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For the Cauchy problem

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[a_j(t,x)\frac{\partial \varphi(t,x,u)}{\partial x_j}\right] + c(t,x)\varphi(t,x,u) + f(t,x) = \frac{\partial u}{\partial t},$$

$$u(0,x) = u_0(x), \quad x \in E_n$$

by assuming suitable conditions on the coefficients of the differential equation and on the initial function u_0 a uniqueness theorem for the generalized solutions of the problem is proved, provided that the initial function u_0 belongs to a certain class of functions.

Secondly, for the Cauchy problem

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(t,x)\frac{\partial \varphi(u)}{\partial x_j}\right] + \sum_{j=1}^{n} b_j(t,x)\frac{\partial \varphi(u)}{\partial x_j} = \frac{\partial u}{\partial t},$$

$$u(0,x) = u_0(x), \quad x \in E_n,$$

under suitable conditions on the coefficients and initial function u_0 an existence and uniqueness theorem for the classical solution of the problem is proved, provided that the initial function u_0 is bounded away from zero. The Cauchy problem with differential equation

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} [a_{ij}(t, x, u) \frac{\partial \varphi(x, u)}{\partial x_j}] + \sum_{j=1}^{n} b_j(t, x, u) \frac{\partial \varphi(x, u)}{\partial x_j} = \frac{\partial u}{\partial t}$$

is also considered.

Thirdly, for the Cauchy problem having as its differential equation the n-dimensional nonsteady gas flow equation:

$$\sum_{j=1}^{n} \frac{\partial^2 u^m}{\partial x_j^2} = \frac{\partial u}{\partial t}$$

the finite difference numerical approximation scheme

$$U(t+k, x) = \sum_{j=1}^{n} \lambda_j [U^m(t, x+h_j e_j) + U^m(t, x-h_j e_j)]$$
$$+ \left[1 - 2\left(\sum_{j=1}^{n} \lambda_j\right) U^{m-1}(t, x)\right] U(t, x)$$

is presented, with a maximum principle property, non-negativeness of solutions property, and consistency with the differential equation property being proved for the scheme.

Finally, a numerical example for the one dimensional isothermal gas flow problem is presented, including tabulated results and graphs obtained through the above approximation scheme.

Mathematical Aspects of the Flow of Gases Underground

by

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MATHEMATICAL ASPECTS OF THE FLOW OF GASES UNDERGROUND

I. INTRODUCTION

Of particular concern to the petroleum industry, and more recently to ecologists, and also of interest to geophysicists, soil scientists, and applied mathematicians are the mathematical problems arising in the non-steady flow of gases through porous media. Indeed, the non-steady flow phenomenon predominates in most cases of actual field production from gas reservoirs (see Muskat [9]). For instance, in the case of gas reservoirs of limited extent, removal of gas depletes the sand pressure, so that flow from wells will decrease if so called "steady" pressures are maintained. In another instance, if a well is bounded by an acquifer (i.e., water saturated sand), it is rather unlikely that the encroachment of water will be sufficient to maintain pressure in the reservoir. A further consideration is that the low viscosity of gases tends to give rise to rapidly changing physical conditions in spite of the high elasticity of gases. These, and other, considerations lead to a mathematical treatment which gives rise to the formulation of a parabolic partial differential equation which is generally non-linear.

In establishing this differential equation, we shall essentially follow the treatment given in Muskat's work cited above. First of all, any gas flowing through a porous medium is subject to an empirical relation known as Darcy's Law.[1] This law, generalized to three dimensions and neglecting gravitational effects states that:

$$q = -K\nabla p \tag{1.1}$$

Here q is the <u>volume flux</u>, or <u>seepage velocity</u>, of the gas, i.e., the volume of gas passing through unit cross-section of the medium per unit time in the direction of the flow, K is the <u>permeability</u> of the medium, i.e., the volume of a fluid of unit viscosity passing through a unit cross-section of the medium per unit time under the action of a unit pressure gradient, p is the gas <u>pressure</u> and ∇ is the <u>gradient operator</u> with respect to the spacial variables.

Next, the law of conservation of mass, or the equation of continuity, is expressed by:

$$\mathrm{div}(\rho q) + \partial(\varphi\rho)/\partial t = 0, \tag{1.2}$$

where ρ is the gas <u>density</u>, φ is the <u>porosity</u>, i.e., a function whose integral over any volume V gives the amount of pore space available to the gas in volume V, <u>div</u> is the <u>divergence operator</u>

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[1] See, however, the paper of Fulks, Guenther, and Roetman [7], which presents a rigorous derivation of the basic laws governing the macroscopic flow of fluids in porous media. Their derivation is within the framework of classical continuum mechanics, and rigorously established Darcy's Law along with other important results.

computed with respect to the spacial variables, and t is time.

To determine the flow of the gas an equation of state relating pressure, density, and temperature is needed. We shall neglect temperature effects and assume an equation of state of the form:

$$\rho = \rho_0 p^m \tag{1.3}$$

where ρ_0 is the gas density at unit pressure, and $m = 1$ if the flow is isothermal, i.e., the temperature of the gas never varies, otherwise $0 < m < 1$ if the flow is adiabatic, i.e., heat never enters or leaves the system.

Combining Equations (1.1), (1.2), and (1.3) we obtain the equation governing the behavior of the gas:

$$\Delta[\rho^{1+1/m}] = \frac{(m+1)\varphi\rho_0^{1/m}}{K} \frac{\partial\rho}{\partial t}, \tag{1.4}$$

where Δ $(= \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)$ is the Laplacian operator. Since the Equation (1.4) is nonlinear, determination of its solution has been quite difficult, and only with the past two decades have rigorous results been obtained.

In 1958, Oleinik, Kalashnikov, and Chzou Yui-Lin' [10] jointly studied a generalized form of Equation (1.4) for the one-dimensional case. They investigated the Cauchy problem and boundary-value problems for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \varphi(t,x,u)}{\partial x^2} , \tag{1.5}$$

where the function $\varphi(t,x,u)$ is defined for all values of the variables t and x, and all non-negative values of the variable u. Both of the functions $\varphi(t,x,u)$ and $\varphi_u(t,x,u)$ are assumed to be strictly positive if variable u is, but they both vanish identically for all values of t and x if variable u vanishes. By taking

$$\varphi(t,x,u) = \frac{K}{(m+1)\varphi\rho_0^{1/m}} u^{1+1/m}$$

(where the porosity constant φ is not to be confused with the function $\varphi(t,x,u)$), the Equation (1.5) reduces to the one-dimensional form of Equation (1.4). By choosing appropriate units of measurement, the coefficient of $u^{1+1/m}$ may be assumed equal to unity, and replacing $(m+1)/m$ by γ, we arrive at a simplified form, namely

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^\gamma}{\partial x^2} . \tag{1.6}$$

Assuming reasonable smoothness conditions for the data, the authors proved existence and uniqueness theorems on a finite time interval, $0 \leq t \leq T$, for both classical and "generalized" solutions of Equation (1.5). A "generalized" solution of Equation (1.5) is a bounded, continuous function $u(t, x)$ satisfying a certain integral equation related to Equation (1.5). We shall give a precise definition in the sequel.

Two years later, Friedman [5] considered the first boundary value problem for the equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 \psi(x, u)}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial \psi(x, u)}{\partial x_j} + c(x)\psi(x, u) + f(x, t), \tag{1.7}$$

where function $\psi(x, 0)$ is non-negative for all values of variable x and function $\partial\psi/\partial u$ is positive if variable u is. He proved the existence of a unique solution of this problem under the assumption that the initial and boundary functions are always positive, also assuming certain smoothness conditions on the data.

In 1961, Sabinina [13] examined the equation

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \frac{\partial^2 \varphi(u)}{\partial x_j^2}, \tag{1.8}$$

with initial condition $0 < m \leq u_0(x) \leq M$, proving existence and uniqueness theorems for classical solutions of the Cauchy problem. Here the function $\varphi(u)$ had to satisfy the conditions that its derivative $\varphi'(u)$ be positive if variable u was, $\varphi'(0)$ be non-negative, and $\varphi(0)$ be zero. Two different methods were given to construct the classical solution.

Kalashnikov [8], in 1967, showed that the derivative $\partial u/\partial x$ of the Cauchy problem solution $u(t,x)$ for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^\gamma}{\partial x^2} , \quad \gamma \geq 2 \tag{1.9}$$

must have discontinuities regardless of how smooth the initial function $u_0(x)$ is assumed to be, provided merely that the initial function $u_0(x)$ vanish on some closed interval of positive length, and be not the identically zero function. Assuming such a condition on the initial function, this behavior of the derivative $\partial u/\partial x$ shows that the Equation (1.9) can not have a solution in the "classical," or ordinary, sense, but only in the "generalized" sense as defined by Oleinik <u>et al</u>. discussed above.

Aronson, in 1969 [2] and 1970 [3, 4], examined the regularity properties of the solution for the Cauchy problem with Equation (1.9), in particular showing that the solution would have the properties expected of physical gas flow. In his 1969 paper he also exhibited an explicit solution of (1.9) due to Pattle [12].

It is the purpose of this thesis to continue the above line of research by investigating the n-dimensional problem. Specifically, we establish a uniqueness theorem for "generalized" solutions provided that the solutions belong to a certain class. Next, we establish an existence and uniqueness theorem for "classical" solutions of the

Cauchy problem by requiring that the initial function be bounded away from zero. If, on the other hand, the initial function be allowed to vanish, then the existence problem for "generalized" solutions remains unsolved except in special cases. This is due to the non-linearity. Since this problem is nevertheless important, we have given a numerical method for computing approximate solutions. Thus we present an n-dimensional finite difference approximation scheme enabling us to obtain a numerical solution for the Cauchy problem whose initial function is non-negative and bounded above. We prove the stability of this scheme. A one-dimensional example of this scheme is presented in the work [14] of Scheidegger from an example of Aronofsky and Jenkins [1]. None of these writers prove stability however. Finally, we present an example, with tables and graphs of a numerical solution of an isothermal gas flow problem using this scheme.

II. A UNIQUENESS THEOREM FOR THE GENERALIZED SOLUTION

A. Preliminaries

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, etc. represent points in n-dimensional Euclidean space E_n, and let t represent the non-negative real-valued time variable. Let G be the region

$$G = \{(t, x): 0 \le t \le T,\ x \in E_n\},$$

and consider in G the partial differential equation

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[a_j(t,x)\,\frac{\partial \varphi(t,x,u)}{\partial x_j}\right] + c(t,x)\varphi(t,x,u) + f(t,x) = \frac{\partial u}{\partial t}, \tag{2.1}$$

together with the initial condition

$$u(0, x) = u_0(x), \tag{2.2}$$

where $0 \le u_0(x) \le M_0$, $x \in E_n$, and the constant M_0 depends only on the data. Such a combination of a partial differential equation with an initial condition is called a <u>Cauchy problem</u>.

If in Equation (2.1) we let $a_j(t,x) \equiv 1$, for $1 \le j \le n$, $c(t,x) \equiv 0$, $f(t,x) \equiv 0$, and $\varphi(t,x,u) \equiv u^\gamma$, for $\gamma > 1$, we obtain the equation

$$\sum_{j=1}^{n} \frac{\partial^2 u^\gamma}{\partial x_j^2} = \frac{\partial u}{\partial t} .$$

Observe that for $\gamma \geq 2$ we have the equation of nonsteady gas flow through a homogeneous, porous, n-dimensional medium. Observing that both u^γ and its derivative $\gamma u^{\gamma-1}$ are positive for positive u, but vanish if variable u vanishes, we wish, accordingly to extend these properties to the general case. Hence we shall assume that

Condition A. For $(t, x) \in G$, and $u \geq 0$, the function $\varphi(t, x, u)$ and its derivative $\varphi_u(t, x, u)$ are strictly positive if variable u is positive, but $\varphi(t, x, 0) \equiv \varphi_u(t, x, 0) \equiv 0$.

In addition, the coefficients $a_j(t, x)$ must satisfy the following mathematical condition to ensure that the differential Equation (2.1) will be of parabolic type. Thus we assume that

Condition B. $0 < \mu_1 \leq a_j(t, x) \leq \mu_2$, for $(t, x) \in G$, $1 \leq j \leq n$, where constants μ_1 and μ_2 are positive real numbers depending only on the data.

Smoothness conditions for the coefficients of Equation (2.1) will be given in the sequel.

Definition 1. Let functions $u(t, x)$ and $v(t, x)$ be integrable on any compact subdomain of region G. If the equation

$$\iint_G [u(t,x)\frac{\partial \zeta(t,x)}{\partial x_j} + v(t,x)\zeta(t,x)]dxdt = 0$$

holds for every continuously differentiable function $\zeta(t,x)$ with compact support in G, the function $v(t,x)$ is called the generalized derivative of $u(t,x)$ with respect to the variable x_j and is written $\partial u/\partial x_j = v$.

Remark. Properties of the generalized derivative are discussed in the work [15] of Sobolev.

Definition 2. A non-negative, bounded, continuous function $u(t,x)$, defined on the region G, is called a generalized solution of the Cauchy problem (2.1), (2.2) if the generalized derivatives

$$\partial\varphi(t,x,u(t,x))/\partial x_j, \quad 1 \le j \le n,$$

exist and are bounded in G, and for every smooth function $\zeta(t,x)$ with compact support in G and vanishing for $t = T$, the integral equation

$$\iint_G \Big\{ u\frac{\partial\zeta}{\partial t} - \sum_{j=1}^{n} a_j(t,x)\frac{\partial\varphi(t,x,u)}{\partial x_j}\frac{\partial\zeta}{\partial x_j} + [c(t,x)\varphi(t,x,u)+f(t,x)]\zeta\Big\} dxdt + \int_{E_n} u_0(x)\zeta(0,x)dx = 0 \tag{2.3}$$

is satisfied.

Remark. It is easy to show that any classical (i.e., ordinary) solution of the Cauchy problem (2.1), (2.2) is also a generalized solution. Thus, if function $u(t,x)$ is a classical solution, we must have:

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[a_j(t,x)\frac{\partial \varphi(t,x,u)}{\partial x_j}\right] + c(t,x)\varphi(t,x,u) + f(t,x) = \frac{\partial u}{\partial t},$$

where $u = u(t,x)$, and also $u(0,x) = u_0(x)$ must hold.

Multiplying both sides of the above differential equation by a test function $\zeta(t,x)$ and integrating with respect to both variables t and x over region G we obtain the integral equation:

$$\iint_G \Big\{ +\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[a_j(t,x)\frac{\partial \varphi(t,x,u)}{\partial x_j}\right]\zeta(t,x) + [c(t,x)\varphi(t,x,u)+f(t,x)]\zeta(t,x)\Big\} dxdt = \iint_G \frac{\partial u}{\partial t}\zeta(t,x)dxdt .$$

Upon applying integration by parts with respect to each variable x_j to the second order term of the above equation, and to the right-hand side of the equation with respect to the variable t, and making use of the properties of the test function $\zeta(t,x)$, we obtain the integral equation:

$$\iint_G \left\{ -\sum_{j=1}^{n} a_j(t,x) \frac{\partial \varphi(t,x,u)}{\partial x_j} \frac{\partial \zeta}{\partial x_j} + [c(t,x)\varphi(t,x,u)+f(t,x)]\zeta \right\} dxdt$$

$$= -\int_{E_n} u(0,x)\zeta(0,x)dx - \iint_G u \frac{\partial \zeta}{\partial t} dxdt,$$

from which Equation (2.3) immediately follows. Since the classical solution $u(t,x)$ is assumed to be bounded, non-negative and continuous with all of its derivatives which appear in Equation (2.1) being continuous and bounded, we must conclude that function $u(t,x)$ must also be a generalized solution of Cauchy problem (2.1, (2.2).

B. The Uniqueness Theorem

In this section we shall extend the uniqueness theorem of Oleinik *et al.*, for the Cauchy problem to second order parabolic partial differential equations with more than one space variable, and having coefficients not necessarily constant. This extension is carried out by proving a uniqueness theorem for the generalized solution of the Cauchy problem (2.1), (2.2).

In order to prove the uniqueness theorem certain assumptions must be made. First of all, we must have

Condition (i). The solution $u(t, x)$ satisfies the inequality

$$u(t, x) \leq M^*/(|x|^{n-1}+1)^{\beta}, \quad (t, x) \in G,$$

where $\beta > n/(n-1)$ if $n \geq 2$ is an integer, otherwise β any real number if $n = 1$ (n is the number of space variables in the differential Equation (2.1)). The constants β and M^* depend only on the data.

Physically, this condition guarantees that we are dealing with a finite amount of gas. Mathematically, some growth condition is needed in the proof of the theorem.

Secondly, we must have

Condition (ii). For any positive real number M, the functions $\varphi(t, x, u)$ and $\varphi_u(t, x, u)$ are continuous and bounded on the region

$$R_M = \{(t, x, u): (t, x) \in G, \ 0 \leq u \leq M\} .$$

In addition, condition A of the Preliminaries is satisfied.

In the physical situation for the gas flow equation, the gas density $u(t, x)$ is always finite, so one can always find a constant M such that $u(t, x) \leq M$. Thus for the case $\varphi(t, x, u) \equiv u^{\gamma}$, condition (ii) is easily satisfied.

Thirdly, we must have the following, strictly mathematical

Condition (iii). The functions $a_j(t,x)$, $(a_j)_t(t,x)$, $c(t,x)$, $c_t(t,x)$, and $f(t,x)$ are all continuous and bounded on G, with the functions $(a_j)_t(t,x)$ being non-negative, and the functions $c(0,x)$ and $c_t(t,x)$ being non-positive for $(t,x) \in G$. In addition, condition B of the Preliminaries is satisfied.

Theorem 1. (Uniqueness of the generalized solution).

There can not be more than one solution $u(t,x)$ of the Cauchy problem (2.1), (2.2) in the region G, provided that the conditions (i)-(iii) above are satisfied.

Proof. The proof of the theorem is essentially an extension of the proof furnished by Oleinik et al. [10] for the uniqueness of the solution of the Cauchy problem in the case of the one-dimensional non-steady gas flow equation. As in their paper, the Equation (2.3) will also hold for any function $\zeta(t,x)$ continuous with compact support in G, and vanishing for $t = T$, and having bounded generalized derivatives $\partial\zeta/\partial t$, $\partial\zeta/\partial x_j$, $1 \leq j \leq n$, in the region G (see Sobolev [15]).

Suppose the Cauchy problem (2.1), (2.2) has two generalized solutions $u(t,x)$ and $v(t,x)$ satisfying condition (i). On subtracting identity (2.3) for the solution v from the identity for solution u, we obtain the identity

$$\iint_G \Big\{ \frac{\partial \zeta}{\partial t}(u-v) - \sum_{j=1}^{n} \frac{\partial \zeta}{\partial x_j} a_j(t,x)\Big[\frac{\partial \varphi(t,x,u)}{\partial x_j} - \frac{\partial \varphi(t,x,v)}{\partial x_j}\Big]$$

$$+ c(t,x)[\varphi(t,x,u)-\varphi(t,x,v)]\zeta(t,x)\Big\} dxdt = 0 . \qquad (2.4)$$

Now let $\chi_m(x)$ be any smooth function defined on E_n, such that

$\chi_m(x) = 1$ for $|x| \leq m$, $\chi_m(x) = 0$ for $|x| \geq m+1$,

$0 \leq \chi_m(x) \leq 1$ for $x \in E_n$,

and the derivatives $\partial\chi_m/\partial x_j$ are bounded uniformly in m, $1 \leq j \leq n$. For the construction of such a sequence of functions χ_m see Friedman [6]. Define the function $\zeta_m(t,x)$ by the equation

$$\zeta_m(t,x) = \chi_m(x)\int_T^t [\varphi(\tau,x,u(\tau,x))-\varphi(\tau,x,v(\tau,x))]d\tau .$$

Then for every positive integer m, it is easy to see that the function $\zeta_m(t,x)$ is continuous with compact support in G, vanishes for $t = T$, and has bounded generalized derivatives $\partial\zeta_m/\partial t$, $\partial\zeta_m/\partial x_j$, $1 \leq j \leq n$ in G. By the remark made at the beginning of the proof, the Equation (2.3) will also hold for this function $\zeta_m(t,x)$, and so will Equation (2.4). Substitution of $\zeta_m(t,x)$ into Equation (2.4) yields the identity

$$\iint_G \chi_m(x)[\varphi(t,x,u)-\varphi(t,x,v)][u-v]dxdt$$

$$-\iint_G \Big\{\sum_{j=1}^{n} \chi_m(x)\int_T^t \Big(\frac{\partial\varphi(\tau,x,u)}{\partial x_j} - \frac{\partial\varphi(\tau,x,v)}{\partial x_j}\Big)d\tau\, a_j(t,x)$$

$$\cdot \Big(\frac{\partial\varphi(t,x,u)}{\partial x_j} - \frac{\partial\varphi(t,x,v)}{\partial x_j}\Big)\Big\}\, dxdt$$

$$-\iint_{Q_m} \Big\{\sum_{j=1}^{n} \frac{\partial\chi_m(x)}{\partial x_j}\int_T^t (\varphi(\tau,x,u)-\varphi(\tau,x,v))d\tau\, a_j(t,x)$$

$$\cdot \Big(\frac{\partial\varphi(t,x,u)}{\partial x_j} - \frac{\partial\varphi(t,x,v)}{\partial x_j}\Big)\Big\}\, dxdt$$

$$+\iint_G \Big\{c(t,x)[\varphi(t,x,u)-\varphi(t,x,v)]\chi_m(x)$$

$$\cdot \int_T^t [\varphi(\tau,x,u)-\varphi(\tau,x,v)]d\tau\Big\}\, dxdt = 0, \qquad (2.5)$$

where Q_m is the region

$$Q_m = \{(t,x): 0 \le t \le T,\ m \le |x| \le m+1\}.$$

Now for the second integral in identity (2.5) we have

$$- \iiint_G \left\{ \sum_{j=1}^{n} \chi_m(x) \int_T^t \left[\frac{\partial \varphi(\tau, x, u)}{\partial x_j} - \frac{\partial \varphi(\tau, x, v)}{\partial x_j} \right] d\tau \, a_j(t,x) \right.$$

$$\left. \cdot \left[\frac{\partial \varphi(t, x, u)}{\partial x_j} - \frac{\partial \varphi(t, x, v)}{\partial x_j} \right] \right\} dxdt$$

$$= -\frac{1}{2} \iiint_G \left\{ \sum_{j=1}^{n} \chi_m(x) a_j(t,x) \frac{\partial}{\partial t} \left(\int_T^t \left[\frac{\partial \varphi(\tau, x, u)}{\partial x_j} - \frac{\partial \varphi(\tau, x, v)}{\partial x_j} \right] d\tau \right)^2 \right\} dxdt$$

$$= \frac{1}{2} \int_{E_n} \left\{ \sum_{j=1}^{n} \chi_m(x) a_j(0,x) \left(\int_T^0 \left[\frac{\partial \varphi(\tau, x, u)}{\partial x_j} - \frac{\partial \varphi(\tau, x, v)}{\partial x_j} \right] d\tau \right)^2 \right\} dx$$

$$+ \frac{1}{2} \iiint_G \left\{ \sum_{j=1}^{n} \chi_m(x) \frac{\partial a_j(t,x)}{\partial t} \left(\int_T^t \left[\frac{\partial \varphi(\tau, x, u)}{\partial x_j} - \frac{\partial \varphi(\tau, x, v)}{\partial x_j} \right] d\tau \right)^2 \right\} dxdt,$$

and the last two integrals in the above expression are non-negative because of condition B, and condition (iii) for function $(a_j)_t(t,x)$. Note that the last step above was obtained through the use of integration by parts with respect to the variable t.

Similarly, for the fourth integral in identity (2.5), by applying integration by parts with respect to the variable t, we obtain

$$\iint_G \left\{ c(t,x)[\varphi(t,x,u) - \varphi(t,x,v)] \chi_m(x) \right.$$

$$\left. \cdot \int_T^t [\varphi(\tau,x,u) - \varphi(\tau,x,v)] d\tau \right\} dxdt \quad =$$

$$= \frac{1}{2}\iint\limits_{G}\left\{\chi_m(x)c(t,x)\frac{\partial}{\partial t}\left(\int_T^t[\varphi(\tau,x,u)-\varphi(\tau,x,v)]d\tau\right)^2\right\}dxdt$$

$$= \int_{E_n}\left\{-\frac{1}{2}\chi_m(x)c(0,x)\left(\int_T^0[\varphi(\tau,x,u)-\varphi(\tau,x,v)]d\tau\right)^2\right\}dx$$

$$+ \iint\limits_{G}\left\{-\frac{1}{2}\chi_m(x)\frac{\partial c(t,x)}{\partial t}\left(\int_T^t[\varphi(\tau,x,u)-\varphi(\tau,x,v)]d\tau\right)^2\right\}dxdt\,,$$

and again the last two integrals are non-negative because of condition (iii) for functions $c(t,x)$ and $c_t(t,x)$.

Hence identity (2.5) can be put into the form

$$\iint\limits_{G}\chi_m(x)[\varphi(t,x,u)-\varphi(t,x,v)][u-v]dxdt$$

$$+\frac{1}{2}\int_{E_n}\left\{\sum_{j=1}^{n}\chi_m(x)a_j(0,x)\left(\int_T^0\left[\frac{\partial\varphi(\tau,x,u)}{\partial x_j}-\frac{\partial\varphi(\tau,x,v)}{\partial x_j}\right]d\tau\right)^2\right\}dx$$

$$+\frac{1}{2}\iint\limits_{G}\left\{\sum_{j=1}^{n}\chi_m(x)\frac{\partial a_j(t,x)}{\partial t}\left(\int_T^t\left[\frac{\partial\varphi(\tau,x,u)}{\partial x_j}-\frac{\partial\varphi(\tau,x,v)}{\partial x_j}\right]d\tau\right)^2\right\}dxdt$$

$$+\int_{E_n}\left\{-\frac{1}{2}\chi_m(x)c(0,x)\left(\int_T^0[\varphi(\tau,x,u)-\varphi(\tau,x,v)]d\tau\right)^2\right\}dx$$

$$+\iint\limits_{G}\left\{-\frac{1}{2}\chi_m(x)\frac{\partial c(t,x)}{\partial t}\left(\int_T^t[\varphi(\tau,x,u)-\varphi(\tau,x,v)]d\tau\right)^2\right\}dxdt =$$

$$= \iiint_{Q_m} \left\{ \sum_{j=1}^{n} \frac{\partial \chi_m(x)}{\partial x_j} \; a_j(t,x) \int_T^t [\varphi(\tau,x,u)-\varphi(\tau,x,v)] d\tau \right.$$

$$\left. \cdot \left[\frac{\partial \varphi(t,x,u)}{\partial x_j} - \frac{\partial \varphi(t,x,v)}{\partial x_j} \right] \right\} dxdt \, . \qquad (2.6)$$

We note that the integrand of the first integral in identity (2.6) is also non-negative since function $\varphi(t,x,u)$ is increasing in the variable u by virtue of condition A. Thus, upon examining the five integrals on the left-hand side of identity (2.6) we find that the integrands are non-negative functions, and are non-decreasing for increasing m, by virtue of the definition of function $\chi_m(x)$. If we are able to show that the right-hand side of identity (2.6) is bounded uniformly with respect to m, then all five integrals on the left-hand side will also be so bounded. In particular, this means that the first integral on the left-hand side tends to a finite limit as m grows without bounds, and therefore the function

$$\psi(t,x) = [\varphi(t,x,u(t,x))-\varphi(t,x,v(t,x))]\,[u(t,x)-v(t,x)]$$

is Lebesque integrable on region G by the monotone convergence theorem.

We shall now show that the right-hand side of identity (2.6) is not only uniformly bounded with respect to m, but indeed vanishes as m grows without bounds. Let A, B, C be positive numbers such that

$$|\partial\chi_m(x)/\partial x_j| \leq A,\ 1 \leq j \leq n,\ m = 1, 2, 3, \ldots, x \in E_n,$$

$$|\partial\varphi(t, x, u(t, x))/\partial x_j| \leq B,\ 1 \leq j \leq n,\ (t, x) \in G,$$

$$|\partial\varphi(t, x, v(t, x))/\partial x_j| \leq B,\ 1 \leq j \leq n,\ (t, x) \in G,$$

$$\varphi_u(t, x, \bar{u}) \leq C, \quad \text{for} \quad (t, x) \in G, \quad \text{and}$$

$$0 \leq \bar{u} \leq \sup_G\{u(t, x), v(t, x)\}$$

$$< +\infty \quad \text{(by condition (i))}.$$

Then the right-hand side of identity (2.6) is majorized by

$$2nAb\mu_2 \iint_{Q_m} \left| \int_T^t [\varphi(\tau, x, u) - \varphi(\tau, x, v)] d\tau \right| dxdt$$

$$\leq 2nAB\mu_2 T \iint_{Q_m} |\varphi(t, x, u) - \varphi(t, x, v)| dxdt$$

(where we changed order of integration)

$$\leq 2nAB\mu_2 T \iint_{Q_m} |\varphi_u(t, x, \hat{u})| \, |u(t, x) - v(t, x)| dxdt$$

(using the mean-value theorem with $\hat{u}$ between $u(t, x)$ and $v(t, x)$)

$$\leq 2nABC\mu_2 T \iint_{Q_m} (|u(t, x)| + |v(t, x)|) dxdt \quad \leq$$

$$\leq 4nABC\mu_2 T^2 \int_{S_m} \frac{M^* dx}{(|x|^{n-1}+1)^{\beta}},$$

where

$$S_m = \{x \in E_n : |x| \geq m\}.$$

But this integral vanishes as m grows without bounds since the function $1/(|x|^{n-1}+1)^{\beta}$, $\beta > n/(n-1)$, is integrable on E_n.

It follows that

$$\iint_G [\varphi(t,x,u)-\varphi(t,x,v)][u-v]dxdt = 0,$$

or (using the mean-value theorem)

$$\iint_G \varphi_u(t,x,\bar{u})[u-v]^2 dxdt = 0, \tag{2.7}$$

where $\bar{u}$ lies strictly between $u(t,x)$ and $v(t,x)$ wherever they differ. From Equation (2.7) we quickly obtain $u(t,x) = v(t,x)$ on G since all functions in the integrand are continuous, and function $\varphi_u(t,x,u)$ must satisfy condition A. Thus we have proved the uniqueness theorem.

C. Concluding Remarks

We wish to conclude this section by pointing out the seemingly insurmountable difficulties which arise in attempting to extend the method of proof given by Oleinik et al [10] of existence of the generalized solution for the one-dimensional Cauchy problem

$$\frac{\partial^2 \varphi(x,u)}{\partial x^2} = \frac{\partial u}{\partial t} ,$$

$$u(0,x) = u_0(x) \geq 0,$$

to the n-dimensional case

$$\sum_{j=1}^{n} \frac{\partial^2 \varphi(u)}{\partial x^2} = \frac{\partial u}{\partial t} ,$$

$$u(0,x) = u_0(x) \geq 0.$$

In the course of their proof they construct a function $u(t,x)$, the proposed generalized solution of the Cauchy problem. It is necessary for them to show that the generalized derivative

$$\partial\varphi(u(t,x))/\partial x$$

exists and is bounded in G. To do this they consider the derivatives of a sequence of functions $\{v_k\}$ satisfying the differential equation

$$\frac{\partial^2 v_k}{\partial x^2} = \Phi'(v_k) \frac{\partial v_k}{\partial t} \qquad (*)$$

and some boundary conditions. Letting $p_k = \partial v_k / \partial x$ and using Equation (*) they find that the functions p_k satisfy the differential equation

$$\frac{\partial^2 p_k}{\partial x^2} = \Phi'(v_k) \frac{\partial p_k}{\partial t} + \frac{\frac{\partial}{\partial x}[\Phi'(v_k)]}{\Phi'(v_k)} \frac{\partial p_k}{\partial x},$$

and that the functions p_k are uniformly bounded with respect to k on the boundaries of their regions of definition. This uniform boundedness on the boundaries of the regions, together with the maximum principle leads to boundedness of the functions p_k uniformly with respect to k throughout the interior of their regions of definition, hence leading to existence and boundedness of the generalized derivatives $\partial\varphi(u(t,x))/\partial x$ in G.

This method of proof will not work in the n-dimensional case, however, for the analogous differential equation for the function $p_k = \partial v_k / \partial x_j$ becomes

$$\sum_{i=1}^{n} \frac{\partial^2 p_k}{\partial x_i^2} = \Phi'(v_k) \frac{\partial p_k}{\partial t} - \frac{\varphi''(u_k)}{\varphi'(u_k)^3} \frac{\partial v_k}{\partial t} p_k,$$

with a <u>zero-th order</u> term in p_k present. Because of the presence of this term, we can no longer conclude that functions p_k are uniformly bounded with respect to k throughout the interiors of their regions of definition, and this lack of uniform boundedness with respect to k of the functions p_k denies us the establishment of existence and boundedness of the generalized derivatives $\partial\varphi(u(t, x))/\partial x_j$ by their method of proof.

III. AN EXISTENCE AND UNIQUENESS THEOREM FOR THE CLASSICAL SOLUTION

A. Preliminaries

In this chapter we wish to prove an existence and uniqueness theorem for the classical solution of the Cauchy problem having differential equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(t,x)\frac{\partial \varphi(u)}{\partial x_j}\right] + \sum_{j=1}^{n} b_j(t,x)\frac{\partial \varphi(u)}{\partial x_j} = \frac{\partial u}{\partial t}, \qquad (3.1)$$

and initial condition

$$u(0,x) = u_0(x). \qquad (3.2)$$

By taking $a_{ij}(t,x) = 0$ for $i \neq j$, $a_{jj}(t,x) = 1$ for $1 \leq j \leq n$, $b_j(t,x) = 0$ for $1 \leq j \leq n$, and $\varphi(u) = u^{\gamma}$, for $\gamma \geq 2$, we again obtain the nonsteady gas flow equation

$$\sum_{j=1}^{n} \frac{\partial^2 u^{\gamma}}{\partial x_j^2} = \frac{\partial u}{\partial t} .$$

Arguing as in Chapter II, we shall assume

Condition C. The function $\varphi(u)$ and its derivative $\varphi'(u)$ are strictly positive if variable u is, but $\varphi(0) = \varphi'(0) = 0$.

Again, to ensure that Equation (3.1) will be of parabolic type we must have

<u>Condition D</u>. For $(t,x) \in G$ we have, $a_{ij}(t,x) = a_{ji}(t,x)$ for $1 \le i \le n$, $1 \le j \le n$, and for every point $y = (y_1, \ldots, y_n) \in E_n$,

$$\mu_1 \sum_{j=1}^{n} y_j^2 \le \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(t,x) y_i y_j \le \mu_2 \sum_{j=1}^{n} y_j^2 ,$$

where μ_1 and μ_2 are positive real constants dependent only on the data.

We must note that the Equation (3.1) can not have a zero-th order term $c(t,x,u)$ such as appears in Equation (2.1) since the presence of such a term will not permit the application of <u>both</u> the maximum and the minimum principle (see Friedman [6]) required in a certain stage of the proof of existence.

It will now be convenient to introduce the following notation. Let Ω be an arbitrary region in E_n and let function $u(t,x)$ be defined on region $Q = \{(t,x): 0 \le t \le T,\ x \in \Omega\}$. For points $P'(t',x')$ and $P''(t'',x'')$ in Q we let

$$d(P', P'') = \sqrt{\left(|x'-x''|^2 + |t'-t''|\right)},$$

$$|u|_0 = \sup_Q |u| ,$$

$$|u|_\alpha = |u_0| + \sup_{P', P'' \in Q} \frac{|u(P') - u(P'')|}{d(P', P'')^\alpha} ,$$

$$|u|_{1+\alpha} = |u|_{\alpha} + \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j} \right|_{\alpha},$$

$$|u|_{2+\alpha} = |u|_{1+\alpha} + \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j} \right|_{1+\alpha} + \left| \frac{\partial u}{\partial t} \right|_{\alpha},$$

where $0 < \alpha < 1$, and α any real number.

We shall say that function $u(t, x)$ is a member of the class of functions $C^q(Q)$, or more briefly $u \in C^q(Q)$ (where $q = 0, \alpha, 1+\alpha, 2+\alpha$), if $|u|_q < +\infty$.

<u>Remark</u>. $u \in C^0(Q)$ is equivalent to saying that u is a bounded function on region Q.

<u>Definition</u>. If the inequality

$$\sup_{P', P'' \in Q} \frac{|u(P') - u(P'')|}{d(P', P'')^{\alpha}} < +\infty$$

holds, then function $u(t, x)$ is said to satisfy a <u>Hölder condition</u> with exponent α on Q (or said to be <u>Hölder continuous</u> with exponent α on Q). If $\alpha = 1$ in the above inequality, then the function $u(t, x)$ is said to satisfy a <u>Lipschitz condition</u> on Q (or be <u>Lipschitz continuous</u> on Q). If function $u(t, x)$ is Hölder continuous, the quantity

$$H(u) = \sup_{P', P'' \epsilon Q} \frac{|u(P') - u(P'')|}{d(P', P'')^{\alpha}}$$

is called the Hölder coefficient of the function $u(t, x)$. For the case when $\alpha = 1$, the Hölder coefficient is called, instead, the Lipschitz coefficient and denoted by $L(u)$.

We shall say that a function $u(t, x)$ on a region D contained in G is a member of the class of functions $C_{q+\alpha}$ (or $u \epsilon C_{q+\alpha}$), with respect to certain of its arguments if the function $u(t, x)$ itself and all of its qth-order partial derivatives with respect to these arguments are bounded in D and satisfy a Hölder condition with exponent α $(0 < \alpha < 1)$ with respect to all of the arguments in region D.

B. The Existence and Uniqueness Theorem

In this section, we shall prove an existence and uniqueness theorem for the classical solution of the Cauchy problem (3.1), (3.2). By a classical solution we mean a non-negative, continuous, bounded function $u(t, x)$ having continuous, bounded first and second order partial derivatives with respect to the space variables x_j, and a continuous, bounded first order partial derivative with respect to the time variable t, such that the function $u(t, x)$ satisfies the Cauchy problem (3.1), (3.2) in region G.

We shall now examine the assumptions that we shall use to prove the theorem. First of all, we shall require

<u>Condition (i)</u>. The initial function $u_0(x) \in C^{2+\alpha}$ for the variable x, and satisfies the inequality $0 < m \leq u_0(x) \leq M$, where the constants m and M depend only on the data.

The inequality above is a reasonable requirement, since the pressure in a gas well can not be infinite at any time, and can not fall below atmospheric pressure, at which time the well is exhausted. The condition $u_0(x) \in C^{2+\alpha}$ is a necessary mathematical one.

Secondly, we must have the necessarily mathematical

<u>Condition (ii)</u>. The functions $a_{ij}(t,x) \in C_{2+\alpha}$, with respect to the variable x, and the functions $b_j(t,x) \in C_{1+\alpha}$ with respect to the variable x.

Clearly, the coefficients of the gas flow equation satisfy Condition (ii) since they are constants.

Thirdly, we must have

<u>Condition (iii)</u>. On the region

$$H = \{u: 0 < u \leq M_1\} ,$$

(where $M_1 = M + \epsilon$, with ϵ any arbitrary positive constant, and

M the constant from Condition (i), the function $\varphi(u)$ has continuous derivatives up to the third order inclusive, and satisfying a Lipschitz condition with respect to variable u in every closed region of the form

$$\{u: 0 < m \leq u \leq M_1\} ,$$

where m is an arbitrary real number such that $0 < m \leq M_1$.

We note that if $\varphi(u) = u^\gamma$, with $\gamma \geq 2$, then Condition (iii) is satisfied, hence we must have it so for the more general case.

Fourthly, we must have

Condition (iv). The function $\varphi(u) \to +\infty$ as $u \to +\infty$, and function $\varphi'(u)$ is bounded away from zero if variable u is bounded away from zero.

Fifthly, we must also have

Condition (v). Functions $\varphi(u)$, and $\varphi'(u)$ are both bounded on the region H.

We note that if $\varphi(u) = u^\gamma$, with $\gamma \geq 2$, then Conditions (iv) and (v) are both satisfied, hence we wish to extend these properties to the general case.

It must be pointed out that Condition (i) is a critical assumption, for if the initial function $u_0(x)$ ever vanishes, then there can be no classical solution of Cauchy problem (3.1), (3.2) as was shown by Kalashnikov in 1967 [8].

Theorem 2. (Existence and Uniqueness of the Classical Solution).

If Conditions (i) through (v), C, and D above are satisfied, then the Cauchy problem (3.1), (3.2) has a unique classical solution $u(t,x)$ in G, with $u(t,x) \in C^{2+\alpha}$ in G, and moreover we have

$$0 < m \leq u(t,x) \leq M$$

holding throughout G.

To prove Theorem 2 we must first prove a lemma for the differential equation

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} [c_{ij}(t,x) \frac{\partial v}{\partial x_j}] + \sum_{j=1}^{n} d_j(t,x) \frac{\partial v}{\partial x_j} = A(v) \frac{\partial v}{\partial t}. \tag{3.3}$$

We shall consider this differential equation on region G, together with the initial condition

$$v(0,x) = \psi(x), \quad x \in E_n \tag{3.4}$$

Lemma. Suppose the following assumptions are satisfied:

i) Function $A(v) \geq a_p > 0$ for $v \geq p > 0$, where p is any positive real number, and a_p depends only on p.

ii) On the closed interval $\{v: 0 < m \leq v \leq M\}$, the function $A(v)$ is bounded and has continuous, bounded derivatives up to the second order inclusive, and satisfying a Lipschitz condition in v. Here m and M are arbitrary real numbers with $0 < m \leq M$.

iii) For $(t, x) \in G$, the functions $c_{ij}(t, x)$ satisfy Condition D, where $a_{ij}(t, x)$ is replaced by $c_{ij}(t, x)$.

iv) For $(t, x) \in G$, we have that the functions $c_{ij}(t, x) \in C_{2+\alpha}$ with respect to the variable x, and the functions $d_j(t, x) \in C_{1+\alpha}$ with respect to the variable x.

v) On the closed region E_n the function $\psi(x) \in C^{2+\alpha}$, and also satisfies the inequality $0 < m_0 \leq \psi(x) \leq M_0$, $x \in E_n$, where the constants m_0 and M_0 depend only on the data.

Then the Cauchy problem (3.3), (3.4) has a unique classical solution $v(t, x)$ in G, with $v(t, x) \in C^{2+\alpha}$ in G, and also satisfying the inequality

$$m_0 \leq v(t, x) \leq M_0 \tag{3.5}$$

throughout the region G.

Remarks. Condition i) asserts that the function $A(v)$ is bounded positively away from zero if variable v is so bounded. Conditions i) and ii) are both needed in the proof of the lemma in order to ensure that a certain change of dependent variable v to another dependent variable w is both one-to-one and smooth. The Conditions ii) to v) are needed to ensure that given data is sufficiently smooth enough to fulfill the hypotheses of a fundamental theorem of Oleinik and Kruzhkov (see Appendix B), their theorem being required at a crucial step in the proof of the lemma.

Proof of Lemma. By applying Conditions i) and ii) to the function $A(v)$ we have that

$$0 < \underline{A} \leq A(v) \leq \overline{A} < +\infty$$

for $m_0 \leq v \leq M_0$, and where

$$\underline{A} = \inf_{m_0 \leq v \leq M_0} A(v), \quad \overline{A} = \sup_{m_0 \leq v \leq M_0} A(v).$$

We next choose a function $A_1(v)$ for $-\infty < v < +\infty$, such that

$$\underline{A}/2 \leq A_1(v) \leq 2\overline{A},$$

$$A_1(v) \equiv A(v) \quad \text{for} \quad m_0 \leq v \leq M_0$$

$$A_1(v) \equiv (\underline{A}+\overline{A})/2 \quad \text{for} \quad v \leq m_0/2 \quad \text{or} \quad 2M_0 \leq v.$$

In addition $A_1(v)$ has bounded, continuous derivatives up to the second order inclusive, satisfying a Lipschitz condition in v uniformly for all values of v.

If in the Cauchy problem

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \left[c_{ij}(t,x) \frac{\partial v}{\partial x_j}\right] + \sum_{j=1}^{n} d_j(t,x) \frac{\partial v}{\partial x_j} = A_1(v) \frac{\partial v}{\partial t}, \tag{3.6}$$

$$v(0,x) = \psi(x), \quad x \in E_n,$$

we introduce the one-to-one change of dependent variable

$$w = F(v), \quad v = G(w)$$

where

$$F(v) = \int_{m_0/2}^{v} A_1(s)ds$$

we obtain the Cauchy problem

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left[C_{ij}(t,x,w) \frac{\partial w}{\partial x_j}\right] + \sum_{j=1}^{n} D_j(t,x,w) \frac{\partial w}{\partial x_j} = \frac{\partial w}{\partial t}, \tag{3.7}$$

$$w(0,x) = \psi_1(x), \quad x \in E_n, \tag{3.8}$$

where we define

$$C_{ij}(t,x,w) = c_{ij}(t,x)/A_1(v),$$

$$D_j(t,x,w) = d_j(t,x)/A_1(v),$$

and

$$\psi_1(x) = F(\psi(x)), \quad x \in E_n.$$

We next wish to show that $C_{ij} \in C_{2+\alpha}$ with respect to the variables x and w, that $D_j \in C_{1+\alpha}$ with respect to the variables x and w, and that $\psi_1 \in C^{2+\alpha}$ on E_n.

First, we observe that the function $A_1(v)$, together with its derivatives up through the second order are also Hölder continuous. This fact follows immediately from assumption ii) of the lemma, and through the use of Proposition 1 of Appendix A, together with the properties of the construction of function $A_1(v)$.

Next, since any function which is Hölder in variable v is also Hölder in variable w by Proposition 5, Appendix A, we conclude that functions $C_{ij}(t, x, w)$ are Hölder continuous in t, x, and w, since the ratios c_{ij}/A_1 are so by virtue of Proposition 4, Appendix A. That the C_{ij} are bounded follows from the fact that the c_{ij} are, and that A_1 is bounded away from zero.

We must now show that all second order derivatives $(C_{ij})_{x_k x_l}$, $(C_{ij})_{x_k, w}$, and $(C_{ij})_{ww}$ are all bounded and Hölder continuous in t, x, and w. Since we have

$$(C_{ij})_{x_k x_l} = (c_{ij})_{x_k x_l} / A_1 ,$$

$$(C_{ij})_{x_k w} = -(c_{ij})_{x_k} A_1' / A_1^3 ,$$

and

$$(C_{ij})_{ww} = c_{ij}[3(A_1')^2 - A_1A_1'']/A_1^5 ,$$

we see that these derivatives must be bounded and Hölder, continuous in the variables t, x, and w since they are rational combinations of such functions, and we may use Propositions 3 and 4, Appendix A. Therefore, $C_{ij} \epsilon C_{2+\alpha}$ with respect to the variables x and w.

Next we must show that $D_j \epsilon C_{1+\alpha}$ with respect to the variables x and w. But this is easily seen to be so since the functions

$$D_j = d_j/A_1 ,$$

$$(D_j)_{x_k} = (d_j)_{x_k}/A_1 ,$$

$$(D_j)_w = -d_jA_1'/A_1^3$$

are all bounded and Hölder continuous in t, x, and w for exactly the same reasons as for the functions C_{ij}. Therefore $D_j \epsilon C_{1+\alpha}$ with respect to the variables x and w.

Next, we wish to show that $\psi_1 \epsilon C^{2+\alpha}$ on the region E_n. Recalling that

$$|u|_{2+\alpha} = |u|_{1+\alpha} + \sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right|_{1+\alpha} + \left|\frac{\partial u}{\partial t}\right|_{\alpha}$$

$$= |u|_{\alpha} + 2\sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right|_{\alpha} + \sum_{i=1}^{n}\sum_{j=1}^{n} \left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|_{\alpha} + \left|\frac{\partial u}{\partial t}\right|_{\alpha},$$

where

$$|u|_{\alpha} = |u|_0 + \sup_{\substack{(t,x)\in G\\(t',x')\in G}} \frac{|u(t,x)-u(t',x')|}{[|x-x'|^2+|t-t'|]^{\alpha/2}}$$

and where

$$|u|_0 = \sup_G |u|,$$

we have that $|u|_{2+\alpha} < +\infty$ if and only if the function u and its derivatives $\partial u/\partial x_j$, $\partial^2 u/\partial x_i \partial x_j$, and $\partial u/\partial t$ are both bounded in G, and Hölder continuous in t and x. Since function $\psi_1(x)$ depends only on x, we need not prove Hölder continuity in t.

First, we prove boundedness, we have

$$|\psi_1(x)| = \left|\int_{m_0/2}^{\psi(x)} A_1(s)ds\right| = |A_1(\bar{s})| \leq 2\bar{A},$$

by using the Mean-value theorem, with

$$m_0/2 < \bar{s} < \psi(x).$$

Next, we have

$$|\partial\psi_1(x)/\partial x_j| = |A_1(\psi(x))\psi_{x_j}(x)|$$

which is bounded, since A_1 is bounded by construction, and $\psi \in C^{2+\alpha}$ on E_n by hypothesis.

Finally, we have

$$|\partial^2\psi_1(x)/\partial x_i\partial x_j| = |A_1'(\psi(x))\psi_{x_i}\psi_{x_j} + A_1(\psi(x))\psi_{x_ix_j}|$$

$$\leq |A_1'(\psi(x))\psi_{x_i}\psi_{x_j}| + |A_1(\psi(x))\psi_{x_ix_j}| ,$$

which is bounded since A_1' is, and $\psi \in C^{2+\alpha}$.

For the proof of Hölder continuity, we first have

$$|\psi_1(x)-\psi_1(x')|/|x-x'|^\alpha = \left|\int_{\psi(x')}^{\psi(x)} A_1(s)ds\right|/|x-x'|^\alpha$$

$$\leq \sup_s |A_1(x)|\,|\psi(x)-\psi(x')|/|x-x'|^\alpha$$

$$\leq 2\bar{A}|\psi|_\alpha < +\infty .$$

Next, since

$$\partial\psi_1(x)/\partial x_j = A_1(\psi(x))\psi_{x_j}(x)$$

is the product of two Hölder continuous functions, it is Hölder continuous by virtue of Propositions 3 and 6, Appendix A.

Finally, since

$$\partial^2\psi_1/\partial x_i \partial x_j = A_1'(\psi)\psi_{x_i}\psi_{x_j} + A_1(\psi)\psi_{x_i x_j},$$

it must be Hölder continuous by Propositions 2, 3, and 6, Appendix A.

Thus $|\psi_1|_{2+\alpha} < +\infty$, and therefore $\psi_1 \in C^{2+\alpha}$.

Next, the functions $C_{ij}(t, x, w)$ satisfy the conditions

$$C_{ij}(t, x, w) = C_{ji}(t, x, w), \quad (t, x) \in G, \quad w \text{ real},$$

and

$$\frac{\mu_1}{\bar{A}} \sum_{j=1}^{n} y_j^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}(t, x, w) y_i y_j \leq \frac{\mu_2}{\underline{A}} \sum_{j=1}^{n} y_j^2$$

for every $y \in E_n$, $(t, x) \in G$, and w real.

Finally, since $D_j \in C_{1+\alpha}$, we conclude that for $(t, x) \in G$, and real w, that the expressions

$$|D_j| + \sum_{j=1}^{n} |(D_j)_{x_k}|, \quad j = 1, 2, \ldots, n$$

are bounded.

From the foregoing discussion we conclude that all conditions in the hypothesis of Theorem 14, of Oleinik and Kruzhkov [11] (see statement of this theorem in Appendix B), are satisfied, hence there exists a unique solution $w(t, x)$ on region G of the Cauchy problem

(3.7), (3.8). In addition $w(t,x) \in C^{2+\alpha}$ on G.

If we define the function $v(t,x)$ on region G by the inverse mapping

$$v(t,x) = G(w(t,x))$$

which was defined at the beginning of this proof, where $w(t,x)$ is the solution of problem (3.7), (3.8) found above, then the function $v(t,x)$ is easily shown to be the unique solution of Cauchy problem (3.6) on region G, and also $v(t,x) \in C^{2+\alpha}$ on G. For using the basic relationship

$$w(t,x) = \int_{m_0/2}^{v(t,x)} A_1(s)ds,$$

we have

$$\frac{\partial v}{\partial x_k} = \frac{1}{A_1(v)} \frac{\partial w}{\partial x_k}$$

and

$$A_1(v)\frac{\partial v}{\partial t} = \frac{\partial w}{\partial t},$$

so that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}[c_{ij}(t,x)\frac{\partial v}{\partial x_j}] + \sum_{j=1}^{n} d_j(t,x)\frac{\partial v}{\partial x_j}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}[C_{ij}(t,x,w)\frac{\partial w}{\partial x_j}] + \sum_{j=1}^{n} D_j(t,x,w)\frac{\partial w}{\partial x_j} = \frac{\partial w}{\partial t} = A_1(v)\frac{\partial v}{\partial t},$$

thereby showing that the function $v(t,x)$ satisfies the differential equation for problem (3.6). Since also

$$v(0,x) = G(w(0,x)) = G(\psi_1(x)) = \psi(x)$$

by virtue of the one-to-oneness of the mapping, the initial condition of problem (3.6) is satisfied.

To show that function $v(t,x) \in C^{2+\alpha}$ it again suffices to show that function v and its derivatives $\partial v/\partial x_j$, $\partial^2 v/\partial x_j \partial x_k$, and $\partial v/\partial t$ all are bounded and Hölder continuous in variables t and x.

First, since the function

$$w = F(v) = \int_{m_0/2}^{v} A_1(s)ds$$

is strictly increasing and continuous in the variable v, then so must the inverse function

$$v = G(w)$$

be also strictly increasing and continuous in the variable w, as is known from elementary calculus. But from these facts we conclude that

$$\begin{aligned} v(t,x) &= G(w(t,x)) \le G(|w(t,x)|) \\ &\le G(\sup_{t,x}|w(t,x)|) = G(|w|_0) < +\infty, \end{aligned}$$

and also

$$v(t,x) = G(w(t,x)) \geq G(-|w(t,x)|)$$

$$\geq G(-\sup_{t,x}|w(t,x)|) \geq G(-|w|_0) > -\infty,$$

since $|w|_0 < +\infty$.

Next, since

$$\frac{\partial v}{\partial x_k} = \frac{1}{A_1(v)} \frac{\partial w}{\partial x_k},$$

we conclude that $\partial v/\partial x_k$ is bounded since the right-hand side is a product of bounded functions.

Further, we have

$$\frac{\partial^2 v}{\partial x_j \partial x_k} = -\frac{A_1'(v)}{A_1(v)^3} \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_k} + \frac{1}{A_1(v)} \frac{\partial^2 w}{\partial x_j \partial x_k},$$

and must be bounded for the same reasons as immediately above.

Finally,

$$\frac{\partial v}{\partial t} = \frac{1}{A_1(v)} \frac{\partial w}{\partial t},$$

which is again easily seen to be bounded.

Now we turn to the proof of Hölder continuity of the function $v(t,x)$, together with its derivatives $\partial v/\partial x_k$, $\partial^2 v/\partial x_j \partial x_k$, and $\partial v/\partial t$.

First, we have

$$|v(t',x')-v(t'',x'')|\,/d(P_1,P_2)^{a}$$

$$= |G(w(t',x'))-G(w(t'',x''))|\,/d(P_1,P_2)^{a}$$

$$= |G'(\overline{w})|\,|w(t',x')-w(t'',x'')|\,/d(P_1,P_2)^{a}$$

(by the mean-value theorem)

$$= (1/|A_1(\overline{v})|)\,|w(t',x'')-w(t'',x'')|\,/d(P_1,P_2)^{a}$$

$$\leq (2/\underline{A})\cdot H(w) < +\infty,$$

thereby showing that the function $v(t,x)$ is Hölder continuous.

Next, since

$$\frac{\partial v}{\partial x_k} = \frac{1}{A_1(v)}\,\frac{\partial w}{\partial x_k}\,,$$

$$\frac{\partial^2 v}{\partial x_j \partial x_k} = -\frac{A_1'(v)}{A_1(v)^3}\,\frac{\partial w}{\partial x_j}\,\frac{\partial w}{\partial x_k} + \frac{1}{A_1(v)}\,\frac{\partial^2 w}{\partial x_j \partial x_k}\,,$$

and

$$\frac{\partial v}{\partial t} = \frac{1}{A_1(v)}\,\frac{\partial w}{\partial t}$$

are all rational combinations of Hölder continuous functions, we conclude that these derivatives of $v(t,x)$ are $\in C^{a}$, and hence $v \in C^{2+a}$ on G.

We now wish to show that the function $v(t,x)$ found above also satisfies the inequality (3.5). Since the function $W = v(t,x)$ satisfies the linear differential equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_i}\left[c_{ij}(t,x)\frac{\partial W}{\partial x_j}\right]+\sum_{j=1}^{n}d_j(t,x)\frac{\partial W}{\partial x_j}$$

$$= A_1(v(t,x))\partial W/\partial t$$

on G, with initial condition

$$W(0,x) = \psi(x),$$

where

$$0 < m_0 \leq \psi(x) \leq M_0 ,$$

we may apply the maximum (and minimum) principle for parabolic partial differential equations (see Friedman [6]) to conclude that $m_0 \leq v(t,x) \leq M_0$ throughout region G.

Since $A_1(v) \equiv A(v)$ if $m_0 \leq v \leq M_0$, we conclude that $v(t,x)$ satisfies the Cauchy problem (3.3), (3.4) on region G.

Remarks. 1. This method of proof is a generalization of one originating with Oleinik, Kalashnikov, and Chzou Yui-Lin' [10].

2. In the construction of function $A_1(v)$ it is not really necessary to require that $\underline{A}/2 \leq A_1(v) \leq 2\overline{A}$, and that $A_1(v) = (\underline{A}+\overline{A})/2$ off the closed interval $[m_0/2, 2M_0]$. If we instead merely require that $A_1(v)$ be positive, bounded, and $A_1(v) \equiv$ any constant $C > 0$ off $[m_0/2, 2M_0]$, it is sufficient.

We may now proceed to the proof of the existence and uniqueness theorem for the Cauchy problem (3.1), (3.2).

Proof of Theorem 2. First, we make the change of dependent variable u to dependent variable v by means of the one-to-one mapping:

$$v = \varphi(u), \quad u = \Phi(v) \tag{3.9}$$

which transforms Equation (3.1) into the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(t,x)\frac{\partial v}{\partial x_j}\right] + \sum_{j=1}^{n} b_j(t,x)\frac{\partial v}{\partial x_j} = \Phi'(v)\frac{\partial v}{\partial t} \tag{3.10}$$

where function $\Phi'(v)$ is positive and bounded for $0 < \mu \leq v \leq M_2$, where

$$M_2 = \varphi(M_1) ,$$

and where μ is any real number such that $0 < \mu \leq M_1$.

We consider Equation (3.10) on the region G, together with the initial condition

$$v(0,x) = v_0(x), \quad x \in E_n \tag{3.11}$$

where

$$v_0(x) = \varphi(u_0(x)) .$$

By virtue of assumption i), we have that

$$0 < \varphi(m) \leq v_0(x) \leq \varphi(M) \leq M_2 .$$

As in the proof of the lemma, we may define a function $A(v)$, for $-\infty < v < +\infty$ by the following equation

$$A(v) = \Phi'(v) \quad \text{for} \quad \varphi(m) \leq v \leq \varphi(M),$$

otherwise $A(v)$ is a bounded, positive-valued function, bounded away from zero. In addition, $A(v)$ has bounded continuous derivatives up to the second order inclusive, satisfying a Lipschitz condition in variable v. In place of Cauchy problem (3.10), (3.11), we consider equation

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i}[a_{ij}(t,x) \frac{\partial v}{\partial x_j}] + \sum_{j=1}^{n} b_j(t,x) \frac{\partial v}{\partial x_j} = A(v) \frac{\partial v}{\partial t} \tag{3.12}$$

together with the initial condition (3.11). We shall now show that all conditions of the lemma are satisfied by Cauchy problem (3.12), (3.11). Since function $A(v)$ is positive-valued and bounded away from zero, it is clear that Condition i) of the lemma is satisfied. From the definition of function $A(v)$ it is clear that Condition ii) is satisfied. From the hypotheses of Theorem 2 it is clear that Conditions iii) and iv) are satisfied by the coefficient functions $a_{ij}(t,x)$ and $b_j(t,x)$.

It now remains to show that Condition v) is satisfied by the initial function $v_0(x)$. We have that

$$v_0(x) = \varphi(u_0(x)),$$

$$\frac{\partial v_0}{\partial x_j} = \varphi'(u_0(x)) \frac{\partial u_0}{\partial x_j},$$

$$\frac{\partial^2 v_0}{\partial x_j \partial x_k} = \varphi''(u_0(x)) \frac{\partial u_0}{\partial x_j} \frac{\partial u_0}{\partial x_k} + \varphi'(u_0(x)) \frac{\partial^2 u_0}{\partial x_j \partial x_k},$$

so that we may immediately conclude that $v_0 \in C^{2+\alpha}$, since v_0, $\partial v_0/\partial x_j$, and $\partial^2 v_0/\partial x_j \partial x_k$ are combinations of functions which are both bounded and Holder continuous, and we use Propositions 2, 3, 4, and 6, Appendix A.

By the lemma, the Cauchy problem (3.12), (3.11) has a unique solution $v(t,x)$ with $v(t,x) \in C^{2+\alpha}$ on region G, and also the inequality

$$\varphi(m) \leq v(t,x) \leq \varphi(M) \tag{3.13}$$

is satisfied throughout region G, and hence

$$A(v(t,x)) \equiv \Phi'(v(t,x)), \quad (t,x) \in G.$$

If we define the function $u(t,x)$ by the relation

$$u(t,x) = \Phi(v(t,x)),$$

we claim that function $u(t,x)$ is the desired classical solution of Cauchy problem (3.1), (3.2). For we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_i}\left[a_{ij}(t,x)\frac{\partial\varphi(u(t,x))}{\partial x_j}\right]+\sum_{j=1}^{n}b_j(t,x)\frac{\partial\varphi(u(t,x))}{\partial x_j}$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_i}\left[a_{ij}(t,x)\frac{\partial v(t,x)}{\partial x_j}\right]+\sum_{j=1}^{n}b_j(t,x)\frac{\partial v(t,x)}{\partial x_j}$$

$$=\Phi'(v(t,x))\frac{\partial v(t,x)}{\partial t}$$

$$=\frac{\partial}{\partial t}\left[\Phi(v(t,x))\right]$$

$$=\frac{\partial u(t,x)}{\partial t},$$

showing that the function $u(t,x)$ satisfies Equation (3.1).

Next, we have

$$u(0,x)=\Phi(v(0,x))=\Phi(v_0(x))=u_0(x),$$

so that the initial condition (3.2) is satisfied.

Further, we have from inequality (3.13)

$$0<m=\Phi(\varphi(m))\le\Phi(v(t,x))\le\Phi(\varphi(M))=M,$$

so that

$$m\le u(t,x)\le M,\quad (t,x)\in G,$$

must hold, since $u(t,x) = \Phi(v(t,x))$ by definition.

Next, we show that function $u(t,x) \in C^{2+\alpha}$ on G. Since $u(t,x) = \Phi(v(t,x))$, and function $\Phi(v)$ is Lipschitz continuous for $\varphi(m) \leq v \leq \varphi(M)$ by Proposition 7 remark, Appendix A, we may apply Proposition 6, Appendix A, to conclude that $u(t,x)$ is Hölder continuous on G. Since we have that

$$\frac{\partial u(t,x)}{\partial x_j} = \frac{1}{\varphi'(u(t,x))} \frac{\partial v(t,x)}{\partial x_j} ,$$

and

$$\frac{\partial^2 u(t,x)}{\partial x_j \partial x_k} = - \frac{\varphi''(u(t,x))}{[\varphi'(u(t,x))]^3} \frac{\partial v(t,x)}{\partial x_j} \frac{\partial v(t,x)}{\partial x_k}$$

$$+ \frac{1}{\varphi'(u(t,x))} \frac{\partial^2 v(t,x)}{\partial x_j \partial x_k} ,$$

together with

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{\varphi'(u(t,x))} \frac{\partial v(t,x)}{\partial t} ,$$

we conclude that all of these derivatives are Hölder continuous by virtue of Propositions 2, 3, 4, and 6, Appendix A, since they are all combinations of Lipschitz and Hölder continuous functions. Since $m \leq u(t,x) \leq M$ on G, the boundedness of the derivatives of function $u(t,x)$ is immediately evident by the boundedness of the functions comprising them. Therefore we have that $u(t,x) \in C^{2+\alpha}$ on G.

Finally, we shall show the uniqueness of the solution $u(t,x)$. If function $u_1(t,x)$ is a second solution of Cauchy problem (3.1), (3.2) with $u_1(t,x) \in C^{2+\alpha}$, then, defining the function $v_1(t,x)$ by the equation

$$v_1(t,x) = \varphi(u_1(t,x)),$$

it is easy to show by methods similar to those used in the proof above that $v_1(t,x) \in C^{2+\alpha}$, and satisfies Cauchy problem (3.10), (3.11). But then $v_1(t,x) \equiv v(t,x)$ by the uniqueness property of the lemma. Hence we conclude that

$$u(t,x) = \Phi(v(t,x)) = \Phi(v_1(t,x)) = u_1(t,x)$$

for all $(t,x) \in G$, and thereby proving uniqueness.

C. Concluding Remarks

We conclude this section by observing that the method of proof used in Theorem 2 may be extended to non-linear parabolic partial differential equations of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} [a_{ij}(t,x,u) \frac{\partial \varphi(x,u)}{\partial x_j}] + \sum_{j=1}^{n} b_j(t,x,u) \frac{\partial \varphi(x,u)}{\partial x_j} = \frac{\partial u}{\partial t},$$

but to do so requires that the method of proof used in the lemma be extended to quasilinear parabolic partial differential equations having the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left[c_{ij}(t,x,v)\frac{\partial v}{\partial x_j}\right] + \sum_{j=1}^{n} d_j(t,x,v)\frac{\partial v}{\partial x_j} = A(x,v)\frac{\partial v}{\partial t} .$$

Proceeding as in the proof of the lemma, we introduce the change of dependent variable

$$w = \int_{m_0/2}^{v} A_1(x,s)ds$$

into the above differential equation and obtain the differential equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left[C_{ij}(t,x,w)\frac{\partial w}{\partial x_j}\right] + \sum_{j=1}^{n} D_j(t,x,w)\frac{\partial w}{\partial x_j} + E(t,x,w) = \frac{\partial w}{\partial t} ,$$

where we define

$$C_{ij}(t,x,w) = c_{ij}(t,x,v)/A_1(x,v),$$

$$D_i(t,x,w) = d_i(t,x,v)/A_1(x,v) - \sum_{j=1}^{n} c_{ij}(A_1)_{x_j}/A_1^2$$

$$+ \sum_{j=1}^{n}\left[\frac{c_{ij}(A_1)_v}{A_1^3} - \frac{(c_{ij})_v}{A_1^2}\right]\int_{m_0/2}^{v}(A_1)_{x_j}(x,s)ds,$$

and

$$E(t,x,w) = -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{c_{ij}}{A_1}\int_{m_0/2}^{v}(A_1)_{x_i x_j}(x,s)ds$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{(c_{ij})_v}{A_1^2}-\frac{c_{ij}(A_1)_v}{A_1^3}\right]\int_{m_0/2}^{v}(A_1)_{x_i}(x,s)ds$$

$$\times\int_{m_0/2}^{v}(A_1)_{x_j}(x,s)ds$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{2c_{ij}(A_1)_{x_i}}{A_1^2}-\frac{(c_{ij})_{x_i}}{A_1}\right]\int_{m_0/2}^{v}(A_1)_{x_j}(x,s)ds$$

$$-\sum_{j=1}^{n}\frac{d_j}{A_1}\int_{m_0/2}^{v}(A_1)_{x_j}(x,s)ds\ .$$

Needless to say, showing that $C_{ij} \in C_{2+\alpha}$, and D_j and $E \in C_{1+\alpha}$ requires further tedious computations, and is not particularly enlightening.

IV. A FINITE DIFFERENCE APPROXIMATION SCHEME FOR NON-STEADY GAS FLOW

In this chapter we present a finite difference approximation scheme for numerical solution of the Cauchy problem for non-steady gas flow. This problem has partial differential equation

$$\sum_{j=1}^{n} \frac{\partial^2 u^m}{\partial x_j^2} = \frac{\partial u}{\partial t}, \quad m \geq 2, \tag{4.1}$$

and initial condition

$$u(0,x) = u_0(x), \tag{4.2}$$

where $0 \leq u_0(x) \leq M_0$.

We shall prove both a maximum principle as well as non-negativeness of solutions for the scheme, then show that the scheme is consistent with the Cauchy problem (4.1), (4.2).

Let e_j be that vector in E_n all of whose components vanish except for the jth component which is unity, with $j = 1, 2, \ldots, n$. We let $U(t,x)$ represent the numerical approximation of the solution $u(t,x)$ of Cauchy problem (4.1), (4.2). If in the differential Equation (4.1) we replace

$$\frac{\partial u}{\partial t} \quad \cancel{\frac{\partial u}{\partial x}} \quad \text{by} \quad \frac{U(t+k,x)-U(t,x)}{k},$$

and

$$\frac{\partial^2 u^m}{\partial x_j^2} \quad \text{by} \quad \frac{U^m(t, x+h_j e_j) - 2U^m(t, x) + U^m(t, x-h_j e_j)}{h_j^2},$$

where k and h_j are positive real numbers, we obtain the finite difference equation:

$$\frac{U(t+k, x) - U(t, x)}{k} = \sum_{j=1}^{n} \frac{U^m(t, x+h_j e_j) - 2U^m(t, x) + U^m(t, x-h_j e_j)}{h_j^2}. \tag{4.3}$$

If we write the difference Equation (4.3) in the form:

$$U(t+k, x) = \sum_{j=1}^{n} \lambda_j [U^m(t, x+h_j e_j) + U^m(t, x-h_j e_j)]$$

$$+ \left[1 - 2\left(\sum_{j=1}^{n} \lambda_j\right) U^{m-1}(t, x)\right] U(t, x), \tag{4.4}$$

where $\lambda_j = k/h_j^2$ (when $h_1 = h_2 = \ldots = h_n = h$, then $\lambda = k/h^2$ is called the grid, or net, or mesh ratio), we see that the numerical solution U is completely and uniquely determined on the $(t+k)$-th time level provided that it is known at all grid points x of the t-th time level.

We shall first prove a maximum principle for the approximation scheme having finite difference Equation (4.4) and initial condition

$$U(0, x) = u_0(x) \tag{4.5}$$

at all grid points x.

A. The Maximum Principle

Theorem 3. (The maximum principle for the numerical approximation scheme.)

Let $U(t, x)$ be the solution of the finite difference problem (4.4), (4.5), and suppose that the following inequalities hold:

$$0 < \lambda_j \leq 1/(2nm M_0^{m-1}), \quad j = 1, 2, \ldots, n,$$

where M_0 is any upper bound for the initial function $u_0(x)$. Then

$$U(t, x) \leq M_0$$

at all grid points (t, x).

Proof. We are given that

$$U(0, x) = u_0(x) \leq M_0.$$

It is sufficient to prove that $U(k, x) \leq M_0$. We first observe that if

$$0 < \lambda_j \leq 1/(2nm M_0^{m-1}),$$

then

$$1 - 2\left(\sum_{j=1}^{n} \lambda_j\right) U^{m-1}(0,x) \geq 1 - 2(1/2m\ M_0^{m-1})M_0^{m-1}$$

$$\geq 1 - 1/m$$

$$\geq 0 ,$$

if $$m \geq 1$$

(even though $m \geq 2$ is required in the gas-flow equation).

Hence if the λ_j satisfy the boundedness conditions of the hypothesis, we have that the coefficient of $U(t,x)$ in Equation (4.4) must be non-negative at time level $t = 0$. Using Equation (4.4) with $t = 0$, we have

$$U(k,x) = \sum_{j=1}^{n} \lambda_j [U^m(0, x+h_j e_j) + U^m(0, x-h_j e_j)]$$

$$+ \left[1 - 2\left(\sum_{j=1}^{n} \lambda_j\right) U^{m-1}(0,x)\right] U(0,x)$$

$$\leq \left(\sum_{j=1}^{n} \lambda_j\right) 2M_0^m + \left[1 - 2\left(\sum_{j=1}^{n} \lambda_j\right) U^{m-1}(0,x)\right] U(0,x)$$

$$= 2n\bar{\lambda} M_0^m + [1 - 2n\bar{\lambda} U^{m-1}(0,x)] U(0,x)$$

(where $\bar{\lambda} = \sum_{j=1}^{n} \lambda_j / n$, and $\bar{\lambda} \leq 1/2nm\ M_0^{m-1}$),

$$= U(0,x) + 2n\bar{\lambda}[M_0^m - U^m(0,x)]$$

$$\leq U(0,x) + 2n[M_0^m - U^m(0,x)]/2nm\, M_0^{m-1}$$

$$= U(0,x) + [M_0^m - U^m(0,x)]/m\, M_0^{m-1} .$$

Let us now consider the auxiliary function

$$f(z) = z + (M_0^m - z^m)/m\, M_0^{m-1}$$

on the interval $0 \leq z$. We have immediately that

$$f'(z) = 1 - z^{m-1}/M_0^{m-1} ,$$

so that function $f(z)$ has an absolute maximum value of M_0 for all $z \geq 0$, and which occurs at the value $z = M_0$. Noting that by condition (4.2), we have

$$0 \leq u_0(x) = U(0,x) \leq M_0 ,$$

and taking $z = U(0,x)$, we conclude that

$$U(0,x) + [M_0^m - U^m(0,x)]/m\, M_0^{m-1} \leq M_0$$

at all grid points x, and therefore

$$U(k, x) \leq M_0 \quad .$$

for all such values of x. By iteration on k we may extend the boundedness property to all time levels.

We next prove the non-negativeness of solutions of the above scheme, assuming non-negativeness of the initial function $u_0(x)$.

B. Non-Negativeness of Solutions

Theorem 4. (Non-negativeness of solutions of the approximation scheme.)

Suppose that the quantities λ_j satisfy the inequality conditions of Theorem 3, and that the inequality

$$0 \leq u_0(x), \qquad x \in E_n \, ,$$

holds. Then

$$0 \leq U(t, x)$$

holds at all grid points (t, x).

Proof. Again, it suffices to prove non-negativeness only for $U(k, x)$. By taking $t = 0$ in Equation (4.4), we have

$$U(k,x) = \sum_{j=1}^{n} \lambda_j [U^m(0,x+h_j e_j) + U^m(0,x-h_j e_j)]$$

$$+ \left[1-2\left(\sum_{j=1}^{n}\lambda_j\right)U^{m-1}(0,x)\right]U(0,x).$$

Since the coefficient of $U(0,x)$ is non-negative by the calculation done in the proof of Theorem 3, and $U(0,x) \geq 0$ at all grid points x, the right-hand side of the above equation is also non-negative, proving that $U(k,x)$ is non-negative. Iteration on k extends the result to all time levels.

We conclude this chapter by proving that the approximation scheme (4.4), (4.5) is consistent with the Cauchy problem (4.1), (4.2). By consistency of the scheme we mean that the scheme is a formal approximation to the differential Equation (4.1) at every point $(t,x) \in G$, in the sense that for function u sufficiently smooth the equation

$$\left[\sum_{j=1}^{n} \frac{\partial^2 u^m(t,x)}{\partial x_j^2} - \frac{\partial u(t,x)}{\partial t}\right]$$

$$-\left[\sum_{j=1}^{n} \frac{u^m(t,x+h_j e_j) - 2u^m(t,x) + u^m(t,x-h_j e_j)}{h_j^2} - \frac{u(t+k,x)-u(t,x)}{k}\right]$$

$$= O(k + \sup_j h_j)$$

holds at every grid point (t, x), where $u(t, x)$ is any function having bounded continuous second order partial derivatives with respect to the variable t and bounded continuous third order partial derivatives with respect to the vector variable x.

C. Consistency

Theorem 5. (Consistency of the finite difference approximation scheme.)

The finite difference approximation scheme (4.4), (4.5) is consistent with the Cauchy problem (4.1), (4.2).

Proof. By Taylor's theorem we have that

$$\frac{u(t+k, x)-u(t, x)}{k} = \frac{1}{k}\left[\frac{\partial u(t, x)}{\partial t} k + \frac{\partial^2 u(\bar{t}, x)}{\partial t^2} \frac{k^2}{2}\right]$$

$$= \frac{\partial u(t, x)}{\partial t} + O(k), \tag{4.6}$$

since $\partial^2 u/\partial t^2$ is everywhere bounded and continuous. Again, we have:

$$\sum_{j=1}^{n} \frac{u^m(t, x+h_j e_j) - 2u^m(t, x) + u^m(t, x-h_j e_j)}{h_j^2}$$

$$= \sum_{j=1}^{n}\left[\frac{u^m(t, x+h_j e_j)-u^m(t, x)}{h_j^2} + \frac{u^m(t, x-h_j e_j)-u^m(t, x)}{h_j^2}\right] =$$

$$= \sum_{j=1}^{n} \frac{1}{h_j^2}\Big[\frac{\partial u^m(t,x)}{\partial x_j} h_j + \frac{\partial^2 u^m(t,x)}{\partial x_j^2}\frac{h_j^2}{2} + \frac{\partial^3 u^m(t,\bar{x}^{(j)})}{\partial x_j^3}\frac{h_j^3}{6}$$

$$- \frac{\partial u^m(t,x)}{\partial x_j} h_j + \frac{\partial^2 u^m(t,x)}{\partial x_j^2}\frac{h_j^2}{2} - \frac{\partial^3 u^m(t,\tilde{x}^{(j)})}{\partial x_j^3}\frac{h_j^3}{6}\Big]$$

$$= \sum_{j=1}^{n}\Big[\frac{\partial^2 u^m(t,x)}{\partial x_j^2} + O(h_j)\Big]$$

$$= \sum_{j=1}^{n}\frac{\partial^2 u^m(t,x)}{\partial x_j^2} + O(\sup_j h_j) . \tag{4.7}$$

From Equations (4.6) and (4.7) we immediately obtain

$$\Big[\sum_{j=1}^{n}\frac{\partial^2 u^m}{\partial x_j^2} - \frac{\partial u}{\partial t}\Big] - \Big[\sum_{j=1}^{n}[u^m(t,x+h_je_j)-2u^m(t,x)+u^m(t,x-h_je_j)]/h_j^2$$

$$- \frac{u(t+k,x)-u(t,x)}{k}\Big]$$

$$= \Big[\sum_{j=1}^{n}\frac{\partial^2 u^m}{\partial x_j^2} - \frac{\partial u}{\partial t}\Big] - \Big[\sum_{j=1}^{n}\frac{\partial^2 u^m}{\partial x_j^2} - \frac{\partial u}{\partial t} + O(\sup_j h_j) - O(k)\Big]$$

$$= O(k + \sup_j h_j) , \tag{4.8}$$

thereby proving that the approximation scheme is consistent.

D. Concluding Remarks

1. In the hypothesis of Theorems 3 and 4 we may replace the condition

$$0 < \lambda_j \leq 1/2nm M_0^{m-1}$$

by the condition

$$0 < \bar{\lambda} \leq 1/2nm M_0^{m-1} ,$$

where

$$\bar{\lambda} = \sum_{j=1}^{n} \lambda_j / n .$$

2. In Theorem 3, if $m = 1$, we have another proof of the maximum principle for the finite difference approximation scheme for the classical heat equation.

3. In Theorem 5, if function $u(t,x)$ has bounded continuous fourth order partial derivatives with respect to x, then $O(k + \sup_j h_j)$ in the conclusion can be replaced by $O(k + \sup_j h_j^2)$.

V. A NUMERICAL EXAMPLE OF THE APPLICATION OF THE FINITE DIFFERENCE APPROXIMATION SCHEME

In this chapter we present an example of use of the approximation scheme for the numerical solution of a one-dimensional isothermal gas flow problem. We seek to find a numerical solution of the non-steady gas flow problem having differential equation

$$\frac{\partial^2 u^2}{\partial x^2} = \frac{\partial u}{\partial t},$$

and initial condition

$$u(0,x) = u_0(x), \quad -\infty < x < +\infty,$$

where

$$u_0(x) = \begin{cases} 0, & \text{for } x < -0.6 \\ 1-100(x+0.3)^2/9, & \text{for } -0.6 \leq x \leq 0 \\ 0, & \text{for } 0 \leq x \leq 1 \\ 2-8(x-1.5)^2, & \text{for } 1 \leq x \leq 2 \\ 0, & \text{for } 2 \leq x. \end{cases}$$

To obtain the numerical solution of the above Cauchy problem we shall use the finite difference approximation with difference equation

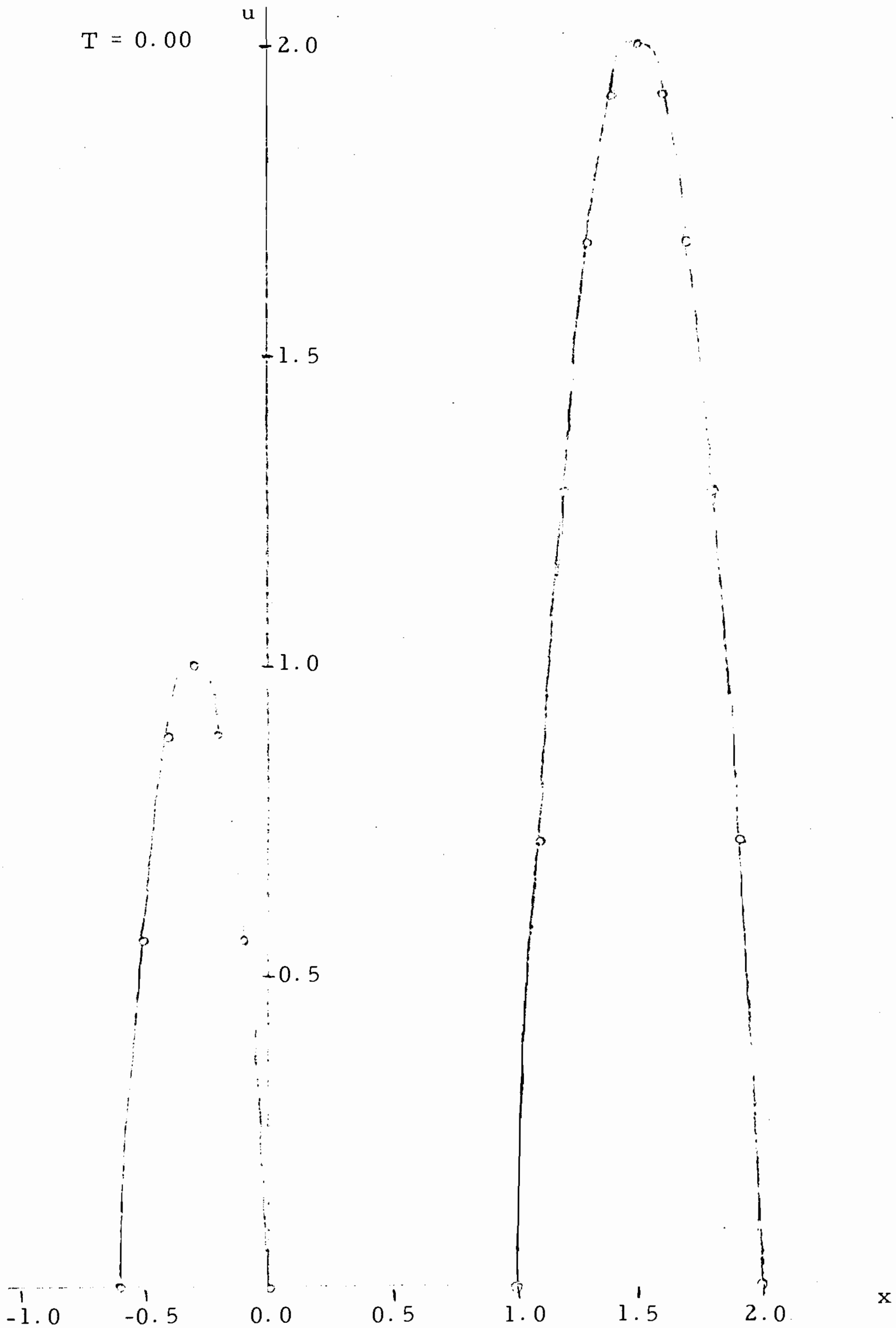
$$U(t+k, x) = \lambda[U^2(t, x+h)+U^2(t, x-h)] + [1-2\lambda U(t, x)]U(t, x),$$

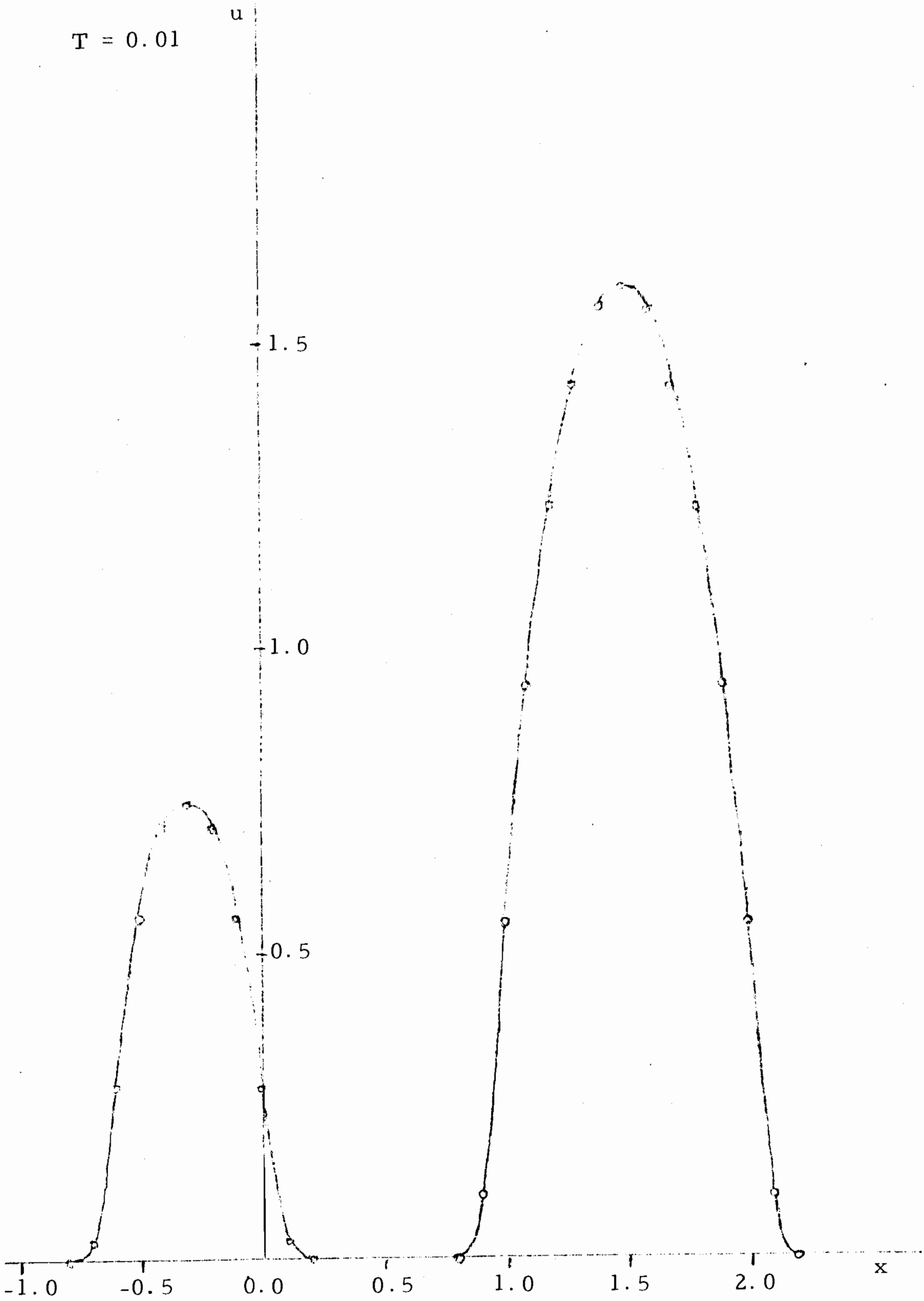
where $\lambda = k/h^2$, and with the initial condition

$$U(0, x) = u_0(x),$$

where $u_0(x)$ is the above defined function. In this approximation scheme we have taken $h = 0.1$ for the spacing on the x-axis. The maximum value then allowed for the grid ratio λ is $\lambda = 1/8$, but we have taken $\lambda = 1/10$ so as to avoid borderline complications, and in order to simplify the computations. We have then taken time steps of size $k = 0.001$. Using these values of λ, h, and k we have computed 81 tables of gas pressure U, doing all computations on the Monroe model number 1655 programmable calculator with nine decimal places carried throughout. We then selected the table of initial values, $t = 0.00$, together with the tables for $t = 0.01, 0.02$, etc., up to $t = 0.08$, and rounded these tables off to the second place, then using these nine tables (see Appendix C) we constructed the nine graphs which we hereby present. The graphs indicate that for positive values of time the pressure curves appear to be smooth up through and including the time when the gases merge at time $t = 0.08$, the behavior of the gases appearing to be almost gelatinous, with the gas pressure most evidently diminishing with respect to increasing time.

We have computed the area under the curves by using Simpson's rule for various times, with initial area under the two parabolas being 2/5 and 4/3 exactly, respectively. Examination has shown that these values for the areas are approximately maintained as time increases up to the last time level inclusive.


T = 0.00
u
2.0
1.5
1.0
0.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
x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u
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0.5
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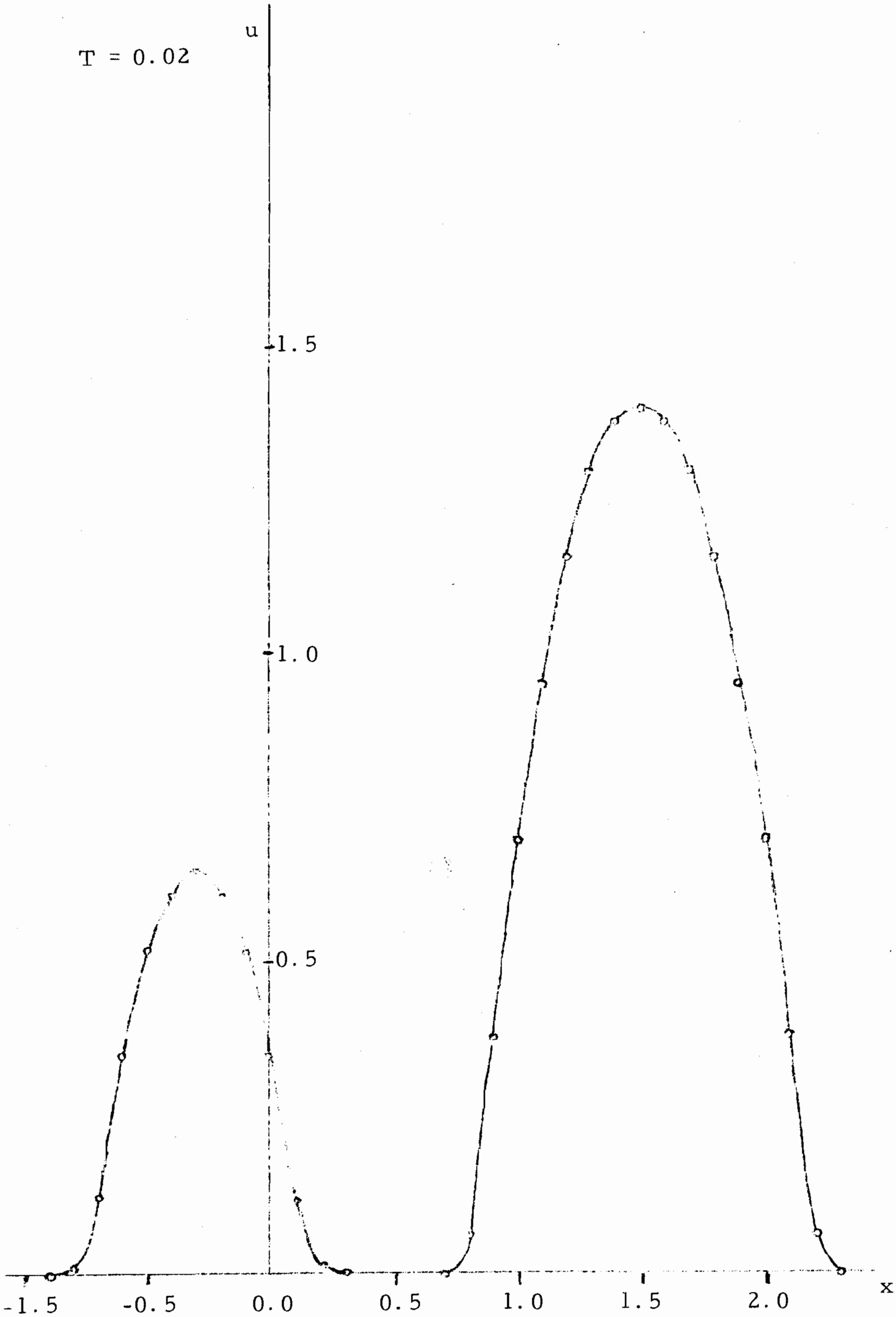
-0.5
0.0
0.5
1.0
1.5
2.0
x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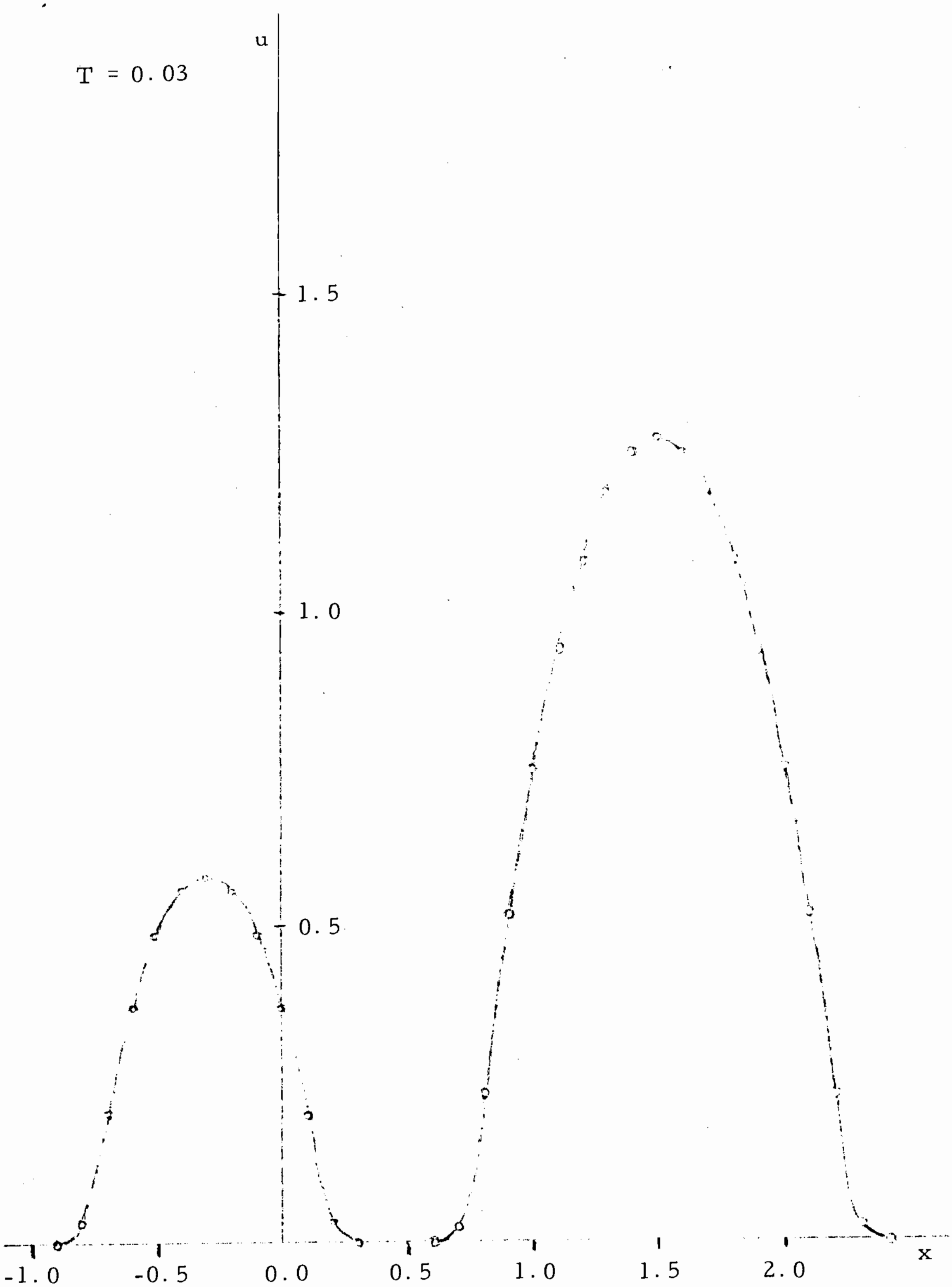
1.0
1.5
2.0
x


u
T = 0.03
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1.0
0.5
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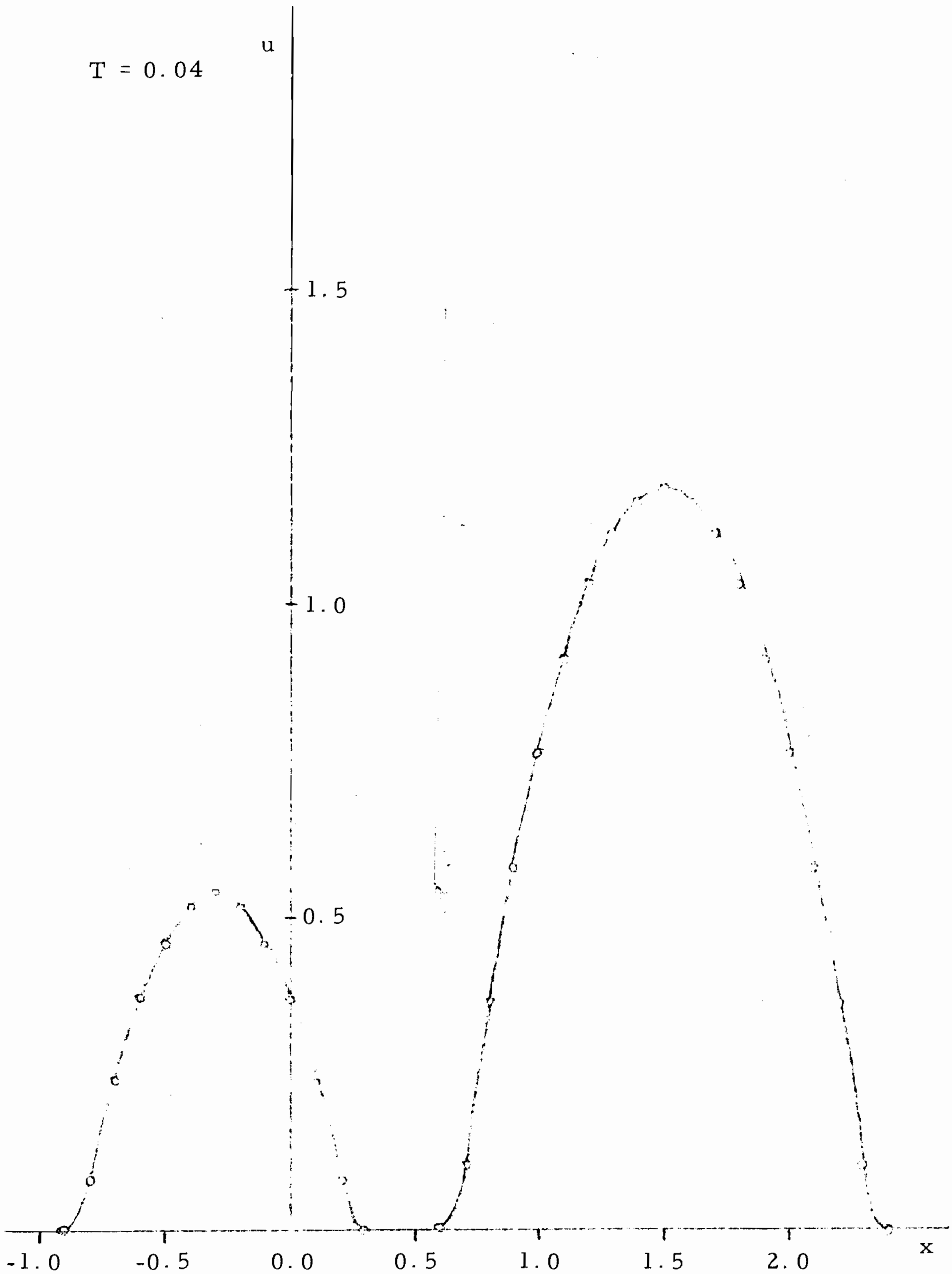
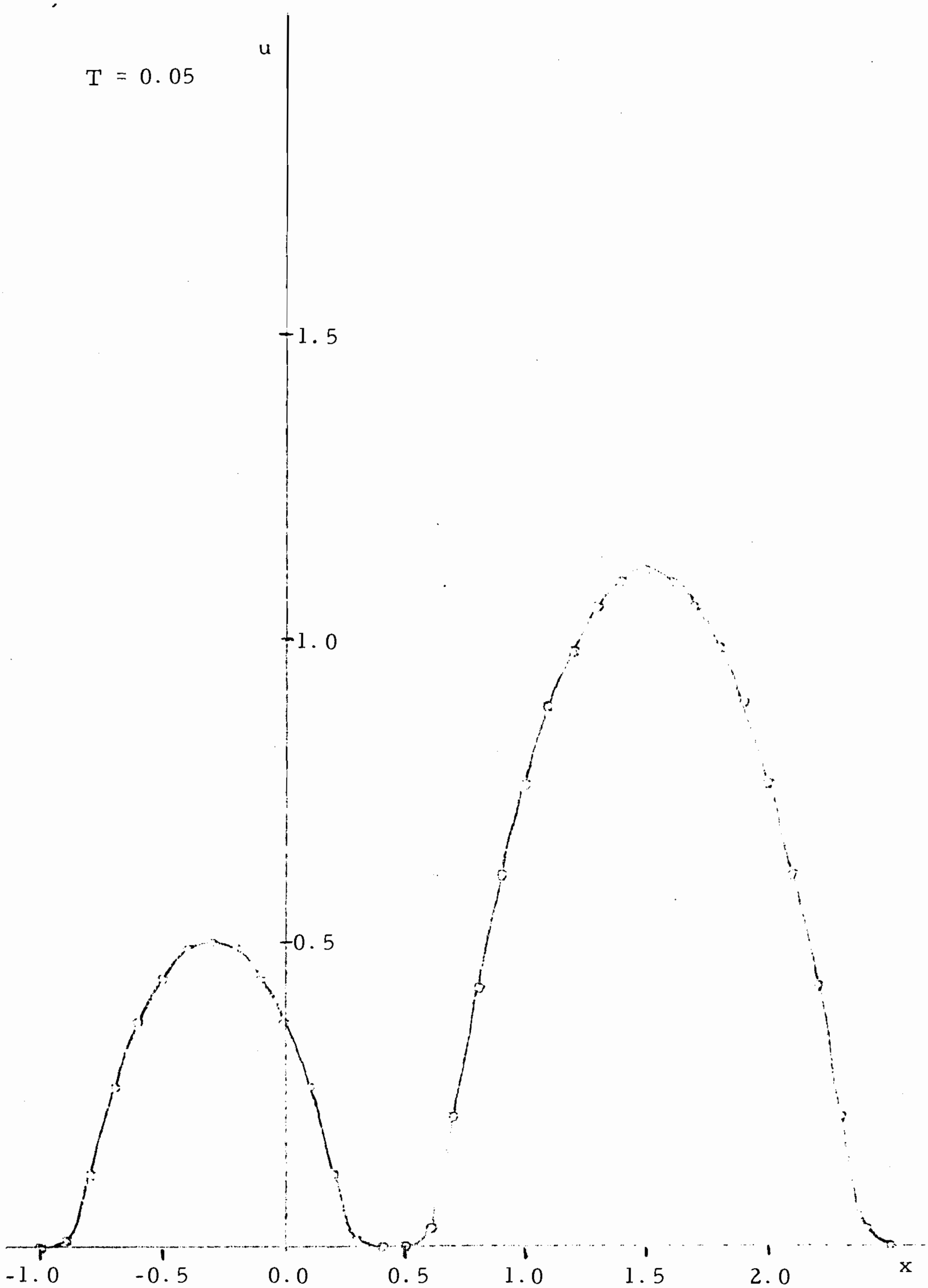
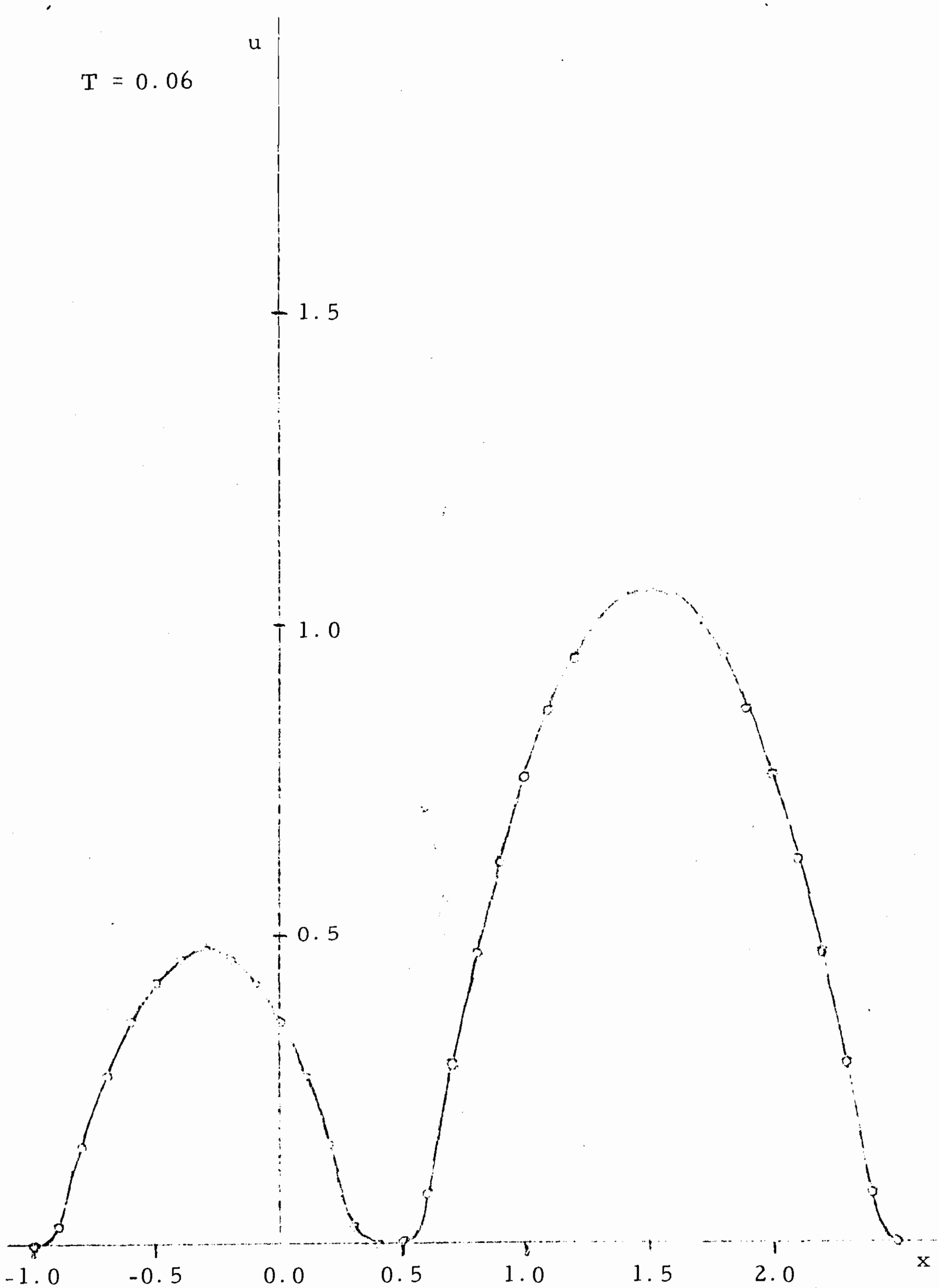
-0.5
0.0
0.5
1.0
1.5
2.0
x


T = 0.04
u
1.5
1.0
0.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
x


T = 0.05
u
1.5
1.0
0.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
x


u
T = 0.06
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1.0
0.5
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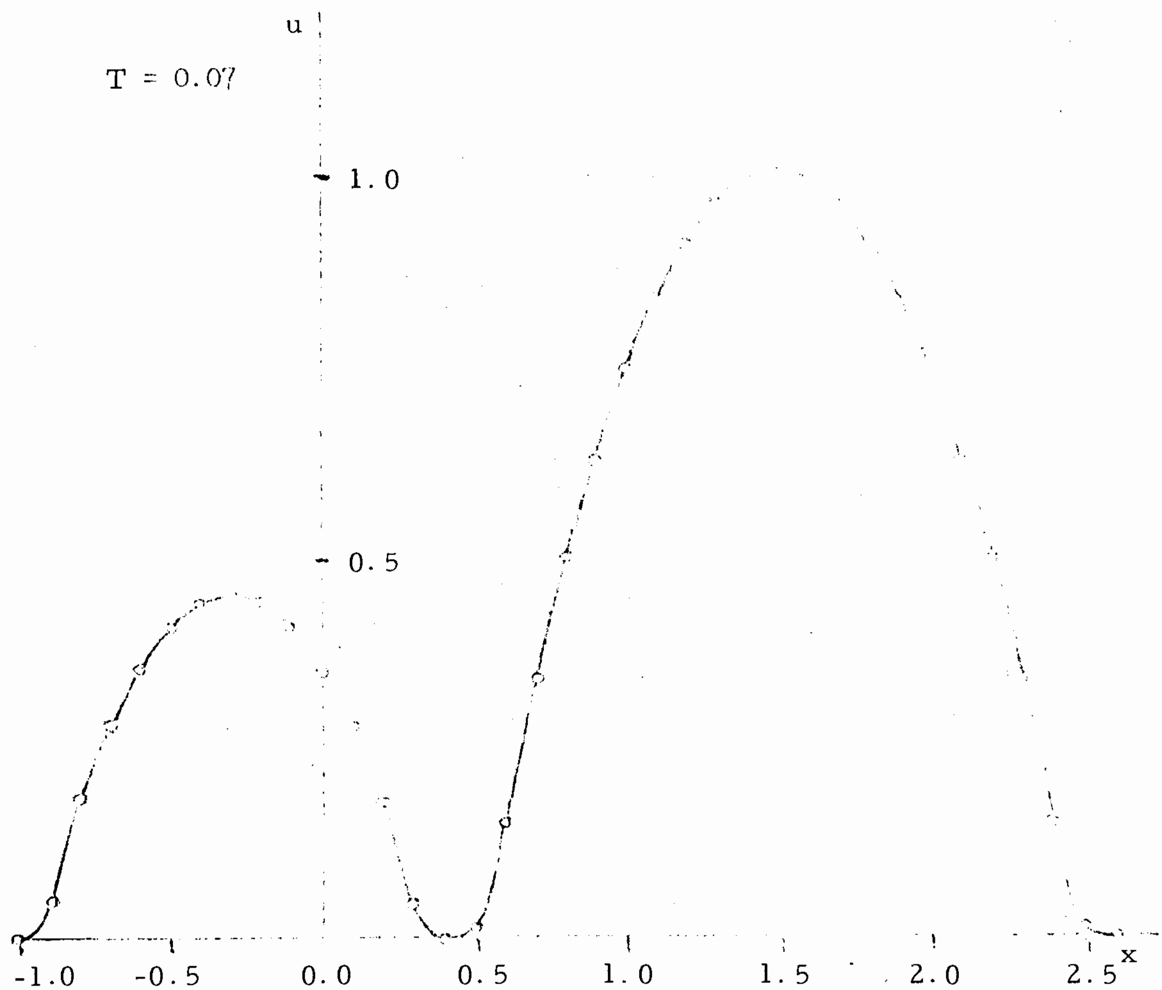
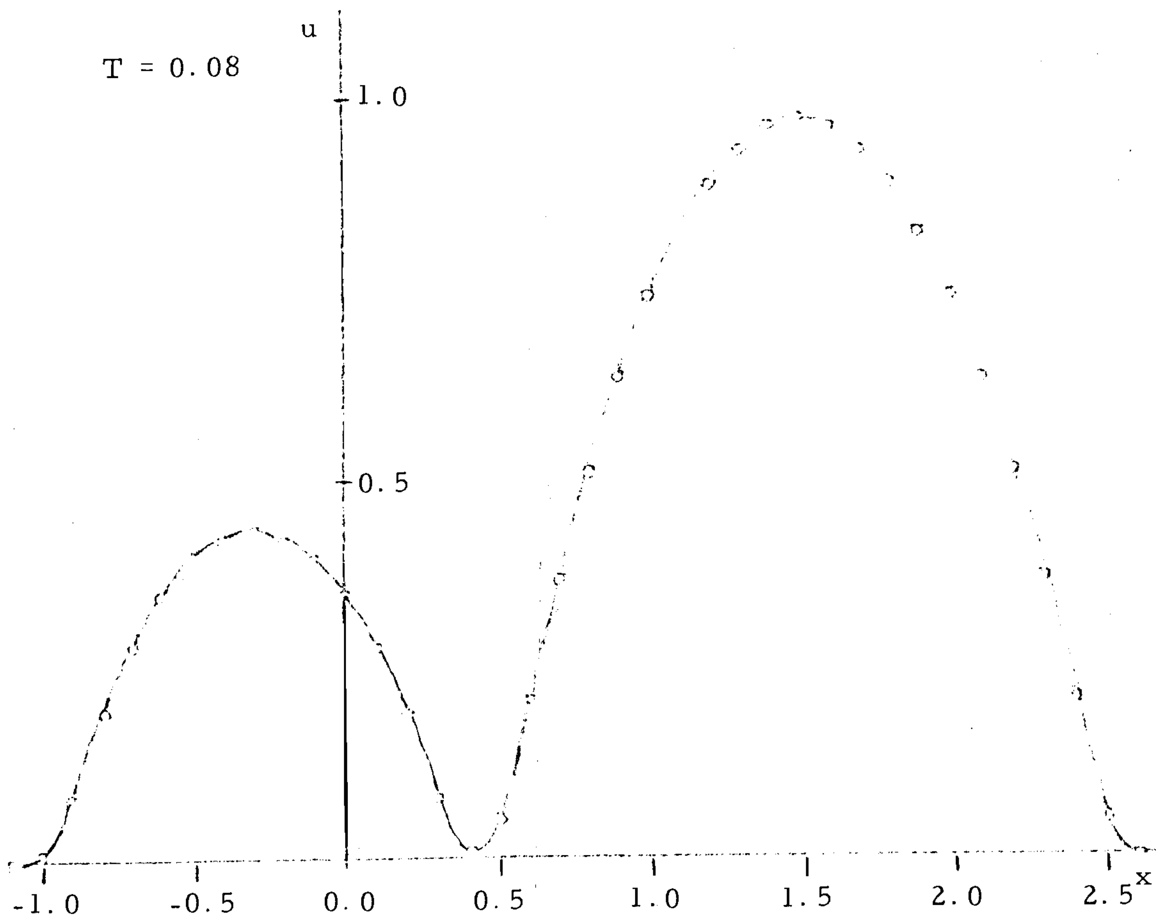
-0.5
0.0
0.5
1.0
1.5
2.0
x


u
T = 0.07
1.0
0.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
2.5
x


u
T = 0.08
1.0
0.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
2.5
x

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APPENDIX

A. Some Properties of Lipschitz and Hölder Functions

Proposition 1. Suppose that function $f(t, x, v)$ is bounded and Lipschitz continuous in the variables t, x, and v, for $0 \leq t \leq T$ (T arbitrary), $x \in$ some arbitrary region in E_n, and $v \in$ some arbitrary interval of the real line, either bounded or infinite. Then the function $f(t, x, v)$ is also Hölder continuous with exponent α, $0 < \alpha < 1$, with respect to all variables t, x, and v.

Proof. Recalling that for points $P' = (t', x')$, $P'' = (t'', x'')$, that

$$d(P', P'') = \sqrt{\left(|x'-x''|^2 + |t'-t''|\right)},$$

there exist constants B_1, B_2, B_3, and B_4 such that

$$\frac{|f(t', x', v)-f(t'', x'', v)|}{d(P', P'')} \leq B_1,$$

$$\frac{|f(t, x, v')-f(t, x, v'')|}{|v'-v''|} \leq B_2,$$

and

$$B_3 \leq f(t, x, v) \leq B_4 ,$$

for all values of the variables under consideration.

We first prove Hölder continuity with respect to the variables t

and x. First, suppose that

$$d(P', P'') \leq 1,$$

then for $0 < \alpha < 1$, we have

$$\frac{|f(t', x', v)-f(t'', x'', v)|}{d(P', P'')^{\alpha}} = \frac{|f(t', x', v)-f(t'', x'', v)|}{d(P', P'')} d(P', P'')^{1-\alpha}$$

$$\leq B_1 d(P', P'')^{1-\alpha} \leq B_1 .$$

On the other hand, if

$$d(P', P'') > 1,$$

we have

$$\begin{aligned} \frac{|f(t', x', v)-f(t'', x'', v)|}{d(P', P'')^{\alpha}} &\leq |f(t', x', v) - f(t'', x'', v)| \\ &\leq |f(t', x', v)| + |f(t'', x'', v)| \\ &\leq 2 \max\{|B_3|, |B_4|\}. \end{aligned}$$

Hence the function $f(t, x, v)$ is Hölder continuous with respect to the variables x and t.

Proof of Hölder continuity with respect to the variable v is analogous to the above proof, but with $d(P', P'')$ replaced by $|v'-v''|$, etc.

Proposition 2. Let functions $f(t, x, v)$ and $g(t, x, v)$ both be Hölder continuous in t, x, and v. Then so are the functions $f + g$ and $f - g$.

Proof. By hypothesis there are constants B_1, B_2 such that

$$|f(t', x', v) - f(t'', x'', v)| / d(P', P'')^{a} \leq B_1 ,$$

$$|g(t', x', v) - g(t'', x'', v)| / d(P', P'')^{a} \leq B_2 .$$

Then we have

$$\frac{|[f+g](t', x', v) - [f+g](t'', x'', v)|}{d(P', P'')^{a}}$$

$$\leq \frac{|f(t', x', v) - f(t'', x'', v)|}{d(P', P'')^{a}} + \frac{|g(t', x', v) - g(t'', x'', v)|}{d(P', P'')^{a}}$$

$$\leq B_1 + B_2 ,$$

and an analogous proof may be used to establish Hölder continuity in the variable v. The proof for the function $f - g$ is similar.

Proposition 3. Let functions $f(t, x, v)$ and $g(t, x, v)$ be both bounded and Hölder continuous in t, x, and v. Then so is the product function fg.

Proof. Proof of boundedness is trivial, and left for the reader. By hypothesis, there are constants B_1, B_2, C_1, C_2 such that

$$\frac{|f(t', x', v) - f(t'', x'', v)|}{d(P', P'')^{a}} \leq B_1 ,$$

$$\frac{|g(t', x', v) - g(t'', x'', v)|}{d(P', P'')^{a}} \leq B_2 ,$$

$$|f(t,x,v)| \leq C_1, \quad |g(t,x,v)| \leq C_2 .$$

It then follows that

$$|[fg](t',x',v)-[fg](t'',x'',v)| / d(P',P'')^{a}$$

$$\leq |f(t',x',v)g(t',x',v)-f(t'',x'',v)g(t',x',v)| / d(P',P'')^{a}$$

$$+ |f(t'',x'',v)g(t',x',v)-f(t'',x'',v)g(t'',x'',v)| / d(P',P'')^{a}$$

$$\leq |g(t',x',v)| \, |f(t',x',v)-f(t'',x'',v)| / d(P',P'')^{a}$$

$$+ |f(t'',x'',v)| \, |g(t',x',v)-g(t'',x'',v)| / d(P',P'')^{a}$$

$$\leq C_2 B_1 + C_1 B_2 ,$$

proving Hölder continuity. Proof for variable v is similar.

Corollary. Let function $f(t,x,v)$ be both bounded and Hölder continuous in the variables t, x, and v, and let n be any positive integer. Then the power function f^n is also bounded and Hölder continuous.

Proof. We need merely apply Proposition 3 and mathematical induction on integer n.

Proposition 4. Let functions $f(t,x,v)$ and $g(t,x,v)$ be both Hölder continuous in the variables t, x, and v. In addition, let function $f(t,x,v)$ be bounded, and function $g(t,x,v)$ be bounded away from zero. Then the quotient function f/g is also Hölder continuous in t, x, and v.

Proof. By hypothesis, there are constants B_1, B_2, C_1, C_2 such that

$$|f(t',x',v)-f(t'',x'',v)|/d(P',P'')^{\alpha} \le B_1 ,$$

$$|g(t',x',v)-g(t'',x'',v)|/d(P',P'')^{\alpha} \le B_2 ,$$

$$|f(t,x,v)| \le C_1, \quad 0 < C_2 \le |g(t,x,v)| .$$

Then we have

$$|[f/g](t',x',v)-[f/g](t'',x'',v)|/d(P',P'')^{\alpha}$$

$$\le \frac{|f(t',x',v)g(t'',x'',v)-f(t'',x'',v)g(t',x',v)|}{|g(t',x',v)g(t'',x'',v)|d(P',P'')^{\alpha}}$$

$$\le \frac{|f(t',x',v)g(t'',x'',v)-f(t',x',v)g(t',x',v)|}{|g(t',x',v)g(t'',x'',v)|d(P',P'')^{\alpha}}$$

$$+ \frac{|f(t',x',v)g(t',x',v)-f(t'',x'',v)g(t',x',v)|}{|g(t',x',v)g(t'',x'',v)|/d(P',P'')^{\alpha}}$$

$$\le \frac{|f(t',x',v)|}{|g(t',x',v)g(t'',x'',v)|} \quad \frac{|g(t'',x'',v)-g(t',x',v)|}{d(P',P'')^{\alpha}}$$

$$+ \frac{1}{|g(t'',x'',v)|} \quad \frac{|f(t',x',v)-f(t'',x'',v)|}{d(P',P'')^{\alpha}}$$

$$\le \frac{C_1}{C_2^2} B_2 + \frac{1}{C_2} B_1 ,$$

proving Hölderness in t and x. Proof for variable v is similar.

Proposition 5. Let function $f(t,x,v)$ be Hölder continuous in the variable v. Then function $f(t, x, v)$ is also Hölder continuous in the variable w (with same exponent α), where w is defined by the relationship

$$w = \int_{m_0/2}^{v} A_1(s)ds ,$$

and where $A_1(v)$ is the function constructed in the proof of the lemma in Chapter III.

<u>Proof</u>. Recalling the notation used in the proof of the lemma, we have

$$w = F(v), \quad v = G(w),$$

where

$$F(v) = \int_{m_0/2}^{v} A_1(s)ds .$$

We then have

$$\begin{aligned} & |f(t, x, G(w)) - f(t, x, G(w'))| / |w-w'|^{\alpha} \\ & \leq H(f)|G(w)-G(w')|^{\alpha}/|w-w'|^{\alpha} \\ & = H(f)|G'(\bar{w})|^{\alpha}, \\ & \qquad \text{(by the mean-value theorem)} \\ & = H(f)/|A_1(\bar{v})|^{\alpha} \leq 2^{\alpha}H(f)/\underline{A}^{\alpha}, \end{aligned}$$

where $H(f)$ is the Hölder coefficient of function $f(t, x, v)$ with respect to variable, v and $\underline{A}$ is a positive constant defined in the proof of the lemma.

<u>Proposition 6</u>. Let function $A(z)$ be Lipschitz continuous in the real variable z, and the real-valued function $B(x)$ be Hölder continuous (with exponent α) in the vector variable x, $x \in E_n$. Then the composite function $A(B(x))$ is also Hölder continuous (with exponent α) in the variable x.

<u>Proof</u>.

$$|A(B(x))-A(B(x'))| / |x-x'|^{\alpha}$$
$$\leq L(A) |B(x)-B(x')| / |x-x'|^{\alpha}$$
$$\leq L(A)\, H(B),$$

where $L(A)$ and $H(B)$ are the Lipschitz and Hölder coefficients of A and B, respectively.

<u>Proposition 7</u>. The function $\Phi(v)$ defined in the proof of Theorem 2 is Lipschitz continuous on all closed intervals of the form $0 < a \leq v \leq b$.

<u>Proof</u>. We have for $a \leq v \leq b$, $a \leq v' \leq b$, that $|\Phi(v)-\Phi(v')| / |v-v'| = |\Phi'(\bar{v})|$, by the mean-value theorem (where $\bar{v}$ is strictly between v and v'). But

$$\Phi'(v) = 1/\varphi'(v),$$

so that if

$$0 < a \leq v \leq b,$$

then

$$0 < \Phi(a) \leq u \leq \Phi(b),$$

so that $1/\varphi'(u)$ is bounded away from zero by its continuity, and positiveness, and

$$\varphi'(u) = 0$$

if and only if $u = 0$. Therefore $|\Phi'(v)|$ is bounded for $a \leq v \leq b$, proving the proposition.

B. An Existence and Uniqueness Theorem for Quasilinear Parabolic Partial Differential Equations

In this section we state the Theorem 14 from the work [11] of Oleinik and Kruzhkov, this theorem being used at a certain step in the proof of our lemma in Chapter III.

We consider in the region

$$G = \{(t, x) : 0 \leq t \leq T, \ x \in E_n\}$$

the quasilinear parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} [a_{ij}(t, x, u) \frac{\partial u}{\partial x_j}]$$

$$+ \sum_{j=1}^{n} b_j(t, x, u, u_x) + c(t, x, u, u_x) , \qquad (A. 1)$$

where

$$u_x = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_n) ,$$

together with the initial condition

$$u(0, x) = \psi_0(x), \quad x \in E_n \qquad (A. 2)$$

where

$$|\psi_0(x)| \leq M_0, \quad x \in E_n ,$$

and constant M_0 depends only on the data.

It is assumed that the coefficients of Equation (A. 1) satisfy the following conditions.

<u>Condition A.</u> For $(t, x) \in G$ and any u, we have

$$c_u(t, x, u, 0) \leq c_0$$

and

$$|c(t, x, 0, 0)| \leq c_1 ,$$

where constants c_1, c_2 depend only on the data.

<u>Condition B.</u> For $(t,x) \in G$ and any u, we have, for all i, j,

$$a_{ij}(t,x,u) = a_{ji}(t,x,u),$$

and for arbitrary $y = (y_1, \ldots, y_n) \in E_n$, we have

$$\mu_1(|u|) \sum_{j=1}^{n} y_j \leq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(t,x,u) y_i y_j \leq \mu_2(|u|) \sum_{j=1}^{n} y_j ,$$

where μ_1 and μ_2 are non-increasing and non-decreasing positive functions respectively, dependent only on the data.

<u>Condition C.</u> For $(t,x) \in G$, $|u| \leq \bar{M}$, and any u_x, where

$$\bar{M} = \max\{M_0 e^{\gamma T}, c_1 e^{\gamma T}/(\gamma - c_0)\},$$

with γ satisfying the condition

$$\gamma - c_0 > 0,$$

the inequalities

$$|b_j| \leq [\lambda(p)+\kappa](p+1), \quad 1 \leq j \leq n,$$

hold, where

$$p = \sqrt{[(\partial u/\partial x_1)^2 + (\partial u/\partial x_2)^2 + \ldots + (\partial u/\partial x_n)^2]},$$

$[\lambda(p)+\kappa](p+1)$ is a positive increasing function for $p \geq 0$, $\lambda(p)$ is bounded,

$$\lim_{p\to\infty} \lambda(p) = 0,$$

and

$$0 \leq \kappa \leq M,$$

where M is a constant depending only on the data. In addition, the following inequalities must hold:

$$|(b_i)_{x_k}| + |(b_i)_u| \leq B_1(p+1),$$

$$|(b_i)_{u_{x_k}}| \leq B_2,$$

$$|c| + |c_{x_k}| + |c_u| + |c_{u_{x_k}}| \leq N,$$

where B_1, B_2, and N are constants depending only on the data.

Theorem. (Of Oleinik and Kruzhkov.)

Suppose that in every portion of the region

$$\{(t,x,u,u_{x_1},u_{x_2},\ldots,u_{x_n}) : (t,x) \in G,\ u \in E_1,\ u_{x_k} \in E_1,\ 1 \leq k \leq n\}$$

which is finite with respect to u, and $u_x (= (u_{x_1}, u_{x_2}, \ldots, u_{x_n}))$, the coefficients $a_{ij} \in C_{2+\alpha}$ with respect to x_k and u, the coefficients c and $b_j \in C_{1+\alpha}$ with respect to the variables x_k, u, and u_{x_k}, and also satisfy Conditions A, B, and C. If also the initial function $\psi_0 \in C^{2+\alpha}$ in E_n, then there exists a unique solution $u(t,x) \in C^{2+\alpha}$ in G of the Cauchy problem (A.1), (A.2).

Remark. If the function b_j is a bounded function then the inequality

$$|b_j| \leq [\lambda(p)+\kappa](p+1), \quad 1 \leq j \leq n,$$

is automatically satisfied if we take

$$\lambda(p) = B/\sqrt{p+1}, \quad p \geq 0,$$

and

$$\kappa = 0,$$

where B is any positive bound for $|b_j|$. For we have

$$\begin{aligned} |b_j(t,x,u)| &\leq B \leq B\sqrt{p+1} \\ &\leq (B/\sqrt{p+1})(p+1) \\ &= \lambda(p)(p+1), \quad \text{with} \quad p \geq 0 . \end{aligned}$$

Clearly $\lambda(p)(p+1) = B\sqrt{p+1}$ is a positive increasing function in p for $p \geq 0$, $\lambda(p) = B/\sqrt{p+1}$ is bounded for $p \geq 0$, and vanishes as $p \to +\infty$. Hence the inequality follows for this selection of function $\lambda(p)$.

C. Tables

x	u	x	u	x	u
T = 0.00					
-0.7	0.00	0.3	0.00	1.3	1.68
-0.6	0.00	0.4	0.00	1.4	1.92
-0.5	0.56	0.5	0.00	1.5	2.00
-0.4	0.89	0.6	0.00	1.6	1.92
-0.3	1.00	0.7	0.00	1.7	1.68
-0.2	0.89	0.8	0.00	1.8	1.28
-0.1	0.56	0.9	0.00	1.9	0.72
0.0	0.00	1.0	0.00	2.0	0.00
0.1	0.00	1.1	0.72	2.1	0.00
0.2	0.00	1.2	1.28		
T = 0.01					
-0.9	0.00	0.2	0.00	1.3	1.43
-0.8	0.00	0.3	0.00	1.4	1.55
-0.7	0.03	0.4	0.00	1.5	1.59
-0.6	0.28	0.5	0.00	1.6	1.55
-0.5	0.56	0.6	0.00	1.7	1.43
-0.4	0.71	0.7	0.00	1.8	1.23
-0.3	0.75	0.8	0.00	1.9	0.94
-0.2	0.71	0.9	0.10	2.0	0.55
-0.1	0.56	1.0	0.55	2.1	0.10
0.0	0.28	1.1	0.94	2.2	0.00
0.1	0.03	1.2	1.23	2.3	0.00
T = 0.02					
-1.0	0.00	0.2	0.01	1.4	1.37
-0.9	0.00	0.3	0.00	1.5	1.39
-0.8	0.01	0.4	0.00	1.6	1.37
-0.7	0.12	0.5	0.00	1.7	1.29
-0.6	0.36	0.6	0.00	1.8	1.15
-0.5	0.52	0.7	0.00	1.9	0.95
-0.4	0.61	0.8	0.06	2.0	0.70
-0.3	0.65	0.9	0.38	2.1	0.38
-0.2	0.61	1.0	0.70	2.2	0.06
-0.1	0.52	1.1	0.95	2.3	0.00
0.0	0.36	1.2	1.15	2.4	0.00
0.1	0.12	1.3	1.29		

x	u	x	u	x	u
T = 0.03					
-1.0	0.00	0.2	0.03	1.4	1.25
-0.9	0.00	0.3	0.00	1.5	1.27
-0.8	0.03	0.4	0.00	1.6	1.25
-0.7	0.20	0.5	0.00	1.7	1.19
-0.6	0.37	0.6	0.00	1.8	1.08
-0.5	0.49	0.7	0.02	1.9	0.94
-0.4	0.56	0.8	0.23	2.0	0.75
-0.3	0.58	0.9	0.52	2.1	0.52
-0.2	0.56	1.0	0.75	2.2	0.23
-0.1	0.49	1.1	0.94	2.3	0.02
0.0	0.37	1.2	1.08	2.4	0.00
0.1	0.20	1.3	1.19	2.5	0.00
T = 0.04					
-1.0	0.00	0.2	0.08	1.4	1.16
-0.9	0.00	0.3	0.00	1.5	1.18
-0.8	0.08	0.4	0.00	1.6	1.16
-0.7	0.24	0.5	0.00	1.7	1.11
-0.6	0.37	0.6	0.00	1.8	1.03
-0.5	0.46	0.7	0.10	1.9	0.91
-0.4	0.52	0.8	0.36	2.0	0.76
-0.3	0.54	0.9	0.58	2.1	0.58
-0.2	0.52	1.0	0.76	2.2	0.36
-0.1	0.46	1.1	0.91	2.3	0.10
0.0	0.37	1.2	1.03	2.4	0.00
0.1	0.24	1.3	1.11	2.5	0.00
T = 0.05					
-1.1	0.00	0.2	0.12	1.5	1.11
-1.0	0.00	0.3	0.01	1.6	1.09
-0.9	0.01	0.4	0.00	1.7	1.05
-0.8	0.12	0.5	0.00	1.8	0.98
-0.7	0.26	0.6	0.03	1.9	0.89
-0.6	0.37	0.7	0.21	2.0	0.76
-0.5	0.44	0.8	0.43	2.1	0.61
-0.4	0.49	0.9	0.61	2.2	0.43
-0.3	0.50	1.0	0.76	2.3	0.21
-0.2	0.49	1.1	0.89	2.4	0.03
-0.1	0.44	1.2	0.98	2.5	0.00
0.0	0.37	1.3	1.05	2.6	0.00
0.1	0.26	1.4	1.09		

x	u	x	u	x	u
T = 0.06					
-1.1	0.00	0.2	0.16	1.5	1.05
-1.0	0.00	0.3	0.03	1.6	1.04
-0.9	0.03	0.4	0.00	1.7	1.00
-0.8	0.16	0.5	0.00	1.8	0.95
-0.7	0.27	0.6	0.08	1.9	0.86
-0.6	0.36	0.7	0.29	2.0	0.76
-0.5	0.42	0.8	0.47	2.1	0.62
-0.4	0.46	0.9	0.62	2.2	0.47
-0.3	0.47	1.0	0.76	2.3	0.29
-0.2	0.46	1.1	0.86	2.4	0.08
-0.1	0.42	1.2	0.95	2.5	0.00
0.0	0.36	1.3	1.00	2.6	0.00
0.1	0.27	1.4	1.04		
T = 0.07					
-1.1	0.00	0.2	0.18	1.5	1.01
-1.0	0.00	0.3	0.05	1.6	1.00
-0.9	0.05	0.4	0.00	1.7	0.97
-0.8	0.18	0.5	0.02	1.8	0.91
-0.7	0.28	0.6	0.15	1.9	0.84
-0.6	0.35	0.7	0.34	2.0	0.75
-0.5	0.41	0.8	0.50	2.1	0.63
-0.4	0.44	0.9	0.63	2.2	0.50
-0.3	0.45	1.0	0.75	2.3	0.34
-0.2	0.44	1.1	0.84	2.4	0.15
-0.1	0.41	1.2	0.91	2.5	0.02
0.0	0.35	1.3	0.97	2.6	0.00
0.1	0.28	1.4	1.00	2.7	0.00

x	u	x	u	x	u
T = 0.08					
-1.2	0.00	0.2	0.19	1.5	0.97
-1.1	0.00	0.3	0.08	1.6	0.96
-1.0	0.01	0.4	0.01	1.7	0.93
-0.9	0.08	0.5	0.05	1.8	0.88
-0.8	0.19	0.6	0.21	1.9	0.82
-0.7	0.28	0.7	0.37	2.0	0.74
-0.6	0.35	0.8	0.51	2.1	0.63
-0.5	0.40	0.9	0.63	2.2	0.51
-0.4	0.43	1.0	0.74	2.3	0.37
-0.3	0.44	1.1	0.82	2.4	0.21
-0.2	0.43	1.2	0.88	2.5	0.05
-0.1	0.40	1.3	0.93	2.6	0.00
0.0	0.35	1.4	0.96	2.7	0.00
0.1	0.28				