A New Solution of a Nonlinear Model of Upwelling

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(Manuscript received 20 February 2004, in final form 7 September 2004)

ABSTRACT

A two-dimensional, frictionless, nonlinear model of coastal upwelling is reexamined. The model has been solved previously at steady state and as an initial-value problem. The previous solution to the initial-value problem is inconsistent with the steady-state solution. A new solution to the spinup problem is presented that approaches the existing steady-state solution. In the new solution, a surface equatorward jet develops more rapidly than a poleward undercurrent, but the surface jet is of limited strength so that the undercurrent velocity eventually surpasses that of the surface flow. Consideration of dimensional scales implies that the magnitude of the wind stress determines how quickly steady state is approached but does not affect the steady-state fields. Exact solutions found with an arbitrary alongshore pressure gradient imply that there is no poleward flow without a poleward pressure gradient.

1. Introduction

Wind-driven upwelling is a fundamentally important and still incompletely understood process in coastal ocean circulation. Theoretical progress has been made to varying degrees on different aspects of this problem. One important line of work has considered the time-dependent response of a continuously stratified ocean to alongshore wind stress in the linear approximation (e.g., Allen 1973; Pedlosky 1974). Although these linear analyses have provided valuable physical insight, little analytical progress has been made on the corresponding nonlinear problem, and few examples exist of explicit dynamical solutions that illustrate the nonlinear upward advection of subsurface isopycnals that is the characteristic signature of observed coastal upwelling.

An exception is the work of Pedlosky (1978a,b), who focused on the nonlinear inviscid dynamics of upwelling below the surface Ekman layer. The effect of upwelling winds was modeled by the requirement that the interior fluid have a mass sink at the surface next to the coast. A constant alongshore pressure gradient was imposed, in order to geostrophically balance a depth-independent onshore flow far from the coast. Alongshore variations of the velocity and density fields were neglected, but dynamical variables were allowed to vary continuously in the vertical coordinate.

The steady-state problem was solved in this framework by rewriting the equations of motion with density as the vertical coordinate (Pedlosky 1978a). The solution was found in closed form for flat and linear cross-shore topography and in integral form for arbitrary cross-shore topography. The solution for flat topography included an equatorward surface jet and poleward undercurrent. Maximum values of the vertical and alongshore velocities were found next to the coast with an offshore decay scale on the order of the internal Rossby radius of deformation.

In contrast, Pedlosky (1978b) addressed the onset of upwelling by solving the initial-value problem under similar assumptions regarding the two-dimensional nature of the flow and with flat across-shore topography. A streamfunction for the cross-shore flow was introduced and the streamfunction was shown to be governed by a nonlinear elliptic partial differential equation. By transforming from physical coordinates to the characteristic coordinates of the elliptic equation, the problem was rendered linear, and a closed-form solution was found. However, in physical coordinates, the upwelling took place within an infinitesimally thin boundary layer next to the coast. The vertical extent of this boundary layer grew with time until, at steady state, no vertical or alongshore currents were present outside the boundary layer. Thus, this solution was not consistent with the steady-state solution found by Pedlosky (1978a). This inconsistency has made the physical interpretation of these solutions difficult and has limited their utility as illustrative models of the nonlinear upwelling response.

A new solution to the initial-value problem is discussed here. It is shown that this solution approaches

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the steady-state solution found by Pedlosky (1978a). The new solution is derived by employing the same transformation to characteristic coordinates as Pedlosky (1978b) used, but different boundary conditions are applied in the new coordinate space.

The model in physical and transformed coordinates is described in section 2. The new solution is presented in section 3, and some features of it are explored in section 4.

2. The model equations

a. Equations and scaling

We examine flow in the inviscid interior away from surface and bottom boundary layers. Our starting point is the following dimensional set of equations expressing the frictionless, hydrostatic motion of a stratified fluid:

$$\frac{\partial u_*}{\partial t_*} + u_*\frac{\partial u_*}{\partial x_*} + v_*\frac{\partial u_*}{\partial y_*} + w_*\frac{\partial u_*}{\partial z_*} - fv_* = -\frac{1}{\rho_0}\frac{\partial p_*}{\partial x_*}, \tag{2.1a}$$

$$\frac{\partial v_*}{\partial t_*} + u_*\frac{\partial v_*}{\partial x_*} + v_*\frac{\partial v_*}{\partial y_*} + w_*\frac{\partial v_*}{\partial z_*} + fu_* = -\frac{1}{\rho_0}\frac{\partial p_*}{\partial y_*}, \tag{2.1b}$$

$$\frac{\partial p_*}{\partial z_*} = -\rho_* g, \tag{2.1c}$$

$$\frac{\partial \rho_*}{\partial t_*} + u_*\frac{\partial \rho_*}{\partial x_*} + v_*\frac{\partial \rho_*}{\partial y_*} + w_*\frac{\partial \rho_*}{\partial z_*} = 0, \quad \text{and} \tag{2.1d}$$

$$\frac{\partial u_*}{\partial x_*} + \frac{\partial v_*}{\partial y_*} + \frac{\partial w_*}{\partial z_*} = 0, \tag{2.1e}$$

where x, y, and z are the eastward (cross-shore), northward (alongshore), and vertical coordinates, respectively; u, v, and w are the velocities in the x, y, and z directions; t is time; p is the pressure; g is the acceleration due to gravity; ρ is the density; and ρ_0 is a constant reference density. The Coriolis parameter f is taken to be constant. The asterisk subscript denotes a dimensional quantity.

The above equations are nondimensionalized according to

$$t_* = Tt, \quad x_* = lx, \quad y_* = Ly, \quad z_* = Hz, \quad u_* = Uu,$$
$$v_* = Vv, \quad \text{and} \quad w_* = Ww, \tag{2.2}$$

where the alongshore scale L will be assumed to be much greater than the cross-shore scale l. The pressure and density are scaled as

$$p_* = -\rho_0 gHz + Pp(x, y, z, t) \quad \text{and}$$
$$\rho_* = \rho_0 + \Delta\rho\rho(x, y, z, t), \tag{2.3}$$

where $\Delta\rho \ll \rho_0$. Assuming all three terms in the continuity equation are of the same order implies $U/l = V/L = W/H$. Denote $\alpha = l/L \ll 1$. Then the continuity equation implies $U/V = \alpha$. Let the scales be related according to

$$P = \rho_0 f_0 UL = \rho_0 f_0 Vl, \quad T = \frac{1}{\alpha f_0}, \quad \text{and} \quad l = \frac{N_0 H}{f_0}, \tag{2.4}$$

where $N_0 = (g\Delta\rho_0/\rho_0 H)^{1/2}$. This step scales the pressure gradient terms like the Coriolis terms, allows $\partial v/\partial t$ to contribute at leading order, and sets the cross-shore length scale l to be the internal Rossby deformation radius.

Nondimensionalizing (2.1) and neglecting $O(\alpha^2, \epsilon\alpha^2)$ terms gives the three-dimensional version of the dynamical model to be studied here:

$$v = \frac{\partial p}{\partial x}, \tag{2.5a}$$

$$\frac{\partial v}{\partial t} + \epsilon u\frac{\partial v}{\partial x} + \epsilon v\frac{\partial v}{\partial y} + \epsilon w\frac{\partial v}{\partial z} + u = -\frac{\partial p}{\partial y}, \tag{2.5b}$$

$$\epsilon\frac{\partial p}{\partial z} = -\rho, \tag{2.5c}$$

$$\frac{\partial \rho}{\partial t} + \epsilon u\frac{\partial \rho}{\partial x} + \epsilon v\frac{\partial \rho}{\partial y} + \epsilon w\frac{\partial \rho}{\partial z} = 0, \quad \text{and} \tag{2.5d}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.5e}$$

where ϵ is the Rossby number:

$$\epsilon = \frac{V}{f_0 l} = \frac{1}{\alpha}\frac{V}{f_0 L} = \frac{1}{\alpha}\frac{U}{f_0 l}. \tag{2.6}$$

The equation set (2.5) is similar to the geostrophic momentum approximation with cross-front geostrophic balance (Hoskins 1975; Hoskins and Bretherton 1972), and some aspects of (2.5) are discussed in the coastal ocean context, for example, by Gan and Allen (2002).

This model is used to study the response of the coastal ocean to an upwelling-favorable alongshore wind stress τ. Thus, the cross-shore velocity scale is set by the strength of the alongshore wind stress, through a balance between the surface offshore Ekman transport and the subsurface onshore flow. Equating the total offshore Ekman flux per meter of coastline, $\tau/(\rho_0 f_0)$, to the onshore mass flux below the Ekman layer, UH, gives

$$U = \frac{\tau}{H\rho_0 f_0}. \tag{2.7}$$

Because the surface boundary layer is not described by this model, the effect of upwelling from the interior into the surface layer is modeled by the fluid exiting our domain at the surface next to the coast.

We further simplify the problem by assuming no alongshore variations in topography or in the wind stress τ so that the velocity components and density field may be assumed to be independent of y. We also assume, following Pedlosky (1978a,b), that the onshore flow far offshore is depth-independent and is geostrophically balanced by a constant alongshore pressure gradient $-\partial p/\partial y = 1$.

The governing equations in nondimensional coordinates are thus

$$v = \frac{\partial p}{\partial x}, \tag{2.8a}$$

$$\frac{\partial v}{\partial t} + \epsilon u \frac{\partial v}{\partial x} + \epsilon w \frac{\partial v}{\partial z} + u = 1, \tag{2.8b}$$

$$\epsilon \frac{\partial p}{\partial z} = -\rho, \tag{2.8c}$$

$$\frac{\partial \rho}{\partial t} + \epsilon u \frac{\partial \rho}{\partial x} + \epsilon w \frac{\partial \rho}{\partial z} = 0, \quad \text{and} \tag{2.8d}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2.8e}$$

which correspond to geostrophic balance of v in the cross-shore momentum equation, momentum conservation in y, hydrostatic balance, density conservation, and mass conservation, respectively. This system of equations is identical to the model discussed by Pedlosky (1978b).

We seek a solution of the initial-value problem for (2.8) in an idealized geometry in which the bottom is flat and the coast is represented by a vertical wall. The boundary conditions are chosen to impose no flow through the surface, bottom, or coastal wall, except for a mass sink at the surface next to the coast, which models the effect of interior fluid upwelling into the surface layer. The initial conditions are that the vertical density gradient is constant, the isopycnals are flat, and the alongshore velocity is zero. Any reader who is not interested in the mathematical details of the derivation of the solution may proceed directly to section 3b after noting that a streamfunction ψ is introduced in (2.15) and that a coordinate transformation is defined in (2.29)–(2.31).

b. The initial-value problem

Let the bottom, surface, and coastal wall be at $z = 0$, $z = 1$, and $x = 0$, respectively, with $x < 0$ offshore. It will be assumed that, at $t = 0$, $\rho = 1 - z$. Thus, the isopycnal surfaces are initially flat. This assumption alternatively means that $\rho \to 1 - z$ as $x \to -\infty$. We define

$$\rho(x, z, t) = 1 - z + \epsilon\rho'(x, z, t) \quad \text{and}$$
$$p(x, z, t) = \overline{p}(z) + p'(x, z, t), \tag{2.9}$$

where $\epsilon d\overline{p}/dz = -1 + z$, to isolate the time-dependent density from the initial condition. In terms of ρ' and p', the governing equations in (2.8) take the form

$$v = \frac{\partial p'}{\partial x}, \tag{2.10a}$$

$$\frac{\partial v}{\partial t} + \epsilon u \frac{\partial v}{\partial x} + \epsilon w \frac{\partial v}{\partial z} + u = 1, \tag{2.10b}$$

$$\frac{\partial p'}{\partial z} = -\rho', \tag{2.10c}$$

$$\frac{\partial \rho'}{\partial t} + \epsilon u \frac{\partial \rho'}{\partial x} + \epsilon w \frac{\partial \rho'}{\partial z} = w, \quad \text{and} \tag{2.10d}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{2.10e}$$

The thermal wind relation is derived by eliminating p' from (2.10a) and (2.10c):

$$\frac{\partial v}{\partial z} = -\frac{\partial \rho'}{\partial x}. \tag{2.11}$$

In the system (2.8) [or (2.10)], potential vorticity is conserved. Using the notation

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \epsilon u \frac{\partial}{\partial x} + \epsilon w \frac{\partial}{\partial z}, \tag{2.12}$$

we may write

$$\frac{dq}{dt} = 0, \tag{2.13}$$

where

$$q = -\left(1 + \epsilon \frac{\partial v}{\partial x}\right)\frac{\partial \rho}{\partial z} + \epsilon \frac{\partial v}{\partial z}\frac{\partial \rho}{\partial x}$$
$$= 1 + \epsilon\left(\frac{\partial v}{\partial x} - \frac{\partial \rho'}{\partial z}\right) + \epsilon^2\left(\frac{\partial \rho'}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial \rho'}{\partial z}\frac{\partial v}{\partial x}\right). \tag{2.14}$$

We will seek solutions in which $v = 0$ and $\rho' = 0$ (i.e., $\rho = 1 - z$) at $t = 0$, so that $q = 1$ at every point in the domain initially. Potential vorticity conservation then implies that $q = 1$ at every point in the domain for all time.

A streamfunction $\psi(x, z, t)$ is introduced to satisfy (2.10e) identically:

$$u = \frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{\partial \psi}{\partial x}. \tag{2.15}$$

With the streamfunction, the equations governing the time evolution of v and ρ' take the form

$$\frac{\partial v}{\partial t} + \left(1 + \epsilon \frac{\partial v}{\partial x}\right) \frac{\partial \psi}{\partial z} - \epsilon \frac{\partial \psi}{\partial x} \frac{\partial v}{\partial z} = 1 \quad \text{and} \tag{2.16}$$

$$\frac{\partial \rho'}{\partial t} + \epsilon \frac{\partial \rho'}{\partial x} \frac{\partial \psi}{\partial z} + \frac{\partial \psi}{\partial x} \left(1 - \epsilon \frac{\partial \rho'}{\partial z}\right) = 0. \tag{2.17}$$

The thermal wind relation (2.11) may be used to eliminate the time derivatives from these equations:

$$(1 + \epsilon v_x)\psi_{zz} + 2\epsilon \rho'_x \psi_{xz} + (1 - \epsilon \rho'_z)\psi_{xx} = 0, \tag{2.18}$$

where subscripts denote partial differentiation. The system (2.16)–(2.18) is to be solved with boundary conditions on ψ that impose a mass sink at the surface next to the coast and zero initial conditions for v and ρ':

$$\begin{aligned} \psi &= z \quad \text{for} \quad x \to -\infty, \\ \psi &= 0 \quad \text{for} \quad x = 0, \\ \psi &= 1 \quad \text{for} \quad z = 1, \\ \psi &= 0 \quad \text{for} \quad z = 0, \\ v(x, z, 0) &= 0 \quad \text{for} \quad t = 0, \quad \text{and} \\ \rho'(x, z, 0) &= 0 \quad \text{for} \quad t = 0. \end{aligned} \tag{2.19}$$

The v momentum equation implies that the quantity

$$\xi(x, z, t) = x + \epsilon(v - t) \tag{2.20}$$

is conserved by each fluid parcel,

$$\frac{d\xi}{dt} = 0. \tag{2.21}$$

Thus the nonlinear problem in physical coordinates may be written in terms of v, ρ', and ψ, as in (2.16)–(2.18), or in terms of ξ, ρ, and ψ:

$$\frac{\partial \xi}{\partial t} + \epsilon \frac{\partial \xi}{\partial x} \frac{\partial \psi}{\partial z} - \epsilon \frac{\partial \psi}{\partial x} \frac{\partial \xi}{\partial z} = 0, \tag{2.22}$$

$$\frac{\partial \rho}{\partial t} + \epsilon \frac{\partial \rho}{\partial x} \frac{\partial \psi}{\partial z} - \epsilon \frac{\partial \psi}{\partial x} \frac{\partial \rho}{\partial z} = 0, \quad \text{and} \tag{2.23}$$

$$\xi_x \psi_{zz} + 2\rho_x \psi_{xz} - \rho_z \psi_{xx} = 0. \tag{2.24}$$

The initial conditions on ξ and ρ are

$$\xi(x, z, 0) = x \quad \text{and} \quad \rho(x, z, 0) = 1 - z. \tag{2.25}$$

The initial conditions imply the governing equation for ψ, (2.24) or (2.18), becomes Laplace's equation at the initial instant:

$$\psi_{xx} + \psi_{zz} = 0 \text{ at } t = 0, \tag{2.26}$$

which can be solved in closed form (Pedlosky 1978b)

$$\begin{aligned} \psi_0(x, z) &= \psi(x, z, t = 0) \\ &= -(2/\pi) \tan^{-1}(\tanh \pi x/2 \tan \pi z/2) \end{aligned} \tag{2.27}$$

(see Fig. 1). The advecting velocities at the initial instant are therefore

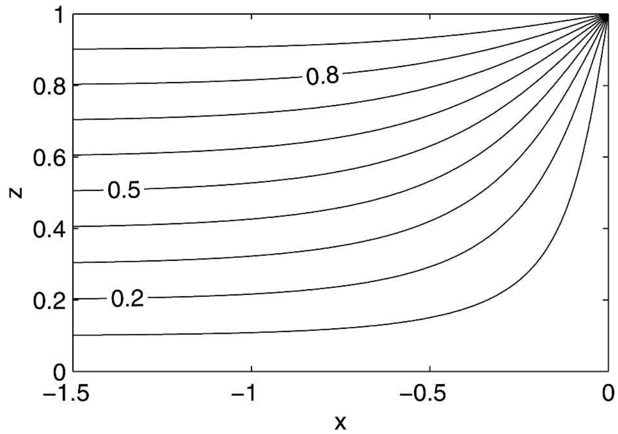

FIG. 1. Solution for ψ at $t = 0$.

$$u = \frac{\partial \psi_0}{\partial z} = -\frac{\sinh \pi x}{\cosh \pi x + \cos \pi z} \quad \text{and}$$

$$w = -\frac{\partial \psi_0}{\partial x} = \frac{\sin \pi z}{\cosh \pi x + \cos \pi z}. \tag{2.28}$$

c. The model in transformed coordinates

Pedlosky (1978b) showed that the variables ξ and z are characteristic coordinates of the elliptic partial differential equation in (2.18). Following Pedlosky (1978b), let us make the change of variables

$$\xi = x + \epsilon(v - t), \tag{2.29}$$

$$Z = z, \quad \text{and} \tag{2.30}$$

$$T = t \tag{2.31}$$

and let us denote functions of the transformed coordinates with a tilde, for example, $\tilde{\psi}(\xi, Z, T) = \psi(x, z, t)$. The governing equation for $\tilde{\psi}$ becomes

$$\tilde{\psi}_{\xi\xi} + \tilde{\psi}_{ZZ} = 0. \tag{2.32}$$

Arbitrary functions of time could be added to ξ and Z and this transformed equation for $\tilde{\psi}$ would remain the same. The auxiliary equations are v momentum,

$$\frac{\partial \tilde{v}}{\partial T} = 1 - \frac{\partial \tilde{\psi}}{\partial Z}, \tag{2.33a}$$

which, by (2.29), may also be expressed as

$$\frac{\partial \tilde{x}}{\partial T} = \epsilon \frac{\partial \tilde{\psi}}{\partial Z}, \tag{2.33b}$$

and density conservation,

$$\frac{\partial \tilde{\rho}}{\partial T} = -\epsilon \frac{\partial \tilde{\psi}}{\partial \xi}. \tag{2.33c}$$

The initial conditions (2.25) transform to

$$\tilde{\upsilon}(\xi, Z, 0) = 0, \tag{2.34a}$$

$$\tilde{x}(\xi, Z, 0) = \xi, \quad \text{and} \tag{2.34b}$$

$$\tilde{\rho}(\xi, Z, 0) = 1 - Z. \tag{2.34c}$$

In these coordinates, the thermal wind relation retains its form

$$\frac{\partial \tilde{\upsilon}}{\partial Z} = -\frac{\partial \tilde{\rho}'}{\partial \xi} \tag{2.35}$$

but the conservation of potential vorticity is simplified considerably:

$$\frac{\partial \tilde{\rho}'}{\partial Z} = \frac{\partial \tilde{\upsilon}}{\partial \xi}. \tag{2.36}$$

The change of coordinates (2.29)–(2.31) therefore transforms the governing equations into a form that is linear. It must be noted that one may, instead of using the transformation (2.29)–(2.31), employ density coordinates to express the system in a linear form. This approach was done by Pedlosky (1978a) for the steady-state case and is shown in appendix A for the time-dependent case.

d. Transformed boundary conditions

The boundary conditions (2.19) must be expressed in transformed coordinates. The transformation $Z = z$ implies that the boundary conditions on $z = 0$ and 1 may be applied on $Z = 0$ and 1, respectively. Because $\upsilon \to 0$ as $x \to -\infty$, the condition applied as $x \to -\infty$ is applied in the transformed space as $\xi \to -\infty$, by (2.29). The boundary at $x = 0$ for $z < 1$ corresponds to $\xi = 0$, as can be seen in the following argument (Pedlosky 1978b). Consider the alongshore momentum equation in the form $d\upsilon/dt = 1 - u$. Because $u = 0$ on $x = 0$ (for $z < 1$), and because $\upsilon = 0$ at $t = 0$, one may anticipate that $\upsilon(0, z, t) = t$ for all times. Then (2.29) implies that $\xi = 0$ on $x = 0$.

Thus, the boundary conditions (2.19) transform unambiguously at all points in the physical domain except for $(x, z) = (0, 1)$:

$$\begin{aligned}
\tilde{\psi} &= Z \quad \text{for} \quad \xi \to -\infty, \\
\tilde{\psi} &= 0 \quad \text{for} \quad \xi = 0, 0 \le Z < 1, \\
\tilde{\psi} &= 1 \quad \text{for} \quad Z = 1, \tilde{x}(\xi, Z) < 0, \quad \text{and} \\
\tilde{\psi} &= 0 \quad \text{for} \quad Z = 0.
\end{aligned} \tag{2.37}$$

At $(x, z) = (0, 1)$, the above argument breaks down because u is multivalued. {Taking the limit of u as (x, z) approaches (0, 1) produces different answers along different limit paths. Consider u at $t = 0$ [(2.28)]. The limit of u as (0, 1) is approached along $x = 0$ is 0, but as (0, 1) is approached along $z = 1$, the limit is ∞.} The statement that u is multivalued is true at the initial instant, and our argument assumes it is true at least for small times, which will be checked a posteriori.

The fact that u is multivalued and the fact that $d\upsilon/dt = 1 - u$ imply that υ is also multivalued, and therefore by (2.29), so is ξ. This means that $(x, z) = (0, 1)$ in physical space corresponds to $Z = 1$ and a range of values of ξ in transformed space. Let us denote that range of values as

$$\xi_0(T) \le \xi \le 0. \tag{2.38}$$

Because $\xi = x$ at $T = 0$, it must be true that $\xi_0(0) = 0$. The parameter $\xi_0(T)$ will be a decreasing function of time as long as $u > 0$ at $Z = 1$. This fact may be seen by considering a physical interpretation of ξ. The variable ξ satisfies $d\xi/dt = 0$, and so it is advected by the cross-shore flow, with $\xi = x$ at the initial instant. The multivalued nature of ξ at (0, 1) arises from the fact that the cross-shore velocity field advects ξ to that point from multiple directions. The parameter ξ_0 is the value of ξ that has been advected to (0, 1) from the farthest location offshore. This value will decrease in time if the advecting velocity u is greater than 0 along $z = 1$.

It therefore remains to specify the boundary conditions on $\tilde{\psi}$ in transformed space along the portion of the boundary at $Z = 1$, $\xi_0 \le \xi \le 0$. The initial and boundary conditions in physical space imply that $\xi = 0$ on $x = 0$ for $z < 1$. We will impose conditions consistent with $(\xi, Z) = (0, 1)$ corresponding to $(x, z) = (0, 1)$. The transformation (2.29) must necessarily remain valid in ξ space:

$$\tilde{\xi} = \tilde{x}(\xi, Z, T) + \epsilon[\tilde{\upsilon}(\xi, Z, T) - T]. \tag{2.39}$$

Then $\tilde{\upsilon}(0, 1, T) = T$ is necessary for $\tilde{x}(0, 1, T) = 0$. As a consequence, (2.33a) implies that $\partial\tilde{\psi}/\partial Z = 0$ at $\xi = 0$ and $Z = 1$. Thus, the appropriate boundary condition is

$$\frac{\partial \tilde{\psi}}{\partial Z} = 0 \text{ in } \xi_0 \le \xi \le 0. \tag{2.40}$$

This equation completes the specification of the boundary conditions in (ξ, Z) space. The function $\xi_0(T)$ will be determined as part of the solution.

3. The solution

a. Solution in transformed coordinates

The solution of Laplace's equation in (2.32) with the boundary conditions (2.37) and (2.40) may be written in closed form as

$$\tilde{\psi}(\xi, Z, T) = \frac{2}{\pi}\sin^{-1}\left[\frac{1}{2}\left(\sqrt{A^+} - \sqrt{A^-}\right)\right], \tag{3.1}$$

where

$$A^{\pm} = \left[\frac{\sinh\pi\xi/2\,\sin\pi Z/2}{\sinh\pi\xi_0(T)/2} \pm 1\right]^2 + \left[\frac{\cosh\pi\xi/2\,\cos\pi Z/2}{\cosh\pi\xi_0(T)/2}\right]^2. \quad (3.2)$$

The time dependence is within ξ_0, which has yet to be specified.

The solution may be written in implicit form as

$$\frac{\sinh^2\pi\xi/2\,\sin^2\pi Z/2}{\sin^2\pi\tilde{\psi}/2} - \frac{\cosh^2\pi\xi/2\,\cos^2\pi Z/2}{\cos^2\pi\tilde{\psi}/2} = \sinh^2\pi\xi_0/2, \quad (3.3)$$

because $x^2/\sin^2 u - y^2/\cos^2 u = 1$ and $[(x+1)^2 + y^2]^{1/2} - [(x-1)^2 + y^2]^{1/2} = 2\sin u$ are equivalent statements that describe a family of hyperbolas. This form of the solution may be used to show $\tilde{\psi}(\xi, Z, T = 0) = \psi_0(x, z)$. If $\xi_0(0) = 0$, then (3.3) gives

$$\tan^2\pi\tilde{\psi}/2 = \tanh^2\pi\xi/2\,\tan^2\pi Z/2 \text{ at } T = 0, \quad (3.4)$$

and, using $\tilde{\psi} \geq 0$, $Z \geq 0$, and $\xi \leq 0$, the streamfunction at the initial instant [(2.27)] is recovered.

It remains to solve for ξ_0, $\tilde{\rho}$, $\tilde{\upsilon}$, and $\tilde{x}$. These variables will be found by integrating the derivatives $\tilde{\psi}_\xi$ and $\tilde{\psi}_Z$ with respect to time [using the auxiliary equations (2.33)]. Note in particular that the evolution equation (2.33b) and initial condition (2.34b) imply

$$\tilde{x}(\xi, Z, T) = \xi + \epsilon\int_0^T \frac{\partial\tilde{\psi}}{\partial Z}\,dT'. \quad (3.5)$$

The normal boundary condition $\tilde{\psi}_Z = 0$ on $\xi_0 < \xi \leq 0$ therefore effectively ensures that all points at $Z = 1$ and $\xi_0 < \xi \leq 0$ in transformed space correspond to the single point $x = 0$ and $z = 1$ in physical space. Because ξ_0 is defined to be the least value of ξ for which $x = 0$, it follows that an equation governing ξ_0 is written by setting $x = 0$, $Z = 1$, and $\xi = \xi_0$ in (3.5):

$$\xi_0(T) = -\epsilon\int_0^T \frac{\partial\tilde{\psi}}{\partial Z}[\xi_0(T), Z = 1, T']\,dT'. \quad (3.6)$$

This is an implicit equation for ξ_0. To solve this equation, it is required to find $\tilde{\psi}_Z$ in the $Z = 1$ limit. In the limit as $Z \to 1$, with $\xi < \xi_0$,

$$\frac{\partial\tilde{\psi}}{\partial Z} = \frac{\cosh\pi\xi/2}{(\sinh^2\pi\xi/2 - \sinh^2\pi\xi_0/2)^{1/2}}. \quad (3.7)$$

The integral equation defining ξ_0 may thus be written in the form

$$\xi_0(T) = -\epsilon\int_0^T \frac{\cosh\pi\xi_0(T)/2}{[\sinh^2\pi\xi_0(T)/2 - \sinh^2\pi\xi_0(T')/2]^{1/2}}\,dT'. \quad (3.8)$$

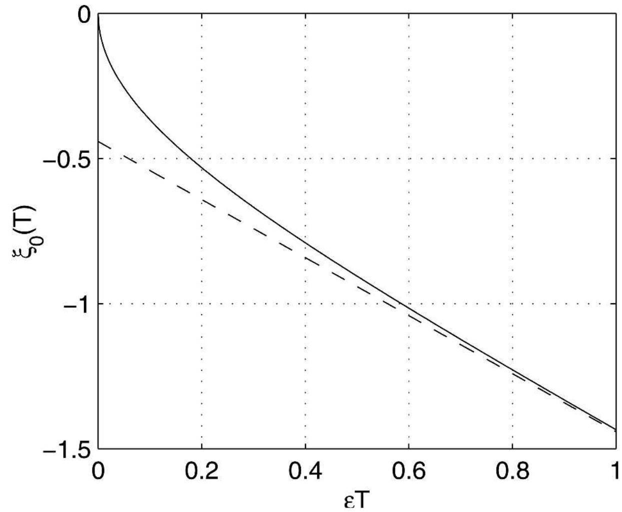

FIG. 2. The solid line is $\xi_0(T)$. The dashed line is $-\epsilon T - 2/\pi \log 2$.

This integral equation has the solution

$$\xi_0(T) = -\frac{2}{\pi}\cosh^{-1}\left[\exp\left(\frac{\pi\epsilon T}{2}\right)\right]. \quad (3.9)$$

See appendix B for a derivation of (3.9). The temporal evolution of ξ_0 is such that $\xi_0(T) \sim -2(\epsilon T/\pi)^{1/2}$ for $T \ll 1$, and $\xi_0(T) \sim -\epsilon T - 2/\pi \log 2$ for $T \gg 1$ (Fig. 2).

The function $\tilde{x}(\xi, Z, T)$ that solves the evolution equation (2.33b) and satisfies the initial condition (2.34b) is

$$\tilde{x}(\xi, Z, T) = -\frac{1}{\pi}\cosh^{-1}\left\{\frac{\cosh\pi\xi - \cos\pi Z}{2\exp(\pi\epsilon T)} + \left[\frac{(\cosh\pi\xi - \cos\pi Z)^2}{4\exp(2\pi\epsilon T)} + \frac{\cosh\pi\xi\cos\pi Z - 1}{\exp(\pi\epsilon T)} + 1\right]^{1/2}\right\}, \quad (3.10)$$

and the function $\tilde{\rho}(\xi, Z, T)$ satisfying (2.33c) and (2.34c) is

$$\tilde{\rho}(\xi, Z, T) = \frac{1}{\pi}\cos^{-1}\left\{\frac{\cosh\pi\xi - \cos\pi Z}{2\exp(\pi\epsilon T)} - \left[\frac{(\cosh\pi\xi - \cos\pi Z)^2}{4\exp(2\pi\epsilon T)} + \frac{\cosh\pi\xi\cos\pi Z - 1}{\exp(\pi\epsilon T)} + 1\right]^{1/2}\right\}. \quad (3.11)$$

At $T = 0$, (3.11) satisfies (2.34c) because $\tilde{\rho}(\xi, Z, 0) = (1/\pi)\cos^{-1}(-\cos\pi Z) = (1/\pi)[\pi - \cos^{-1}(\cos\pi Z)] = 1 - Z$. The form of $\tilde{\upsilon}(\xi, Z, T)$ may now be expressed in terms of $\tilde{x}$ using (2.39):

$$\tilde{\upsilon}(\xi, Z, T) = T + \epsilon^{-1}[\xi - \tilde{x}(\xi, Z, T)]. \quad (3.12)$$

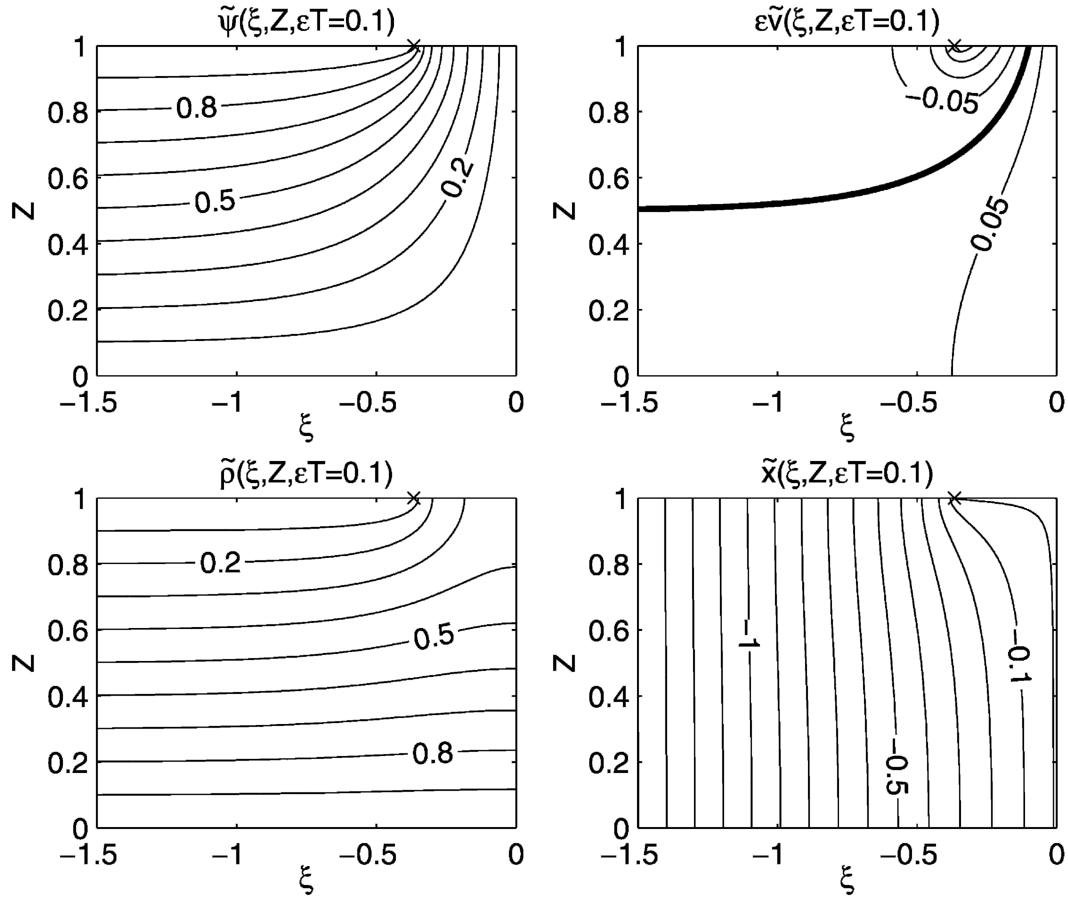

FIG. 3. The solution in (ξ, Z) space at $\epsilon T = 0.1$. Contour intervals are 0.05 for $\tilde{v}$ and 0.1 for the other variables, with a -0.01 contour added to the plot of $\tilde{x}$ to show the behavior near the upper-right-hand corner. The thick solid line is the zero contour of $\tilde{v}$. A cross marks the location of ξ_0 at $Z = 1$ at this time.

This completes the statement of the solution in transformed coordinates. Note that, in transformed coordinates, the solution is not multivalued in any part of the domain for $T > 0$ (Fig. 3).

b. Solution in physical coordinates

The function $\tilde{x}(\xi, Z, T)$ [(3.10)] may, with some manipulation, be expressed in terms of the solution (3.1) for $\tilde{\psi}(\xi, Z, T)$,

$$\tilde{x}(\xi, Z, T) = -\frac{2}{\pi}\tanh^{-1}\left\{\tan\left[\frac{\pi}{2}\tilde{\psi}(\xi, Z, T)\right]\cot\left(\frac{\pi}{2}Z\right)\right\}. \tag{3.13}$$

This must also hold in physical coordinates, because $Z = z$,

$$x = -\frac{2}{\pi}\tanh^{-1}\left\{\tan\left[\frac{\pi}{2}\psi(x, z, t)\right]\cot\left(\frac{\pi}{2}z\right)\right\}, \tag{3.14}$$

which can be rearranged to give the solution for the streamfunction

$$\psi(x, z, t) = \psi_0(x, z) = -\frac{2}{\pi}\tan^{-1}\left(\tanh\frac{\pi x}{2}\tan\frac{\pi z}{2}\right). \tag{3.15}$$

In physical coordinates, ψ is independent of time. Note that this means that the y component of the vorticity $u_z - w_x = \psi_{zz} + \psi_{xx}$ remains zero.

Solving (3.10) for ξ, using $Z = z$ and $T = t$, yields

$$\xi(x, z, t) = -\frac{1}{\pi}\cosh^{-1}\left\{\cosh\pi x + \frac{\sinh^2\pi x[\exp(\pi\epsilon t) - 1]}{\cos\pi z + \cosh\pi x}\right\}. \tag{3.16}$$

The solution for $v(x, z, t)$ follows from (2.29) and (3.16):

$$v(x, z, t) = t - \frac{x}{\epsilon} - \frac{1}{\epsilon\pi}\cosh^{-1}\left\{\cosh\pi x + \frac{\sinh^2\pi x[\exp(\pi\epsilon t) - 1]}{\cos\pi z + \cosh\pi x}\right\}. \tag{3.17}$$

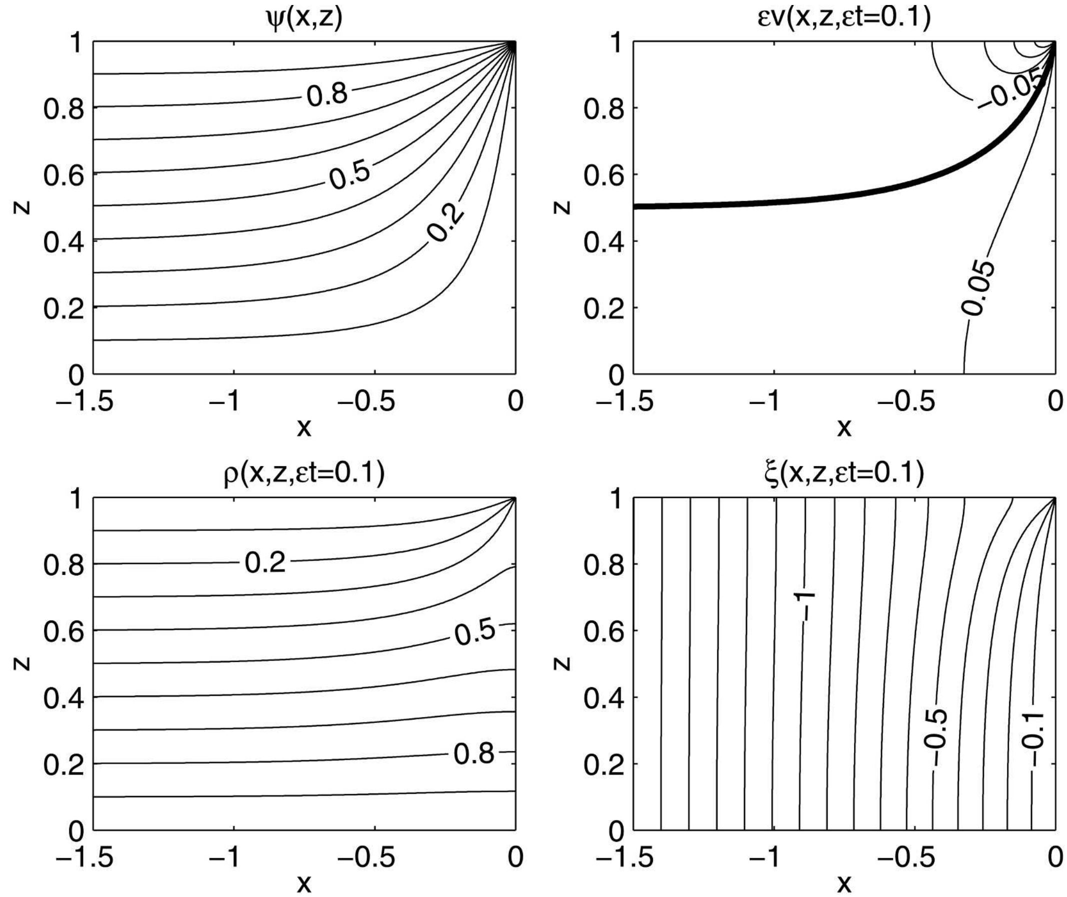

FIG. 4. The solution in (x, z) space at $\epsilon t = 0.1$. Contour intervals are 0.05 for v and 0.1 for the other variables. The thick solid line is the zero contour of v.

Using the expression for $\xi(x, z, t)$ in (3.11) yields

$$\rho(x, z, t) = \frac{1}{\pi}\cos^{-1}\left\{-\cos\pi z - \frac{\sin^2\pi z[1 - \exp(-\pi\epsilon t)]}{\cos\pi z + \cosh\pi x}\right\}. \quad (3.18)$$

Expressions (3.15), (3.17), and (3.18) constitute the complete time-dependent solution to the problem, as expressed in the original system of coordinates. It can be directly verified that these expressions solve the system (2.10b), (2.10d), (2.10e), and (2.11). The solution describes, in an exact analytical form, the nonlinear upward advection of subsurface isopycnals (Fig. 4).

c. Solution at steady state

In this section, we demonstrate that the solution (3.15), (3.17), and (3.18) tends to the solution found by Pedlosky (1978a) for the steady-state case. First, we show that the streamfunction in the steady-state solution is identical to (3.15). The vertical coordinate of Pedlosky (1978a) is $\tilde{z} = z - 1$ so that the steady-state solution there is

$$z = \frac{2}{\pi}\tan^{-1}[\tanh(\pi x/2)\tan(\pi\rho/2)] + 1, \quad (3.19)$$

which may be written

$$-\cot(\pi z/2) = \tanh(\pi x/2)\tan(\pi\rho/2). \quad (3.20)$$

To express this in terms of ψ, the relation between ρ and ψ at steady state must be used. At steady state, (2.23) reduces to $\rho_x\psi_z - \rho_z\psi_x = 0$, which implies that ρ may be expressed as some function of ψ. Far offshore, $\psi = z$ and $\rho = 1 - z$ so that the steady-state relation between ρ and ψ must be $\rho = 1 - \psi$. Therefore the solution of Pedlosky (1978a) may be expressed in the form

$$-\cot(\pi z/2) = \tanh(\pi x/2)\tan[\pi(1 - \psi)/2], \quad (3.21)$$

which gives

$$\tan(\pi\psi/2) = -\tanh(\pi x/2)\tan(\pi z/2). \quad (3.22)$$

Thus, the streamfunction (3.15) is the same as the one implicit in the steady-state solution of Pedlosky (1978a).

To show $\rho(x, z, t)$ tends toward the steady-state ρ, it is sufficient to show that $\rho \rightarrow 1 - \psi$. Note that the streamfunction (3.15) may be expressed in the form

$$\psi_0(x, z) = \frac{1}{\pi}\cos^{-1}\left(\frac{1 + \cos\pi z \cosh\pi x}{\cos\pi z + \cosh\pi x}\right), \qquad (3.23)$$

so that

$$1 - \psi_0(x, z) = \frac{1}{\pi}\cos^{-1}\left(-\frac{1 + \cos\pi z \cosh\pi x}{\cos\pi z + \cosh\pi x}\right). \qquad (3.24)$$

From (3.18) for $t \gg 1$:

$$\rho \sim \frac{1}{\pi}\cos^{-1}\left(-\cos\pi z - \frac{1 - \cos^2\pi z}{\cos\pi z + \cosh\pi x}\right) \qquad (3.25)$$

$$\sim \frac{1}{\pi}\cos^{-1}\left(-\frac{1 + \cos\pi z \cosh\pi x}{\cos\pi z + \cosh\pi x}\right). \qquad (3.26)$$

Thus $\rho \rightarrow 1 - \psi$ as $t \rightarrow \infty$ and therefore tends to the solution found by Pedlosky (1978a).

The steady-state υ found by Pedlosky (1978a) that satisfies $\upsilon \rightarrow 0$ as $x \rightarrow -\infty$ may be written in the form

$$\upsilon = -\epsilon^{-1}x - (\epsilon\pi)^{-1}\log(2\cosh\pi x + 2\cos\pi\rho). \qquad (3.27)$$

Given the form of the $t \gg 1$ limit of ρ (3.26),

$$2\cosh\pi x + 2\cos\pi\rho = \frac{2\sinh^2\pi x}{\cos\pi z + \cosh\pi x}, \qquad (3.28)$$

so that the steady-state υ expressed in (x, z) is

$$\upsilon = -\frac{x}{\epsilon} - \frac{1}{\epsilon\pi}\log\left(\frac{2\sinh^2\pi x}{\cos\pi z + \cosh\pi x}\right). \qquad (3.29)$$

Now we show that our solution (3.17) for υ tends toward the steady υ of Pedlosky (3.29). As a preliminary step, consider the function $y = \cosh u$ for $y \gg 1$ and $u \gg 1$:

$$y = \cosh u = (e^u + e^{-u})/2 \sim e^u/2, \quad y \gg 1, \quad u \gg 1, \qquad (3.30)$$

so that

$$u = \cosh^{-1}y \sim \log(2y), \quad y \gg 1, \quad u \gg 1. \qquad (3.31)$$

Then, for $t \gg 1$ and $x < 0$, (3.17) gives

$$\upsilon(x, z, t) \sim t - \frac{x}{\epsilon} - \frac{1}{\epsilon\pi}\cosh^{-1}\left[\frac{\sinh^2\pi x \exp(\pi\epsilon t)}{\cos\pi z + \cosh\pi x}\right] \qquad (3.32a)$$

$$\sim t - \frac{x}{\epsilon} - \frac{1}{\epsilon\pi}\log\left[\frac{2\sinh^2\pi x \exp(\pi\epsilon t)}{\cos\pi z + \cosh\pi x}\right] \qquad (3.32b)$$

$$\sim -\frac{x}{\epsilon} - \frac{1}{\epsilon\pi}\log\left(\frac{2\sinh^2\pi x}{\cos\pi z + \cosh\pi x}\right), \qquad (3.32c)$$

which is the steady-state υ in (3.29). Note that at $x = 0$, $\upsilon = t$ identically. The steady-state υ has a logarithmic singularity in the $x \rightarrow 0$ limit, consistent with this unbounded growth of υ at $x = 0$.

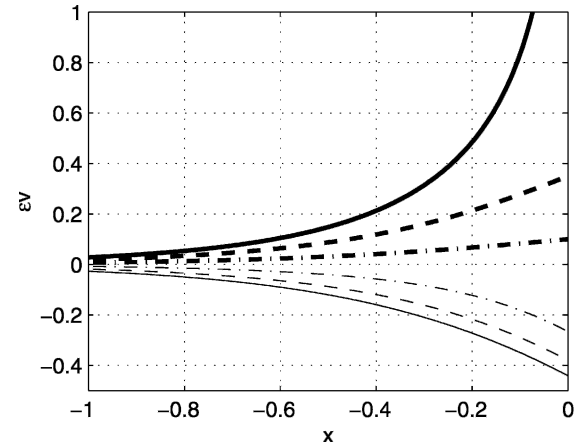

FIG. 5. Alongshore velocity $\epsilon\upsilon$ at the surface and bottom plotted vs offshore distance for times $\epsilon t = 0.1$ (dash–dot lines) and 0.35 (dashed lines) and at steady state (solid lines). The thick lines are the bottom velocities, and the thin lines are the surface velocities.

4. Discussion of time-dependent solutions

The alongshore velocity υ is initially zero and grows monotonically to its steady-state value, except at the coastal wall, where the velocity grows indefinitely with time. A southward flow develops along the surface, and northward flow develops below the surface, with the strength of flow decaying in the offshore direction (Fig. 5).

Most of the flow is within one Rossby deformation radius of the coast. The initial growth of the surface jet is more rapid than the undercurrent, but after approximately $\epsilon t = 0.4$, the undercurrent flows more rapidly than the surface current (Fig. 6).

Steady state is reached by $\epsilon t \approx 1$ for $|x| \gtrsim 1$. For $|x| \ll 1$, more time is needed to reach steady state, consistent with the requirement $[\sinh^2\pi x \exp(\pi\epsilon t)] \gg 1$ in the derivation of the expression of steady-state υ (3.32).

Investigation of the alongshore momentum equation [(2.8b)] reveals that the initial growth of υ is due to the combination of Coriolis and pressure gradient terms, because $\upsilon = 0$ implies the nonlinear terms vanish at the initial instant. The Coriolis and pressure gradient terms are steady in time (recall u is steady); therefore, steady state is achieved exactly when the cross-shore advection of alongshore momentum balances the Coriolis and pressure gradient terms. Thus, the surface jet is accel-

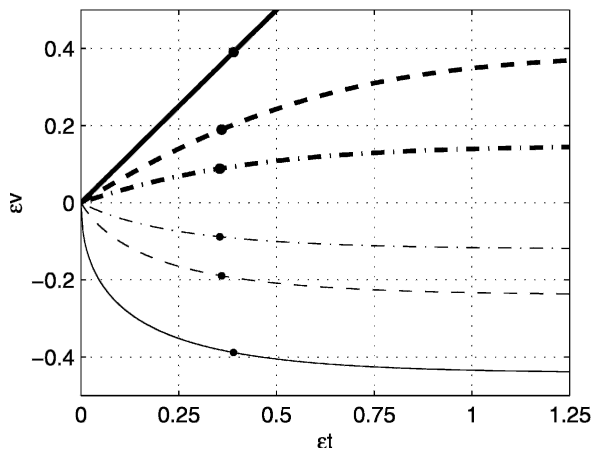

FIG. 6. Temporal evolution of the alongshore velocity ϵv at the surface and bottom at $x = 0$ (solid lines), $x = -0.25$ (dashed lines), and $x = -0.5$ (dash–dot lines). The thick lines are the bottom velocities and the thin lines are the surface velocities. The time when the bottom velocity surpasses the surface velocity is marked on each line with a dot.

erated by the Coriolis force from the rapid near-surface onshore flow, while the undercurrent is driven by the alongshore pressure gradient, due to the weak onshore flow at lower levels. The rapid onshore flow near the surface also allows the nonlinear terms to grow quickly there, causing the surface alongshore flow to approach steady conditions faster than the deep flow.

The density ρ is advected by the steady cross-shore velocity field. The advecting velocity near the surface is large so that the isopycnals close to the surface are advected into the surface Ekman layer quickly (Fig. 7).

The exact expression for density (3.18) permits us to calculate the time required for any given isopycnal to upwell to the surface. Taking the limits $x \to 0$, then $z \to 1$ in (3.18) yields the maximum density that has been upwelled to the surface at a given time (Fig. 7b). Solving this expression for time yields

$$t_\rho = -\frac{1}{\epsilon\pi}\log\left[\frac{1}{2}(\cos\pi\rho + 1)\right], \tag{4.1}$$

the length of time an isopycnal $0 \le \rho < 1$ will take to reach the surface.

The alongshore transport of this solution, calculated by an adaptive quadrature method as the integral of ϵv over the domain $0 \le z \le 1$ and $-10 \le x \le 0$, remains bounded (Fig. 8), because the singularity in the steady-state v at $x = 0$ is logarithmic. The net transport is always poleward, despite the rapid growth of the equatorward surface jet at early times.

A constant alongshore pressure gradient $\partial p/\partial y = -1$ has been assumed to exist in this model. The exact solution to the initial-value problem, given by (3.15), (3.17), and (3.18), for an arbitrary constant alongshore pressure gradient $\partial p/\partial y = -P_G$ is unchanged except for the alongshore velocity v, which is replaced by

$$v(x, z, t) = P_G t - \frac{x}{\epsilon} - \frac{1}{\epsilon\pi}\cosh^{-1}\left\{\cosh\pi x + \frac{\sinh^2\pi x[\exp(\pi\epsilon t) - 1]}{\cos\pi z + \cosh\pi x}\right\}. \tag{4.2}$$

The solution for ξ remains (3.16), although the expression relating v and ξ now appears as

$$\xi = x + \epsilon(v - P_G t). \tag{4.3}$$

The solution tends to a steady-state solution only for $P_G = 1$. Changing the pressure gradient changes only the alongshore velocity field by a spatially uniform value that evolves linearly in time. If the alongshore pressure gradient is set to zero, the poleward undercurrent vanishes (Fig. 9). In this case, the alongshore velocity $v = 0$ at the coast (except at the surface), while $v = -t$ far offshore.

If the alongshore pressure gradient is constant in space but is a function of time $P_G(t)$, then the same

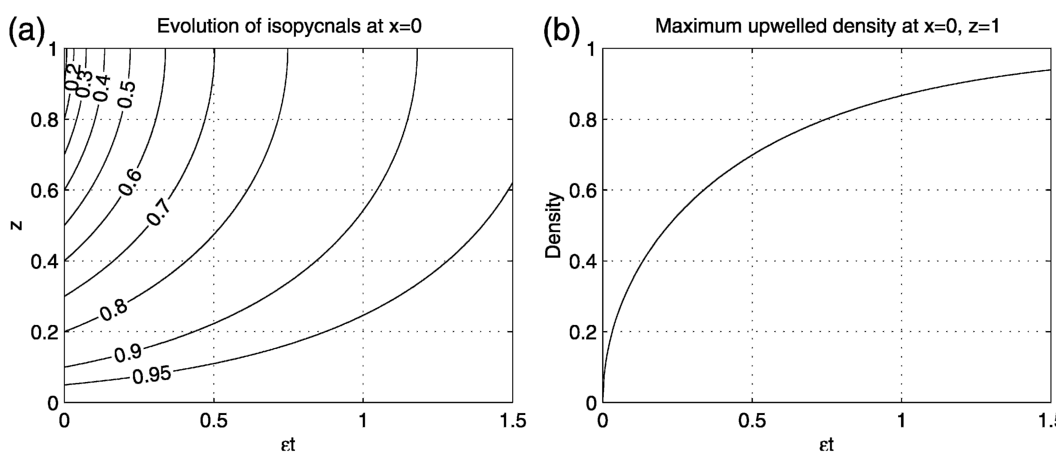

FIG. 7. The evolution of the density next to the coast: (a) isopycnals at $x = 0$ vs time, and (b) maximum upwelled density vs time.

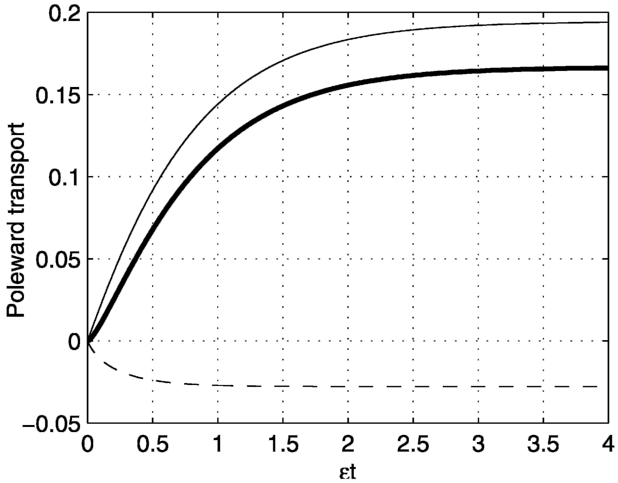

FIG. 8. Alongshore transport, calculated as the area integral of ϵv over the domain by numerically integrating exact solution. Positive values correspond to northward flow. Thick solid line is net poleward transport, thin solid line is northward-flowing contribution, and thin dashed line is southward-flowing contribution. The net transport is positive for all time.

solution is obtained except that the terms $P_G t$ in (4.2) and (4.3) must be replaced by time integrals:

$$v(x, z, t) = \int_0^t P_G(t')\,dt' - \frac{x}{\epsilon} - \frac{1}{\epsilon\pi}\cosh^{-1}\left\{\cosh\pi x + \frac{\sinh^2\pi x[\exp(\pi\epsilon t) - 1]}{\cos\pi z + \cosh\pi x}\right\} \quad \text{and} \tag{4.4}$$

$$\xi = x + \epsilon\left[v - \int_0^t P_G(t')\,dt'\right]. \tag{4.5}$$

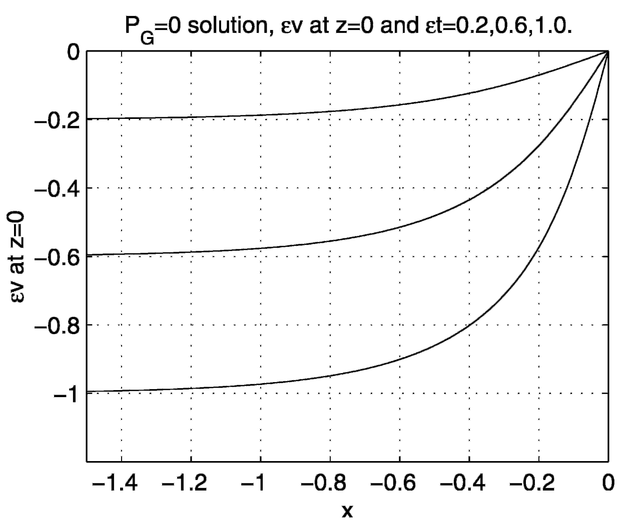

FIG. 9. Alongshore velocity at the bottom, with no alongshore pressure gradient. The alongshore velocity ϵv at $z = 0$ is shown for times ϵt = 0.2, 0.6, and 1.0. Velocity is increasingly negative at subsequent times.

Let us consider the dimensional values of motion in the $P_G \equiv 1$ solution expressed by (3.15), (3.17), and (3.18). As noted by Pedlosky (1978a), the dimensional value of the steady-state alongshore flow v_* is actually independent of the choice of V as defined in section 2a because of the dependence of the solution on ϵ. The solution for v (3.17) implies that the natural dimensionless variables for v and t are ϵv and ϵt. Let the scale of these variables be denoted by $v_* = V_0 \epsilon v$ and $t_* = T_0 \epsilon t$. The appropriate characteristic dimensional scales for v_* and t_* are then

$$V_0 = V/\epsilon = NH \quad \text{and} \tag{4.6}$$

$$T_0 = T/\epsilon = l/U = \rho_0 N H^2/\tau, \tag{4.7}$$

respectively. Thus, the alongshore velocity scales with the internal Kelvin wave speed, and the time scales with the cross-shore advective time scale. The existence of the natural dimensionless variables ϵv and ϵt is also indicated by the fact that the Rossby number ϵ can be completely scaled out of the equations of motion (2.8) by the rescaling

$$\hat{v} = \epsilon v, \quad \hat{t} = \epsilon t, \quad \text{and} \quad \hat{p} = \epsilon p. \tag{4.8}$$

Therefore, we specify V and T according to their characteristic values, $V = NH$ and $T = \rho_0 N H^2/\tau$. This effectively sets the Rossby number $\epsilon = 1$, determines the horizontal aspect ratio $\alpha = U/V = \tau/(\rho_0 f_0\, NH^2)$, and defines the alongshore length scale $L = l/\alpha = \rho_0 N^2 H^3/\tau$.

The consideration of dimensional scales shows that, while the steady-state v_* is independent of the wind stress τ, the time scale does depend on τ. That is, the wind stress determines how quickly steady state is approached, but not the magnitude of the steady-state alongshore velocity.

Let us choose dimensional scales appropriate for the Oregon coastal ocean:

$$\tau = 0.1\ \mathrm{N\,m^{-2}}, \quad H = 200\ \mathrm{m}, \quad \text{and} \quad \Delta\rho/\rho_0 = 5 \times 10^{-4}. \tag{4.9}$$

Note that H is not the total ocean depth but is chosen to represent the depth of the fluid flowing onshore in response to upwelling winds. The remaining scales are then

$$l = 10\ \mathrm{km}, \quad U = 5\ \mathrm{mm\,s^{-1}}, \quad V = 1\ \mathrm{m\,s^{-1}}, \quad \text{and}$$
$$T = 2 \times 10^6\ \mathrm{s} = O(20\ \text{days}), \tag{4.10}$$

and the horizontal aspect ratio is $\alpha = 0.005$. At one-half of deformation radius offshore after $\epsilon t = 0.5$, these scales give an equatorward surface flow $O(10\ \mathrm{cm\ s^{-1}})$ and a poleward undercurrent $O(10\text{–}20\ \mathrm{cm\ s^{-1}})$ (see Figs. 5 and 6). The transport (Fig. 8) has a dimensional scale of $VlH = 2$ Sv (where 1 Sv $\equiv 10^6\ \mathrm{m^3\ s^{-1}}$) so that the net poleward transport at steady state is approximately 0.3 Sv.

5. Summary

The nonlinear upwelling model of Pedlosky (1978b) is revisited, and a new time-dependent solution to the initial-value problem is derived. This solution is found by expressing the model in the same characteristic coordinates as in Pedlosky (1978b), but new time-dependent boundary conditions are applied in the transformed coordinate frame. Transforming back to physical coordinates reveals that the streamfunction is time-independent so that the vorticity component $u_z - w_x$ remains zero. In mathematical terms, analytic expressions are written for the streamfunction, the density, and the alongshore velocity, completing the solution of the nonlinear, time-dependent system of coupled partial differential equations. In physical terms, this solution represents a self-consistent description of the cross-shore advection of momentum and density that describes the characteristic lifting of isopycnals typically observed during upwelling. It is demonstrated that the new solution tends toward the steady-state solution of Pedlosky (1978a).

In the solution, it is the inviscid interior flow that reaches a steady state. Interactions of the surface boundary layer and the interior flow are not explicitly represented in the model. Unsteady surface boundary-layer features, such as the wind-driven offshore movement of an upwelling front, which in general may arise in the steadily forced problem, are assumed to have a relatively small influence on the interior solution.

Alongslope variations, effects of sloping topography, the presence of a pycnocline, and the bottom Ekman layer have all been neglected, and the effect of the surface Ekman layer has been parameterized as a mass sink in the corner of our domain. These idealizations must be kept in mind while interpreting the solution.

Inasmuch as there is an interior response to upwelling winds that can be described by frictionless, hydrostatic, and semigeostrophic ($l \ll L$) dynamics, the solution does provide insight and understanding about the upwelling process. It demonstrates that the inviscid interior response to coastal Ekman transport divergence can progressively bring deep isopycnals to the surface, a result that is suggested by classical linear theory but not previously represented explicitly in an analytical solution. It suggests that the advective timescale T [(4.7)] of this interior response depends on the magnitude of the wind stress τ, even though the amplitude of the steady solution for the alongshore current is independent of τ. It suggests also that an equatorward jet will develop rapidly in the upper interior and then soon equilibrate, while a poleward undercurrent below will grow slowly to a significant velocity, if a poleward pressure gradient force is present to drive it.

The equation set (2.5) should have wide applicability to coastal flow dynamics. Further investigation of behavior in these equations is planned.

Acknowledgments. This research was supported by a postdoctoral fellowship from the Natural Sciences and Engineering Research Council of Canada to PFC, by the National Science Foundation Coastal Ocean Processes (CoOP) program under Grant OCE-9907854, and by the Cooperative Institute for Oceanographic Satellite Studies, with funding from the NOAA/NESDIS Office of Research and Applications Oceanographic Research and Applications Division.

APPENDIX A

An Alternate Set of Characteristic Coordinates

One can derive an alternate set of canonical coordinates of the system by first using potential vorticity conservation [(2.14), with $q = 1$] and the thermal wind relation (2.11) to write the partial differential equation in (2.18) for ψ in the form

$$(1 - \epsilon\rho'_z)^2\psi_{xx} + (1 + \epsilon^2\rho'^2_x)\psi_{zz} + (1 - \epsilon\rho'_z)2\epsilon\rho'_x\psi_{xz} = 0. \tag{A.1}$$

The following characteristic coordinates may be found

$$X = x, \tag{A.2}$$

$$\zeta = z - \epsilon\rho'(x, z, t), \quad \text{and} \tag{A.3}$$

$$T = t. \tag{A.4}$$

These are similar to the coordinates used in Pedlosky (1978a) for study of the steady-state problem, where density was used as the vertical coordinate. The existence of two independent sets of coordinate transformations that render the semigeostrophic system linear has been noted by Hoskins and Draghici (1977) and discussed by Gill (1981).

Starting from the equations of motion in physical (x, z, t) coordinates and transforming to the above set of characteristic coordinates changes the system (2.16), (2.17), (2.14), and (2.11) to

$$\frac{\partial\hat{\xi}}{\partial T} + \epsilon\frac{\partial\hat{\psi}}{\partial\zeta} = 0, \tag{A.5}$$

$$\frac{\partial\hat{z}}{\partial T} + \epsilon\frac{\partial\hat{\psi}}{\partial X} = 0, \tag{A.6}$$

$$\frac{\partial\hat{\xi}}{\partial X} - \frac{\partial\hat{z}}{\partial\zeta} = 0, \quad \text{and} \tag{A.7}$$

$$\frac{\partial\hat{\xi}}{\partial\zeta} + \frac{\partial\hat{z}}{\partial X} = 0. \tag{A.8}$$

These equations are the v momentum, density conservation, potential vorticity conservation, and thermal wind, respectively. In analogy with ξ defined in the body of the paper, we have defined

$$\hat{\xi}(X, \zeta, T) = X + \epsilon[\hat{v}(X, \zeta, T) - T]. \tag{A.9}$$

The resulting equation for the streamfunction is Laplace's equation,

$$\frac{\partial^2 \hat{\psi}}{\partial X^2} + \frac{\partial^2 \hat{\psi}}{\partial \zeta^2} = 0. \tag{A.10}$$

The solution of the problem in this coordinate system involves a variable analogous to ξ_0, let us call it $\zeta_0(T)$, where the boundary condition on ψ is different for $\zeta > \zeta_0$ than it is for $\zeta < \zeta_0$. It turns out that ζ_0 starts at $X = 0$ and $\zeta = 1$ when $T = 0$ and moves down the line $X = 0$, approaching the point $X = 0$ and $\zeta = 0$ as $T \to \infty$.

The solution in this coordinate system may be written

$$\hat{\psi}(X, \zeta, T) = \frac{1}{\pi}\cos^{-1}\left[\frac{1 + C(X, \zeta, T)\cosh\pi X}{C(X, \zeta, T) + \cosh\pi X}\right], \tag{A.11}$$

$$\hat{z}(X, \zeta, T) = \frac{1}{\pi}\cos^{-1}[C(X, \zeta, T)], \quad \text{and} \tag{A.12}$$

$$\hat{\xi}(X, \zeta, T) = -\frac{1}{\pi}\cosh^{-1}\left\{-\frac{\cos\pi\zeta - \cosh\pi X}{2\exp(-\pi\epsilon T)} + \left[\frac{(\cos\pi\zeta - \cosh\pi X)^2}{4\exp(-2\pi\epsilon T)} + \frac{(\cosh\pi X\cos\pi\zeta - 1)}{\exp(-\pi\epsilon T)} + 1\right]^{1/2}\right\}, \tag{A.13}$$

where

$$C(X, \zeta, T) = \frac{\cos\pi\zeta - \cosh\pi X}{2\exp(-\pi\epsilon T)} + \left[\frac{(\cos\pi\zeta - \cosh\pi X)^2}{4\exp(-2\pi\epsilon T)} + \frac{(\cosh\pi X\cos\pi\zeta - 1)}{\exp(-\pi\epsilon T)} + 1\right]^{1/2}. \tag{A.14}$$

This demonstrates that either of the two different coordinate transformations used by Pedlosky (1978a,b) can be applied to the time-dependent system to convert it to linear form, and an exact time-dependent solution can then be written in either set of transformed coordinates.

APPENDIX B

Derivation of $\xi_0(T)$

The solution (3.9) of the integral equation in (3.8) is obtained as follows. Assume a parcel of fluid in (x, z) space starts at $x = \xi_0$ and $z = 1$ and is advected to $x = 0$ in a time $T(\xi_0)$. The inverse of this function is $\xi_0(T)$. Along $z = 1$, the vertical velocity vanishes so that the parcel remains on $z = 1$ and is advected in one dimension. Here we calculate $T(\xi_0)$ under the assumption that $\partial\psi/\partial z \approx \partial\psi_0/\partial z$ along $z = 1$. This assumption will be valid for $t = 1$, and the time at which this assumption breaks down must be checked a posteriori. The one-dimensional advection problem may be written (see 2.28):

$$\frac{dx}{dt} = \epsilon u = -\frac{\epsilon\sinh\pi x}{\cosh\pi x - 1}, \tag{B.1}$$

which is a separable differential equation,

$$\epsilon dt = -\frac{\cosh\pi x - 1}{\sinh\pi x}\,dx. \tag{B.2}$$

Integrating gives

$$\epsilon T(\xi_0) = \int_{\xi_0}^{0}\left(\frac{1 - \cosh\pi x}{\sinh\pi x}\right)dx \tag{B.3}$$

$$= \frac{2}{\pi}\log\left(\cosh\frac{\pi\xi_0}{2}\right). \tag{B.4}$$

Inverting this equation to express ξ_0 as a function of T yields (3.9). Direct substitution of (3.9) into the integral equation in (3.8) shows that (3.9) is the exact solution, despite the use of the approximation $u(x, z = 1, t) \approx u(x, z = 1, t = 0)$ in the derivation.

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