THE INVERSE WEIERSTRASS p-FUNCTION: NUMERICAL SOLUTION, RELATED PROPERTIES AND APPLICATIONS

by

VERNER EMIL HOGGATT, JR.

A THESIS

submitted to

OREGON STATE COLLEGE

in partial fulfillment of
the requirements for the
degree of

DOCTOR OF PHILOSOPHY

June 1956

APPROVED:

Redacted for Privacy

Professor of Mathematics

In Charge of Major

Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy

Chairman of School of Science Graduate Committee

Redacted for Privacy

Dean of Graduate School

Date thesis is presented August 8, 1955

Typed by Louis Rall

ACKNOWLEDGMENT

From the date of inception of this thesis problem Professor C. L. Clark has tirelessly persevered with me. The aid given by K. O. Wolf in the drawings and calculations has been substantial. The fine and friendly advice of Professor C. D. Olds of San Jose State College is duly appreciated. Were it not for the patience and understanding of my wife Herta this thesis could not have been written.

TABLE OF CONTENTS

THE INVERSE WEIERSTRASS p-FUNCTION: NUMERICAL SOLUTION, RELATED PROPERTIES AND APPLICATIONS

CHAPTER I

INTRODUCTION

A doubly-periodic function of a complex variable is a function satisfying the functional equation

$$\xi(z) = \xi(z + mw + m'w'),$$

whose periods w and w' are two complex numbers which are not collinear with the origin, while m and m' are integers. This equation must hold for all z and for any integers m and m'. In the complex z-plane draw the line L determined by the origin and w and similarly the line L' determined by the origin and w'. Next draw lines parallel to L' through mw, $m = 0, \pm 1, \ldots, \pm n, \ldots$, and lines parallel to L through $m'w'$, $m' = 0, \pm 1, \ldots, \pm n, \ldots$. These lines divide the plane into a network of parallelograms. The double periodicity means that ξ assumes the same value, or exhibits the same singularity at the corresponding (or congruent) points of each parallelogram of the network. Thus in studying a doubly-periodic function, it is sufficient to study it in one parallelogram of the network. We shall call the parallelogram whose vertices are 0, w, $w + w'$, w', the fundamental or period parallelogram of

$\zeta(z)$. The region belonging to the fundamental parallelogram is the interior and the boundaries $z = \delta w$ and $z = \delta w'$ for $0 \leq \delta < 1$.

A <u>doubly-periodic function</u> is called an <u>elliptic function</u> if it has no singularities other than poles in the period parallelogram. In other words: A meromorphic doubly-periodic function is called an elliptic function belonging to this fundamental parallelogram.

The Weierstrass p-function is perhaps the simplest of the non-constant elliptic functions. The p-function has poles of order two at $\Delta = mw + m'w'$. A well-known theorem of elliptic functions is (8, p. 346): An elliptic function of order k takes on every value in the complex plane exactly k times in the fundamental parallelogram. Hence the equation $p(z) = B$, B an arbitrary complex number, has exactly two solutions in the fundamental parallelogram.

The main problem of this thesis is to find these two solutions given the periods w and w'. This is accomplished by inverting the p-function.

One method for inverting an analytic function is the inversion of its Taylor series expansion. A second is the use of an inversion integral. For example, two well-known inversion integrals are

$$\int_0^x (1 - t^2)^{-1/2} dt = \sin^{-1} x, \text{ and} \int_1^x \frac{dt}{t} = \ln x.$$

Both of the functions here satisfy differential equations of a special type. The function $\sin x$ satisfies the differential equation

$$\{\frac{d}{dx}\sin x\}^2 = 1 - \sin^2 x,$$

and e^x satisfies

$$\frac{d}{dx}e^x = e^x.$$

In both cases some positive integral power of the first derivative is a plain polynomial in the function itself. The p-function also satisfies a differential equation of this type.

The Weierstrass p-function satisfies the differential equation

$$\{p'(z)\}^2 = 4p^3(z) - g_2 p(z) - g_3,$$

where g_2 and g_3 are complex constants dependent only on the periods w and w'. The Mittag-Leffler expansion of the p-function reveals that p(z) is a complex function of really three complex variables, namely z, w, and w',

$$p(z|w,w') = \frac{1}{z^2} + \sum_{m'=-\infty}^{\infty} \sum_{m=-\infty}^{\infty}{}' \{\frac{1}{(z-mw-m'w')^2} - \frac{1}{(mw+m'w')^2}\}.$$

The prime on the above summation indicates that m and m' are not both zero in any one term.

The p-function considered as a function of three complex variables z, w, w' is homogeneous in those variables of degree −2. The constants $g_2(w,w')$ and $g_3(w,w')$ are homogeneous functions of w and w' of degrees −4 and −6 respectively, and are called the invariants of p(z) because they are invariant under the modular group of transformations (3, p. 152).

It is well known that the p-function can be inverted by means of the elliptic integral:

$$z = p^{-1}(B) = \int_B^{\infty} (4t^3 - g_2 t - g_3)^{-1/2} dt,$$

where the path of integration does not pass through any of the singularities of the integrand. The branch-points are e_1, e_2, e_3, and ∞, where e_1, e_2, and e_3 are the roots of $4t^3 - g_2 t - g_3 = 0$. The value assigned to the contour integral depends upon how the path of integration winds around the branch-points (11, p. 437 and 7, pp. 167–169).

The particular path of integration we choose gives us the values of z in the fundamental parallelogram. With the periods known, we can write out a formula for all the solutions of p(z) = B in the complex plane (8, p. 348).

While the contour integral described above formally solves the problem of inverting the p-function, the problem remains as actually how to evaluate the integral along some path. If we expand the integrand in a Taylor series about the origin, the series will converge for

$$|t| < \min \{|e_1|, |e_2|, |e_3|\}.$$

It is impossible to evaluate the integral with term by term integration if both limits do not lie in or on the circle of convergence. Since our upper limit is infinity and the radius of convergence of the Taylor series is always finite, the Taylor series expansion method is not feasible.

If we make a Taylor expansion about infinity by means of $t = 1/t'$, the transformed integrand has a singularity of $t'^{-1/2}$ at the origin. Since the path must approach the origin, therefore this method also lacks promise.

It is well known that there is an algebraic relationship (8, p. 448) between the Jacobi elliptic function $\operatorname{sn} z$ and the Weierstrass p-function. If we could invert one, we could invert the other. The integrand of the inversion integral for the Jacobi elliptic function can be expanded in a Taylor series and integrated term by term. For those values inside the circle of convergence for which this is valid, we could use these results to invert the Weierstrass p-function.

We seek however a more general method which, while not completely conquering our difficulties, allows us more freedom for B, the values for which we can actually effect the inversion of the p-function.

Copson (1, pp. 361–362) remarks that if a cubic polynomial $P(x)$ has repeated roots, the $\int \{P(x)\}^{-1/2} dx$ can be evaluated by elementary functions. We shall make good use of this fact.

If $g_2 \neq 0$, we define $g_2 = 12\lambda^4$, $g_3 = 4(2 + h)\lambda^6$, $t = \lambda^2 x$, then the inversion integral for the p-function (1, p. 361 and 11, p. 437) becomes

$$z = p^{-1}(B) = \int_{B/\lambda^2}^{\infty} \frac{1}{2\lambda}(x^3 - 3x - 2 - h)^{-1/2} dx.$$

We expand the integral in a Maclaurin series in the variable h, getting successive derivatives by differentiating under the integral sign. The series obtained is of the form

$$z = \frac{1}{2\lambda} \sum_{n=0}^{\infty} S_n K_n h^n,$$

where $S_n = (2n)!/2^{2n}(n!)^2$, and $K_n = \int_{B/\lambda^2}^{\infty} (x^3 - 3x - 2)^{-\frac{2n+1}{2}} dx.$

We note here that $x^3 - 3x - 2 = (x + 1)^2(x - 2)$ so that $x = -1$ is a repeated root. It turns out that the K_n can be expressed in terms of elementary functions if we alter the form slightly. Let $x - 2 = r^2$, then

$$K_n = \int_A^{\infty} \frac{2dr}{r^{2n}(r^2+3)^{2n+1}} ,$$

where $$A = -(B/\lambda^2 - 2)^{1/2}.$$

The integrals K_n can now be evaluated by integrating by parts n times and using standard integration formulas. The K_n are thus expressible as a finite number of elementary functions of B and λ^2.

The main results of this thesis are:

Theorem I. If the Weierstrass p-function has invariants g_2 and g_3 such that $g_2 = 12\lambda^4 \neq 0$ and $g_3 = 4(2+h)\lambda^6$, then an inverse of the p-function can be obtained by

$$p^{-1}(B) = \frac{1}{2\lambda} \sum_{n=0}^{\infty} S_n K_n h^n, \tag{3.01}$$

where $S_n = \dfrac{(2n)!}{2^{2n}(n!)^2}$ and $K_n = \displaystyle\int_A^{\infty} (v^3 - 3v - 2)^{-\frac{2n+1}{2}} dv$,

provided the path of integration does not pass through or terminate in the regions $|v^3-3v-2| \leq |h|$.

Theorem II. If the Weierstrass p-function has invariants $g_2 = 12\lambda^4 \neq 0$ and $g_3 = 4(2+h)\lambda^6$, and further the periods w and w' are known, then the p-functions for which $|h|$ is sufficiently large or less than or equal to 2 can be completely inverted for all values in the complex plane by means of the series (3.01).

Corollary I. If the periods of the p-function are such that their ratio is pure imaginary and the periods are known to be $w = b > 0$ and $w' = i$, then the p-function can be inverted completely by means of the series (3.01), in spite of the fact that $|h| \leq 2$ is not always satisfied, by solving $p(-biz|b,i) = -B/b^2$ whenever $-4 < h < -2$.

Corollary II. Any elliptic function $F(z)$ of order k having the same periods as a p-function which can be inverted completely by series (3.01) can itself be inverted completely by series (3.01) by solving k p-function inversions for each desired inverted value of $F(z)$.

Theorem III. If a Weierstrass p-function has invariants g_2 and g_3 such that $|h| < 4$ and $|4+h| < 4$, then the periods are given by

$$w = \frac{1}{2\lambda_0} \sum_{n=0}^{\infty} B_n h^n, \quad w' = \frac{1}{2\lambda_0} \sum_{n=0}^{\infty} B_n (-4-h)^n,$$

where $B_n = 2\pi(2n+1)(2n+3)\ldots(6n-1)/3^{1/2}(n!)^2(-432)^n$. This includes finding the periods of the rectangular case for which $0 > h > -4$, which is discussed in (5, pp. 357-368).

CHAPTER II

THE DERIVATION OF THE SERIES FOR $p^{-1}(B)$

The well-known inversion integral for the Weierstrass p-function is

$$z = p^{-1}(B) = \int_{B}^{\infty} (4u^3 - g_2 u - g_3)^{-1/2} du.$$

Since p(z) takes on each value in the complex plane twice in each period parallelogram, the value assigned to the inverse function depends upon the manner in which the path of integration winds around the branch-points. To show this let D_1 and D_2 be two distinct paths of integration starting at B. In the u-plane, consider a circle $|u| = R$ (see Figure 1). Since D_1 and D_2 are continuous curves both extending to infinity, they both cut $|u| = R$, say in u_1 and u_2, respectively. If the path of integration C from B along D_1 to u_1, then along $|u| = R$ to u_2, and back along D_2 to B does not pass through or enclose on the left any branch-points of the integrand, then the integrand is analytic within contour C and continuous on C. Since the conditions of the Cauchy theorem are satisfied, the value of the contour integral around contour C is zero. We next

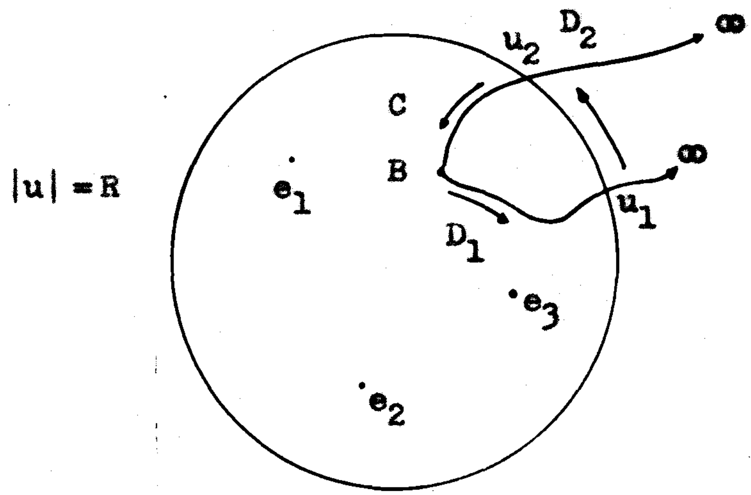


Figure 1

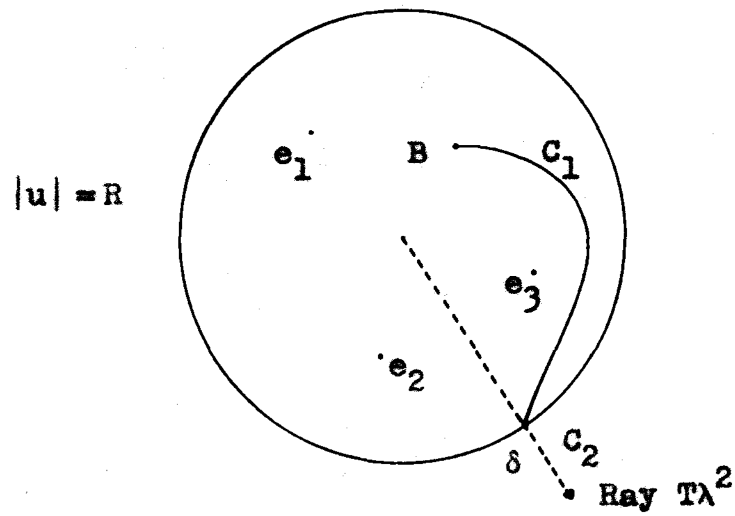


Figure 2

show that the contribution from part of C on the circle $|u| = R$ tends to zero as R becomes large without bound. First let R be large enough so that the circle $|u| = R$ contains all the finite branch-points of the integrand. Using the integral inequalities concerning absolute values, we are able to write

$$\left| \int_{u_1}^{u_2} \frac{dz}{\{4(z-e_1)(z-e_2)(z-e_3)\}^{1/2}} \right| < \int_{T_1}^{T_2} \frac{|dT|}{2R_1^{3/2}} = \frac{R|T_1-T_2|}{2R_1^{3/2}} = \frac{RT'}{2R_1^{3/2}},$$

where the integrals are taken along the curve $|u| = R$ and R_1 is the minimum of the three distances $|z - e_1|$, $|z - e_2|$, and $|z - e_3|$. Clearly the ratio R/R_1 tends to one as R becomes large without bound. The maximum of RT' is of course $2\pi R$. Hence the limit of $RT'/2R_1^{3/2}$ is zero as R becomes large without bound. This shows that the value of the integral along D_1 or D_2 is the same since

$$0 = \lim_{R\to\infty} \{ \int_{\substack{B \\ D_1}}^{u_1} + \int_{\substack{u_1 \\ |u|=R}}^{u_2} + \int_{\substack{u_2 \\ -D_2}}^{B} \} = \int_{D_1} + \int_{-D_2} = 0 \text{ and } \int_{D_1} = \int_{D_2} .$$

Therefore the value of the integral along any two paths of integration that do not enclose any branch-points is the same. If, however, the two paths do enclose one or more branch-points then the two values of the integral are not the same (7, pp. 167–169).

The value of $z = p^{-1}(B)$ is thus independent of the path of integration within the foregoing restrictions and we choose a particular path of integration with which to evaluate the integral. Let $|u| = R$ be a circle containing all the finite branch-points e_1, e_2, and e_3 of the integrand. Let $g_2 = 12\lambda^4 \neq 0$ and let T be real and positive. Let C be any regular contour joining B to δ, the point where the ray $T\lambda^2$ intersects the circle $|u| = R$. Then (see Figure 2)

$$(2.01) \quad z = \int_C \frac{du}{(4u^3 - g_2u - g_3)^{1/2}} + \lim_{T\to\infty} \int_\delta^{T\lambda^2} \frac{du}{(4u^3 - g_2u - g_3)^{1/2}}.$$

We now make the following change of variables. Let $g_2 = 12\lambda^4 \neq 0$, $g_3 = 4(2+h)\lambda^6$, $u = \lambda^2 v$. This is justified by the following theorem from Pierpont (8, p. 193):

Theorem A. Let $f(u)$ be continuous on the curve C. When u ranges over C, let $v = g(u)$ range over a curve D which corresponds unipunctually to C. Let the inverse function $u = \eta(v)$ have a continuous derivative on D. Then,

$$\int_C f(u)\,du = \int_D f\{\eta(v)\}\eta'(v)\,dv.$$

By unipunctually, it is meant that as u traces out the curve C, the variable $v = g(u)$ traces out a curve D once and only once. In the proof of Pierpont's theorem the contours C and D are assumed to be rectifiable or finite, conditions which are satisfied in our case. Hence we may write

$$\int_C \frac{du}{(4u^3-g_2u-g_3)^{1/2}} = \frac{1}{2\lambda}\int_D \frac{dv}{(v^3-3v-2-h)^{1/2}},$$

where we have made the change of variable $u = \lambda^2 v$, $g_2 = 12\lambda^4$, $g_3 = 4(2+h)\lambda^6$. For each finite T, we may write

$$\int_{\delta}^{T\lambda^2} \frac{du}{(4u^3-g_2u-g_3)^{1/2}} = \frac{1}{2\lambda}\int_{\frac{R}{|\lambda^2|}}^{T} \frac{dv}{(v^3-3v-2-h)^{1/2}},$$

hence

$$\lim_{T\to\infty}\int_{\delta}^{T\lambda^2} \frac{du}{(4u^3-g_2u-g_3)^{1/2}} = \lim_{T\to\infty}\frac{1}{2\lambda}\int_{\frac{R}{|\lambda^2|}}^{T} \frac{dv}{(v^3-3v-2-h)^{1/2}}.$$

Thus equation (2.01) can be written as

$$z = \frac{1}{2\lambda}\int_D \frac{dv}{(v^3-3v-2-h)^{1/2}} + \lim_{T\to\infty}\frac{1}{2\lambda}\int_{\frac{R}{|\lambda^2|}}^{T} \frac{dv}{(v^3-3v-2-h)^{1/2}}.$$

This leads to an integral of the form

(2.02) $$z = \frac{1}{2\lambda}\int_{\zeta}^{\infty} \frac{dv}{(v^3-3v-2-h)^{1/2}}$$

where the path of integration in the v-plane goes from $\delta' = \zeta = B/\lambda^2$ along D to the real positive $R/|\lambda^2| = T_1$ then from T_1 out the positive real axis to infinity.

We wish to consider next the integral part of equation (2.02) as a function of the variable h. Suppose that the integral on the right-hand side of equation (2.02) represents an analytic function of the variable h, whose derivatives of all orders with respect to h may be obtained by differentiating under the integral sign. Suppose we do this formally and thus obtain a Maclaurin series in the variable h,

$$f(h) = \int_{\delta'}^{\infty} \frac{dv}{(v^3-3v-2-h)^{1/2}} = \sum_{n=0}^{\infty} S_n K_n h^n,$$

where $S_n = \dfrac{(2n)!}{2^{2n}(n!)^2}$ and $K_n = \displaystyle\int_{\delta'}^{\infty} \frac{dv}{(v^3-3v-2)^{\{2n+1\}/2}}$.

We will find a region of convergence of this formal series and by a continuity argument, we will show that this series multiplied by $1/2\lambda$, evaluated at the proper lower limit of integration and the proper value of h, represents an inverse of the p-function.

CHAPTER III

REGION OF CONVERGENCE OF THE FORMAL SERIES

In Chapter II we derived in a formal manner the series

$$(3.01) \qquad f(h) = \int_{\delta'}^{\infty} \frac{dv}{(v^3-3v-2-h)^{1/2}} = \sum_{n=0}^{\infty} S_n K_n h^n,$$

where $S_n = (2n)!/2^{2n}(n!)^2$ and $K_n = \int_{\delta'}^{\infty} (v^3-3v-2)^{-\frac{2n+1}{2}} dv$, by assuming we could obtain the successive derivatives of the function represented by the integral in equation (2.02) by differentiating under the integral sign. We now seek the region of convergence with respect to the variable of expansion h.

The contour integrals K_n, which are coefficients in the formal series (3.01), are evaluated along a contour consisting of two pieces, namely C_1, any regular contour joining ζ to T_1, a large real positive number, and C_2, a contour along the positive real axis from T_1 to infinity.

We now investigate the region of absolute convergence of the formal series (3.01). Using some integral inequalities concerning absolute values and the fact that the maximum of the sum of two functions is less than or equal to the sum of the maximums of the individual functions, we may write the following sequence of inequalities: Since $(v^3-3v-2) = (v+1)^2(v-2)$,

$$|K_n h^n| \leq \int_{C_1} \frac{|h^n|\,|dv|}{|v+1|^{2n+1}\,|v-2|^{\frac{2n+1}{2}}} + \int_{C_2} \frac{|h^n|\,|dv|}{|v+1|^{2n+1}\,|v-2|^{\frac{2n+1}{2}}}.$$

Furthermore,

$$\int_{C_1} \frac{|h^n|\,|dv|}{|v+1|^{2n+1}\,|v-2|^{\frac{2n+1}{2}}} \leq \frac{|h^n|}{\min\limits_{\substack{\text{on}\\ C_1}} |(v+1)^2(v-2)|^n} \int_{C_1} \frac{|dv|}{|v+1|\,|v-2|^{1/2}}$$

and

$$\int_{C_2} \frac{|h^n|\,|dv|}{|v+1|^{2n+1}\,|v-2|^{\frac{2n+1}{2}}} \leq \frac{|h^n|}{\min\limits_{\substack{\text{on}\\ C_2}} |(v+1)^2(v-2)|^n} \int_{C_2} \frac{|dv|}{|v+1|\,|v-2|^{1/2}}.$$

Therefore,

$$(3.02) \quad |K_n h^n| \leq \frac{|h^n|}{\min\limits_{\substack{\text{on}\\ C_1+C_2}} |(v+1)^2(v-2)|^n} \int_{C_1+C_2} \frac{|dv|}{|v+1|\,|v-2|^{1/2}} \; .$$

The last integral on the right is finite if $T_1 > 2$ and if the path of integration does not pass through -1 or $+2$. This also prevents the minimum of $|(v+1)^2(v-2)|$ over C_1+C_2

from being zero, providing path C_1 avoids -1 and $+2$ also.

Consider the series of positive terms $\sum_{n=0}^{\infty} u_n$, where the u_n are the right side of equation (3.02). By the ratio test, the series $\sum_{n=0}^{\infty} u_n$ converges provided that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ is < 1. In our case this leads to

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{|h|}{\min_{C_1+C_2} \text{ on } |(v+1)^2(v-2)|} . \tag{3.03}$$

In the v-plane the level curves $|(v+1)^2(v-2)| = |h|$ of the polynomial $(v+1)^2(v-2)$ are continuous closed curves. They divide the plane into two open regions. These level curves may be two disjoint closed curves or joined together at a single point or a single closed curve. Everywhere inside these level curves the strict inequality

$$|(v+1)^2(v-2)| < |h| \tag{3.04}$$

holds by the maximum modulus theorem for analytic functions. It will be shown later that outside of these curves the strict inequality holds

$$|(v+1)^2(v-2)| > |h|. \tag{3.05}$$

We therefore conclude that the test ratio for $\sum u_n$ is less than one if no part of the contour C_1+C_2 passes inside or on one of the level curves $|(v+1)^2(v-2)| = |h|$. By the comparison test with $\sum u_n$, the formal series (3.01) converges absolutely and uniformly since the $S_n \leq 1$ for all n (9, p. 3).

Thus the region of absolute and uniform convergence of the formal series (3.01) is defined by the level curves

$$|(v+1)^2(v-2)| = |h| \tag{3.06}$$

of the polynomial $v^3-3v-2 = (v+1)^2(v-2)$. Outside these level curves the formal series (3.01) converges absolutely and uniformly to an analytic function, but does it represent the integral from which it was obtained in a formal fashion? In the next part, we shall show that the series does indeed represent the integral for every lower limit $v = \zeta = B/\lambda^2$ satisfying the equation (3.05).

We shall in a still later section study these level curves in some detail. These together with certain addition formulas for the Weierstrass p-function allow us to complete the inversion of the p-function for $|h| \leq 2$. At our present state we have a formal series which converges but not for all possible lower limits of integration. This deficiency we can only partly avoid.

In our next chapter we will see that the conditions for absolute and uniform convergence of the formal series (3.01) to an analytic function are sufficient for the series (3.01) to represent the integral.

CHAPTER IV

THE INTEGRAL AND THE FUNCTION

In this chapter we will discuss the function defined by the integral

$$(4.01) \qquad f(h) = \int_A^{\infty} F(z,h)\,dz,$$

where $F(z,h) = (z^3-3z-2-h)^{-1/2}$. We shall prove the following most important theorem:

Theorem I. The function f(h) defined by the integral

$$f(h) = \int_{C_1+C_2} (z^3-3z-2-h)^{-1/2}dz$$

is an analytic function of h providing the path of integration does not intersect the closed regions defined in the z-plane by equation (4.05), and furthermore derivatives of all orders of f(h) with respect to variable h may be found by differentiating under the integral sign.

It was assumed in Chapters II and III that f(h) could be represented by the series (3.01) which was discussed there. To prove this we make use of the following theorems:

Theorem B. Let F(z,h) be a continuous function of the complex variables z and h, where h ranges over a region D, and z lies on a contour C. Let F(z,h) be an analytic function of h, then for every value of z on C,

$$f(h) = \int_C F(z,h)\,dz \tag{4.02}$$

is an analytic function of h in region D, and

$$f'(h) = \int_C \frac{\partial F(z,h)}{\partial h}\,dz, \tag{4.03}$$

and similarly for derivatives of higher orders (9, p. 99).

Theorem C. Let F(z,h) satisfy the following conditions:

(i) It is a continuous function of both variables when h lies inside a closed contour D, and $T \leq z \leq T'$, for every finite value of T';

(ii) for each such value of z, it is an analytic function of h, regular within D;

(iii) the integral $f(h) = \int_T^{\infty} F(z,h)\,dz$ is convergent

when h lies within D and uniformly convergent when h lies in any closed domain D' within D.

Then f(h) is an analytic function of h, regular within D, whose derivatives of all orders may be found by differentiating under the integral sign (1, p. 110).

Test for uniform convergence of an infinite integral: Let $F(t,h)$ be a continuous function of t when h lies in a bounded closed domain D and $t \geq a$ and which satisfies at each point of D the inequality

$$|F(t,h)| \leq M(t), \tag{4.04}$$

where $M(t)$ is a positive function independent of h. Then, if $\int_a^{\infty} M(t)dt$ converges, the integral $\int_a^{\infty} F(t,h)dt$ is uniformly and absolutely convergent in D (1, p. 111).

We saw in Chapter III that if the contour C_1+C_2 used in defining $f(h)$ in equation (3.01) does not intersect the closed regions of the complex plane defined by

$$|z^3-3z-2| \leq |h|, \tag{4.05}$$

then the Maclaurin series in h, where the successive derivatives of $f(h)$ were obtained by differentiating under the integral sign, is absolutely and uniformly convergent.

We shall now prove that the function

$$f(h) = \int_{C_1+C_2} \frac{dz}{(z^3-3z-2-h)^{1/2}} \tag{4.06}$$

does have precisely this expansion in a Maclaurin series in the variable h under the condition that the contour C_1+C_2 does not intersect the closed regions in the complex z-plane defined by equation (4.05).

To show this, let the contour C_1 start at z_0; we choose the region D in Theorem B to be $|h| < |h_0| < |z_0^3-3z_0-2|$, where the variable z ranges over contour C_1, the contour C of Theorem B.

Clearly, any singularities of $F(z,h)$ can occur only where $h = z^3-3z-2$. But this is impossible if C_1 does not intersect the closed regions of equation (4.05), because the equation $h = z^3-3z-2$ can only be satisfied on the level curves defined by the equality in equation (4.05). Therefore, since the level curves for $|h| < |h_0|$ lie inside those of $|h_0|$, it follows that $F(z,h)$ is continuous in both variables for z on C_1 for all $|h| < |h_0|$. Therefore, by Theorem B, the conclusion of Theorem I holds for contour C_1.

We now apply Theorem C to the integral in equation (4.06) where we are interested only in the path C_2, the path C_1 having been taken care of in the last paragraph. Clearly, if the conditions of Theorem B are met, then so are the first two conditions of Theorem C. We are therefore interested in the uniform convergence of the integral in (iii) of Theorem C. We now apply the test for uniform convergence of an infinite integral. Let $T_1 > 3 + |h_0|$, then clearly $|T_1^3-3T_1-2| > |h_0|$, and furthermore $T_1^3 > 2(3T_1+2+|h_0|) > 2(3T_1+2+|h|)$, so that finally $T_1^3/2 < |T_1^3-3T_1-2-h|$ for all $|h| \leq |h_0|$. From this it follows that $|T_1^3-3T_1-2-h|^{-1/2} < 4T_1^{-3/2}$ for all $|h| \leq |h_0|$, which is region D, and is closed and bounded. Since the integral $\int_{T_1>0}^{\infty} 4T^{-3/2}dT$ converges for every $T_1 > 0$, it follows by the test for uniform convergence of an infinite integral that the integral

$$\int_{C_2} F(t,h)\,dt$$

converges absolutely and uniformly for all $|h| \leq |h_0|$. This completes the proof of the satisfaction of condition (iii) of Theorem C. Therefore, by Theorem C, we may draw the conclusion of Theorem I for the contour C_2, provided C_2 does not intersect the closed regions defined by equation (4.05). This completes the proof of Theorem I.

Therefore, series (3.01) is the Maclaurin series of the function f(h), which by Theorem I is analytic and continuous. Combining results now we have our main theorem:

Theorem II. If $g_2 = 12\lambda^4 \neq 0$ and $g_3 = 4(2+h)\lambda^6$, then an inverse of the Weierstrass p-function having g_2 and g_3 as invariants is given by

$$(4.07) \qquad z = p^{-1}(B) = \frac{1}{2\lambda}\sum_{n=0}^{\infty} S_n K_n h^n \text{ for all } B = \lambda^2 v,$$

where $S_n = (2n)!/2^{2n}(n!)^2$ and $K_n = \int_{C_1+C_2} (v^3-3v-2)^{-\frac{2n+1}{2}} dv$,

and the contours C_1 and C_2 do not intersect the closed regions bounded by the level curves in the v-plane of the polynomial v^3-3v-2 given by

$$|v^3-3v-2| \leq |h|.$$

The constant λ and the value of the variable h are both determined by the conditions $g_2 = 12\lambda^4 \neq 0$ and $g_3 = 4(2+h)\lambda^6$. In Chapter II, we derived by change of variables that the inverse p-function was given by equation (2.02) when λ and h have the values determined above. The series (3.01) which is the right side of equation (4.07) is a continuous function of h and represents the integral of equation (2.02) when h and λ are determined by the conditions of Theorem II. This concludes the proof of Theorem II.

Still to be determined are the values of λ and h under various hypotheses. We wish to assume that the periods w and w' of the p-function are given. From this we must derive the necessary invariants g_2 and g_3, from which λ and h may be determined, so that an inverse may be computed using series (3.01). Also, the level curves of equation (4.05) limit the use of series (3.01), and the K_n are yet to be determined.

CHAPTER V

THE INVARIANTS g_2 AND g_3 OF THE p-FUNCTION

The Weierstrass p-function satisfies the differential equation

$$(5.01) \qquad \{p'(z|w,w')\}^2 = $$
$$= 4p^3(z|w,w')-g_2(w,w')p(z|w,w')-g_3(w,w'),$$

where w and w' are periods of the p-function. The quantities $g_2(w,w')$ and $g_3(w,w')$ are called the invariants of $p(z|w,w')$. These invariants are expressed in terms of the periods w and w' as follows

$$(5.02) \qquad g_2(w,w') = \sum_m \sum_{m'}{}' 60(mw+m'w')^{-4},$$

$$(5.03) \qquad g_3(w,w') = \sum_m \sum_{m'}{}' 140(mw+m'w')^{-6},$$

where the summations extend with respect to both m and m' over all the integers and the prime indicates that m and m' are not both zero in any one term.

Both series are absolutely convergent, hence they can be rearranged at our convenience (3, pp. 152-153). In this section we wish to derive a more practical series for computing the invariants g_2 and g_3 when the periods are given. In fact we will derive:

$$(5.04) \qquad g_2(w,w') = \frac{4\pi^4}{3w^4}(1+240\sum_{n=1}^{\infty}\delta_3(n)q^{2n})$$

$$(5.05) \qquad g_3(w,w') = \frac{8\pi^6}{27w^6}(1-504\sum_{n=1}^{\infty}\delta_5(n)q^{2n}),$$

where $q = \exp(\pi i\varepsilon)$, $\varepsilon = w'/w$, such that $I(\varepsilon) > 0$, and $\delta_k(n)$ is the sum of the kth powers of the divisors of n, including n and one.

Following (3, p. 154), we begin with the Mittag-Leffler partial fraction expansion of

$$(5.06) \qquad \pi \cot \pi u = \frac{1}{u} + \sum\{\frac{1}{u+m} - \frac{1}{m}\},$$

where the summation extends with respect to m over all the positive and negative integers but not zero; this partial fraction expansion is valid only for non-integral u. We borrow from complex variables another representation of the same function

$$(5.07) \qquad \pi \cot \pi u = \pi i(w+1)/(w-1),$$

where $w = \exp(2\pi iu)$; this expansion is also valid only for non-integral u, or what is the same, $|w| < 1$.

Differentiating the right-hand sides of equations (5.06) and (5.07) each with respect to variable u, three and five times respectively, using each time $dw/du = 2\pi i w$, we obtain the following after equating

$$(5.06)' \quad -6\sum_{-\infty}^{+\infty}(m+u)^{-4} = -16\pi^4(w+2^3w^2+\ldots+n^3w^n+\ldots),$$

$$(5.07)' \quad -120\sum_{-\infty}^{+\infty}(m+u)^{-6} = 64\pi^6(w+2^5w^2+\ldots+n^5w^n+\ldots),$$

where u^{-4} and u^{-6} have been put under the summation signs. Setting $u = m'\varepsilon$, $(m' > 0)$, so that $w = \exp(2\pi i m'\varepsilon) = t^{m'}$, we obtain

$$(5.06)'' \quad \sum_{-\infty}^{+\infty}(m+m'\varepsilon)^{-4} = \frac{8}{3}\pi^4(t^{m'}+2^3t^{2m'}+\ldots+n^3t^{nm'}+\ldots),$$

$$(5.07)'' \quad \sum_{-\infty}^{+\infty}(m+m'\varepsilon)^{-6} = -\frac{8}{15}\pi^6(t^{m'}+2^5t^{2m'}+\ldots+n^5t^{nm'}+\ldots).$$

Factoring out a w from equations (5.02) and (5.03), we have

$$(5.08) \qquad g_2(w,w') = w^{-4}\sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty}{}' 60(m+m'\varepsilon)^{-4}$$
$$= w^{-4}g_2(1,\varepsilon),$$

$$(5.09) \qquad g_3(w,w') = w^{-6}\sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty}{}' 140(m+m'\varepsilon)^{-6}$$
$$= w^{-6}g_3(1,\varepsilon).$$

Noting that the sums remain the same if m' is replaced by

$-m'$, from equation (5.06)" we may then write

$$(5.10) \qquad g_2(1,\varepsilon) = 120\sum_{m'=1}^{\infty}\sum_{-\infty}^{+\infty}{}'(m+m'\varepsilon)^{-4} + 240\sum_{m=1}^{\infty}m^{-4}$$

$$= \frac{4}{3}\pi^4 + 320\pi^4\sum_{m'=1}^{\infty}(t^{m'}+2^3t^{2m'}+\ldots+n^3t^{nm'}+\ldots)$$

$$= \frac{4}{3}\pi^4\{1 + 240\sum_{n=1}^{\infty}\delta_3(n)t^n\},$$

which is precisely equation (5.04), if $t = q^2$.

Similarly, from equation (5.07)",

$$(5.11) \qquad g_3(1,\varepsilon) = 280\sum_{m'=1}^{\infty}\sum_{-\infty}^{+\infty}{}'(m+m'\varepsilon)^{-6} + 560\sum_{m=1}^{\infty}m^{-6}$$

$$= \frac{8}{27}\pi^6(1 - 504\sum_{n=1}^{\infty}\delta_5(n)t^n),$$

which is precisely equation (5.05) if $t = q^2$.

The validity of rearranging the terms of the double series for the invariants g_2 and g_3 follows from the absolute convergence of the double series given in equations (5.02) and (5.03) (10, p. 204). The sums $\sum_{m=1}^{\infty}m^{-4}$ and $\sum_{m=1}^{\infty}m^{-6}$ were obtained from (3, p. 154). The fact that w and w' can be chosen so that $I(\varepsilon) > 0$ is given in (6, p. 63).

We shall now prove the resultant series are absolutely convergent for $|q^2| = |t| < 1$. Let $\delta_k(n)$ be the sum of the kth powers of the divisors of n, including n and one; then

$$\delta_k(n) < n^k \sum_{S=1}^{\infty} S^{-k} < 2n^k \text{ for } k \geq 2.$$

Therefore,

$$\left| \sum_{n=1}^{\infty} \delta_k(n) t^n \right| < \sum_{n=1}^{\infty} (2n^k) |t^n| \text{ for } k \geq 2.$$

Applying the ratio test to the series on the right above for fixed k, we readily find that the series converges absolutely for $|t| < 1$. Therefore, by the comparison test, our series given in equations (5.04) and (5.05) are absolutely convergent for $|q^2| < 1$.

CHAPTER VI

THE VARIABLES h AND λ

In this chapter we wish to discuss in some detail the variables h and λ which are defined by the transformations $g_2 = 12\lambda^4 \neq 0$ and $g_3 = 4(2+h)\lambda^6$, which we used in a previous chapter to transform the inversion integral for the Weierstrass p-function into a more tractable form. Let it be supposed that we are given values for g_2 and g_3, then there are four permissible values of λ and two permissible values for h. Let λ_0 be the principal fourth root of $g_2/12$, then corresponding to λ_0 and $-\lambda_0$ we set $h + 2 = (108)^{1/2} g_3/g_2^{3/2}$, and for $i\lambda_0$ and $-i\lambda_0$ the negative value of h + 2 above, where the principal square root of g_2 is used. We shall have need for both combinations of h and λ. We should note here that $h(\lambda_0) + h(-i\lambda_0) = -4$, to be proved later.

Either combination of $h(\lambda_0)$ and λ_0 or $h(-i\lambda_0)$ and $-i\lambda_0$ used in series (3.01) gives a value of the inverse p-function. By direct calculations in the last chapter it will be found that $h(\lambda_0)$ and λ_0 gives a value of $p^{-1}(B)$ in the fundamental parallelogram, but we shall have occasion to use the other values also.

To properly set the stage for a more detailed study of the variable h, we wish to discuss automorphic functions.

Basically, there are functions which are periodic like $\sin z$ which satisfies $\sin z = \sin(z + 2n\pi)$. In other words, they are invariant under a certain group of linear translations. The Weierstrass p-function is invariant under all the linear translations $z' = z + mw + m'w'$, where m and m' are integers and w and w' the periods of the p-function. Since both $\sin z$ and $p(z)$ are analytic everywhere in the complex plane except for poles and are invariant under certain groups of transformations, they are said to be automorphic with respect to their respective groups.

Here we wish to deal with certain functions which are automorphic with respect to the modular group. This group is a subgroup of the group of bilinear fractional transformations

$$z' = (az+b)/(cz+d), \quad ad - bc \neq 0.$$

In the modular group a, b, c, d are integers with the further restriction that $ad - bc = 1$. If w and w' are a pair of primitive periods of $p(z)$, the complete set of pairs of primitive periods for this p-function is given by

(6.01) $$w_1 = aw + bw' \text{ and } w_1' = cw + dw',$$

where a, b, c, d are integers satisfying $ad - bc = 1$. For proof see (3, p. 150).

Let us form the function $H(w,w') = \{h(w,w') + 2\}^2$. Clearly both of the previous h's discussed satisfy

$$H(w,w') = \{h(\lambda_0)+2\}^2 = \{h(-i\lambda_0)+2\}^2 = 108g_3^2/g_2^3.$$

We use w and w' to emphasize the periods as invariants g_2 and g_3 were defined originally in terms of these periods. Using the homogeneity of $g_2(w,w')$ and $g_3(w,w')$, we may write

$$g_2(w,w') = w^{-4}g_2(1,\varepsilon),\ g_3(w,w') = w^{-6}g_3(1,\varepsilon),$$

where $\varepsilon = w'/w$ and is chosen as in Chapter V such that $I(\varepsilon) > 0$. Forming the function

$$H(w,w') = 108\frac{g_3^2(w,w')}{g_2^3(w,w')} = 108\frac{g_3^2(1,\varepsilon)}{g_2^3(1,\varepsilon)} = H(1,\varepsilon),$$

where the period $w \neq 0$ has cancelled out and this shows that H is a function of ε alone.

Since the new periods w_1 and w_1' obtained from applying equations (6.01) give us the same lattice points as before, then it follows that g_2 and g_3, whose series are absolutely convergent, merely have their sums rearranged and consequently have the same value. Hence,

$$g_2(w,w') = g_2(w_1,w_1') \text{ and } g_3(w,w') = g_3(w_1,w_1').$$

Thus we assert that

$$H(\varepsilon) = H\{(a\varepsilon+b)/(c\varepsilon+d)\}$$

for integers a, b, c, d satisfying $ad - bc = 1$. Thus $H(\varepsilon)$ takes on the same values or displays the same singularities at congruent points under the modular group of transformations. These transformations map the half-plane $I(\varepsilon) > 0$ into itself.

It is shown in (3, p. 151) that $g_2(1,\varepsilon)$ and $g_3(1,\varepsilon)$ are analytic in the half-plane $I(\varepsilon) > 0$. The poles of H are taken on at $\varepsilon = \frac{1}{2} + i3^{1/2}/2$ and points congruent to ε under the modular group of transformations since the invariant g_2 has zeros at ε and points congruent to it under the modular group and no others. Invariant g_3 has no singularities in the finite complex plane. The only singularities of $H(\varepsilon)$ are therefore at the zeros of g_2 or at the poles of g_3. Therefore, $H(\varepsilon)$ is a meromorphic function of ε in this half-plane.

From Ford (3, p. 83) we use the definition of an automorphic function: A function $f(z)$ is automorphic with respect to a group of bilinear fractional transformations T_n provided:

1. $f(z)$ is a single-valued analytic function.

2. If z lies in the domain of existence (the points where $f(z)$ is analytic or has poles), so shall $T_n z$.

3. $f\{T_n(z)\} = f(z)$.

Thus we may assert that $H(\varepsilon)$ is automorphic with respect to the modular group.

The modular group has a fundamental region in the half-plane in which every function which is automorphic with respect to it takes on all of its values at least once. We shall now describe the fundamental region R_0 for the modular group in the half-plane $I(\varepsilon) > 0$.

The region R_0 shall consist of the open region above the circle $|\varepsilon| = 1$ and between the two vertical lines $R(\varepsilon) = \pm 1/2$. To this we add the boundary curves $R(\varepsilon) = 1/2$ above $|\varepsilon| = 1$, and the closed minor arc of $|\varepsilon| = 1$ joining $\varepsilon = \frac{1}{2} + i\frac{3^{1/2}}{2}$ and $\varepsilon = i$. The maps of R_0 under the modular group of transformations cover the open half-plane without overlapping.

Ford's automorphic function $J(\varepsilon)$ (3, p. 151) is related to our function $H(\varepsilon)$. They are both automorphic with respect to the same group. We will therefore make good use of the properties of $J(\varepsilon)$ given in Ford to analyze our $H(\varepsilon)$ (3, pp. 151–163).

The function $J(\varepsilon) = g_2^3/(g_2^3-27g_3^2)$ is related to $H(\varepsilon) = 108g_3^2/g_2^3$ by the functional relationship

$$J(\varepsilon) = \frac{4}{4 - H(\varepsilon)} . \tag{6.02}$$

It can be shown without too much trouble that if $\bar{\varepsilon}$ is the complex conjugate of ε, then

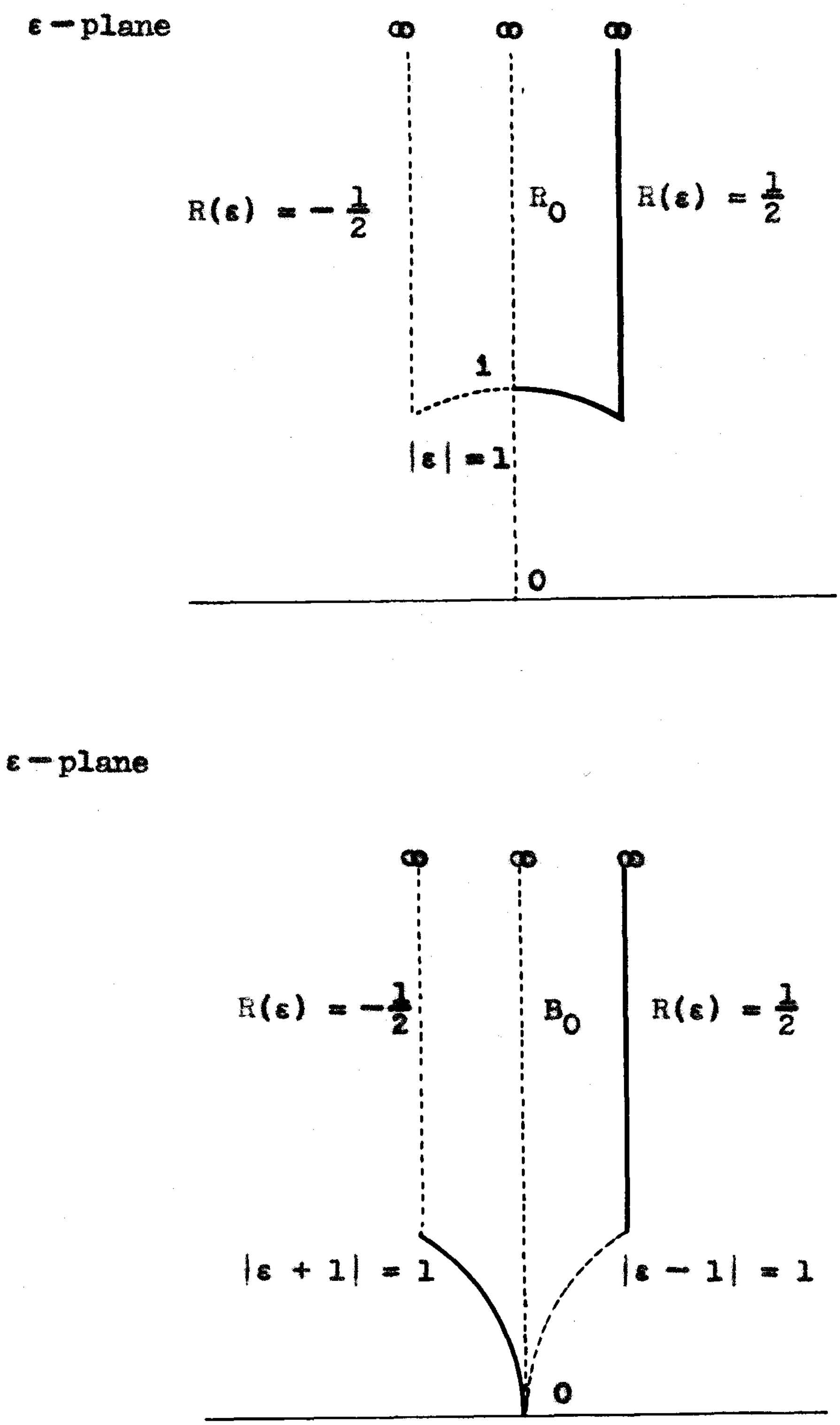


Figure 3

$$g_2(1,-\bar{\varepsilon}) = \overline{g_2(1,\varepsilon)},\quad g_3(1,-\bar{\varepsilon}) = \overline{g_3(1,\varepsilon)},$$

from which it readily follows that $J(-\bar{\varepsilon}) = \overline{J(\varepsilon)}$, and $H(-\bar{\varepsilon}) = \overline{H(\varepsilon)}$. This means that $J(\varepsilon)$ and $H(\varepsilon)$ as well as g_2 and g_3 take on conjugate values at ε and at $-\bar{\varepsilon}$, which is the reflection of ε across the imaginary axis. It remains to assign the exceptional values to the exceptional points of the function $H(\varepsilon)$. As ε moves outward from $\varepsilon = i$ along the imaginary axis, the function $z = J(\varepsilon)$ varies steadily from $z = 1$ to $z = \infty$ through positive values. As $J(\varepsilon) = 1$ at $\varepsilon = i$, it follows that $H(i) = 0$, and further that $H(\varepsilon)$ increases steadily to 4. We therefore assign $H(\infty) = 4$. At $\varepsilon = \frac{1}{2} + \frac{3^{1/2}}{2}i$, J has the value 0, whence H cannot be finite. We therefore assign $H\{\frac{1}{2}+i\frac{3^{1/2}}{2}\} = \infty$.

Since $J(\varepsilon+1) = J(\varepsilon)$, $H(\varepsilon+1) = H(\varepsilon)$, which follows both from the fact that $\varepsilon' = \varepsilon + 1$ is a member of the modular group and from the unique relationship between J and H. This plus the fact that both J and H take on conjugate values at ε and $-\bar{\varepsilon}$ shows that J and H are real on the vertical boundary of R_0. The modular group transformation $\varepsilon' = -1/\varepsilon$ maps the minor arc of $|\varepsilon| = 1$ joining i and $\frac{1}{2} + i\frac{3^{1/2}}{2}$ into the minor arc of $|\varepsilon| = 1$ joining i and $\frac{1}{2} - i\frac{3^{1/2}}{2}$ in such a manner that on those arcs the corresponding points are reflections of one another across the imaginary axis. It therefore follows that on these arcs that J and H are real. The function J takes on each value in the complex plane once and only once in R_0, and from the functional relationship between H and J it follows that H takes on each value once and only once. In traversing the boundary of R_0 clockwise along $|\varepsilon| = 1$ through i and then out the imaginary axis the function $z = J(\varepsilon)$ increases steadily from $-\infty$ to $+\infty$. It follows therefore from the functional relationship (6.02) that $H(\varepsilon)$, starting at $\varepsilon = \frac{1}{2} + i\frac{3^{1/2}}{2}$ going outward along $R(\varepsilon) = \frac{1}{2}$ and returning from the far extent of the positive imaginary axis through i and along the minor arc of $|\varepsilon| = 1$ to our starting point, goes through all the real values from $+\infty$ to $-\infty$. $H(\varepsilon)$ is negative in R_0 only on the open minor arc of $|\varepsilon| = 1$ joining i and $\frac{1}{2} + i\frac{3^{1/2}}{2}$.

Since J has a positive imaginary part in the left half of R_0, it follows by a little algebra that H has the same property, and further that H has a negative imaginary part in the right half of R_0. These halves are separated by the imaginary axis.

We next look at a region B_0 which consists of R_0 and the map of R_0 under $\varepsilon' = -1/\varepsilon$, a member of the modular group. Since H is automorphic with respect to the modular group and takes on each value once and only once in R_0, it follows that H takes on each value twice and only twice in

B_0, and for $\varepsilon_1 \neq \varepsilon_2$, then $H(\varepsilon_1) = H(\varepsilon_2)$ implies $\varepsilon_1 = -1/\varepsilon_2$ if both ε_1 and ε_2 lie in B_0.

Let ε_1 and ε_2 be inverse points with respect to the circle $|\varepsilon| = 1$, then it follows that ε_1 and $-1/\varepsilon_2$ are reflections of one another across the imaginary axis and hence $H(\varepsilon_1) = \overline{H(\varepsilon_2)}$, since $H(\varepsilon_2) = H(-1/\varepsilon_2)$.

Since $\{h(\varepsilon)+2\}^2 = H(\varepsilon)$, it follows that at each point of B_0, either $h(\varepsilon) = -2 + \{H(\varepsilon)\}^{1/2}$ or $h(\varepsilon) = -2-\{H(\varepsilon)\}^{1/2}$. If we are using the conventional principal square root of $H(\varepsilon)$, to make $h(\varepsilon)$ continuous except for singularities at the zeros of g_2 in the half-plane $\mathrm{I}(\varepsilon) > 0$, we use one formula in R_0 and the other in the map of R_0 under $\varepsilon' = -1/\varepsilon$, a member of the modular group. The continuity follows in the region R_0 by the continuity of the principal square root, however, where H is negative the principal square root is discontinuous, hence we must switch to the negative value of the principal square root in crossing a curve of negative values. This occurs where H is negative, namely, on the common boundary of R_0 and the lower half of B_0. The function h defined by two separate functional elements in two halves of B_0 takes on each value in the complex plane once and only once, for suppose it took on the same value twice in R_0; this would imply H also did, which is a contradiction. Further suppose that it took on one value in

R_0 and the same value again in the lower half of B_0; this implies that $H = 0$, but this can happen only at $\varepsilon = i$, which lies on the boundary of R_0. We therefore assert that the equation

$$h(\varepsilon_1) + h(\varepsilon_2) = -4 \tag{6.03}$$

has solutions in B_0 if and only if $\varepsilon_1 = -1/\varepsilon_2$. This equation is of importance in the next chapter.

We note that the exceptional values for h which are 0 and -4 are taken on at ∞ and 0 respectively.

Some numerical calculations of hand computer type of h and λ_0 are given in Chapter XIII, for either w and w' or g_2 and g_3 given. In either case,

$$h = -2 + \frac{108^{1/2} g_3}{g_2^{3/2}} \text{ and } h' = -2 - \frac{108^{1/2} g_3}{g_2^{3/2}},$$

where the series of Chapter V will compute g_2 and g_3 from the periods w and w'.

CHAPTER VII

THE PERIODS w AND w' OF THE p-FUNCTION

In this chapter we are interested in deriving a series which will give a pair of primitive periods of a p-function with g_2 and g_3 the only prior knowledge.

From Chapter VI, we recall from equation (6.03) that if $h(\varepsilon_1) + h(\varepsilon_2) = -4$ in B_0, then $\varepsilon_1 = -1/\varepsilon_2$. Let $h(\lambda_0) = h(w,w')$. The pair of primitive periods w and w' are carried into a pair of primitive periods by equations

$$(7.01) \qquad \begin{aligned} w_1' &= -w = (0)w' + (-1)w \\ w_1 &= w' = (1)w' + (0)w, \end{aligned}$$

where a = 0, b = −1, c = +1, d = 0, whence ad − bc = +1. Therefore, w_1 and w_1' are a pair of primitive periods of the same p-function.

We may suppose $\varepsilon = w'/w$ to lie in B_0, since R_0 and its maps under the modular group cover the half-plane $I(\varepsilon) > 0$ without overlapping. Therefore, there exists a member of the modular group which will map $\varepsilon = w'/w$, any point in $I(\varepsilon) > 0$ not in B_0, into B_0. Since ε is in B_0, $\varepsilon' = -1/\varepsilon$ lies also in B_0, and $h(\varepsilon) + h(\varepsilon') = -4$. Therefore,

$$(7.02) \qquad h(w,w') + h(w',-w) = -4.$$

If $p(\frac{w}{2}|w,w') = e_1$, $p(\frac{w+w'}{2}|w,w') = e_2$, and $p(\frac{w'}{2}|w,w') = e_3$, then e_1, e_2, e_3 are the roots of $4t^3-g_2t-g_3 = 0$, and

$$(7.03) \quad w = 2\int_{e_3}^{e_2} \frac{dt}{(4t^3-g_2t-g_3)^{1/2}} = \int_C \frac{dt}{(4t^3-g_2t-g_3)^{1/2}},$$

where C is a closed loop enclosing only roots e_2 and e_3 (7, p. 169).

Correspondingly, then, for $p(z|W,W')$, with $W = w'$, $W' = -w$, $p(\frac{w'}{2}|w',-w) = e_1' = e_3$, $p(\frac{-w+w'}{2}|w',-w) = e_2' = e_2$, and $p(\frac{w}{2}|w',-w) = e_3' = e_1$, since $p(z|w,w') = p(z|w',-w)$. Further,

$$(7.04) \qquad w' = 2\int_{e_3'}^{e_2'} \frac{dt}{(4t^3-g_2t-g_3)^{1/2}} = \int_C \frac{dt}{(4t^3-g_2t-g_3)^{1/2}},$$

where C is a closed loop enclosing only e_2' and e_3'.

Corresponding to $h = h(\lambda_0) = h(w,w')$ is λ_0, and to $h' = h(-i\lambda_0) = h(w',-w)$ is $-i\lambda_0$. To see this we recall that the transformation $\varepsilon' = -1/\varepsilon$, associated with equations (7.01) is a member of the modular group so that

$$(7.05) \qquad \lambda_0^4 = \frac{g_2(w,w')}{12} = \frac{g_2(w',-w)}{12} = \lambda_0'^4.$$

Since $h = h(w,w')$ is transformed into $h' = h(w',-w)$ by $\varepsilon' = -1/\varepsilon$, then from equation (7.02) we may use $h' = -4-h$ in $g_3(w',-w) = 4(2+h')\lambda_0'^6 = g_3(w,w') = 4(2+h)\lambda_0^6$ to obtain

$$\lambda_0'^6 = \frac{g_3}{4(-2-h)} = \frac{-g_3}{4(2+h)} = -\lambda_0^6 . \tag{7.06}$$

Collecting results, we have from equations (7.05) and (7.06) $\lambda_0^4 = \lambda_0'^4$ and $\lambda_0^6 = -\lambda_0'^6$, whence

$$\lambda_0' = \pm i\lambda_0. \tag{7.07}$$

Since p(z) is an even function, its inverse function is also even so that the choice of the sign in equation (7.07) is arbitrary. If $|h| < 4$, it will be shown in Chapter X that the "unsolvable regions" defined by

$$|v^3-3v-2| \leq |h|$$

are bounded by distinct closed curves which are disjoint. We characterize them as S_1 having E_2 and E_3 on it and $v = -1$ inside, and S_2 having E_1 on it and $v = +2$ inside. Therefore, we can enclose S_1 with a closed loop at every point of which the series (3.01) represents the integral of equation (2.02). Now the integrals which are the coefficients are essentially of a different character than the inversion integral.

Therefore, we can employ series (3.01) to find the periods of the p-function if the path of integration stays out of the "unsolvable regions". A series for one period w is

$$w = \tfrac{1}{2}\sum_{n=1}^{\infty} \lambda_0^{-1} S_n A_n h^n, \tag{7.08}$$

where $S_n = (2n)!/2^{2n}(n!)^2$ and $A_n = \int_C (v^3-3v-2)^{-\frac{2n+1}{2}} dv$ and the path of integration C encloses the level curve S_1. Now C can be chosen close enough around S_1 so that it never crosses the line $R(v) = +1$, and hence does not cut S_2. The function $(v-2)^{-\frac{2n+1}{2}}$ can be rendered analytic in the v-plane if the plane is cut from +2 to ∞ along the positive axis of the reals. With this understanding we may now employ the Cauchy derivative formulas. If $f(z)$ identifies a function which is analytic in and continuous on C, then

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{k+1}} .$$

Therefore,

$$A_n = \int_C \frac{f(v)\,dv}{\{v-(-1)\}^{2n+1}} = \frac{2\pi i}{(2n)!} f^{(2n)}(-1) ,$$

where $f(v) = (v-2)^{-\frac{2n+1}{2}}$ and C does not cross the branch cut joining + 2 to ∞, which is the case if C does not cross $R(v) = 1$.

Actually performing the differentiation and evaluating those derivatives at $v = -1$ gives

$$(7.09)\qquad A_n = \frac{(2n+1)(2n+3)\dots(6n-1)}{2^{2n}(-3)^{3n}(-3)^{1/2}(2n)!}2\pi i.$$

The ratio test reveals that the resulting series converges absolutely for $|h| < 4$, so that the series for period w is given by

$$(7.10)\qquad w = \frac{\lambda_0^{-1}}{2}\frac{2\pi}{3^{1/2}}\sum_{n=0}^{\infty}\frac{(2n+1)(2n+3)\dots(6n-1)}{(n!)^2(432)^n}(-1)^n h^n,$$

for $|h| < 4$. Next we obtain an analogous form for w'.

Armed with the knowledge that $\lambda_0' = \pm i\lambda_0$, we shall proceed from equation (7.04). Making the substitutions $g_2 = 12\lambda_0'^4$, $g_3 = 4(2+h')\lambda_0'^6$ and $t = \lambda_0'^2 x$, the integral in equation (7.04) becomes

$$(7.11)\qquad w' = \frac{1}{2\lambda_0'}\int_{e_3'/\lambda_0'^2}^{e_2'/\lambda_0'^2}\frac{dx}{(x^3-3x-2-h')^{1/2}}\,.$$

But $\lambda_0'^2 = -\lambda_0^2$, hence the limits become $E_2' = e_2'/(-\lambda_0^2)$ and $E_3' = e_3'/(-\lambda_0^2)$. Expanding in a series,

$$(7.12)\qquad w' = \frac{1}{2\lambda_0'}\sum_{n=0}^{\infty} S_n C_n h'^n,$$

where $C_n = \displaystyle\int_{C'}\frac{dx}{(x^3-3x-2)^{\{2n+1\}/2}}$, and loop C' encloses E_2'

and E_3'.

The roots E_1, E_2, and E_3 of $x^3-3x-2-h = 0$ lie on two level curves, E_2 and E_3 on S_1 and E_1 on S_2. These roots are continuous functions of h. Roots E_2 and E_3 lie on S_1 and E_1 lies on S_2 for $|h| < 4$. Now let $|h'| < 4$ so that E_2' and E_3' lie on S_1'. Loop C' will enclose E_2' and E_3' if it encloses S_1'. If $|h'| \leq |h| < 4$, then the level curve S_1' lies inside of or on S_1, and if $|h| \leq |h'| < 4$, then S_1' contains S_1. The polynomial x^3-3x-2 has no zeros within S_1 or S_1' except $x = -1$ which lies within both. Therefore, since

$$A_n = \int_C (v^3-3v-2)^{-\frac{2n+1}{2}} dv,$$

where C encloses S_1, and

$$C_n = \int_{C'} (x^3-3x-2)^{-\frac{2n+1}{2}} dx,$$

where C' encloses S_1', then $A_n = C_n$. Therefore, equations (7.08) and (7.12) can be written

$$w = \frac{\lambda}{2}_0^{-1} \sum_{n=0}^{\infty} S_n A_n h^n \text{ and } w' = \frac{\lambda}{2}_0'^{-1} \sum_{n=0}^{\infty} S_n A_n h'^n.$$

But $h(1,i) = h(i,-1) = -2$, since $g_3(1,i) = 0$. Therefore,

$$i = \frac{\lambda_0^{-1}}{2}\sum_{n=0}^{\infty} S_n A_n (-2)^n \text{ and } i = \frac{\lambda_0'^{-1}}{2}\sum_{n=0}^{\infty} S_n A_n (-2)^n.$$

Both converge since $|h'| = |h| = 2 < 4$. Therefore, $\lambda_0' = -i\lambda_0$, and we have the final form for our series:

$$(7.13) \qquad w = \frac{\lambda_0^{-1}}{2}\sum_{n=0}^{\infty} B_n h^n \text{ and } w' = \frac{i\lambda_0^{-1}}{2}\sum_{n=0}^{\infty} B_n (-4-h)^n,$$

where $B_n = S_n A_n = \dfrac{2\pi(2n+1)(2n+3)\ldots(6n-1)}{3^{1/2}(n!)^2(-432)^n}$, as in (7.10).

Hereafter, $h(w,w')$ belongs with λ_0 and $h(w',-w)$ belongs with $-i\lambda_0$.

CHAPTER VIII

THE ADDITION FORMULAS FOR p(z)

The Weierstrass p-function has an addition formula

$$(8.01)\qquad p(u+v) = \frac{1}{4}\left\{\frac{p'(u)-p'(v)}{p(u)-p(v)}\right\}^2 - p(u) - p(v)$$

which on specialization of v gives three well-known addition formulas. Let e_1, e_2, e_3 be the values assumed by p(z) at $z = w_1$, w_2, and w_3 respectively, where $w_1 = w/2$, $w_2 = (w+w')/2$, $w_3 = w'/2$, and w and w' are periods of p(z); then

$$(8.02)\qquad p(u+w_1) - e_1 = \frac{(e_1-e_2)(e_1-e_3)}{p(u)-e_1}$$

$$(8.03)\qquad p(u+w_2) - e_2 = \frac{(e_2-e_1)(e_2-e_3)}{p(u)-e_2}$$

$$(8.04)\qquad p(u+w_3) - e_3 = \frac{(e_3-e_1)(e_3-e_2)}{p(u)-e_3} .$$

By the homogeneity of $p(z|w,w')$, namely that

$$p(\lambda z|\lambda w,\lambda w') = p(z|w,w')/\lambda^2,\ \lambda \neq 0,$$

we may express formulas (8.01), (8.02), and (8.03) in a more useful form. For example, using formula (8.01), we write

$$p\{\lambda(u+w_1)\} = \frac{p(u+w_1)}{\lambda^2} = \frac{e_1}{\lambda^2} + \frac{(e_1/\lambda^2-e_2/\lambda^2)(e_1/\lambda^2-e_3/\lambda^2)}{p(u)/\lambda^2-e_1/\lambda^2},$$

or

$$p\{\lambda(u+w_1)\} = E_1 + \frac{(E_1-E_2)(E_1-E_3)}{z-E_3}, \tag{8.05}$$

where $\lambda^2 z = p(u)$, $E_1 = e_1/\lambda^2$, $E_2 = e_2/\lambda^2$, $E_3 = e_3/\lambda^2$. Treating formulas (8.03) and (8.04) in the same way, we obtain three more useful formulas

$$p\{\lambda(u+w_1)\} = E_1 + (E_1-E_2)(E_1-E_3)/(z-E_1) = r, \tag{8.06}$$

$$p\{\lambda(u+w_2)\} = E_2 + (E_2-E_3)(E_2-E_1)/(z-E_2) = s, \tag{8.07}$$

$$p\{\lambda(u+w_3)\} = E_3 + (E_3-E_1)(E_3-E_2)/(z-E_3) = t. \tag{8.08}$$

We saw before in Chapter IV that the series (3.01) converges to the inverse of the p-function if $p(u) = \lambda^2 z$ and z satisfies in the complex z-plane the inequality of equation (4.05), namely, that $|z^3-3z-2| > |h|$. In what follows the proper value of λ and h prescribed by Theorem II will be assumed, namely $g_2 = 12\lambda^4 \neq 0$ and $g_3 = 4(2+h)\lambda^6$.

It is convenient to say that a region is "solvable" by means of series (3.01) in the z-plane provided the inequality $|z^3-3z-2| > |h|$ is satisfied for each z of the region; otherwise, we say it is "unsolvable". Thus the values of z satisfying the inequality $|z^3-3z-2| \leq |h|$ constitute the "unsolvable" regions. The boundaries of these regions are the level curves $|z^3-3z-2| = |h|$, which may be two continuous but disjoint curves, or one continuous closed curve.

Suppose we have obtained a solution to $p(u) = \lambda^2 z = B$ by means of series (3.01). We have simultaneously obtained solutions to

A. $p(u+w_1) = B$; B. $p(u+w_2) = B$; C. $p(u+w_3) = B$,

provided w_1, w_2, and w_3 are known. Also, we have obtained indirectly by means of formulas (8.06), (8.07), and (8.08) solutions to

$$\text{D.}\quad p(u+w_1) = e_1 + (e_1-e_2)(e_1-e_3)/(B-e_1),$$

$$\text{E.}\quad p(u+w_2) = e_2 + (e_2-e_3)(e_2-e_1)/(B-e_2),$$

$$\text{F.}\quad p(u+w_3) = e_3 + (e_3-e_1)(e_3-e_2)/(B-e_3),$$

provided w_1, w_2, and w_3 and also e_1, e_2, and e_3 are known. There are known formulas for computing e_1, e_2, and e_3 from w and w', which will be found in (8, p. 449). Thus the "solvable" regions can be extended. The prime question now arising is: Can the solvable region be the entire complex plane? We consider next the answer to this question. The answer will not be complete nor will it be simple.

It is convenient to consider four planes z, r, s, t. We associate the planes z, r, s, and t with the p-functions as follows:

$$p(u) = \lambda^2 z;\ p(u+w_1) = \lambda^2 r;\ p(u+w_2) = \lambda^2 s;\ p(u+w_3) = \lambda^2 t.$$

Equations (8.06), (8.07), and (8.08) now become mappings of the complex z-plane onto the complex r, s, and t-planes which are to be labeled as follows:

$$T_1:\quad r-E_1 = (E_1-E_2)(E_1-E_3)/(z-E_1), \tag{8.09}$$

$$T_2:\quad s-E_2 = (E_2-E_1)(E_2-E_3)/(z-E_2), \tag{8.10}$$

$$T_3:\quad t-E_3 = (E_3-E_1)(E_3-E_2)/(z-E_3). \tag{8.11}$$

These transformations carry the extended complex z-plane into the extended complex r, s, and t-planes.

In each plane, z, r, s, and t, the series (3.01) can be used to invert the corresponding p-function, giving

$$u = p^{-1}(B), \text{ where } z = B/\lambda^2, \text{ if } |z^3-3z-2| > |h|,$$

$$u+w_1 = p^{-1}(B), \text{ where } r = B/\lambda^2, \text{ if } |r^3-3r-2| > |h|,$$

$$u+w_2 = p^{-1}(B), \text{ where } s = B/\lambda^2, \text{ if } |s^3-3s-2| > |h|,$$

$$u+w_3 = p^{-1}(B), \text{ where } t = B/\lambda^2, \text{ if } |t^3-3t-2| > |h|.$$

We have therefore identical level curves in each of the z, r, s, and t-planes for fixed $|h|$. The "unsolvable" regions in the respective planes are given by

$$|v^3-3v-2| \leq |h| \text{ for } v = z, r, s, \text{ or } t. \tag{8.12}$$

Formulas (8.09), (8.10), and (8.11), however, give interconnections between the "solvable" regions and the "unsolvable" regions in the z-plane and each other plane. If a point $z_0 = B_0/\lambda^2$ is in an "unsolvable" region in the z-plane, it need not necessarily be mapped into an "unsolvable" region of the r, s, or t-planes by transformations

T_1, T_2, or T_3, respectively. Let z_0 be mapped into a "solvable region in the r-plane. In such a case, instead of solving directly $p(u) = B_0$ for u by series (3.01), we solve instead the equation $p(u+w_1) = B_1 = T_1(B_0)$.

We wish now to investigate what happens geometrically to a typical point in the z-plane under one of the transformations. It is convenient here to consider the transformations as mappings of the extended complex z-plane into itself.

In particular, let us consider the transformation T_1 given by (8.09), where E_1, E_2, and E_3 are distinct complex numbers. We may translate the origin to point E_1, so that (8.09) becomes

$$(8.13) \qquad r = R_1^2 \exp(2i\xi) z^{-1}, \quad z = \eta \exp(i\zeta),$$

where $(E_1-E_2)(E_1-E_3) = R_1^2\exp(2i\xi)$. Now if $z = \eta\exp(i\zeta)$, then

$$(8.14) \qquad r = (R_1^2/\eta)\exp(2i\xi - i\zeta) = \eta_1 \exp(i\zeta_1).$$

The relations $R_1^2 = \eta\eta_1$, $\zeta_1 = 2\xi - \zeta$, clearly show that under T_1 the point z is first inverted in the circle with center at $z = E_1$ and radius R_1 and then is reflected in the internal angle bisector of the angle $E_2E_1E_3$.

The inversion is conformal, but the reflection is isogonal (angle-reversing), so that the net result is isogonal (12, p. 240). Under an inversion, the points lying on the circle of inversion are rearranged but still lie on the same circle. Under a reflection in a line the points on the line are not changed.

In the next chapter we shall discuss the transformations T_1, T_2, and T_3 collectively and find some important geometrical results.

CHAPTER IX

THE TRANSFORMATIONS T_1, T_2, T_3 AND T

From Chapter VIII we see that the transformations T_1, T_2, and T_3 have a key connection in finding the solution to the functional equation p(u) = B. In order to further study the possible values of B for which the foregoing equation may be solved by means of series (3.01), it is necessary to investigate the overall effect of these transformations.

In order to discuss all three transformations, it is convenient to introduce the following notation: Let K_1, K_2, K_3 of radii R_1, R_2, R_3 and centers E_1, E_2, E_3 be the circles of inversion of transformations T_1, T_2, and T_3. We denote by $T_i(K_j)$ the map of the boundary of circle K_j under transformation T_i, and further $T_iT_j(x) = T_i\{T_j(x)\}$. The circles K_i and K_j always intersect in points P_{ij} and Q_{ij}, and the line L_{ij} passes through E_i and E_j. The point Q_{ij} lies on the same side of L_{ij} as does E_k. The point P_{ij} is the reflection of Q_{ij} in line L_{ij}.

Since e_1, e_2, and e_3 are distinct roots of $4u^3-g_2u-g_3$,

(9.01) $$4u^3-g_2u-g_3 = 4(u-e_1)(u-e_2)(u-e_3) = 0.$$

Under the transformation $u = \lambda^2 v$, $g_2 = 12\lambda^4 \neq 0$,

$g_3 = 4(2+h)\lambda^6$, equation (9.01) is transformed into

$$(9.02) \qquad 0 = 4\lambda^6(v-E_1)(v-E_2)(v-E_3) = 4\lambda^6(v^3-3v-2-h),$$

so that, since $\lambda \neq 0$, the roots of $v^3 - 3v - 2 - h = 0$ are $E_1 = e_1/\lambda^2$, $E_2 = e_2/\lambda^2$, and $E_3 = e_3/\lambda^2$.

We consider first that triangle $E_1E_2E_3$ is scalene, so we order the sides of the triangle C_1, C_2, and C_3 as follows:

$$(9.03) \qquad C_3 = |E_2-E_1| > C_2 = |E_3-E_1| > C_1 = |E_3-E_2|.$$

Therefore $R_1 = (C_2C_3)^{1/2} > R_2 = (C_1C_3)^{1/2} > R_3 = (C_1C_2)^{1/2}$. By saying $T_iT_j(x)$, we mean $T_i\{T_j(x)\}$, and by $T_i(K_j) = T_k(K_j)$ we mean that the two maps are circles having the same radii and the same center. In other words, they are both point sets which bound the same circle. It does not necessarily follow that $T_i(x) = T_k(x)$ for all x on the boundary of K_j.

Theorem I. The transformations T_1, T_2, T_3 together with the transformation T: $w = z$ form a commutative group of transformations which map the extended z-plane into itself.

Proof: This follows directly from the relations

$$T = T_1^2 = T_2^2 = T_3^2,$$

$$T_1 = TT_1 = T_1T = T_2T_3 = T_3T_2,$$

$$T_2 = TT_2 = T_2T = T_1T_3 = T_3T_1,$$

$$T_3 = TT_3 = T_3T = T_1T_2 = T_2T_1.$$

We next state our key mapping theorem:

Theorem II. Let region R consist of all the interior points of K_1, K_2, and K_3, then any point of R is mapped outside of R by T_1, T_2, or T_3.

These transformations belong to the bilinear fractional transformations which carry the family of straight lines and circles into itself. For example, a straight line through the center of inversion E_i goes over into another straight line through E_i, a well-known result of inversive geometry. The following lemma is inserted without proof to aid the reader in following the diagrams. It follows readily from the geometric properties of the transformations T_1, T_2, and T_3.

Lemma I. Under the assumption that triangle $E_1E_2E_3$ is scalene, let $T_i(K_j)$ be the map of the boundary points of K_j under transformation T_i. Then the map $T_i(K_j)$ is a circle, orthogonal to L_{ik}, with a radius $C_jR_j/(C_k-C_i)$. If the direction from E_i to E_k along L_{ik} be taken as positive, then the center of $T_i(K_j)$ is on L_{ik} and at a distance $C_kC_j/(C_k-C_i)$ from E_i.

We next consider the maps of some special points under transformation T_i. In Figure 4, consider $\Delta P_{ij}E_iE_j$ with sides C_k, $R_j = (C_kC_i)^{1/2}$, $R_i = (C_kC_j)^{1/2}$, and $\Delta E_kE_iP_{ik}$ with sides $R_i = (C_kC_j)^{1/2}$, $R_k = (C_iC_j)^{1/2}$, C_j. They are both

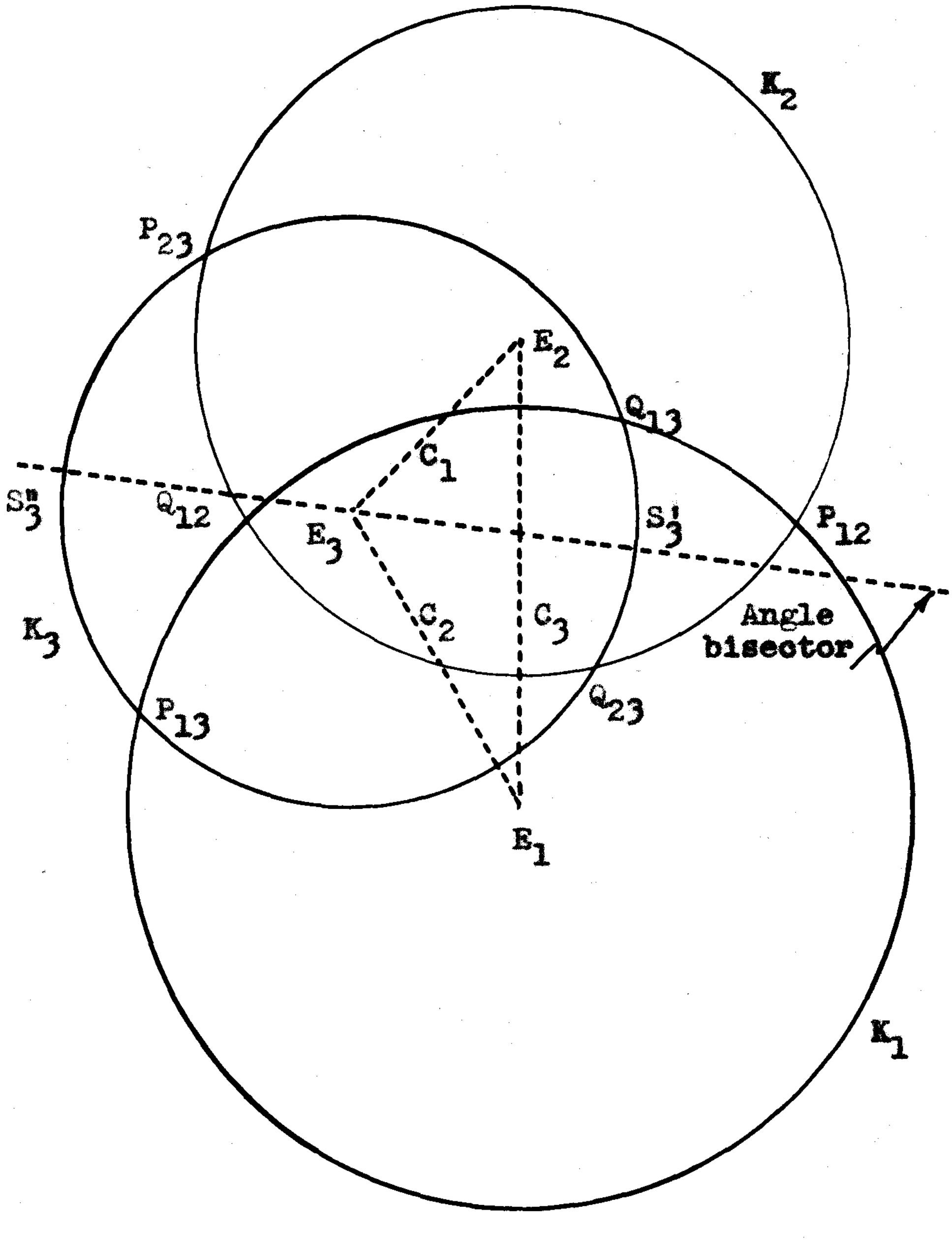

K_2
P_{23}
E_2
Q_{13}
C_1
S_3''
Q_{12}
E_3
S_3'
P_{12}
Angle
bisector
K_3
C_2
C_3
Q_{23}
P_{13}
E_1
K_1

Figure 4

similar to a triangle with sides $C_k^{1/2}$, $C_i^{1/2}$, $C_j^{1/2}$, and hence to each other. Further, angle $P_{ij}E_iE_j = \measuredangle P_{ik}E_iE_k$. The line containing E_i and P_{ij} and the line containing E_i and P_{ik} are mutual inverses under T_i. Points P_{ij} and P_{ik} both lie on K_i. Therefore, P_{ij} and P_{ik} are inverse points under T_i. The same is true of Q_{ij} and Q_{ik}.

Lemma II. Circle $T_i(K_k)$ passes through P_{ij} and Q_{ij}.

Proof: Circle K_k passes through P_{ik} and Q_{ik} by definition, and hence $T_i(K_k)$ passes through the inverses of these points under T_i.

Lemma III. Circles $T_i(K_k)$ and $T_j(K_k)$ are the same set of points.

Proof: Since $T_k(K_k) = K_k$, then $T_i(K_k) = T_iT_k(K_k) =$ $= T_j(K_k)$ from group property $T_iT_k = T_j$.

Let K_k' be the map of K_k under T_i or T_j, then K_k' is invariant under T_k, for $T_k(K_k') = T_kT_j(K_k) = T_i(K_k) = K_k'$. Since K_k and K_k' are both invariant under T_k, they must intersect in the invariant points S_k' and S_k'' of T_k. Since K_k and K_k' are mutually inverse under both T_i and T_j, it follows that S_k' and S_k'' are also mutually inverse under both T_i and T_j. The points S_k' and S_k'' lie at the intersections of the bisector of $\measuredangle E_iE_kE_j$ and circle K_k.

Circles K_i, K_j, and K_k' intersect in P_{ij} and Q_{ij}. Under T_k the maps of these circles must again intersect in two points P' and Q'. We now make a table of inverses to go with Figure 5. In the column under T_i in the row of, say, P_{ij} is the inverse of P_{ij} under T_i:

Point	T_1	T_2	T_3	Point	T_1	T_2	T_3
S_1'	S_1'	S_1''	S_1''	P_{12}	P_{13}	P_{23}	P'
S_1''	S_1''	S_1'	S_1'	P_{13}	P_{12}	P'	P_{23}
S_2'	S_2''	S_2'	S_2''	P_{23}	P'	P_{12}	P_{13}
S_2''	S_2'	S_2''	S_2'	Q_{12}	Q_{13}	Q_{23}	Q'
S_3'	S_3''	S_3''	S_3'	Q_{13}	Q_{12}	Q'	Q_{23}
S_3''	S_3'	S_3'	S_3''	Q_{23}	Q'	Q_{12}	Q_{13}
P'	P_{23}	P_{13}	P_{12}	Q'	Q_{23}	Q_{13}	Q_{12}

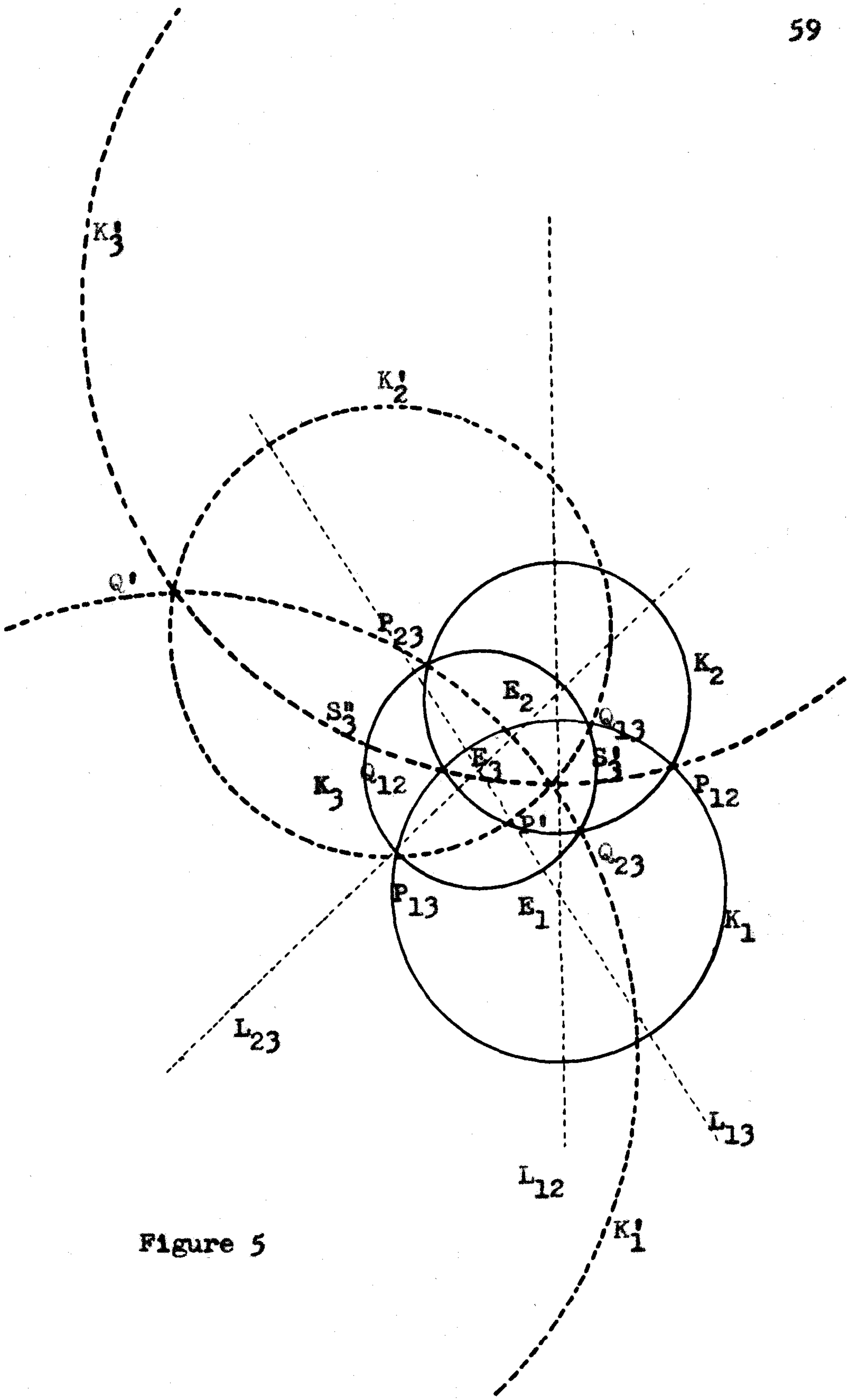
A word or two about interior points under mappings. Under T_i, the interior of K_j maps into the interior of K_j' if K_j does not contain E_i; otherwise, the interior of K_j is mapped exterior to K_j'.

We now need another lemma.

Lemma IV. Let R_0 be the points interior to K_1 or K_2 which are exterior to K_3 together with the open major arc of K_3 between P_{13} and P_{23}. If $C_3 > C_2 > C_1$, then every point of R_0 is mapped exterior to R as defined in Theorem II by either T_1 or T_2.

Proof: In what follows, curvilinear triangles will carry the symbol Δ (see Figure 5). Under T_1:

a) The interior of $\Delta P_{12}S_3'Q_{23}$ is carried into the interior of $\Delta P_{13}S_3''Q'$ and outside of R,



K_3'
K_2'
Q'
P_{23}
K_2
E_2
S_3''
Q_{13}
S_3'
E_3
K_3
Q_{12}
P_{12}
S'
Q_{23}
P_{13}
E_1
K_1
L_{23}
L_{13}
L_{12}
K_1'

Figure 5

b) the interior of $\Delta P_{13}Q_{23}P_{12}$ being interior to K_1, exterior to K_2, and exterior to K_3 is mapped exterior to K_1, exterior to K_2', and exterior to K_3', hence, outside of R,

c) arc $P_{13}Q_{23}$ of K_3 interior to K_1 goes into the major arc $P_{12}Q'$ of K_3' outside of R,

d) minor arc $S_3'Q_{23}$ goes into minor arc $S_3''Q'$ of K_3' outside of R.

Under T_2:

a) The interior of $\Delta P_{12}S_3'Q_{13}$ is carried into the interior of $\Delta P_{23}S_3''Q'$ and outside of R,

b) the interior of $\Delta P_{23}S_3'P_{12}$ being interior to K_2, exterior to K_1, and exterior to K_3 is mapped exterior to K_2, exterior to K_1', and interior to K_3', hence, outside of R,

c) arc $P_{23}Q_{13}$ of K_3 interior to K_2 goes into the major arc $P_{12}Q'$ of K_3',

d) minor arc $S_3'Q_{13}$ goes into minor arc $S_3''Q'$ of K_3' outside of R.

This completes Lemma IV. We can now complete the proof of Theorem II.

Transformation T_3 carries the interior of K_3 exterior to K_3, part on R_0 and the remainder clear of R. If a point x of the interior of K_3 is mapped into a subregion R_0' of R_0, while R_0' is mapped exterior to R by T_1, then x is mapped exterior to R by T_2. If $T_1(T_3x)$ is not in R, then $T_2(x) = T_1T_3(x)$ is not in R. This concludes the scalene case.

Now if $\Delta E_1E_2E_3$ is not scalene but such that $C_3 = C_2 > C_1$, then K_1' is a straight line through P_{13} and Q_{13}. The proof goes through unaltered, except for a remark about interiors. A circle through the center of inversion is mapped into a straight line which is the radical axis of the circle and the circle of inversion. The interior of such a circle is mapped into the half-plane not containing the center of inversion.

If $C_3 > C_2 = C_1$, then K_3' is a straight line through P_{12} and Q_{12} and the remark on interiors above makes the proof go through unaltered.

The two cases of isosceles triangles just discussed cover any eventuality. The equilateral case cannot occur. This completes the proof of Theorem II.

Corollary to Theorem II. If the level curves $|z^3-3z-2| \le |h_0|$ lie interiqr to region R of Theorem II, then all p-functions having $|h| \le |h_0|$ may be completely inverted by use of series (3.01), provided the periods w and w' are known.

We note here the region R depends on h. The level curves are independent of amplitude of h, but not the magnitude $|h|$. Hence the level curves referred to must lie in all regions R as h varies over the circle $|h| = k > 0$.

Since every point of the "unsolvable regions" of series (3.01) is then mapped into a "solvable region" by T_1, T_2, or T_3 we conclude that each such p-function can be completely inverted.

In Chapters X and XI we shall find two examples where this Corollary applies.

CHAPTER X

THE LEVEL CURVES AND THEIR BOUNDS

Of utmost importance in the absolute and uniform convergence of series (3.01) are the level curves

$$|z^3-3z-2| = |h|.$$

The regions R of Chapter IX depend upon the variable h. The level curves are independent of the angle of h, hence if we are to prove that the level curves lie inside of R, it is sufficient to show that a circle contained within R covers a circle enclosing the level curves independent of the angle of h. For $|h|$ sufficiently large and for $|h| \leq 2$, we are able to show that the level curves lie within R.

The case $|h| \leq 2$ is considered in Chapter XI. We deal first with the case where $|h|$ is large.

We wish to minimize and maximize the function $f(z) = |z^3-3z-2|$ on the circles $|z| = R > 2$. We state without proof:

Theorem I. The function $f(z) = |z^3-3z-2|$, on the circle $|z| = R > 2$, achieves its minimum at $z = R$ and its maximum at $R(z) = 1/2$.

For a given level curve S: $|z^3-3z-2| = |h|$, the smallest bounding circle, $|z| = R$, enclosing S has radius $R > 2$ satisfying $(R+1)^2(R-2) = |h|$. No part of the level curve S lies outside of this circle. Similarly, the largest circle, $|z| = r$, lying inside S has radius r satisfying $(r^2+2)^3 = |h|^2$.

Hence the level curve S lies between

$$|z| = R > 2,\ (R+1)^2(R-2) = |h| \qquad (10.01)$$
$$|z| = r,\ (r^2+2)^3 = |h|^2.$$

We next approximate the positive roots of these equations.

Theorem II. If $a^3 = |h|$, then for r and R of (10.01),

$$a + 2/a > R > a + 1/a \qquad (10.02)$$
$$a - 1/a > r > a - 2/a,\ \text{for } a > 2^{1/2}.$$

Observe that for $R > 2$, $(R+1)^2(R-2)$ is a monotone increasing continuous function of R. Direct substitution and the inequality $a + 1/a \geq 2$; valid for $a > 0$, gives the first result. Clearly $(r^2+2)^3$ is also a strictly increasing function of r, so direct substitution and choosing a so that $a > 2^{1/2}$ gives the result.

The roots of $z^3-3z-2 = h$ are E_1, E_2, E_3 and lie on level curve S and hence between the two circles, center at the origin, described in (10.01). Since the second degree term of the cubic polynomial is zero, it follows that the centroid of $\Delta E_1E_2E_3$ is at the origin. This gives us control over the angles and the length of the sides of of this triangle.

We also note that as $|h|$ becomes large the difference $R - r$ becomes small for $4/a > R - r > 0$. If $|h| = 1000$, then grossly

(10.03) $$10.2 > R > 10.1, \quad 9.9 > r > 9.8$$

and any angle A of $\Delta E_1E_2E_3$ satisfies $62^\circ > A > 57^\circ$.

As $|h|$ becomes large, the $\Delta E_1E_2E_3$ becomes nearly equilateral with its center near the origin (centroid) and its circumcircle will have its center near the origin. If the triangle were equilateral, region R of Chapter IX would contain a circle most generously covering our level curve S. In the next few theorems we describe such a circle.

Theorem III. The circumcircle of $\Delta P_{12}P_{23}P_{13}$ has its center at the incenter of $\Delta E_1E_2E_3$.

Since P_{ij} and P_{ik} are inverses under T_i and both lie on K_i, then the bisector of $\measuredangle E_jE_iE_k$ is the perpendicular bisector of the side joining P_{ij} and P_{ik}. These bisectors meet at the incenter of $\Delta E_1E_2E_3$.

Theorem IV. The radius of the circumcircle of $\Delta P_{12}P_{23}P_{13}$ is given by $R = r + 2s$, where r is the inradius of $\Delta E_1E_2E_3$ and s is the area of triangle T having sides $C_1^{1/2}$, $C_2^{1/2}$, $C_3^{1/2}$.

The radius drawn from the incenter, I, of $\Delta E_1E_2E_3$ to P_{ij} is perpendicular to side C_k. Triangle $P_{ij}E_iE_j$ is similar to T by similarity ratio $C_k^{1/2}$ to 1. The area of $\Delta P_{ij}E_iE_j = C_k(s)$, where s is the area of T. The altitude of $\Delta P_{ij}E_iE_j$ is $R - r$ and the base is C_k. Therefore $C_k(s) = C_k(R-r)/2$ or $R = r + 2s$.

Lemma I. If the centroid of a triangle is at the origin and the vertices lie in the annular region between $|z| = R$ and $|z| = r$, where

$$a + 2/a > R > a + 1/a \text{ and } a - 1/a > r > a - 2/a,$$

the sides of the triangle, C_i, satisfy

$$(10.04)\quad 3(a^2-8) < 3(a-4/a)^2 < C_i^2 < 3(a^2+8) < 3(a+4/a)^2,$$

for $a > 3$.

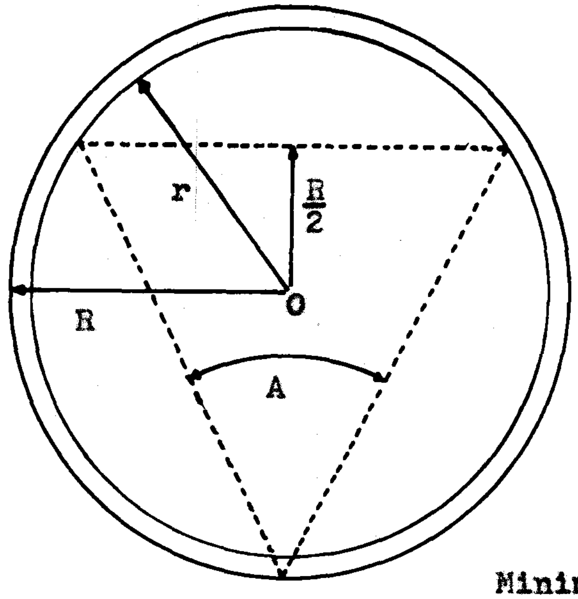
From Figure 6, since the centroid of the triangle is at the origin, the minimum side can be achieved when both endpoints lie on $|z| = r$, then the midpoint of this minimum side cannot lie further from the origin than $R/2$ for otherwise the third vertex would lie outside of $|z| = R$. Similarly, the maximum side can be achieved when its two ends lie on $|z| = R$, and its midpoint cannot lie closer to the origin than $r/2$.

Applying these conditions we get

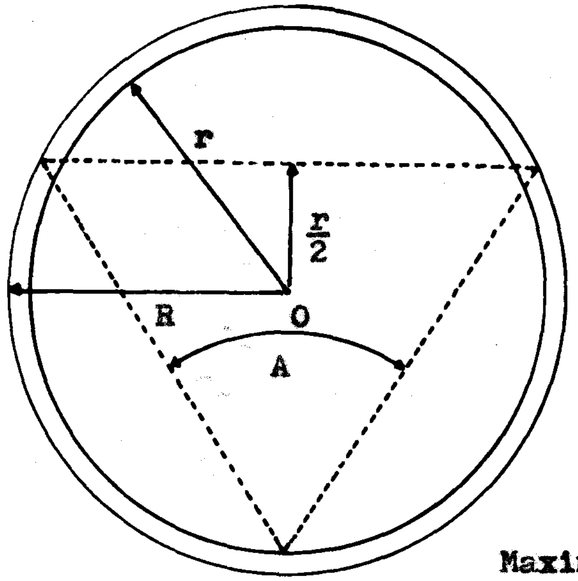
$$4r^2 - R^2 < C_i^2 < 4R^2 - r^2.$$

By elementary inequalities the results follow from Theorem II.

Using Lemma I and the law of cosines we find for any angle A of $\Delta E_1E_2E_3$



Minimum A



Maximum A

Figure 6

$$(a^2-24)/(2a^2+16) < \cos A < (a^2+24)/(2a^2-16)$$

(10.05)

$$(3a^2-8)/(2a^2+16) < 1+\cos A < (3a^2+8)/(2a^2-16).$$

We seek a boundary inequality for the distance between the incenter and centroid of the $\Delta E_1E_2E_3$ referred to in Lemma I.

Without loss of generality let side C_i be parallel to the x-axis and below it, centroid at the origin, then the coordinates of the incenter are given by

$$x_i = \frac{C_iC_j(1+\cos A_k)}{C_j\{1+\cos A_k\}+C_k\{1+\cos A_j\}} - \frac{C_j\cos A_k+C_i}{3}$$

(10.06)

$$y_i = \frac{C_iC_j\sin A_k}{C_j\{1+\cos A_k\}+C_k\{1+\cos A_j\}} - \frac{C_j\sin A_k}{3}.$$

Let D be the minimum of C_i, C_j, or C_k, then replacing those quantities by D in the negative terms and in the denominators of the positive terms and further denoting these new quantities by X_i and Y_i, we note

$$(X_i^2+Y_i^2)^{1/2} = Y_i\csc(A_k/2) > (x_i^2+y_i^2)^{1/2}$$

since $X_i = Y_i\cot(A_k/2)$.

In Y_i we first replace $1 + \cos A_k$ and $1 + \cos A_j$ by $(1 + \cos A)$, and then $(1 + \cos A)$ by the left side of (10.05). Replace D by the far left side of (10.04) then

C_i and C_j in the numerator by the middle right side of (10.04). In the denominator, after getting a common denominator and factoring out $3^{-1/2}$, replace $(a^2-8)^{1/2}$ by the lesser quantity $19a/20$, valid for $a \geq 10$. The net result is

$$Y_1 \csc(A_k/2) < 2\cos(A_k/2)\frac{20}{3^{1/2}19}\{\frac{80a^2+128}{3a^3-8a}\},$$

which for $a \geq 10$, can be further reduced by inequalities of (10.03) where $\cos(A/2) < 0.88$, to our final form

$$(x_1^2 + y_1^2)^{1/2} < (X_1^2 + Y_1^2)^{1/2} < 30/a.$$

We state our results.

Theorem V. If the centroid of $\Delta E_i E_j E_k$ lies at the origin and vertices lie in the annular region between $|z| = R$ and $|z| = r$, where $a + 2/a > R > a + 1/a$, and $a - 1/a > r > a - 2/a$, then the distance between the centroid and the incenter satisfies

$$(x_1^2 + y_1^2)^{1/2} < 30/a$$

for $a \geq 10$.

Theorem VI. The radius of the circumcircle of $\Delta P_{ij} P_{jk} P_{ki}$ satisfies the inequality

$$R = r + 2s > 2a - 79/a, \text{ for } a \geq 10.$$

Let M be the semiperimeter of $\Delta E_i E_j E_k$ then r, the inradius is given by

$$r = \{(M-C_i)(M-C_j)(M-C_k)/M\}^{1/2}.$$

Using appropriate inequalities from (10.04) and (10.05), we find after slight reduction

$$r > \frac{(a-20/a)^{1/2}(a-20/a)}{2(a+4/a)^{1/2}}.$$

Let $0 < \frac{a-20/a}{a+4/a} < 1$, then $\frac{a-20/a}{a+4/a} < \{\frac{a-20/a}{a+4/a}\}^{1/2}$ so that

$$r > \frac{1}{2}(a - \frac{20}{a})(1 - \frac{24}{a^2+4}) > \frac{1}{2}(a - \frac{44}{a})$$

for $a > 7$.

The area of triangle T from Theorem IV is needed next. Triangle T has sides which are square roots of the sides of $\Delta E_i E_j E_k$. Therefore if M' is the semiperimeter of T

$$s = \{M'(M'-C_i')(M'-C_j')(M'-C_k')\}^{1/2}$$

where $C_n' = C_n^{1/2}$ for $n = i,j,k$. Using appropriate inequalities from (10.04) and by analogous reasoning we find

$$s > \frac{3}{4}(a - \frac{38}{a}).$$

Therefore $r + 2s > \frac{1}{2}(a-44/a) + \frac{3}{2}(a-38/a) = 2a - 79/a$.

Theorem VII. The circumcircle of $\Delta P_{ij}P_{jk}P_{ki}$ encloses $|z| = R$, where $a + 2/a > R$, for $a \geq 11$.

The center of the circumcircle cannot lie further from the origin than $30/a$, for $a \geq 10$, by Theorem III and Theorem V, and the radius of this circumcircle is greater than $2a - 79/a$, for $a \geq 10$, by Theorem VI. But

$$2a - 79/a > a + 2/a + 30/a$$

for $a \geq 11$. This gives the theorem.

We may now apply the Corollary of Theorem II of Chapter IX. For all $|h| \geq 11^3$ the circumcircle of $\Delta P_{ij}P_{jk}P_{ki}$, which lies in region R of Theorem II of Chapter IX, covers the circle which in turn encloses the level curve S. Thus level curve S lies in the region R. Therefore all p-functions for which $|h| \geq 11^3$ may be completely inverted by series (3.01). This gives:

Theorem VIII. All p-functions for which $|h| \geq 11^3$ can be completely inverted by series (3.01) provided w and w' are known.

CHAPTER XI

ANOTHER APPLICATION OF THE COROLLARY

In Chapter X we discussed an application of the Corollary to Theorem II of Chapter IX, which is effective for $|h|$ large enough. We shall find another region in the h-plane where the Corollary has application.

It is easy to verify that on the line $R(z) = x_0$ the function $f(z) = |z^3-3z-2|$ is an increasing function of $|y|$, where $z = x_0 + iy$. It follows that, since $f(1) = 4$, that $f(z) \geq 4$ on $R(z) = +1$. The level curves $f(z) = |h| <$ < 4 do not cross the line $R(z) = 1$. It follows from a study of the polynomial $P(x) = x^3-3x-2$ (see Figure 7) that if $|h| < 4$, then on the line $R(z) = x_0$, where $x_0 < x_1$ or $x_0 > x_4$, the equation $f(z) = |h|$ has no solutions. If $x_0 = x_1, x_2, x_3$, or x_4, then $f(z) = |h|$, and if $x_1 < x_0 <$ $< x_2$ or $x_3 < x_0 < x_4$, then $f(x_0) < |h|$. Also $f(z) = |h|$ has two symmetric solutions, since $f(\overline{z}) = f(z)$, where $\overline{z}$ is the complex conjugate of z. The function $f(z)$, which equals $|P(x)|$ on the real axis, has an absolute maximum of $+4$ on $-1 \leq x \leq 2$. Therefore, if $|h| > 4$, then x_2 and x_3 shown in Figure 7 do not exist, and if $|h| = 4$, we have a point of tangency. We therefore conclude that there are two level curves S_1, enclosing $z = -1$, and S_2, enclosing $z = +2$, for $|h| < 4$. If $|h| = 4$, the level curve has a double point (9, p. 121). If $|h| > 4$, there is but one level curve S.

We have shown that outside of the level curves $f(z) = |h|$ that $f(z) > |h|$ and inside of them that $f(z) < |h|$.

On the circles $|z + 1| = R < 2$, the function $f(z)$ assumes a maximum of $R^2(R+3)$ at $z = -R-1$, and a minimum of $R^2(3-R)$ at $z = -1+R$. Therefore the level curve S_1 lies between circles

$$|z+1| = R, \text{ where } R^2(3-R) = |h| < 4; \text{ (outside of } S_1\text{)},$$

(11.01)

$$|z+1| = r, \text{ where } r^2(r+3) = |h| < 4; \text{ (inside of } S_1\text{)}.$$

On the circles $|z-2| = R < 2$, function $f(z)$ assumes a maximum of $(R+3)^2R$ at $z = R + 2$, and assumes a minimum of $(3-R)^2R$ at $z = 2 - R$. Therefor the level curve S_2 lies between circles

$$|z-2| = R, \text{ where } (3-R)^2R = |h| < 4; \text{ (outside of } S_2\text{)},$$

(11.02)

$$|z-2| = r, \text{ where } (r+3)^2r = |h| < 4; \text{ (inside of } S_2\text{)}.$$

These are the best possible fits for circles with these centers. The radii R and r as defined by the equations of (11.01) and (11.02) are uniquely determined since each defining equation has one root between 0 and 2.

From Figure 7, the R and r of (11.01) are given by

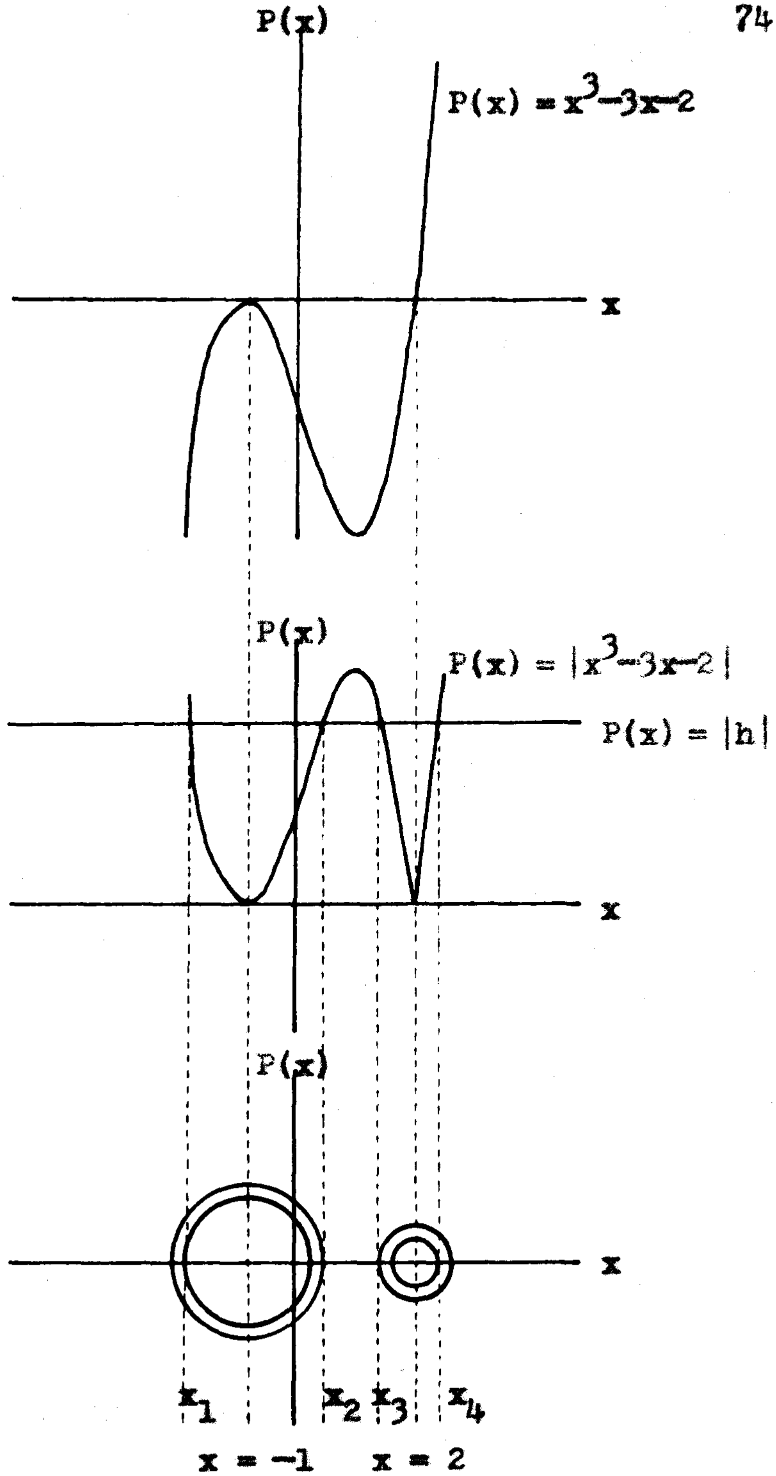


Figure 7

$R = x_2 + 1$ and $r = -(x_1+1)$, and the R and r of (11.02) are given by $R = 2 - x_3$ and $r = x_4 - 2$. Roots x_1, x_2, x_3 satisfy $x^3-3x-2+|h| = 0$ and the roots of $x^3-3x-2-|h| = 0$ are x_4, $-(x_4/2) \pm iy_4$.

The points $z = x_1$, x_2 and the line $R(z) = -x_4/2$ divide S_1 into four arcs. The equation $z^3-3z-2 = h$ has solutions E_1, E_2, E_3 with E_1 on S_2 and E_2 and E_3 on S_1. We shall prove:

Lemma I. The equation $z^3-3z-2 = h$ has two roots on S_1, and one root on S_2. Further one root E_2 or E_3 lies on or to the left of $R(z) = -x_4/2$.

From Figure 8, let $z + 1 = p_1 e^{i\eta_1}$ and $z - 2 = p_2 e^{i\eta_2}$, where $p_1^2 p_2 = |h|$, then

$$z^3-3z-2 = (z+1)^2(z-2) = |h|e^{i(2\eta_1+\eta_2)} = h.$$

In Figure 8, angles η_1 and η_2 increase as z_0 moves along S_1 counterclockwise from $-x_4/2 + iy_4$ to $-x_4/2 - iy_4$. Since h is real for these two values of z, the argument of h has changed by a multiple of 2π. Inspection of Figure 8 shows this change is more than zero and less than 4π. Therefore $z^3-3z-2-h = 0$ has one solution on S_1 to the left of or on $R(z) = -x_4/2$.

The roots of $z^3-3z-2 = h$ are continuous functions of complex h. For $h = -2$, the roots $-(3^{1/2})$ and 0 lie on S_1 and root $3^{1/2}$ lies on S_2. Since the roots cannot cross

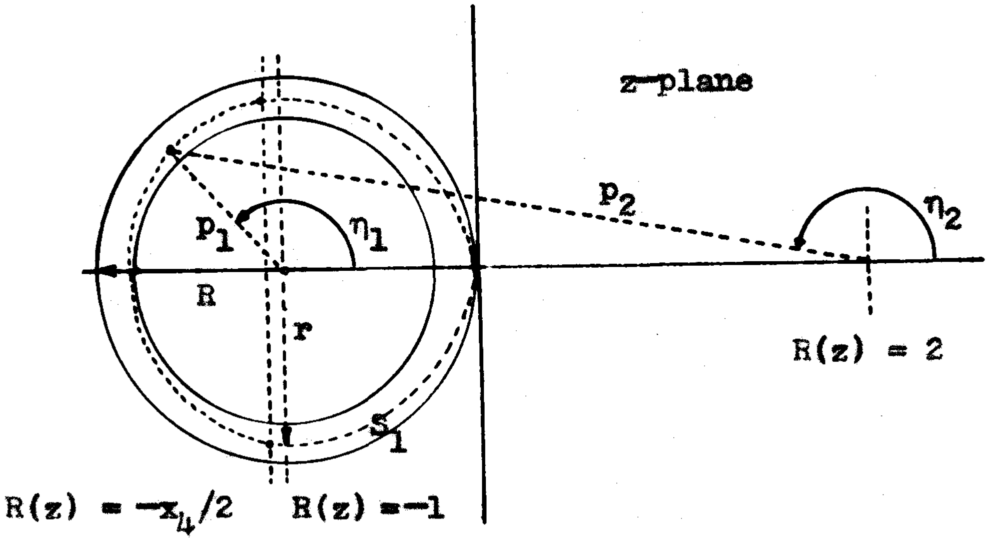


Figure 8

the line $R(z) = 1$ for $|h| < 4$, then E_2 and E_3 always lie on S_1 and E_1 always lies on S_2.

We are now ready to prove also:

Lemma II. For $|h| \leq 2$, the circle of inversion K_1 of the transformation T_1, which has center E_1 on S_2, has radius $R_1 > 3^{1/2}$. Further the level curve S_2 is entirely within K_1.

Proof: For $|h| < 2$, then $-(3^{1/2}) \leq x_1 < -1$, $-1 < x_2 \leq 0$, and $3^{1/2} \leq x_3 < 2$. Since E_2 and E_3 are on S_1 which lies across the imaginary axis from S_2, it follows that $|E_1 - E_2| \geq 3^{1/2}$ and $|E_1 - E_3| \geq 3^{1/2}$ where equality cannot hold for both at once. Therefore $R_1 > 3^{1/2}$. Since the circle $|z - 2| = 2 - x_3$ encloses S_2, hence encloses E_1, the center of K_1, and $3^{1/2} \leq x_3 < 2$, then the diameter of $|z - 2| = 2 - x_3$ is $4 - 2x_3 < 3^{1/2} < R_1$. Thus K_1 encloses the circle $|z - 2| = 2 - x_3$ which encloses S_2.

We wish to prove a similar lemma about K_3 and S_1. We need first some inequalities for the circles bounding both S_1 and S_2.

From equations (11.02)

$$R(3-R)^2 = |h| \text{ (enclosing } S_2)$$
$$r(r+3)^2 = |h| \text{ (enclosed by } S_2)$$

with $|h| = 2$, then $R = 2 - 3^{1/2}$, $r < 0.2$. For $|h| \leq 2$, we have the inequalities

$$\frac{|h|}{9} < R \le \frac{|h|}{4+2^{1/2}} < \frac{|h|}{7},$$

(11.03)

$$\frac{|h|}{10.24} < r < \frac{|h|}{9}.$$

From equations (11.01),

$$R^2(3-R) = |h| \text{ (enclosing } S_1)$$

$$r^2(r+3) = |h| \text{ (enclosed by } S_1)$$

with $|h| = 2$, then $R = 1$ and $r = 3^{1/2} - 1$. For $|h| \le 2$,

$$\{|h|/3\}^{1/2} < R \le \{|h|/2\}^{1/2}$$

(11.04)

$$\{|h|/(2+3^{1/2})\}^{1/2} < r < \{|h|/3\}^{1/2}.$$

The roots E_2 and E_3 lie on S_1 and the root E_1 lies on S_2. Since the second degree term of $z^3-3z-2 = h$ is missing, then $E_1 + E_2 + E_3 = 0$, and further $(E_2+E_3)/2 = -E_1/2$. Therefore the midpoint P of the line segment joining E_2 and E_3 is at $-E_1/2$. As E_1 traces out S_2, then $-E_1/2$ traces out a mirror image of S_2 but half as large. We thereby conclude from (11.03) that the midpoint P lies inside the circle

$$|z+1| = R/2 \text{ where } |h|/9 < R < |h|/7.$$

The endpoints E_2 and E_3 lie on S_1 which from (11.04) lies outside of $|z+1| = r > |h|^{1/2}(3.74)^{-1/2}$. Therefore a lower bound for the length of segment

$$|E_2 - E_3| > 2\{(|h|/14)\cot A\},$$

where $\sin A < \frac{\{3.74|h|\}^{1/2}}{14}$.

Therefore

(11.05) $|E_2-E_3| > 2\cos A|h|^{1/2}/(3.74)^{1/2}$, where $\cos A > 0.98$.

From Lemma I, E_1 lies to the right of or on $R(z) = x_3$ or at least the distance $2 - R$ from the imaginary axis, where R is of (11.03). Root E_3 lies to the left of or on $R(z) = -x_4/2 = -(2+r)/2$, where r is of (11.03).

Therefore
$$|E_3 - E_1| \geq (2-R) + \tfrac{1}{2}(2+r)$$
$$> 3 - |h|(\tfrac{1}{7} - \tfrac{1}{20.48})$$
$$> 3 - |h|/10,$$

(11.06) $$|E_3-E_1| > 2.8 \text{ for all } |h| \leq 2.$$

The radius of inversion R_3 is given by
$$R_3^2 = |E_3-E_1||E_3-E_2|.$$
From (11.05) and (11.06) we get the combined inequalities which are close, but adequate

(11.07) $$R_3^2 > \frac{(2.8)(2)(.98)|h|^{1/2}}{(3.74)^{1/2}} > 2(2|h|)^{1/2}.$$

From (11.04), the circle enclosing S_1 has radius $R < (|h|/2)^{1/2}$. Therefore
$$R_3^2 > 2(2|h|)^{1/2} = 4(|h|/2)^{1/2} > 4R^2 = (2R)^2.$$
This results in

Lemma III. Circle K_3 encloses S_1.

Therefore Lemmas I and II combined give

Theorem I. If $|h| \leq 2$, the corresponding p-functions can be completely inverted by using series (3.01).

Lemmas I and II place the level curves S_1 and S_2 within the region R of Theorem II of Chapter IX, hence the Corollary there is again applicable.

CHAPTER XII

THE EVALUATION OF THE K_n

The integrand of the inversion integral (2.02) has three finite branch-points E_1, E_2, E_3 and an infinite one. The value assigned to the integral depends on how the path of integration loops around the finite branch-points. These roots are the roots of $v^3 - 3v - 2 - h = 0$, hence they lie on the level curves (see Chapters X and XI) $|v^3-3v-2| = |h|$. In Chapter XI we assigned E_2 and E_3 to lie on S_1 which encloses $v = -1$, and E_1 to lie on S_2 which encloses $v = 2$. In Chapter VII we derived the periods of the p-function by taking a loop integral around S_1, thereby enclosing E_2 and E_3.

The path of integration may not intersect the closed regions enclosed by the level curves. As far as concerns us we may render the inverse p-function single-valued by cutting the v-plane from $v = 2$ to minus infinity along the axis of the reals. This imposes no extra restriction since to render the integrand single-valued one would normally join minus infinity to say E_3 thence to E_2 then to E_1. Between minus infinity and E_3, we designate an impassable barrier, between E_3 and E_2 the branch-cut is erased, and between E_2 and E_1 it is impassable again. But the cut between E_3 and E_2 could easily have been considered interior to level curve S_1 since E_3 and E_2 lie on its boundary. Since we couldn't have entered that region anyhow with the path of integration, we suffer no loss. The region S_2 encloses the point $v = 2$ so in reality the branch-cut could have terminated anywhere in or on the level curve S_2, as the path cannot enter that region.

From Chapter II we saw that the coefficients in the formal series (3.01) were contour integrals

$$K_n = \int_{\zeta}^{\infty} (v+1)^{-2n-1}(v-2)^{-\frac{2n+1}{2}}\,dv$$

which under the transformation $v = r^2 + 2$ become

$$K_n = 2\int_{A}^{\infty} (r^2+3)^{-2n-1} r^{-2n}\,dr, \quad A = -(\zeta-2)^{1/2},$$

and can readily be expressed as a finite number of rational functions. For evaluation we wish to use two reduction formulas, (C) and (D) of (4, p. 307). Applying formula (D) twice and then formula (C) we find

$$K_{n+1} = \frac{\{1 - \frac{6n+5}{6(2n+1)}(r^4+5r^2+6)\}}{6(n+1)r^{2n+1}(r^2+3)^{2n+2}} - \frac{(6n+5)(6n+1)}{72(n+1)(2n+1)}K_n,$$

which is a recursive type relationship between the K_n.

Therefore in the calculation of the K_n each is calculated in a fixed fashion from the one before it.

In order for the value of the integrals K_n to be unaltered by the change of variables and consequently a change of contour path under the mapping, it is a sufficient condition that the transformation $v = r^2+2$ possess a continuously differentiable inverse function. This is the case if the v-plane is cut with a branch-cut from $v = 2$ to minus infinity along the axis of the reals which will give us the principal square root in the inverse function and the inverse is analytic in the cut plane.

Employing the same technique as in Chapter II relative to passing to the limit of a finite contour (over which Theorem A of Chapter II applies), we see that each K_n has a term involving K_0. All other terms of each vanish at the upper limit of integration, namely infinity. However K_0 is an arctangent and hence gets a contribution from the upper limit. Therefore the values of r obtained form the lower limits of integrations after the various changes of variables and can be directly substituted into the coefficients K_n with the proper value of K_0. We note each K_n is an odd function of r. This completes the theoretical justification of the expansion and the restrictions on the paths. It remains to actually write out the series for a few different cases and do some calculations.

CHAPTER XIII

CONCLUSIONS AND COMPUTATIONS

In this chapter we indicate the use of the theorems in previous chapters to obtain the introductory theorems. We also give some conclusions and calculations.

Theorem I in the Introduction is Theorem II of Chapter IV. Theorem II of the Introduction follows from Theorem VIII of Chapter X and Theorem I of Chapter XI. Theorem III of the Introduction is proved in Chapter VII. These results are more general than those of Innes (5, pp. 357–368). Corollary II of the Introduction follows from the known results that if two elliptic functions have the same periods then there is an algebraic relationship between them (6, p. 91). It remains to solve the algebraic equation for the values of p(z) once the value A of the elliptic function F(z) to be inverted is given. The inverse functional values z_i, where $p(z_i) = B_i$, are the values of z where $F(z) = A$. This is an application.

Corollary I of the Introduction requires a little background material. If the periods w and w' have a real positive ratio, then the parallelograms are rectangles, and we shall call it the real or rectangular case, otherwise the skew case.

In the rectangular case the roots of $v^3-3v-2 = h$ are all real and distinct, and h is real. From the discriminant of a real cubic polynomial we find $0 > h > -4$. Let $w = b > 0$ and $w' = i$, then, $h(b,i) = -2$ at $b = 1$, and $-2 < h(b,i) < 0$ for $0 < b < 1$. This follows since $\varepsilon = i/b$ lies in R_0 of Chapter VI. If $1 < b < \infty$, then $\varepsilon = i/b$ lies in the lower half of B_0 and $-4 < h(b,i) < -2$.

To solve the equation $p(z|b',i) = B$ for $b' > 1$, we solve instead the problem $p(-biz|b,i) = -B/b^2$, where $b = 1/b'$, and find

$$z = p^{-1}(B|b',i) = \frac{i}{b}p^{-1}\left(-\frac{B}{b^2}\middle|b,i\right).$$

To see this we use first the homogeneity of $p(z)$, then a change of periods. First

$$\lambda^2 p(\lambda z|\lambda w,\lambda w') = p(z|w,w'),$$

let $\lambda = -u = -i/b'$, $w = b'$, $w' = i$, then it becomes

$$b'^{-2}p(-iz/b'|-i,1/b') = p(z|b',i).$$

Changing periods with $\varepsilon' = -1/\varepsilon$, as in Chapter VI, and letting $b = 1/b'$, we see that

$$-b^2p(-biz|b,i) = p(z|b',i).$$

Hence to solve $p(z|b',i) = B$, we can solve instead $p(-ibz|b,i) = -B/b^2$. If $b' > 1$, then $-4 < h(b',i) < -2$ but $h(b',i) + h(b,i) = -4$ so that $-2 < h(b,i) < 0$ and series (3.01) gives complete inversion of the rectangular case by Theorem I of Chapter XI.

We believe that the Corollary of Theorem II of Chapter IX has wider application than the gross bounds of Chapters X and XI give us. One could for example for a particular p-function of known periods find roots E_1, E_2, E_3, plot them on the graphed level curve, and draw the circles of inversion to give a comparison of the level curves and the region R. Trials by geometrical methods indicate that the level curve is in R for $|h| > 14$. This concerns the composite of the four regions of convergence in the various planes, and the complete inversion for every value in the complex plane. For individual values of particular p-functions a check may be made through (8.12) as to whether or not it is invertible. We have yet to find a point that does not lie in a solvable region in some one of the four planes. It is possible that the coverage is complete except for $h = \infty$, which is the equianharmonic case when $g_2 = 0$. We can guarantee complete invertibility of the p-function if $|h| \leq 2$ or $|h| \geq 11^3$.

Once z_1, a value of the inverse p-function in the fundamental parallelogram, is found and the periods are known, then all the roots in the plane are given by $z_2 = \pm z_1 + mw + m'w'$. If $m = m' = 1$ and the minus sign is chosen, then z_2 is in the fundamental parallelogram.

We shall outline the computation for a zero of a p-function in the rectangular case.

For simplicity let $g_2(w,w') = \frac{4}{3}(\pi/w)^4\eta_2$ and $g_3(w,w') = \frac{8}{27}(\pi/w)^6\eta_3$, where η_2 and η_3 are the parts in parentheses in equations (5.04) and (5.05).

Using this convention we may write from Chapter VI

$$h = 2\{\eta_3\eta_2^{-3/2} - 1\}, \quad \frac{1}{\lambda} = \frac{3^{1/2}w}{\pi}\,\eta_2^{-1/4}.$$

Let $w = 1/2$, $w' = i$, whence $q = e^{-2\pi}$, substitute this value of q in the above two expressions to obtain

$$h = -\,0.00601\ 95772\ 338,$$

$$\frac{1}{\lambda} = 0.27560\ 67958\ 78.$$

The lower limit of integration in the K_n is $r = -2^{1/2}i$, which gives the following recursion formula for the K_n

$$K_{n+1} = \frac{2^{1/2}(-1)^{n+2}i}{6(n+1)2^{n+1}} - \frac{(6n+5)(6n+1)}{72(n+1)(2n+1)}K_n.$$

The first four K_n are:

$$K_0 = \pi 3^{-1/2} - 2(3^{-1/2})\tan^{-1}\{-i(2/3)^{1/2}\}$$

$$= 2\{0.90689\ 96821\ 171 + i(0.66176\ 80207\ 72102)\},$$

$$K_1 = 2\{-0.06297\ 91446\ 384 + i(0.01296\ 94525\ 4526)\},$$

$$K_2 = 2\{0.01122\ 54493\ 816 - i(0.01704\ 30761\ 0339)\},$$

$$K_3 = 2\{-0.00229\ 70595\ 55 + i(0.00839\ 79821\ 09)\}.$$

The first four S_n are:

$$S_0 = 1,\ S_1 = 1/2,\ S_2 = 3/8,\ S_3 = 5/16.$$

The resulting inverted value of $p(z|\frac{1}{2},i) = 0$ is to seven places

$$z = 0.2500000 + i(0.1823769).$$

The first four entries in the following table were obtained in this way from the power series.

Periods		Zero	
w	w'	real part	imaginary part
1/5	i	0.1000000	0.0729704
2/5	i	0.2000000	0.1459390
3/5	i	0.3000000	0.2184317
4/5	i	0.4000000	0.2824636
1	i	0.5000000	0.5000000

The errors in the inverted values can be bounded by the geometric series with

$$r = \frac{|h|}{|z^3-3z-2|},$$

and a is the integral on the right-hand side of equation (3.02).

BIBLIOGRAPHY

1. Copson, E. T. An introduction to the theory of functions of a complex variable. London, Oxford, 1935. 448p.

2. Court, N. A. College geometry. 2d ed. New York, Barnes and Noble, 1952. 313p.

3. Ford, L. R. Automorphic functions. 2d ed. New York, Chelsea, 1952. 333p.

4. Granville, W. A., P. A. Smith, and W. R. Longley. Elements of the differential and integral calculus. Rev. ed. New York, Ginn, 1946. 556p.

5. Innes, R. T. A. On the periods of the elliptic functions of Weierstrass. Proceedings of the Royal Society of Edinburgh 27:357-368. 1907.

6. Knopp, Konrad. Theory of functions. Part II. New York, Dover, 1947. 150p.

7. MacRobert, T. M. Functions of a complex variable. 3d ed. Macmillan, 1914. 390p.

8. Pierpont, James. Functions of a complex variable. Boston, Ginn, 1914. 583p.

9. Titchmarsh, E. C. The theory of functions. 2d ed. London, Oxford, 1939. 454p.

10. Townsend, E. J. Functions of a complex variable. New York, Henry Holt, 1915. 384p.

11. Whittaker, E. T. and G. N. Watson. A course of modern analysis. 4th ed. Cambridge, Cambridge, 1950. 608p.

12. Wolfe, H. E. Introduction to non-Euclidean geometry. New York, Dryden, 1948. 247p.