\# [I_Y(\omega)](\sigma) &= [I_Y(\omega)](f \circ \sigma) = \int_{f \circ \sigma} \omega \\
&= \int_{\Delta^p} (f \circ \sigma)^\# \omega = \int_{\Delta^p} \sigma^\# \circ f^\# \omega = \int_{\sigma} f^\# \omega = [I_X(f^\# \omega)](\sigma)
\end{aligned}$$

For a given map  $f : X \rightarrow Y$ , the naturality of  $I$  implies that  $I_X$  restricts to a map between the images of  $f^\#$  and  $f_d^\#$ . Denote this restriction of  $I_X$  by  $\hat{I}_X : f^\# F(Y) \rightarrow f_d^\# C(Y; \mathbb{R})$ . Hence we obtain the following commutative diagram for a differentiable map  $f : X \rightarrow Y$ .

$$(10) \quad \begin{array}{ccc}
F(Y) & \xrightarrow{f^\#} & f^\# F(Y) \\
\downarrow I_Y & & \downarrow \hat{I}_Y \\
{}_d C(Y; \mathbb{R}) & \xrightarrow{f_d^\#} & f_d^\# C(Y; \mathbb{R})
\end{array}$$

In the case of the two functors  ${}_s C(*; \mathbb{R})$  and  ${}_d C(*; \mathbb{R})$  we also obtain a natural transformation  $R$  induced by the inclusion map of the smooth singular chains into the singular chains. Hence in the same way we obtain a commutative diagram similar to (10).

$$(11) \quad \begin{array}{ccc}
{}_s C(Y; \mathbb{R}) & \xrightarrow{f_s^\#} & f_s^\# C(Y; \mathbb{R}) \\
\downarrow R_Y & & \downarrow \hat{R}_Y \\
{}_d C(Y; \mathbb{R}) & \xrightarrow{f_d^\#} & f_d^\# C(Y; \mathbb{R})
\end{array}$$

Applying the homology functor to these diagrams we obtain the

basic commutative diagram of this chapter.

$$(12) \quad \begin{array}{ccc} H_s^*(Y) & \xrightarrow{\tilde{f}_s} & H^*(f_s^\# C(Y;R)) \\ \downarrow R^* & & \downarrow \hat{R}^* \\ H_d^*(Y) & \xrightarrow{\tilde{f}_d} & H^*(f_d^\# C(Y;R)) \\ \uparrow I^* & & \uparrow \hat{I}^* \\ H_{d,R}^*(Y) & \xrightarrow{\tilde{f}} & H^*(f^\# F(Y)) \end{array}$$

Because the cup product of singular cohomology restricts, via  $R$ , to give a product on the smooth singular cohomology, we see that  $R^*$  is a graded algebra homomorphism. The following theorem is a well known fact, first proved by Eilenberg [4] in 1947.

Theorem 3. 1: If  $Y$  is a paracompact manifold, then  $R^*$  is a graded algebra isomorphism.

We already know by Theorem 2. 3 that for a surjective Sard map,  $\tilde{f}$  is a graded algebra isomorphism. Bredon [1] has given the following sufficient condition for  $\tilde{f}_s$  to be a graded algebra isomorphism.

Theorem 3. 2: If  $f : X \rightarrow Y$  is a ductile map onto a paracompact space  $Y$ , then  $\tilde{f}_s$  is a graded algebra isomorphism.

Bredon proves this result by sheaf theoretic techniques. Because the smooth singular chains admit the usual subdivision operation, Bredon's proof is also valid in this case, provided the ductility assumption is strengthened to differentiably ductile.

Theorem 3.3: If  $f : X \rightarrow Y$  is a differentiably ductile map of manifolds where  $Y$  is paracompact, then  $\tilde{f}_d$  is a graded algebra isomorphism.

The classical de Rham theorem was first proved by de Rham in a restricted case [8]. The theorem is most easily proved via sheaf theory as given in Godement's book [5].

Theorem 3.4 (de Rham): If  $Y$  is a paracompact manifold, then  $R^{*-1} \circ I^* : H_{d,R}^*(Y) \rightarrow H_s^*(Y)$  is a graded algebra isomorphism.

From Theorem 3.1,  $R^*$  is an isomorphism hence  $R^{*-1}$  is well defined.

Now by combining all the theorems of this chapter with some from Chapters I and II we obtain the main theorem of Chapter III.

Theorem 3.5 (Generalized de Rham Theorem): If  $f : X \rightarrow Y$  is a radially ductile map onto a paracompact manifold  $Y$ , then  $\hat{R}^{*-1} \circ \hat{I}^* : H^*(f^{\#}F(Y)) \rightarrow H^*(f^{\#}_s C(Y;R))$  is a graded algebra isomorphism.

Proof: Because  $f$  is radially ductile,  $f$  is (i) differentiably ductile, (ii) ductile, and (iii) a surjective Sard map (by Theorem 1.6). By Theorems 3.1, 3.2, 3.3 and diagram (12) we see that  $\hat{R}^*$  is a graded algebra isomorphism and hence  $\hat{R}^{*-1}$  is defined. From the diagram (12) and Theorems 2.3 and 3.4 the result follows. |

In the following theorem we will identify the cohomology of the cokernal complex with that of the mapping cone cohomology.

Theorem 3.6: If  $f : X \rightarrow Y$  is a surjective Sard map between paracompact manifolds then

$$R_f^{*-1} \circ I_f^* \circ \rho^{*-1} : H^* \left( \frac{F(X)}{f^\# F(Y)} \right) \rightarrow H^* ({}_s C(f))$$

is a graded module isomorphism, where  ${}_s C(f)$  is the mapping cone complex for singular cohomology.

Proof: By Theorem 2.6 we see that  $\rho^*$  is an isomorphism of graded modules.

$$\rho^* : H^* (F(f)) \rightarrow H^* \left( \frac{F(X)}{f^\# F(Y)} \right)$$

Hence  $\rho^{*-1}$  is well defined.

Now define a homomorphism  $I_f : F(f) \rightarrow {}_d C(f)$  by

$I_f = I_X \oplus I_Y$ . The following calculation shows that  $I_f$  is a cochain

map. Assume that  $\omega \in F^q(X)$  and  $\theta \in F^{q+1}(Y)$  then using that fact that  $I$  is a natural transformation between cochain functors, we have:

$$\begin{aligned}
 I_f(d_f(\omega, \theta)) &= I_f(-d\omega + f^\# \theta, d\theta) \\
 &= (I_X(-d\omega + f^\# \theta), I_Y(d\theta)) \\
 &= (-dI_X\omega + f^\# I_Y\theta, dI_Y\theta) \\
 &= d_f(I_X\omega, I_Y\theta)
 \end{aligned}$$

In an analogous manner define

$$R_f: {}_s C(f) \rightarrow {}_d C(f) \quad \text{as} \quad R_f = R_X \oplus R_Y.$$

Then the same type of calculation shows that  $R_f$  is a cochain map.

We note that for both the singular and smooth singular cohomology there are short exact sequences analogous to (7) which induce long exact sequences analogous to (8). From the definition of  $I_f$  and  $R_f$  and from the naturality of the transformations  $I$  and  $R$ , we obtain the following commutative diagram of long exact sequences of type (8).

$$\begin{array}{ccccccccc}
\rightarrow & H_s^p(Y) & \rightarrow & H_s^p(X) & \rightarrow & H_s^p(C(f)) & \rightarrow & H_s^{p+1}(Y) & \rightarrow & H_s^{p+1}(X) & \rightarrow \\
& \downarrow R_Y^* & & \downarrow R_X^* & & \downarrow R_f^* & & \downarrow R_Y^* & & \downarrow R_X^* & \\
\rightarrow & H_d^p(Y) & \rightarrow & H_d^p(X) & \rightarrow & H_d^p(C(f)) & \rightarrow & H_d^{p+1}(Y) & \rightarrow & H_d^{p+1}(X) & \rightarrow \\
& \uparrow I_Y^* & & \uparrow I_X^* & & \uparrow I_f^* & & \uparrow I_Y^* & & \uparrow I_X^* & \\
\rightarrow & H_{d.R}^p(Y) & \rightarrow & H_{d.R}^p(X) & \rightarrow & H^p(F(f)) & \rightarrow & H_{d.R}^{p+1}(Y) & \rightarrow & H_{d.R}^{p+1}(X) & \rightarrow
\end{array}$$

From Theorem 3.4 (de Rham) we see that  $R^{*-1} \circ I^*$  is an isomorphism, thus by the 5-lemma the result follow.  $\square$

## BIBLIOGRAPHY

1. Bredon, Glen E. On the continuous image of a singular chain complex. *Pacific Journal of Mathematics* 15:1115-1118. 1965.
2. Brown, Arthur B. Functional dependence. *Transactions of the American Mathematical Society* 38:379-394. 1935.
3. Ehresmann, Charles. Sur les espaces fibrés différentiables. *Comptes Rendus des Seances de L'Academie des Sciences* 224:1611-1612. 1947.
4. Eilenberg, Samuel. Singular homology in differentiable manifolds. *Annals of Mathematics* 48:670-682. 1947.
5. Godement, Roger. *Topologie algébrique et théorie des faisceaux*. Paris, Hermann, 1964. 283 p.
6. Lang, Serge. *Introduction to differentiable manifolds*. New York, Interscience, 1967. 126 p.
7. Marcus, Marvin and Henry Minc. *A survey of matrix theory and matrix inequalities*. Boston, Allyn and Bacon, 1964. 180 p.
8. Rham, Georges de. Sur l'analysis situs des variétés  $n$  dimensionales. *Journal de Mathématiques Pures et Appliquées* 10:115-200. 1931.
9. Sard, Arthur. The measure of the critical values of differentiable maps. *Bulletin of the American Mathematical Society* 48:883-890. 1942.
10. Spanier, Edwin H. *Algebraic topology*. New York, McGraw-Hill, 1966. 528 p.
11. Taussky, Olga. A recurring theorem on determinants. *American Mathematical Monthly* 56:672-675. 1949.
12. Whitney, Hassler. A function not constant on a connected set of its critical points. *Duke Mathematical Journal* 1:514-517. 1935.

APPENDIX

## APPENDIX I

Let  $V \subset \mathbb{R}^n$  be an open set containing the origin whose tangent bundle  $TV$  is trivial. Then we will define the map  $g : TV - \{0\} \rightarrow S^{n-1}$  as the composite of the following maps,

$$TV - \{0\} \xrightarrow{\tau} V \times (\mathbb{R}^n - \{0\}) \xrightarrow{\text{pr}_2} \mathbb{R}^n - \{0\} \xrightarrow{\frac{x}{\|x\|}} S^{n-1}$$

where  $\tau$  is the trivialization.

Lemma A: Let  $p \in V$ , and  $\xi$  and  $\eta$  be nonzero vector fields on  $V$ , then if  $g \circ \xi(p) = g \circ \eta(p)$  there exists a positive constant  $a$  such that  $a\xi(p) = \eta(p)$ .

Proof: Let  $x = \text{pr}_2 \circ \tau \circ \xi(p)$  and  $y = \text{pr}_2 \circ \tau \circ \eta(p)$  then  $\frac{x}{\|x\|} = \frac{y}{\|y\|}$ . Thus letting  $a = \frac{\|y\|}{\|x\|}$  we have  $a(\text{pr}_2 \circ \tau \circ \xi(p)) = \text{pr}_2 \circ \tau \circ \eta(p)$ . But since both  $\text{pr}_2$  and  $\tau$  are fibre isomorphisms, we have  $a\xi(p) = \eta(p)$ . |

Theorem B: Let  $S$  be a compact, connected orientable hypersurface in  $\mathbb{R}^n$ , let  $\eta : S \rightarrow T\mathbb{R}^n$  be the normal vector field to  $S$ , then  $g \circ \eta : S \rightarrow S^{n-1}$  is a surjective map.

Proof: Let  $e_i$  be the  $i^{\text{th}}$  unit co-ordinate vector field on  $\mathbb{R}^n$ . Choose a point  $p \in S^{n-1}$  and assume that  $p$  represents the

constant vector field  $e_1$  via the map  $g$ . Let  $\pi: S \rightarrow \mathbb{R}$  be the height function with respect to the first co-ordinate axis. Since  $S$  is compact and connected, we see that  $\pi(S)$  is also compact and connected. Thus  $\pi(S)$  is a closed interval of  $\mathbb{R}$ , say  $[a, b]$ . Let  $\bar{S} = \pi^{-1}b \cap S$ . Then  $\bar{S}$  is a closed nonempty set. By the construction we see that  $p \in \bar{S}$  is an absolute maximum with respect to the map  $\pi$ . Since  $S$  is a  $C^\infty$  manifold,  $\pi$  is a  $C^\infty$  map. Hence for  $p \in \bar{S}$  we have  $\frac{\partial \pi}{\partial x_i}(p) = 0$  for  $2 \leq i \leq n$  where the  $x_i$  are the co-ordinate functions. Thus  $T_p S$  is parallel to the hyperplane spanned by  $\{e_i(p) : 2 \leq i \leq n\}$  and hence  $e_1(p)$  is parallel to  $\eta(p)$ . |