A NEW MATHEMATICAL THEORY OF DETECTION BY ELECTRONIC INSTRUMENTS

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BACKGROUND

In any communication system the detection of a received signal is made difficult by the presence of system and presystem noise. In recent years the necessity for communicating under extreme physical conditions, the demands for greater accuracy in the handling of information, and such other factors as the need for engineering economy in design have each contributed to the importance of optimum detection of weak signals in the presence of background noise. The evaluation of system performance is again more complex under these conditions because the random nature of the noise, and frequently signal, make impossible an analytical description and force a statistical approach.

In general, the statistical analysis of electronic circuitry results in equations that describe the statistical behavior of the output as completely as possible. The input to the system is a desired signal which may contain both random and nonrandom components plus the unwanted background noise. The system response depends upon the input and in particular upon the character of the random process involved. The response further depends upon the system itself, which may be linear or nonlinear. Each of these conditions must be specified if the performance of a given system is to be evaluated. When the system is linear and the random input is Gaussian a complete statistical description is always possible (7, p. 6). However if a nonlinearity is involved or the input is not Gaussian, only certain classes of problems can be solved and these will be solved by approximation[1].

Electronic detection systems generally contain a predetection filter, whose function is to enhance the strength of the signal relative to that the noise gives. A post-detection filter also may be employed to good advantage. At one extreme the system may be specified and the problem is one of analysis to determine the performance under a given set of conditions. At the other extreme the problem may be one of synthesis in which an optimum system is to be composed subject to some criterion of goodness; again, the problem may contain a mixture of these elements, requiring some synthesis and some analysis. In any event, research efforts in this field fall readily into two broad classifications accordingly as they are strongly probabilistic or weakly probabilistic in character, referred to

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1. These remarks assume the random process involved is stationary with time, a valid assumption in a majority of the physical situations of interest.

as probabilistic or nonprobabilistic.

Regardless of the specific form of the detection system, some criterion must be established which allows one to make a decision that a signal is present. In a pulsed radar, for example, it is customary for purposes of analysis to consider the returned pulses of energy as samples taken from a hypothetical continuous sine wave. The random noise is of course present as background. In the simplest form a decision regarding the presence of signal might be based on a single sample. A more realistic approach is that of taking into account the concept of the integration of N pulses before a decision is reached. In this spirit J. I. Marcum (15, pp. 1-114) has studied the statistical effect of receiver noise applied to radar range performance. Marcum's work has been extended by W. M. Stone to include the effect of a randomly modulated carrier (22, pp. 935-939) and later to include also integration of N pulses (21, pp. 1543-1548).

These efforts are probabilistic in nature, ultimately requiring determination of the probability distribution of signal plus noise. Specifically, the criterion of detection is that the envelope of signal plus noise shall exceed a predetermined bias level x_0, or, in the case of integration, that the sum of N samples of the envelope shall exceed a certain bias level. It is thus required to find the probability density function f(x) associated with the appropriate variable from which the probability of detection is

$$P(x > x_0) = \int_{x_0}^{\infty} f(x)dx. \tag{1.1}$$

The choice of bias level x_0 depends on the prescribed false alarm time, the average time interval between false signal indications in the absence of signal.

The Marcum process of integration applies to an ideal integrating circuit device which actually performs the addition without weighting the N original variates in any way. From this point of view the process is somewhat artificial. However it is significant that an electronic filter is itself an integrating device, and although the integration is not performed in the manner described, it has been possible to establish a correlation between Marcum's N and filter bandwidths for certain systems (5, p. 1175 and 23, p. 50).

Another probabilistic approach is that of M. Kac and A. J. F. Siegert (10, pp. 383-397) and later R. C. Emerson (5, pp. 1168-1176). The physical model employed here is shown in Figure 1. This system is of great practical importance since the conventional superheterodyne receiver is of just this type. The intermediate frequency amplifier constitutes the first filter while the audio

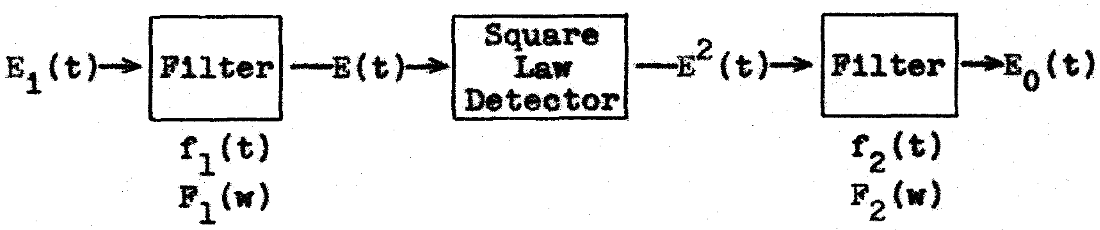

Figure 1. Block diagram of detection system.

(or video) amplifier constitutes the second filter. The square law detector is of course the so-called second detector in the receiver. While the filters thus defined could be assumed linear, the system of figure 1 is nonlinear because of the presence of the detector. This nonlinearity followed by a filter makes the analysis of this system extremely difficult, and approximation techniques are required in all but special cases. Kac and Siegert were able to obtain an exact expression for the first probability distribution of the output voltage of this system. Their expression, however, depends on inverting a somewhat complicated characteristic function whose explicit form further depends on determination of the eigenvalues and eigenfunctions of the integral equation

$$\int_0^{\infty} K(t)r(s-t)f(t)dt = \lambda f(s), \tag{1.2}$$

where $r(T)$ is the Fourier transform of the intermediate frequency power spectrum and $K(t)$ is the response frequency function of the audio filter. The difficulties of obtaining useful results arises, first because inversion of the characteristic function of the output may not be possible in closed form, and second because the integral equation can be solved only in special cases. Kac and Siegert were able to obtain a solution of (1.2) for the noise only special case of a single tuned intermediate frequency amplifier and a simple low pass filter for the audio amplifier. However, a closed form expression for the output probability distribution was possible for only one particular ratio of audio bandwidth to intermediate frequency bandwidth.

Emerson, recognizing these disadvantages, outlined a procedure for approximating the probability density function of the output directly, thus avoiding both the eigenvalue problem and inversion of the characteristic function. The method depends on use of one of the orthonormal systems for expanding density functions (Gram-Charlier or Laguerre) and takes advantage of relations existing between the cumulants of the output distribution and the system kernel. This procedure will be discussed more in detail in a following chapter.

Emerson applied his technique to the system of Figure 1 with the filters each specified to have Gaussian pass characteristics. Employing the same general approach, W. M. Stone and Robert L. Brook (23, pp. 1-71) extended this work to include more general filter combinations as well as to include the concept of a slowly varying random input signal. In each the criterion of detection is that of equation (1.1), often referred to as the threshold detector.

Of the probabilistic approaches to the detection problem, the one drawing most heavily from mathematical statistics is that which utilizes the classical concept of hypothesis testing. Here it is recognized that to determine the presence or absence of a signal in noise is statistically equivalent to testing the hypothesis H_0 that noise alone is present against the alternative hypothesis H_1 of a signal and noise. In practice the decision is necessarily based on a finite amount of information and hence is subject to two kinds of error, known to statisticians as type I and type II errors. In the type I error H_0 is rejected when it is in fact true, that is, noise is called a signal when actually noise alone is present. If H_0 is accepted when H_1 is really true, that is, signal is called noise alone when signal is present, the error is of type II. The problem then is to determine whether a sample of a finite number of values of incoming data arises from the probability distribution associated with H_0 or from that associated with H_1.

Although not the earliest, the most comprehensive paper applying the methods of statistical decision theory is that by D. Middleton (16, pp. 372-391). Middleton is concerned with optimum detection of constant amplitude pulsed carriers in narrow-band random noise. In considerable generality he develops several types of optimum tests of a statistical hypothesis based on three types of observer: The Neyman-Pearson, the ideal, and the sequential observer (16, pp. 372-373). In reaching a decision, the observer in effect makes a "bet", which in the limiting case of an infinite number of repeated trials has a definite probability of faulty decision. The process of observation is then specified by the way in which the observer uses the data to make his bet.

The observer is to decide on the basis of n observations of the envelope of the narrow-band noise and pulsed carrier, whether or not a signal is actually present. Again the two types of error, I and II with probabilities of occurrence denoted by P and Q respectively, are possible. The problem is to test the statistical hypothesis H_0 against the alternative hypothesis H_1. A decision can then be reached by any one of the following three observation processes (16, p. 373).

The probability P and the number of observations n are fixed, and the probability Q of a type II error is minimized. This is the Neyman-Pearson test of stastical hypothesis H_0 against H_1 and an observer operating in this way is called a Neyman-Pearson observer by Middleton. Significantly, other research workers have also applied the Neyman-Pearson criterion to the question of optimum detection and filtering, notably Thomas Slattery (20, pp. 1232-1236) and M. Schwartz (19, p. 3).

With the ideal observer, the observation space for the n samples is divided such that for fixed n the probability of a correct decision is maximized for all possible signal strengths. Thus the sum (P + Q) is minimized. The concept of the ideal observer was originally introduced and applied to radar problems by A. J. F. Siegert (13, pp. 167-173).

The third process, suggested for the first time in connection with electronic detection processes by Middleton, rest squarely on the methods of sequential analysis introduced by A. Wald (27, pp. 1-212). In this process both the probabilities P and Q are fixed while the number of observations n remains adjustable. The observations are continued until certain limits that depend on P and Q are exceeded for the first time. At this point the hypothesis H_0 is accepted or rejected on the basis of this test. An observer following this procedure is called by Middleton a sequential observer.

Employing these concepts Middleton (16, p. 372) is able to uniquely define a minimum dectable signal. For each of the three observers it is verified that the best second detector is a half-wave nonlinear device which is closely approximated by the usual half-wave linear detector for suitable signal levels (16, p. 377). Slattery, on the other hand, applies the Neyman-Pearson procedure to predict a best nonlinear filter, which is defined once the required probability of detection and tolerable false alarm probability are specified (20, pp. 1233-1234).

In the probabilistic approaches the mathematics is greatly simplified if it is assumed that the data samples are statistically independent. This assumption generally has been made, and the performance is then described in terms of the first probability density functions of the output signal and noise. The realization that this assumption is not valid in all cases has been recognized by various writers[2] (16, pp. 371-372 and 20, p. 1234). The great advantage of the probabilistic over the nonprobabilistic approach is that the former represents a mathematical model capable of including the sample dependency case when it is important, although it is very difficult to obtain the required higher order joint density functions for random processes other than Gaussian.

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2. These questions have been discussed in general terms by this writer in a mathematical note written in connection with Boeing Airplane Company Contract No. 460-39-98.

The nonprobabilistic approach is inherently simpler to handle mathematically. Less statistical information is required, generally a knowledge of the correlation function and spectral density being sufficient. The probability density function associated with noise and signal is not explicitly evaluated. The sample dependency question is avoided through the use of mathematical tools that are completely derived from the second order joint density function for signal and noise. In this sense, this method is not as efficient as the probabilistic method; however, it is not overly complex to use and in many cases includes all the information that is required.

In an early paper on determination of optimum filters, D. O. North (17, pp. 1-27) bases performance on the criterion of maximization of the predetection signal to noise ratio. The noise is considered to be purely random white noise and a postdetection filter is not included in the system. J. H. Van Vleck and D. Middleton (26, pp. 943-944) obtained essentially the same result as North.

The results of correlation analysis have been applied directly by Y. W. Lee, T. P. Cheatham, and J. B. Wiesner (14, pp. 1165-1171) to improve the signal to noise ratio in the detection process. Here too, a postdetection filter was not considered. Also the criterion for best is again maximization of signal to noise ratio. More recently L. A. Zadeh and J. R. Ragozzini (31, pp. 1223-1231) have employed correlation analysis with somewhat different criteria of performance to optimize filters used in the detection process.

These writers and others employing the nonprobabilistic approach have been concerned primarily with optimum filters subject to certain criteria of performance. Such efforts, based on the correlation function of input signal and noise, can lead only to linear filters as optimum. However, it has been proved formally by H. Bode and Claude Shannon (1, p. 424) that linear filters are optimum of all classes of filters when the input signal and noise are Gaussian, a situation that exists in many practical cases. Furthermore, nonlinear filters, as prescribed by optimization based on higher order density functions, are difficult to mechanize. Also, the higher order density functions are readily attainable only for Gaussian processes. For these reasons electronic circuits used in practice generally employ linear filters.

There is then considerable justification for analyzing the performance of electronic detection circuits, such as that of Figure 1, which utilize only nonlinear filters by the nonprobabilistic method. In this thesis the performance of the circuit of Figure 1 is analyzed and subsequently a detection criterion is established which is based on the expected number of crossings per second of an arbitrary bias line by the output signal;

the application of this method requires only a knowledge of the correlation function of the output voltage $E_0(t)$. This nonprobabilistic approach will be referred to as the crossing theory.

In the classic paper, "Mathematical Analysis of Random Noise", S. O. Rice (18, pp. 57–63) develops a formula for the expected number of zeros per second of a normally distributed stationary process,

$$(1.3) \qquad N_0 = 2\left\{\frac{\int_0^\infty f^2 A(f)df}{\int_0^\infty A(f)df}\right\}^{\frac{1}{2}},$$

where A(f) is the power density spectrum (or spectral density) associated with the random variable and f is the frequency. As an example, Rice applied this to an isolated ideal band pass filter with Gaussian input and hence Gaussian output. In a second application involving a first order condenser and resistance in series filter, the integral in the numerator of (1.3) failed to converge. Significantly, we show here that convergence is obtained when the same filter form is employed in the realistic detection circuit of Figure 1.

A slight generalization of a formula presented by M. Kac (8, p. 609) leads Rice to an expression for the expected number of crossings of an arbitrary bias line x_0 which is (18, p. 61)

$$N_0 e^{-\frac{x_0^2}{2\sigma^2}}, \tag{1.4}$$

where σ^2 is the variance of the normally distributed random variable whose mean is zero. This expression, which is essentially that which will be used here, will be derived formally in the following chapter.

The study of zeros of random functions has been pointed out by M. Kac (9, pp. 24-25) to be an extremely interesting and difficult problem, and one where much further work needs to be done. It is known that considerable information about the power spectrum of the random process can be obtained from a knowledge of the zeros. The work here demonstrates that an extension to include the number of crossings of an arbitrary line can be applied successfully to the process of detection of signal in the presence of noise in a practical system.

THE CROSSING THEORY

Before developing equation (1.4), the principal tool in the crossing theory, it is desirable to consider the concept of correlation function and associated spectral density. The correlation function appears to have been introduced by G. I. Taylor (24, pp. 196–212) in connection with studies on the motion of a turbulent fluid. Further significant advances have been made by N. Wiener (29, pp. 118–242), A. Khintchine (12, pp. 604–615), H. Cramér (4, pp. 215–230), and much later by K. Karhunen (11, pp. 7–79).

We shall need some preliminary definitions. Consider a real-valued random variable $X(t)$ where t is a parameter representing time. If observations are made at times $t_1, \ldots, t_n$, different values of $X(t)$ result, and in fact one can only observe certain probability distributions. The random process $X(t)$ is said to be completely defined by the simultaneous probability distribution of the variables $X(t_1), \ldots, X(t_n)$. This distribution may be characterized by the corresponding joint distribution function

(2.1) $F(x_1,\ldots,x_n;t_1,\ldots,t_n) = P(X(t_1)\leq x_1;\ldots;X(t_n)\leq x_n)$,

where P represents the probability of the indicated set of inequalities. The system of distribution functions

$F(x_1,..,x_n;t_1,..,t_n)$ for $n = 1,2,...$ must be such that every F is symmetric in all pairs (x_k,t_i) and satisfies the consistency relation

(2.2) $F(x_1,..,x_j,\infty,..,\infty;t_1,..,t_n) = F(x_1,..,x_j;t_1,..,t_j)$

for all $j < n$. The requirement (2.2) is necessary since each function $F(x_1,..,x_n;t_1,..,t_n)$ must imply all previous $F(x_1,..,x_j;t_1,..,t_j)$ with $j < n$. This definition of a one-dimensional random process is essentially that given by Cramér (4, p. 215). Equation (2.1) will be considered to represent an nth order joint distribution function.

In computing average values of the process $X(t)$ it is necessary to distinguish between two kinds of average; an ensemble average and a time average. Consider determination of the functions F experimentally. It is then necessary to obtain a great number of records $X(t)$ obtained from a great number of experiments. There results then an ensemble of observations $X_1(t),X_2(t),...$. For instance, to obtain the first probability distribution $F(x_1,t_1)$ it is necessary to determine at a definite time t_1 how often in the different experiments $X(t)$ occurs in a given interval $(X,X+\Delta X)$. The <u>statistical average</u> of X is the usual mean value or expectation of X given by

(2.3) $$E\{X(t)\} = \int x dF(x,t).$$

This average value will in general depend on time. It is realized by averaging, at the time t, the values $X_1(t)$, $X_2(t)$,... of the ensemble of observations. The other method of averaging is the familiar <u>time average</u> defined by

$$(2.4) \qquad E_t\{X(t)\} = \lim_{\theta\to\infty} \frac{1}{2\theta}\int_{-\theta}^{\theta} X(t)dt.$$

This average, while of course independent of time, will in general differ for the various functions $X_1(t), X_2(t)$, ... of the ensemble.

We will be concerned in this thesis with random processes that are stationary. Physically this means that all transients have died down and the detection instrument is in a steady state and further that the underlying sources of the random variations do not statistically change with time. Mathematically the random process is defined to be stationary in the following way (4, p. 216 and 12, p. 606). Let it be assumed that the mean values $E\{X^2(t)\}$ are finite for all t. This implies that the mean values $E\{X(t)\}$ and $E\{X(t)X(t+T)\}$ are also finite for all t. The process X(t) then defines a <u>stationary random process</u> if the following two conditions are satisfied:

$$(2.5) \qquad E\{X(t)\} = m \text{ is independent of } t;$$

$$(2.6) \quad E\{X(t)X(t+T)\} = \Psi_1(T) \text{ is a function of } T \text{ only.}$$

Without loss of generality we assume

$$(2.7) \quad E\{X(t)\} = m = 0,\ E\{X^2(t)\} = \Psi_1(0) = \sigma^2 > 0.$$

Having made this assumption, then $\Psi_1(T)$ as defined by (2.6) is denoted the <u>correlation function</u> of the stationary random process defined by X(t). It is clear from (2.6) that $\Psi_1(t)$ is an even valued function. The definition given here is a restricted form of that given by Khintchine (12, p. 607) and Cramér (4, p. 216) in that we have here considered only a single random process which is required to be real. In the more general definition the function defined by (2.6) is often referred to as the autocorrelation function.

The process X(t) will be called a <u>continuous stationary random process</u> when $\lim_{T \to 0} \Psi_1(T) = \Psi_1(0) = \sigma^2$. For such a process Khintchine (12, p. 607) has shown that $\Psi_1(T)$ is itself a continuous and bounded function. We shall be concerned throughout this thesis with continuous stationary random processes.

Khintchine proves an important theorem regarding the correlation function (12, p. 608) which we shall state here. A necessary and sufficient condition for the function $\Psi_1(T)$ to be the correlation function of a continuous stationary random process is that it be expressible in the form

$$\Psi_1(T) = \int_{-\infty}^{\infty} \cos Ty \, dF(y), \tag{2.8}$$

where F(y) represents a certain distribution function. This theorem, stated in more general terms, is given by Cramer (4, p. 221) with the condition that $\Psi_1(T)$ corresponding to the single random process discussed here be represented by the Fourier-Stieltjes integral

$$(2.9) \qquad \Psi_1(T) = \int_{-\infty}^{\infty} e^{iTy} dF(y).$$

Neither Khintchine nor Cramér further persued the question of the function F(y). However, the seeds of the relationship existing between the correlation function $\Psi_1(T)$ and the spectral density are present in (2.9). S. O. Rice (18, p. 32) has shown that F(y) corresponds to the total spectral intensity of X(t). If F(y) is continuous everywhere and $dF = f(y)dy$, then

$$(2.10) \qquad \Psi_1(T) = \int_{-\infty}^{\infty} e^{iTy} f(y)dy,$$

where f(y) represents the spectral density of X(t) and the variable y thus has the units of frequency. Clearly, $\Psi_1(T)$ and f(y) are related by Fourier integral transforms and

$$(2.11) \qquad f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_1(T) e^{-iTy} dT.$$

An equivalent relationship between the spectral density and a function R(T) was established earlier by Wiener (29, pp. 132-137) in his paper on Generalized Harmonic Analysis.

Wiener's R(T) is defined rather simply. Let f(t) be a measurable function such that

(2.12) $$R(T) = \lim_{\Theta\to\infty} \frac{1}{2\Theta}\int_{-\Theta}^{\Theta} f(t+T)f(t)dt$$

exists for every T. Using this definition, Wiener shows that the total spectral intensity S(u) exists when defined as

(2.13) $$S(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(T)\frac{1-e^{-iuT}}{iT}dT.$$

This direct formulation of the total spectral intensity has the advantage of retaining meaning in those cases where the spectral density contains singular peaks of the well known Dirac δ-function type. By differentiating (2.13) with respect to u, the spectral density S'(u) of the continuous portion of the spectrum of f(t) is obtained as

(2.14) $$S'(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(T)e^{-iuT}dT.$$

By direct application of Plancherel's theorem (30, p. 69), it follows that

(2.15) $$R(T) = \int_{-\infty}^{\infty} S'(u)e^{iTu}du.$$

The correlation function is often defined in a more general way. Consider the random process X(t) which may contain hidden periodicities. The process is not specified as stationary. From the second probability distribution $F(x_1,x_2;t_1,t_2)$ may be formed the average

(2.16) $$E\{X(t_1)X(t_2)\} = \iint x_1x_2dF(x_1,x_2;t_1,t_2)$$

which is in general a function of t_1 and t_2. Setting $t_2 = t_1 + T$, and performing an additional time average over t_1 yields a function of T,

(2.17) $$\Psi(T) = \lim_{\theta\to\infty}\frac{1}{2\theta}\int_{-\theta}^{\theta}E\{X(t_1)X(t_1+T)\}dt_1.$$

The same function $\Psi(T)$ is obtained if the time average is taken first followed by the statistical average. The function $\Psi(T)$ is the correlation function of the random processX(t). It is entirely equivalent to $\Psi_1(T)$, as well as to R(T), when the process is stationary. For, when the process is stationary, $E\{X(t)X(t+T)\}$ is independent of time from (2.6), thus from (2.17)

(2.18) $$\Psi(T) = E\{X(t)X(t+T)\} = \Psi_1(T).$$

To show the equivalence of $\Psi_1(T)$ and R(T) we appeal to the Ergodic theorem of statistical mechanics (28, p. 324) which states that when the process is stationary the time average defined by (2.4) yields the same result as the statistical average defined by (2.3). Replacing the statistical average of (2.18) then with the time average gives

(2.19) $$\Psi(T) = E_t\{X(t)X(t+T)\} = R(T).$$

This establishes the equivalence of the three correlation functions $\Psi(T)$, $\Psi_1(T)$, and $R(T)$ for the stationary process.

It is thus reasonable to speak of the correlation function of both periodic and random time functions. Correlation functions considered in this manner are a measure of the mean relationship existing between the product of all pairs of time points of the time series involved. Henceforth we shall write the correlation function for a nonrandom or random stationary process of zero mean as

$$\Psi(T) = E\{X(t)X(t+T)\}, \tag{2.20}$$

where $E\{\ \}$ will mean the statistical average for random functions and the time average for nonrandom functions.

The correlation function defined by (2.20) may be used in conjunction with the Fourier transform relations (2.14) and (2.15), written again in the form

$$\Psi(T) = \int_{-\infty}^{\infty} A(w) e^{iTw} dw \tag{2.21}$$

and

$$A(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(T) e^{-iTw} dT, \tag{2.22}$$

to determine the spectral density $A(w)$ appearing in equations (1.3) and (1.4), or their equivalent, which will now be developed.

The method used by S. O. Rice (18, pp. 57-61) to derive expression (1.3) for the expected number of zeros per second will be followed here in arriving at the expression for the expected number of crossings of an arbitrary bias line by a signal to be detected. Consider X defined by

(2.23) $$X = F(\theta_1, \theta_2, \ldots, \theta_N; t),$$

where the θ's are random variables. For a given set of θ's, equation (2.23) expresses a "random" curve of X versus t. In any short interval (t_1, t_1+dt) many random curves corresponding to different sets of θ's are possible. Rice (18, p. 58 and p. 61) shows the probability that the curve obtained by putting these θ's in (2.23) will pass through the value X_0 in (t_1, t_1+dt) to be

(2.24) $$dt\int_{-\infty}^{\infty}|\eta|\,p(X_0, \eta; t_1)\,d\eta,$$

and the expected number of crossings of the value x_0 in the interval (t_1, t_2) to be

(2.25) $$\int_{t_1}^{t_2} dt\int_{-\infty}^{\infty}|\eta|\,p(x_0, \eta; t)\,d\eta.$$

These relations will be assumed in what follows. The probability density function $p(\xi, \zeta; t)$ applies to the random variables

(2.26) $$\xi = F(\theta_1, \ldots, \theta_N; t), \quad \eta = \frac{dF}{dt},$$

where t is a parameter of the distribution.

Let it be assumed that X as defined by (2.23) is a normally distributed random variable with mean zero. According to Rice (18, p. 48), X may be written explicitly as

$$(2.27) \qquad X(t) = \sum_{n=1}^{N} c_n \cos(w_n t - \theta_n), \; c_n^2 = 4A(w_n)\Delta w,$$

where $\theta_1, \ldots, \theta_N$ are angles distributed at random over the range $(0, 2\pi)$ and $A(w)$ is the spectral density. For N sufficiently large the sum in (2.27) approaches a normal law according to the central limit theorem of probability applied to a sum of independent random variables having identical distributions.

To evaluate (2.25) it is required to obtain the joint probability density function of the two random variables

$$(2.28) \qquad \begin{aligned} \xi &= \sum_{n=1}^{N} c_n \cos(w_n t_1 - \theta_n) \\ \eta = X'(t_1) &= -\sum_{n=1}^{N} c_n w_n \sin(w_n t_1 - \theta_n) \end{aligned} ,$$

where the prime denotes differentiation with respect to t. The problem is essentially that of determining the distribution of the sum of N independent two-dimensional random vectors, all having identical two-dimensional distributions. The resultant vector (ξ, η) has the components ξ and η of (2.28). The central limit theorem (3, pp. 286-287) applied to this case states that if the second moments of the random vector (ξ, η) exist then the distribution of

(ξ,η) approaches the normal law as N approaches infinity. When the moments of a normal distribution are known we may write down the probability density at once,

$$(2.29)\quad p(\xi,\eta;t) = \frac{(m_{20}m_{02}-m_{11}^2)^{-\frac{1}{2}}}{2\pi}\exp\left[\frac{-m_{02}\xi^2-m_{20}\eta^2+2m_{11}\xi\eta}{2(m_{02}m_{20}-m_{11}^2)}\right].$$

The central moments of the distribution are (3, pp. 263-265)

$$m_{11} = E\{\xi\eta\} = -\sum_{n=1}^{N}\sum_{j=1}^{N} w_j c_j c_n E\{\cos(w_n t_1-\theta_n)\sin(w_j t_1-\theta_j)\}$$

$$= -\sum_{n=1}^{N} w_n c_n^2 E\{\cos(w_n t_1-\theta_n)\sin(w_n t_1-\theta_n)\} = 0$$

$$m_{20} = E\{\xi^2\} = \Psi(0)$$

$$m_{02} = E\{\eta^2\} = \sum_{n=1}^{N} c_n^2 w_n^2 E\{\sin^2(w_n t_1-\theta_n)\}$$

$$= \sum_{n=1}^{N} c_n^2 w_n^2 E\{\tfrac{1}{2} - \tfrac{1}{2}\cos(2w_n t_1-2\theta_n)\} = \sum_{n=1}^{N} c_n^2 w_n^2 \tfrac{1}{2}$$

$$= \sum_{n=1}^{N} w_n^2 A(w_n)\Delta w \xrightarrow[N\to\infty]{} \int_0^{\infty} w^2 A(w)dw = -\Psi''(0).$$

The extreme right hand result in the expression for m_{02} is obtained by differentiating (2.21) with respect to T, when differentiation is permissible. The function $\Psi(T)$ is of course the correlation function of the random variable X(t).

The probability density of (2.29) then is written as

$$(2.30)\quad p(\xi,\eta;t_1) = \frac{\{-\Psi(0)\Psi''(0)\}^{-\frac{1}{2}}}{2\pi}\exp\{-\frac{\xi^2}{2\Psi(0)} + \frac{\eta^2}{2\Psi''(0)}\},$$

where $\Psi''(0)$ is negative. The expression on the right side of (2.30) is independent of t, hence the probability of crossing the value x_0 in (t_1, t_1+dt) is from (2.24)

$$(2.31)\quad dt\int_{-\infty}^{\infty}|\eta|p(x_0,\eta;t_1)d\eta =$$

$$= \frac{dt}{\pi}\,\frac{\{-\Psi(0)\Psi''(0)\}^{-\frac{1}{2}}}{2}\,e^{-\frac{x_0^2}{2\Psi(0)}}\int_{-\infty}^{\infty}|\eta|e^{\frac{\eta^2}{2\Psi''(0)}}d\eta$$

$$= \frac{dt}{\pi}\{-\frac{\Psi''(0)}{\Psi(0)}\}^{\frac{1}{2}}e^{-\frac{x_0^2}{2\Psi(0)}}.$$

The result in (2.31) substituted in (2.25) and integrated over a time interval of one second yields for the expected number of crossing per second of an arbitrary bias line x_0

$$(2.32)\quad N_x = \frac{1}{\pi}\{-\frac{\Psi''(0)}{\Psi(0)}\}^{\frac{1}{2}}e^{-\frac{x_0^2}{2\Psi(0)}}$$

$$= \frac{1}{\pi}(\exp\{-\frac{x_0^2}{2\Psi(0)}\})\left(\frac{\int_0^{\infty}w^2A(w)dw}{\int_0^{\infty}A(w)dw}\right)^{\frac{1}{2}}.$$

This is the desired expression which will be applied to the circuit of Figure 1 for three specific filter forms.

Significantly, the derivation of (2.32) required that the variable $X(t)$ be normally distributed. In the applications to follow, the output $E_0(t)$ of the system shown in Figure 1 plays the rôle of $X(t)$. It has been established (5, p. 1173 and 23, p. 47) by independent methods that $E_0(t)$ approximates a normal law under conditions of greatest physical interest, namely for rations of first to second filter bandwidths in excess of about ten. The formula of (2.32) thus may be applied effectively to a study of this system under the conditions described. The form of N_x involving the correlation function $\Psi(T)$ and its derivatives will be employed whenever the second derivatives exist. Otherwise the procedure is to obtain $A(w)$ by Fourier inversion of $\Psi(T)$ followed by evaluation of the integrals in the right hand expression.

THE SYSTEM OPERATOR

Referring to Figure 1, it is required to obtain an expression for $E_0(t)$, the output of the detection system. The filters are to contain no time-varying components and are to be linear. By linear filter we imply a filter characterized by a complex transfer function $F(w)$ such that if a pure sine wave of angular frequency w_1 and amplitude E is applied to the filter input, the output is also a pure sine wave of frequency w_1 and amplitude $|F(w_1)|E$. The phase of the output is also advanced by the angle of $F(w_1)$.

Consider the response $E(t)$ of the first filter which is thus related to the input $E_1(t)$ by the constant coefficient linear differential equation

$$(3.1) \qquad \frac{d^nE}{dt^n} + a_{n-1}\frac{d^{n-1}E}{dt^{n-1}} + \ldots + a_1\frac{dE}{dt} + a_0 = E_1(t).$$

By straightforward application of the Laplace transform theory we obtain

$$(3.2) \qquad e(s) = F_1(s)e_1(s),$$

where $e(s)$, $e_1(s)$, and $F_1(s)$ are the Laplace transforms of $E(t)$, $E_1(t)$, and $f_1(t)$ respectively. The quantity $F_1(s)$ is the complex voltage-frequency transfer function of the filter. This is a steady state result and does not involve initial conditions.

Equation (3.2) is inverted formally by application of the convolution integral theorem (2, p. 37) giving

$$E(t) = \int_0^t f_1(t-x)E_1(x)dx. \tag{3.3}$$

A more useable form results if we observe that $E_1(y) = 0$ for $y < 0$, and that for a physically realizable filter we must also require $f_1(y) = 0$ for $y < 0$. Equation (3.3) is then written

$$E(t) = \int_{-\infty}^{\infty} f_1(t-x)E_1(x)dx. \tag{3.4}$$

Using the fact that $f_1(t) = 0$ for $t < 0$ and replacing s with iw, the functions $f_1(t)$ and $F_1(s)$ form a Fourier transform pair

$$f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(w)e^{iwt}dw,$$

(3.5)

$$F_1(w) = \int_{-\infty}^{\infty} f_1(t)e^{-iwt}dt,$$

where w has the dimensions of angular frequency.

The output of the nonlinear detector is obtained simply by squaring the input E(t) as given by (3.4). Thus the detector output, which is also the second filter input, is written

$$E^2(t) = \int_{-\infty}^{\infty} f_1(t-x)E_1(x)dx \int_{-\infty}^{\infty} f_1(t-y)E_1(y)dy \tag{3.6}$$

$$= \iint_{-\infty}^{\infty} f_1(t-x)E_1(x)E_1(y)f_1(t-y)dxdy.$$

The output of the second filter is readily obtained in the same manner as that of the first filter, with $f_2(t)$ and $F_2(w)$ for the second filter also related by (3.5). Thus

$$E_0(t) = \int_{-\infty}^{\infty} f_2(t-w)E^2(w)dw \tag{3.7}$$

$$= \int_{-\infty}^{\infty} f_2(t-w) \iint_{-\infty}^{\infty} f_1(w-x)E_1(x)E_1(y)f_1(w-y)dxdy\ dw.$$

If we make the substitution $x = t - u$, $y = t - v$, $z = t - w$, equation (3.7) takes the more convenient form

$$E_0(t) = \iint_{-\infty}^{\infty} E_1(t-u)g(u,v)E_1(t-v)dudv, \tag{3.8}$$

where

$$g(u,v) = \int_{-\infty}^{\infty} f_1(u-z)f_2(z)f_1(v-z)dz \tag{3.9}$$

is a symmetrical function in u and v called the system kernel. The right side of (3.8) is the system operator. It is significant that the system kernel depends only on the filter characteristics! The interchange of the order of integration in (3.8) is permissible since the integrand is bounded everywhere and g(u,v) will always contain negative exponential factors such that uniform convergence of the integrals in question is assured. Hereafter, the interchange of order will be made without comment.

It is desirable at this point to draw attention to the sharp sifference between the Emerson approach (5, pp. 1169–1171 and 23, pp. 1–71) and the crossing theory developed here. Proceeding from (3.8) and (3.9), Emerson expands $g(u,v)$ into a uniformly convergent bilinear series

$$(3.10) \qquad g(u,v) = \sum_{j=1}^{\infty} \lambda_j h_j(u) h_j(v),$$

where the $h_j(x)$ and λ_j are respectively the jth normal orthogonal eigenfunction and corresponding eigenvalue of the integral equation

$$(3.11) \qquad \lambda h(x) = \int_{-\infty}^{\infty} g(x,y) h(y) dy.$$

Substitution of (3.10) into (3.8) yields an expression for $E_0(t)$ from which it is possible to evaluate the characteristic function $\Phi(\xi)$ of the distribution of $E_0(t)$ in terms of an infinite product. In order to avoid the problem of inverting $\Phi(\xi)$ to obtain the first probability density of $E_0(t)$, Emerson considers the characteristic function in the following identity

$$(3.12) \qquad \exp\left[\sum_{n=1}^{\infty} K_n \frac{(i\xi)^n}{n!}\right] = \Phi(\xi).$$

From (3.12) and the explicit result for $\Phi(\xi)$ Emerson obtains an expression for the cumulants K_n of the output distribution involving

$$\sum_{j=1}^{\infty} \lambda_j^n \text{ and } \sum_{j=1}^{\infty} \lambda_j^n S_j^2(t),$$

where $S_j(t)$ represents the jth component of the input signal. Then by virtue of (3.10), these infinite sums are expressed in terms of integrals whose integrands involve g(u,v) and the input signal. It is thus possible to evaluate the K_n for a particular filter and specified signal. It remains then to approximate the output probability density by expansion in an orthonormal series such as the Gram-Charlier. Thus

$$(3.13) \qquad P(E_0) \cong \frac{1}{K_2^{\frac{1}{2}}} \sum_{j=0}^{\infty} \delta_j \phi^{(j)}\left(\frac{E_0 - K_1}{K_2^{\frac{1}{2}}}\right),$$

where

$$(3.14) \qquad \phi^{(j)}(x) = \frac{d^j}{dx^j} \left\{ \frac{1}{(2\pi)^{\frac{1}{2}}} \exp(-\frac{x^2}{2}) \right\}.$$

In this way Emerson avoids specific solution of the integral equation (3.11) as well as the inversion of a characteristic function which appears as an infinite product.

The point of departure of the crossing theory is with equations (3.8) and (3.9). In considerable contrast to the probabilistic methods, such as that of Emerson, we do not seek a probability density function but rather we seek the explicit form of N_x as given by (2.31). To achieve this it is necessary to evaluate the correlation function of the output $E_0(t)$, and this is immediately possible once the system kernel $g(u,v)$ has been evaluated for the particular filters specified. The corresponding spectral density $A(w)$ is then obtained by Fourier inversion of $\Psi(T)$. As mentioned earlier, determination of $A(w)$ may be avoided when the second derivative of $\Psi(T)$ exists. The expected number of crossings of an arbitrary bias line then follows by direct substitution in (2.32). This result is then used to determine the proper bias setting to achieve desired detection results.

THE CORRELATION FUNCTION FOR $E_0(t)$

In this chapter we shall develop a general expression for the correlation function of the output signal plus noise which is then readily evaluated for each of the three filter forms examined. The input signal plus noise is expressed as

(4.1) $$E_1(t) = S(t) + N(t),$$

where N(t) is the input noise which is assumed to be an uncorrelated Gaussian random process with zero mean, and S(t) is the signal input specified as

(4.2) $$S(t) = Q\cos w_0 t.$$

Q is the peak voltage amplitude and w_0 is the angular frequency of the carrier. The random process is of course assumed stationary.

Recalling that the correlation function was defined for a random process with zero mean, we write for the correlation function of the system output

(4.3) $$\Psi(T) = E\{[E_0(t)-m][E_0(t+T)-m]\},$$

where the mean m is

$$E\{E_0(t)\} = E\{E_0(t+T)\} = m$$

and $E_0(t)$ is given by (3.8). Expanding the product in (4.3) and taking termwise expectations yields

(4.5) $$\Psi(T) = E\{E_0(t)E_0(t+T)\} - (E\{E_0(t)\})^2.$$

Setting T = 0 in (4.3) gives

(4.6) $$\Psi(0) = E\{(E_0(t)-m)^2\} = \sigma^2,$$

which is the variance by definition and also the second cumulant of the distribution. This relation will be useful for checking later results.

Consider first the evaluation of the mean given by (4.4). Referring to (3.8) we have

(4.7) $$E\{E_0(t)\} =$$

$$= \iint_{-\infty}^{\infty} E\{[Q\cos w_0(t-u)+N(t-u)]g(u,v)[Q\cos w_0(t-v)+N(t-v)]\}dudv$$

$$= \iint_{-\infty}^{\infty} Q^2 E\{\cos w_0(t-u)\cos w_0(t-v)\}g(u,v)dudv +$$

$$+ \iint_{-\infty}^{\infty} E\{N(t-u)N(t-v)\}g(u,v)dudv,$$

since the signal and noise are statistically independent, thus permitting use of the product theorem for expectations.

Since the input noise is uncorrelated we have

(4.8) $$E\{N(t-u)N(t-v)\} = A_0\delta(u-v),$$

where A_0 is the noise power per unit frequency and $\delta(u-v)$ is the unit impulse function

$$\delta = \begin{cases} 1 \text{ for } u = v \\ 0 \text{ for } u \neq v \end{cases}.$$

Recalling that nonrandom functions are to be averaged in the sense of (2.4), equation (4.7) becomes

(4.9) $$E\{E_0(t)\} = \frac{Q^2}{2}\iint\limits_{-\infty}^{\infty}\cos w_0(u-v)g(u,v)\,du\,dv + A_0\int\limits_{-\infty}^{\infty}g(u,u)\,du,$$

and

(4.10) $$(E\{E_0(t)\})^2 =$$

$$= \frac{Q^4}{4}\iiiint\limits_{-\infty}^{\infty}\cos w_0(u-v)\cos w_0(x-y)g(u,v)g(x,y)\,du\,dv\,dx\,dy$$

$$= Q^2A_0\iint\limits_{-\infty}^{\infty}\cos w_0(u-v)g(u,v)\,du\,dv\int\limits_{-\infty}^{\infty}g(x,x)\,dx +$$

$$+ \left[A_0\int\limits_{-\infty}^{\infty}g(x,x)\,dx\right]^2 .$$

For each of the filters to be specified the kernel $g(u,v)$ contains a factor $\cos w_0(u-v)$, hence for convenience we write

(4.11) $$g(u,v) = \cos w_0(u-v)G(u,v).$$

Making this substitution in the first integral of (4.10) gives

(4.12) $$(E\{E_0(t)\})^2 = \frac{Q^4}{4}\iiiint\limits_{-\infty}^{\infty}\tfrac{1}{4}G(u,v)G(x,y)\,du\,dv\,dx\,dy +$$

$$+ Q^2A_0\iint\limits_{-\infty}^{\infty}\cos w_0(u-v)g(u,v)\,du\,dv\int\limits_{-\infty}^{\infty}g(x,x)\,dx +$$

$$+ \left[A_0\int\limits_{-\infty}^{\infty}g(x,x)\,dx\right]^2 .$$

The identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ has been used, and the terms involving $2w_0$ have been neglected since the extent to which these high frequency components will be passed by the low-pass filter is negligible.

The next step is that of evaluating the first term of (4.5). Thus

(4.13) $$E\{E_0(t)E_0(t+T)\} =$$

$$= \iiiint_{-\infty}^{\infty} E\{E_i(t-u)E_i(t-v)E_i(t+T-x)E_i(t+T-y)\}g(u,v)g(x,y)dV,$$

$$dV = dudvdxdy.$$

Observing the form of $E_i(t)$ as given by (4.1) and (4.2), we write the expectation in the symbolic form

(4.14) $$E\{(S_1+n_1)(S_2+n_2)(S_3+n_3)(S_4+n_4)\},$$

where the subscripts represent different instants of time. The S's are nonrandom and the n's are Gaussian random. We shall need a theorem (28, p. 332) on Gaussian random processes which states that if f(t) is a Gaussian process then the following averages hold:

(4.15) $$E\{f(t_1)f(t_2)\ldots f(t_{2n+1})\} = 0$$

$$E\{f(t_1)f(t_2)\ldots f(t_{2n})\} = \sum_{\substack{\text{all}\\ \text{pairs}}} E\{f(t_i)f(t_j)\}E\{f(t_k)f(t_l)\}\ldots,$$

where the sum is to be taken over all the different ways in which the 2n time points $t_1, t_2, \ldots, t_{2n}$ can be divided into n pairs. With this theorem and the fact that the S's and n's are statistically independent, we have by expanding the indicated product in (4.14)

$$(4.16) \quad E\{(S_1+n_1)(S_2+n_2)(S_3+n_3)(S_4+n_4)\} =$$

$$= E\{S_1S_2S_3S_4\} + E\{S_1S_2\}E\{n_3n_4\} + E\{S_1S_3\}E\{n_2n_4\} +$$
$$+ E\{S_1S_4\}E\{n_2n_3\} + E\{S_2S_3\}E\{n_1n_4\} + E\{S_1S_4\}E\{n_1n_3\} +$$
$$+ E\{S_3S_4\}E\{n_1n_2\} + E\{n_1n_2\}E\{n_3n_4\} + E\{n_1n_3\}E\{n_2n_4\} +$$
$$+ E\{n_1n_4\}E\{n_2n_3\}.$$

With this result, equation (4.13) becomes

$$(4.17) \quad E\{E_0(t)E_o(t+T)\} =$$

$$= \iiiint_{-\infty}^{\infty}\Big[Q^4E\{\cos w_0(t-u)\cos w_0(t-v)\cos w_0(t+T-x)\cos w_0(t+T-y)\} +$$
$$+ Q^2E\{\cos w_0(t-u)\cos w_0(t-v)\}E\{N(t+T-x)N(t+T-y)\} +$$
$$+ Q^2E\{\cos w_0(t-u)\cos w_0(t+T-x)\}E\{N(t-v)N(t+T-y)\} +$$
$$+ Q^2E\{\cos w_0(t-u)\cos w_0(t+T-y)\}E\{N(t-v)N(t+T-x)\} +$$
$$+ Q^2E\{\cos w_0(t-v)\cos w_0(t+T-x)\}E\{N(t-u)N(t+T-y)\} +$$
$$+ Q^2E\{\cos w_0(t-v)\cos w_0(t+T-y)\}E\{N(t-u)N(t+T-x)\} +$$
$$+ Q^2E\{\cos w_0(t+T-x)\cos w_0(t+T-y)\}E\{N(t-u)N(t-v)\} +$$
$$+ E\{N(t-u)N(t-v)\}E\{N(t+T-x)N(t+T-y)\} +$$
$$+ E\{N(t-u)N(t+T-x)\}E\{N(t-v)N(t+T-y)\} +$$
$$+ E\{N(t-u)N(t+T-y)\}\{E\{N(t-v)N(t+T-x)\}\Big]g(u,v)g(x,y)dV,$$

$$DV = dudvdxdy.$$

With the exception of the first term, the averages indicated in (4.17) are readily obtained. The first term expectation is

$$(4.18) \quad E\{\cos w_0(t-u)\cos w_0(t-v)\cos w_0(t+T-x)\cos w_0(t+T-y)\} =$$
$$E\{\tfrac{1}{2}[\cos w_0(u+T-x)+\cos w_0(2t+T-u-x)]\tfrac{1}{2}[\cos w_0(v+T-y)+$$
$$+\cos w_0(2t+T-v-y)]\} =$$

$$= \frac{\cos w_0(u+T-x)\cos w_0(v+T-y)}{4} + 0 + 0 + \tfrac{1}{4}E\{\tfrac{1}{2}\cos w_0(u+x-v-y) +$$

$$+ \tfrac{1}{2}\cos w_0(4t+2T-u-x-v-y)\} =$$

$$= \frac{\cos w_0(u+T-x)\cos w_0(v+T-y)}{4} + \frac{\cos w_0(v+y-u-x)}{8} .$$

From the results of (4.18) and $g(u,v)$ written according to (4.11), the first term of 4.17 becomes

$$\frac{Q^2}{4}\iiiint_{-\infty}^{\infty}[\cos w_0(u+T-x)\cos w_0(v+T-y)+\tfrac{1}{2}\cos w_0(v+y-u-x)]$$

$$[\cos w_0(u-v)\cos w_0(x-y)]G(u,v)G(x,y)dudvdxdy.$$

The trigonometric factors in the integrand can be reduced to $\frac{1}{4}$ through the use of trigonometric identites and by again neglecting all high frequency terms involving twice the argument. Having done this and carrying out the indicated averages we obtain for (4.17)

$$(4.19)\quad E\{E_0(t)E_0(t+T)\} = \frac{Q^4}{4}\iiiint_{-\infty}^{\infty}\tfrac{1}{4}G(u,v)G(x,y)dudvdxdy +$$

$$+ \frac{Q^2A_0}{2}\iiint_{-\infty}^{\infty}\cos w_0(u-v)g(u,v)g(x,x)dxdudv +$$

$$+ \frac{Q^2A_0}{2}\iiint_{-\infty}^{\infty}\cos w_0(x-T-u)g(x,v+T)g(u,v)dudvdx +$$

$$+ \frac{Q^2A_0}{2}\iiint_{-\infty}^{\infty}\cos w_0(y-T-v)g(v+T,y)g(u,v)dudvdy +$$

$$+ \frac{Q^2A_0}{2}\iiint_{-\infty}^{\infty}\cos w_0(x-T-v)g(x,u+T)g(u,v)dudvdx +$$

$$+ \frac{Q^2A_0}{2}\iiint_{-\infty}^{\infty}\cos w_0(y-T-v)g(u+T,y)g(u,v)dudvdy +$$

$$+ \frac{Q^2A_0}{2}\iiint_{-\infty}^{\infty}\cos w_0(x-y)g(x,y)g(u,u)dudxdy +$$

$$+ A_0^2\iint_{-\infty}^{\infty}g(u,u)g(x,x)dudx + A_0^2\iint_{-\infty}^{\infty}g(u+T,v+T)g(u,v)dudv +$$

$$+ A_0^2\iint_{-\infty}^{\infty}g(v+T,u+T)g(u,v)dudv.$$

The system kernel g(u,v) is always symmetrical in u and v, hence we combine terms in (4.19) to get

$$(4.20)\quad E\{E_0(t)E_0(t+T)\} = \frac{Q^4}{4}\iiiint_{-\infty}^{\infty}\tfrac{1}{4}G(u,v)G(x,y)dudvdxdy +$$

$$+ Q^2A_0\iint_{-\infty}^{\infty}\cos w_0(u-v)g(u,v)dudv\int_{-\infty}^{\infty}g(x,x)dx +$$

$$+ 2Q^2A_0\iiint_{-\infty}^{\infty}\cos w_0(x-u-T)g(u,v)g(x,v+T)dvdudx +$$

$$+ 2A_0^2\iint_{-\infty}^{\infty}g(u,v)g(u+T,v+T)dudv + \left[A_0\int_{-\infty}^{\infty}g(x,x)dx\right]^2$$

From (4.5), (4.12), and (4.20) we obtain for the correlation function of $E_0(t) - m$

$$(4.21)\quad \Psi(T) = 2Q^2A_0 \iint_{-\infty}^{\infty} \cos w_0(x-u-T)\left[\int_{-\infty}^{\infty} g(u,v)g(x,v+T)dv\right]dudx$$

$$+\ 2A_0^2 \iint_{-\infty}^{\infty} g(u,v)g(u+T,v+T)dudv.$$

Expression (4.21) is general for the system of Figure 1 with the specified signal plus noise input and as such it will be employed in the study of each of the specific filters. The specialization for any particular filter comes about in the particular form of the kernel g(u,v) as given by (3.9). For certain filters, g(u,v) will be such that the integration range in (4.21) is from zero to plus infinity rather than over the doubly infinite range. However, as this does not invalidate the general development here, equation (4.21) is then applicable for any of the filters providing the proper integration limits are taken.

GAUSSIAN FILTERS

As a first application of the crossing theory we consider the case in which the first filter has a Gaussian pass band of width b_1 centered at the very high frequency f_0. Similarly, let the second filter have a pass band of width b_2 centered at zero frequency. The voltage transfer functions are then

$$(5.1) \qquad F_1(f) = e^{-\frac{(f+f_0)^2}{2b_1^2}} + e^{-\frac{(f-f_0)^2}{2b_1^2}} ,$$

$$(5.2) \qquad F_2(f) = e^{-\frac{f^2}{2b_2^2}} .$$

For specific evaluation of the system kernel we require the Fourier transforms of (5.1) and (5.2). From (3.5) we have

$$(5.3) \quad f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{-\frac{(w+w_0)^2}{8\pi^2 b_1^2}} + e^{-\frac{(w-w_0)^2}{8\pi^2 b_1^2}}\right)e^{iwt}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{z^2}{8\pi^2 b_1^2}} e^{i(z-w_0)t} dz + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{z^2}{8\pi^2 b_1^2}} e^{i(z+w_0)t} dz$$

$$= \frac{1}{\pi}\cos w_0 t \int_{-\infty}^{\infty} e^{-\frac{z^2}{8\pi^2 b_1^2} + izt} dz,$$

where $w = 2\pi f$ is the angular frequency. This last integral is evaluated readily by completed squaring and using the well known result

$$(5.4) \qquad \int_0^{\infty} e^{-a^2x^2} dx = \pi^{\frac{1}{2}}/2a, \; a > 0.$$

This results in

$$(5.5) \qquad f_1(t) = 2(2\pi b_1^2)^{\frac{1}{2}} \cos w_0 t \; e^{-2\pi^2 b_1^2 t^2}.$$

In a similar manner we have from (5.2)

$$(5.6) \qquad f_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{w^2}{8\pi^2 b_2^2}} e^{iwt} dw$$

$$= (2\pi b_2^2)^{\frac{1}{2}} e^{-2\pi^2 b_2^2 t^2}.$$

Substitution of these results in (3.9) yields for the system kernel

$$(5.7) \quad g(u,v) = C(u,v)\int_{-\infty}^{\infty} \cos w_0(u-z)\cos w_0(v-z)\exp(-2\pi^2 \{(2b_1^2+b_2^2)z^2 - 2b_1^2(u+v)z\})dz,$$

$$C(u,v) = 8\pi b_1^2(2\pi b_2^2)^{\frac{1}{2}}\exp\{-2\pi^2 b_1^2(u^2+v^2)\}.$$

Since w_0 is large we are justified (5, p. 1172) in neglecting the term in (5.7) involving $\cos w_0(u+v-2z)$ which represents the high frequency residue and which arises from the identity

$$2\cos w_0(u-z)\cos w_0(v-z) = \cos w_0(u-v)+\cos w_0(u+v-2z).$$

With this approximation and with completion of the square in the exponential factor of (5.7) we obtain

(5.8)

$$g(u,v) = 4\pi b_1^2(2\pi b_2^2)^{\frac{1}{2}}\cos w_0(u-v)\exp\{-2\pi^2 b_1^2(u^2+v^2) + $$

$$+ \frac{2\pi^2 b_1^4(u+v)^2}{2b_1^2+b_2^2}\}\int_{-\infty}^{\infty}\exp(-2\pi^2(2b_1^2+b_2^2)\{z - \frac{b_1^2(u+v)^2}{2b_1^2+b_2^2}\})dz$$

$$= \frac{4\pi b_1^2 b_2\cos w_0(u-v)}{(2b_1^2+b_2^2)^{\frac{1}{2}}}\exp(-2\pi^2 b_1^2\{u^2+v^2+ \frac{b_1^2(u+v)^2}{2b_1^2+b_2^2}\}).$$

Let δ be the ratio of bandwidths defined as

(5.9) $\delta = b_1/b_2.$

Then (5.8) may be expressed in the more convenient form

$$(5.10)\quad g(u,v) = \frac{4\pi b_1^2\cos w_0(u-v)}{(1+2\delta^2)^{\frac{1}{2}}}\exp(- \frac{(2\pi b_1)^2}{4}\{(u-v)^2 + $$

$$+ \frac{(u+v)^2}{1+2\delta^2}\}),$$

from which it is clear that $g(u,v)$ is symmetric about the line $u = v$.

We consider next evaluation of the correlation function given by (4.21). Consider first

$$(5.11)\qquad I_1 = \int_{-\infty}^{\infty} g(u,v)g(v+T,x)dv$$

$$= \frac{(4\pi b_1^2)^2}{1+2\delta^2}\int_{-\infty}^{\infty}\cos w_0(u-v)\cos w_0(v+T-x)\exp(-\frac{2\pi b_1^2}{4}\{(u-v)^2 +$$

$$+ \frac{(u+v)^2}{1+2\delta^2} + (v+T-x)^2 + \frac{(v+T+x)^2}{1+2\delta^2}\})dv$$

$$= \frac{(4\pi b_1^2)^2}{2(1+2\delta^2)}\cos w_0(u+T-x)\exp(-\frac{(2\pi b_1)^2}{2(1+2\delta^2)}\{(u^2+x^2)(1+\delta^2) -$$

$$- 2\delta^2 Tx - \frac{(T+\delta^2T-\delta^2u-\delta^2x)^2}{2(1+\delta^2)}\})\int_{-\infty}^{\infty}\exp(-\frac{(2\pi b_1)^2(1+\delta^2)}{1+2\delta^2}\{v +$$

$$+ \frac{T+\delta^2T-\delta^2u-\delta^2x}{2(1+\delta^2)}\}^2)dv,$$

where again we have neglected the high frequency term involving $\cos w_0(u+x-2v-T)$. The integral in (5.11) is readily evaluated with the aid of (5.4) as

$$(5.12)\ I_1 = \frac{4\pi^{3/2}b_1^3\cos w_0(u+T-x)}{(1+2\delta^2)^{\frac{1}{2}}(1+\delta^2)^{\frac{1}{2}}}\ \exp(-\frac{(2\pi b_1)^2}{2(1+2\delta^2)}\{(u^2+x^2+$$

$$+T^2)(1+\delta^2) - 2\delta^2 Tx - \frac{(T+\delta^2T-\delta^2u-\delta^2x)^2}{2(1+\delta^2)}\}).$$

Using this result the first term of (4.21), which we denote $\Psi_S(T)$, becomes

$$(5.13)\quad \Psi_S(T) = \frac{8Q^2A_0\pi^{3/2}b_1^3}{(1+2\delta^2)^{\frac{1}{2}}(1+\delta^2)^{\frac{1}{2}}}\iint_{-\infty}^{\infty}\cos^2 w_0(u+T-x)\cdot$$

$$\cdot\exp(-\frac{(2\pi b_1)^2}{2(1+2\delta^2)}\{(u^2+x^2+T^2)(1+\delta^2) - 2\delta^2 Tx -$$

$$-\frac{(T+\delta^2T-\delta^2u-\delta^2x)^2}{2(1+\delta^2)}\})dudx.$$

Again, neglecting the high frequency components involving $2w_0$ and complete the square first in u then in x we obtain

$$(5.14)\quad \overline{\Psi}_S(T) = \frac{4A_0Q^2\pi^{3/2}b_1^3}{(1+2\delta^2)^{\frac{1}{2}}(1+\delta^2)^{\frac{1}{2}}}\iint_{-\infty}^{\infty}\exp(-\frac{(2\pi b_1)^2(2+4\delta^2+\delta^4)}{4(1+2\delta^2)(1+\delta^2)}\cdot$$

$$\cdot\{u+\frac{\delta^2T+\delta^4T-\delta^4x}{2+4\delta^2+\delta^4}\}^2)\exp(-\frac{(2\pi b_1)^2(2+4\delta^2+\delta^4)}{4(1+2\delta^2)(1+\delta^2)}\{x^2 -$$

$$-\frac{2\delta^2T+2\delta^4T}{2+4\delta^2+\delta^4}x+\frac{T^2+2T^2\delta^2+\delta^4T^2}{2+4\delta^2+\delta^4}-\frac{(\delta^2T+\delta^4T-\delta^4x)^2}{(2+4\delta^2+\delta^4)^2}\})dudx =$$

$$=\frac{4\pi b_1^2A_0Q^2}{(2+4\delta^2+\delta^4)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\exp(-\frac{(2\pi b_1)^2(2+4\delta^2+\delta^4)}{4(1+2\delta^2)(1+\delta^2)}\{x^2-\frac{2\delta^2T+2\delta^4T}{2+4\delta^2+\delta^4}x +$$

$$+\frac{T^2+2T^2\delta^2+\delta^4T^2}{2+4\delta^2+\delta^4}-\frac{(\delta^2T+\delta^4T-\delta^4x)^2}{(2+4\delta^2+\delta^4)^2}\})dx =$$

$$=\frac{4\pi b_1^2A_0Q^2}{(2+4\delta^2+\delta^4)^{\frac{1}{2}}}\exp(-\frac{(2\pi b_1)^2(1+4\delta^2+5\delta^4+2\delta^6)}{(1+2\delta^2)(1+\delta^2)(2+4\delta^2+\delta^4)}\{\frac{T^2}{2} -$$

$$-\frac{(-\delta^2T-3\delta^4T-2\delta^6T)^2}{4(1+4\delta^2+5\delta^4+2\delta^6)^2}\})\int_{-\infty}^{\infty}\exp(-\frac{(2\pi b_1)^2(1+4\delta^2+5\delta^4+2\delta^6)}{(1+2\delta^2)(1+\delta^2)(2+4\delta^2+\delta^4)}\{x +$$

$$+(-\Delta^2T-3\delta^4T-2\delta^6T)/2(1+4\delta^2+5\delta^4+2\delta^6)\}^2)dx =$$

$$= \frac{2\pi^{\frac{1}{2}} b_1 A_0 Q^2 (1+2\delta^2)^{\frac{1}{2}} (1+\delta^2)^{\frac{1}{2}}}{(1+4\delta^2+5\delta^4+2\delta^6)^{\frac{1}{2}}} \exp\left\{-\frac{(2\pi b_1)^2 T^2}{4(1+\delta^2)}\right\},$$

where (5.4) has been used in the integration. Finally

$$(5.15)\quad \Psi_S(T) = \frac{2\pi^{\frac{1}{2}} b_1 A_0 Q^2}{(1+\delta^2)^{\frac{1}{2}}} \exp\left\{-\frac{(2\pi b_1)^2 T^2}{4(1+\delta^2)}\right\}.$$

Denoting the second term of (4.21) by $\Psi_N(T)$ we have

$$(5.16)\quad \Psi_N(T) = \frac{2A_0^2 (4\pi b_1)^2}{1+2\delta^2} \iint_{-\infty}^{\infty} \cos^2 w_0(u-v) \exp\Big(-\frac{(2\pi b_1)^2}{4} \cdot$$

$$\cdot \left\{2(u-v)^2 + \frac{(u+v)^2}{1+2\delta^2} + \frac{(2T+u+v)^2}{1+2\delta^2}\right\}\Big)\, du\, dv.$$

Again approximating $\cos^2 w_0(u-v)$ by $\frac{1}{2}$, completing the square first in u then in v, and using (5.4) we obtain

$$(5.17)\quad \Psi_N(T) = \frac{4\pi b_1^2 A_0^2}{(1+2\delta^2)^{\frac{1}{2}}}\, e^{-(2\pi b_1 T)^2/2(1+2\delta^2)}.$$

Combining (5.15) and (5.17) we obtain for the correlation function

$$(5.18)\quad \Psi(T) = \frac{2\pi^{\frac{1}{2}} b_1 A_0 Q^2}{(1+\delta^2)^{\frac{1}{2}}} \exp\left\{-\frac{(2\pi b_1)^2 T^2}{4(1+\delta^2)}\right\} +$$

$$+ \frac{4\pi b_1^2 A_0^2}{(1+2\delta^2)^{\frac{1}{2}}} \exp\left\{-\frac{(2\pi b_1)^2 T^2}{2(1+2\delta^2)}\right\}.$$

As an algebraic check we recall from (4.6) that the second cumulant of the output distribution is given by

$$(5.19)\quad \overline{\Psi}(0) = \frac{2\pi^{\frac{1}{2}} b_1 A_0 Q^2}{(1+\delta^2)^{\frac{1}{2}}} + \frac{4\pi b_1^2 A_0^2}{(1+2\delta^2)^{\frac{1}{2}}}$$

$$= \frac{2zN^2}{(1+\delta^2)^{\frac{1}{2}}} + \frac{N^2}{(1+2\delta^2)^{\frac{1}{2}}},$$

where we have defined the noise power admitted to the system as $N = 2\pi^{\frac{1}{2}} b_1 A_0$ and the signal to noise power ratio as $z = Q^2/2N$. Equation (5.19) agrees with Emerson's expression for the second cumulant K_2 (5, p. 1173).

Significantly, for the Gaussian filters as well as for those to follow in the other applications, the second derivative of $\overline{\Psi}(T)$ will exist and we may use the first form in equation (2.33) to compute N_x, the expected number of crossings of an arbitrary bias line. The reason for this is that the high frequency periodic input is not passed to any extent by the second filter. This is taken into account mathematically by neglecting these higher frequency components, as we have done in all calculations. Hence, of the two forms in (2.33) we may use whichever appears to be more convenient.

Letting primes denote derivatives with respect to T we have

$$(5.20)\quad \overline{\Psi}'(T) = \frac{\pi^{\frac{1}{2}} b_1 A_0 Q^2}{(1+\delta^2)^{\frac{1}{2}}}\left\{-\frac{(2\pi b_1)^2 T}{1+\delta^2}\right\}\exp\left\{-\frac{(2\pi b_1)^2 T^2}{4(1+\delta^2)}\right\} +$$

$$+ \frac{4\pi b_1^2 A_0^2}{(1+2\delta^2)^{\frac{1}{2}}}\{- \frac{(2\pi b_1)^2 T}{1+2\delta^2}\}\exp\{- \frac{(2\pi b_1)^2 T^2}{2(1+2\delta^2)}\},$$

and

$$(5.21) \quad \Psi''(0) = - \{\frac{4\pi^{\frac{1}{2}}\pi^2 A_0 Q^2 b_1^3}{(1+\delta^2)^{3/2}} + \frac{16\pi^3 b_1^4 A_0^2}{(1+2\delta^2)^{3/2}}\}$$

$$= - \{\frac{4N^2 z\pi^2 b_1^2}{(1+\delta^2)^{3/2}} + \frac{4\pi^2 b_1^2 N^2}{(1+2\delta^2)^{3/2}}\}.$$

We observe from (5.20) that $\Psi'(0) = 0$ as is required since $\Psi(T)$ is an even function by definition.

Substituting (5.19) and (5.21) into equation (2.33) yields for the expected number of crossings of the bias line x_0 by the output signal plus noise

$$(5.22) \quad N_x = 2b_1 \left[\frac{\dfrac{z}{(1+\delta^2)^{3/2}} + \dfrac{1}{(1+2\delta^2)^{3/2}}}{\dfrac{2z}{(1+\delta^2)^{\frac{1}{2}}} + \dfrac{1}{(1+2\delta^2)^{\frac{1}{2}}}}\right]^{\frac{1}{2}} \exp\{- x_0^2/\Delta\},$$

$$\Delta = \frac{4zN^2}{(1+\delta^2)^{\frac{1}{2}}} + \frac{2N^2}{(1+2\delta^2)^{\frac{1}{2}}}.$$

For noise only we have $z = 0$ and

$$(5.23) \quad N_x = \frac{2b_1}{(1+2\delta^2)^{\frac{1}{2}}}\exp\{- \frac{x_0^2(1+2\delta^2)^{\frac{1}{2}}}{2N^2}\}.$$

This last result is useful in determining the proper bias setting for a given false alarm time.

FIRST ORDER FILTERS

For a second application let both filters be low pass filters composed of resistance and capacitance elements such as in Figure 2. Assuming the output current

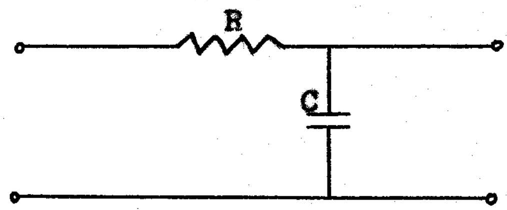

Figure 2. A simple first order filter.

to be negligible, the first order linear system of Figure 2 is described by

$$RC\frac{de_0}{dt} + e_0 = e_1, \tag{6.1}$$

where e_1 is the input function and e_0 is the corresponding output. For the steady state situation

$$e_0 = E_0e^{iwt}, \quad e_1 = E_1e^{iwt}, \tag{6.2}$$

which together with (6.1) yield for the voltage transfer function

$$F(w) = \frac{E_0}{E_1} = \frac{1}{1+iwCR} . \tag{6.3}$$

Introducing the concept of negative frequency, thereby gaining the advantage of symmetry, we write the voltage transfer functions for the two filters as

(6.4) $$F_1(w) = \{1 + i\frac{w-w_0}{w_1}\}^{-1} + \{1 + i\frac{w+w_0}{w_1}\}^{-1}$$

and

(6.5) $$F_2(w) = \{1 + i\frac{w}{w_2}\}^{-1},$$

where w_0 is the high frequency carrier and $w_j = 1/R_jC_j$, $j = 1,2$, are the half power bandwidths of the first and second filters. The transfer function $F_1(w)$ corresponding to the first filter is actually a fictitious function and does not precisely describe the circuit of Figure 2. However, it is mathematically expedient to use $F_1(w)$ as defined, and it is justified physically in that probabilistic studies of more realistic second order filters may be interpreted in terms of probabilistic studies based on this fictitious first order filter.

According to (3.5) the Fourier transform of (6.4) is

(6.6) $$\begin{aligned} f_1(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\{(1+i\frac{w-w_0}{w_1})^{-1}+(1+i\frac{w+w_0}{w_1})^{-1}\}e^{iwt}dw \\ &= \frac{w_1}{2\pi}\int_{-\infty}^{\infty}e^{it(zw_1+w_0)}\frac{dz}{1+iz} + \frac{w_1}{2\pi}\int_{-\infty}^{\infty}e^{it(zw_1-w_0)}\frac{dz}{1+iz} \\ &= \frac{w_1}{\pi}\cos w_0t\int_{-\infty}^{\infty}(\cos w_1tz+i\sin w_1tz)\frac{1-iz}{1+z^2}dz. \end{aligned}$$

The last integral may be evaluated with the help of the established results

$$(6.7)\quad \int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}dx = \int_{-\infty}^{\infty} \frac{x\sin ax}{1+x^2}dx = \pi e^{-a}.$$

Taking advantage of symmetry and noting that the imaginary terms integrate to zero we have for (6.6)

$$(6.8)\qquad f_1(t) = \begin{cases} 2w_1 \cos w_0 t\, e^{-w_1 t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

For the second filter we obtain in analogous manner

$$(6.9)\qquad f_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwt}\left(1+i\frac{w}{w_2}\right)^{-1}dw$$

$$= \frac{w_2}{2\pi}\int_{-\infty}^{\infty} e^{iw_2tz}\frac{dz}{1+iz}$$

$$= \begin{cases} w_2 e^{-w_2 t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

The system kernel may be written now from (6.8), (6.9), and (3.9) as

$$(6.10)\quad g(u,v) = 4w_1^2 w_2 \int_{-\infty}^{\infty} \cos w_0(u-z)\cos w_0(v-z)\cdot$$

$$\cdot\exp\{-w_1(u-z)-w_2z-w_1(v-z)\}dz,$$

where it is to be remembered that each f(t) function goes to zero for negative values of its argument. We again neglect the contribution of the high frequency term involving $\cos w_0(u+v-2z)$ in the identity

$$2\cos w_0(u-z)\cos w_0(v-z) = \cos w_0(u-v) + \cos w_0(u+v-2z).$$

Taking $v \geq u \geq 0$ we have for (6.10)

$$(6.11)\quad g(u,v) = 2w_1^2 w_2 \cos w_0(u-v)\int_0^u e^{-w_1(u-z)-w_2 z-w_1(v-z)}dz$$

$$= \frac{2w_1^2 w_2}{2w_1-w_2}\cos w_0(u-v)e^{-w_1(u+v)}\{e^{(2w_1-w_2)u}-1\}.$$

The final result for $g(u,v)$ is then written

$$(6.12)\quad g(u,v) = \begin{cases} \dfrac{2w_1^2 w_2}{2w_1-w_2}\cos w_0(u-v)\{e^{(w_1-w_2)u-w_1 v} - \\ \qquad\qquad - e^{-w_1(u+v)}\}, \quad 0\leq u\leq v, \\ 0, \quad u,v < 0. \end{cases}$$

For $u \geq v \geq 0$ the u and v are interchanged in (6.12) and $g(u,v)$ is clearly symmetric in u and v.

The next step is that of evaluating the correlation function given by (4.21). Designating the first term of (4.21) by $\Psi_S(T)$ we write from (6.12)

$$(6.13)\quad \Psi_S(T) = \frac{2Q^2 A_0 (w_1^2 w_2)^2}{(2w_1-w_2)^2}\iiint_{-\infty}^{\infty} f(u,v)f(x,v+T)\,du\,dx\,dv,$$

where $f(u,v)$ represents the exponential factor in braces in (6.12). The disposal of the three cosine factors, resulting in a factor of $\frac{1}{4}$, is achieved as before by neglecting the high frequency contributions in the trigonometric identities.

Since $g(u,v)$ is zero for negative values of either argument, the integration in (6.13) is over the first octant in the uXv space. Observing that u appears only in f(u,v) in (6.13) we evaluate first

$$(6.14)\quad G_1(v) = \int_0^v \{e^{(w_1-w_2)u-w_1v} - e^{-w_1(u+v)}\}du +$$

$$+ \int_v^\infty \{e^{(w_1-w_2)v-w_1u} - e^{-w_1(u+v)}\}du$$

$$= \frac{e^{-w_2v} - e^{-w_1v}}{w_1-w_2} + \frac{e^{-2w_1v} - e^{-w_1v}}{w_1} +$$

$$+ \frac{e^{-w_2v} - e^{-2w_1v}}{w_1}$$

$$= \frac{2w_1-w_2}{w_1(w_1-w_2)}\{e^{-w_2v} - e^{-w_1v}\}.$$

The two ranges of integration are required since u = v is a plane of symmetry for f(u,v). Similarly, x appears only in f(x,v+T), hence we integrate next with respect to x to obtain a function of v+T which we denote G_2(v+T). Paying due attention to the x = v + T plane which is a plane of symmetry for f(x,v+T) we obtain G_2(v+T) from the result in (6.4) simply by replacing v with v+T. Thus

$$(6.15)\quad G_2(v+T) = \frac{2w_1-w_2}{w_1(w_1-w_2)}\{e^{-w_2(v+T)} - e^{-w_1(v+T)}\}.$$

Then $\overline{\Psi}_S(T)$ is given by

$$(6.16)\quad \bar{\Psi}_S(T) = \frac{2Q^2A_0(w_1^2w_2)^2}{(2w_1-w_2)^2}\int_0^\infty G_1(v)G_2(v+T)dv$$

$$= \frac{2Q^2A_0w_1^2w_2^2}{(w_1-w_2)^2}\int_0^\infty \{e^{-w_2v} - e^{-w_1v}\}\{e^{-w_2(v+T)}-e^{-w_1(v+T)}\}dv$$

$$= \frac{Q^2A_0w_1w_2}{(w_1-w_2)(w_1+w_2)}\{w_1e^{-w_2T} - w_2e^{-w_1t}\}.$$

We have yet to evaluate the second term of (4.21) in order to obtain the correlation function. Designating this term $\bar{\Psi}_N(T)$ and referring to (6.12) we have

$$(6.17)\quad \bar{\Psi}_N(T) = 4A_0^2\left(\frac{2w_1^2w_2}{2w_1-w_2}\right)^2 2\int_0^\infty\int_u^\infty \cos^2 w_0(u-v)\{e^{(w_1-w_2)u-w_1v} -$$

$$- e^{-w_1(u+v)}\}\{e^{(w_1-w_2)(u+T)-w_1(v+T)} - e^{-w_1(u+v+2T)}\}dvdu,$$

where the limits of integration are again determined from the consideration that $g(u,v)$ is zero for negative values of either argument and $0 \le u \le v$. The factor of 2 enters because $g(u,v)$ is symmetric with respect to the line $u = v$ and the integration is over the first quadrant in the uv plane.

Again the high frequency components are neglected by taking only the $\frac{1}{2}$ term in the identity

$$\cos^2 w_0(u-v) = \tfrac{1}{2} + \tfrac{1}{2}\cos 2w_0(u-v).$$

Integrating (6.17) we obtain

$$(6.18)\quad \Psi_N(T) = \frac{A_0^2}{w_1}\left(\frac{2w_1^2w_2}{2w_1-w_2}\right)^2\int_0^\infty\{e^{-w_2u}-e^{-2w_1u}\}\{e^{-w_2(u+T)} - e^{-2w_1(u+T)}\}du$$

$$= \frac{A_0^2w_1^2w_2}{4w_1^2-w_2^2}\{2w_1e^{-w_2T} - w_2e^{-2w_1T}\}.$$

Combining (6.16) and (6.18), the correlation function is

$$(6.19)\quad \Psi(T) = \frac{Q^2A_0w_1w_2}{(w_1-w_2)(w_1+w_2)}\{w_1e^{-w_2T} - w_2e^{-w_1T}\} + \frac{A_0^2w_1^2w_2}{4w_1^2-w_2^2}\{2w_1e^{-w_2T} - w_2e^{-2w_1T}\}.$$

We observe that the first derivative of $\Psi(T)$ evaluated at $T=0$ is zero as is required since $\Psi(T)$ is an even valued function by definition.

As a further check we evaluate the second cumulant which is according to (4.6)

$$(6.20)\quad \Psi(0) = \frac{Q^2A_0w_1w_2}{w_1+w_2} + \frac{A_0^2w_1^2w_2}{2w_1+w_2} = \frac{2N^2z}{1+\delta} + \frac{N^2}{1+2\delta},$$

where the noise power N is $N = w_1A_0$ and z is the signal to noise power ratio defined $z = Q^2/2N$. The bandwidth ratio δ is here defined as $\delta = w_1/w_2$. This result for the second cumulant agrees with that obtained independently from the output probability distribution (23, p. 47).

As before, we compute the second derivative of

(6.19) evaluated at $T = 0$ to be

$$(6.21)\quad \overline{\Psi}''(0) = \frac{2A_0^2 w_1^3 w_2 (2w_1 - w_2)}{-4w_1^2 + w_2^2} - \frac{Q^2 A_0 w_1^2 w_2^2}{w_1 + w_2}$$

$$= - w_1 w_2 N^2 \{\frac{2}{1+2\delta} + \frac{2z}{1+\delta}\}.$$

Substitution of (6.20) and (6.21) into equation (2.32) yields for the expected number of crossings of the bias line x_0 by the output signal plus noise

$$(6.22)\quad N_x = \frac{1}{\pi}\sqrt{2w_1 w_2 \frac{\frac{1}{2\delta+1} + \frac{z}{\delta+1}}{\frac{1}{2\delta+1} + \frac{2z}{\delta+1}}}\, e^{-x_0^2/2\overline{\Psi}(0)} .$$

For noise only we have $z = 0$ and the useful result

$$(6.23)\quad N_x = \frac{1}{\pi}(2w_1 w_2)^{\frac{1}{2}} e^{-x_0^2(1+2\delta)/2N^2} .$$

SECOND ORDER FILTER

In this final application the predetection filter is to be second order and the postdetection filter is to be first order in the system of Figure 1. The Fourier transform of the voltage transfer function for the postdetection filter is tendered by (6.9). The second order filter is characterized by the voltage transfer function (23, p. 19)

$$(7.1)\quad F_2(w) = \{1-(\frac{w-w_0}{w_1})^2+2bi\frac{w-w_0}{w_1}\}^{-1} + \\ + \{1-(\frac{w+w_0}{w_1})^2+2bi\frac{w+w_0}{w_1}\}^{-1}.$$

This transfer function applies to the second order filter composed of two resistance-capacitance elements of the type in Figure 2 connected in tandem in the same way that (6.4) applies to the circuit of Figure 2. More significantly, equation (7.1) also applies to the conventional double tuned inductively coupled circuit when the carrier frequency w'_0 is large compared to the filter bandwidth, a condition that is generally true in a physical system. For the double tuned circuit the parameters are defined by

$$(7.2)\quad w_1 = \frac{w_0'}{2Q_0}\left(1+\frac{KQ_0}{1-K^2}\right)^{\frac{1}{2}},\quad b = \left(1+\frac{KQ_0}{1-K^2}\right)^{-\frac{1}{2}} < 1,$$

$$w_0 = w_0'\left\{1 + \frac{K}{2(1-K^2)}\right\},$$

where K is the coefficient of coupling and Q_0 is the parallel circuit Q at the resonant frequency w_0'. The quantity w_1 is a measure of the filter bandwidth.

The Fourier transform of (7.1) is somewhat clumsy but straightforward to obtain. Consider the first term of (7.1). Using equation (3.5) and letting $uw_1 = w - w_0$ we have for the transform of the first term after rearrangement

$$\frac{w_1 e^{iw_0t}}{4\pi(b^2-1)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{iw_1ut}\left(\{b-(b^2-1)^{\frac{1}{2}}+iu\}^{-1} - \{b+(b^2-1)^{\frac{1}{2}}+iu\}^{-1}\right)du.$$

This integral is readily evaluated using the known results

$$\int_{-\infty}^{\infty}\frac{a\cos wu}{a^2+u^2}du = \int_{-\infty}^{\infty}\frac{\cos wat}{1+t^2}dt = \begin{cases}\pi e^{-wa}, & w > 0\\ \pi, & w = 0\\ \pi e^{wa}, & w < 0\end{cases}$$

and

$$\int_{-\infty}^{\infty}\frac{u\sin wu}{a^2+u^2}du = \int_{-\infty}^{\infty}\frac{t\sin wat}{1+t^2}dt = \begin{cases}\pi e^{-wa}, & w > 0\\ 0, & w=0\\ -\pi e^{wa}, & w < 0\end{cases},$$

and noting that the real part of $b - (b^2-1)^{\frac{1}{2}}$ is greater than zero. The second term in (7.1) is treated similarly, and the resulting complete Fourier transform of (7.1) is then

$$(7.3)\quad f_2(t) = \frac{w_1}{c}\cos w_0 t \begin{cases} e^{-w_1(b-c)t} - e^{-w_1(b+c)t}, & t \geq 0 \\ 0, & t < 0, \end{cases}$$

where for convenience we have set $c = (b^2-1)^{\frac{1}{2}}$. For the realistic double tuned filter c is imaginary since $b < 1$. For the second order resistance-capacitance filter $b > 1$ hence c is real. For either case the crossing theory development is valid. This is a distinct advantage this method maintains over the probability density approach of Emerson.

To simplify the notation we write

$$(7.4)\qquad w_{11} = w_1(b-c), \quad w_{12} = w_1(b+c),$$

where w_{11} and w_{12} may both be complex but their sum and product are real. Referring to (3.9), (6.9), and (7.3) the system kernel may be written for $v \geq u \geq 0$ as

$$(7.5)\ g(u,v) = \frac{w_1^2 w_2}{2c^2}\cos w_0(u-v)\int_0^u \{e^{-w_{11}(u-z)} -$$

$$- e^{-w_{12}(u-z)}\}e^{-w_2 z}\{e^{-w_{11}(v-z)} - e^{-w_{12}(v-z)}\}dz,$$

where again the high frequency contribution has been neglected. The integration limits result from the fact that each f(t) function is zero for negative values of its argument. The kernel g(u,v) is of course zero for u,v<0.

For $u \geq v \geq 0$ the u and v are interchanged in (7.5) and $g(u,v)$ is clearly symmetric in u and v as expected. We also observe from (7.4) that when c is imaginary the function $g(u,v)$ remains a real function.

Carrying out the integration in (7.5) and rearranging the result we have

$$(7.6)\quad g(u,v) = \frac{w_1^2 w_2 (w_{12}-w_{11}) \cos w_0(u-v)}{2c^2(w_{11}+w_{12}-w_2)} \Big(\frac{1}{2w_{11}-w_2}\{e^{-w_{11}(u+v)}\cdot$$

$$\cdot (e^{(2w_{11}-w_2)u}-1)\} - \frac{1}{2w_{12}-w_2}\{e^{-w_{12}(u+v)}(e^{(2w_{12}-w_2)u}-1)\}\Big) +$$

$$+ \frac{w_1^2 w_2 \cos w_0(u-v)}{2c^2(w_{11}+w_{12}-w_2)}\Big(e^{-w_{11}(u+v)}\{e^{-(w_{12}-w_{11})u}-1\} +$$

$$+ e^{-w_{12}(u+v)}\{e^{(w_{12}-w_{11})u}-1\}\Big).$$

The evaluation of the correlation function in (4.21) is somewhat difficult because of the complexity of this second order system kernel. While the integrals in (4.21) are still elementary in principle, the algebraic complexity of the evaluation is quite great; this is particularly true for the first integral which involves the signal. If the input to the system is noise only then the problem is reduced to evaluation of the second term in (4.21). This result is a useful one since it enables one to determine the false alarm time and subsequently to choose a suitable bias level x_0.

From equations (4.21) and (7.6) we have for the noise only case

$$(7.7)\quad \overline{\Psi}_N(T) = 4A_0^2\int_0^{\infty}\int_u^{\infty}\cos^2 w_0(u-v)(K_1\{e^{-w_{11}(u+v)}(e^{(2w_{11}-w_2)u}-$$

$$-1)\}-K_2\{e^{-w_{12}(u+v)}(e^{(2w_{12}-w_2)u}-1)\}+K_3\{e^{-w_{11}(u+v)}\cdot$$

$$\cdot(e^{-(w_{12}-w_{11})u}-1)\}+K_4\{e^{-w_{12}(u+v)}(e^{(w_{12}-w_{11})u}-1)\})(K_1\cdot$$

$$\cdot\{e^{-w_{11}(u+v+2T)}(e^{-(w_{11}2-w_2)(u+T)}-1)\}+K_2\{e^{-w_{12}(u+v+2T)}\cdot$$

$$\cdot(e^{(2w_{12}-w_2)(u+T)}-1)\}+K_3\{e^{-w_{11}(u+v+2T)}(e^{-(w_{12}-w_{11})(u+T)}-$$

$$-1)\}+K_4\{e^{-w_2(u+v+2T)}(e^{(w_{12}-w_{11})(u+T)}-1)\})dvdu,$$

where

$$(7.8)\qquad K_1 = \frac{w_1^2 w_2(w_{12}-w_{11})}{2c^2(w_{11}+w_{12}-w_2)(2w_{11}-w_2)},$$

$$K_2 = \frac{w_1^2 w_2(w_{12}-w_{11})}{2c^2(w_{11}+w_{12}-w_2)(2w_{12}-w_2)},$$

$$K_3 = K_4 = \frac{w_1^2 w_2}{2c^2(w_{11}+w_{12}-w_2)}.$$

The limits of integration are determined by recalling that $g(u,v)$ is zero for negative values of either argument and $0 \leq u \leq v$. The factor of two enters from symmetry of $g(u,v)$ in u and v and the fact that the integration is to cover the first quadrant.

The evaluation of the integrals in (7.7) is quite straightforward but requires many pages to record step by step, hence only the result will be stated. As before, we neglect the high frequency contribution through approximation of $\cos^2 w(u-v)$ by $\frac{1}{2}$. The final result is

$$(7.9)\quad \Psi_N(T) = \frac{A_0^2 w_1^4 w_2^2 (w_{12}-w_{11})^2}{8c^4(w_{11}+w_{12})}\Big\{(w_{11}+w_{12}-w_2)^{-1}\cdot$$

$$\cdot\frac{e^{-(w_{11}+w_{12})T}}{(w_{11}+w_{12}+w_2)(w_{11}+w_{12})w_{11}w_{12}} - \frac{e^{-2w_{11}T}}{2w_{11}^2(w_{11}+w_{12})(4w_{11}^2-w_2^2)} -$$

$$- \frac{e^{-2w_{12}T}}{2w_{12}^2(w_{11}+w_{12})(4w_{12}^2-w_2^2)} +$$

$$+ \frac{(w_{12}-w_{11})e^{-w_2T}(4w_{11}+2w_{12}+w_2)}{w_2w_{11}(w_{11}+w_{12}-w_2)(w_{11}+w_{12}+w_2)(2w_{12}+w_2)(4w_{11}^2-w_2^2)} -$$

$$- \frac{(w_{12}-w_{11})e^{-w_2T}(2w_{11}+4w_{12}+w_2)}{w_2w_{12}(w_{11}+w_{12}-w_2)(w_{11}+w_{12}+w_2)(2w_{11}+w_2)(4w_{12}^2-w_2^2)}\Big\}.$$

The correlation function $\Psi_N(T)$ is necessarily a real function although this is not immediately apparent from (7.9). From (7.9) we obtain for $\Psi_N(0)$ after simplification

$$(7.10)\quad \Psi_N(0) = \frac{A_0^2 w_1^4 w_2\{2(w_{11}^2+3w_{11}w_{12}+w_{12}^2)+3w_2(w_{11}+w_{12})+w_2^2\}}{(w_{11}+w_{12})^2(w_{11}+w_{12}+w_2)(2w_{11}+w_2)(2w_{12}+w_2)}.$$

It is apparent from equation (7.4) that this result is real as required. It is not possible in this particular application to use $\Psi_N(0)$ as a check on the result (7.9) since the cumulants have not been evaluated independently as in the earlier applications.

As a last result we need the second derivative of (7.9) evaluated at $T = 0$. This is readily obtained and after much algebraic simplification is written

$$(7.11)\quad \Psi_N''(0) = -\frac{(w_{11}+w_{12})^{-2}\,2A_0^2 w_1^6 w_2^2(2w_{12}+2w_{11}+w_2)}{(w_{11}+w_{12}+w_2)(2w_{11}+w_2)(2w_{12}+w_2)}\,.$$

Equations (7.10) and (7.11) together with (2.32) yield for the expected number of crossings of the bias line x_0 by the output noise

$$(7.12)\quad N_x = \frac{1}{\pi}\left\{\frac{2w_1^2(2w_{11}+2w_{12}+w_2)w_2}{2(w_{11}^2+3w_{11}w_{12}+w_{12}^2)+3w_2(w_{11}+w_{12})+w_2^2}\right\}^{\frac{1}{2}} e^{-\frac{x_0^2}{2\Psi_N(0)}}$$

where $\Psi_N(0)$ is given by equation (7.10).

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