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Title EIGENFUNCTION EXPANSIONS FOR SELF-ADJOINT BILINEAR OPERATORS IN HILBERT SPACE

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This thesis is concerned with certain nonlinear operators and the eigenfunctions and eigenvalues associated with them. The formal setting is in an abstract Hilbert space H.

We single out for special attention a subset of the class of compact bilinear operators on H. Here, for simplicity, the subset is defined for real H only. Let B denote a bilinear operator on H. For all $x, y \in H$, let $B_1xy = Bxy$ and $B_2xy = Byx$. Then, the subset is defined as the set of all B for which B_1x and B_2x, acting as linear operators on H, are self-adjoint for all $x \in H$. Hence, the concept of a self-adjoint nonlinear operator is introduced. We also consider the quadratic operator induced by B.

Properties analogous to those for self-adjoint linear operators are investigated. In particular, we consider the existence of eigenvalues, the existence of a maximal orthonormal set of eigenelements, and an eigenfunction expansion for an induced quadratic operator.

The theory for operators of this type requires more elaborate analysis than the corresponding theory for linear operators. However, new results are obtained and new questions are raised.

The theory is applied to certain quadratic integral equations and a formal solution is presented. Finally, the concept of self-adjointness is extended to quadratic differential equations and more general nonlinear operators.

EIGENFUNCTION EXPANSIONS FOR SELF-ADJOINT BILINEAR OPERATORS IN HILBERT SPACE

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TABLE OF CONTENTS

EIGENFUNCTION EXPANSIONS FOR SELF-ADJOINT BILINEAR OPERATORS IN HILBERT SPACE

CHAPTER 1

INTRODUCTION

§1.1 Preliminary Remarks

We begin with a formal description of some spaces and operators which are prerequisites for much of the theory.

Let H denote an abstract Hilbert space over the real field R or the complex field C. The elements (or vectors) of H are denoted by $x, y, z, \cdots$ and the scalars by $\alpha, \beta, \gamma, \cdots$. Let (x, y) be the inner product and $\|x\|$ the norm. We write $x_n \to x$ iff $\|x_n - x\| \to 0$.

To illustrate the theory in later chapters we shall use several concrete spaces. These are listed next.

Example 1.1.1 Let ℓ_2^n denote the (n-dimensional) space of all n-tuples $x = (\xi_1, \cdots, \xi_n)$ of real or complex numbers. The inner product and norm are, respectively,

$$(x, y) = \sum_{k=1}^{n} \xi_k \overline{\eta}_k, \qquad \|x\| = \left(\sum_{k=1}^{n} |\xi_k|^2 \right)^{\frac{1}{2}} .$$

Example 1.1.2 Let ℓ_2 denote the (infinite-dimensional) space of all infinite sequences $\{\xi_1, \cdots, \xi_n, \cdots\}$ of real or complex numbers such that

$$\sum_{k=1}^{\infty} |\xi_k|^2 < \infty.$$

The inner product and norm are defined as in Example 1.1.1 with n replaced by ∞.

Example 1.1.3 Let $L_2(0,1)$ denote the space of measurable square integrable, real or complex functions $x(s)$ on $(0,1)$. The inner product and norm are given by

$$(x,y) = \int_0^1 x(s)\overline{y(s)}\,ds,$$

$$\|x\| = \left(\int_0^1 |x(s)|^2 ds\right)^{\frac{1}{2}}.$$

Next we consider some familiar mappings. A linear operator L on H is a mapping $L: H \to H$ which is additive and homogeneous:

$$(1.1.1) \qquad L(\alpha x+\beta y) = \alpha Lx + \beta Ly .$$

Similarly, a bilinear operator B on H is a mapping $B: H \times H \to H$ which is additive and homogeneous in each argument:

$$B(\alpha x + \beta y)z = \alpha Bxz + \beta Byz , \tag{1.1.2}$$

$$Bx(\alpha y + \beta z) = \alpha Bxy + \beta Bxz . \tag{1.1.3}$$

Also B is symmetric iff

$$Bxy = Byx \quad \text{for all} \quad x, y \in H . \tag{1.1.4}$$

On a real Hilbert space a bilinear operator induces a quadratic operator $Q: H \rightarrow H$. For set $x = y$; then Q is defined by

$$Qx \equiv Bxx . \tag{1.1.5}$$

Conversely, if B is symmetric then B can be expressed in terms of Q by

$$Bxy = \tfrac{1}{2}[Q(x+y) - Qx - Qy] . \tag{1.1.6}$$

The algebraic nature of Q becomes apparent if we apply the homogeneity of B to (1.1.5). Thus

$$Q(\alpha x) = \alpha^2 Qx ,$$

Many of the results in later chapters will be expressed in terms of an induced quadratic operator Q; however, as we shall see, the associated bilinear operator B will be required for much of the theory.

§1.2 Topological Preliminaries

Let T be a linear or nonlinear operator on H (into H). Then T is continuous iff $x_n \to x$ implies $Tx_n \to T_x$, and T is bounded iff T maps bounded sets into bounded sets. For linear operators these concepts are equivalent.

Now we turn to compact operators. An operator T is compact provided that T maps bounded sets into precompact sets, or by virtue of the completeness of H, into relatively compact sets. In other words, T is compact iff for each bounded $S \subset H$, $\overline{T(S)}$ is compact. It is also useful to recall that this is equivalent to $T(S)$ being sequentially compact, that is, for any sequence $\{x_n\} \subset S$ there exists a subsequence $\{x_{n_i}\} \subset S$ such that $\{Tx_{n_i}\}$ converges in H. Clearly, a compact operator is bounded, and conversely if the range space is finite-dimensional.

Finally recall that T is completely continuous iff T is compact and continuous.

Of course, much more can be said when T is linear $(T = L)$. For instance, a linear operator L is bounded iff

(1.2.1) $$\sup_{\|x\| \leq 1} \|Lx\| < \infty.$$

This agrees with the more general notion of a bounded operator. For linear operators, L is completely continuous iff L is compact.

We shall denote the set of all bounded linear operators on H by $\mathcal{L}(H,H)$ or merely $\mathcal{L}(H)$. We recall that $\mathcal{L}(H)$ is itself a normed linear space, with norm

$$(1.2.2) \qquad \|L\| = \sup_{\|x\| \leq 1} \|Lx\| = \sup_{\|x\| = 1} \|Lx\| .$$

Let us consider some familiar examples of linear operators on concrete Hilbert spaces.

Example 1.2.1 For H we take ℓ_2^n (Ex. 1.1.1). The bounded linear operators on ℓ_2^n are represented by $n \times n$ matrices of the form $\{a_{ij}\}$. Since ℓ_2^n is finite-dimensional the operators are compact and, hence, completely continuous.

Example 1.2.2 Consider the Hilbert space ℓ_2 (Ex. 1.1.2). Each bounded linear operator L on ℓ_2 is represented by an infinite matrix of the form $\{a_{ij}\}$ with the rows and columns in ℓ_2. For the so-called Hilbert-Schmidt operators,

$$\sum_i \sum_j |a_{ij}|^2 < \infty ,$$

which implies that L is compact.

<u>Example 1.2.3</u> Here we consider the linear integral operators (Hilbert-Schmidt) on the space $L_2(0,1)$, i.e. operators of the form

$$Lx(s) = \int_0^1 K(s,t)x(t)dt \tag{1.2.3}$$

where the kernel $K(s,t)$ satisfies the condition

$$\int_0^1 \int_0^1 |K(s,t)|^2 ds dt < \infty . \tag{1.2.4}$$

As is well known, L is compact and hence completely continuous.

Next we consider the case when T is bilinear $(T = B)$. Even though B is now nonlinear, the topological properties of B agree, in most respects, with those for linear operators. For instance, B is bounded provided that

$$\sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} \|Bxy\| < \infty, \tag{1.2.5}$$

and the norm of B is given by

$$\|B\| = \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} \|Bxy\| = \sup_{\substack{\|x\| = 1 \\ \|y\| = 1}} \|Bxy\| . \tag{1.2.6}$$

Furthermore, B is continuous iff B is bounded. Hence, completing the parallelism, B is completely continuous iff B is compact.

We denote by $\mathcal{B}(H)$ the set of all bounded bilinear operators on H. Note that $\mathcal{B}(H)$ can be identified with $\mathcal{L}(H, \mathcal{L}(H))$ and thus $\mathcal{B}(H)$ is a normed linear space with norm (1. 2. 6).

We now consider some bilinear operators which are analogous to the linear operators in the foregoing examples.

<u>Example 1. 2. 4</u> The bilinear operators on ℓ_2^n are represented by $n \times n \times n$ matrices of the form $\{b_{ijk}\}$. Again, the finite-dimensionality of ℓ_2^n insures that these operators are compact, and, hence, completely continuous.

<u>Example 1. 2. 5</u> Let $H = \ell_2$. The bounded bilinear operators on ℓ_2 are represented by matrices of the form $\{b_{ijk}\}$ $(i, j, k = 1, 2, \cdots)$ with $\sum_{i=1}^{\infty} |b_{ijk}|^2 < \infty$ for $(j, k = 1, 2, \cdots)$.

<u>Example 1. 2. 6</u> For $H = L_2(0, 1)$ we consider the integral operator B which is defined by

$$Bxy = \int_0^1 \int_0^1 K(s, t, u)x(t)y(u)dudt \ .$$

The kernel $K(s, t, u)$ is assumed to satisfy the (generalized Hilbert-Schmidt) condition

$$\int_0^1 \int_0^1 \int_0^1 |K(s,t,u)|^2 ds\,dt\,du .$$

Under these conditions B is compact and, therefore, completely continuous.

§ 1.3 Self-adjoint Operators

Let $L \epsilon \mathcal{L}(H)$. By L^* we mean the adjoint of L, that is, the bounded linear operator $L^* \epsilon \mathcal{L}(H)$ such that

(1. 3. 1) $$(Lx, y) = (x, L^*y) \quad \text{for all} \quad x, y \epsilon H.$$

We recall that L is self-adjoint iff $L = L^*$, in which case L is symmetric:

(1. 3. 2) $$(Lx, y) = (x, Ly), \quad \text{for all} \quad x, y \epsilon H.$$

In the concrete Hilbert spaces ℓ_2^n and ℓ_2 the given bounded linear operators are self-adjoint iff $a_{ij} = \overline{a}_{ji}$ for all i, j. For the integral operator of Hilbert-Schmidt type the condition is $K(s,t) = \overline{K(t,s)}$.

Let λ be an eigenvalue of $L \epsilon \mathcal{L}(H)$ and $x \neq 0$ the corresponding eigenelement, so that

(1. 3. 3) $$Lx = \lambda x.$$

Suppose, in addition to being self-adjoint, L is compact and nonzero $(L \neq 0)$. Then L possesses a finite or countably infinite set of eigenvalues $\{\lambda_n\}$, and a corresponding set of eigenelements $\{x_n\}$. If $\{\lambda_n\}$ is infinite then $\lambda_n \to 0$. Furthermore, there exists a maximal (complete) orthonormal system of eigenelements. Thus,

$$(1.3.4) \qquad (x_i, x_j) = \delta_{ij}, \quad (\delta_{ij} \text{ Kronecker delta}).$$

The eigenmanifolds corresponding to nonzero eigenvalues are finite-dimensional.

This brings us to the focal point of the paper, or more precisely, to the analogy which will serve as a focal point. This is the eigenfunction expansion for a compact self-adjoint L. In view of our previous notation, we write the expansion, for arbitrary $x \in H$, as

$$(1.3.5) \qquad Lx = \sum_k (Lx, x_k)x_k = \sum_k \lambda_k (x, x_k)x_k,$$

where the sum may be finite or infinite.

Now we pass to the nonlinear problem. For the remainder of the chapter assume H to be real (the complex case appears in Chapter 2).

It is convenient to introduce the bilinear operators B_1 and

B_2 which are defined by

(1.3.6) $$B_1xy = Bxy, \qquad B_2xy = B_1xy = Byx$$

for all $x, y \in H$. Thus $B_1 = B$. Note that according to (1.1.4) B is symmetric iff $B_1 \equiv B_2$.

With this notation we introduce the concept of a self-adjoint nonlinear operator.

Definition 1.3.1 A bounded bilinear operator B is said to be self-adjoint iff, for all $x \in H$, B_1x and B_2x are self-adjoint linear operators.

In other words, by (1.3.2), the two conditions

(1.3.7) $$(B_1xy, z) = (y, B_1xz),$$

(1.3.8) $$(B_2xy, z) = (y, B_2xz),$$

must hold for all $x, y, z \in H$. Observe that a self-adjoint B is necessarily bounded and thus continuous. We also have

Lemma 1.3.1 If B is self-adjoint then B is symmetric.

Proof: Equations (1.3.7) and (1.3.8) imply

$$(B_1xy, z) = (y, B_1xz) = (y, B_2zx) = (B_2zy, x) = (B_1yz, x) = (z, B_1yx).$$

Thus, since H is real, $(B_1xy, z) = (B_1yx, z)$ for all $z \epsilon H$, which implies $B_1xy=B_1yx$ for all $x,y \epsilon H$. Hence by (1.3.6) and (1.1.4) $B_1 = B_2$.

The converse of Lemma 1. 3. 1 is obviously false. However, we have the following equivalent formulation for self-adjointness.

Lemma 1. 3. 2 A bilinear operator is self-adjoint iff

(a) B is symmetric for all $x, y \epsilon H$, and

(b) B_1x (or B_2x) is a self-adjoint linear operator for all $x \epsilon H$.

Proof: If B is self-adjoint then Lemma 1. 3. 2 follows directly from the definition and Lemma 1. 3. 1. Conversely, assume that $B_1 = B_2$, and B_1x is self-adjoint for all $x \epsilon H$. Then,

$$(B_2xy, z) = (B_1xy, z) = (y, B_1xz) = (y, B_2xz)$$

for all $x, y, z \epsilon H$. Thus B_2x is a self-adjoint linear operator for all $x \epsilon H$. The alternative in (b) follows by symmetry.

Definition 1. 3. 2 A quadratic operator Q is said to be self-adjoint iff the (symmetric) bilinear operator (1. 1. 6) which induces Q is self-adjoint.

An eigenvalue problem corresponding to (1. 3. 3) can be formulated easily for quadratic operators Q. We seek scalars (eigenvalues) λ and nonzero eigenelements $x \in H$ such that

$$(1.3.9) \qquad Qx = \lambda x, \qquad (Qx \equiv Bxx).$$

Corresponding to each eigenelement x there is only one eigenvalue. On the other hand, to each eigenvalue there may be one or more eigenelements; moreover, the relationship here may not be the same as it is for linear operators, as Example 4.2.1 will illustrate.

In analogy with the linear problem assume that Q is a self-adjoint quadratic operator. Furthermore, assume that Q possesses a finite or countably infinite set of eigenvalues $\{\lambda_n\}$ and corresponding maximal orthonormal system of eigenelements $\{x_n\}$. Carrying the analogy with the linear problem one stepfurther, we consider, for an arbitrary $x \in H$, the expansion

$$(1.3.10) \qquad Qx = \sum_k (Qx, x_k)x_k ,$$

where the summation may be finite or infinite. What we now seek is the analog of the third member in (1.3.5).

Since Q is self-adjoint, we have, by definition

$$(1.3.11) \qquad (Qx, x_k) = (Bxx, x_k) = (x, Bxx_k)$$

for all $x \in H$. And, since $x \in H$ and $\{x_n\}$ is maximal

$$x = \sum_j a_j x_j, \qquad a_j = (x, x_j). \tag{1.3.12}$$

For convenience now assume

$$Bx_j x_k = 0 \quad \text{for} \quad j \neq k. \tag{1.3.13}$$

Then, (1.3.11), (1.3.12) and the continuity of B imply

$$(Qx, x_k) = (x, B(\sum_j a_j x_j) x_k) = \sum_j a_j (x, Bx_j x_k). \tag{1.3.14}$$

But (1.3.13), (1.3.14), and (1.3.9) yield

$$(Qx, x_k) = a_k (x, Bx_k x_k) = a_k (x, Qx_k) = a_k \lambda_k (x, x_k). \tag{1.3.15}$$

Consequently, in view of (1.3.15) and the definition of a_k, the expansion (1.3.10) assumes a very natural form, namely,

$$Qx = \sum_k (Qx, x_k) x_k = \sum_k \lambda_k (x, x_k)^2 x_k. \tag{1.3.16}$$

This is clearly the analog of (1.3.5).

Finally, we use the expansion for quadratic operators to obtain a more general expansion for bilinear operators. A simple application of (1.3.16) to (1.1.6) produces the result:

$$\text{(1.3.17)} \qquad Bxy = \sum_k (Bxy, x_k)x_k = \sum_k \lambda_k (x, x_k)(y, x_k)x_k .$$

Since (1.3.16) is valid for arbitrary x, the expansion (1.3.17) is valid for all $x, y \epsilon H$.

In subsequent chapters we shall establish the validity of (1.3.16). As a general procedure, we shall present the theory as an analog of the existing theory for self-adjoint linear operators. However, when the nonlinearity of the operator comes into play we shall find it necessary to depart from this procedure and pursue an independent course. In this way the essential nature of the bilinear operator comes into view.

The complex case (H complex) is confined to Chapter 2. It will be observed that some of the subsequent results are obviously applicable to the complex case, but a complete analysis must await further study.

Applications to integral and differential equations are indicated in Chapter 4, and the final chapter discusses some generalizations.

As a final comment in this chapter we write expansions (1.3.5) and (1.3.16) as a single equation. Let T denote either a linear (L) or quadratic (Q) self-adjoint operator. Then,

$$Tx = \sum_k (Tx, x_k)x_k = \sum_k \lambda_k t_k(x) x_k \tag{1.3.18}$$

where

$$t_k(x) = \begin{Bmatrix} (x, x_k) & \text{if} \quad T = L \\ \\ (x, x_k)^2 & \text{if} \quad T = Q \end{Bmatrix} .$$

CHAPTER 2

BILINEAR OPERATORS ON COMPLEX SPACES

§ 2.1 Preliminary Remarks

In the first chapter we defined the concept of a self-adjoint bilinear operator for a real Hilbert space only. Throughout this chapter H is a complex Hilbert space.

We now observe, by (1.3.7) and (1.3.8), that

$$i(Bxy, z) = (Bixy, z) = (y, Bixz) = -i(y, Bxz) = -i(Bxy, z) ,$$

for all $x, y, z \in H$. This shows that the only self-adjoint bilinear operator on H in sense of Def. 1.3.1 is the zero operator. Consequently, we shall extend the definition to the complex case in a less direct manner.

We introduce the conjugate $\overline{x}$ of an element in H. For $H = \ell_2^n$, let $\overline{x} = (\overline{\xi}_1, \cdots, \overline{\xi}_n)$ for $x = (\xi_1, \cdots, \xi_n)$ and similarly for ℓ_2. In $L_2(0, 1)$ or in a more general L_2 space $\overline{f}(t) = \overline{f(t)}$. In an abstract Hilbert space, suppose that $\{x_n\}$ is any orthonormal basis. Then $x = \sum (x, x_k)x_k$ and we can define $\overline{x} = \sum \overline{(x, x_k)}x_k$. It should be recognized that this definition depends on the particular basis chosen. Thus, the definition of $\overline{x}$ depends on the fact that the Hilbert space may be regarded (in many ways) as a space of scalar functions or equivalence classes of functions. Thus, to obtain $\overline{x}$ we merely conjugate the values of these functions.

Assume henceforth in this chapter that the Hilbert space is a space of complex functions in terms of which $\overline{x}$ has been defined. We also introduce $x_r = (x + \overline{x})/2$ and $x_i = (x - \overline{x})/2i$ for the real and imaginary parts of x. Thus

$$(2.1.1) \qquad x = x_r + i x_i .$$

From this we may now write the following:

$$(2.1.2) \qquad \overline{x_1 + x_2} = \overline{x_1} + \overline{x_2}, \quad \overline{\alpha x} = \overline{\alpha}\,\overline{x},$$

$$(2.1.3) \qquad (\overline{x}, \overline{y}) = \overline{(x, y)},$$

$$(2.1.4) \qquad \|\overline{x}\| = \|x\|,$$

$$(2.1.5) \qquad \overline{\overline{x}} = x.$$

Now let B denote a bilinear operator on H. We assert that

$$(2.1.6) \qquad Bxy = \overline{B\,\overline{x}\,\overline{y}} .$$

To prove this substitute $x = x_r + i x_i$ and $y = y_r + i y_i$ and simplify.

§ 2.2 The Adjoints of B

As in (1. 3. 6), associate to each $B \in \mathcal{B}(H)$ the operators B_1 and B_2 defined by

(2. 2. 1) $$B_1 xy = Bxy, \qquad B_2 xy = Byx$$

for all $x, y \in H$.

Recall (1. 1. 4) that B is symmetric iff $Bxy = Byx$ for all $x, y \in H$; equivalently, B is symmetric iff

(2. 2. 3) $$B_1 = B_2 .$$

We now introduce the adjoints B_1^* and B_2^* If for all $x, y, z \in H$

(2. 2. 4) $$(B_1 xy, z) = (y, B_1^* \bar{x} z)$$

(2. 2. 5) $$(B_2 xy, z) = (y, B_2^* \bar{x} z)$$

then B_1^* and B_2^* are called the adjoints of B_1 and B_2 respectively. Thus $B_1^* \bar{x}$ is the adjoint of the linear operator $B_1 x$ and similarly for B_2.

Example 2. 2. 1 Consider the space ℓ_2^n (Ex. 1. 1. 1). (The results are also valid for ℓ_2.) Denote the vectors by

$x = (\xi_1, \cdots, \xi_n)$, $y = (\eta_1, \cdots, \eta_n)$, $z = (\zeta_1, \cdots, \zeta_n)$ and the matrix of B by $\{b_{ijk}\}$ (Ex. 1.2.4). Then (2.2.4) shows that the matrix of B_1^* is given by $\{\overline{b_{ijk}}\}$. Similarly, the matrix of B_2^* is given by $\{\overline{b_{kji}}\}$.

Example 2.2.2 Let $H = L_2(0,1)$ and suppose B is the integral operator of Example 1.2.6. Then, (2.2.4) and (2.2.5) imply the adjoint of B_1,

$$B_1 xy = \int_0^1 \int_0^1 K(s,t,u)x(u)y(t)\,du\,dt$$

is given by

$$B_1^* xy = \int_0^1 \int_0^1 \overline{K(t,s,u)}x(u)y(t)\,du\,dt \ .$$

Similarly, the adjoint of B_2 is

$$B_2^* xy = \int_0^1 \int_0^1 \overline{K(u,t,s)}x(u)y(t)\,du\,dt \ .$$

Next we consider some properties of the adjoints B_1^* and B_2^* To simplify notation, let B denote either B_1 or B_2 and similarly for C.

Theorem 2.2.1 The adjoint has the following properties:

(2.2.6) $$(B + C)^* = B^* + C^*$$

(2.2.7) $$(\alpha B)^* = \bar{\alpha} B^*$$

(2.2.8) $$(B^*)^* = B$$

(2.2.9) $$\|B^*\| = \|B\| .$$

Proof: The first three are proved in essentially the same manner. We illustrate with (2.2.7). From (2.2.4) and the properties of the inner product it follows that

$$(y, (\alpha B)^* \bar{x} z) = ((\alpha B)xy, z) = \alpha(Bxy, z) = \alpha(y, B^* \bar{x} z) = (y, \bar{\alpha} B^* \bar{x} z)$$

for all $x, y, z \in H$.

Property (2.2.9) is proved by observing that

$$\|B^*xy\|^2 = (B^*xy, B^*xy) = (B\bar{x}B^*xy, y) \leq \|B\bar{x}B^*xy\| \, \|y\| \leq \|B\| \, \|B^*xy\| \, \|x\| \, \|y\|$$

implies $\|B^*\| \leq \|B\|$. By (2.2.8) $\|B\| = \|(B^*)^*\| \leq \|B^*\|$. Hence $\|B^*\| = \|B\|$.

§ 2.3 Self-adjoint B on Complex Spaces

For complex Hilbert spaces the self-adjoint bilinear operators are defined in a natural way.

Definition 2.3.1 A bilinear operator $B \in \mathcal{B}(H)$ is said to be self-adjoint iff

(2.3.1) $$B_1 = B_1^* \quad \text{and} \quad B_2 = B_2^*.$$

Example 2.3.1 Let $H = \ell_2^n$ (or ℓ_2). Then Example 2.2.1 shows that B is self-adjoint iff

(2.3.2) $$b_{ijk} = \overline{b_{jik}} \quad \text{and} \quad b_{ijk} = \overline{b_{kij}}$$

for all i, j, k.

Example 2.3.2 The integral operators on $L_2(0, 1)$ (Ex. 2.2.2) are clearly self-adjoint iff

(2.3.3) $$K(s,t,u) = \overline{K(t,s,u)} \quad \text{and} \quad K(s,t,u) = \overline{K(u,t,s)}$$

for all s, t, and u in the unit cube.

Corresponding to Lemma 1.3.1 we have

Lemma 2.3.1 If B is self-adjoint then B is symmetric.

Proof: The proof follows essentially the same as in Lemma 1.3.1. Apply (2.3.1) to (2.2.4) and (2.2.5). Then

$$\begin{aligned}(B_1xy, z) &= (y, B_1^*\overline{x}z) = (y, B_1\overline{x}z) = (y, B_2z\overline{x})\\ &= (B_2^*\overline{z}y, \overline{x}) = (B_2\overline{z}y, \overline{x}) = (B_1y\overline{z}, \overline{x})\\ &= (\overline{z}, B_1^*\overline{y}\overline{x}) = (\overline{z}, B_1\overline{yx}) = \overline{(B_1\overline{yx}, \overline{z})} \quad .\end{aligned}$$

Now apply (2. 1. 3) and (2. 1. 6) to the last quantity.

$$\overline{(B_1\overline{yx}, \overline{z})} = \overline{(B_1\overline{yx}, z)} = (B_1 yx, z).$$

Therefore, $(B_1 xy, z) = (B_1 yx, z)$ for all $z \in H$. The remainder of the proof is the same as Lemma 1. 3. 1.

Example 2. 3. 3 Suppose B is self-adjoint on ℓ_2^n (or ℓ_2). Then (2. 3. 2) implies

$$b_{ijk} = \overline{b_{kji}} = b_{jki} = \overline{b_{ikj}} \quad \text{for all} \quad i, j, k.$$

Thus, a self-adjoint B is symmetric on ℓ_2^n iff

(2. 3. 4) $$b_{ijk} = \overline{b_{ikj}} \quad \text{for all} \quad i, j, k.$$

Example 2. 3. 4 It is clear that the integral operator on $L_2(0, 1)$ of Example 2. 3. 2 is symmetric iff

$$K(s, t, u) = \overline{K(s, u, t)}$$

for all s, t, and u in the unit cube.

The last two examples show that an operator may be symmetric without being self-adjoint. However, we may formulate an equivalent definition for self-adjointness in terms of symmetry if we add another condition. We will do this after the next result.

Lemma 2.3.2 B is symmetric $(B_1 = B_2)$ iff the adjoints are symmetric $(B_1* = B_2*)$.

Proof: If B is symmetric then by (2.2.1) ff

$$(B_1*xy, z) = (y, B_1\bar{x}z) = (y, B_2\bar{x}z) = (B_2^* xy, z)$$

for all $x, y, z \in H$. Hence $B_1* = B_2*$. The converse is similar.

Theorem 2.3.1 B is self-adjoint iff B is symmetric and either $B_1 = B_1^*$ or $B_2 = B_2^*$.

Proof: If B is self-adjoint then the theorem is obvious from Lemma 2.3.1. Conversely, suppose $B_1 = B_2$ and $B_1 = B_1*$ (case 1). Lemma 2.3.2 gives $B_1* = B_2*$, thus for all $x, y, z \in H$

$$(B_2xy, z) = (B_1xy, z) = (y, B_1^*\bar{x}z) = (y, B_2^*\bar{x}z),$$

which shows that $B_2 = B_2^*$. Hence B is self-adjoint. The proof for the second case $(B_1 = B_2$ and $B_2 = B_2^*$ is the same.

There are several other interesting results which strengthen the analogy with the linear case. These are presented next.

Lemma 2.3.3 Every $B \in \mathcal{B}(H)$ can be expressed in the form

$$B = C + iD, \qquad B^* = C - iD$$

where C and D are self-adjoint.

Proof: For arbitrary $B \in \mathcal{B}(H)$ define $C = (B+B^*)/2$ and $D = (B - B^*)/2i$. Then (2. 2. 6) ff shows that C and D are self-adjoint. Since $B = C + iD$ the lemma is proved.

Lemma 2. 3. 4 The set of all self-adjoint bilinear operators is a real-linear manifold.

Proof: Let O denote the zero operator. Then $O^* = O$, and O belongs to the set. Next, for any $\alpha, \beta \in R$, and any self-adjoint C and D it is clear from (2. 2. 6) ff that

$$(\alpha C + \beta D)^* = \bar{\alpha} C^* + \bar{\beta} D^* = \alpha C + \beta D.$$

which completes the proof.

Lemma 2. 3. 5 If $\{B_n\} \subset \mathcal{B}(H)$ is a sequence of self-adjoint operators which converge to an operator $B \in \mathcal{B}(H)$ then B is self-adjoint.

Proof: Applying (2. 2. 6) and (2. 2. 9) to

$$\|B-B^*\| \leq \|B-B_n\| + \|B_n - B_n^*\| + \|B_n^*-B^*\|$$

$$= \|B-B_n\| + \|B_n^*-B^*\| = \|B-B_n\| + \|(B_n-B)^*\|$$

$$= 2\|B-B_n\| \to 0$$

shows that $B = B^*$.

<u>Theorem 2.3.2</u> The set of all self-adjoint bilinear operators is a (closed) real-linear subspace of $\mathcal{B}(H)$.

<u>Proof:</u> The last two lemmas provide the proof.

§2.4 Eigenfunction Expansions on Complex Spaces

In Chapter 1 we saw that each symmetric operator B induced a quadratic operator Q. We also said that Q was self-adjoint iff the corresponding B was self-adjoint. Clearly, these ideas apply to complex spaces as well.

Furthermore, the eigenvalue problem is formulated in precisely the same manner. However, some new features enter into the problem. For example, let λ be an eigenvalue and x an eigenelement of Q. Then (2.1.8) ff implies

$$Q\bar{x} = B\bar{x}\bar{x} = \overline{\overline{B\bar{x}\bar{x}}} = \overline{Bxx} = \overline{\lambda x} = \bar{\lambda}\bar{x}$$

so that

(2.4.1) $$Q\bar{x} = \bar{\lambda}\bar{x} .$$

Let us now proceed to expansion (1.3.10) for complex H, that is, (2.4.2) $$Qx = \sum_k (Qx, x_k)x_k.$$

We make the same assumption regarding the existence and nature of the eigenvalues and eigenelements of Q. We note that in addition to (1.3.12) we also have

(2.4.3) $$\overline{x} = \sum_j \overline{\alpha}_j \overline{x}_j, \qquad \overline{\alpha}_j = \overline{(x, x_j)}.$$

Applying (1.3.12), (1.3.13) to (Qx, x_k) yields

$$(Qx, x_k) = \lambda_k (x, x_k)^2,$$

which is exactly the same result as in the real case. Consequently,

(2.4.4) $$Qx = \sum_k (Qx, x_k)x_k = \sum_k \lambda_k (x, x_k)^2 x_k.$$

This is as far as the theory goes for the complex case. It is probably not in a definitive form. Henceforth, we consider the real case only.

CHAPTER 3

EXISTENCE OF EIGENVALUES FOR QUADRATIC OPERATORS

§ 3.1 Preliminary Remarks

In the two previous chapters we made some rather broad assumptions. In particular, we assumed the existence of a maximal orthonormal set of eigenelements $\{x_n\}$ and a corresponding sequence of eigenvalues $\{\lambda_n\}$. The validity of this assumption, of course, depends upon the properties of the invariant subspaces of the operators in question. If the operators are bounded, self-adjoint and linear, then the properties are easily deduced. If, on the other hand, the operators are bounded, self-adjoint, and bilinear, then more elaborate analysis is required. However, as a result of these investigations, we gain insight into some of the properties which are due to the nonlinearity of the operator. These results are essential to the existence theorems which follow.

As an example (and counter example) for the theory, we shall consider a simple operator on ℓ_2^n.

Let $\{b_{ijk}\}$ be an $n \times n \times n$ matrix of an operator B. We list all of the elements of the matrix $\{b_{ijk}\}$ by writing

(3.1.1) $$\{b_{ijk}\} = (b_{i1k} | b_{i2k} | \cdots | b_{ink}),$$

where b_{ilk} represents an ordinary $n \times n$ matrix all of whose elements have a middle subscript equal to one. The rule is now clear. Also, note that the vector $z = Bxy$ is given by

$$(3.1.2) \qquad \zeta_i = \sum_{j=1}^{n} \sum_{k=1}^{n} b_{ijk} \xi_k \eta_j, \quad (i = 1, \cdots, n)$$

where $z = (\zeta_1, \cdots, \zeta_n)$, $y = (\eta_1, \cdots, \eta_n)$, and $x = (\xi_1, \cdots, \xi_n)$.

Example 3.1.1 Let ℓ_2^n be real and 2-dimensional. We consider the $2 \times 2 \times 2$ matrix of an operator B written according to (3.1.1) in the form

$$(3.1.3) \qquad \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) .$$

It is readily verified, by using (3.1.1) and (2.3.2), that the matrix is self-adjoint.

Throughout this chapter B denotes a bilinear operator and Q the induced quadratic operator. The Hilbert space is real here and henceforth.

§3.2 Invariant Subspaces

Let L be a bounded linear operator, and Q a bounded quadratic operator on H. Recall that a closed linear subspace $M \subseteq H$ is invariant under L iff $L(M) \subseteq M$. This idea applies equally well to the operator Q; hence, M is invariant under Q iff $Q(M) \subseteq M$. Thus, Q may be regarded as operating on M alone. If, in addition, the orthogonal complement $M^{\perp}$ of M is also invariant under Q then M is said to reduce Q. This, of course, is our principal goal, for if M reduces Q then the study of Q on H can be reduced to the study of Q on M and $M^{\perp}$ separately.

As a matter of convenience we introduce for bilinear operators the notation $B(M_1, M_2) = \{Bxy: x \in M_1, y \in M_2\}$, where M_1 and M_2 are closed linear subspaces of H. If the subspaces are not distinct then the notation is shortened to $B(M)$.

A standard result from the theory of linear operators states that: $L(M) \subseteq M$ iff $L^*(M^{\perp}) \subseteq M^{\perp}$. Furthermore, if L is self-adjoint then $L(M) \subseteq M$ iff $L(M^{\perp}) \subseteq M^{\perp}$. This simple result is no longer true for nonlinear operators, as the next example shows.

<u>Example 3. 2. 1</u> Consider the matrix of the self-adjoint operator B of Example 3. 1. 1. It can be easily verified that $\lambda = 1$ is an eigenvalue and $x = (1, 0)$ is an eigenvector of the induced quadratic operator Q. Consequently, the (one-dimensional) subspace M spanned by x is invariant under Q. However, for $y = (0, 1)$ we have $y \in M^{\perp}$ and $Byy \notin M^{\perp}$. Hence, $M^{\perp}$ is

not invariant under Q even though B is self-adjoint.

The next lemmas clarify some of the difficulties. We shall assume (here and henceforth) that B is a bounded self-adjoint bilinear operator.

<u>Lemma 3.2.1.</u> If $B(M) \subseteq M$ then $B(M, M^{\perp}) \subseteq M^{\perp}$.

Proof: Since B is self-adjoint, we have for any $x \in M$, $y \in M^{\perp}$, and $z \in M$,

$$(Bxy, z) = (y, Bxz) = 0.$$

Therefore, $Bxy \in M^{\perp}$.

<u>Lemma 3.2.2</u> If $B(M_1) \subseteq M_1$, $B(M_2) \subseteq M_2$ and $M_1 \perp M_2$ then $B(M_1, M_2) \subseteq M_1^{\perp} \cap M_2^{\perp}$.

Proof: Observe that $M_2 \subseteq M_1^{\perp}$. Hence, by the previous lemma, it follows that $B(M_1, M_2) \subseteq M_1^{\perp}$. Now, interchange the role of M_1 and M_2 and repeat the arguments. Then $B(M_2, M_1) \subseteq M_2^{\perp}$. The result now follows from the symmetry of B.

<u>Lemma 3.2.3</u> If $B(M_1) \subseteq M_1$, $B(M_2) \subseteq M_2$, and $M_2 \supset M_1^{\perp}$ then $B(M_1, M_2) = \{0\}$.

Proof: This is obvious from the last lemma.

Next we shall consider the eigenvalue problem for a quadratic operator Q induced by a self-adjoint B.

<u>Theorem 3.2.1</u> Let λ be an eigenvalue of Q corresponding to the eigenelement x, and denote by M the (invariant) subspace spanned by x. Then $M^{\perp}$ is invariant under Q iff

(3.2.1) $$(Bx - \lambda P_M)z = 0 \quad \text{for all} \quad z \in H,$$

where P_M is the orthogonal projection of H onto M.

<u>Proof:</u> First, suppose $B(M) \subseteq M$ and $B(M^{\perp}) \subseteq M^{\perp}$. Observe that, for all $z \in H$

$$z = P_M z + (I - P_M)z$$

where $P_M z \in M$ and $(I - P_M)z \in M^{\perp}$. Hence, by Lemma 3.2.3,

(3.2.2) $$Bxz = Bx(P_M z + (I - P_M)z) = BxP_M z.$$

But, $P_M z = \alpha x$. Thus

(3.2.3) $$BxP_M z = \alpha Bxx = \alpha\lambda x = \lambda P_M z.$$

Now (3.2.2) and (3.2.3) combine to give (3.2.1)

Conversely, assume (3.2.1). Let $z \in M$ to see that $B(M) \subseteq M$. Next, observe that the self-adjointness of B and (3.2.1) yield for $z, w \in M^{\perp}$ and $x \in M$,

$$(Bzw, x) = (w, Bzx) = (w, \lambda P_M z) = (w, 0) = 0.$$

Hence, $B(M^{\perp}) \subseteq M^{\perp}$. The proof is now complete.

Another standard result from the theory of self-adjoint linear operators states that: if the eigenvalues λ_1 and λ_2 are different then the corresponding eigenelements are orthogonal. Once again, the result fails for self-adjoint bilinear operators, as the next example shows.

Example 3. 2. 2 Consider the self-adjoint operator B with matrix (3. 1. 1). A simple calculation shows that $\lambda_2 = \sqrt{2}$ is an eigenvalue corresponding to the eigenvector $x_2 = (1/\sqrt{2}, 1/\sqrt{2})$. But, from Example 3. 2. 1, we have $\lambda_1 = 1$ and $x_1 = (1, 0)$. Hence $\lambda_1 \neq \lambda_2$ and $(x_1, x_2) \neq 0$.

The next theorem shows that Example 3. 2. 2 is the general rule rather than the exception. Even though this appears to be discouraging for obtaining orthogonal sets of eigenelements, it has, in fact, no bearing on the problem. We will clarify this assertion in the next section.

Theorem 3. 2. 2 Let M_1 and M_2 be one-dimensional closed linear subspaces of H (eigenspaces) such that $M_1 \perp M_2$. Let $M = M_1 \oplus M_2$. If M_1, M_2 and $M^{\perp}$ are invariant under Q, then there exists a unique (proper) subspace $N \subset M$ such that $Q(N) \subseteq N$, $N \neq M_1$, and $N \neq M_2$.

Proof: The proof is by construction. Let λ_1 and λ_2 be eigenvalues of Q, and $x_1 \epsilon M_1$ and $x_2 \epsilon M_2$ the corresponding eigenelements. Then

$$(3.2.4) \qquad Qx_1 = \lambda_1 x_1, \qquad Qx_2 = \lambda_2 x \;.$$

Let $y \epsilon M$ such that $y \epsilon M_1$ and $y \notin M_2$. Then y has the unique representation

$$(3.2.5) \qquad y = \alpha x_1 + \beta x_2 \qquad \alpha \neq 0, \; \beta \neq 0.$$

We shall show that scalars α and β can be determined in such a way that

$$(3.2.6) \qquad Byy = \gamma y.$$

It follows from $Q(M^{\perp}) \subseteq M^{\perp}$, $Q(M_2) \subseteq M_2$, and $M_1^{\perp} = M_2 \oplus M^{\perp}$ that $Q(M_1^{\perp}) \subseteq M_1^{\perp}$. Consequently, since $M_2 \subset M_1^{\perp}$, Lemma 3.2.3 shows that $B(M_1, M_2) = \{0\}$. In particular $Bx_1x_2 = 0$. Thus, with the aid of (3.2.4),

$$(3.2.7) \qquad Byy = \alpha^2 Bx_1x_1 + \beta^2 Bx_2x_2 = \alpha^2 \lambda_1 x_1 + \beta^2 \lambda_2 x_2 \;.$$

In order for (3.2.6) to hold, it is necessary that

$$(3.2.8) \qquad \alpha^2 \lambda_1 x_1 + \beta^2 \lambda_2 x_2 = \gamma(\alpha x_1 + \beta x_2).$$

But, x_1 and x_2 are linearly independent in (3.2.8) thus,

(3.2.9) $$\alpha(\alpha\lambda_1 - \gamma) = 0 \text{ and } \beta(\beta\lambda_2 - \gamma) = 0 .$$

But, (3.2.9) and (3.2.5) imply that $\alpha = \lambda_1/\gamma$ and $\beta = \lambda_2/\gamma$. Consequently, (3.2.5) becomes

(3.2.10) $$y = \gamma(x_1/\lambda_1 + x_2/\lambda_2).$$

Hence, (3.2.6) is satisfied.

Now let $w = y/\|y\|$. Then $Qw = \gamma w/\|y\|$, $w \in M$, $w \notin M_1$, and $w \notin M_2$. Therefore, let N be the subspace spanned by w. From this and (3.2.6) it follows that $B(N) \subseteq N$, $N \neq M_1$, $N \neq M_2$, and $N \subset M$. This completes the proof.

<u>Corollary 3.2.1</u> Let y be given by (3.2.10). Then,

(3.2.11) $$Qw = \varepsilon w, \quad \varepsilon = 1/\|x_1/\lambda_1 + x_2/\lambda_2\|,$$

where $w = y/\|y\|$.

<u>Corollary 3.2.2</u> The orthogonal complement $N^{\perp}$ is not invariant under B.

<u>Proof:</u> Let $w \in N$ be defined as in (3.2.11). It suffices to show that $(Bw'w', w) \neq 0$ for some $w' \in N^{\perp}$.

Since $M^{\perp} \subset N^{\perp}$ and $B(M^{\perp}) \subseteq M^{\perp}$ (Thm. 3.3.2) $w' \notin M^{\perp}$. Hence, choose $w' \in M \cap N^{\perp}$ as a candidate. Set $x = w$ in (3.2.1), which is a necessary condition for $B(N^{\perp}) \subseteq N^{\perp}$. Thus, for $z = w'$ in (3.2.1),

$$(3.2.12) \qquad Bww' = \varepsilon P_N w' = \varepsilon(w', w)w.$$

It now follows from (3.2.12) and the self-adjointness of B that

$$(3.2.13) \qquad (Bw'w', w) = (w', Bw'w) = \varepsilon^2[(x_1 w')^2 + (x_2, w')^2] \geq 0.$$

But, (3.2.13) is strictly greater than zero, since $x_1, x_2, w' \in M$ and $(x_1, x_2) = 0$. The proof is now complete.

The next corollary is of paramount interest, especially for Chapter 4.

Corollary 3.2.3 Let λ_1, λ_2, and ε be defined as in Thm. 3.2.2. Then,

$$|\varepsilon| < |\lambda_1| \quad \text{and} \quad |\varepsilon| < |\lambda_2|.$$

Proof: Observe that

$$1/\varepsilon^2 = \| x_1/\lambda_1 + x_2/\lambda_2 \|^2 = \|x_1\|^2/\lambda_1^2 + \|x_2\|^2/\lambda_2^2 = 1/\lambda_1^2 + 1/\lambda_2^2.$$

Hence,

$$|\varepsilon| = (1/\lambda_1^2 + 1/\lambda_2^2)^{-\frac{1}{2}} < (1/\lambda_1^2)^{-\frac{1}{2}} = |\lambda_1|$$

since $\lambda_2 \neq 0$. Interchanging the role of λ_1 and λ_2 yields the second inequality.

§3.3 Existence Theorems

In the previous sections we assumed the existence of eigenvalues. Now we shall remove this assumption by showing that a self-adjoint quadratic operator always possesses at least one eigenvalue and a corresponding eigenelement.

Two existence theorems are given: one for finite-dimensional spaces and another for infinite-dimensional spaces.

Theorem 3.3.1 If Q is a self-adjoint quadratic operator on a finite dimensional Hilbert space H, then there exists at least one eigenvalue λ_1 and a corresponding eigenvector x_1 such that

$$Qx_1 = \lambda_1 x_1, \qquad \lambda_1 = (Qx_1, x_1) .$$

Proof: We define the functional F by

$$F(x) = \frac{(Qx, x)}{\|x\|^3}, \quad \|x\| \neq 0, \tag{3.3.1}$$

and note that it is continuous. Furthermore, it is clear that

$F(x) = (Qx, x)$ on the unit sphere. But the unit sphere is a closed bounded subset of H (finite-dimensional); therefore, F assumes a maximum on $\|x\| = 1$, that is,

$$F(x) \leq F(x_1) \tag{3.3.2}$$

for some x_1 on $\|x\| = 1$.

Let y be an arbitrary vector and t a real number. Let

$$f(t) = F(x_1 + ty). \tag{3.3.3}$$

By (3.3.2) it follows that f assumes a maximum at $t = 0$. Thus, it is necessary that

$$f'(0) = 0. \tag{3.3.4}$$

By (3.3.3) and (3.3.1),

$$f(t) = \frac{(Q(x_1 + ty),\ x_1 + ty)}{\|x_1 + ty\|^3}. \tag{3.3.5}$$

Calculating the derivative of (3.3.5) we have, after applying the self-adjointness of Q and (3.3.4),

$$f'(0) = 3(Qx_1, y) - 3(x_1, Qx_1)(x_1, y) = 0.$$

In other words,

$$(Qx_1 - (Qx_1, x_1)x_1, y) = 0$$

for all $y \in H$. Hence, x_1 is an eigenvector and $\lambda_1 = (Qx_1, x_1)$ is the corresponding eigenvalue of Q.

The existence of eigenvalues for the infinite-dimensional case will be considered after several lemmas. For convenience, let S be the unit sphere.

Lemma 3.3.1 Let B be a bounded bilinear operator (not necessarily self-adjoint). Then,

$$\|B\| = \sup_{x,y,z \in S} |(Bxy, z)|. \tag{3.3.6}$$

Proof: If B is the zero operator (3.3.6) is obvious; therefore, suppose B is nonzero. By (1.2.6) and the Schwarz inequality

$$|(Bxy, z)| \leq \|Bxy\| \, \|z\| \leq \|B\| \, \|x\| \, \|y\| \, \|z\|$$

for all $x, y, z \in H$. Therefore,

$$\sup_{x,y,z \in S} |(Bxy, z)| \leq \|B\| .$$

In the above replace z by $Bxy/\|Bxy\|$ for $x, y \in S$ to obtain $(Bxy, z) = \|Bxy\|$. Then, by (1.2.6)

$$\sup_{x,y,z\in S} |(Bxy,z)| \geq \sup_{x,y\in S} \|Bxy\| = \|B\| .$$

This completes the proof.

Lemma 3.3.2 If B is a self-adjoint bilinear operator on H then $\|B\| = \|Q\|$.

Proof: Note that Bx is a linear operator for all $x \in H$. From this and the self-adjointness of B it follows that

$$\|B\| = \sup_{x,y\in S} \|Bxy\| = \sup_{x\in S} (\sup_{y\in S} \|Bxy\|)$$

$$= \sup_{x\in S} \|Bx\| = \sup_{x\in S} (\sup_{y\in S} |(Bxy,y)|)$$

$$= \sup_{x,y\in S} |(Qy,x)| = \sup_{y\in S} \|Qy\| = \|Q\| .$$

The quantity $\sup_{x\in S} (Qx,x)$ plays a fundamental role in what follows.

We note that $(Q(\alpha x), \alpha x) = \alpha^3 (Qx,x)$. In particular $(Q(-x),-x) = -(Qx,x)$. Thus

$$\sup_{x\in S} (Qx,x) = \sup_{x\in S} |(Qx,x)| \geq 0 .$$

We conjecture that for a self-adjoint Q, $\sup_{x\in S} (Qx,x) = \|Q\|$ $(= \|B\|$ by Lemma 3.3.2). At present this is an open question. However, we can prove the following weaker result.

Lemma 3.3.3 Let B be a self-adjoint bilinear operator on H. Then $B = Q = 0$ iff $(Qx, x) = 0$ for all $x \in S$.

Proof: First note that

$$(Bxy, z) = \sum_{i,j,k} b_{ijk}\, \xi_k \eta_j \zeta_i$$

for any orthonormal basis $\{u_k\}$ (countable or not) where $b_{ijk} = (Bu_k u_j, u_i)$, $\xi_k = (x, u_k)$ etc. Since B is self-adjoint $b_{ijk} = b_{jik} = b_{kji}$. In particular,

$$(Bxx, x) = \sum_{i,j,k} b_{ijk} \xi_k \xi_j \xi_i \, .$$

Suppose now that $(Bxx, x) = 0$ for all $x \in H$. Let $x = u_n$ to get $b_{nnn} = 0$ for all n. Let $x = \alpha u_m + u_n$ to get $b_{mmn} = 0$ and $b_{mnn} = 0$ for all m, n. Finally, let $x = u_\ell + u_m + u_n$ to prove that $b_{\ell mn} = 0$ for all ℓ, m, n. Hence, $B \equiv 0$, since $\|B\| = \|Q\| = 0$. The converse is easy.

Theorem 3.3.2 If Q is a nonzero compact self-adjoint quadratic operator on H, then there exists at least one nonzero eigenvalue λ and a corresponding eigenelement x_0 ($\|x_0\| = 1$) such that

(3.3.7) $$Qx_0 = \lambda x_0, \qquad \lambda = (Qx_0, x_0) = \sup_{x \in S} (Qx, x).$$

Proof: Let

(3.3.8) $$\lambda = \sup_{x \in S} (Qx, x)$$

By the previous results $\lambda > 0$ and

$$(Qx, x) \leq \lambda \|x\|^3,$$

for all $x \in H$. Thus,

(3.3.9) $$(Qx, x)^2 \leq \lambda^2 (x, x)^3 .$$

From (3.3.8) it follows that there exists a sequence $\{x_n : n \geq 1\} \subset S$ such that

(3.3.10) $$\lambda^2 - (Qx_n, x_n)^2 < 1/n^2, \quad \|x_n\| = 1.$$

Without loss of generality $(Qx_n, x_n) > 0$.

Next, let $y_n = Qx_n - \lambda x_n$. Then it follows from what we have just said and from (3.3.10) that

$$|(y_n, x_n)| = |(Qx_n - \lambda x_n, x_n)| = \lambda - (Qx_n, x_n)$$

$$= \frac{\lambda^2 - (Qx_n, x_n)^2}{\lambda + (Qx_n, x_n)} < \frac{\lambda^2 - (Qx_n, x_n)^2}{\lambda} = 0(1/n^2).$$

Hence,

$$(3.3.11) \qquad (y_n, x_n) = 0(1/n^2).$$

Now we shall show that $y_n \to 0$. Replace x by $x_n + y_n/n$ in (3.3.9). Then,

$$(3.3.12) \qquad (Q(x_n + y_n/n), x_n + y_n/n)^2 \leq \lambda^2 (x_n + y_n/n, x_n + y_n/n)^3.$$

The quantity on the left can be rewritten. Express (3.3.12) in terms of the self-adjoint bilinear operator B which induced Q, and apply the properties of the inner product. After some manipulation we obtain

$$(3.3.13) \qquad (Qx_n, x_n)^2 + 6(Qx_n, x_n)(Qx_n, y_n)/n + 0(1/n^2).$$

The right member of (3.3.12) can also be rewritten. The result is

$$(3.3.14) \quad \lambda^2[(x_n, x_n)^3 + 6(x_n, x_n)^2 (y_n, x_n)/n + 0(1/n^2)] \leq \lambda^2 + 0(1/n^2).$$

Therefore,

$$6(Qx_n, x_n)(Qx_n, y)/n \leq \lambda^2 - (Qx_n, x_n)^2 + 0(1/n^2).$$

Apply (3. 3. 10) to the last inequality. Then

$$(Qx_n, x_n)(Qx_n, y_n) = 0(1/n),$$

(3. 3. 15) $$(Qx_n, y_n) = 0(1/n).$$

It now follows from (3. 3. 15) and (3. 3. 11) that

$$0 \leq \| y_n \|^2 = (y_n, y_n) = (Qx_n - \lambda x_n, y_n)$$

$$= |(Qx_n, y_n) - \lambda (x_n, y_n)|$$

$$\leq |(Qx_n, y_n)| + \lambda |(x_n, y_n)| = 0(1/n).$$

Hence,

(3. 3. 16) $$\| Qx_n - \lambda x_n \|^2 = \| y_n \|^2 \to 0 .$$

Since Q is compact the sequence $\{Qx_n\}$ contains a convergent subsequence $\{Qx_{n_i}\}$; furthermore, since $\lambda \neq 0$, $\{x_{n_i}\}$ converges to a limit, say x_0. Thus, by the continuity of Q, $Qx_{n_i} \to Qx_0$. Hence, by (3. 3. 16) $Qx_0 = \lambda x_0$ and by (3. 3. 8) $\lambda = (Qx_0, x_0)$.

Thus we have proved the existence of a positive eigenvalue and corresponding eigenelement of a nonzero compact self-adjoint quadratic operator Q. This is one of our principal results.

In closing this chapter, we note that if λ is any nonzero eigenvalue of Q then

$$Q(\alpha x) = (\alpha\lambda)(\alpha x).$$

Thus, every real number is an eigenvalue. We single out particular eigenvalues by the requirement that the corresponding eigenelements have norm one. This was done, for example, in Thm. 3.3.2.

CHAPTER 4

EXPANSION THEOREM AND APPLICATIONS

§ 4.1 Preliminary Remarks

Here we consider the question of the existence of a maximal orthonormal sequence of eigenelements. As already indicated and as shown in Example 4.4.2, a compact self-adjoint quadratic operator does not necessarily possess even two orthonormal eigenelements. Therefore, it is of fundamental interest to obtain a general existence theorem.

§ 4.2 Existence Theorem

The extremal property of the functional (Qx, x) in § 3.3, can be exploited in order to determine a sequence of eigenvalues $\{\lambda_k\}$. This is done in a manner similar to the linear problem.

First we establish the notation. Denote by λ_k an eigenvalue of the quadratic operator Q, and by x_k a corresponding normalized eigenelement. Denote by M_k the closed linear subspace (eigenspace) spanned by x_k. Let P_k be the orthogonal projection onto M_k. Finally, let $H_1 = H$ and $H_k = (\sum_{i=1}^{k-1} M_i)^{\perp}$ for $k \geq 2$.

Criterion 4.2.1 A (nonzero) compact, self-adjoint quadratic operator Q possesses a sequence of eigenvalues $\{\lambda_k\}$ and a corresponding orthonormal system of eigenelements $\{x_k\}$ iff for each k

$$(4.2.1) \qquad (Bx_k - \lambda_k P_k)z = 0 \quad \text{for all} \quad z \in H_k$$

where B is the bilinear operator corresponding to Q.

Proof: Suppose (4.2.1) is satisfied for each k. We will show that the sequence $\{\lambda_k\}$ and the orthonormal system $\{x_k\}$ can be determined successively.

According to Thm. 3.3.2, for a nonzero Q, there exists a nonzero eigenvalue λ_1 and a corresponding normalized eigenelement x_1. Since (4.2.1) holds for $k = 1$, Thm. 3.2.1 shows that $H_2 = M_1^{\perp}$ is invariant under B, and hence under Q. Therefore, M reduces Q, and we may regard Q as acting on M_1 and H_2 separately.

Next, consider the restriction of Q to H_2. Since Q is compact and self-adjoint on $H_1 = H$, it is compact and self-adjoint on $H_2 \subset H_1$. But, H_2 is a Hilbert space in its own right. Thus, we may replace H_1 with H_2 and repeat the arguments. If $\lambda_2 = 0$ then Lemma 3.3.3 shows that Q is the zero operator on H_2, and the process terminates. If, on the other hand, $\lambda_2 \neq 0$

then a normalized eigenelement x_2 exists; furthermore, it is clear that $(x_1, x_2) = 0$. Induction provides the remaining eigenvalues and eigenelements. The converse is immediate from Thm. 3.2.1. This completes the proof.

We remark in passing that we have avoided the nonorthogonal eigenelements of Corollary 3.2.1. This is always possible because the eigenvalues are determined by successively solving an extremal problem, and because the eigenvalues of the nonorthogonal eigenelements satisfy Corollary 3.2.3. Nevertheless, we must not assume that all compact self-adjoint quadratic operators give rise to orthonormal systems. A simple example will suffice.

<u>Example 4.2.1</u> As in Ex. 3.1.1 we define the matrix of the bilinear operator B on the real Hilbert space ℓ_2^2 by

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{array}\right) \quad . \tag{4.2.2}$$

This is a compact self-adjoint operator. Furthermore, it is easy to see that the following are eigenvalues and normalized eigenvectors of the induced quadratic operator Q:

$$\begin{array}{ll} \lambda_1 = -1 & x_1 = (1/2, \sqrt{3}/2) \\ \lambda_2 = 1 & x_2 = (-1/2, \sqrt{3}/2) \\ \lambda_3 = 1 & x_3 = (1, 0) \quad . \end{array} \tag{4.2.3}$$

But, $\{x_k\}$ does not contain an orthogonal pair.

Corollary 4.2.1 The sequence of eigenvalues determined by Criterion 4.2.1 has the property that, for each k

$$|\lambda_{k+1}| \leq |\lambda_k| . \tag{4.2.4}$$

Proof: From $H_{k+1} \subset H_k$ and Thm. 3.3.2 it follows that

$$|\lambda_{k+1}| = \sup_{x \in H_{k+1} \cap S} |(Qx, x)| \leq \sup_{x \in H_k \cap S} |(Qx, x)| = |\lambda_k| .$$

As in the case of linear operators we obtain by induction a finite or infinite sequence of eigenvalues. If the sequence does not terminate then we have a familiar result.

Theorem 4.2.1 If the compact self-adjoint quadratic operator Q has an infinite sequence of nonzero eigenvalues $\{\lambda_k\}$ corresponding to the orthonormal system $\{x_k\}$, then

$$\lim_{k \to \infty} \lambda_k = 0. \tag{4.2.5}$$

Proof: The proof is by contradiction. Assume that the nonincreasing sequence $\{\lambda_k\}$ converges to a number $c > 0$. Observe for $m \neq n$

$$(4.2.6) \qquad \| Qx_n - Qx_m \|^2 = \| \lambda_n x_n - \lambda_m x_m \|^2 = \lambda_n^2 + \lambda_m^2 \geq 2c^2.$$

Thus $\{Qx_n\}$ does not contain a convergent subsequence, which contradicts the compactness.

Before proceeding to the next theorem we recall several familiar relations.

Let $\{x_k\}$ be any finite or infinite orthonormal set. Then, for all $x \epsilon H$, Bessel's inequality is

$$(4.2.7) \qquad \sum_k (z, x_k)^2 \leq \| z \|^2.$$

If $\{x_k\}$ is maximal then we have the Parseval relation,

$$(4.2.8) \qquad \sum_{k=1}^{\infty} (z, x_k)^2 = \| z \|^2.$$

In particular, (4.2.7) and (4.2.8) apply when the orthonormal system consists of eigenelements of Q.

<u>Theorem 4.4.2</u> Let Q be a compact self-adjoint quadratic operator on H. Then, for any $z \epsilon H$,

$$(4.4.9) \qquad Qz = \sum_k \lambda_k (z, x_k)^2 x_k$$

iff Criterion 4.2.1 is satisfied. The series may have a finite or infinite number of terms.

Proof: Suppose (4.2.1) is satisfied. Then a sequence $\{\lambda_n\}$ and an orthonormal system $\{x_n\}$ exist.

First, consider the case of only a finite number of nonzero eigenvalues. Let $M = \sum_{k=1}^{n} M_k$ where M_k is the (closed) linear subspace spanned by x_k. Let B denote the bilinear operator corresponding to Q.

Let $z \in H$ be arbitrary. Then z has the unique representation

(4.2.10) $$z = x + y, \qquad x \in M, \; y \in M^{\perp}.$$

Furthermore, since $x \in M$ and $\{x_k\}$ is orthonormal,

(4.2.11) $$x = \sum_{k=1}^{n} a_k x_k$$

where

(4.2.12) $$a_k = (x, x_k) = (z-y, x_k) = (z, x_k).$$

From Criterion 4.2.1 and Lemma 3.3.3 the restriction of Q to $M^{\perp}$ is the zero operator; thus, $Qy = 0$ for all $y \in M^{\perp}$. In particular,

(4. 2. 13) $$Qy = Q(z-x) = Qz - 2Bzx + Qx = 0.$$

By (4. 2. 1) and (4. 2. 11),

(4. 2. 14) $$Bzx = Bz(\sum_{k=1}^{n} \alpha_k x_k) = \sum_{k=1}^{n} \alpha_k Bzx_k = \sum_{k=1}^{n} \alpha_k \lambda_k P_k z = \sum_{k=1}^{n} \lambda_k \alpha_k^2 x_k,$$

(4. 2. 15) $$Qx = Bx(\sum_{k=1}^{n} \alpha_k x_k) = \sum_{k=1}^{n} \alpha_k Bxx_k = \sum_{k=1}^{n} \alpha_k \lambda_k P_k x = \sum_{k=1}^{n} \lambda_k \alpha_k^2 x_k .$$

It is now evident, from (4. 2. 13)ff, that

(4. 2. 16) $$Qz = \sum_{k=1}^{n} \lambda_k \alpha_k^2 x_k, \quad \alpha_k = (z, x_k),$$

for all $z \in H$. Thus, (4. 2. 16) gives expansion (4. 2. 9) when the sequence $\{\lambda_k\}$ has only n elements.

Next consider the case of an infinite sequence $\{\lambda_k\}$ of nonzero eigenvalues.

Let $M^p = \sum_{k=1}^{p} M_k$, $x^n \in M^n$, $y^n \in (M^n)^{\perp}$, $x^m \in M^m$, and $y^m \in (M^m)^{\perp}$ where $m > n$. Then, as in (4. 2. 10), $z = x^n + y^n$ and $z = x^m + y^m$. By (4. 2. 14) and (4. 2. 15),

(4. 2. 17) $$Qy^n - Qy^m = \sum_{k=n+1}^{m} \lambda_k a_k^2 x_k .$$

Now it follows from (4. 2. 4) and (4. 2. 7) that

(4. 2. 18) $$\| Qy^n - Qy^m \|^2 = \sum_{k=n+1}^{m} \lambda_k^2 a_k^4 \leq \lambda_{n+1}^2 \sum_{k=n+1}^{m} a_k^4 \leq \lambda_{n+1}^2 \| z \|^4 .$$

Thus $\| Qy^n - Qy^m \|^2 \to 0$ as $m, n \to \infty$. And, since H is complete, $\{ Qy^n \}$ converges to some element in H. We shall show that this element is the zero element.

Observe with the aid of (4. 2. 1) that, for each j,

$$(Qy^n, x_j) = (Q(z - x^n), x_j) = (Qz, x_j) - \sum_{k=1}^{n} \lambda_k a_k^2 (x_k, x_j) = (z, Bzx_j) - \lambda_j a_j^2$$

$$= (z, \lambda_j P_j z) - \lambda_j a_j^2 = 0.$$

Thus, since $\{ Qy^n \}$ converges and $\{x_k\}$ is maximal, it follows that $Qy^n \to 0$ as $n \to \infty$. Therefore,

(4. 2. 19) $$\| Qy^n \|^2 = \| Q(z - x^n) \|^2 = \| Qz - \sum_{k=1}^{n} \lambda_k a_k^2 x_k \|^2 \to 0$$

as $n \to \infty$. Hence,

(4. 2. 20) $$Qz = \sum_{k=1}^{\infty} \lambda_k a_k^2 x_k , \quad a_k = (z, x_k).$$

Therefore, expansion (4. 2. 10) is verified.

Conversely, if expansion (4. 2. 10) exists, where $\{x_k\}$ is orthonormal, then (4. 2. 1) is immediate from Thm. 3. 2. 1.

Corollary 4. 2. 2 Expansion (4. 2. 10) may be written in the form:

(4. 2. 21)
$$Qz = \sum c_k x_k, \qquad c_k = (Qz, x_k).$$

Proof: From 4. 2. 1,

$$(Qz, x_k) = (z, Bzx_k) = (z, \lambda_k P_k z) = (z, \lambda_k (z, x_k) x_k) = \lambda_k \alpha_k^2 .$$

Thus,

(4. 2. 22)
$$c_k = (Qz, x_k) = \lambda_k \alpha_k^2, \quad \alpha_k = (x, x_k).$$

In view of (4. 2. 21), it is tempting to call the numbers c_k — the Fourier coefficients of the quadratic "function" Qz .

Corollary 4. 2. 3 Let x and y be arbitrary elements of H. If expansion (4. 2. 10) holds for Q then

(4. 2. 23)
$$Bxy = \sum_k \lambda_k \alpha_k \beta_k x_k, \qquad \alpha_k = (x, x_k), \quad \beta_k = (y, x_k).$$

Proof: Apply (4. 2. 10) to the right member of (1. 1. 6) and simplify.

§ 4.3 Quadratic Integral Equations

The previous theory provides a means for solving certain nonlinear integral equations. Consider the equation for $x(s)$,

$$(4.3.1) \qquad x(s) = y(s) + \int_0^1 \int_0^1 K(s,t,u)x(u)x(t)dudt.$$

The integral operator

$$(4.3.2) \qquad (Qx)(s) = \int_0^1 \int_0^1 K(s,t,u)x(u)x(t)dudt$$

is quadratic. Assume the conditions of Examples (2.3.2), (2.2.2), and (1.2.6). Then Q is a compact self-adjoint quadratic operator on $L_2(0,1)$. Let us write (4.3.1) in the form

$$(4.3.3) \qquad x = y + Qx .$$

Suppose that Q satisfies Criterion 4.2.1. Then, for any $x \in L_2(0,1)$, Qx may be expanded in terms of its eigenvalues and eigenfunctions in accordance with (4.2.10).

Denote the eigenvalues by λ_k and the normalized eigenfunctions by x_k. Then, from (4.2.22) and (4.3.3),

$$(4.3.4) \qquad (x-y, x_k) = (Qx, x_k) = \lambda_k \alpha_k^2, \quad \alpha_k = (x, x_k).$$

On the other hand, if we let $\beta_k = (y, x_k)$ then

(4.3.5) $$(x-y, x_k) = (x, x_k) - (y, x_k) = \alpha_k - \beta_k .$$

Thus, by (4.3.4) and (4.3.5),

(4.3.6) $$\lambda_k \alpha_k^2 - \alpha_k + \beta_k = 0.$$

We may now solve (4.3.6) for α_k in terms of the known quantities λ_k and β_k. This produces two possible values of α_k, namely,

$$\alpha_k^+ = (-1 + \sqrt{1-4\lambda_k \beta_k})/2\lambda_k,$$

$$\alpha_k^- = (-1 - \sqrt{1-4\lambda_k \beta_k})\ /2\lambda_k .$$

Let α_k denote either value, α_k^+ or α_k^-. Then, using (4.2.10), we may write

(4.3.7) $$x = y + \sum_{k=1}^{\infty} \lambda_k \alpha_k^2 x_k \qquad \alpha_k = \alpha_k^+ \text{ or } \alpha_k^- .$$

Since α_k can be calculated from known quantities (4.3.7) is a solution of the quadratic integral equation (4.3.2) provided, upon substitution, it satisfies (4.3.3). We will show that this is indeed the case.

Let $c_k = \lambda_k a_k^2$ in (4.3.7). Then, by (4.3.7),

$$(4.3.8) \qquad Qx = Q(y + \sum_{k=1}^{\infty} c_k x_k)$$

$$= Qy + 2By\,(\sum_{k=1}^{\infty} c_k x_k) + Q(\sum_{k=1}^{\infty} c_k x_k).$$

Since $y \in L_2(0, 1)$ we may expand Qy using (4.2.10), i.e.

$$(4.3.9) \qquad Qy = \sum_{k=1}^{\infty} \lambda_k (y, x_k)^2 x_k = \sum_{k=1}^{\infty} \lambda_k \beta_k^2 x_k .$$

Also, with the aid of (4.2.1),

$$(4.3.10) \qquad By\,(\sum_{k=1}^{\infty} c_k x_k) = \sum_{k=1}^{\infty} c_k Byx_k = \sum_{k=1}^{\infty} \lambda_k c_k \beta_k x_k ,$$

and

$$(4.3.11) \qquad Q(\sum_{k=1}^{\infty} c_k x_k) = \sum_{k=1}^{\infty} c_k^2 Qx_k = \sum_{k=1}^{\infty} \lambda_k c_k^2 x_k .$$

Substituting (4.3.9)ff into (4.3.8) we have

$$(4.3.12) \qquad Qx = \sum_{k=1}^{\infty} \lambda_k (\beta_k + c_k)^2 x_k = \sum \lambda_k a_k^2 \, x_k ,$$

where the last equality follows from (4.3.6) and the fact that $\beta_k + c_k = \beta_k + \lambda_k a_k^2$. Thus,

$$Qx = \sum_{k=1}^{\infty} \lambda_k a_k^2 x_k = x-y$$

which shows that (4.3.7) satisfies (4.3.3) and thus solves (4.3.1).

§ 4.4 Quadratic Differential Equations

In this section we define a quadratic self-adjoint differential operator. The operators in this section are in general unbounded. A complete theory is as yet undeveloped.

First, we recall the form of a linear self-adjoint differential operator; that is, an operator of the form

$$\text{(4.4.1)} \qquad Lx = (p(t)x'(t))' - q(t)x(t)$$

where p and q are certain real valued functions. It is assumed that the domain of the operator is a suitable subspace of $L_2(a,b)$, such as the twice continuously differentiable functions. Also, q is assumed to be continuous and p continuously differentiable. Finally, the boundary conditions are of the form:

$$\alpha_1 x(a) + \alpha_2 x'(a) = 0; \qquad \beta_1 x(b) + \beta_2 x'(b) = 0 \ .$$

Under these circumstances L is self-adjoint, that is,

$$(Lx, y) = (x, Ly)$$

for all x and y in the domain of L. We seek an analogous result for quadratic operators.

Let x, y, and z be twice continuously differentiable functions of t, and let $\alpha = \alpha(t)$ be continuous and $\beta = \beta(t)$, $\gamma = \gamma(t)$, and $\delta = \delta(t)$ be continuously differentiable. Thus, as in the linear case, we confine our attention to a suitable subspace of $L_2(a, b)$.

We define the bilinear operators A_1 and A_2 as follows:

$$A_1[x, y] = \delta x' y' + \gamma(xy)' + \beta xy, \tag{4.4.2}$$

$$A_2[x, y] = \gamma x' y' + \beta(xy)' + \alpha xy. \tag{4.4.3}$$

Now, consider the expression

$$T(x, y, z) = A_1[x, y] z' + A_2[x, y] z . \tag{4.4.4}$$

Observe that $T(x, y, z)$ is symmetric in all arguments. Hence,

$$\int_a^b T(x, y, z)dt - \int_a^b T(x, z, y)dt = 0. \tag{4.4.5}$$

But, upon integrating by parts we have

$$(4.4.6)\qquad \int_a^b T(x,y,z)dt = A_1[x,y]\,z\,\Big|_a^b - \int_a^b (A_1'[x,y] - A_2[x,y])z\,dt$$

$$(4.4.7)\qquad \int_a^b T(x,z,y)dt = A_1[x,z]y\,\Big|_a^b - \int_a^b (A_1'[x,z] - A_2[x,z])y\,dt$$

where the prime denotes differentation with respect to t. It is clear, from (4.4.5) ff, that

$$(4.4.8)\qquad \int_a^b [y(Bxz)-(Bxy)z]\,dt = A_1[x,z]\,y - A_1[x,y]\,z\,\Big|_a^b$$

where the bilinear operator B is defined by

$$(4.4.9)\qquad Bxy = A_1'[x,y] - A_2[x,y]\ .$$

It now follows from (4.4.3) and

$$(4.4.10)\qquad A_1'[x,y] = (\delta x'y')'+(\gamma(xy)')'+\beta'xy+\beta(xy)'\ ,$$

that B is symmetric. Furthermore, the left member of (4.4.8) is simply

$$(4.4.11)\qquad (y, Bxz) - (Bxy, z).$$

Thus, if the boundary conditions are chosen so that the right member (4.4.8) is zero, then

$$(Bxy, z) = (y, Bxz). \tag{4.4.12}$$

Thus we may say that B is self-adjoint.

The analogy with the linear problem becomes apparent when we consider the induced quadratic operator Q. First, we rewrite (4.4.9) using (4.4.10) and (4.4.3) in the form:

$$Bxy = (\delta x'y')' - \gamma x'y' + (\gamma(xy)')' + (\beta' - \alpha)xy \ . \tag{4.4.13}$$

Next, let $p(t) = \delta(t)$, $q(t) = \gamma(t)$, $r(t) = \alpha(t)$, and $\lambda s(t) = \beta'(t)$. Then, we may write a homogeneous quadratic differential equation from (4.4.13) in the form:

$$(p(t)(x')^2)' - q(t)(x')^2 + (q(t)(x^2)')' + (\lambda s(t) - r(t))x^2 = 0 \ . \tag{4.4.14}$$

Suppose the boundary conditions are such that the right member of (4.4.8) vanishes. Then,

$$Qx = (p(t)(x')^2)' - q(t)(x')^2 + (q(t)(x^2)')' - r(t)x^2 \tag{4.4.15}$$

is the analog of (4.4.1). We may also introduce the parameter λ of (4.4.14) and write

$$Qx = -\lambda\, s(t)\, x^2 \ . \tag{4.4.14}$$

This is a homogeneous eigenvalue problem for quadratic operators.

Our purpose here is to show that the concept of self-adjointness is applicable to a class of nonlinear differential operators. The operator of (4.4.15) shows that this is indeed the case. A complete analysis of this problem awaits further study.

CHAPTER 5

GENERALIZATIONS

§ 5.1 Summary

Our principal goal was to obtain an eigenfunction expansion for certain quadratic and bilinear operators acting in a Hilbert space. Expansions (4. 2. 10) and (4. 2. 23) are the final realizations of this goal. Looking at the expansion (4. 2. 10) for a quadratic operator Q we see that it is the analog of the familiar Hilbert expansion (1. 3. 5) for self-adjoint linear operators. As such, we recognized the possibility of applying it to quadratic integral equations. With this in mind, we were able to present a formal solution for a quadratic integral equation which is an analog of a Fredholm equation of the second kind. Moreover, an immediate observation of these results will show that, in special cases, the number of solutions may be determined. In particular, if the range of the operator is finite-dimensional, say n-dimensional, then there are precisely 2^n solutions according to (4. 3. 7).

Several new ideas were necessary before the expansion for quadratic operators could be written. In particular, the concept of a self-adjoint bilinear operator was essential. As we have seen, this concept turned out to be a very natural extension of the concept of self-adjointness for linear operators.

In spite of the analogy between the bilinear and linear operators, it was necessary, in Chapter 3, to pursue an independent course. In this case, we found that the nonlinearity of the operator introduced new invariant subspaces. In particular, we saw under appropriate conditions that in the subspace spanned by any two orthogonal invariant subspaces of a bilinear operator there was always another, which was uniquely defined. However, due to the nature of these subspaces, and also to the nature of the functional (Qx, x) we were still able to determine an orthonormal system of eigenelements, and thus the eigenfunction expansion.

Finally, even though a quadratic operator is compact and self-adjoint there may not exist an orthonormal system of eigenelements as shown by Example 4.2.1. Thus, Criterion 4.2.1 is of paramount concern since it provides the necessary and sufficient conditions for such a system to exist. This, of course, raises some questions as to the nature of the self-adjoint quadratic operators which do not possess an orthonormal system. But this is only one of many questions we may raise. For, in general, one may raise any question which already applies to the theory for linear self-adjoint operators.

§5.2 Generalizations

Much of the theory could have been carried out for more general nonlinear operators. This, of course, requires a further extension of the concept of self-adjointness. We give an informal definition next.

Let T be a trilinear operator on H. We define the operators T_k $(k = 1, 2, 3)$ by the following relations: for all $x, y, z \in H$, let

$$T_1xyz = Txyz; \qquad T_2xyz = Tyxz; \qquad T_3xyz = Tzyx$$

which is similar to (1.3.6). Thus, we say that T is symmetric iff $T_1 = T_2 = T_3$. Following the procedure in Definition 1.3.1, we say that T is self-adjoint iff T_kx, acting as a bilinear operator, is self-adjoint for all $x \in H$. Thus, we could proceed inductively to define self-adjoint multilinear operators on H. Hopefully, under appropriate conditions this would give rise to expansions similar to (1.3.18). For instance, for a cubic operator we could expect an expansion of the form:

$$Tx = \sum_k (Tx, x_k)x_k = \sum \lambda_k (x, x_k)^3 x_k \, .$$

In view of this, generalizations to other multilinear operators is obvious. However, passage to a general nonlinear operator is a question of a different magnitude.

We now turn to a different topic. It is clear that the linear and quadratic self-adjoint operators can be combined in the form:

$$Qx + Lx .$$

In fact, the integral equation of §4.3 is of this form with L equal to the identity. Thus we could consider self-adjoint operator equations of the form

$$Qx + Lx = y , \tag{5.2.1}$$

where y is a given element of H. In view of the quadratic and linear differential operators of §4.4, this means that the eigenvalue problem for a homogeneous differential equation of form 5.2.1 (with $y = 0$) could be written as:

$$Qx + Lx = \lambda\,(a_2(t)x^2 + a_1(t)x), \tag{5.2.2}$$

where $a_1(t)$ and $a_2(t)$ are known functions. Generalizations to multilinear operators are implicit in (5.2.1) and (5.2.2).

Finally, we consider one more area of vital interest. We present this as a conjecture.

Let the quadratic operator Q possess a maximal orthonormal sequence of eigenelements $\{x_k\}$ and a corresponding sequence of eigenvalues. Then, the conjecture is that Q has a spectral representation of the form

$$Q = \sum_k \lambda_k P_k$$

where the P_k are bilinear "projections" rather than linear projections (for example, motivated by (4. 2. 23), $P_k xy = (x, x_k)(y, x_k)x_k$). It is also conjectured that the P_k are self-adjoint; that they are orthogonal in the sense that $P_j P_k$ is the zero (trilinear) operator for $j \neq k$; and finally, that $\| P_k \| = 1$ for each k.

Other questions will undoubtedly occur to the reader. It is hoped that we have opened up as many fruitful areas of investigations as we have analyzed.

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