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SIMPLY SUPPORTED SANDWICH BEAM

A Nonlinear Theory

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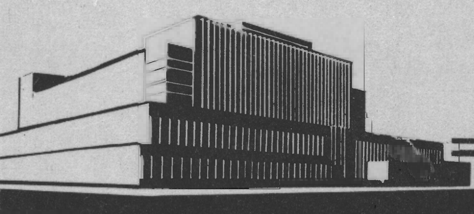

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ABSTRACT

Stresses in a simply supported sandwich beam under symmetric loading are obtained by a non linear theory. With the assumption of anti-plane core, the differential equations of equilibrium of the core are solved for core displacements in terms of four infinite sequences of integration constants. The facings are treated as cylindrically bent thin plates subject to large deflections. Stresses and displacements are matched at the interfaces of core and facings and the resulting non-linear differential equations are solved by Fourier analysis.

Since the differential equations are non-linear, the integration constants result from the simultaneous solution of an infinite set of non linear algebraic equations. Numerical solutions must therefore be obtained by successive approximations.

A single example is worked out for the case of a central load and the results are shown in a series of curves. Numerical work was performed with the aid of the IBM 650.

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SIMPLY SUPPORTED SANDWICH BEAM

A Nonlinear Theory[1]

by

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INTRODUCTION

This is an investigation of the stresses in a simply supported sandwich beam. The sandwich is assumed to consist of two thin strong facings bonded to a soft lightweight core which serves primarily to separate the facings. The investigation is restricted to those core materials, such as metal honeycomb, which are relatively stiff in the direction perpendicular to the facings and have negligible stiffness in the other two directions.

This study differs from the work of previous authors in that large deflections of the facings are considered--a matter of importance if the facings are very thin and the core material quite soft. Since the advent of high strength alloy steels and titanium alloys, such constructions have been employed in aircraft. Even for such constructions, the consideration of large deflections is not necessary unless the load is locally concentrated. Hence this theory is developed primarily to investigate the stresses in the neighborhood of a concentrated load. To simplify the mathematics, only symmetric loading is considered. The extension to a general loading is indicated but not carried out.

[1] A thesis submitted in partial fulfillment of the requirements for the degree of master of science (Mechanics) at the University of Wisconsin.

[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

A theory is derived which yields approximate numerical results by a rather lengthy iterative process. A secondary aim is to study the practicability of such a solution.

NOTATION

A_n, B_n, C_n, D_n	Integration constants.
α_{nmr}, δ_{nmrs}, γ_{nmrs}	Functions of the indices n, m, r, and s; defined by equations (29A).
b	Total length of the beam.
c	Core thickness.
$\tilde{c}$	$\frac{c}{b}$
E_1, E_2	Modulus of elasticity of the facings.
$\tilde{E}$	$\frac{E_2 \lambda_1}{E_1 \lambda_2}$
E_c	Modulus of elasticity of the core in the direction perpendicular to the facings.
$\tilde{E}_c$	$\frac{E_c \lambda_1}{E_1}$
f_1, f_2	Facing thickness.
$\tilde{f}_1$	$\frac{f_1}{b}$
$\tilde{f}_2$	$\frac{f_2}{b}$
G	Shear modulus of the core.
$i = 1, 2$	Subscripts denoting upper and lower facings respectively.
k_n	Fourier expansion coefficients of a unit load.
λ_1	$1 - \mu_1^2$
λ_2	$1 - \mu_2^2$
μ_1, μ_2	Poisson's ratio of the facings.

M_1, M_2	Bending moment in the individual facings.
$\tilde{M}_{1\,max}$	$\left.\dfrac{6M_1\lambda_1}{f_1^2\ E_1}\right\vert_{y=0}$ dimensionless stress parameter.
N_1, N_2	Membrane force in the facings. Tension is taken to be positive.
$\tilde{N}_{1\,max}$	$\left.\dfrac{N_1\ \lambda_1}{f_1\ E_1}\right\vert_{y=0}$ dimensionless stress parameter.
P	Total load in pounds per inch of width.
$\tilde{P}$	$\dfrac{q_{ave}\ \lambda_1}{E_1}$
ϕ_n, Ψ_n	Defined by equations (14) and (15).
q	Intensity of loading in pounds per square inch.
$\tilde{q}$	$\dfrac{q\ \lambda_1}{E_1}$
Q_1, Q_2	Vertical shear in the individual facings.
σ	Normal stress in the core. Tension is taken to be positive.
σ_1, σ_2	Boundary value of σ at the interfaces of core and facings.
$\tilde{\sigma}_{1\,max}$	$\left.\dfrac{\sigma_1}{E_c}\right\vert_{y=0}$ dimensionless core stress parameter.
τ	Core shear stress.
v, w	Core displacements in the y- and z-directions respectively.
v_1, w_1, v_2, w_2	Displacements of the middle planes of the facings.
$\tilde{w}_{1\,max}$	$\left.\dfrac{w_1}{b}\right\vert_{y=0}$

V_i, W_i, Y_i, Z_i Combinations of the integration constants given by equations (16).

y, z Coordinate axes.

ANALYSIS[3]

The present analysis of a sandwich beam follows closely the analysis of a simply supported sandwich plate by Milton E. Raville, (3)[4] differing primarily in that certain non-linear terms are considered. Specifically, a more exact expression for facing curvature is used and a non-linear term is included in the equations of facing equilibrium. This term results from a consideration of the vertical component of membrane force. Two other non-linear effects are also considered, but they are shown to mutually cancel.

Since the numerical solution of non-linear equations is often quite lengthy, a beam rather than a plate has been analysed so as to reduce the amount of computation. The loading is considered to be symmetric about the center of the span for the same purpose.

The Core

The difficulty of analysing a sandwich with thin facings and thick core is that a thick-plate theory should be used in the core. This

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[3]Details of the mathematical analysis are in the Appendix. Formula numbers ending in A refer to the appendix.

[4]Underlined numbers in parentheses refer to Literature Cited at the end of the thesis.

difficulty can be avoided if certain assumptions are made as to the nature of the core material. Specifically, the material is assumed to be stiff in only one direction. Goodier calls such materials "anti-plane" because their state of stress is the opposite of plane stress[5] (2). The resulting stress distribution in the core is quite simple and was known earlier to British investigators, who discovered it by another method.[6] (7)

With the assumption of anti-plane core and with coordinate axes as shown in figure 1, the following expressions for core displacements satisfy equilibrium at all points in the core:

$$w = b \sum_{\substack{n \\ \text{odd}}} \left\{ \frac{n\pi}{2} A_n \left(\frac{z}{c}\right)^2 + B_n \frac{z}{c} + C_n \right\} \cos \frac{n\pi y}{b} \tag{1}$$

$$v = c \sum_{\substack{n \\ \text{odd}}} \left\{ (n\pi) \left[\frac{n\pi}{6} A_n \left(\frac{z}{c}\right)^3 + \frac{1}{2} B_n \left(\frac{z}{c}\right)^2 + C_n \frac{z}{c} \right] - \frac{b^2}{c^2} \frac{E_c}{G} A_n \frac{z}{c} + D_n \right\} \sin \frac{n\pi y}{b} \tag{2}$$

The corresponding stress distribution in the core is:

$$\sigma = E_c \frac{b}{c} \sum_{\substack{n \\ \text{odd}}} \left\{ n\pi A_n \frac{z}{c} + B_n \right\} \cos \frac{n\pi y}{b} \tag{3}$$

$$\tau = - E_c \frac{b^2}{c^2} \sum_{\substack{n \\ \text{odd}}} A_n \sin \frac{n\pi y}{b} \tag{4}$$

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[5] That is, the components which are non-zero in a state of anti-plane stress are those which are zero in plane stress and vice versa.

[6] The "tilting method" of Williams, Leggett and Hopkins.

The expressions for core displacements and stresses contain four infinite sequences of integration constants: $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, and $\{D_n\}$. From equation (4) it is seen that $\{A_n\}$ is associated with shear stress in the core, which varies from zero at the center to a maximum at the supports. Thus the primary function of $\{A_n\}$ is to provide support reactions. Equation (3) shows that σ is a linear function of z. Evidently the primary function of $\{B_n\}$ is to provide the average core compressive stress at mid-span. Similarly, equation (1) shows that $\{C_n\}$ provides the center deflection of the middle plane of the core. Finally, $\{D_n\}$ provides for axial movement of the supports, as is seen in equation (2).[7]

The Support

The expression for v shows that plane cross-sections of the core do not remain plane since v is a non-linear function of z. The nature of the assumed support, then, is that every point on the cross-section at a support is restrained against vertical displacement but free to displace laterally, and the supported section is free to warp.

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[7] Since the supports are free to move axially, the facings contain membrane forces only by virtue of the shear rigidity of the core. If this shear rigidity were not present and E_c were infinite, the only resistance to deflection would be that provided by the bending stiffnesses of the individual facings. Conversely, if the shear rigidity of the core were also infinite, plane sandwich cross-sections would remain plane and the beam could be analysed by the elementary Bernoulli-Euler analysis of flexure.

The Facings

Since the facings are thin, they can be analysed by the usual assumptions of thin-plate theory (5). The equations of equilibrium are derived in the Appendix, taking into account the vertical components of membrane force, shear stress transmitted to the facing from the core, and variation in membrane force along the facing. These last two effects mutually cancel by virtue of equilibrium in the horizontal direction. The resulting equations are

$$\frac{dN_1}{dy} + \tau = 0 \tag{5}$$

$$\frac{d^2M_1}{dy^2} + \frac{f_1}{2}\frac{d\tau}{dy} + N_1\frac{d^2w_1}{dy^2} + \sigma_1 + q = 0 \tag{6}$$

$$\frac{dN_2}{dy} - \tau = 0 \tag{7}$$

$$\frac{d^2M_2}{dy^2} + \frac{f_2}{2}\frac{d\tau}{dy} + N_2\frac{d^2w_2}{dy^2} - \sigma_2 = 0 \tag{8}$$

where the subscripts 1 and 2 denote the upper and lower facings respectively. Core shear stress is not subscripted because it is independent of z.

Forces and moments in the facings are related to facing displacements by thin-plate theory, with one additional refinement: instead of using $\frac{d^2w}{dy^2}$ as an approximate expression for curvature, the following somewhat better approximation is made:

$$\frac{\frac{d^2w}{dy^2}}{[1 + (\frac{dw}{dy})^2]^{3/2}} \approx \frac{d^2w}{dy^2} [1 - \frac{3}{2} (\frac{dw}{dy})^2] \tag{9}$$

Thus the equations of facing equilibrium are written in terms of facing displacements (16A) (18A).

Symmetric Loading

The intensity of loading is represented by its Fourier cosine expansion

$$\tilde{q} = \sum_{n\ \mathrm{odd}} \tilde{P} k_n \cos \frac{n\pi y}{b}$$

where $\tilde{q} = \frac{q \lambda_1}{E_1}$ and $\tilde{P} = \frac{q_{ave} \lambda_1}{E_1}$

The magnitude is determined by a convenient dimensionless parameter, $\tilde{P}$, and the distribution is determined by the k_n.

Extension to General Loading

Any load can be represented as the sum of its symmetric and anti-symmetric parts. A general load can be analysed by adding anti-symmetric parts to the foregoing expressions for core displacements and applied loading. The added expressions will be similar to the ones given except that $\cos \frac{n\pi y}{b}$ will be replaced by $\sin \frac{n\pi y}{b}$ and the summations will be carried out over n even. There will then be eight sequences of integration constants, four for the symmetric part and four for the anti-symmetric part. It should be noted that symmetric and anti-symmetric loads cannot be solved independently and superimposed for this is a non-linear theory and there will be cross products from all eight sequences of integration constants in equations (10) through (13).

Continuity of Stresses and Displacements

Stresses and displacements must be continuous at the interfaces of core and facings, i.e. at $z = \pm \frac{c}{2}$. By matching displacements at this boundary, facing displacements are obtained in terms of the core displacements--hence ultimately in terms of the integration constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, and $\{D_n\}$. Continuity of stress is insured by identifying the boundary values of facing stress with the boundary values of core stress at the interfaces of core and facings.

From equations (3) and (4) the boundary values of core stress are obtained in terms of the integration constants of the core. Thus the four equations of facing equilibrium are the conditions from which the four sequences of integration constants can be evaluated. In terms of integration constants, these four equations are (23A), (24A), (25A), and (26A) in the Appendix.

Fourier Analysis

Equations (23A), (24A), (25A), and (26A) are reduced to algebraic equations in $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, and $\{D_n\}$ by taking advantage of the orthogonality of $\{\cos \frac{n\pi y}{b}\}$ and of $\{\sin \frac{n\pi y}{b}\}$ over the interval $(-\frac{b}{2}, \frac{b}{2})$. The equations for the determination of the integration constants are thus

$$\left\{\frac{\tilde{E}_c}{\tilde{c}^3 \tilde{f}_1} \cdot \frac{1}{(n\pi)^3} - (n\pi)\left[\frac{1}{48} + \frac{\tilde{f}_1}{16\tilde{c}}\right] + \frac{1}{2\tilde{c}^2}\frac{\tilde{E}_c}{\tilde{G}} \cdot \frac{1}{n\pi}\right\} A_n$$

$$+\left\{\frac{1}{8} + \frac{\tilde{f}_1}{4\tilde{c}}\right\} B_n - \frac{1}{2}\left\{1 + \frac{\tilde{f}_1}{\tilde{c}}\right\} C_n + \frac{1}{n\pi} D_n = 0 \tag{10}$$

$$\left\{\frac{\tilde{E}_c}{\tilde{E}\tilde{c}^3 \tilde{f}_2} \cdot \frac{1}{(n\pi)^3} - (n\pi)\left[\frac{1}{48} + \frac{\tilde{f}_2}{16\tilde{c}}\right] + \frac{1}{2\tilde{c}^2}\frac{\tilde{E}_c}{\tilde{G}}\frac{1}{n\pi}\right\} A_n$$

$$-\left\{\frac{1}{8} + \frac{\tilde{f}_2}{4\tilde{c}}\right\} B_n - \frac{1}{2}\left\{1 + \frac{\tilde{f}_2}{\tilde{c}}\right\} C_n - \frac{1}{n\pi} D_n = 0 \tag{11}$$

$$\left\{\frac{\tilde{f}_1^{\,3}(n\pi)^5}{96} + \frac{1}{2}\frac{\tilde{f}_1}{\tilde{c}^2}\tilde{E}_c(n\pi) + \frac{1}{2}\frac{\tilde{E}_c}{\tilde{c}}(n\pi)\right\} A_n$$

$$-\left\{\frac{\tilde{f}_1^{\,3}}{24}(n\pi)^4 + \frac{\tilde{E}_c}{\tilde{c}}\right\} B_n + \frac{\tilde{f}_1^{\,3}}{12}(n\pi)^4 C_n = \phi_n \tag{12}$$

$$\left\{\frac{\tilde{E}\tilde{f}_2^{\,3}(n\pi)^5}{96} + \frac{1}{2}\frac{\tilde{f}_2}{\tilde{c}^2}\tilde{E}_c(n\pi) + \frac{1}{2}\frac{\tilde{E}_c}{\tilde{c}}(n\pi)\right\} A_n$$

$$+\left\{\frac{\tilde{E}\tilde{f}_2^{\,3}}{24}(n\pi)^4 + \frac{\tilde{E}_c}{\tilde{c}}\right\} B_n + \frac{\tilde{E}\tilde{f}_2^{\,3}}{12}(n\pi)^4 C_n = \Psi_n \tag{13}$$

where

$$\phi_n = \tilde{P} k_n - \tilde{f}_1 \tilde{c} \sum_m \sum_r (m\pi) V_m (r\pi)^2 W_r \alpha_{nmr}$$

$$+ \frac{\tilde{f}_1{}^3}{8} \sum_m \sum_r \sum_s (m\pi)^2 (r\pi)(s\pi) W_m W_r W_s \Big\{ [(m\pi)^2$$

$$+ 6(s\pi)^2] \gamma_{nmrs} - 2(r\pi)(s\pi) \delta_{nmrs} \Big\} \tag{14}$$

$$\Psi_n = - \tilde{E} \tilde{f}_2 \tilde{c} \sum_m \sum_r (m\pi) Y_m (r\pi)^2 Z_r \alpha_{nmr}$$

$$+ \frac{\tilde{E}\tilde{f}_2{}^3}{8} \sum_m \sum_r \sum_s (m\pi)^2 (r\pi)(s\pi) Z_m Z_r Z_s \Big\{ [(m\pi)^2$$

$$+ 6(s\pi)^2] \gamma_{nmrs} - 2(r\pi)(s\pi) \delta_{nmrs} \Big\} \tag{15}$$

$$\left.\begin{aligned}
V_i &= (i\pi) \Big[\Big(- \frac{i\pi}{48} - \frac{\tilde{f}_1}{2\tilde{c}} \frac{i\pi}{8} + \frac{\tilde{E}_c}{2\tilde{G}\tilde{c}^2} \cdot \frac{1}{i\pi} \Big) A_i + \Big(\frac{1}{8} + \frac{\tilde{f}_1}{4\tilde{c}} \Big) B_i \\
&\quad - \Big(\frac{1}{2} + \frac{\tilde{f}_1}{2\tilde{c}} \Big) C_i \Big] + D_i \\
W_i &= \frac{i\pi}{8} A_i - \frac{1}{2} B_i + C_i \\
Y_i &= (i\pi) \Big[\Big(\frac{i\pi}{48} + \frac{\tilde{f}_2}{2\tilde{c}} \frac{i\pi}{8} - \frac{\tilde{E}_c}{2\tilde{G}\tilde{c}^2} \cdot \frac{1}{i\pi} \Big) A_i + \Big(\frac{1}{8} + \frac{\tilde{f}_2}{4\tilde{c}} \Big) B_i \\
&\quad + \Big(\frac{1}{2} + \frac{\tilde{f}_2}{2\tilde{c}} \Big) C_i \Big] + D_i \\
Z_i &= \frac{i\pi}{8} A_i + \frac{1}{2} B_i + C_i
\end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned}
\widetilde{E} &= \frac{E_2\lambda_1}{E_1\lambda_2} \\
\widetilde{E}_c &= \frac{E_c\lambda_1}{E_1} \\
\widetilde{G} &= \frac{G\lambda_1}{E_1} \\
\widetilde{f}_1 &= \frac{f_1}{b} \\
\widetilde{f}_2 &= \frac{f_2}{b} \\
\widetilde{c} &= \frac{c}{b}
\end{aligned}\right\} \qquad (17)$$

and α_{nmr}, δ_{nmrs}, and γ_{nmrs} are given by equations (29A) of the Appendix.

Equations (10) through (13) are an infinite set of simultaneous equations, four for each value of n, in an infinite number of unknowns, also four for each value of n. To obtain a particular numerical result, a finite subset may be used, the required number of terms depending upon the particular loading and sandwich construction considered. From the nature of the problem, it is known that only one real solution will exist if the beam is elastically stable. Furthermore, the real solutions will converge as the size of the set is increased.

Solution by Successive Approximations

If n terms are taken, there are 4n unknowns and 4n equations, half of which contain non-linear terms on the right hand side. A first approximation can be obtained by ignoring the non-linear terms and solving the remaining linear set. This is Raville's solution. To obtain each succeeding approximation, the non-linear part can be treated as a constant evaluated from the previous approximation and the resulting linear set solved as before, etc. The final value of an unknown is then the limit of a sequence of values of successive approximations.

If this sequence should prove to be divergent, or too slowly convergent, the iterative process can be amended: Evaluate the non-linear terms from a weighted average of the two previous approximations. For example, if x_j is the j-th approximate value of the unknown x, the non-linear terms are evaluated from the $\hat{x}_j$, where

$$\hat{x}_j = (x_j - x_{j-1})\, p + x_{j-1}$$

and p is a weight factor.

The proper value of the weight factor can speed up slow convergence or force a divergent sequence to converge. It is dependent upon the magnitude and distribution of the load, and must be chosen by trial.

EXAMPLE: The Simply Supported Beam Under Central Load

The following sandwich construction was chosen:

$$\tilde{E} = 1, \quad \tilde{E}_c = \frac{1}{50}, \quad \tilde{G} = \frac{1}{300}, \quad \tilde{f}_1 = \frac{5}{6,000}, \quad \tilde{f}_2 = \frac{5}{6,000}, \quad \tilde{c} = \frac{5}{60}$$

This could be, for example, a six inch beam with steel facings of 0.005 inch thickness bonded to a core, 0.50 inch thick, of square-cell stainless steel honeycomb. A linear theory would be expected to give poor results for such a construction.

Load Distribution

Let P be the total load in pounds per inch of width. To approximate actual loading more closely, the load was taken to be distributed about the neighborhood of the center rather than concentrated. A suitable function $\tilde{q}(y)$ is the following adaptation of the Witch of Agnesi as used by Smith and Voss (4).

$$\tilde{q}(y) = \tilde{P}\,\frac{\delta b}{\pi} \sum_{j=-\infty}^{\infty} \frac{1}{(y + jb)^2 + \delta^2} \tag{18}$$

The Fourier cosine expansion of $\tilde{q}(y)$ over $(-\frac{b}{2}, \frac{b}{2})$ is

$$\tilde{q}(y) = \tilde{P} \sum_{\substack{n \\ \text{odd}}} k_n \cos \frac{n\pi y}{b} \tag{19}$$

where

$$\tilde{P} = \frac{P\lambda_1}{bE_1} \tag{20}$$

and, by Fourier analysis,

$$\left.\begin{aligned} k_n &= \frac{2}{b} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[\frac{\delta b}{\pi} \sum_{j=-\infty}^{\infty} \frac{1}{(y + jb)^2 + \delta^2} \right] \cos \frac{n\pi y}{b} \, dy \\ &= \frac{2\delta}{\pi} \int_{-\infty}^{\infty} \frac{\cos \frac{n\pi y}{b}}{y^2 + \delta^2} \, dy \\ &= 2e^{-\frac{n\pi\delta}{b}} \end{aligned}\right\} \quad (21)$$

The parameter $\frac{\delta}{b}$ determines the degree of concentration of the load, smaller values giving a more concentrated load than larger values.

Seven terms were taken. That is, a sub-set of equations (10), (11), (12), and (13) corresponding to the first seven odd integral values of n was used to obtain approximate values for the first 28 unknowns A_1 through D_{13}. Thus the seventh partial sum of (19) is the actual load distribution applied. Accordingly, $\frac{\delta}{b}$ was adjusted by trial to make the seventh partial sum as smooth as possible. This yielded the value $\frac{\delta}{b} = 0.03$. The corresponding seventh partial sum of (19) is shown in figure 2.

Effect of Non-Linearity

If it is considered that $k_n = 0$ for n greater than 13, the loading shown in figure 2 is not only the seventh partial sum of $\tilde{q}$, but the exact expression for $\tilde{q}$. However, the corresponding expressions for stresses and displacements are not exact unless an infinite number of terms are taken, because $\cos \frac{m\pi y}{b} \cos \frac{s\pi y}{b}$, m and s odd, is not orthogonal to $\cos \frac{n\pi y}{b}$, n odd, over the interval $\left(-\frac{b}{2}, \frac{b}{2} \right)$.

Whence the appearance of α_{nmr} in equations (12) and (13). Similarly, δ_{nmrs} and γ_{nmrs} arise because of non-orthogonality. Through the agency of α_{nmr}, δ_{nmrs}, γ_{nmrs} and the double- and triple-summations in equations (12) and (13), each integration constant is made to depend on <u>all</u> other integration constants, infinite in number, regardless of how many k_n are non-zero.

Simultaneous Solution

The 28 non-linear equations were solved simultaneously by successive approximations as described in the Analysis. In each approximation, seven fourth-order linear systems had to be solved. To minimize the round-off error, the 4 x 5 array was arranged as follows to obtain a strong principal diagonal:

	A_n	B_n	C_n	D_n =	$\phi_n + \Psi_n$	$\phi_n - \Psi_n$
(12) + (13)	β_{11}	β_{12}	...		1	0
(12) - (13)	β_{21}	β_{22}	...		0	1
(10) + (11)	.	.			0	0
(10) - (11)	.	.			0	0

That is, β_{11} is the coefficient of A_n in the equation obtained by adding equation (13) to equation (12), etc. The array indicated above has many off-diagonal terms equal to zero for all constructions which have equal similar facings. For constructions with greatly unequal facing thicknesses and greatly dissimilar facing properties, the off-diagonal terms would be appreciable and a greater round-off error would result. For such constructions, it may be necessary to find a better array.

Data Obtained

For each of five values of $\widetilde{P}$, the 28 integration constants were obtained by successive approximations. From the integration constants, the force and moment in the upper facing, the deflection of the upper facing, and the maximum core compressive stress were each evaluated at the center of the beam. The results are presented in figures 3-6 where the following dimensionless parameters are shown plotted against $\widetilde{P}$:

$$\widetilde{N}_{1\,max} = \left.\frac{N_1 \lambda_1}{f_1 E_1}\right|_{y=0}, \qquad \widetilde{M}_{1\,max} = \left.\frac{6 M_1 \lambda_1}{f_1^2 E_1}\right|_{y=0}$$

$$\widetilde{\sigma}_{1\,max} = \left.\frac{\sigma_1}{E_c}\right|_{y=0}, \qquad \widetilde{w}_{1\,max} = \left.\frac{w_1}{b}\right|_{y=0} \tag{22}$$

In terms of the above dimensionless groups, the extreme fiber stress in the top facing is given by

$$(\widetilde{N}_1 \pm \widetilde{M}_1)\,\frac{E_1}{\lambda_1}$$

For comparison, the curves in figures 3-6 also show the solution by Raville's linear theory as dashed lines.

DISCUSSION OF RESULTS

The load parameter $\widetilde{P}$ was varied from zero to a value well beyond that expected to produce failure. For example, if the top facing were steel, figure 3 would show direct facing stress up to 870,000 psi. If the top facing were aluminum, the same figure would show stress up to 290,000 psi. In either case, the curve extends well beyond failure. Similarly, figure 5 shows core compressive stress to two or three times the probable strength of metal honeycomb.

Comparison With Linear Theory

A comparison of figures 3 through 6 shows that the non-linear effects here considered produce only negligible corrections to facing stress and maximum deflection but a significant correction to core compressive stress. This indicates that a linear theory is probably sufficient for the design of sandwich beams since the core may be intentionally overdesigned without much increase in overall weight.

Relative Importance of Effects Considered

The two second order effects considered are, in the order of their importance:

Effect (1)--The vertical component of membrane force. This introduces the term $N \frac{d^2w}{dy^2}$ in equations (7A) and (11A) and the double summations on the right in equations (12) and (13).

Effect (2)--The more exact expression for facing curvature. This introduces the bracket on the right in equations (16A) and (18A) and the triple summations on the right in equations (12) and (13).

To investigate these effects separately, a load of $\widetilde{P} = 7.22 \times 10^{-6}$ was applied and equations (10) through (13) solved twice, considering first both effects and then effect (1) alone. The resulting stresses and displacements, to five significant figures, were identical. Effect (2) may be more important, however, for a more locally concentrated load. The degree of concentration cannot be increased simply by decreasing the value of $\frac{\delta}{b}$, since then the seventh partial sum of $\widetilde{q}$ is no longer smooth but has many positive and negative peaks. To obtain a smooth loading with increased concentration, higher terms with shorter wavelengths must be used. Accordingly, ten terms were taken, $\frac{\delta}{b}$ was adjusted to a value of 0.021 and the above comparison was repeated. Again, the results were identical to five significant figures. A plot of the ten-term load distribution with $\frac{\delta}{b} = 0.021$ is shown in figure 7. The curves in figures 3 through 6 were obtained with seven terms.

<u>Rate of Convergence</u>

Convergence forcing was not required. For each integration constant, the sequence of successive approximations was monotone and rapidly convergent. Table 1 shows the number of iterations at various loads required to obtain four significant figures in the expressions for stress.

Elastic Buckling

Since this is a non-linear theory, it should exhibit the phenomenon of elastic buckling. For lack of time, this was not presently investigated. It may be pointed out, however, that as the buckling range is approached, the method will converge much more slowly or may even diverge, necessitating recourse to some means of convergence forcing. More concentrated loadings are expected to yield lower buckling loads, since a local dent in the upper facing would facilitate buckling. The subject is, however, academic, since the beam, if it buckles at all, buckles plastically.

CONCLUSIONS AND RECOMMENDATIONS

One of the chief advantages of a non-linear theory is that no linear theory pictures the core compressive stress accurately. However, the principal load carrying members of a sandwich are the facings, and facing stress is predicted with sufficient accuracy by a linear theory. Nevertheless, an accurate knowledge of the core compressive stress could be of value in certain situations.

For example, this theory could be used in conjunction with an experimental program utilizing simply supported sandwich beams under a vibrating central load to study core fatigue properties. An advantage of this test is that the fabricated sandwich is tested rather than the core material alone, permitting an evaluation of the effect of different methods of fabrication on core fatigue life. The program could include constructions with very thin facings such as are currently in use.

Even in this theory, however, the accuracy of core compressive stress is contingent upon two assumptions: (1) that the exact load distribution is known and can be adequately represented by a Fourier series partial sum, and (2) that the core material is anti-plane, an assumption that no real material exactly satisfies.

The usefulness of this theory is limited by the large number of parameters involved. It may be possible to reduce this number by assuming that facings which are otherwise dissimilar and unequal have the same Poisson's ratio. Chang and Ebcioglu succeeded in reducing the number of parameters in their solution of sandwich panels under compressive edge load in this manner (1). Of course, if the facings are equal and similar, the number of parameters reduces immediately from six to four.

The accumulation of round-off error and the time and expense of securing an answer put a practical limit on the number of terms which can profitably be considered. Yet locally concentrated load distributions are difficult to represent with a small number of terms, and it is for just such loadings that a non-linear theory is most needed. The example shows that a highly concentrated central load can only be smoothly represented by a Fourier series partial sum of more than ten times.

It is suggested that more terms be taken to permit the investigation of more concentrated loads. The amount of computation could be reduced byconsidering only the first non-linear effect and iterating until the system converged. With this as a starting approximation, the iteration could then be continued considering both non-linear effects. This would reduce the computing time by a factor of nearly n, when n terms are taken.

An investigation of elastic buckling should be pursued as a matter of academic interest.

Although the non-linear terms from large deflections are relatively unimportant below the elastic limit, they would be of more importance in a limit design analysis in which inelastic action was considered.

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Table 1.

Number of terms	Effects considered (1)	Effects considered (2)	$\widetilde{P} \times 10^6$	Number of iterations
7	x	x	1.944	4
7	x	x	2.777	4
7	x	x	5.555	5
7	x	x	7.222	6
7	x	x	9.444	6
7	x		7.222	6
10	x		7.222	6
10	x	x	7.222	6

APPENDIX

Details of the Mathematics

The Core

The core is assumed to be in a state of anti-plane stress. That is,

$$\sigma_x = 0$$

$$\sigma_y = 0$$

$$\tau_{xy} = 0$$

Since the beam is taken to lie in the y-z plane, τ_{xz} is also zero. Hence there are only two components of stress in the core: σ_z and τ_{yz}. Hereafter the subscripts are dropped and it is understood that unsubscripted variables refer to the core. Subscripts 1 and 2 are used to denote the boundary values of stress at the junctions with the upper and lower facings respectively.

Equilibrium. -- The differential equations of equilibrium are (6)

$$\left.\begin{aligned} &\frac{\partial \tau}{\partial z} = 0 \\ &\frac{\partial \tau}{\partial y} + \frac{\partial \sigma}{\partial z} = 0 \end{aligned}\right\} \qquad (1A)$$

Hooke's law. --With the substitution of Hooke's law,

$$\left.\begin{aligned} &\sigma = E_c \frac{\partial w}{\partial z} \\ &\tau = G \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \end{aligned}\right\} \qquad (2A)$$

the differential equations of equilibrium are written in terms of core displacements:

$$\left.\begin{aligned} &\frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 v}{\partial z^2} = 0 \\ &G\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial z} \right) + E_c \frac{\partial^2 w}{\partial z^2} = 0 \end{aligned}\right\} \qquad (3A)$$

<u>Boundary conditions</u>.--The boundary conditions at $y = \pm \frac{b}{2}$ are those of simple support:

$$w \Big|_{y = \pm \frac{b}{2}} = 0 \; ; \qquad \frac{\partial^2 w}{\partial y^2}\Big|_{y = \pm \frac{b}{2}} = 0$$

<u>Displacements</u>.--The solution of equations (3A) which satisfies the conditions of simple support can be obtained by separation of variables. It is

$$w = b \sum_{n \text{ odd}} \left\{ \frac{n\pi}{2} A_n \left(\frac{z}{c}\right)^2 + B_n \frac{z}{c} + C_n \right\} \cos \frac{n\pi y}{b}$$

$$\begin{aligned} v = c \sum_{n \text{ odd}} &\left\{ (n\pi) \left[\frac{n\pi}{6} A_n \left(\frac{z}{c}\right)^3 + \frac{1}{2} B_n \left(\frac{z}{c}\right)^2 + C_n \frac{z}{c} \right] \right. \\ &\left. - \frac{b^2}{c^2} \frac{E_c}{G} A_n \frac{z}{c} + D_n \right\} \sin \frac{n\pi y}{b} \end{aligned} \qquad (4A)$$

where the summations are carried out over n odd only.

<u>Stresses</u>.--From equations (4A) and (2A) the corresponding expressions for core stresses are found to be

$$\left.\begin{aligned} \sigma &= E_c \frac{b}{c} \sum_{n\ \mathrm{odd}} \left\{ n\pi A_n \frac{z}{c} + B_n \right\} \cos \frac{n\pi y}{b} \\ \tau &= - E_c \frac{b^2}{c^2} \sum_{n\ \mathrm{odd}} A_n \sin \frac{n\pi y}{b} \end{aligned}\right\} \qquad (5A)$$

The Facings

The width of the beam is surely great in comparison to the thickness of the facings. Hence, the facings are treated as cylindrically bent thin plates.

Equilibrium of the upper facing. --See figure 8. Neglecting the horizontal component of Q_1, the differential equations of equilibrium are

$$[\Sigma M = 0] \curvearrowleft + \qquad \frac{dM_1}{dy} + \frac{f_1}{2} \tau - Q_1 = 0 \qquad (6A)$$

$$[\Sigma F_z = 0] \downarrow + \qquad \frac{dQ_1}{dy} + N_1 \frac{d^2 w_1}{dy^2} + \sigma_1 + \frac{dN_1}{dy} \frac{dw_1}{dy}$$

$$+ \tau \frac{dw_1}{dy} + q = 0 \qquad (7A)$$

$$[\Sigma F_y = 0] \xrightarrow{+} \qquad \frac{dN_1}{dy} + \tau = 0 \qquad (8A)$$

By virtue of equation (8A), the fourth and fifth terms of equation (7A) drop out. Eliminating Q_1 between equations (6A) and (7A) yields

$$\frac{d^2 M_1}{dy^2} + \frac{f_1}{2} \frac{d\tau}{dy} + N_1 \frac{d^2 w_1}{dy^2} + \sigma_1 + q = 0 \qquad (9A)$$

Thus equations (8A) and (9A) describe equilibrium of the upper facing.

Equilibrium of the lower facing.--See figure 9. Neglecting the horizontal component of Q_2, the differential equations of equilibrium are

$$[\Sigma M = 0] \; + \qquad \frac{dM_2}{dy} + \frac{f_2}{2} \tau - Q_2 = 0 \tag{10A}$$

$$[\Sigma F_z = 0] \downarrow + \qquad \frac{dQ_2}{dy} + N_2 \frac{d^2 w_2}{dy^2} - \sigma_2 + \frac{dN_2}{dy} \frac{dw_2}{dy} - \tau \frac{dw_2}{dy} = 0 \tag{11A}$$

$$[\Sigma F_y = 0] \xrightarrow{+} \qquad \frac{dN_2}{dy} - \tau = 0 \tag{12A}$$

By virtue of equation (12A), the fourth and fifth terms of equation (11A) drop out. Eliminating Q_2 between equations (10A) and (11A) yields

$$\frac{d^2M_2}{dy^2} + \frac{f_2}{2}\frac{d\tau}{dy} + N_2 \frac{d^2 w_2}{dy^2} - \sigma_2 = 0 \tag{13A}$$

Thus equations (12A) and (13A) describe equilibrium of the lower facing.

Displacements.--In order to match displacements of core and facings at the boundaries $z = \pm \frac{c}{2}$, the equations of equilibrium of the facings must be written in terms of facing displacements. Let w_1, v_1, w_2, and v_2 be displacements of the middle planes of the facings.

For cylindrical bending of a thin plate (5),

$$\left.\begin{aligned} N_1 &= \frac{E_1 f_1}{\lambda_1}\frac{dv_1}{dy} ; \qquad M_1 = -\frac{E_1 f_1^3}{12\lambda_1}\frac{d^2 w_1}{dy^2}\left[1 - \frac{3}{2}\left(\frac{dw_1}{dy}\right)^2\right] \\ N_2 &= \frac{E_2 f_2}{\lambda_2}\frac{dv_2}{dy} ; \qquad M_2 = -\frac{E_2 f_2^3}{12\lambda_2}\frac{d^2 w_2}{dy^2}\left[1 - \frac{3}{2}\left(\frac{dw_2}{dy}\right)^2\right] \end{aligned}\right\} \qquad (14A)$$

where the usual approximation for curvature has been replaced by a somewhat better one.

Substituting (14A) into equations (8A), (9A), (12A), and (13A) yields

$$\frac{E_1 f_1}{\lambda_1}\frac{d^2 v_1}{dy^2} + \tau = 0 \qquad (15A)$$

$$-\frac{E_1 f_1^3}{12\lambda_1}\frac{d^4 w_1}{dy^4} + \frac{f_1}{2}\frac{d\tau}{dy} + \sigma_1 = -q - \frac{E_1 f_1}{\lambda_1}\frac{dv_1}{dy}\frac{d^2 w_1}{dy^2}$$
$$-\frac{E_1 f_1^3}{8\lambda_1}\left[\frac{d^4 w_1}{dy^4}\left(\frac{dw_1}{dy}\right)^2 + 6\frac{dw_1}{dy}\frac{d^2 w_1}{dy^2}\frac{d^3 w_1}{dy^3} + 2\left(\frac{d^2 w_1}{dy^2}\right)^3\right] \qquad (16A)$$

$$\frac{E_2 f_2}{\lambda_2}\frac{d^2 v_2}{dy^2} - \tau = 0 \qquad (17A)$$

$$-\frac{E_2 f_2^3}{12\lambda_2}\frac{d^4 w_2}{dy^4} + \frac{f_2}{2}\frac{d\tau}{dy} - \sigma_2 = -\frac{E_2 f_2}{\lambda_2}\frac{dv_2}{dy}\frac{d^2 w_2}{dy^2}$$

$$-\frac{E_2 f_2^3}{8\lambda_2}\left[\frac{d^4 w_2}{dy^4}\left(\frac{dw_2}{dy}\right)^2 + 6\frac{dw_2}{dy}\frac{d^2 w_2}{dy^2}\frac{d^3 w_2}{dy^3} + 2\left(\frac{d^2 w_2}{dy^2}\right)^3\right]$$
$$(18A)$$

Stresses. --From thin plate theory, the extreme fiber stress in a facing is given by

$$\frac{N_i}{f_i} \pm \frac{6M_i}{f_i^2} , \qquad i = 1, 2$$

Matching Displacements at the Interfaces of Core and Facings

Continuity of displacements at $z = \pm \frac{c}{2}$ requires that

$$w_1 = w\Big|_{\frac{z}{c} = -\frac{1}{2}}$$

$$v_1 = v\Big|_{\frac{z}{c} = -\frac{1}{2}} + \frac{f_1}{2}\frac{dw_1}{dy}$$

$$w_2 = w\Big|_{\frac{z}{c} = \frac{1}{2}}$$

$$v_2 = v\Big|_{\frac{z}{c} = \frac{1}{2}} - \frac{f_2}{2}\frac{dw_2}{dy}$$

Therefore,

$$w_1 = b \sum_{n\ odd} \left\{ \frac{n\pi}{8} A_n - \frac{1}{2} B_n + C_n \right\} \cos \frac{n\pi y}{b} \tag{19A}$$

$$v_1 = c \sum_{n\ odd} \left\{ n\pi \left[\left(-\frac{n\pi}{48} - \frac{f_1}{2c}\frac{n\pi}{8} + \frac{b^2}{c^2}\frac{E_c}{G}\frac{1}{2n\pi} \right) A_n \right. \right.$$

$$\left. \left. + \left(\frac{1}{8} + \frac{f_1}{4c} \right) B_n - \left(\frac{1}{2} + \frac{f_1}{2c} \right) C_n \right] + D_n \right\} \sin \frac{n\pi y}{b} \tag{20A}$$

$$w_2 = b \sum_{n\ odd} \left\{ \frac{n\pi}{8} A_n + \frac{1}{2} B_n + C_n \right\} \cos \frac{n\pi y}{b} \tag{21A}$$

$$v_2 = c \sum_{n\ odd} \left\{ n\pi \left[\left(\frac{n\pi}{48} + \frac{f_2}{2c}\frac{n\pi}{8} - \frac{b^2}{c^2}\frac{E_c}{G}\frac{1}{2n\pi} \right) A_n \right. \right.$$

$$\left. \left. + \left(\frac{1}{8} + \frac{f_2}{4c} \right) B_n + \left(\frac{1}{2} + \frac{f_2}{2c} \right) C_n \right] + D_n \right\} \sin \frac{n\pi y}{b} \qquad (22A)$$

The foregoing expressions for facing displacements provide continuity of stresses and displacements at $z = \pm \frac{c}{2}$ and satisfy equilibrium of the core.

The Load

The intensity of load is expressed as a Fourier cosine series:

$$q = \frac{E_1}{\lambda_1} \sum_{n\ odd} \tilde{P}\, k_n \cos \frac{n\pi y}{b}$$

where $\tilde{P} = \frac{q_{ave}\,\lambda_1}{E_1}$ is a convenient dimensionless parameter which determines the magnitude of the load.

Fourier Analysis

Substitution of expressions (5A) and (19A) through (22A) into equations (15A) through (18A) gives the following four equations:

$$\sum_{n\ odd} \left\{ \left[\frac{\tilde{E}_c}{\tilde{c}^3 \tilde{f}_1} \frac{1}{(n\pi)^3} - n\pi \left(\frac{1}{48} + \frac{\tilde{f}_1}{16\tilde{c}} \right) + \frac{\tilde{E}_c}{2\tilde{c}^2 \tilde{G}} \frac{1}{n\pi} \right] A_n \right.$$

$$+ \left[\frac{1}{8} + \frac{\tilde{f}_1}{4\tilde{c}} \right] B_n - \frac{1}{2} \left[1 + \frac{\tilde{f}_1}{\tilde{c}} \right] C_n$$

$$\left. + \frac{1}{n\pi} D_n \right\} \sin \frac{n\pi y}{b} = 0 \qquad (23A)$$

$$\sum_{n\ \mathrm{odd}} \Bigg\{ \left[\frac{\tilde{f}_1^{\,3}}{96} (n\pi)^5 + \frac{\tilde{f}_1}{2\tilde{c}^2} \tilde{\tilde{E}}_c\, n\pi + \frac{\tilde{\tilde{E}}_c}{2\tilde{c}}\, n\pi \right] A_n$$

$$- \left[\frac{\tilde{f}_1^{\,3}}{24} (n\pi)^4 + \frac{\tilde{\tilde{E}}_c}{\tilde{c}} \right] B_n + \frac{\tilde{f}_1^{\,3}}{12} (n\pi)^4\, C_n \Bigg\} \cos \frac{n\pi y}{b}$$

$$= - \sum_{n\ \mathrm{odd}} \tilde{P}\, k_n \cos \frac{n\pi y}{b}$$

$$- \tilde{f}_1 \tilde{c} \sum_{n\ \mathrm{odd}} \sum_{m\ \mathrm{odd}} n\pi V_n\, (m\pi)^2\, W_m \cos \frac{n\pi y}{b} \cos \frac{m\pi y}{b}$$

$$+ \frac{\tilde{f}_1^{\,3}}{8} \Bigg[\sum_{n\ \mathrm{odd}} \sum_{m\ \mathrm{odd}} \sum_{r\ \mathrm{odd}} (n\pi)^4\, W_n\, (m\pi)\, W_m\, (r\pi)\, W_r$$

$$\cdot \cos \frac{n\pi y}{b} \sin \frac{m\pi y}{b} \sin \frac{r\pi y}{b}$$

$$+ 6 \sum_{\substack{n\\ \mathrm{odd}}} \sum_{\substack{m\\ \mathrm{odd}}} \sum_{\substack{r\\ \mathrm{odd}}} (n\pi)\, W_n\, (m\pi)^2\, W_m\, (r\pi)^3\, W_r \cos \frac{n\pi y}{b}$$

$$\cdot \sin \frac{m\pi y}{b} \sin \frac{r\pi y}{b}$$

$$- 2 \sum_{\substack{n\\ \mathrm{odd}}} \sum_{\substack{m\\ \mathrm{odd}}} \sum_{\substack{r\\ \mathrm{odd}}} (n\pi)^2\, W_n (m\pi)^2\, W_m\, (r\pi)^2\, W_r \cos \frac{n\pi y}{b}$$

$$\cdot \cos \frac{m\pi y}{b} \cos \frac{r\pi y}{b} \Bigg] \qquad (24A)$$

$$\sum_{n\ \mathrm{odd}} \Bigg\{ \Big[\frac{\tilde{E}_c}{\tilde{E}\tilde{c}^3\tilde{f}_2} \frac{1}{(n\pi)^3} - n\pi \Big(\frac{1}{48} + \frac{\tilde{f}_2}{16\tilde{c}} \Big) + \frac{\tilde{E}_c}{2\tilde{c}^2\,\tilde{G}} \frac{1}{n\pi} \Big] A_n$$

$$- \Big[\frac{1}{8} + \frac{\tilde{f}_2}{4\tilde{c}}\Big] B_n - \frac{1}{2}\Big[1 + \frac{\tilde{f}_2}{\tilde{c}}\Big] C_n - \frac{1}{n\pi} D_n \Bigg\} \sin\frac{n\pi y}{b} = 0 \quad (25A)$$

$$\sum_{n\ \mathrm{odd}} \Bigg\{ \Big[\frac{\tilde{f}_2{}^3}{96} (n\pi)^5 + \frac{\tilde{f}_2}{2\tilde{c}^2} \tilde{E}_c\, n\pi + \frac{\tilde{E}_c}{2\tilde{c}}\, n\pi \Big] A_n + \Big[\frac{\tilde{f}_2{}^3}{24} (n\pi)^4$$

$$+ \frac{\tilde{E}_c}{\tilde{c}}\Big] B_n + \frac{\tilde{f}_2{}^3}{12} (n\pi)^4\, C_n \Bigg\} \cos\frac{n\pi y}{b}$$

$$= - \tilde{E}\tilde{f}_2\tilde{c} \sum_{n\ \mathrm{odd}} \sum_{m\ \mathrm{odd}} n\pi\, Y_n\, (m\pi)^2 Z_m \cos\frac{n\pi y}{b} \cos\frac{m\pi y}{b}$$

$$+ \frac{\tilde{E}\tilde{f}_2{}^3}{8} \Bigg[\sum_{n\ \mathrm{odd}} \sum_{m\ \mathrm{odd}} \sum_{r\ \mathrm{odd}} (n\pi)^4 Z_n\, (m\pi) Z_m\, (r\pi) Z_r$$

$$\cdot \cos\frac{n\pi y}{b} \sin\frac{m\pi y}{b} \sin\frac{r\pi y}{b} + 6 \sum_{n\ \mathrm{odd}} \sum_{m\ \mathrm{odd}} \sum_{r\ \mathrm{odd}}$$

$$\cdot (n\pi) Z_n\, (m\pi)^2 Z_m\, (r\pi)^3 Z_r \cos\frac{n\pi y}{b} \sin\frac{m\pi y}{b} \sin\frac{r\pi y}{b}$$

$$- 2 \sum_{n\ \mathrm{odd}} \sum_{m\ \mathrm{odd}} \sum_{r\ \mathrm{odd}} (n\pi)^2 Z_n\, (m\pi)^2 Z_m\, (r\pi)^2 Z_r$$

$$\cdot \cos\frac{n\pi y}{b} \cos\frac{m\pi y}{b} \cos\frac{r\pi y}{b} \Bigg] \quad (26A)$$

where

$$\left.\begin{aligned}
V_i &= i\pi\left[\left(-\frac{i\pi}{48} - \frac{\tilde{f}_1}{16\tilde{c}}\, i\pi + \frac{\tilde{E}_c}{\tilde{c}^2\,\tilde{G}}\,\frac{1}{2i\pi}\right) A_i \right. \\
&\quad \left. + \left(\frac{1}{8} + \frac{\tilde{f}_1}{4\tilde{c}}\right) B_i - \left(\frac{1}{2} + \frac{\tilde{f}_1}{2\tilde{c}}\right) C_i\right] + D_i \\
W_i &= \frac{i\pi}{8} A_i - \frac{1}{2} B_i + C_i \\
Y_i &= i\pi\left[\left(\frac{i\pi}{48} + \frac{\tilde{f}_2}{16\tilde{c}}\, i\pi - \frac{\tilde{E}_c}{\tilde{c}^2\,\tilde{G}}\,\frac{1}{2i\pi}\right) A_i \right. \\
&\quad \left. + \left(\frac{1}{8} + \frac{\tilde{f}_2}{4\tilde{c}}\right) B_i + \left(\frac{1}{2} + \frac{\tilde{f}_2}{2\tilde{c}}\right) C_i\right] + D_i \\
Z_i &= \frac{i\pi}{8} A_i + \frac{1}{2} B_i + C_i
\end{aligned}\right\} \qquad (27A)$$

and

$$\left.\begin{aligned}
\tilde{E} &= \frac{E_2\lambda_1}{E_1\lambda_2} \\
\tilde{E}_c &= \frac{E_c\lambda_1}{E_1} \\
\tilde{G} &= \frac{G\lambda_1}{E_1} \\
\tilde{f}_1 &= \frac{f_1}{b} \\
\tilde{f}_2 &= \frac{f_2}{b} \\
\tilde{c} &= \frac{c}{b}
\end{aligned}\right\} \qquad (28A)$$

Equations (23A) and (25A) are multiplied by $\frac{2}{b}$ sin $\frac{s\pi y}{b}$ and integrated with respect to y from $-\frac{b}{2}$ to $+\frac{b}{2}$ to obtain equations (10) and (11) of the Analysis. Equations (24A) and (26A) are multiplied by $\frac{2}{b}$ $\cos\frac{s\pi y}{b}$ and integrated with respect to y from $-\frac{b}{2}$ to $+\frac{b}{2}$ to obtain equations (12) and (13) of the analysis, where

$$\left.\begin{aligned} \alpha_{mnr} &= \frac{2}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \cos\frac{n\pi y}{b}\cos\frac{m\pi y}{b}\cos\frac{r\pi y}{b}\,dy \\ \gamma_{nmrs} &= \frac{2}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \cos\frac{n\pi y}{b}\sin\frac{m\pi y}{b}\sin\frac{r\pi y}{b}\cos\frac{s\pi y}{b}\,dy \\ \text{and}\quad \delta_{nmrs} &= \frac{2}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \cos\frac{n\pi y}{b}\cos\frac{m\pi y}{b}\cos\frac{r\pi y}{b}\cos\frac{s\pi y}{b}\,dy \end{aligned}\right\} \quad (29A)$$

After integration of the foregoing expressions, it can be shown that they reduce to

$$\alpha_{nmr} = \frac{(-1)^{\frac{n+m+r+1}{2}}\ \frac{8}{\pi}\ n\ m\ r}{(n+m+r)(m+r-n)(n+r-m)(n+m-r)}$$

and

$$\delta_{nmrs} = \frac{1}{4}\left[e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7\right]$$

$$\gamma_{nmrs} = \frac{1}{4}\left[-e_1 + e_2 + e_3 - e_4 - e_5 + e_6 + e_7\right]$$

where

$$e_1 = \begin{cases} 1 & \text{if } n + s = m + r \\ 0 & \text{otherwise} \end{cases}$$

$$e_2 = \begin{cases} 1 & \text{if } n + r = m + s \\ 0 & \text{otherwise} \end{cases}$$

$$e_3 = \begin{cases} 1 & \text{if } n + m = r + s \\ 0 & \text{otherwise} \end{cases}$$

and the remaining $e_i = 0$, unless $e_1 = e_2 = e_3 = 0$ and $\frac{n+m+r+s}{2}$ is odd, in which case at most one of the remaining e_i equals one:

$$e_4 = \begin{cases} 1 & \text{if } n = m + r + s \\ 0 & \text{otherwise} \end{cases}$$

$$e_5 = \begin{cases} 1 & \text{if } s = n + m + r \\ 0 & \text{otherwise} \end{cases}$$

$$e_6 = \begin{cases} 1 & \text{if } m = n + r + s \\ 0 & \text{otherwise} \end{cases}$$

$$e_7 = \begin{cases} 1 & \text{if } r = n + m + s \\ 0 & \text{otherwise} \end{cases}$$

Note that if $\delta_{nmrs} = 0$, $\gamma_{nmrs} = 0$ but not vice versa. Therefore δ_{nmrs} should be computed first.

The solution of the infinite set of equations (10) through (13) is the solution of the problem. This last step cannot be carried out in general, but an approximate solution can be obtained in each particular case. Further discussion of particular solutions is contained in the Analysis.

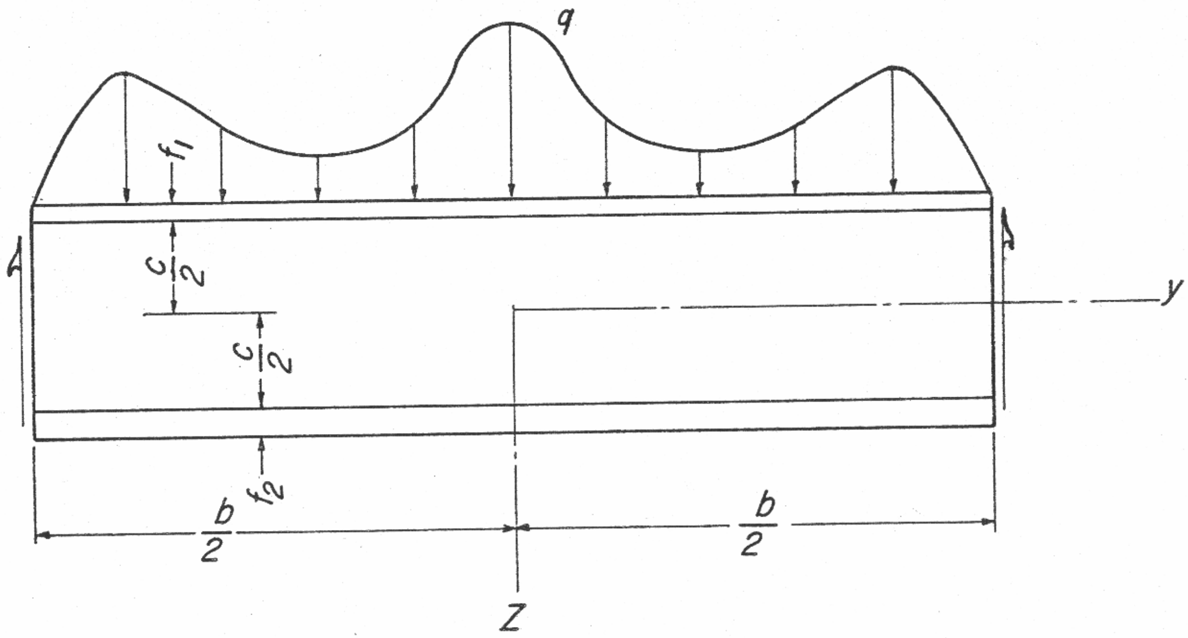

Figure 1. --The beam under symmetric loading.

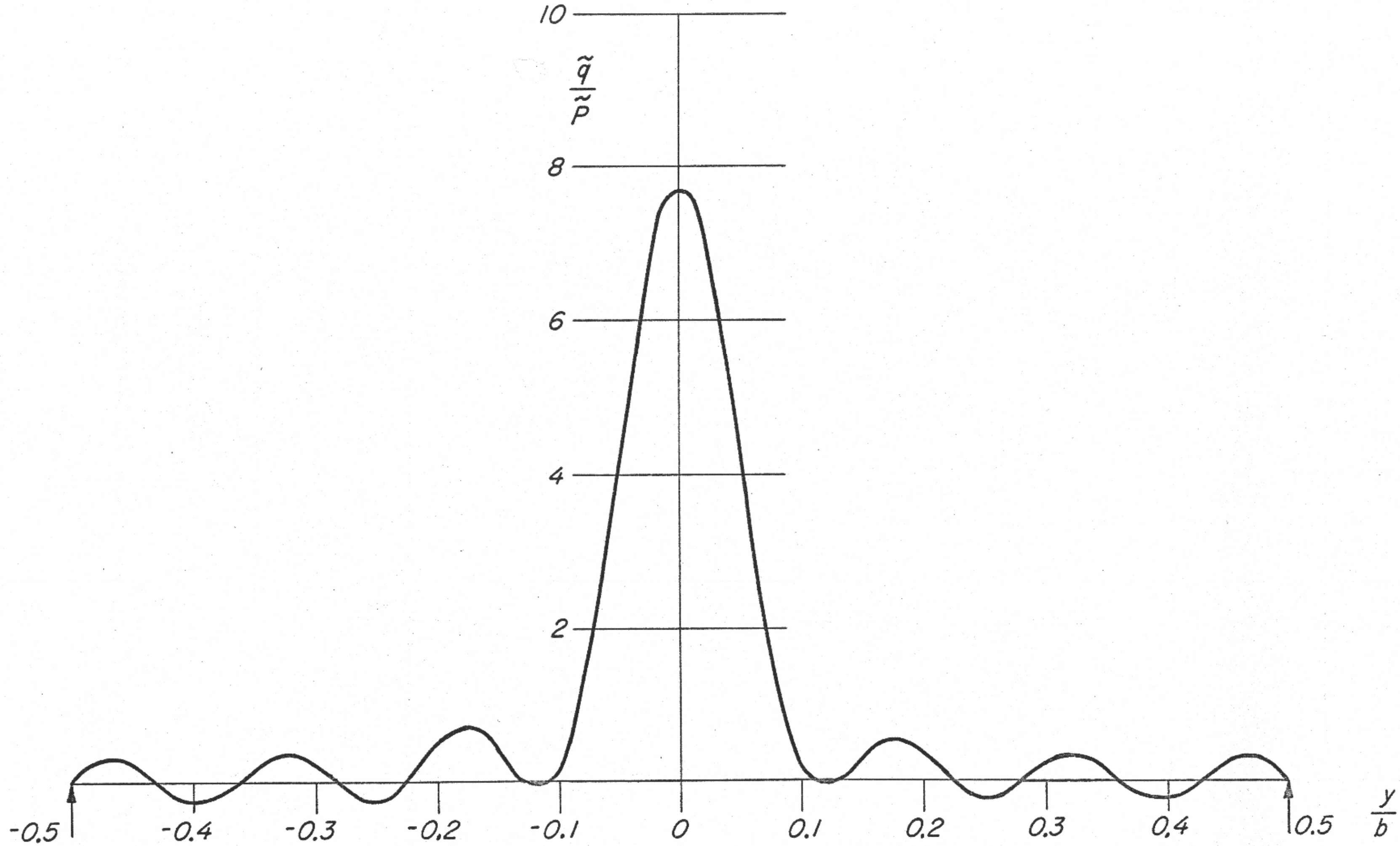

Figure 2. --Seventh partial sum of equation (19), with $\frac{\delta}{b} = 0.03$.

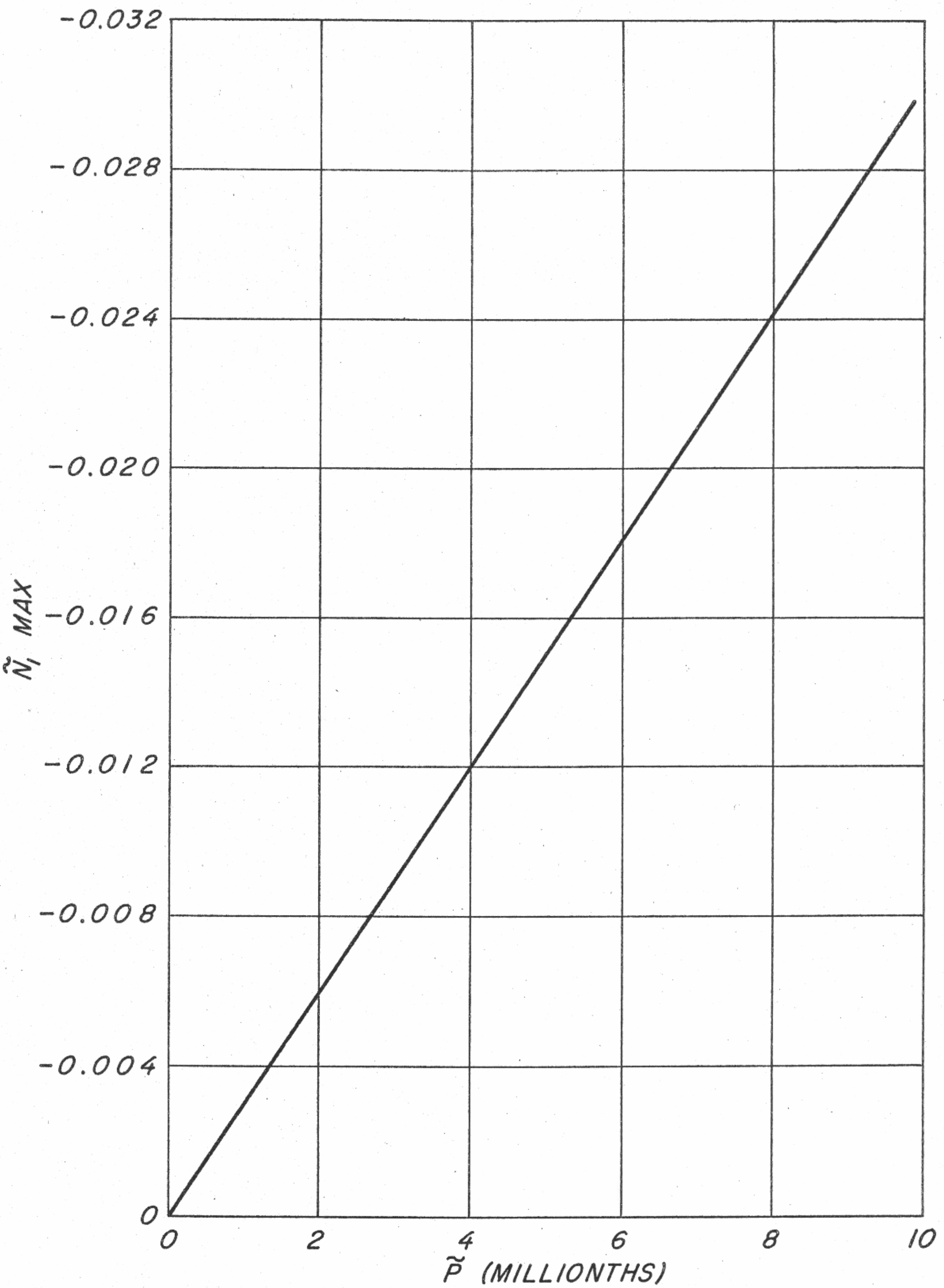

Figure 3. -- Maximum stress due to membrane force in upper facing.

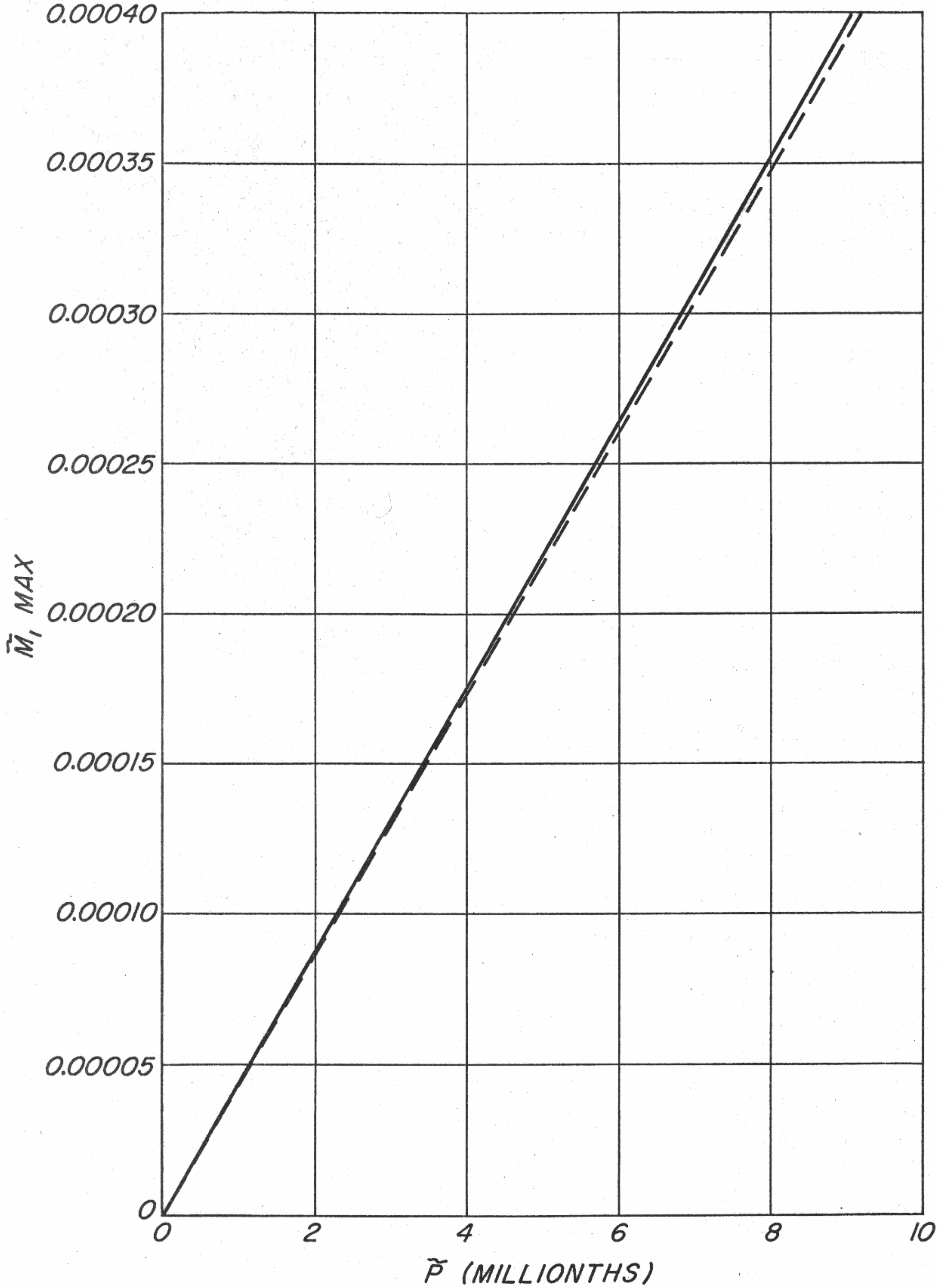

Figure 4. --Maximum stress due to moment in upper facing.

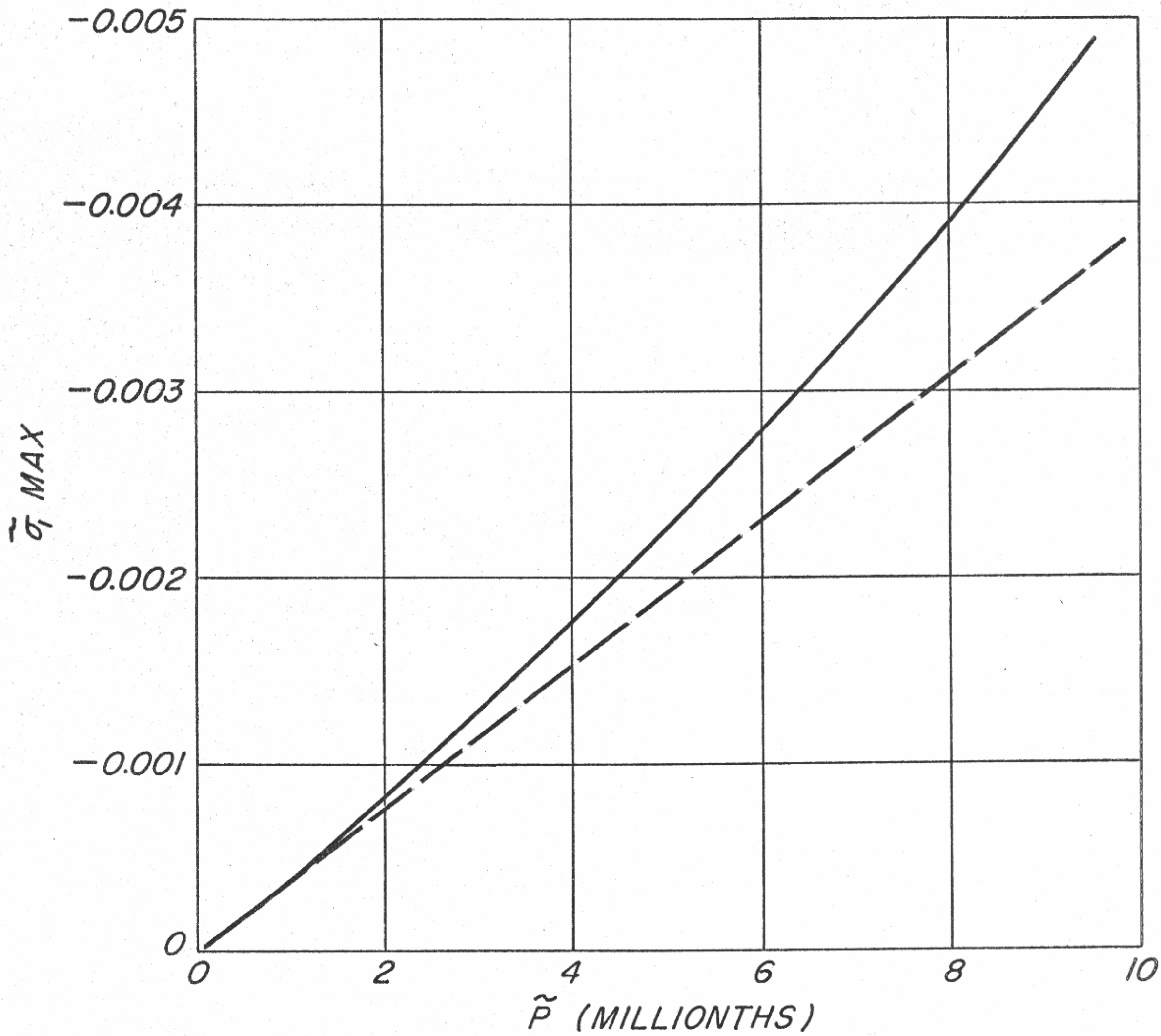

Figure 5. --Maximum core normal stress.

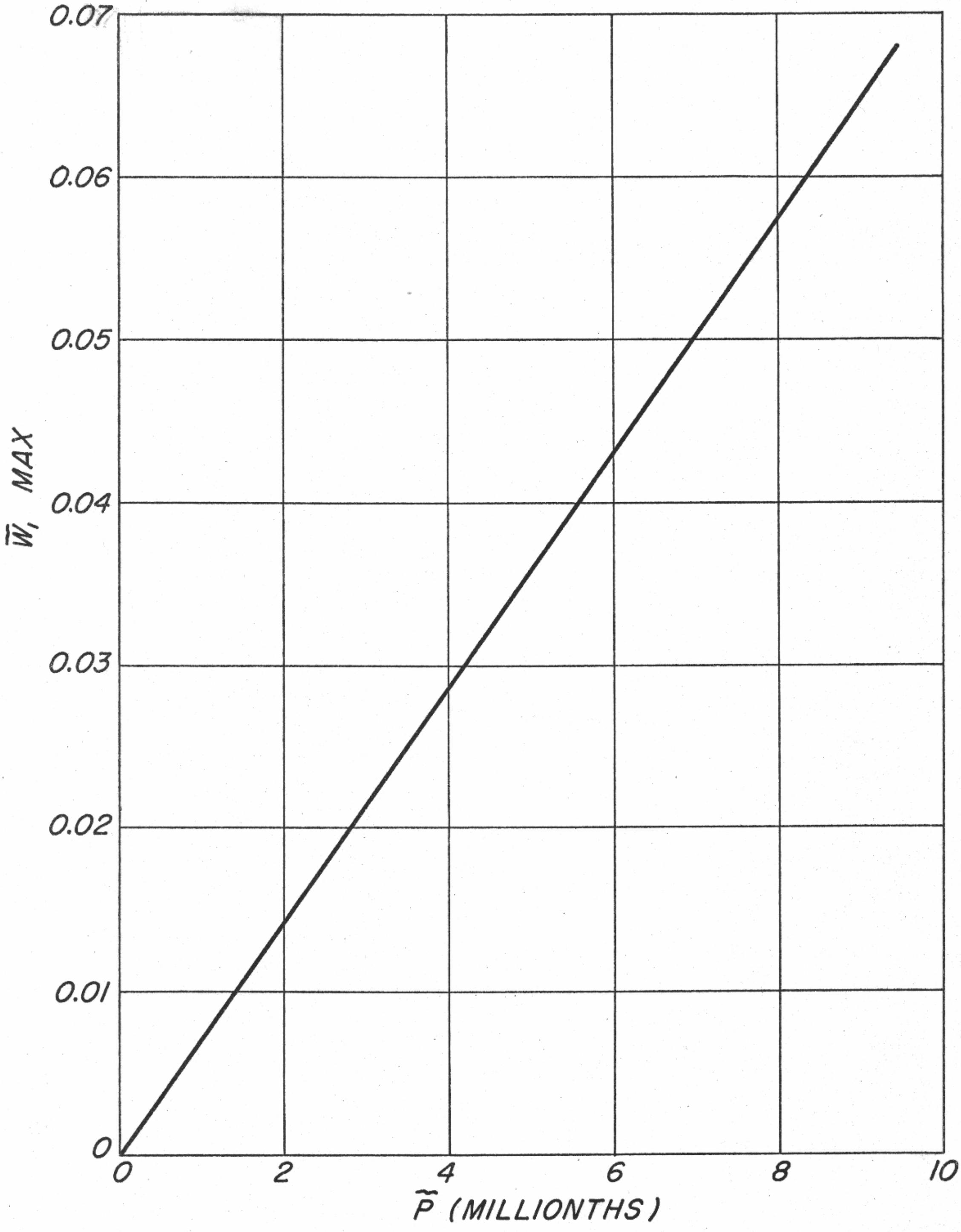

Figure 6. --Maximum deflection of upper facing.

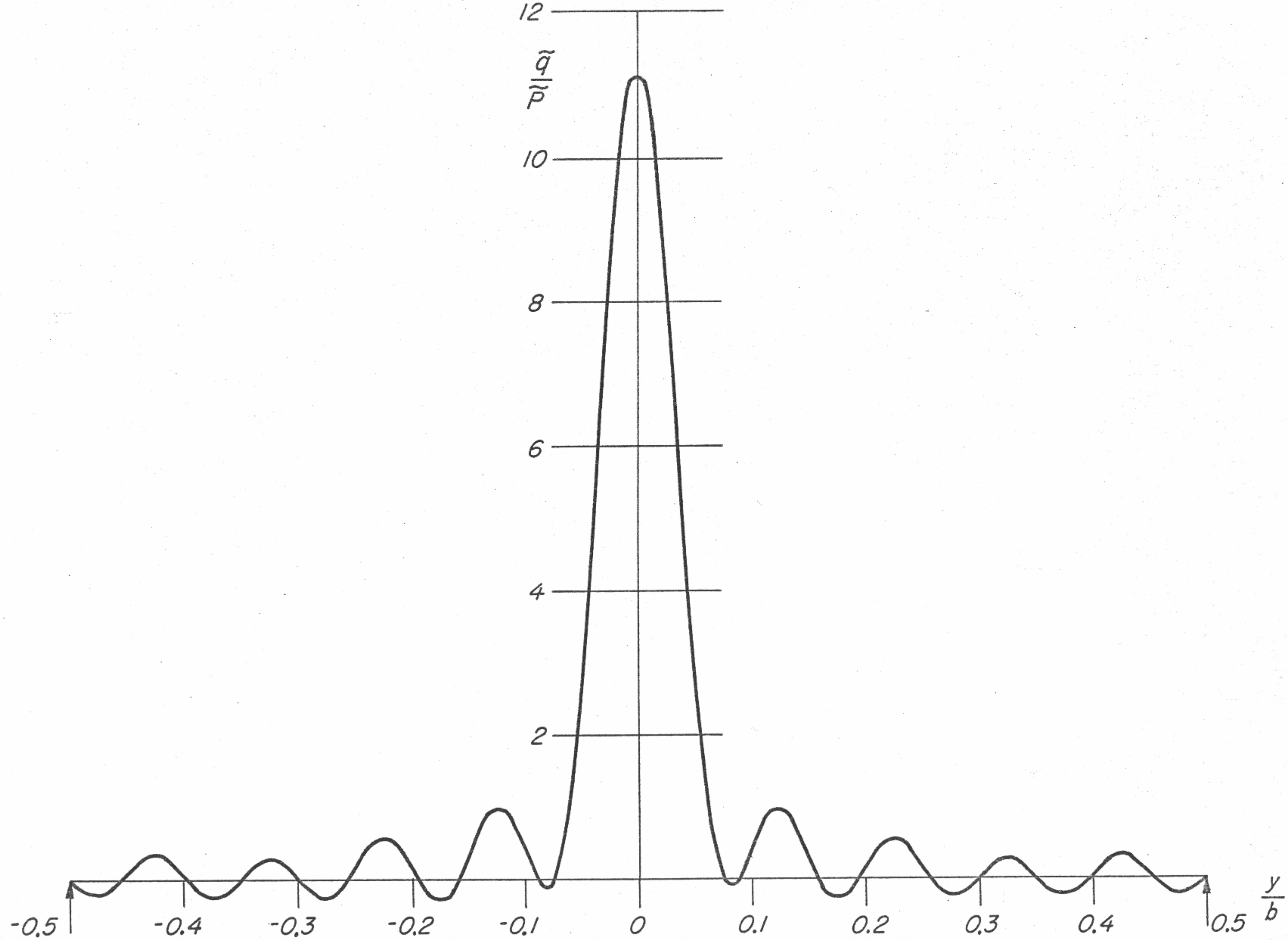

Figure 7. --Tenth partial sum of equation (19) with $\frac{\delta}{b} = 0.021$.

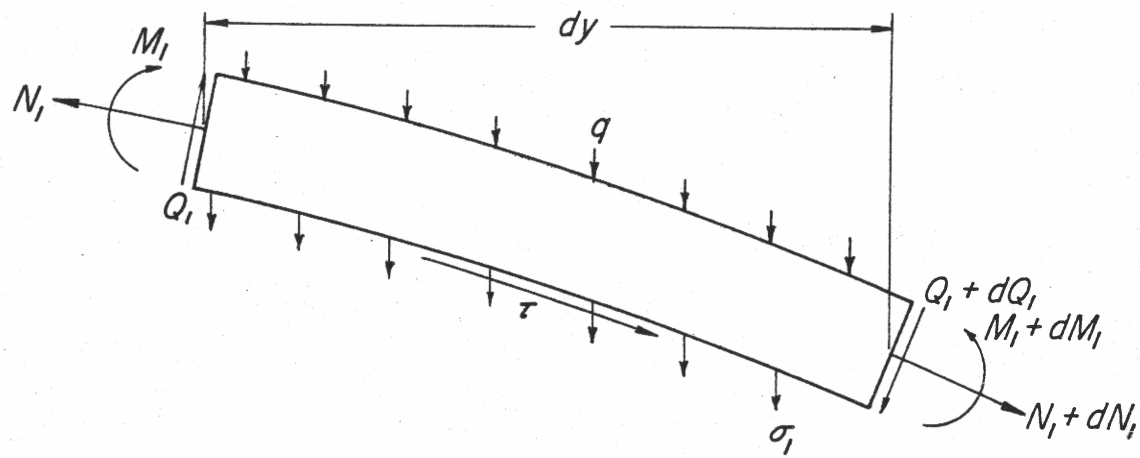

Figure 8. --Differential element of upper facing.