

AN ABSTRACT OF THE THESIS OF

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Abstract approved: _____
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Micropolar Theory, a specialization of the Microcontinuum Theory advanced by A. C. Eringen in the early 1960's has been used successfully in predicting the behavior of materials whose inner structure is composed of rod-like elements. Liquid crystals fall into this class of materials and their wide-spread use in industry and in medical diagnostic devices requires a detailed analysis of their response to various external stimuli. To this end we have undertaken a detailed study of smectic liquid crystals using the micropolar theory.

The first step was to develop a mathematical foundation for irreversible and nonlinear thermodynamics of liquid crystals and mathematical methods for obtaining thermodynamical restrictions on the material response coefficients of smectic liquid crystals.

Full symmetry properties of liquid crystals have been employed in order to simplify the constitutive equations.

This theory has then been applied to the physical problem of predicting orientational effects in smectic liquid crystals due to boundary influences, superposed shear flows, and thermal gradients. Since most experiments are carried out for viscometric flows we have considered the flow of smectic liquid crystals between two rotating coaxial cylinders. The velocity field obtained is found to include the classical Couette flow as a special case. We then compare our results with experiments and identify the material coefficients of the theory with physically measurable quantities such as apparent viscosity. The shear-rate dependence of viscosity at various temperatures of the smectic range is determined and comparisons with experimental results yield good agreement.

Finally, with the aim of constructing a viscoelastic theory suitable for description of materials with stretchable rod-like elements we first develop the theory of micropolar elasticity with stretch which takes into account the uniform stretching of the material's microelements. Future work involving the construction of the theory of micropolar viscoelasticity with stretch and its application to the study of viscoelastic waves is indicated.

Microcontinuum Theory and Orientational Effects
in Smectic Liquid Crystals

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MICROCONTINUUM THEORY AND ORIENTATIONAL EFFECTS IN SMECTIC LIQUID CRYSTALS

I. INTRODUCTION

1. Micropolar Theory

Classical continuum mechanics had its beginnings mainly in the works of Euler over two centuries ago and later in the works of Cauchy. The fundamental concepts of the continuum field approach to the treatment of matter are based on the idea that all material bodies possess continuous mass densities and that the laws of motion and the axioms of constitution are valid for every part of the body regardless of its size. Thus continuous media are viewed as dense collections of concentrated infinitesimal masses devoid of internal structure.

The classical continuum theory has enjoyed wide success and acceptance especially in the engineering sciences. However, the classical continuum theory has drastic limitations on the extent to which its description of macroscopic behavior can successfully reflect the effects of the fine structure of matter. Increasingly sophisticated experiments and accurate measurements led the experimentalists and theorists alike to the conclusion that the fine structure of materials needed to be considered.

According to classical continuum theory a small volume ΔV enclosed within a surface ΔS possesses a mass density ρ defined by

$$\rho \equiv \lim_{\Delta V \rightarrow 0} (\Delta m / \Delta V) \quad (1.1)$$

where Δm is the total mass contained in ΔV . Here, ρ is independent of the size of ΔV and depends only on the position vector \underline{x} of a point in ΔV and time t . However, experimentally, one finds that as the size of the material volume element is allowed to approach zero the sequence of ratios $\Delta m / \Delta V$ fails to possess a limit and exhibits a marked dependence on the volume size as the latter decreases below a certain critical volume ΔV^c . The continuity assumption for mass density is no longer applicable as the size of the volume element approaches a characteristic material dimension such as grain size in a solid or particle size in a fluid suspension.

Another case to consider is when the fine structure of a material is set in motion without application of a macroscopic force. This is accomplished when the applied external physical force has length scale comparable to the average grain size or molecular size of the material. Thus any theory which purports to predict adequately the response of the material in this situation must be capable of taking into account the intrinsic motions of the material constituents of continuum microelements.

Hence we see that engineering scientists were confronted with the problem of describing the behavior of rheologically complex fluids such as liquid crystals, polymeric suspensions, and blood. What was required was a field theory approach that could adequately describe the macroscopic manifestations of microscopic events, namely the motion and deformation of the microelements. The new theories of microcontinua developed to fulfill this need regarded continuous media as sets of structured particles possessing not only mass and velocity but also a substructure which has associated with it a moment-of-inertia density and a microdeformation tensor. These new theories required a complete reappraisal of classical concepts such as the symmetry of the stress tensor and the absence of couple stresses in view of local structural aspects and micro-motions. Thus while many of the principles of classical continuum mechanics remain valid for this new class of materials, additional balance laws and constitutive relations were necessitated by the presence of microscopic elements in these materials which gave rise not only to the usual Cauchy stresses, but also to couple stresses due to microelement interaction. The orientation of the microelement also has to be considered, as well as the production of internal angular momentum due to their rotational motions.

The concept of couple stresses acting in a medium independent of tractions is not a recent development. Bernoulli and Euler

considered them in connection with their respective works on beam theories. As early as 1893 Duhem [7] suggested the concept of an oriented medium wherein a body is regarded as a collection of points each of which is associated with a set of directors or vectors which can undergo deformations independent of the deformation of the points. The Cosserat brothers [4] utilized the concept of a three-dimensional director-oriented continuum to construct a unified theory for deformable bars, surfaces, and bodies which has since been generalized by Ericksen and Truesdell [10]. Thus in a Cosserat medium the orientation of a given point can be expressed mathematically by the values of three mutually perpendicular unit vectors called the 'directors' of an oriented medium. The drawback of the 'director' description is that there is no mechanism for calculating the moment of inertia of a spinning microvolume element.

The theory of 'simple microelastic solids' put forth by A. C. Eringen and E. S. Suhubi [12,13] remedies this situation by developing a physical model in which each continuum element is assigned a substructure containing microvolume elements which can translate, rotate, and deform independently of the motion of the macrovolume; however, each deformation of the macrovolume element can be expected to produce a subsequent deformation of the microvolume element. In this manner a mechanism is provided

in the theory to treat materials which are capable of supporting local stress moments and body moments and are influenced by the microelement spin inertia.

The development of Eringen and Suhubi's general theory of microcontinuum mechanics proceeded in the following manner. The field equations involving balance laws of momenta, conservation of mass and energy, etc. were formulated by applying Cauchy's laws of motion and the classical conservation laws of mass and energy expressed as volume and surface integral operators to each microelement contained in a material macrovolume element. The fundamental laws of microcontinua resulted when simple statistical averages of the global balance and conservation laws were taken over the entire macrovolume element. In addition to the usual laws of classical continuum mechanics two additional equations were obtained; the first accounts for conservation of microinertia moments and the second expresses the balance of first stress moments which arise due to consideration of the material's microstructure. The point should be made here that while a statistical averaging process is employed in the development, the theory itself is based solely on the principles of continuum mechanics and not on molecular or statistical mechanics.

In a series of papers [11, 16] Eringen has reiterated and extended his initial theory of micromorphic continua. He has

obtained Master equations in the form of integral operators from which all order volume and surface moments are derivable. This leads to a hierarchy of balance laws and the micromorphic material is assigned a specific grade $(1, 2, \dots, N)$ according to the number of statistical moments taken of the momenta equations, energy equation, and continuity equation which are necessary to develop the balance laws governing the material's dynamic behavior. One of the most important developments of Eringen's microcontinuum theory was the introduction of new kinematic variables, namely, the gyration tensor and microinertia tensor and the addition of the concepts of body moments, stress moments, and microstress averages to classical continuum mechanics.

In a subsequent paper [11] Eringen defined the class of simple microfluids as fluent media whose properties and behavior are affected by the local motions of the material particles in each of its volume elements. The main thrust of the paper was the mathematical development of thermodynamically admissible constitutive equations for the theory of simple microfluids. Thus the simple microfluids which are viscous isotropic fluids are characterized in the simplest case of a constitutively linear theory by 22 viscosity and material coefficients. However, when this theory is applied to a real nontrivial flow problem the resulting system of 19 partial differential equations in 19 unknowns presents a serious difficulty in that the

underlying mathematical problem is not easily solved. This difficulty has led Eringen to consider subclasses of these simple microfluids which, while simplified enough to allow for the solution of the resulting system of equations, they still are able to describe adequately the dynamical behavior of the material.

Thus Eringen considered the subclass of simple microfluids called micropolar fluids [14] wherein the microgyration tensor is required to be skew-symmetric and the material is considered to be microisotropic although not necessarily macroisotropic. Physically, these assumptions amount to saying that the microelements do not deform, but microrotational effects are still present and surface and body couples are permitted. That is, the microelements may be thought of as rigid rods which may undergo rotations and translations. This situation prevails in many different materials such as polymeric additives, solid rocket propellants, animal blood, colloidal suspensions, liquid crystals, materials with macromolecules and in fibrous and granular media. Mathematically these conditions reduced the simple microfluid system of 19 equations to seven equations in seven unknowns. Two basic and independent kinematic vector fields were also introduced: (1) the classical vector field representing the translational velocity of the fluid particles and (2) the vector field representing the angular (or spin) velocity of the particles, namely the microgyration vector.

Hence, the local rotational motion of the particles and the usual rigid body motion of the entire volume element is taken into account by the micropolar theory.

The thermodynamics of such fluids figured prominently in the development of the theory of micropolar fluids [14]. Through the conservation law of energy and the entropy inequality the generalized Clausius-Duhem inequality was obtained and then used to find restrictions governing the material coefficients. Further work [38] treated the heat-flux vector as a constitutive response variable rather than as being externally imposed on the material. Thus we have at our disposal a theory which can represent fluids consisting of rigid, randomly oriented microelements suspended in a viscous medium. We now turn our attention to an examination of liquid crystals in order to see that the micropolar theory can be suitably adapted to describe the dynamical behavior of liquid crystals.

2. Liquid Crystals

The transition of substances from the solid state to the isotropic liquid state is familiar to all and the temperature at which this transition takes place for a certain substance is an essential piece of information for the material classification. However, this transition is oftentimes preceded by other transitions and a substance can go through several phases as it is heated from a

crystalline solid to an isotropic liquid. In 1888 while looking for the solid to isotropic transition temperature for some esters of cholesterol, F. Reinitzer [47] noticed a new and peculiar melting phenomenon. The crystals of cholesteryl benzoate were melting perfectly sharply at 145.5°C , but the melt itself was opaque. It was not until the sample was heated to 178.5°C that the opacity disappeared giving the true isotropic liquid. Shortly after this, in 1889, O. Lehmann [33] reported similar properties with silver iodide between the temperatures of 146° and 450°C . Although this "plastic" behavior of silver iodide was later found to be of a different class than the phenomenon observed by Reinitzer, Lehmann did report in 1890 [34] that both aluminum oleate and p-azoxyphenetole passed through these turbid states when heated from the true crystalline state to the isotropic liquid state.

Both Lehmann and Reinitzer continued their initial work in observing and classifying the properties of these new states or mesophases. The striking thing about these mesophases was that while clearly fluid-like they also exhibited anisotropic properties when viewed in thin section between crossed polaroids. The characteristics were therefore partly those of the crystalline solid and partly those of the liquid. Thus these substances came to be known as liquid crystals, an apparent contradiction in terms, but nonetheless, descriptive of the materials' behavior.

During the next 30-40 years liquid crystals received extensive study, most notably by Lehmann and Vorlander who found many different liquid crystals and helped establish the fact that the mesophase is a true state of the material between the solid and isotropic liquid states. In 1922 Friedel [22] published the results of his detailed optical studies of liquid crystals and his classification of liquid crystals into three main types according to symmetry properties due mainly to molecular ordering. They were:

- (a) The smectic mesophase--A turbid viscous state, with certain properties reminiscent of soaps. In fact, the term smectic is derived from the Greek word *smectos* meaning soap-like.
- (b) The nematic mesophase--A turbid but mobile state. Once again the word nematic is derived from the Greek word *nematos* meaning thread-like. This term is used since this mesophase when deposited on a glass surface frequently adopts a characteristic "threaded" texture clearly visible between crossed polaroids.
- (c) The cholesteric mesophase--A turbid and mobile mesophase, exhibiting some unique optical characteristics such as birefringence (double refraction) and circular dichroism. The name derives from the fact that the majority of compounds exhibiting these characteristics are derived from cholesterol or other sterol systems.

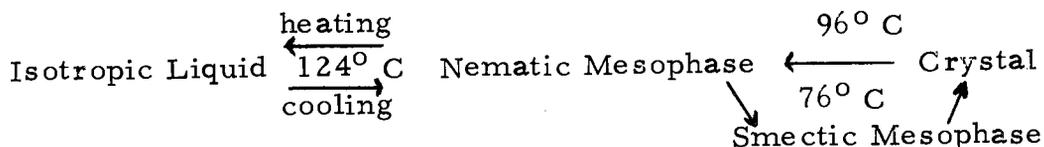
Liquid crystals are also classified according to how they are formed [46]. Most mesophases are observed when a suitable compound is heated to a temperature above that at which the crystal lattice is unstable. This type is referred to as a thermotropic liquid crystal. Materials exhibiting thermotropic liquid crystal phases are usually organic substances with molecular structures typified by those of cholesteryl nonanoate and N-(p-methoxybenzylidene)-p'-n-butylaniline (MBBA). Axial ratios of 4-8 and molecular weights of 200-500 gm/mol are typical for thermotropic liquid crystals.

It is not uncommon for a thermotropic liquid crystal to exhibit more than one type of liquid crystal mesophase. For example, the following transitions have been observed upon heating the appropriate solid:

- (i) solid \rightarrow smectic B \rightarrow smectic C \rightarrow smectic A \rightarrow
nematic \rightarrow isotropic
- (ii) solid \rightarrow smectic A \rightarrow cholesteric \rightarrow isotropic
- (iii) solid \rightarrow cholesteric \rightarrow isotropic

where the transitions in (ii) and (iii) have been observed in substances having both cholesteric and smectic mesophases. Polymorphism involving both nematic and cholesteric mesophases has not been shown at present [46]. In addition, whether a substance is heated or cooled has a direct bearing on the transitions the substance

undergoes as evidenced by the following transition diagram for ethyl-p-(4 ethoxybenzylidene-amino)- α -methylcinnamate [24]:



The other type of liquid crystal consists of rod-like entities in a normally isotropic solvent with a sufficiently high solute concentration. These anisotropic solution mesophases are called lyotropic liquid crystals. Compared to the lengths of the rod-like elements of thermotropic liquid crystals the lengths of the rod-like entities of lyotropic liquid crystals are quite large although axial ratios seldom go higher than 15. Deoxyribonucleic acid (DNA), certain viruses (e. g. tobacco mosaic virus (TMV)), and many synthetic peptides all form lyotropic mesophases when dissolved in an appropriate solvent (usually water) in suitable concentration. These liquid crystals are rather unstable with respect to temperature changes, a fact which essentially eliminates thermally induced transitions. Transitions are usually produced by varying the solvent concentration.

We return now to our discussion of the nematic, cholesteric, and smectic mesophases with particular attention to the molecular order of the liquid crystal. The nematic liquid crystals may be represented schematically by Figure 1a. We note that the rod-like

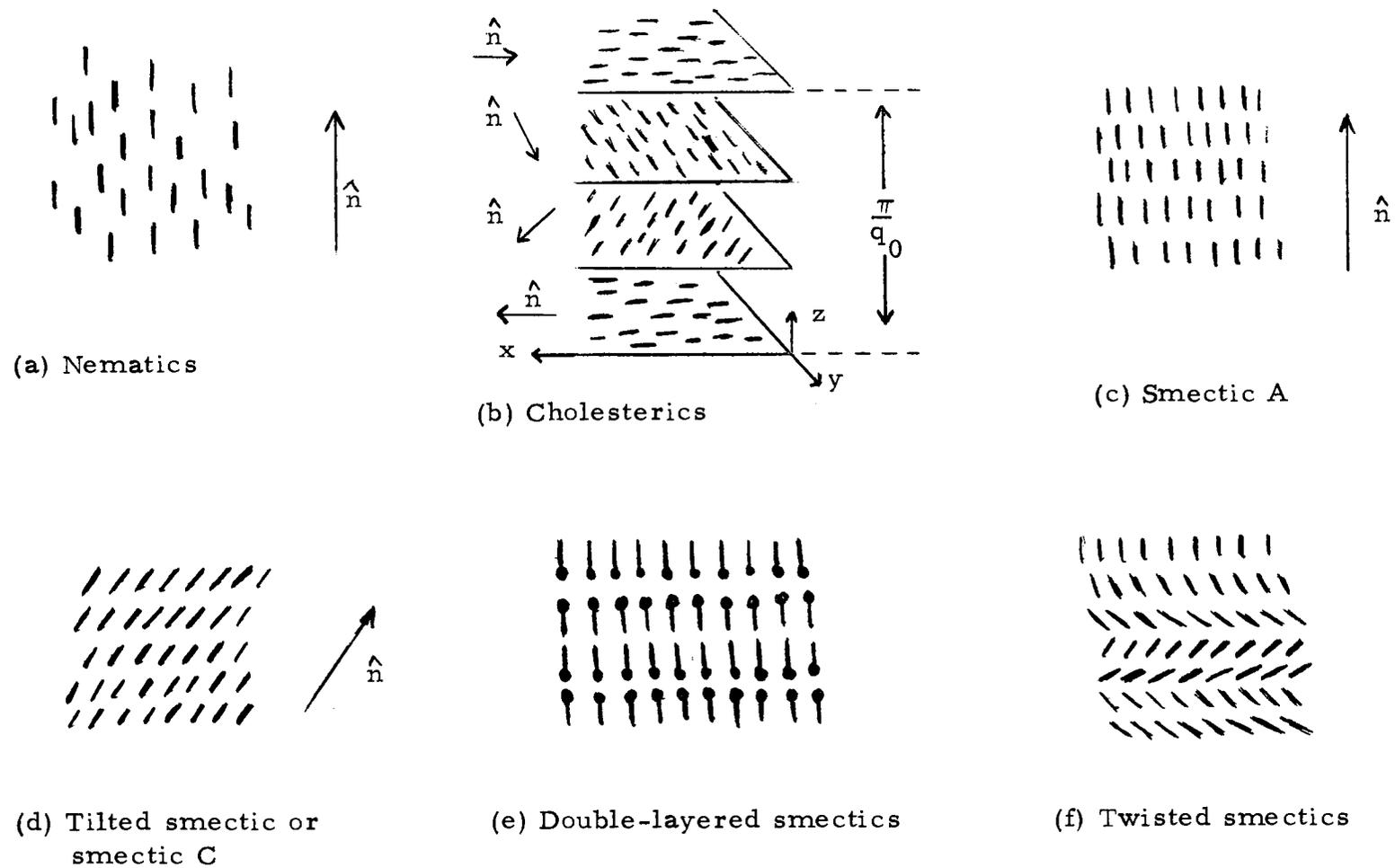


Figure 1. Schematic representations of different types of liquid crystal orderings.

molecules exhibit a long range orientational order in their overall tendency to align parallel to each other. The nematic phase in a state of thermal equilibrium exhibits uniaxial symmetry. The direction of the principal axis of orientation is arbitrary in space. Some substances having the nematic mesophase are given in Figure 2a.

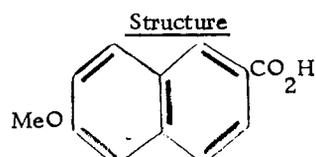
As is apparent from Figure 1b the cholesteric and nematic mesophases are similar on a local scale. However, on a larger scale in the case of the cholesterics, the principle axis of orientation undergoes a rotation that follows a helix of the form

$$\begin{aligned} n_x &= \cos(q_0 z + \phi) , \\ n_y &= \sin(q_0 z + \phi) , \\ n_z &= 0 \end{aligned} \tag{1.2}$$

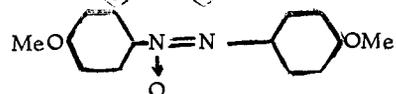
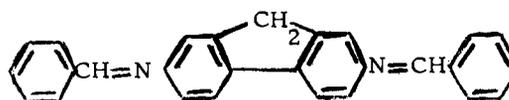
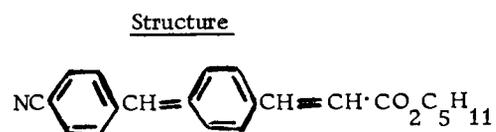
where both the direction of the helix axis z in space and the magnitude of the phase angle ϕ are arbitrary. Thus the cholesteric phase structure exhibits a periodicity $L = \pi / |q_0|$ where the sign of q_0 differentiates between right and left helices and its magnitude determines the spatial period. When L is comparable to optical wavelengths, the periodicity results in strong Bragg scattering and if the wavelength of the scattered light happens to be in the visible spectrum then the cholesteric phase will appear brightly colored.

Compound (Nematic)

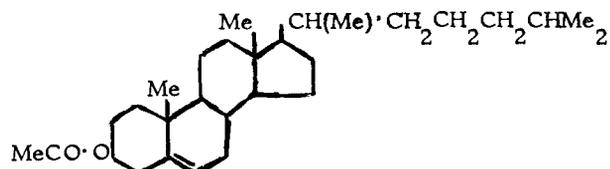
6-Methoxy-2-naphtholic acid



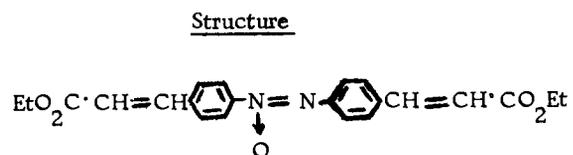
p-Azoxyanisole

2,7-Di-(Benzylidene amino)
FluorineCompound (Cholesteric)Amyl p-(4-cyanobenzylidene amino)
Cinnamate

Cholesteryl Acetate

Compound (Smectic)

Ethyl-p-azoxycinnamate



Ethyl-p-azoxybenzoate

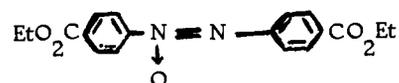


Figure 2. Liquid crystal compounds and their chemical structures.

We note here that the nematic liquid crystal can be considered as a cholesteric liquid crystal with $q_0 = 0$ (infinite pitch) and, indeed, the cholesteric liquid crystal is sometimes referred to as a twisted-nematic. Some substances exhibiting the cholesteric mesophase are given in Figure 2b.

There have been at least eight smectic phases tentatively identified, however, we shall concern ourselves with only three of the better known phases: smectic A, smectic B, and smectic C. There is still some uncertainty as to the molecular structure of the other smectic phases. The smectic A mesophase consists of layers of rod-like molecules with the molecules within each layer aligned parallel to that layer's normal. The layers are individually fluid-like since the interaction between the layers is slight compared to interactions between molecules within the same layer. As illustrated in Figure 1c the layer thickness is essentially one molecular length. At thermal equilibrium the smectic A phase is optically uniaxial due to the infinite fold symmetry about an axis parallel to the layer normal.

The smectic C ordering has a layer thickness significantly less than the molecular length. This fact, based on x-ray scattering data, has been interpreted to mean that there is a uniform tilting of the molecular axes with respect to the layer normals. A tilt angle of up to 45° has been observed and in some substances the

tilt angle is temperature dependent. As in smectic A liquid crystals, smectic C phases have individually fluid layers with diffusion between layers possible, but less probable than that for the smectic A phase.

The smectic B phase exhibits ordering within each layer as well as the usual smectic layered structure and tilting of the molecular axes within each layer. Thus we have a 3-dimensional ordering, but the mechanical properties exhibited do not really reflect this. It has been suggested that the ordering is similar to that found in a plastic crystal rather than that found in a solid crystal. A number of models have been suggested for smectic B substances, namely that of a disordered-crystal and ordered-fluid mesophase and that of a collection of 2-dimensional solid layers coupled by very weak forces, thus allowing the layers to slip over each other. For a detailed discussion of the smectic B phase the reader is referred to [25].

As further evidence of its layered structure the smectic mesophase adopts certain textures when subjected to appropriate conditions. The most commonly observed is the focal-conic texture and is usually obtained by cooling the isotropic liquid although it may be obtained occasionally by heating a smectic compound or by displacing a cover slip on a smectic compound. When viewed through a microscope the focal-conic texture of the

smectic gives the impression of a crystalline solid since it appears as a collection of tiny crystals that seem to be immobile. However, it is found that a displacement of the cover slip may be made quite easily, once again suggesting that the structure is one of layers that slip over one another.

With proper preparation the smectic liquid crystal takes the form of a stepped drop. This occurs when a drop of smectic fluid is stretched over a hole drilled in a glass surface in order to minimize the effects of temperature and contact with a boundary such as glass since either of these gives rise to the focal-conic texture. The stepped drop is homeotropic [24], that is, when viewed through crossed polaroids almost the entire field of view is optically extinct. The drop consists of a number of flat planes which are exactly parallel to one another and to the plane of the supporting surface in which the hole is drilled. Each plane terminates in a sharp edge or step and only at the perimeter of these planes does a small section of focal-conic texture disturb the homeotropy of the drop. These stepped drops were first observed by Grandjean [23] and hence are often called Grandjean terraces. Besides showing the layered structure of the smectic mesophase the method of obtaining the stepped drop shows that the layered ordering of the smectic is inherent to the mesophase itself and not produced by contact with the boundary.

We have seen that liquid crystals combine molecular ordering and fluidity in a single phase, a situation which allows them to exhibit certain properties unique to liquid crystals. The anisotropy in the molecular structure manifests itself in various macroscopic responses such as brilliant color changes and anisotropic changes in the dielectric constant. The responses may be brought about by external changes in temperature, pressure, chemical environment and electromagnetic field. Hence, liquid crystals have been employed as temperature sensors, crack detectors, vapor detectors, in display devices, and in medical diagnostic devices. Brown and Shaw [3] have done extensive research on the use of smectic liquid crystals in non-destructive testing and the use of smectics in computer memory banks has also been investigated [28].

Liquid crystals are also in evidence in biological systems, most notably in the human body itself [1]. Nematics in the form of microfilaments just beneath the plasma membrane of mammalian cells such as fibroblasts and microphages, and microtubules formed from globular elements which aggregate in spiral filament form to generate hollow tubes have been found. Cholesteric liquid crystals have been found in solutions of synthetic polypeptides. Smectic liquid crystal structure is found in nerve myelin and the plasma membrane formed mainly from the lipid-bilayer may be thought of as a two layer smectic liquid crystal. This lipid-bilayer is

important in the study of cancer since the cancer cells must migrate through this layer to attach themselves to clusters of lectin molecules which are restricted to move only in the plane of the membrane.

3. Continuum Theories and Liquid Crystals

The continuum approach to the study of liquid crystals can be traced back to Oseen [41], Zocher [51], and Frank [21] who developed the static theory. An attempt at a dynamical theory was made by Anzelius [2]. Recent attempts using the continuum approach while making provisions for the molecular structure of the liquid crystal include the anisotropic theory of Ericksen [8, 9] involving a director vector associated with each particle, denoting its preferred direction inherent in the liquid crystal, thus allowing for additional degrees of freedom of the ordered molecules. Leslie [35, 36], using what is basically an extension of the nonlinear theory of Ericksen has considered some thermal effects in cholesteric liquid crystals. Davison [5] and Davison and Amos [6] have used a linear theory based on Ericksen's nonlinear theory to discuss heat-conduction in nematic liquid crystals. The equations of Davison become complicated and very difficult to solve since the introduction of a temperature field affects the orientation field by

introducing an intrinsic body moment on the director coupled with moments already prescribed on the director due to boundaries.

The micropolar theory of Eringen has enjoyed great success in its application to liquid crystals. We have seen that liquid crystals are fluids containing more or less elongated molecules which may be thought of as having a definite orientation and they may undergo rigid rotations about their long and transverse axes. The micropolar theory of Eringen describes just such a substance, that is, the micro-elements of the micropolar theory correspond to the molecules of the liquid crystal. Thus the micropolar theory contains the mechanism to describe both the orientation and the intrinsic body moment associated with the molecules of the liquid crystal. In order to be clear about what is meant by the orientation of a sample of liquid crystal we use the following definitions put forth by Eringen and Lee [30].

Definition 1. Axis of Orientation: The axis of orientation is an imaginary line such that the statistical average of $\sin \theta$ (the sine of the angle between this line and the orientation of each elongated molecule) taken over the whole sample is zero, i. e.

$$\langle \sin \theta \rangle = 0 . \quad (4.1)$$

Definition 2. Degree of Order. It is a measure of the extent of ordering and is defined as the statistical average of $(3 \cos^2 \theta - 1)/2$ which gives the average deviation of orientation of the molecular axis, i. e.

$$S \equiv \frac{1}{2} \langle 3 \cos^2 \theta - 1 \rangle . \quad (4.2)$$

For a nematic liquid crystal S ranges from .3 to .8 (depending on temperature), $S = 0$ for a random arrangement, and $S = 1$ for a solid.

Eringen and Lee [20, 29, 32] have put forth theories for non-heat conducting nematic, smectic, and cholesteric liquid crystals. The starting point for these theories is Eringen's micropolar viscoelasticity theory [17]. Extensive use of the particular liquid crystal's symmetry properties and of thermodynamics result in a simplified constitutively linear theory that bears relevance to the actual physical situation. Eringen and Lee then investigated boundary and shear effects on the orientation field as well as wave propagation and magnetic field effects [31]. Their predicted results are in close agreement with experimental observations.

Narasimhan and Eringen [38] then extended the range of application to heat-conducting nematic liquid crystals in order to determine the effect of the temperature field on the orientation. Use of the Clausius-Duhem inequality and full material symmetry

conditions produced restrictions on the material moduli. They then used this theory to study orientation effects in heat-conducting nematic liquid crystals [39] and found their predictions to be in excellent agreement with experimental observations. In addition, they were able to determine certain material coefficients of the theory; this further facilitates the use of the theory by experimental workers.

In contrast with the extensive theoretical and experimental work on nematic and cholesteric liquid crystals we have a relative paucity of work on smectic liquid crystals. The study of smectics has been restricted on two counts. First of all it has proven difficult to obtain an adequate continuum theory for their rheological description and secondly the experimental work done is relatively scarce. Among the experimental work we have that of Southam and Puddington [48], Powell and Puddington [45], Vorlander [50], Ostwald [42], and Porter and Johnson [44].

The best effort at providing a continuum theory for smectic liquid crystals to date is the work of Eringen and Lee [32]. They have considered smectic A liquid crystals and developed the concept of the reference state for smectic liquid crystals and treated problems of bending, torsion, and wave-propagation. Narasimhan [37] has extended this theory to include heat-conducting smectic liquid crystals. It is from this point that we take up our

study of heat-conducting smectic liquid crystals.

Our first step in this thesis will be to continue the development of the theory of heat-conducting smectic liquid crystals. We use the full material symmetry properties of the liquid crystal, as well as thermodynamics to obtain restrictions on both the elastic and dissipative material moduli. Application of the symmetry properties considerably reduce the number of nonvanishing material moduli making the theory even more amenable to practical applications. The thermodynamic restrictions on the nonvanishing material coefficients will serve as a guide to the experimentalist for determining the range of applicability of the theory.

In Chapter III we continue our study of heat-conducting smectic liquid crystals by examining in detail the flow of these liquid crystals between two concentric rotating circular cylinders. Besides obtaining explicit analytical expressions for the velocity, microgyration, heat-flux, and orientation fields we develop an expression for the apparent coefficient of viscosity. We then turn to a comparison of our theoretical predictions with available experimental results reaching gratifying agreement. In addition to building confidence in our theory the comparison leads to the actual determination of some of the material coefficients in our theory.

Encouraged by the success of applying micropolar viscoelasticity theory to media such as liquid crystals we turn to the development of a theory of viscoelasticity that allows the microelement to undergo deformations as well as rigid rotations. In Chapter IV we construct the theory of micropolar elasticity with stretch obtaining the appropriate constitutive equations and field equations for the theory. In addition we use the principle of non-negative internal energy to obtain thermodynamic restrictions on the new material moduli. Further work involving the construction of a theory of micropolar viscoelasticity with stretch and its application to the study of wave propagation in such media is indicated.

In the sequel, a symbol such as (II.2.5) refers to formula 5 in section 2 of Chapter II and (2.5) refers to formula 5 in section 2 of the current chapter.

II. THERMOMICROPOLAR THEORY OF HEAT- CONDUCTING SMECTIC LIQUID CRYSTALS

1. Preliminary Remarks

In this chapter we continue the development of a continuum theory of heat-conducting smectic liquid crystals. Eringen and Lee [32] have developed a continuum theory of smectic liquid crystals using micropolar theory. In their work they have introduced the important concept of the reference state for smectic liquid crystals. That is, they have found that [32], "Smectic liquid crystals are materials which assume any state that leaves density, axis of orientation, and layered structure unchanged as reference state, in which both stress tensor and moment stress vanish. Moreover, even in the dynamic case, as long as density, axis of orientation, and layered structure remain unchanged, one can take the instantaneous state to be the reference state." Narasimhan [37] has introduced the temperature gradient as an additional argument in the constitutive equations proposed by Eringen and Lee and has constructed a constitutively linear theory.

In the present work these constitutive equations are subjected to the invariance requirements of the principle of spatial objectivity and full material symmetry of smectic liquid crystals. This results in a simplified set of constitutive equations. Next we obtain

thermodynamic restrictions on both the dissipative and nondissipative material moduli. These thermodynamic restrictions, together with the restrictions due to symmetry considerations, further reduce the number of material coefficients, thus simplifying the constitutively linear theory and enhancing its usefulness for practical applications.

2. Balance Laws and Kinematics

The basic laws of motion of a micropolar continuum are [14]:

Conservation of Mass:

$$\dot{\rho} + \rho v_{k,k} = 0 ; \quad (2.1)$$

Conservation of Microinertia:

$$i_{kl} = I_{KL} \chi_{kK} \chi_{lL} ; \quad (2.2)$$

Balance of Linear Momentum:

$$t_{kl,k} + \rho f_l = \rho \dot{v}_l ; \quad (2.3)$$

Balance of Moment of Momentum:

$$t_{rs,r} + \epsilon_{spq} t_{pq} + \rho l_s = \rho \frac{D}{Dt} [(i_{pp} \delta_{sr} - i_{rs}) v_r] ; \quad (2.4)$$

Conservation of Energy:

$$\rho \dot{\xi} = t_{kl} (v_{l,k} - \epsilon_{klm} v_m) + m_{kl} v_{l,k} + q_{k,k} + \rho h ; \quad (2.5)$$

Entropy Inequality:

$$\rho \dot{\eta} - (q_k/T)_{,k} - \rho h/T \geq 0. \quad (2.6)$$

In equations (2.1) to (2.6) we have ρ = mass density, v_k = velocity, $i_{kl} = i_{lk}$ = microinertia tensor in deformed state, t_{kl} = stress tensor, f_k = body force, m_{kl} = couple stress, l_k = body couple, v_k = gyration velocity, ϵ = internal energy density, q_k = heat flux, h = heat source, η = entropy density, T = absolute temperature, ϵ_{klm} = permutation tensor, δ_{kl} = Kronecker delta, and χ_{kK} represents the microrotation with the constraints

$$(\chi_{kK})^{-1} = (\chi_{kK})^T. \quad (2.7)$$

We shall employ a rectangular Cartesian coordinate system and assume the summation convention over repeated indices. Indices following a comma represent partial differentiation and a superposed dot represents material differentiation, e. g.

$$v_{k,1} \equiv \partial v_k / \partial x_1, \quad x_{k,K} \equiv \partial x_k / \partial X_K, \quad \dot{f} \equiv \frac{\partial f}{\partial t} + f_{,k} v_k,$$

where the x_k denote the spatial position of a material point X_K at time t , both being referred to a rectangular frame of reference.

The kinematic tensors of deformation and their rates are given by [14]

$$\begin{aligned}
\nu_k &= -\frac{1}{2} \epsilon_{klm} \dot{\chi}_{lK} \chi_{mK} \quad , \quad C_{KL} = x_{k,K} \chi_{kL} \quad , \\
\Gamma_{KL} &= \frac{1}{2} \epsilon_{KMN} \chi_{kM,L} \chi_{kN} \quad , \quad \dot{C}_{KL} = (v_{l,k} - \epsilon_{klm} \nu_m) x_{k,K} \chi_{lL} \quad , \\
\dot{\Gamma}_{KL} &= \nu_{k,l} x_{l,L} \chi_{kK} . \quad (2.8)
\end{aligned}$$

3. Constitutive Equations

Eringen and Lee [32] derive the constitutive equations for micropolar viscoelastic smectic liquid crystals to which we shall introduce the variable $T_{,k}$ the temperature gradient, since we are considering heat-conducting smectic liquid crystals. The constitutive equations become:

$$\begin{aligned}
\psi &= \psi(C_{KL}, \dot{C}_{KL}, \Gamma_{KL}, \dot{\Gamma}_{KL}, T, T_{,K}) \quad , \\
\eta &= \eta(C_{KL}, \dot{C}_{KL}, \Gamma_{KL}, \dot{\Gamma}_{KL}, T, T_{,K}) \quad , \\
t_{kl} &= \frac{\rho}{\rho_0} T_{KL} (C_{KL}, \dot{C}_{KL}, \Gamma_{KL}, \dot{\Gamma}_{KL}, T, T_{,K}) x_{k,K} \chi_{lL} \quad , \\
m_{kl} &= \frac{\rho}{\rho_0} M_{KL} (C_{KL}, \dot{C}_{KL}, \Gamma_{KL}, \dot{\Gamma}_{KL}, T, T_{,K}) x_{k,L} \chi_{lL} \\
q_k &= \frac{\rho}{\rho_0} Q_K (C_{KL}, \dot{C}_{KL}, \Gamma_{KL}, \dot{\Gamma}_{KL}, T, T_{,K}) x_{k,K} \quad , \quad (3.1)
\end{aligned}$$

where $\psi = \mathcal{E} - T\eta \equiv$ free energy density and $q_k =$ heat flux vector.

We now eliminate h between (2.5) and (2.6) in order to obtain a generalized form of the Clausius-Duhem inequality:

$$-\rho(\dot{\psi} + \dot{T}\eta) + t_{kl}(v_{l,k} - \epsilon_{klm}v_m) + m_{kl}v_{l,k} + T^{-1}q_k T_{,k} \geq 0. \quad (3.2)$$

Using (3.1) in (3.2) we obtain

$$\begin{aligned} -\rho_0 \left[\left(\frac{\partial \psi}{\partial T} + \eta \right) \dot{T} + \frac{\partial \psi}{\partial T_{,K}} \dot{T}_{,K} + \frac{\partial \psi}{\partial \dot{C}_{KL}} \ddot{C}_{KL} + \frac{\partial \psi}{\partial \dot{\Gamma}_{KL}} \dot{\Gamma}_{KL} \right. \\ \left. + \frac{\partial \psi}{\partial \ddot{\Gamma}_{KL}} \ddot{\Gamma}_{KL} \right] + T_{KL} \dot{C}_{KL} + M_{KL} \dot{\Gamma}_{KL} + T^{-1}Q_K T_{,K} \geq 0. \quad (3.3) \end{aligned}$$

This inequality is postulated to be valid for all independent variations of T , \dot{T} , $T_{,K}$, $\dot{T}_{,K}$, \dot{C}_{KL} , $\dot{\Gamma}_{KL}$ and $\ddot{\Gamma}_{KL}$. Since (3.3) is linear in \dot{T} , $\dot{T}_{,K}$, \ddot{C}_{KL} , and $\ddot{\Gamma}_{KL}$ it cannot be maintained for all values of these quantities unless their coefficients vanish.

Thus, we obtain

$$\eta = -\frac{\partial \psi}{\partial T}, \quad \psi = \psi(C_{KL}, \Gamma_{KL}, T), \quad (3.4)$$

and (3.3) may be rewritten as

$$d^T_{KL} \dot{C}_{KL} + d^M_{KL} \dot{\Gamma}_{KL} + T^{-1}Q_K T_{,K} \geq 0 \quad (3.5)$$

where

$$d^T_{KL} = T_{KL} - e^T_{KL} = T_{KL} - \rho_0 \frac{\partial \psi}{\partial C_{KL}}, \quad (3.6)$$

$$d^M_{KL} = M_{KL} - e^M_{KL} = M_{KL} - \rho_0 \frac{\partial \psi}{\partial \Gamma_{KL}}. \quad (3.7)$$

We note that if d^T_{KL} , d^M_{KL} , and Q_K are continuous functions of \dot{C}_{KL} , $\dot{\Gamma}_{KL}$, and T, K , then from (3.5) it follows that

$$\dot{C}_{KL} = \dot{\Gamma}_{KL} = T, K = 0 \text{ implies } d^T_{KL} = d^M_{KL} = Q_K = 0 . \quad (3.8)$$

Thus we have proved:

Theorem: The general form of constitutive equations (3.1), of smectic liquid crystals is thermodynamically admissible if and only if (3.4) to (3.7) are satisfied.

4. Constitutively Linear Theory

In this section our main objective is to construct suitable constitutive equations for the smectic substance. To this end we employ two of the most important principles of constitutive theory, namely, the principle of objectivity and the principle of material symmetry [15] (since the smectic compounds that we are interested in possess uniaxial symmetry as well as a center of inversion and other possible symmetries). As a result, we find that our constitutive relations, when subjected to these principles undergo considerable simplification.

In order to obtain a set of linear constitutive equations, we expand ψ as follows:

$$\begin{aligned}
\rho_0 \psi = & \Sigma_0 + A_{KL} (C_{KL} - \delta_{KL}) + B_{KL} \Gamma_{KL} + \frac{1}{2} A_{KLMN} \{(C_{KL} - \delta_{KL}) \\
& \cdot (C_{MN} - \delta_{MN})\} + \frac{1}{2} B_{KLMN} \Gamma_{KL} \Gamma_{MN} + \{C_{KLMN} \\
& \cdot (C_{KL} - \delta_{KL}) \Gamma_{MN}\}, \tag{4.1}
\end{aligned}$$

where Σ_0 , A_{KL} , B_{KL} , A_{KLMN} and C_{KLMN} are material moduli which are functions of temperature only. The elastic parts of T_{KL} and M_{KL} are then obtained as

$$e T_{KL} = \frac{\partial(\rho_0 \psi)}{\partial C_{KL}} = A_{KL} + A_{KLMN} (C_{MN} - \delta_{MN}) + C_{KLMN} \Gamma_{MN}, \tag{4.2}$$

$$e M_{KL} = \frac{\partial(\rho_0 \psi)}{\partial \Gamma_{KL}} = B_{KL} + B_{KLMN} \Gamma_{MN} + C_{MNKL} (C_{MN} - \delta_{MN}).$$

Linear expressions for $d T_{KL}$, $d M_{KL}$, and Q_K can now be written in terms of their arguments C_{KL} , \dot{C}_{KL} , $\dot{\Gamma}_{KL}$ and T_K as follows:

$$\frac{\rho}{\rho_0} d T_{KL} = a_{KLMN} \dot{C}_{MN} + \alpha_{KLMN} \dot{\Gamma}_{MN}, \tag{4.3}$$

$$\frac{\rho}{\rho_0} d M_{KL} = b_{KLMN} \dot{\Gamma}_{MN} + \beta_{KLMN} \dot{C}_{MN} + \gamma_{KLMN} G_{MN},$$

$$\frac{\rho}{\rho_0} Q_{KL} = D_{KLMN} \dot{C}_{MN} + d_{KLMN} \dot{\Gamma}_{MN} + \delta_{KLMN} G_{MN},$$

where the material moduli \underline{a} , \underline{b} , \underline{D} , $\underline{\alpha}$, $\underline{\beta}$, \underline{d} , $\underline{\gamma}$ and $\underline{\delta}$ are functions of temperature only and where

$$G_{KL} = \frac{1}{2} T^{-1} \epsilon_{KLM} T, M, \quad (4.4)$$

$$Q_{KL} = \epsilon_{KLM} Q_M .$$

In equation (4.3)₁ the absence of the temperature gradient term is due to the theorem proved by Eringen [11] which states that an even order objective tensor can only be an even tensor function of odd order tensors. Hence the linear term $\underline{\nabla}T$, which is an odd order tensor does not appear in the expression for the second order tensor $\underline{d}T$. We also note that in the natural state, i.e. when $x_{k,K} = \delta_{kK} = x_{kK}$, the stress and the couple stress tensors must vanish. However, the hydrostatic pressure in a viscoelastic medium is given by the mean pressure, $\bar{p} = \frac{1}{3} T_{ii}$.

Material Symmetry

A smectic liquid crystal possesses uniaxial symmetry about its axis of orientation and a center of inversion (for simplicity we only consider those substances with center of inversion). With this in mind we examine the results of performing rigid rotations and inversion on equations (4.2) to (4.4) and requiring that these equations remain invariant under these rotations about the smectic

liquid crystal's axis of orientation and after inversions through its center of inversion. The orthogonal group which characterizes this symmetry has the following forms [29].

$$(S_{KL}) = \begin{bmatrix} \cos \varnothing & -\sin \varnothing & 0 \\ \sin \varnothing & \cos \varnothing & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \quad (4.5)$$

where \varnothing is the orientation, which is given by the angle made by the molecular axis of orientation with the axis of orientation of the liquid crystal. The principle of material invariance [15] states that under the group of transformations of the material frame of reference X_K ,

$$X_K^* = S_{KL} X_L \quad (4.6)$$

Equations (4.2) to (4.4) should remain invariant. Hence, we have the following restrictions on the material moduli:

$$(i) \quad C_{KLMN} = 0 = \alpha_{KLMN} = \beta_{KLMN} = D_{KLMN} \quad (4.7)$$

$$(ii) \quad \text{The non-vanishing components of } A_{KLMN} \text{ (} B_{KLMN} \text{) are} \quad (4.8)$$

$$A_{1111} = A_{2222}, \quad A_{1122} = A_{2211}, \quad A_{1133} = A_{2233} = A_{3311} = A_{3322}$$

$$A_{3333}, \quad A_{1313} = A_{2323}, \quad A_{3131} = A_{3232}, \quad A_{1331} = A_{3113} = A_{2332} =$$

$$A_{3223}, \quad A_{1212} = A_{2121}, \quad A_{1221} = A_{2112} = A_{1111} - A_{1122} - A_{1212} \cdot$$

(iii) The non-vanishing components of a_{KLMN} (b_{KLMN} , d_{KLMN} , δ_{KLMN}) are

$$a_{1111} = a_{2222}, a_{1122} = a_{2211}, a_{1133} = a_{2233}, a_{3311} = a_{3322},$$

$$a_{3333}, a_{1313}, = a_{2323}, a_{3131} = a_{3232}, a_{1331} = a_{2332}, \quad (4.9)$$

$$a_{3113} = a_{3223}, a_{1212} = a_{2121}, a_{1221} = a_{2112} = a_{1111} - a_{1122} - a_{1212}.$$

Using equations (4.7) to (4.9) and making use of the fact that the displacements and microrotations involved are assumed small, we obtain the linear constitutive equations for heat-conducting smectic liquid crystals with center of inversion.

$$t_{kl} = A_{klmn} (u_{n,m} - \epsilon_{mnp} \phi_p) + a_{klmn} (v_{n,m} - \epsilon_{mnp} \nu_p),$$

$$m_{kl} = B_{lkmn} \phi_{m,n} + b_{lkmn} \nu_{m,n} + \gamma_{lkmn} g_{mn}, \quad (4.10)$$

$$q_b = (d_{lkmn} \nu_{m,n} + \delta_{lkmn} g_{mn}) \frac{1}{2} \epsilon_{lkb},$$

where u_k and ϕ_k are, respectively, components of displacement and microrotation,

$$g_{mn} = \frac{1}{2} T^{-1} \epsilon_{mnq} T_{,q}, \quad (4.11)$$

and

$$A_{klmn} = A_{KLMN} \delta_{kK} \delta_{lL} \delta_{mM} \delta_{nN}. \quad (4.12)$$

Similar relations hold for a_{klmn} , B_{lkmn} , b_{lkmn} , etc.

We have already noted that the reference state of smectic liquid crystals is any state that leaves density, axis of orientation, and layered structure unchanged. For heat-conducting smectic liquid crystals we need only add that the reference state can be taken as any state in which the temperature field remains unchanged, besides the other field properties mentioned above.

5. Thermodynamic Restrictions

In this section we formulate the thermodynamic restrictions governing the material coefficients both in the reversible and irreversible parts of the constitutive equations.

To obtain the thermodynamic restrictions for the material coefficients present in the reversible parts of the stress and couple stress we substitute the appropriate terms from the right-hand sides of equations (4.10)₁ and (4.10)₂, namely

$$e_{kl}^t = A_{klmn} (u_{n,m} - \epsilon_{mnp} \phi_p) \quad (5.1)$$

and

$$e_{kl}^m = B_{lkmn} \phi_{m,n} \quad (5.2)$$

into the energy equation (2.5) with $q_{k,k} = 0$ and $h = 0$. Thus we obtain for the non-dissipative part of the energy production

$$\begin{aligned}
\rho_e \dot{\xi} &= A_{klmn} (u_{n,m} - \epsilon_{mnp} \phi_p) (v_{l,k} - \epsilon_{klr} v_r) + B_{lkmn} \phi_{m,n} v_{l,k} \\
&= A_{klmn} \{e_{mn} + \epsilon_{mnp} (r_p - \phi_p)\} \{d_{kl} + \epsilon_{klr} (\omega_r - v_r)\} + \\
&\quad + B_{lkmn} \phi_{m,n} v_{l,k}, \tag{5.3}
\end{aligned}$$

where

$$e_{kl} = \frac{1}{2} (u_{k,l} + u_{l,k}), \tag{5.4}$$

$$\dot{e}_{kl} = d_{kl} \equiv \frac{1}{2} (v_{k,l} + v_{l,k}), \tag{5.5}$$

$$r_{kl} = \frac{1}{2} (u_{k,l} - u_{l,k}), \tag{5.6}$$

$$r_k = \frac{1}{2} \epsilon_{klm} u_{m,l}, \tag{5.7}$$

$$\dot{r}_{kl} = w_{kl} \equiv \frac{1}{2} (v_{k,l} - v_{l,k}), \tag{5.8}$$

$$\dot{r}_k = \omega_k \equiv \frac{1}{2} \epsilon_{klm} v_{m,l}, \tag{5.9}$$

where (5.7) and (5.9) follow from the skew-symmetry of (5.6)

and (5.8), respectively. We require for a non-dissipative medium

that equation (5.3) be integrable, so we have, upon integration

of (5.3),

$$\begin{aligned}
\rho_e \xi - \rho_e \xi_0 &= \frac{1}{2} A_{klmn} \{e_{mn} + \epsilon_{mnp} (r_p - \phi_p)\} \{e_{kl} + \epsilon_{klq} (r_q - \phi_q)\} \\
&\quad + \frac{1}{2} B_{lkmn} \phi_{m,n} \phi_{l,k} \tag{5.10}
\end{aligned}$$

where ξ_0 is the energy density function for a given reference configuration. We note that without loss of generality, we may take $\xi_0 = 0$ in view of ξ_0 being referred to a fixed (but arbitrary) reference configuration, and thus (5.10) becomes

$$\begin{aligned} \rho \xi_e = & \frac{1}{2} A_{klmn} \{e_{mn} + \epsilon_{mnp} (r_p - \phi_p)\} \{e_{kl} + \epsilon_{klq} (r_q - \phi_q)\} \\ & + \frac{1}{2} B_{lkmn} \phi_{m,n} \phi_{l,k} . \end{aligned} \quad (5.11)$$

We require that $\rho \xi$ be nonnegative for all possible variations of the independent variables e_{mn} , $r_p - \phi_p$, and $\phi_{m,n}$. Thus if we take $r_p - \phi_p = 0$ and $\phi_{m,n} = 0$ we are led to the following restrictions on the nonvanishing components of A_{klmn} :

$$\begin{aligned} A_{1111} \geq 0, \quad A_{1111} \geq |A_{1122}|, \quad A_{3333} \geq 0, \quad A_{1111} + A_{3333} \geq \\ |A_{1133}|, \quad A_{1313} + A_{3131} \geq -2A_{1331}, \quad A_{1212} + A_{1221} \geq 0. \end{aligned} \quad (5.12)$$

Further restrictions on the A_{klmn} are obtained by letting $e_{kl} = 0$ and $\phi_{m,n} = 0$ and letting $e_{23} \neq 0$, $e_{13} \neq 0$, $r_1 - \phi_1 \neq 0$, $r_2 - \phi_2 \neq 0$, all other e_{kl} , $r_3 - \phi_3$, and $\phi_{m,n}$ vanish. Then

$$A_{1313} + A_{3131} \geq 2A_{1331}, \quad A_{1212} \geq A_{1221}, \quad A_{1313} \geq 0. \quad (5.13)$$

We note here that the first two inequalities in (5.13) and the last two in (5.12) combine to yield

$$A_{1313} + A_{3131} \geq |2A_{1331}|, \quad A_{1212} \geq |A_{1221}|. \quad (5.14)$$

Finally, if we set $e_{kl} = 0$ and $r_i - \phi_i = 0$ we obtain the following set of restrictions for the B_{lkmn} :

$$\begin{aligned} B_{1111} \geq 0, \quad B_{1212} \geq 0, \quad B_{1313} \geq 0, \quad B_{1212} \geq B_{1221}, \quad B_{1111} \geq B_{1122}, \\ B_{3131} B_{1313} \geq (B_{1331})^2, \quad B_{3333} (B_{1111} + B_{1122}) \geq (B_{1133})^2. \end{aligned} \quad (5.15)$$

The results in (5.15) were obtained by investigating the conditions under which the following quadratic form would be nonnegative.

$$\pi_{ij} \xi_i \xi_j \geq 0, \quad i, j = 1, 2, \dots, 9, \quad (5.16)$$

where

$$\xi_1 = \phi_{1,1}, \quad \xi_2 = \phi_{1,2}, \quad \xi_3 = \phi_{1,3}, \quad \xi_4 = \phi_{2,1}, \quad \xi_5 = \phi_{2,2}$$

$$\xi_6 = \phi_{2,3}, \quad \xi_7 = \phi_{3,1}, \quad \xi_8 = \phi_{3,2}, \quad \xi_9 = \phi_{3,3}$$

$$\pi_{11} = \pi_{55} = \frac{1}{2} B_{1111}, \quad \pi_{15} = \pi_{51} = B_{1122}, \quad \pi_{19} = \pi_{91} = \pi_{59} = \pi_{95} = \frac{1}{2} B_{1133}$$

$$\pi_{22} = \pi_{44} = \frac{1}{2} B_{1212}, \quad \pi_{24} = \pi_{42} = \frac{1}{2} B_{1221}, \quad \pi_{33} = \pi_{66} = \frac{1}{2} B_{1313},$$

(5.17)

$$\pi_{37} = \pi_{73} = \pi_{68} = \pi_{86} = \frac{1}{2} B_{1331}, \quad \pi_{77} = \pi_{88} = \frac{1}{2} B_{3131}, \quad \pi_{99} = \frac{1}{2} B_{3333},$$

All other π_{ij} vanish.

The relations (5.12) to (5.15) represent the thermodynamic restrictions on the nondissipative material moduli of heat-conducting smectic liquid crystals.

The A_{klmn} and B_{klmn} appearing in (5.12) to (5.15) are given by equations of the type (4.12). They depend on the choice of spatial coordinates x_k with respect to the material coordinates X_K in the natural state and hence are not material coefficients, whereas A_{KLMN} , B_{KLMN} are indeed the material coefficients. Thus for the relations (5.12) to (5.15) to represent the thermodynamic restrictions on the nondissipative material moduli we would have to choose the x_k coincident with the X_K for all $k, K = 1, 2, 3$ and thus A_{klmn} , B_{klmn} become the actual material coefficients.

We now turn our attention to obtaining restrictions on the dissipative material moduli. To this end we substitute equations (4.3) into the inequality (3.5) and use the restrictions on the material moduli due to symmetry properties of the smectic compound to obtain the following inequality:

$$\begin{aligned}
 & a_{KLMN} \dot{C}_{MN} \dot{C}_{KL} + b_{KLMN} \dot{\Gamma}_{MN} \dot{\Gamma}_{KL} + 2v_{KLMN} G_{MN} \dot{\Gamma}_{KL} \\
 & + \delta_{KLMN} G_{MN} G_{KL} \geq 0.
 \end{aligned}
 \tag{5.18}$$

where we have defined

$$v_{KLMN} = \frac{1}{2} (v_{KLMN} + d_{MNKL}).$$

This inequality must be satisfied for all possible processes involving $\dot{\underline{C}}$, $\dot{\underline{\Gamma}}$, and \underline{G} . In particular, for $\underline{G} = \underline{0}$ we have

$$\dot{c}_{KL} \dot{c}_{MN} a_{KLMN} + \dot{\Gamma}_{KL} \dot{\Gamma}_{MN} b_{KLMN} \geq 0. \quad (5.19)$$

Since $\dot{\underline{C}}$ and $\dot{\underline{\Gamma}}$ are independent variables, we can reduce separately each term on the left-hand side of (5.19) into a canonical form.

Since

$$\begin{aligned} \dot{c}_{KL} \dot{c}_{MN} a_{KLMN} &= \frac{1}{2} [\dot{c}_{KL} \dot{c}_{MN} a_{KLMN} + \dot{c}_{MN} \dot{c}_{KL} a_{MNKL}] \\ &= \dot{c}_{KL} \dot{c}_{MN} \frac{1}{2} [a_{KLMN} + a_{MNKL}] \end{aligned} \quad (5.20)$$

we let

$$S_{KLMN} = \frac{1}{2} [a_{KLMN} + a_{MNKL}] = S_{MNKL} \quad (5.21)$$

and set

$$\dot{c}_{11} = \xi_1, \quad \dot{c}_{12} = \xi_2, \quad \dot{c}_{13} = \xi_3, \quad \dot{c}_{21} = \xi_4, \quad \dot{c}_{22} = \xi_5, \quad \dot{c}_{23} = \xi_6,$$

$$\dot{c}_{31} = \xi_7, \quad \dot{c}_{32} = \xi_8, \quad \dot{c}_{33} = \xi_9,$$

and

$$\begin{aligned}
\pi_{11} = S_{1111} = \pi_{55}, \quad \pi_{22} = S_{1212} = \pi_{44}, \quad \pi_{33} = S_{1313} = \pi_{66}, \\
\pi_{37} = S_{1313} = \pi_{68}, \quad \pi_{91} = S_{3311} = \pi_{95}, \quad \pi_{99} = S_{3333}, \\
\pi_{19} = S_{1133} = \pi_{59}, \quad \pi_{15} = S_{1133} = \pi_{51}, \quad \pi_{24} = S_{1221} = \pi_{42}, \\
\pi_{73} = S_{3113} = \pi_{86}, \quad \pi_{77} = S_{3131} = \pi_{88}.
\end{aligned} \tag{5.22}$$

Thus when $\dot{\underline{\Gamma}} = \underline{0}$, we can write equation (5.19) as

$$\dot{C}_{KL} \dot{C}_{MN} a_{KLMN} = \pi_{ij} \xi_i \xi_j \geq 0 \text{ with } \pi_{ij} = \pi_{ji}. \tag{5.23}$$

We now seek the necessary and sufficient conditions for the quadratic form (5.23) to be positive definite. This is accomplished by examining the determinant of the matrix (π_{ij}) and requiring that the determinants $\Delta_1, \Delta_2, \dots, \Delta_9$, where $\Delta_k = \det(\pi_{ij})$, $i, j = 1, \dots, k$, be positive. Adjoining to these conditions the necessary and sufficient conditions required to satisfy $\pi_{ij} \xi_i \xi_j = 0$ for all ξ we have

$$a_{1111} \geq 0, \quad a_{1212} \geq 0, \quad a_{1313} \geq 0, \quad a_{1212} \geq a_{1221}, \quad a_{1111} \geq a_{1122},$$

$$a_{3131} a_{1313} \geq \frac{1}{4} (a_{1331} + a_{3113})^2, \text{ and } a_{3333} (a_{1111} + a_{1122}) \geq$$

$$\frac{1}{2} (a_{1133} + a_{3311})^2. \tag{5.24}$$

Next by taking $\dot{C}_{KL} = 0$ in (5.19) we can obtain similar results for the coefficients b_{KLMN} by an identical process. So we have

$$\begin{aligned} b_{1111} \geq 0, \quad b_{1212} \geq 0, \quad b_{1313} \geq 0, \quad b_{1212} \geq b_{1221}, \quad b_{1111} \geq b_{1122}, \\ b_{3131} a_{1313} \geq \frac{1}{4} (b_{1331} + b_{3113})^2, \quad \text{and } b_{3333} (b_{1111} + b_{1122}) \geq \\ \frac{1}{2} (b_{1133} + b_{3311})^2. \end{aligned} \quad (5.25)$$

Now consider $\dot{C}_{KL} = 0$ in (5.18); then we have

$$\dot{\Gamma}_{MN} \dot{\Gamma}_{KL} b_{KLMN} + 2G_{MN} \dot{\Gamma}_{KL} v_{KLMN} + G_{MN} G_{KL} \delta_{KLMN} \geq 0. \quad (5.26)$$

We wish to write this as a quadratic inequality of the form

$$A_{ij} \mu_i \mu_j \geq 0, \quad i, j, = 1, 2, \dots, 18; \quad A_{ij} = A_{ji}. \quad (5.27)$$

To this end we make the following identifications:

$$\begin{aligned} \mu_1 &= \dot{\Gamma}_{11}, \quad \mu_2 = \dot{\Gamma}_{12}, \quad \mu_3 = \dot{\Gamma}_{13}, \\ \mu_4 &= \dot{\Gamma}_{21}, \quad \mu_5 = \dot{\Gamma}_{22}, \quad \mu_6 = \dot{\Gamma}_{23}, \\ \mu_7 &= \dot{\Gamma}_{31}, \quad \mu_8 = \dot{\Gamma}_{32}, \quad \mu_9 = \dot{\Gamma}_{33}, \\ \mu_{10} &= G_{11}, \quad \mu_{11} = G_{12}, \quad \mu_{12} = G_{13}, \\ \mu_{13} &= G_{21}, \quad \mu_{14} = G_{22}, \quad \mu_{15} = G_{23}, \\ \mu_{16} &= G_{31}, \quad \mu_{17} = G_{32}, \quad \mu_{18} = G_{33}, \end{aligned} \quad (5.28)$$

$$t_{KLMN} = \frac{1}{2} (b_{KLMN} + b_{MNKL}) = t_{MNKL},$$

$$u_{KLMN} = \frac{1}{2} (\delta_{KLMN} + \delta_{MNKL}) = u_{MNKL},$$

$$A_{11} = A_{55} = t_{1111}, \quad A_{15} = A_{51} = t_{1122}, \quad A_{19} = A_{91} = A_{59} = A_{95} = t_{1133},$$

$$A_{1,10} = A_{10,1} = A_{5,14} = A_{14,5} = v_{1111},$$

$$A_{1,14} = A_{14,1} = A_{10,5} = A_{5,10} = v_{1122},$$

$$A_{1,18} = A_{18,1} = A_{18,5} = A_{5,18} = v_{1133},$$

$$A_{22} = A_{44} = t_{1212}, \quad A_{24} = A_{42} = t_{1221},$$

$$A_{2,11} = A_{11,2} = A_{13,4} = A_{4,13} = v_{1212},$$

$$A_{2,13} = A_{13,2} = A_{4,11} = A_{11,4} = v_{1221},$$

$$A_{33} = A_{66} = t_{1313}, \quad A_{3,12} = A_{12,3} = A_{6,15} = A_{15,6} = v_{1313},$$

$$A_{3,16} = A_{16,3} = A_{6,17} = A_{17,6} = v_{1331}, \quad A_{68} = A_{86} = A_{37} = A_{73} = t_{1331},$$

$$A_{77} = A_{88} = t_{3131}, \quad A_{7,12} = A_{12,7} = A_{8,15} = A_{15,8} = v_{3113},$$

$$A_{7,16} = A_{16,7} = A_{8,17} = A_{17,8} = v_{3131},$$

$$A_{99} = t_{3333}, \quad A_{9,10} = A_{10,9} = A_{9,14} = A_{14,9} = v_{3311},$$

$$A_{9,18} = A_{18,9} = v_{3333}, \quad A_{10,10} = A_{14,14} = u_{1111},$$

(5.30)

$$A_{10,18} = A_{18,10} = A_{14,18} = A_{18,14} = u_{1133} ,$$

$$A_{11,11} = A_{13,13} = u_{1212} , A_{11,13} = A_{13,11} = u_{1221} ,$$

$$A_{12,12} = A_{15,15} = u_{1313} , A_{12,16} = A_{16,12} = A_{15,17} = u_{1331} ,$$

$$A_{16,16} = A_{17,17} = u_{3131} , A_{18,18} = u_{3333} ,$$

and all other $A_{ij} = 0$.

By using equations (5.28) to (5.30) we can obtain the appropriate canonical form (5.27) for the inequality in (5.26). We then employ the exchange step method [40] in order to obtain the necessary and sufficient conditions for (5.27) to be positive definite and adjoin these to the necessary and sufficient conditions for

$$A_{ij} \xi_i \xi_j = 0 , \quad i, j = 1, \dots, 18 , \quad (5.31)$$

and obtain finally the following restrictions on the material moduli:

$$(\delta_{1212} b_{1212} - v_{1212}^2) (b_{1212}^2 - b_{1221}^2) \geq (v_{1221} b_{1212} - b_{1221} v_{1212})^2 ,$$

$$(\delta_{1313} b_{1313} - v_{1313}^2) [4b_{1313} b_{3131} - (b_{1331} + b_{3113})^2] \geq$$

$$[2v_{1313} b_{1313} - v_{1313} (b_{1331} + b_{3113})]^2 , \quad (5.32)$$

$$[S_{16,16}^{(1)} S_{7,7}^{(1)} - (S_{7,16}^{(1)})^2] [S_{12,12}^{(1)} S_{7,7}^{(1)} - (S_{7,12}^{(1)})^2] \geq$$

$$\begin{aligned}
& [S_{12,16}^{(1)} S_{77}^{(1)} - S_{7,12}^{(1)} S_{7,16}^{(1)}]^2, \quad S_{13,13}^{(2)} S_{11,11}^{(2)} \geq (S_{11,13}^{(2)})^2, \\
& S_{10,10}^{(3)} S_{9,9}^{(3)} \geq (S_{9,10}^{(3)})^2, \quad \{ [S_{14,14}^{(3)} S_{9,9}^{(3)} - (S_{9,14}^{(3)})^2] \\
& \times [S_{10,10}^{(3)} S_{9,9}^{(3)} - (S_{9,10}^{(3)})^2] \} \geq [S_{10,14}^{(3)} S_{9,9}^{(3)} - S_{9,10}^{(3)} S_{9,14}^{(3)}]^2, \\
& [S_{17,17}^{(4)} S_{8,8}^{(4)} - (S_{8,17}^{(4)})^2] [S_{15,15}^{(4)} S_{8,8}^{(4)} - (S_{8,15}^{(4)})^2] \geq \\
& [S_{15,17}^{(4)} S_{8,8}^{(4)} - S_{8,15}^{(4)} S_{8,17}^{(4)}]^2, \quad \{ [S_{18,18}^{(5)} S_{10,10}^{(5)} - (S_{10,18}^{(5)})^2] \\
& \times [S_{14,14}^{(5)} S_{10,10}^{(5)} - (S_{10,14}^{(5)})^2] \} \geq [S_{14,18}^{(5)} S_{10,10}^{(5)} - S_{10,14}^{(5)} S_{10,18}^{(5)}]^2
\end{aligned}$$

where

$$S_{p,q}^{(1)} = A_{pq} A_{3,3} - A_{3,p} A_{3,q} \quad \text{for } p, q = 7, 12, 16;$$

$$\begin{aligned}
S_{p,q}^{(2)} &= M(A_{p,q} A_{2,2} - A_{2,p} A_{2,q}) - (A_{4,p} A_{2,2} - A_{2,4} A_{2,p}) \\
&\quad \times (A_{4,q} A_{2,2} - A_{2,4} A_{2,q}) \quad \text{for } p, q = 11, 13;
\end{aligned}$$

$$\begin{aligned}
S_{p,q}^{(3)} &= N(A_{p,q} A_{11} - A_{1,p} A_{1,q}) - (A_{s,p} A_{1,1} - A_{1,s} A_{1,p}) \\
&\quad \times (A_{s,q} A_{1,1} - A_{1,s} A_{1,q}) \quad \text{for } p, q = 9, 10, 14, 18;
\end{aligned}$$

$$S_{p,q}^{(4)} = A_{p,q} A_{6,6} - A_{6,p} A_{6,q} \quad \text{for } p, q = 8, 15, 17; \quad (5.33)$$

$$S_{p,q}^{(5)} = [S_{p,q}^{(3)} S_{9,9}^{(3)} - S_{9,p}^{(3)} S_{p,q}^{(3)}] / N^2 A_{11}^2 \quad \text{for } p, q = 10, 14, 18;$$

$$M = b_{1212}^2 - b_{1221}^2 \quad \text{and} \quad N = b_{1111}^2 - b_{1122}^2.$$

Thus we have proved the following theorem:

Theorem. The constitutive equations (4.10) are thermodynamically admissible if and only if the restrictions (5.12) to (5.15) and (5.24) to (5.33) hold for the elastic and dissipative material moduli, respectively.

We note here that the thermodynamic restrictions for the dissipative material coefficients are similar to those obtained for heat-conducting nematic liquid crystals [38]. This is physically reasonable since both the smectics and nematics are made up of rod-like or cigar-shaped molecules with an overall parallel alignment. This similarity manifests itself in the expression for the dissipative stress, dissipative couple stress, and the temperature gradient bivector; all of which appear in the Clausius-Duhem inequality from which the thermodynamic restrictions on dissipative material moduli are obtained.

The difference between the nematics and smectics, that is the layered structure which is present in smectics but not in nematics, is featured in the elastic parts of the stress and couple stress. Here we have determined the thermodynamic restrictions on these material moduli characterizing the elastic properties of the smectics.

In the next chapter we shall examine the flow of smectic liquid crystals between two concentric rotating cylinders to determine the

shear-rate dependence of viscosity and the effects of heat-conduction as well as those due to boundaries on the orientation field. We shall also see that the thermodynamic restrictions obtained in this chapter are necessary to determine the solution of the viscometric flow problem in Chapter III.

III. ORIENTATIONAL EFFECTS IN HEAT-CONDUCTING SMECTIC LIQUID CRYSTALS

1. Preliminary Remarks

As noted in Chapter I there is a relative scarcity of work, both theoretical and experimental, concerning the rheological behavior of smectic liquid crystals. In this chapter we employ the theory developed in Chapter II to investigate a viscometric flow of heat-conducting smectic liquid crystals. Besides the contribution to the theoretical knowledge concerning the behavior of smectic liquid crystals which we make here, we are also providing a stimulus for further experimental work in order to verify our theoretical predictions arising from the solution of the problem and subsequent comparison with the admittedly scant experimental data.

Thus we consider the flow of heat-conducting smectic liquid crystals between two rotating coaxial circular cylinders. We obtain analytical expressions for the velocity, microgyration, orientation, and heat-conduction vector fields. In addition we derive an expression for the apparent coefficient of viscosity and show that it depends on the shear-rate. Comparison with experimental data is undertaken in order to confirm our theory and to

identify and evaluate certain material coefficients of the theory so that the experimentalist can employ the theory with confidence.

2. Basic Equations

The equations of motion of a general micropolar continuum [14] and constitutive equations for heat-conducting smectic liquid crystals (II. 4.10) reduce in the linear incompressible case to the following:

Equation of Continuity

$$v_{k,k} = 0 \quad , \quad (2.1)$$

Microinertia

$$\frac{\partial i_{kl}}{\partial t} + i_{kl,m} v_m + e_{lmr} v_r i_{km} + e_{kmr} v_r i_{ml} = 0 \quad , \quad (2.2)$$

Linear Momentum

$$t_{ji,j} + \rho (f_i - \dot{v}_i) = 0 \quad , \quad (2.3)$$

Moment of Momentum

$$m_{ji,j} + e_{ikl} t_{kl} + \rho (l_i - \dot{\sigma}_i) = 0 \quad , \quad (2.4)$$

Conservation of Energy

$$\rho \dot{\epsilon} = t_{kl} (v_{l,k} - e_{klm} v_m) + m_{kl} v_{l,k} + q_{k,k} + \rho h \quad , \quad (2.5)$$

Stress Constitutive Equation

$$t_{kl} = A_{klmn} (u_{n,m} - e_{mnp} \phi_p) + a_{klmn} (v_{n,m} - e_{mnp} v_p), \quad (2.6)$$

Couple Stress Equation

$$m_{kl} = B_{klmn} \phi_{m,n} + b_{klmn} v_{m,n} + \gamma_{klmn} g_{mn}, \quad (2.7)$$

Heat-Flux Vector

$$q_b = (d_{lkmn} v_{m,n} + \delta_{lkmn} g_{mn}) \frac{1}{2} e_{lkb}, \quad (2.8)$$

where t_{ij} , m_{ij} , and q_b are respectively, the stress tensor, the moment stress tensor and the heat-flux vector, f_i is the body force, v_i is the velocity vector, u_n is the displacement vector, \mathcal{E} is the internal energy density per unit mass, h is the heat supply per unit mass, l_i is the body couple, ρ is the mass density, ϕ_m is the angular microrotation vector, $v_r = \dot{\phi}_r$ is the microgyration vector and g_{mn} given by

$$g_{mn} = \frac{1}{2} T^{-1} e_{mnq} T, q \quad (2.9)$$

is the temperature gradient bivector, T is the temperature, e_{mnq} is the alternating tensor and the spin-inertia $\dot{\sigma}_i$ is given by

$$\dot{\sigma}_i = (i_{mn} \delta_{ij} - i_{ij}) \dot{v}_j - e_{kij} i_{km} v_m v_j \quad (2.10)$$

in which $i_{kl} = i_{lk}$ is the microinertia tensor satisfying (2.2) and δ_{ij} is the Kronecker delta. The generalized Clausius-Duhem inequality has been derived and used in Chapter II of this thesis to obtain the thermodynamic restrictions governing the constitutive equations.

Throughout this chapter we employ spatial and material rectangular coordinate systems x_i ($i = 1, 2, 3$) and X_I ($I = 1, 2, 3$). The coefficients a_{klmn} in the spatial system are related to the corresponding a_{KLMN} in the material coordinate system by the relation

$$a_{klmn} = a_{KLMN} \delta_{kK} \delta_{lL} \delta_{mM} \delta_{nN} \quad (2.11)$$

and are functions of temperature only, δ_{kK} being the direction cosines between the spatial and material coordinates. Similar statements as (2.11) hold for the coefficients A_{klmn} , b_{klmn} , B_{klmn} , γ_{klmn} , d_{klmn} and δ_{klmn} . Thermodynamic and material symmetry restrictions on these material moduli have been derived previously in Chapter II. The motion of a material point X_K is given by $x_k = x_k(X_K, t)$ and χ_{kK} denotes the micromotion.

3. Formulation of the Problem

We consider a steady Couette flow of heat-conducting smectic liquid crystals between two long concentric circular cylinders of

radii r_1 and r_2 ($r_1 < r_2$) which are rotating with constant angular velocities Ω_1 and Ω_2 respectively. We choose cylindrical polar coordinates (r, θ, z) such that the z -axis coincides with the long axis of the cylinders. Let (x, y, z) and (X, Y, Z) represent the spatial and material coordinates. We assume that the material and spatial frames are coincident and hence in view of (2.17) the coefficients a_{klmn} , etc., become the actual material coefficients. In addition the material frame is chosen in such a way that the X -axis coincides with the axis of orientation of the liquid crystal. The velocity field and the microgyration velocity field are given by

$$v_r = 0, \quad v_\theta = r w(r, \theta), \quad v_z = 0; \quad (3.1)$$

$$v_r = 0, \quad v_\theta = 0, \quad v_z = \nu(r, \theta). \quad (3.2)$$

In the velocity field, the z -dependence may be excluded due to infinite channel length and noting that the equation of continuity in cylindrical coordinates has the form

$$\frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{r \partial \theta} + \frac{v_r}{r} = 0,$$

which reduces to $0 = \partial w / \partial \theta$ by use of (3.1). Thus we have w as a function of r only. The temperature field is $T = T(r)$ and we allow arbitrary spatial dependence for the microgyration velocity ν to depend on both r and θ .

The micromotion is given by

$$\chi_{kK} = \begin{pmatrix} \cos \vartheta & -\sin \vartheta & 0 \\ \sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.3)$$

where

$$\vartheta = (0, 0, \vartheta(r, \theta)) \quad (3.4)$$

is an axial vector normal to the plane of motion (i. e. normal to the $r\theta$ -plane). Finally, the displacement field u is given by

$$u_r = 0, \quad u_\theta = u(r, \theta), \quad u_z = 0. \quad (3.5)$$

In cartesian coordinates (3.1), (3.2), and (3.5) become

$$v_x = rw(r)\sin \theta, \quad v_y = rw(r)\cos \theta, \quad v_z = 0; \quad (3.6)$$

$$\dot{v}_x = 0, \quad \dot{v}_y = 0, \quad \dot{v}_z = \dot{\vartheta} = \dot{v}(r, \theta); \quad (3.7)$$

$$u_x = -u \sin \theta, \quad u_y = u \cos \theta, \quad u_z = 0. \quad (3.8)$$

We shall now substitute (3.4), (3.6), (3.7), and (3.8) into equations (2.6) - (2.8) and make use of the material symmetry conditions (II. 4.8) to (II. 4.9) on the A_{klmn} , B_{klmn} , a_{klmn} , b_{klmn} , γ_{klmn} , d_{klmn} , and δ_{klmn} , in order to obtain the following results:

The stress components are:

$$t_{xx} = t_{11} = (A_{1111} - A_{1122}) \sin \theta \cos \theta \left[\frac{u}{r} - \frac{\partial u}{\partial r} \right] + (A_{1111} \sin^2 \theta + A_{1122} \cos^2 \theta) \frac{\partial u}{r \partial \theta} - \frac{1}{2} (a_{1111} - a_{1122}) r w' \sin 2 \theta ,$$

$$t_{yy} = t_{22} = - (A_{1111} - A_{1122}) \sin \theta \cos \theta \left[\frac{u}{r} - \frac{\partial u}{\partial r} \right] + (A_{1111} \cos^2 \theta + A_{1122} \sin^2 \theta) \frac{\partial u}{r \partial \theta} + \frac{1}{2} (a_{1111} - a_{1122}) r w' \sin 2 \theta ,$$

$$t_{xy} = t_{12} = (A_{1212} \cos^2 \theta - A_{1221} \sin^2 \theta) \frac{\partial u}{\partial r} + (A_{1212} \sin^2 \theta - A_{1221} \cos^2 \theta) \frac{u}{r} - (A_{1212} + A_{1221}) \cos \theta \sin \theta \left(\frac{\partial u}{r \partial \theta} \right) - (A_{1212} - A_{1221}) \phi + r w' \left[\frac{1}{2} (a_{1212} - a_{1221}) + \frac{1}{2} (a_{1111} - a_{1122}) \cos 2 \theta \right] + (a_{1212} - a_{1221}) (w - v),$$

$$t_{yx} = t_{21} = (A_{1221} \cos^2 \theta - A_{1212} \sin^2 \theta) \frac{\partial u}{\partial r} + (A_{1221} \sin^2 \theta - A_{1212} \cos^2 \theta) \frac{u}{r} - (A_{1212} + A_{1221}) \cos \theta \sin \theta \left(\frac{\partial u}{r \partial \theta} \right) + (A_{1212} - A_{1221}) \phi + r w' \left[\frac{1}{2} (a_{1111} - a_{1122}) \cos 2 \theta - \frac{1}{2} (a_{1212} - a_{1221}) \right] - (a_{1212} - a_{1221}) (w - v),$$

$$t_{zz} = t_{33} = A_{3311} \left(\frac{\partial u}{r \partial \theta} \right) ,$$

$$t_{xz} = t_{13} = 0 = t_{zx} = t_{yz} = t_{zy} . \quad (3.9)$$

The moment-stress components are:

$$m_{xx} = m_{11} = 0 , m_{yy} = m_{22} = 0 , m_{zz} = m_{33} = 0 , m_{xy} = m_{12} = 0 ,$$

$$m_{yx} = m_{21} = 0 , \quad (3.10)$$

$$m_{zx} = m_{31} = B_{3131} \phi_{,x} + b_{3131} \nu_{,x} + \frac{1}{2} T^{-1} (\gamma_{1331} - \gamma_{1313}) T_{,y} ,$$

$$m_{xz} = m_{13} = B_{3131} \phi_{,x} + b_{3131} \nu_{,x} + \frac{1}{2} T^{-1} (\gamma_{3131} - \gamma_{3113}) T_{,y} ,$$

$$m_{zy} = m_{32} = B_{2332} \phi_{,y} + b_{2332} \nu_{,y} + \frac{1}{2} T^{-1} (\gamma_{2323} - \gamma_{2332}) T_{,x} ,$$

$$m_{yz} = m_{23} = B_{3232} \phi_{,y} + b_{3232} \nu_{,y} + \frac{1}{2} T^{-1} (\gamma_{3223} - \gamma_{3232}) T_{,x} .$$

The heat-flux vector components are:

$$q_x = q_1 = \frac{1}{2} (d_{2332} - d_{3232}) \nu_{,y} + \frac{1}{4} T^{-1} (\delta_{3232} + \delta_{2323} - \delta_{3223} - \delta_{2332}) T_{,x} ,$$

$$q_y = q_2 = -\frac{1}{2} (d_{2332} - d_{3232}) \nu_{,x} + \frac{1}{4} T^{-1} (\delta_{3232} + \delta_{2323} - \delta_{3223} - \delta_{2332}) T_{,y} , \quad (3.11)$$

$$q_z = q_3 = 0 .$$

We now define the following parameters involving the material coefficients and temperature:

$$\begin{aligned}
 k_1 &= \frac{1}{2} (a_{1111} - a_{1122}) = k_4, \quad k_2 = a_{3311}, \quad k_3 = \frac{1}{2} (a_{1212} - a_{1221}), \\
 h_1 &= A_{1111}, \quad h_2 = A_{1122}, \quad h_3 = A_{1133}, \quad h_4 = A_{3333}, \quad h_5 = A_{1212}, \\
 h_6 &= A_{1221}, \quad h_7 = \frac{1}{2} (A_{1111} - A_{1122}) = \frac{1}{2} (A_{1212} + A_{1221}), \quad h_8 = A_{1313}, \\
 h_9 &= A_{3113} = A_{1331}, \quad h_{10} = (A_{1212} - A_{1221}), \quad (3.12) \\
 k_5 &= B_{1331}, \quad k_6 = b_{1331}, \quad k_7 = \frac{1}{2} T^{-1} (\gamma_{1331} - \gamma_{3113}), \\
 k_{11} &= \frac{1}{2} (d_{2332} - d_{3232}), \quad k_{12} = \frac{1}{4} T^{-1} (\delta_{3232} + \delta_{2323} - \delta_{3223} - \delta_{2332}).
 \end{aligned}$$

On using (3.12) in (3.9) to (3.11) the stress, the couple stress and the heat-flux components reduce to the following simplified forms:

$$\begin{aligned}
 t_{xx} = t_{11} &= 2h_7 \sin \theta \cos \theta \left(\frac{u}{r} - \frac{\partial u}{\partial r} \right) + (h_1 \sin^2 \theta + h_2 \cos^2 \theta) \frac{\partial u}{r \partial \theta} \\
 &\quad - k_1 r w' \sin 2 \theta,
 \end{aligned}$$

$$\begin{aligned}
 t_{yy} = t_{22} &= -2h_7 \sin \theta \cos \theta \left(\frac{u}{r} - \frac{\partial u}{\partial r} \right) + (h_1 \cos^2 \theta + h_2 \sin^2 \theta) \frac{\partial u}{r \partial \theta} \\
 &\quad + k_1 r w' \sin 2 \theta,
 \end{aligned}$$

$$\begin{aligned}
t_{xy} = t_{12} &= (h_5 \cos^2 \theta - h_6 \sin^2 \theta) \frac{\partial u}{\partial r} + (h_5 \sin^2 \theta - h_6 \cos^2 \theta) \frac{u}{r} \\
&\quad - 2h_7 \sin \theta \cos \theta \left(\frac{\partial u}{r \partial \theta} \right) - h_{10} \phi + rw' (k_3 + k_1 \cos 2\theta) \\
&\quad + 2k_3 (w - v), \tag{3.13}
\end{aligned}$$

$$\begin{aligned}
t_{yx} = t_{21} &= (h_6 \cos^2 \theta - h_5 \sin^2 \theta) \frac{\partial u}{\partial r} + (h_6 \sin^2 \theta - h_5 \cos^2 \theta) \frac{u}{r} \\
&\quad - 2h_7 \sin \theta \cos \theta \left(\frac{\partial u}{r \partial \theta} \right) + h_{10} \phi + rw' (k_1 \cos 2\theta - k_3) \\
&\quad - 2k_3 (w - v),
\end{aligned}$$

$$t_{xz} = t_{13} = 0 = t_{zx} = t_{yz} = t_{zy},$$

$$t_{zz} = t_{33} = h_3 \frac{\partial u}{r \partial \theta}.$$

$$m_{xx} = 0 = m_{yy} = m_{zz} = m_{xy} = m_{yx},$$

$$m_{zx} = m_{31} = k_5 \phi_{,x} + k_6 v_{,x} + k_7 T_{,y},$$

$$m_{xz} = m_{13} = k_8 \phi_{,x} + k_9 v_{,x} + k_{10} T_{,y}, \tag{3.14}$$

$$m_{zy} = m_{32} = k_5 \phi_{,y} + k_6 v_{,y} - k_7 T_{,x},$$

$$m_{yz} = m_{23} = k_8 \phi_{,y} + k_9 v_{,y} - k_{10} T_{,x}.$$

$$q_x = q_1 = k_{11} v_{,y} + k_{12} T_{,x},$$

$$q_y = q_2 = -k_{11} v_{,x} + k_{12} T_{,y} , \quad (3.15)$$

$$q_z = q_3 = 0 .$$

Equations (3.13) - (3.15) are now substituted into the balance equations of motion and energy (2.3) - (2.5) to obtain the field equations governing $w(r)$, $T(r)$, $v(r, \theta)$, and $u(r, \theta)$ (in the absence of external body forces, body couples, and heat sources):

$$\begin{aligned} h_5 \left(\frac{u}{r^2} - \frac{\partial u}{r \partial r} - \frac{\partial^2 u}{\partial r^2} \right) - h_1 \frac{\partial^2 u}{r^2 \partial \theta^2} - (h_1 + h_5) \cot \theta \left(\frac{\partial u}{r^2 \partial \theta} \right) \\ + (h_1 - h_5) \left(\frac{\partial^2 u}{r \partial \theta \partial r} \right) \cot \theta - (k_1 + k_3) (r w'' + 3 w') \\ + 2 k_3 \left(\frac{\partial v}{\partial r} + \frac{\cot \theta}{r} \frac{\partial v}{\partial \theta} \right) = 0 , \end{aligned} \quad (3.16)$$

$$\begin{aligned} -h_5 \left(\frac{u}{r^2} - \frac{\partial u}{r \partial r} - \frac{\partial^2 u}{\partial r^2} \right) + h_1 \frac{\partial^2 u}{r^2 \partial \theta^2} - (h_1 + h_5) \tan \theta \left(\frac{\partial u}{r^2 \partial \theta} \right) \\ + (h_1 - h_5) \tan \theta \left(\frac{\partial^2 u}{r \partial \theta \partial r} \right) + (k_1 + k_3) (r w'' - 3 w') - 2 k_3 \left(\frac{\partial v}{\partial r} - \frac{\tan \theta}{r} \frac{\partial v}{\partial \theta} \right) \\ = 0 , \end{aligned} \quad (3.17)$$

$$\begin{aligned} B_{3131} \nabla^2 \phi + b_{3131} \nabla^2 v + 2k_3 (r w' + 2w) - 4k_3 v - \rho j \dot{v} \\ - 4h_7 \phi + h_{10} \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) = 0 , \end{aligned} \quad (3.18)$$

$$\nabla^2 T = 0, \quad (3.19)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \quad j = i_{11} - i_{22}. \quad (3.20)$$

In order for a solution set to exist for equations (3.16) to (3.19) taking into account the micromotions of local substructures under appropriate boundary conditions it will be necessary to have

$$k_3 = \frac{1}{2} (a_{1212} - a_{1221}) \neq 0 \text{ and } a_{1212} \neq 0 \quad (3.21)$$

which is permitted by the thermodynamic restrictions on a_{1212} and a_{1221} obtained in Chapter II, that is

$$a_{1212} \geq 0, \quad a_{1212} - a_{1221} \geq 0. \quad (3.22)$$

Therefore we require $a_{1212} - a_{1221} > 0$ and $a_{1212} > 0$.

The coefficient B_{3131} is subject to the thermodynamic restrictions derived in Chapter II, namely, it is nonnegative, but we note that its vanishing would eliminate part of the explicit expression for microrotation ϕ in equation (3.18). In view of this we should require that $B_{3131} = 0$ and $h_7 = \frac{1}{2} (A_{1111} - A_{1122}) = \frac{1}{2} (A_{1212} + A_{1221}) = 0$ not occur simultaneously. Thus we must consider three cases: (i) $B_{3131} \neq 0$ and $A_{1212} \neq A_{1221}$,

(ii) $B_{3131} \neq 0$ and $A_{1212} = -A_{1221}$, and (iii) $B_{3131} = 0$ and $A_{1212} \neq -A_{1221}$. We note also that the coefficient b_{3131} appearing in equation (3.18) is required to be nonnegative by the thermodynamic restrictions in Chapter II.

As for equation (3.19) governing the temperature field we now show that it can be obtained in the following manner from the energy equation (2.5). The term $\rho \dot{\epsilon}$ on the left-hand side of equation (2.5) has the form

$$\rho \dot{\epsilon} = \rho (\dot{\psi} + \dot{\eta} T + \eta \dot{T}) \simeq \rho \dot{\psi}_0(\rho^{-1}, T) + \rho \dot{\eta} T = 0 \quad (3.23)$$

in the linear micropolar theory which we are using. Here we have used the fact that in the steady state case owing to the geometry of the channel the material time derivative denoted by D/Dt or by a superposed dot is given by

$$\frac{D}{Dt} = w(r) \frac{\partial}{\partial \theta} = w \frac{\partial}{\partial \theta}, \quad \frac{D \phi(r, \theta)}{Dt} = \dot{\phi} = w \frac{\partial \phi}{\partial \theta}, \quad \dot{\nu}(r, \theta) = w \frac{\partial \nu}{\partial \theta} \simeq 0. \quad (3.24)$$

The last equation in (3.24) is due to the fact that in the linear theory w and ν are taken to be first order, small quantities. Also, from the linear theory we have

$$\begin{aligned} \dot{\psi} &= \dot{\psi}_0(\rho^{-1}, T) + \text{Higher order terms} \simeq \dot{\psi}(\rho^{-1}, T) \\ &= w \frac{\partial}{\partial \theta} [\psi_0(\rho^{-1}, T(r))] = 0 \end{aligned} \quad (3.25)$$

and

$$\dot{\eta} = \frac{D}{Dt} \left(-\frac{\partial \psi}{\partial t} \right) \simeq w \frac{\partial}{\partial \theta} [\psi_0(\rho^{-1}, T(r))] + \text{Higher order terms} = 0, \quad (3.26)$$

where we make use of the fact that ρ^{-1} is constant due to the incompressibility condition and $T = T(r)$. The right-hand side of equation (2.5) contains only one term of the first order, $q_{k,k}$, the other terms may be neglected in comparison with w , v , and ∇T in the linear theory. Use of equations (3.15), and (3.23) to (3.26) in the energy equation will then yield (3.19).

4. Solution of the Field Equations

From equations (3.16) and (3.17) we obtain the following two equations

$$2k_3 \frac{\partial v}{\partial \theta} = (h_1 + h_5) \frac{\partial u}{r \partial \theta} + (h_5 - h_1) \frac{\partial^2 u}{\partial r \partial \theta} \quad (4.1)$$

and

$$2k_3 \frac{\partial v}{\partial r} = a_{1212} (rw'' + 3w') + h_5 \frac{\partial}{\partial r} \left(\frac{u}{r} + \frac{\partial u}{\partial r} \right) + h_1 \frac{\partial^2 u}{r^2 \partial \theta^2} \quad (4.2)$$

Differentiation of both sides of equation (4.1) with respect to r and of equation (4.2) with respect to θ and the use of the facts that $w = w(r)$ and the interchange of partial derivatives is allowed lead to the following differential equation for the displacement u .

$$h_1 \left[\frac{\partial^2 \tau}{\partial r^2} - \frac{\partial}{\partial r} \left(\frac{\tau}{r} \right) + \frac{\partial^2 \tau}{r^2 \partial \theta^2} \right] = 0 \quad (4.3)$$

where

$$\tau = \frac{\partial u}{\partial \theta} \quad . \quad (4.4)$$

Equation (4.3) requires that $h_1 = A_{1111} \neq 0$ since we require a nontrivial solution to our problem and that the following differential equation be satisfied by τ .

$$\frac{\partial^2 \tau}{\partial r^2} - \frac{\partial}{\partial r} \left(\frac{\tau}{r} \right) + \frac{\partial^2 \tau}{r^2 \partial \theta^2} = 0 \quad (4.5)$$

Equation (4.5) may be solved using the technique of separation of variables and the requirement that τ must be real-valued leads to

$$\tau = Lr \quad (4.6)$$

where L is a constant. We now make use of the following relation between the displacement $u(r, \theta)$ and the tangential velocity $v_\theta(r) = rw(r)$, namely

$$\frac{Du}{Dt} = v_\theta \quad . \quad (4.7)$$

However, since we are using the constitutively linear theory we have $D/Dt = w \frac{\partial}{\partial \theta}$ and hence (4.7) may be rewritten

$$w \frac{\partial u}{\partial \theta} = w\tau = v_\theta = rw \quad (4.8)$$

Thus we must have $L = 1$ and τ is then written as

$$\tau = \frac{\partial u}{\partial \theta} = r \quad (4.9)$$

Integration of (4.9) with respect to θ now yields

$$u(r, \theta) = r\theta + U(r) \quad , \quad \theta \in [-\pi, \pi) \quad . \quad (4.10)$$

Without loss of generality we may choose $U(r) = 0$ since the fluid particles are in continuous rotation about the common axis of the cylinders describing circles with centers on the cylinders' common axis (z - axis). We now define the displacement field which satisfies (4.10) in the interval $[-\pi, \pi)$ as well as any extended interval for all θ . Since the circular cylinders in our problem are performing continuous rotations about their common axis the displacement field $u(r, \theta)$ must be a periodic function of θ with periodicity 2π . Thus we define the displacement field to be bounded and have a finite number of maxima and minima in $[-\pi, \pi)$.

we have

$$u(r, \theta) = u(r, \theta_0) \quad , \quad \theta_0 = \theta \bmod 2\pi \quad , \quad \theta_0 \in [-\pi, \pi) \quad (4.11)$$

and then by Dirichlet's theorem, the Fourier series for $u(r, \theta)$ converges to $\frac{1}{2} [u(r, \theta^+) + u(r, \theta^-)]$ at every value of θ . Here the notations $u(r, \theta^+)$ and $u(r, \theta^-)$ are defined by $u(r, \theta^+) =$

$$\lim_{\epsilon \rightarrow 0} u(r, \theta \pm \epsilon) \quad \text{where } \epsilon \text{ is an arbitrary positive number.}$$

Equations (4.1) and (4.2) may now be rewritten as

$$\frac{\partial v}{\partial r} = \frac{1}{\alpha} \left[\frac{d}{dr} (rw' + 2w) \right], \quad (4.12)$$

$$\frac{\partial v}{\partial \theta} = \gamma_1, \quad (4.13)$$

where

$$\alpha = \frac{a_{1212} - a_{1221}}{a_{1212}} > 0 \quad \text{and} \quad \gamma_1 = \frac{2A_{1212}}{a_{1212} - a_{1221}}. \quad (4.14)$$

In addition we use the expression for u in equation (3.18) and differentiate both sides of the resulting equation with respect to θ to obtain:

$$B_{3131} \nabla^2 \left(\frac{\partial \phi}{\partial \theta} \right) - 2(A_{1212} + A_{1221}) \frac{\partial \phi}{\partial \theta} = 2(A_{1212} + A_{1221}) \quad (4.15)$$

It is clear from equation (4.15) that we must consider three different cases depending on which of the coefficients in (4.15) are taken to vanish. Thus we must consider:

$$\text{Case i.} \quad B_{3131} = 0 \quad \text{and} \quad A_{1212} \neq -A_{1221} \quad (4.16)$$

$$\text{Case ii.} \quad B_{3131} \neq 0 \quad \text{and} \quad A_{1212} = -A_{1221} \quad (4.17)$$

$$\text{Case iii.} \quad B_{3131} \neq 0 \quad \text{and} \quad A_{1212} \neq -A_{1221} \quad (4.18)$$

We now seek the expression for the angular velocity $w = w(r)$ for each of the three cases (4.16) - (4.17). For case (i) equation (4.15) reduces to

$$\frac{\partial \phi}{\partial \theta} = -1 \quad (4.19)$$

and we immediately obtain

$$v = \frac{D\phi}{Dt} = w \frac{\partial \phi}{\partial \theta} = -w \quad (4.20)$$

Since $w = w(r)$ equations (4.13) and (4.14)₂ requires that

$$A_{1212} = 0 \quad (4.21)$$

The condition $A_{1212} = 0$ together with the thermodynamic restriction obtained in Chapter II, namely $A_{1212} \geq |A_{1221}|$ (II. 5.14) requires that

$$A_{1212} = A_{1221} = 0 \quad (4.22)$$

In view of (4.22) we must conclude that case (i) cannot occur for smectic liquid crystals in this type of viscometric flow.

In order to solve for w in case (ii) we must first solve

$$\nabla^2 \left(\frac{\partial \phi}{\partial \theta} \right) = 0 \quad (4.23)$$

for $\partial\theta/\partial\theta$. Equation (4.23) has as solution

$$\frac{\partial\theta}{\partial\theta} = C_0 + C_1 \ln r + \theta (C_3 + C_4 \ln r) \quad (4.24)$$

where C_0 , C_1 , C_3 and C_4 are constants to be determined.

We now employ the relation $v = w \partial\theta/\partial\theta$ and equation (4.13) to obtain the following expression for w .

$$w = \gamma_1 / [C_3 + C_4 \ln r]. \quad (4.25)$$

The constants C_3 and C_4 in (4.25) may now be determined by use of the boundary conditions on w at the inner and outer cylinder, respectively. Thus we have

$$w(r) = \Omega_2 \ln R / [\bar{\Omega} \ln R + (1 - \bar{\Omega}) \ln \bar{r}] \quad (4.26)$$

where

$$R = \frac{r_2}{r_1}, \quad \bar{r} = \frac{r}{r_1}, \quad \bar{\Omega} = \frac{\Omega_2}{\Omega_1} \quad (4.27)$$

Now equation (4.26) is immediately suspect since we note that if (4.26) holds then the only way for w to vanish is for $w(r_2) = \Omega_2 = 0$ and then $w \equiv 0$ for all $r \in [r_1, r_2]$; a situation which is not borne out experimentally. Likewise we expect that when the two cylinders are rotating in opposite senses that there be an $r_0 \in (r_1, r_2)$ such that $w(r_0) = 0$. Our misgivings about equation (4.26) are justified mathematically by substituting for w from the

latter and checking the expression for w in equation (4.12) at the endpoints $r = r_1$ and $r = r_2$. We obtain

$$C_0 + C_1 \left(\frac{\ln r_1 - \bar{\Omega} \ln r_2}{1 - \bar{\Omega}} \right) = \frac{2}{\alpha} \left(1 - \frac{1 - \bar{\Omega}}{\bar{\Omega} \ln R} \right) \quad (4.28)$$

and

$$C_0 + C_1 \left(\frac{\ln r_1 - \bar{\Omega} \ln r_2}{1 - \bar{\Omega}} \right) = \frac{2}{\alpha} \left(1 - \frac{1 - \bar{\Omega}}{\ln R} \right) \quad (4.29)$$

which when subtracted yield

$$\bar{\Omega} = 1, \text{ that is } \Omega_1 = \Omega_2. \quad (4.30)$$

But we have required that $\Omega_1 \neq \Omega_2$ since for $\Omega_1 = \Omega_2$ we would have simply a rigid rotation. Thus we conclude that $C_3 = C_4 = 0$ and $A_{1212} = 0$, in view of $\nu = \omega \partial \phi / \partial \theta$ and (4.24) and (4.13) which leads to $A_{1221} = 0$ since $A_{1212} = -A_{1221}$. The expression for $\partial \phi / \partial \theta$ is now given by

$$\frac{\partial \phi}{\partial \theta} = a + b \ln \bar{r}$$

where we have defined

$$a = \frac{\partial \phi}{\partial \theta}(r_1); \quad b = \frac{\hat{b} - a}{\ln R}; \quad \hat{b} = \frac{\partial \phi}{\partial \theta}(r_2), \quad \bar{r} = \frac{r}{r_1} \quad (4.32)$$

We note that the vanishing of C_3 and C_4 means that $\partial \phi / \partial \theta$ and hence ν is a function of r only. Using this fact, equation (4.12) is integrated with respect to r on both sides to obtain

$$w (a + b \ln \bar{r}) = \frac{1}{\alpha} (r w' + 2w) + C \quad (4.33)$$

where C is the constant of integration. Now equation (4.33) has the solution

$$w(r) = r^{(A+B \ln r)} \{ \Delta_0 \operatorname{erf}(\eta) + \Delta_1 \} \quad (4.34)$$

where

$$A = -2 - \alpha (a - b \ln r_1), \quad B = -\frac{1}{2} \alpha b, \quad \eta = B^{1/2} \ln r + \frac{1}{2} AB^{-1/2},$$

$$\eta_1 = B^{1/2} \ln r_1 + \frac{1}{2} AB^{-1/2} = -(2 + \alpha a) \left[\frac{-\ln r}{2\alpha(b-a)} \right]^{1/2},$$

$$\eta_2 = B^{1/2} \ln r_2 + \frac{1}{2} AB^{-1/2} = -(2 + \alpha b) \left[\frac{-\ln R}{2\alpha(b-a)} \right]^{1/2},$$

$$\operatorname{erf}(\eta) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{2}\eta} \exp(-x^2/2) dx, \quad E = \operatorname{erf}(\eta_2) - \operatorname{erf}(\eta_1), \quad (4.35)$$

$$\begin{aligned} \Delta_0 &= \frac{\Omega_2 r_2^{-(A+B \ln r_2)} - \Omega_1 r_1^{-(A+B \ln r_1)}}{E} \\ &= \frac{\Omega_2}{r_1^{-(A+B \ln r_1)}} \cdot \frac{R^{[2+\alpha(\frac{a+b}{2})]} - \Omega_0}{E}, \end{aligned}$$

$$\Omega_0 = \Omega_2 / \Omega_1,$$

$$\begin{aligned} \Delta_1 &= \frac{\Omega_1 r_1^{-(A+B \ln r_1)} \operatorname{erf}(\eta_2) - \Omega_2 r_2^{-(A+B \ln r_2)} \operatorname{erf}(\eta_1)}{E} \\ &= \frac{\Omega_2}{r_1^{-(A+B \ln r_1)}} \cdot \frac{\Omega_0 \operatorname{erf}(\eta_2) - R^{[2+\alpha(\frac{a+b}{2})]} \operatorname{erf}(\eta_1)}{\operatorname{erf}(\eta_2) - \operatorname{erf}(\eta_1)}, \end{aligned}$$

We remark here that the third equation in (4.35) requires $B > 0$. Since $B = -\frac{1}{2} \alpha b$ and $\alpha > 0$ we must have $b < 0$ which, in view of (4.32) requires that

$$\hat{b} = \frac{\partial \phi}{\partial \theta}(r_2) < \frac{\partial \phi}{\partial \theta}(r_1) = a. \quad (4.36)$$

If it should happen that $b = 0$, i. e. $\hat{b} = a$ then we have from (4.31)

$$\frac{\partial \phi}{\partial \theta} = a \quad (4.37)$$

and so equation (4.33) would have the solution

$$w(r) = \frac{\Omega_1 r_1^k - \Omega_2 r_2^k}{r_1^k - r_2^k} + \frac{r_1^k r_2^k (\Omega_2 - \Omega_1)}{r_1^k - r_2^k} \cdot \frac{1}{r^k}, \quad (4.38)$$

where

$$k = 2 - \alpha a. \quad (4.39)$$

Note that if $a = b = 0$ then $k = 2$ and we have the usual classical solution for the angular velocity w due to Couette flow.

We now consider case (iii) where we have $B_{3131} \neq 0$, $A_{1212} \neq -A_{1221}$ so we must solve

$$\nabla^2 \left(\frac{\partial \phi}{\partial \theta} \right) - p^2 \left(\frac{\partial \phi}{\partial \theta} \right) - p^2 = 0 \quad (4.40)$$

where we have defined

$$p^2 = \frac{2(A_{1212} + A_{1221})}{B_{3131}} \quad (4.41)$$

This equation has solution

$$\frac{\partial \theta}{\partial r} = \bar{a} I_0(pr) + \bar{b} K_0(pr) - 1 + \theta \{a_1 I_0(pr) + b_1 K_0(pr)\} \quad (4.42)$$

where I_0 and K_0 are zero-order modified Bessel functions of the first and second kind, respectively, and \bar{a} , \bar{b} , a_1 , b_1 are constants to be determined by the boundary conditions. If the same procedure applied in case (ii) is employed we find that the last term on the right-hand side of (4.42) must vanish, i.e. $a_1 = b_1 = 0$ and we also must conclude that $A_{1212} = 0$. Once again the thermodynamic restriction (II. 5.14) leads to $A_{1221} = 0$ and hence case (iii) does not occur for smectic liquid crystals in this viscometric flow.

The temperature field $T(r)$ is obtained by solving equation (3.19) subject to the boundary conditions

$$T(r_1) = T_1 \quad \text{and} \quad T(r_2) = T_2 \quad (4.43)$$

Thus we obtain for the temperature field

$$\bar{T} = \frac{T(r)}{T_1} = \frac{\ln R + (\mathcal{K} - 1) \ln \bar{r}}{\ln R} \quad (4.44)$$

where

$$R = r_2/r_1, \quad \mathcal{K} = T_2/T_1 \quad (4.45)$$

Heat-conduction can be obtained from the equations (3.11) by using the appropriate expressions for $v(r, \theta)$ and $T(r)$. The components of the heat-flux vector are

$$q_r = \frac{1}{2} d_0 \frac{\partial v}{r \partial \theta} + \frac{1}{4} T^{-1} \delta_0 \frac{\partial T}{\partial r}, \quad q_\theta = -\frac{1}{2} d_0 \frac{\partial v}{\partial r} \quad (4.46)$$

where

$$d_0 = d_{2332} - d_{3232}, \quad \delta_0 = \delta_{3232} + \delta_{2323} - \delta_{3223} - \delta_{2332} \quad (4.47)$$

Thus, the solution set for the governing field equations can be expressed in non-dimensional form where $\Omega_1 \neq 0$, $T_1 \neq 0$, $R \neq 1$ and $\alpha \neq 0$.

Angular Velocity Field

For $a \neq \hat{b}$:

$$\bar{w} = \frac{w(r)}{\Omega_1} = \frac{\bar{r}^{-[2 + \alpha(a + \frac{\hat{b}-a}{2 \ln R} \ln \bar{r})]}}{E} \{ \bar{\Delta}_0 \operatorname{erf}(\eta) + \bar{\Delta}_1 \}, \quad (4.48)$$

where

$$\bar{\Delta}_0 = \bar{\Omega} R^{2 + \frac{\alpha}{2}(a + \hat{b})} - 1, \quad ,$$

$$\bar{\Delta}_1 = \operatorname{erf}(\eta_2) - \bar{\Omega} \operatorname{erf}(\eta_1) R^{[2 + \frac{\alpha}{2}(a + \hat{b})]} \quad (4.49)$$

For $a = \hat{b}$:

$$\bar{w} = \frac{w(r)}{\Omega_1} = \frac{1 - \bar{\Omega} R^k}{1 - R^k} + \frac{(\bar{\Omega} - 1) R^k}{1 - R^k} \cdot \frac{1}{\bar{r}^k}, \quad (4.50)$$

where

$$k = 2 - \alpha a. \quad (4.51)$$

Microgyration Field

For $a \neq \hat{b}$:

$$\bar{v} = \frac{\nu}{\Omega_1} = \frac{a + b \ln \bar{r}}{E} \{ \bar{\Delta}_0 \operatorname{erf}(\eta) + \bar{\Delta}_1 \} \bar{r}^{-[2 + \alpha(a + \frac{b}{2} \ln \bar{r})]}, \quad (4.52)$$

where

$$b = \frac{\hat{b} - a}{\ln R}. \quad (4.53)$$

For $a = \hat{b}$:

$$\bar{v} = \frac{\nu}{\Omega_1} = \frac{a}{1 - R^k} \left\{ 1 - \bar{\Omega} R^k + \frac{(\bar{\Omega} - 1) R^k}{\bar{r}^k} \right\}. \quad (4.54)$$

Macro-displacement Field

$$\bar{u} = \frac{u(r, \theta)}{r_1} = \bar{r} \theta, \quad \theta \in [-\pi, \pi]. \quad (4.55)$$

Temperature Field

$$\bar{T} = \frac{T(r)}{T_1} = \frac{(K - 1) \ln \bar{r} + \ln R}{\ln R}. \quad (4.56)$$

Heat-Conduction

$$\bar{q}_r = q_r/q_1 = \frac{\delta T^{-1}}{4\bar{r}} \cdot \frac{\chi-1}{\ln R} \quad (4.57)$$

where

$$q_1 = \frac{1}{\delta_0 r_1} \quad (4.58)$$

For $a \neq \hat{b}$:

$$\bar{q}_\theta = q_\theta/q_0 = \frac{\bar{w}}{\bar{r}} [b-2(a+b \ln \bar{r}) - \alpha(a+b \ln \bar{r})^2] + \frac{a+b \ln \bar{r}}{\bar{r}} \cdot \frac{\Delta_0 R^{\frac{2}{\alpha(\hat{b}-a)}} [1+\alpha a+(\alpha a)^2]}{E} \cdot \sqrt{\frac{-2\alpha b}{\pi}} \quad (4.59)$$

where

$$q_0 = \frac{-\Omega_1 d_0}{2r_1} \quad (4.60)$$

For $a = \hat{b}$:

$$\bar{q}_\theta = q_\theta/q_0 = \frac{a k}{\bar{r}^{k+1}} \frac{R^k (\bar{\Omega} - 1)}{1 - R^k} \quad (4.61)$$

5. Orientation Field

In section 4 we have found an expression for $\partial\phi/\partial\theta$ with, as yet, undetermined boundary conditions at the inner and outer cylinder walls. We can expect a variety of boundary conditions here which are physically meaningful since smectic liquid crystals orient themselves at the boundary in a way that depends on the prior treatment of the boundary material. Hence, we have left the expressions for boundary conditions flexible, preferring to denote

$$\frac{\partial\phi}{\partial\theta} (r_1) = a \quad \text{and} \quad \frac{\partial\phi}{\partial\theta} (r_2) = \hat{b} \quad (5.1)$$

We now proceed to develop an analytical expression for the orientation field $\phi(r, \theta)$.

Integration of equation (4.31) with respect to θ yields

$$\phi(\bar{r}, \theta) = \theta [a + b \ln \bar{r}] + H(\bar{r}) \quad (5.2)$$

where $H(\bar{r})$ is an arbitrary function of integration and $\bar{r} = r/r_1$.

In order to determine $H(\bar{r})$ we shall use the expression for the torque on the cylinders. The torque per unit length about the z -axis on the cylinder of radius r is denoted by τ_0 and besides the usual contribution to the torque from the shear stress $t_{r\theta}$ we will also have a contribution from the couple stress, namely m_{rz} .

Thus we have

$$\tau_0 = \int_0^{2\pi} (r^2 t_{r\theta} + r m_{rz}) d\theta \quad (5.3)$$

where

$$t_{r\theta} = a_{1212} r w' + 2 k_3 (w - v) \quad (5.4)$$

$$m_{rz} = B_{3131} \frac{\partial \phi}{\partial r} + b_{3131} \frac{\partial v}{\partial r}$$

Upon carrying out the θ -integration in (5.3), using the known expressions for w , v , and ϕ , and making use of the law of balance of torques which requires that the torque remain constant for any cylinder of radius r , $r_1 \leq r \leq r_2$, we obtain the following differential equation in the dimensional form for the unknown function $H(r)$:

$$H'(r) = \frac{1}{r} \left[\frac{\tau_0}{2\pi B_{3131}} - 2 b \pi \right] - \frac{b_{3131}}{B_{3131}} \left[\frac{1}{\alpha} (r w'' + 3 w') \right] + (a_{1212} + a_{1221}) r w - \frac{2 a_{1212} \Delta_0 (\sqrt{B/\pi}) r}{B_{3131} \exp(A^2/4B)}, \quad (5.5)$$

$$H'(r) = \frac{1}{r} \left[\frac{\tau_0}{2\pi B_{3131}} \right] - \frac{b_{3131}}{B_{3131}} \left[\frac{1}{\alpha} (r w'' + 3 w') \right] + r a_{1221} \frac{\Omega_2 r_2^k - \Omega_1 r_1^k}{r_2^k - r_1^k} + \frac{1}{r^{k-1}} \left[(a_{1212} + a_{1221}) \frac{(\Omega_1 - \Omega_2) r_1^k r_2^k}{r_2^k - r_1^k} \right], \quad (5.6)$$

where the equation in (5.5) is for $a \neq \hat{b}$ and the equation in (5.6) is for $a = \hat{b}$. Integration of (5.5) and (5.6) with respect to r yields

$$H(r) = (\ln r) \left[\frac{\tau_0 - 4b\pi^2 B_{3131}}{2\pi B_{3131}} \right] - \frac{b_{3131}}{B_{3131}} \left[\frac{1}{\alpha} (rw' + 2w) \right] - r^2 \{ \exp(-A^2/4B) \\ \times a_{1212} \Delta_0 \sqrt{B/\pi} \} + (a_{1212} + a_{1221}) \int_0^r tw(t) dt + C_A. \quad (5.7)$$

$$H(r) = \ln r \left[\frac{\tau_0}{2\pi B_{3131}} \right] - \frac{b_{3131}}{B_{3131}} \left[\frac{1}{\alpha} (rw' + 2w) \right] + \left\{ \frac{a_{1221} r^2}{2} \right. \\ \left. \times \frac{\Omega_2 r_2^k - \Omega_1 r_1^k}{r_2^k - r_1^k} \right\} - \frac{(a_{1212} + a_{1221})}{\alpha a r} \cdot \frac{(\Omega_1 - \Omega_2) r_1^k r_2^k}{r_2^k - r_1^k} + C_B \quad (5.8)$$

where C_A and C_B are arbitrary constants of integration for the cases $a \neq b$ and $a = b$, respectively. However, the presence of C_A and C_B in the expressions for the orientation field derived from putting (5.7) and (5.8) into (5.2) does not affect the behavior of the orientation and its dependence on the spatial coordinates r and θ , the radius ratio R and the angular velocity ratio of the cylinders and neither does C_A or C_B appear in any of the solutions given by equations (4.48) to (4.63). In view of these facts we may as well omit the constants from the expressions for the orientation field

for the two cases. The torque τ_0 appearing in (5.7) and (5.8) can be physically measured in a given experiment and hence is considered to be known. The classical expression for τ_0 in the case of Couette flow between two concentric rotating cylinders is given by

$$\tau_0 = \frac{4\pi r_1^2 (\Omega_2 - \Omega_1) \mu}{1 - R^2} \quad (5.9)$$

where μ is the viscosity of the Newtonian fluid between the cylinders. Finally we remark that the integral in (5.7) may be integrated numerically for a given experiment and hence the expressions for the orientation fields are completely known for each case.

6. Apparent Viscosity

In order to gain some idea of the non-Newtonian behavior of the smectic liquid crystal we define an apparent viscosity. The apparent viscosity is defined such that it is equal to the constant coefficient of viscosity for the flow of an incompressible Newtonian fluid. Therefore, we choose:

$$\eta_{\text{app}} = \left[\frac{\text{Shear stress}}{\text{Shear-rate on the outer cylinder}} \right]_{\text{at } \theta = 0, r = r_2} \quad (6.1)$$

which becomes, on using the known solutions for w and v , and also the expressions for shear stress and shear-rate:

$$\eta_{\text{app}} = a_{1212} + \frac{1}{\gamma} [2k_3 \Omega_2 (1-\hat{b})] \quad , \quad (6.2)$$

where

$$\gamma = \frac{2 \Delta_0 \sqrt{B/\pi}}{e^{A/4B}} - \Omega_2 (2 + \alpha \hat{b}) \quad , \quad a \neq \hat{b} \quad , \quad (6.3)$$

$$\gamma = \frac{(2 + \alpha a) (\Omega_2 - \Omega_1)}{R^k - 1} \quad , \quad a = \hat{b} \quad , \quad (6.4)$$

$$2k_3 = a_{1212} - a_{1221} \quad . \quad (6.5)$$

We recall that a and \hat{b} in equations (6.3) and (6.4) involve the boundary conditions on the orientation field at the cylinder walls.

Determination of the Material Coefficients

We shall now proceed to demonstrate a method by which the material coefficients a_{1212} and a_{1221} can be calculated for a given smectic substance. In order to do this we utilize the experimental work of Tamamushi and Matsumoto [49], wherein the authors investigated the non-Newtonian behavior of the viscosity (as a function of shear-rate and temperature) of ammonium laurate which exhibits the smectic mesophase in the temperature range from 106° to 112° C. The apparatus used for determination of the viscosity at

given constant temperature was a viscometer of Couette type which worked in the range of shear-rate up to $5000 \text{ dyn. cm}^{-2}$. The actual shear-rates used in the experiments ranged from 41 sec^{-1} to 325 sec^{-1} while the corresponding shear force ranged from 0 to $1500 \text{ dyn. cm}^{-2}$.

To determine a_{1212} we note that according to equation (6.2) the main contribution to η_{app} at high shear-rates comes from the first term on the right-hand side. Indeed, the experimental results indicate a leveling off of the viscosity values for shear-rates above 325 sec^{-1} and an examination of ratios of apparent viscosities at successive experimental shear-rates leads us to make the following estimate for a_{1212} :

$$a_{1212} = 0.9 [\eta_{\text{app}} \text{ for } 325 \text{ sec}^{-1}] \quad (6.6)$$

The determination of a_{1221} proceeds in the following manner. Since only the shear-rate γ and η_{app} are given in Tamamushi and Matsumoto's experiments we make the following choices for Ω_2 and $1 - \hat{b}$ based on experimental evidence and the thermodynamic restrictions on a_{1212} and a_{1221} obtained in Chapter II. Peter and Peters [43] have conducted Couette viscometer experiments with nematic liquid crystals with an outer cylinder radius of 4.00 cm and an inner cylinder radius of 3.96 cm. The outer cylinder was given an angular velocity of 11.84 rad./sec. in their experiments.

Thus a reasonable choice for Ω_2 in (6.2) would be 19 rad./sec.

Our choice of a value for $1 - \hat{b}$ must be guided by the thermodynamic restrictions on a_{1212} and a_{1221} since little experimental data concerning the boundary conditions on the orientation field are available. The thermodynamic restrictions on a_{1212} and a_{1221} are

$$a_{1212} \geq 0, \quad a_{1212} - a_{1221} > 0, \quad a_{1212} > 0. \quad (6.7)$$

We rewrite equation (6.2) using our value chosen above for Ω_2 and obtain

$$\frac{1}{19} (\eta_{\text{app}} - a_{1212}) = \frac{1 - \hat{b}}{\gamma} (a_{1212} - a_{1221}). \quad (6.8)$$

Now the left-hand side of equation (6.8) is positive and attains its largest values for low shear-rates. Using the values for a_{1212} and η_{app} at $\dot{\gamma} = 41 \text{ sec}^{-1}$, and the restrictions (6.7) we obtain the following value for $1 - \hat{b}$ which satisfies equation (6.8) as well as the restrictions in (6.7):

$$1 - \hat{b} = 10 \quad (6.9)$$

The value for $1 - \hat{b}$ reflects the change in the orientation field at the outer cylinder wall and can be maintained by the use of external means such as the imposition of a magnetic field of suitable strength or a thermal gradient.

Now the experimental data for $\gamma = 41 \text{ sec}^{-1}$ and the values for a_{1212} , Ω_2 and $1 - \hat{b}$ as given above, can be used in the following equation

$$\eta_{\text{app}} (\text{at } 41 \text{ sec}^{-1}) = a_{1212} + 4.63 (a_{1212} - a_{1221}), \quad (6.10)$$

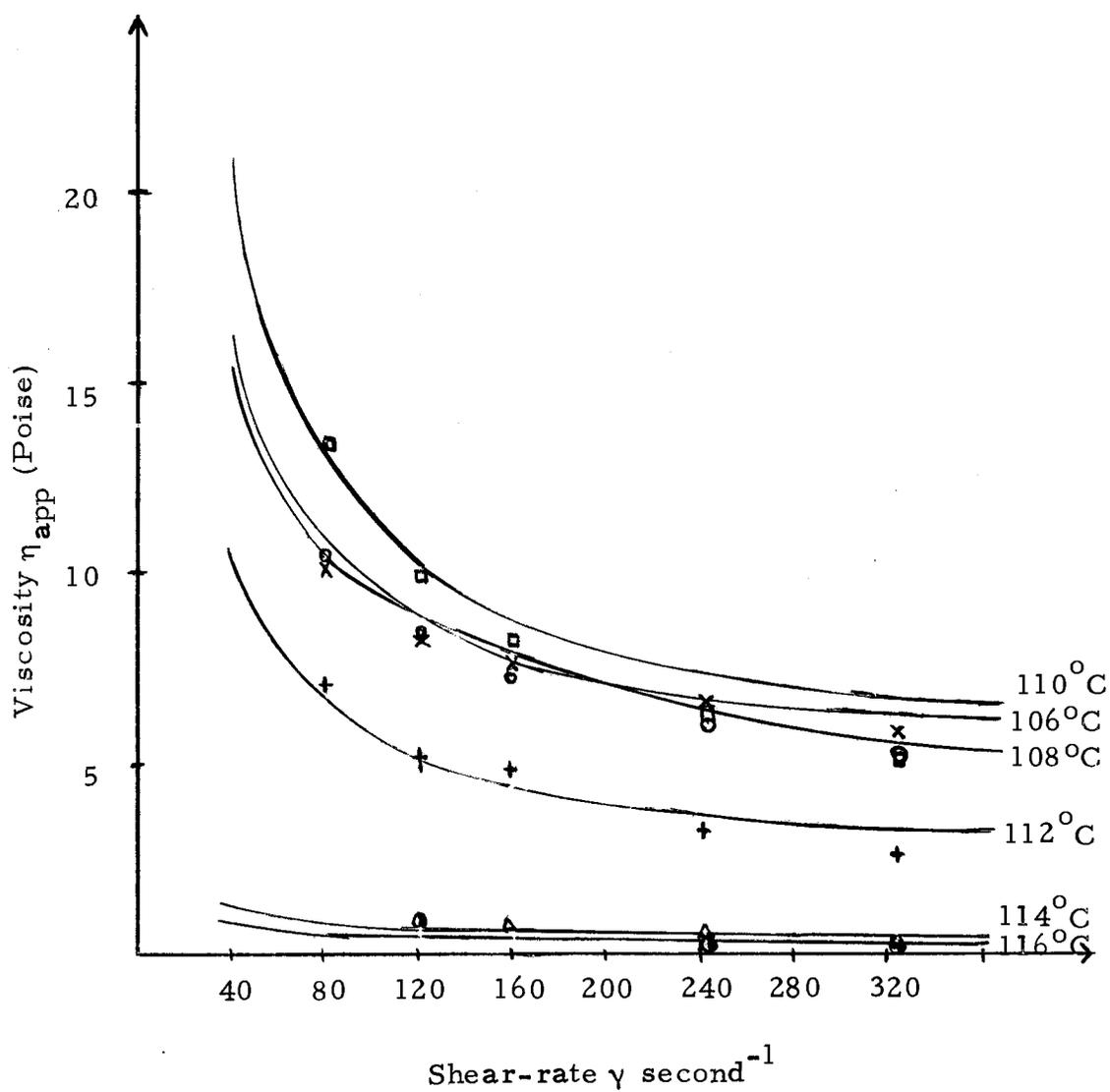
to determine a_{1221} for various temperatures in the smectic range.

Thus we obtain the following table of values for a_{1212} and a_{1221} .

Table 1. Values for a_{1212} and a_{1221} .

	106°	108°	110°	112°	114°	116°	°C
a_{1212}	4.68	5.22	4.59	2.25	.27	.18	Poise
a_{1221}	2.15	2.98	1.06	.45	.07	.01	Poise

These values for a_{1212} and a_{1221} given in the above table can now be used in equation (6.8) in order to obtain our analytical formula for η_{app} in the case of the smectic compound, ammonium laurate. The results of our analytical computation of η_{app} versus the shear-rate for different smectic temperatures are shown in Figure 3. The solid line denotes our theoretical results and the symbols denote experimental results obtained from the graphs of Tamamushi and Matsumoto [49]. It is clear from the graph that our



- + 106° Experimental results of Tamamushi and Matsumoto
- x 108° Experimental results of Tamamushi and Matsumoto
- 110° Experimental results of Tamamushi and Matsumoto
- + 112° Experimental results of Tamamushi and Matsumoto
- Δ 114° Experimental results of Tamamushi and Matsumoto
- 116° Experimental results of Tamamushi and Matsumoto

Figure 3. Behavior of apparent viscosity with shear-rate.

theoretical results are found to be in close agreement with the above experimental results.

IV. MICROPOLAR ELASTICITY WITH STRETCH

1. Preliminary Remarks

The theory of heat-conducting smectic liquid crystals was developed from the micropolar theory of viscoelasticity due to Eringen [17]. As noted in Chapter I, micropolar fluids form a subclass of the simple microfluids. In micropolar media the microelements are allowed to undergo only rigid rotations without stretch. In this way the second rank tensor ν becomes skew-symmetric and hence can be written in terms of an axial vector. Media consisting of dumbbell molecules, short rigid cylindrical elements, or elongated molecules are properly represented by this model. Indeed, a great deal of successful work using this theory has already been done on the rheological properties as well as wave propagation phenomena in a variety of media such as polymeric additives, liquid crystals, and animal blood.

If we now turn our attention to blood plasma and certain liquid crystals we realize that the microelements of these media are deformable and hence, this fact should be properly reflected in our analytical models as well. For this purpose we consider that the dipolar microelements can undergo uniform stretch or contraction in addition to rigid rotation. This physical assumption

once again provides an elegant mathematical way to reduce the complexity present in the general theory of simple microelastic solids and thus obtain a more manageable theory that still encompasses a wide range of real materials.

In this chapter we develop the theory of micropolar elastic solids with stretch. This includes development of the constitutive equations and of the field equations for this theory. In addition, the energy equation is employed to obtain thermodynamic restrictions on the elastic material moduli of the theory thus facilitating the determination of the range of applicability of this new theory. Finally, we discuss the construction of a theory for micropolar viscoelastic media with stretch and the application of such a theory to the study of wave propagation in such media.

2. Balance Laws

In this section the basic balance laws of motion are given for simple microfluids and microelastic solids. They are derived in detail in [12].

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_k = 0. \quad (2.1)$$

Conservation of Microinertia

$$\frac{\partial i_{km}}{\partial t} + i_{km,r} v_r - i_{rm} v_{rk} - i_{kr} v_{rm} = 0. \quad (2.2)$$

Balance of Momentum

$$t_{kl,k} + \rho (f_l - a_l) = 0. \quad (2.3)$$

Balance of First Stress Moments

$$t_{km} - s_{km} + \lambda_{lmk,l} + \rho (l_{mk} - \dot{\sigma}_{mk}) = 0. \quad (2.4)$$

Conservation of Energy

$$\rho \dot{\epsilon} = t_{kl} v_{l,k} + (s_{kl} - t_{kl}) v_{kl} + \lambda_{klm} v_{ml,k} + q_{k,k} + \rho h. \quad (2.5)$$

Principle of Entropy

$$\rho \dot{\Gamma} = \rho \dot{\eta} - \left(\frac{q_k}{\theta} \right)_{,k} - \rho \frac{h}{\theta} \geq 0. \quad (2.6)$$

Inequality (2.6) is postulated to be valid for all independent processes and in (2.1) to (2.6) we have ρ as the mass density, v_k as the velocity vector, $i_{kl} = i_{lk}$ as the microinertia tensor, v_{kl} as the microgyration tensor, t_{kl} as the stress tensor, f_l as the body force per unit mass, a_l as the acceleration vector, s_{kl} as the microstress average tensor, λ_{klm} as the first stress moments, l_{mk} as the first body moment per unit mass, $\dot{\sigma}_{lm}$ as the inertial spin tensor,

\mathcal{E} as the internal energy density per unit mass, q_k as the heat vector (directed outward from the body), h as the heat source per unit mass, η as the entropy, and θ as the temperature. As in previous chapters the summation convention over repeated indices occurring in a term is assumed as well as the definition of material derivative and the use of cartesian tensors.

We note that (2.1), (2.3), and (2.6) are the usual classical laws whereas (2.2), (2.4), and (2.5) are new, but they reduce to their classical counterparts when the microstructural components i_{kl} , ν_{kl} , l_{km} and λ_{klm} vanish identically. A complete discussion of these new tensors is given by Eringen in [12]. Finally, we have the following kinematical relation for $\dot{\sigma}_{kl}$ as given in [12]

$$\dot{\sigma}_{kl} = i_{ml} (\dot{\nu}_{mk} + \nu_{nk} \nu_{mn}) . \quad (2.7)$$

The expressions (2.1) to (2.7) are valid for all parts of the body B having volume V and surface S , except at a finite number of discontinuity surfaces, lines, and points. At the surface of the body the following traction boundary conditions hold

$$\begin{aligned} t_{kl} n_k &= t_l && \text{on } S , \\ \lambda_{klm} n_k &= \lambda_{lm} && \text{on } S . \end{aligned} \quad (2.8)$$

If the phenomenon of heat-conduction is excluded in the present theory the determination of motion requires the determination of the following nineteen unknowns

$$\rho(x, t), i_{kl}(x, t), v_k(x, t), v_{kl}(x, t) \quad (2.9)$$

as opposed to the four unknowns ρ and v_k of classical theory.

3. Constitutive Equations of the Microcontinuum Theory

The constitutive equations in their linearized form are obtained from [13] and are as follows:

$$t_{kl} = [(\lambda_1 + \tau) e_{rr} + \eta_1 \epsilon_{rr}] \delta_{kl} + 2(\mu_1 + \sigma_1) e_{kl} + \kappa_1 \epsilon_{lk} + \nu_1 \epsilon_{kl} \quad (3.1)$$

$$s_{kl} = [(\lambda_1 + 2\tau) e_{rr} + (2\eta_1 - \tau) \epsilon_{rr}] \delta_{kl} + 2(\mu_1 + \sigma_1) e_{kl} + \{(\nu_1 + \kappa_1 - \sigma_1) \times (\epsilon_{kl} + \epsilon_{lk})\} \quad (3.2)$$

$$\begin{aligned} \lambda_{klm} = & \tau_1 (\phi_{kr, r} \delta_{ml} + \phi_{rr, l} \delta_{mk}) + \tau_2 (\phi_{rk, r} \delta_{ml} + \phi_{rr, m} \delta_{kl}) \\ & + \tau_3 \phi_{rr, k} \delta_{ml} + \tau_4 \phi_{lr, r} \delta_{mk} + \tau_5 (\phi_{rl, r} \delta_{mk} + \phi_{mr, r} \delta_{kl}) \\ & + \tau_6 \phi_{rm, r} \delta_{kl} + \tau_7 \phi_{lm, k} + \tau_8 (\phi_{mk, l} + \phi_{kl, m}) \\ & + \tau_9 \phi_{lk, m} + \tau_{10} \phi_{ml, k} + \tau_{11} \phi_{km, l} \end{aligned} \quad (3.3)$$

where $\lambda_1, \tau, \eta_1, \mu_1, \sigma_1, \kappa_1, \nu_1$ and τ_1 through τ_{11} are material coefficients and e_{kl} and ξ_{kl} are respectively, the strain tensor and the microstrain tensor of the linear theory, defined by

$$e_{kl} \equiv \frac{1}{2} (u_{k,1} + u_{1,k}) \quad (3.4)$$

$$\xi_{kl} \equiv u_{k,1} + \phi_{kl}$$

where u_k is the macrodisplacement vector and ϕ_{kl} is the microdisplacement tensor.

4. Micropolar Elastic Solids with Stretch

In this section we define a micropolar elastic solid with stretch and use the definition to effect simplification in the conservation laws and balance equations and in the constitutive equations.

A body will be called a micropolar elastic solid with stretch if for all motions,

$$\phi_{kl} = \phi \delta_{kl} + \epsilon_{klm} \phi_m, \quad (4.1)$$

$$\lambda_{klm} = \lambda_k \delta_{lm} - \frac{1}{2} \epsilon_{lmr} m_{kr}. \quad (4.2)$$

These solids differ from the micropolar solids in that a scalar microstretch ϕ is added to the microdisplacement gradient ϕ_{kl} and the corresponding first stress moment vector λ_k is added to

λ_{klm} . $\dot{\phi}_m$ is the microrotation vector, m_{kr} is the couple stress of the micropolar theory, and ϵ_{klm} is the well-known permutation tensor. The presence of $\dot{\phi}$ is attributed to the stretchability of the microelement and λ_k is the necessary nondeviatoric first stress moment accompanying this extra stretch.

Since we are using a linear theory we may use the result that [14]

$$\nu_{kl} = \dot{\phi}_{kl} , \quad (4.3)$$

so that we may write

$$\nu_{kl} = \nu \delta_{kl} + \epsilon_{klm} \nu_m \quad (4.4)$$

where $\nu = \dot{\phi}$ and $\nu_r = \dot{\phi}_r$. In addition the present theory is concerned with micro-isotropic solids so

$$i_{kl} = \frac{1}{2} j \delta_{kl} , \quad (4.5)$$

where j is a scalar-valued function. This does not necessarily imply macro-isotropy, that is, the medium can be micro-isotropic but macro-anisotropic.

When (3.5) is substituted into (2.2) we obtain

$$\frac{Dj}{Dt} - 2\nu j = 0 . \quad (4.6)$$

To calculate $\dot{\sigma}_{kl}$ we use (4.4) and (4.5) in (2.7) to obtain

$$\dot{\sigma}_{(kl)} = \frac{j}{2} [(\dot{v} + v^2 - v_r v_r) \delta_{kl} + v_k v_l] , \quad (4.7)$$

$$\dot{\sigma}_{[kl]} = \frac{j}{2} [\epsilon_{klr} (\dot{v}_r + 2v_r v_r)] , \quad (4.8)$$

where indices within a parenthesis (bracket) indicate the symmetric (anti- or skew-symmetric) part of the quantity. In the linear theory (quasi-linear), we write

$$\dot{\sigma}_{(kl)} = \frac{j}{2} \dot{v} \delta_{kl} , \quad (4.9)$$

$$\dot{\sigma}_{[kl]} = -\frac{j}{2} \epsilon_{klr} \dot{v}_r . \quad (4.10)$$

Constitutive Equations

We now derive the constitutive equations of micropolar media with stretch by using (4.1) and (4.2). The stress tensor t_{kl} and the stress-average tensor $s_{kl} = s_{lk}$ given by (3.1) and (3.2), on using (4.1) reduce to

(4.11)

$$t_{kl} = [\lambda e_{rr} + (3\eta_1 + \nu_1 + \mathcal{K}_1) \phi] \delta_{kl} + (2\mu + \mathcal{K}) e_{kl} + \mathcal{K} \epsilon_{klm} (r_m - \phi_m) , \quad (4.12)$$

$$s_{kl} = [(\lambda + \eta_1) e_{rr} + \{3(2\eta_1 - \tau) + 2(\nu_1 + \mathcal{K}_1 - \sigma_1)\} \phi] \delta_{kl} + 2(\mu + \mathcal{K}_1) e_{kl} ,$$

where

$$\lambda \equiv \lambda_1 + \eta_1 + \tau , \quad \mathcal{K} \equiv \mathcal{K}_1 - \nu_1 , \quad \mu \equiv \mu_1 + \sigma_1 + \nu_1 . \quad (4.13)$$

Again, substitution of (4.1) into (3.3) will give rise to the following expressions which constitute λ_{klm} :

$$\begin{aligned}
 2\lambda_{k(lm)} = & 2\delta_{lm}(\beta_1\phi_{,k} + (\tau_1 - \tau_2)\epsilon_{krs}\phi_{s,r}) + \delta_{km}[(\beta_2 + \beta_3)\phi_{,1} \\
 & + (\tau_4 - \tau_6)\epsilon_{lrs}\phi_{s,r}] + \delta_{kl}[(\beta_2 + \beta_3)\phi_{,m} \\
 & + (\tau_4 - \tau_6)\epsilon_{mrs}\phi_{s,r}] + (\tau_9 - \tau_{11}) \\
 & \times [\epsilon_{lks}\phi_{s,m} + \epsilon_{mks}\phi_{s,l}], \tag{4.14}
 \end{aligned}$$

$$\begin{aligned}
 2\lambda_{k[lm]} = & \delta_{km}[(\beta_2 - \beta_3)\phi_{,1} + 2(\tau_4 - \tau_5)\epsilon_{lrs}\phi_{s,r}] + \delta_{kl}[(\beta_3 - \beta_2)\phi_{,m} \\
 & + 2(\tau_5 - \tau_4)\epsilon_{mrs}\phi_{s,r}], \tag{4.15} \\
 & + 2(\tau_7 - \tau_{10})\epsilon_{lms}\phi_{s,k} + (\tau_8 - \tau_9)[\epsilon_{mks}\phi_{s,l} + \epsilon_{kls}\phi_{s,m}],
 \end{aligned}$$

where

$$\beta_1 \equiv \tau_1 + \tau_2 + \tau_7 + \tau_{10} + 3\tau_3, \quad \beta_2 \equiv 3\tau_1 + \tau_4 + \tau_5 + \tau_8 + \tau_{11},$$

$$\beta_3 \equiv 3\tau_2 + \tau_5 + \tau_6 + \tau_8 + \tau_9. \tag{4.16}$$

Condition (4.2) gives

$$\lambda_{k(lm)} = \lambda_{kml}^{\delta}, \tag{4.17}$$

and this enables us to write

$$\lambda_{k} = \beta_1\phi_{,k} + (\tau_1 - \tau_2)\epsilon_{krs}\phi_{s,r} \tag{4.18}$$

which also requires that $\beta_2 + \beta_3 = 0$, $\tau_4 = \tau_6$, and $\tau_9 = \tau_{11}$. It should be noted here that $\tau_4 = \tau_6$ and $\tau_9 = \tau_{11}$ have been used in the derivation of equation (4.15).

Condition (4.2) also yields

$$\lambda_{k[lm]} = -\frac{1}{2} \epsilon_{lmr} m_{kr} \quad , \quad (4.19)$$

and so we can solve for the couple stress tensor m_{kl}

$$m_{kl} = -\epsilon_{lrs} \lambda_{k[rs]} \quad . \quad (4.20)$$

Next we substitute (4.15) into (4.20) to obtain the constitutive equation

$$m_{kl} = \alpha \phi_{r,r} \delta_{kl} + \beta \phi_{k,l} + \gamma \phi_{l,k} - \tau_0 \epsilon_{kls} \phi_{,s} \quad (4.21)$$

where

$$\gamma = 2 (\tau_5 - \tau_4 + \tau_8 - \tau_9 + \tau_{10} - \tau_7), \quad \beta = 2 (\tau_4 - \tau_5) \quad ,$$

$$\alpha = 2 (\tau_9 - \tau_8), \quad \tau_0 = 3 (\tau_1 - \tau_2) \quad (4.22)$$

Field Equations

We now examine the symmetric part of the stress moment equations (2.4), which on using (4.2) yield

$$\lambda_{k,k} \delta_{lm} - (\eta_1 e_{rr} + \{3(\eta_1 - \tau) + \nu_1 + \kappa_1 - 2\sigma_1\} \phi) \delta_{lm} - (\nu_1 + \kappa_1) e_{lm} \\ + (\beta_1 \phi_{,kk} + \tau_0 \epsilon_{krs} \phi_{s,rk}) \delta_{lm} + \rho (1 - j \frac{\dot{\nu}}{2}) \delta_{lm} = 0 \quad (4.23)$$

where

$$1_{(km)} = 1 \delta_{km} \text{ and } \dot{\sigma}_{lm} = \dot{\sigma} \delta_{lm} = j \frac{\dot{\nu}}{2} \delta_{lm} . \quad (4.24)$$

The symmetric part of the body couple takes the form (4.24)₁ in view of compatibility with equations (4.2). We also note that for $l \neq m$ (4.23) reduces to

$$(\nu_1 + \kappa_1) e_{lm} = 0 , \quad (4.25)$$

which requires that $\nu_1 + \kappa_1 = 0$ since the tensor e_{lm} may certainly have nonzero off-diagonal terms. Thus, (4.23) may be rewritten as

$$\lambda_{k,k} - \eta_1 e_{rr} - (3\eta_1 + 3\tau - 2\sigma_1) \phi + \beta_1 \{ \phi_{,kk} + \tau_0 \epsilon_{krs} \phi_{s,rk} + \rho (1 - j \frac{\dot{\nu}}{2}) \} = 0 . \quad (4.26)$$

The skew-symmetric part of (2.4) can be expressed in terms of suitable axial vectors and tensors which yield the following simplification

$$\epsilon_{lmk} t_{[lm]} + \epsilon_{lmk} \lambda_{r[ml],r} + \rho \epsilon_{lmk} (l_{[ml]} - \sigma_{[ml]}) = 0 . \quad (4.27)$$

By use of (4.20) we find

$$m_{rk, r} + \epsilon_{lkr} t_{[lr]} + \rho (l_k - \dot{\sigma}_k) = 0, \quad (4.28)$$

where

$$l_k \equiv -\epsilon_{lmk} l_{[lm]} \quad \text{and} \quad \dot{\sigma}_k \equiv -\epsilon_{lmk} \dot{\sigma}_{[lm]}. \quad (4.29)$$

We now use (4.10) in (4.29)₂ to obtain

$$\dot{\sigma}_k = j \dot{\nu}_k \quad (4.30)$$

The field equations for micropolar solids with stretch are therefore obtained by substituting the expression for stress (4.11) into (2.3) and by combining (4.20) and (4.29) with (4.28). Equation (4.26) is the symmetric part of equation (2.4). Thus we have

$$\begin{aligned} & (\lambda e_{rr, k} + 3\eta_1 \phi_{, k}) \delta_{kl} + (2\mu + \kappa) e_{kl, k} + \kappa \epsilon_{klm} (r_{m, k} - \phi_{m, k}) \\ & + \rho (f_1 - a_1) = 0, \end{aligned} \quad (4.31)$$

$$\begin{aligned} & \lambda_{k, k} - \eta_1 e_{rr} - (3\eta_1 + 3\tau - 2\sigma_1) \phi + \beta_1 \phi_{, kk} + \tau_0 \epsilon_{krs} \phi_{s, rk} \\ & + \rho (1 - j \frac{\dot{\nu}}{2}) = 0, \end{aligned} \quad (4.32)$$

$$\gamma \phi_{k,rr} + \beta \phi_{r,kr} + \alpha \phi_{s,sk} - \tau_0 \epsilon_{rks} \phi_{s,r} + 2\kappa (r_k - \theta_k) + \rho (l_k - \dot{\sigma}_k) = 0 \quad (4.33)$$

In addition to the above equations, we have the microinertia conservation equation

$$\frac{Dj}{Dt} - 2\nu_j = 0 \quad (4.34)$$

and also the equation of continuity (2.1). So, if we are given f_k , l_k and l we have $3 + 1 + 3 + 1 + 1 = 9$ equations for the determination of the nine field components u_k , ϕ_k , θ , j , and ρ .

Boundary Conditions

The tractions and moments on the surface S of the body can be calculated from (2.8) and the basic assumptions (3.1) and (3.2). Through the use of (3.18), the boundary conditions (2.8)₁ may be written as

$$m_{kl} n_k = m_l \quad \text{and} \quad \lambda_k n_k = m \quad \text{on } S \quad (4.35)$$

where

$$m_k = -\epsilon_{klm} \lambda_{lm} \quad \text{and} \quad m = \frac{1}{3} \lambda_{rr} \quad (4.36)$$

are the surface couple and the stress moment resultant per unit area acting on the surface of the body.

5. Thermodynamic Restrictions on Material Moduli

In this section the energy equation will be used to obtain restrictions on the material moduli. First we may rewrite equation (2.5) as

$$\begin{aligned} \rho \dot{\mathcal{E}} = & t_{(kl)} v_{1,k} + (s_{kl} - t_{(kl)}) v_{kl} + \lambda_{k(lm)} v_{ml,k} \\ & + t_{[kl]} (v_{1,k} - v_{kl}) + \lambda_{k[lm]} v_{ml,k} \quad , \end{aligned} \quad (5.1)$$

where we have $q_{k,k} = 0$ and $h = 0$. If we use equations (4.11) - (4.14) in equation (5.1) then the latter takes the following form:

$$\begin{aligned} \rho \dot{\mathcal{E}} = & \lambda e_{rr} \dot{e}_{ll} + (2\mu + \mathcal{K}) e_{kl} \dot{e}_{kl} + 3\eta_0 (\theta \dot{e}_{ll} + \dot{\theta} e_{ll}) \\ & + 2\mathcal{K} (r_m - \theta_m) (\dot{r}_m - \dot{\theta}_m) \\ & + \alpha \theta_{k,k} \dot{\theta}_{1,1} + \beta \theta_{k,1} \dot{\theta}_{1,k} + \gamma \theta_{1,k} \dot{\theta}_{1,k} + \beta_0 \theta_{,k} \dot{\theta}_{,k} \\ & + 3\tau_{\epsilon} \epsilon_{0krs} (\theta_{s,r} \dot{\theta}_{,k} - \dot{\theta}_{s,r} \theta_{,k}) \quad , \end{aligned} \quad (5.2)$$

where

$$\eta_0 = 3\eta_1 + \nu_1 + \mathcal{K}_1 \quad , \quad \tau_0 = 3(\tau_1 - \tau_2) \quad , \quad \beta_0 = 3(\tau_1 + \tau_2 + \tau_{10} + \tau_7 + 3\tau_3) \quad . \quad (5.3)$$

We further restrict our attention to the infinitesimal strains and rotations; we have for our kinematic quantities,

$$\dot{e}_{kl} = d_{kl} \equiv \frac{1}{2} (v_{k,1} + v_{1,k}) , \quad (5.4)$$

$$r_{kl} = \frac{1}{2} (u_{k,1} - u_{1,k}) , \quad (5.5)$$

$$r_k = \frac{1}{2} \epsilon_{klm} u_{m,1} , \quad (5.6)$$

$$\dot{r}_{kl} = w_{kl} \equiv \frac{1}{2} (v_{k,1} - v_{1,k}) , \quad (5.7)$$

$$\dot{r}_k = w_k \equiv \frac{1}{2} \epsilon_{klm} v_{m,1} , \quad (5.8)$$

Where (5.6) and (5.8) follow from the skew-symmetry of (5.5) and (5.7) respectively.

We require for an elastic medium (non-dissipative) that the equation (5.2) be integrable. Hence we introduce the following theorem and proof:

Theorem 1. The necessary and sufficient condition for a micropolar medium with stretch to be elastic and hence non-dissipative is that the material coefficients τ_1 and τ_2 be equal.

Proof. The non-dissipative elastic medium must possess a strain-energy [26, 27] (or internal energy) function whose material time derivative is precisely given by equation (5.2). Consequently, (5.2) should be an exact differential. All the terms in (5.2) are

found to be integrable, except for the last one on the right-hand side. Thus, the integrability conditions for (5.2) lead to $\tau_1 = \tau_2$ which is a restriction on the material coefficients characterizing the micropolar, micro-isotropic elastic solid with stretch; conversely, if $\tau_1 = \tau_2$, (5.2) becomes integrable and yields the strain-energy function.

Now setting $\tau_1 = \tau_2$ and carrying out the integration we have

$$\begin{aligned} \rho \mathcal{E} - \rho \mathcal{E}_0 &= \frac{1}{2} [\lambda e_{rr} e_{ll} + (2\mu + \mathcal{K}) e_{kl} e_{lk} + 6\eta_1 \phi e_{ll}] \\ &+ \mathcal{K} (r_m - \phi_m) (r_m - \phi_m) \\ &+ \frac{1}{2} [\alpha \phi_{k,k} \phi_{l,l} + \beta \phi_{k,l} \phi_{l,k} + \gamma \phi_{l,k} \phi_{l,k} + \beta_0 \phi_{,k} \phi_{,k}] , \end{aligned} \quad (5.9)$$

where \mathcal{E}_0 is the energy density function for a given reference configuration. Equation (5.9) leads to the following forms of energy:

$$\text{Strain Energy} \equiv \rho \mathcal{E}_E = \frac{1}{2} [\lambda e_{rr} e_{ll} + (2\mu + \mathcal{K}) e_{kl} e_{lk} + 6\eta_1 \phi e_{ll}] , \quad (5.10)$$

$$\text{Rotational Energy} \equiv \rho \mathcal{E}_R = \mathcal{K} (r_m - \phi_m) (r_m - \phi_m) , \quad (5.11)$$

$$\begin{aligned} \text{Micro-Energy} \equiv \rho \mathcal{E}_M &= \frac{1}{2} [\alpha \phi_{k,k} \phi_{l,l} + \beta \phi_{k,l} \phi_{l,k} + \gamma \phi_{l,k} \phi_{l,k} \\ &+ \beta_0 \phi_{,k} \phi_{,k}] . \end{aligned} \quad (5.12)$$

We note that without loss of generality, we may take $\mathcal{E}_0 = 0$ in view

of ξ_0 being referred to a fixed (but arbitrary) reference configuration, and thus (5.9) becomes

$$\rho \xi = \rho [\xi_E + \xi_R + \xi_M] . \quad (5.13)$$

We require that $\rho \xi$ be nonnegative and since each type of energy involves independent processes we must have

$$\rho \xi_E \geq 0 , \quad \rho \xi_R \geq 0 , \quad \rho \xi_M \geq 0 . \quad (5.14)$$

The restrictions in (5.14) will lead to corresponding restrictions on the material moduli. For example, if $\rho \xi_R$ and $\rho \xi_M$ are taken to be zero and we let $\emptyset = 0$ we must have

$$\frac{1}{2} [\lambda e_{rr} e_{ll} + (2\mu + \kappa) e_{kl} e_{lk}] \geq 0 . \quad (5.15)$$

If, in (5.15) we take $e_{kl} = e \delta_{kl}$ where e is a scalar then we obtain

$$3\lambda + (2\mu + \kappa) \geq 0. \quad (5.16)$$

If, for example, we now take the diagonal components of e_{kl} to be zero while taking the off-diagonal components to be equal to e , we obtain from (5.15)

$$2\mu + \kappa \geq 0. \quad (5.17)$$

By proceeding in this manner we can determine the following

thermodynamic restrictions for the material moduli:

$$3\lambda + 2\mu + \mathcal{K} \geq 0, \quad 2\mu + \mathcal{K} \geq 0, \quad \mathcal{K} \geq 0, \quad \eta_1 \geq 0, \quad (5.18)$$

$$\alpha + \beta + \gamma \geq |\alpha|, \quad \gamma \geq |\beta|, \quad 3\alpha + \beta + \gamma \geq 0, \quad \beta_1 \geq 0. \quad (5.19)$$

The results in (5.19) were obtained by setting $\rho \mathcal{E}_E = \rho \mathcal{E}_R = 0$ and writing $\rho \mathcal{E}_M$ as a quadratic form

$$\rho \mathcal{E}_M = a_{ij} y_i y_j, \quad i, j, = 1, 2, \dots, 12, \quad (5.20)$$

where

$$y_1 = \emptyset_{1,1}, \quad y_2 = \emptyset_{2,2}, \quad y_3 = \emptyset_{3,3}, \quad y_4 = \emptyset_{1,2}, \quad y_5 = \emptyset_{2,1},$$

$$y_6 = \emptyset_{2,3}, \quad y_7 = \emptyset_{3,2}, \quad y_8 = \emptyset_{3,1}, \quad y_9 = \emptyset_{1,3},$$

$$y_{10} = \emptyset_{,1}, \quad y_{11} = \emptyset_{,2}, \quad y_{12} = \emptyset_{,3}, \quad (5.21)$$

$$a_{11} = a_{22} = a_{33} = \alpha + \beta + \gamma = \xi,$$

$$a_{12} = a_{21} = a_{13} = a_{31} = a_{23} = a_{32} = \alpha,$$

$$a_{44} = a_{55} = a_{66} = a_{77} = a_{88} = a_{99} = \gamma, \quad (5.22)$$

$$a_{45} = a_{54} = a_{67} = a_{76} = a_{89} = a_{98} = \beta,$$

$$a_{10,10} = a_{11,11} = a_{12,12} = \beta_0, \quad \text{and all other } a_{ij} = 0,$$

and requiring that the quadratic form in (5.20) be non-negative.

We have therefore proved the following theorem:

Theorem 2. The necessary and sufficient conditions for the thermodynamic admissibility of the constitutive equations (4.11), (4.12), (4.18), and (4.21) are the relations (5.18) and (5.19).

6. Micropolar Viscoelasticity with Stretch

In this section we outline the procedure to follow in constructing the theory of micropolar viscoelasticity with stretch. The first step is to supplement the equations of balance and conservation laws (2.1) through (2.5) and the entropy inequality (2.6) with the constitutive equations appropriate to each medium. We have:

(a) Micropolar elastic solid with stretch,

$$t_{kl} = [\lambda e_{rr} + (3\eta_1 + \nu_1 + \kappa_1) \phi] \delta_{kl} + (2\mu + \kappa) e_{kl} + \kappa \epsilon_{klm} (r_m - \phi_m), \quad (6.1)$$

$$s_{kl} = [(\lambda + \eta_1) e_{rr} + \{3(2\eta_1 - \tau) + 2(\nu_1 + \kappa_1 - \sigma_1)\} \phi] \delta_{kl} + 2(\mu + \kappa_1) e_{kl}, \quad (6.2)$$

$$m_{kl} = \alpha \phi_{r,r} \delta_{kl} + \beta \phi_{k,l} + \gamma \phi_{l,k}, \quad (6.3)$$

$$\lambda_k = \beta_1 \phi_{,k}. \quad (6.4)$$

(b) Micropolar fluids with stretch [18] ,

$$t_{kl} = (-\pi + \lambda_v v_{r,r} + \lambda_0 v) \delta_{kl} + \mu_v (v_{k,1} + v_{1,k}) + \\ + \kappa_v (v_{1,k} - \epsilon_{klm} v_m) , \quad (6.5)$$

$$s_{kl} = (-\pi + \lambda_v v_{r,r} + \eta_0 v) \delta_{kl} + (\mu_v + \frac{\kappa_v}{2}) (v_{k,1} + v_{1,k}) , \quad (6.6)$$

$$m_{kl} = \alpha_v v_{r,r} \delta_{kl} + \beta_v v_{k,1} + \gamma_v v_{1,k} + \beta_c \epsilon_{klr} v_r , \quad (6.7)$$

$$\lambda_k = \alpha_0 v_{,k} + \alpha_1 \epsilon_{klm} v_{m,1} . \quad (6.8)$$

The new elastic moduli in equations (6.1) to (6.4) are subject to the thermodynamic restrictions (5.18) and (5.19) derived earlier in this chapter. The viscosity coefficients appearing in equations (6.5) to (6.8) are subject to the following thermodynamic restrictions [18]:

$$3\lambda_v + 2\mu_v + \kappa_v \geq 0 , \quad 2\mu_v + \kappa_v \geq 0 , \quad \kappa_v \geq 0 , \\ 3\alpha_v + \beta_v + \gamma_v \geq 0 , \quad -\gamma_v \leq \beta_v \leq \gamma_v , \quad \gamma_v \geq 0 , \quad (6.9) \\ \alpha_0 \geq 0 , \quad \eta_0 - \lambda_0 \geq 0 , \quad (\eta_0 - \lambda_0) (3\lambda_v + 2\mu_v + \kappa_v) \geq \frac{\lambda_0^2}{4}$$

We now wish to obtain a set of constitutive equations for a class of viscoelastic materials in which the constitutive response

functions depend on the strain measures, microrotation, microrotation gradient, micro-stretch and micro-stretch gradient, and their first order material-time rates. These constitutive equations would yield a description of the media corresponding to the generalization of the Kelvin-Voigt viscoelastic solids. This construction may be accomplished by taking a linear combination of constitutive equations governing micropolar elasticity with stretch and micropolar fluids with stretch analogous to the method followed in the classical Kelvin-Voigt theory of viscoelasticity.

However, for a more rigorous treatment of this theory we would first have to determine a set of objective variables to construct the theory, basing our selection of these variables on equations (6.1) to (6.8). Accordingly we could choose $\tilde{\epsilon}_{kl}$, Δ_{kl} , $\phi_{k,l}$, $\nu_{k,l}$, $\phi_{,k}$, $\nu_{,k}$, ϕ , ν , θ where $\tilde{\epsilon}_{kl}$ and Δ_{kl} are given by

$$\begin{aligned}\tilde{\epsilon}_{kl} &= e_{kl} + \epsilon_{klm} (r_m - \phi_m) - \phi \delta_{kl} , \\ \Delta_{kl} &= d_{kl} + \epsilon_{klm} (w_m - \nu_m) - \nu \delta_{kl} = \dot{\tilde{\epsilon}}_{kl} .\end{aligned}\tag{6.10}$$

Next we would define a micropolar viscoelastic solid with stretch by the following constitutive equations:

$$\begin{aligned}t_{kl} &= F_{kl} (\tilde{\epsilon}_{rs}, \Delta_{rs}, \phi_{r,s}, \nu_{r,s}, \phi, \nu, \phi_{,k}, \nu_{,k}, \theta) \\ s_{kl} &= S_{kl} (\tilde{\epsilon}_{rs}, \Delta_{rs}, \phi_{r,s}, \nu_{r,s}, \phi, \nu, \phi_{,k}, \nu_{,k}, \theta)\end{aligned}$$

$$\begin{aligned}
m_{kl} &= M_{kl} (\tilde{\epsilon}_{rs}, \Delta_{rs}, \phi_{r,s}, \nu_{r,s}, \theta, \nu, \phi_k, \nu_k, \theta) \\
\lambda_k &= \Lambda_k (\tilde{\epsilon}_{rs}, \Delta_{rs}, \phi_{r,s}, \nu_{r,s}, \theta, \nu, \phi_k, \nu_k, \theta) \\
q_k &= H_k (\tilde{\epsilon}_{rs}, \Delta_{rs}, \phi_{r,s}, \nu_{r,s}, \theta, \nu, \phi_k, \nu_k, \theta) \\
\psi &= \Psi (\tilde{\epsilon}_{rs}, \Delta_{rs}, \phi_{r,s}, \nu_{r,s}, \theta, \nu, \phi_k, \nu_k, \theta) \\
\eta &= N (\tilde{\epsilon}_{rs}, \Delta_{rs}, \phi_{r,s}, \nu_{r,s}, \theta, \nu, \phi_k, \nu_k, \theta) \quad (6.11)
\end{aligned}$$

As is customary in current literature [12, 17], in order to obtain a constitutively linear theory for micropolar materials with stretch suitable for practical applications, one could eliminate the heat supply term from the energy equation and the thermodynamic inequality, thereby obtaining the appropriate Clausius-Duhem inequality. Postulating the validity of this inequality for all independent thermodynamic processes would lead one to the desired constitutive equations for such media. Next, by expanding the free energy function in terms of its arguments, retained up to quadratic terms and using the constitutive forms obtained above, linear expressions for stress and couple stress can be constructed.

We shall not pursue this work further in this thesis, but state that we are now in the process of constructing such a constitutive theory. Our aim in pursuing this work is to apply this theory to problems of wave propagation in micropolar viscoelastic materials with stretch.

V. SUMMARY, DISCUSSION AND SCOPE OF FUTURE WORK

In the Introduction to this thesis we remarked on the relative scarcity of work, both experimental and theoretical, concerning smectic liquid crystals. The theoretical work which we have undertaken here is only our first step in filling this void and we are certain that it will stimulate others to attempt further experimental and theoretical studies.

The thermodynamic restrictions developed in Chapter II and the simplification of the constitutively linear theory due to the material symmetry properties of the smectic substance are invaluable aids to the experimentalist and theoretician alike. We point out, once again, that the thermodynamic restrictions were used extensively in obtaining the solution to the flow problem of heat-conducting smectic liquid crystals contained between two concentric rotating circular cylinders. Not only did the thermodynamic restrictions serve to simplify the mathematical problem, but they were instrumental in our efforts to identify a number of the material coefficients present in our theory. This identification of material coefficients is vital to any theoretical work, for if the theory is to have a practical value for the experimentalist then there must be a means for measuring the value of the material

coefficients. Thus we have demonstrated in Chapter III how one can determine a number of the material coefficients of our theory. Further work is also strongly indicated here, for there are still other material coefficients to be determined. However, this problem awaits further experimental work as well as further theoretical studies.

We have already embarked on providing a theory which will give an even better description of the smectic liquid crystal's rheological behavior as is evidenced by our work on micropolar elasticity with stretch in Chapter IV. We have indicated in that chapter how we are using this theory to aid us in the construction of the theory of micropolar viscoelasticity with stretch. After constructing this theory we plan to study wave propagation in substances such as smectic liquid crystals wherein the microelements can now undergo uniform stretching or contracting as well as rigid rotations.

Thus we see that the work accomplished in this thesis has, in effect, two purposes. One is to fill a void in the existing theoretical work and the other is to stimulate further theoretical and experimental work on smectic liquid crystals.

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