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Abstract approved: ______________________________

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The purpose of this descriptive case study analysis was to provide portraits of the methods college students used to solve probability problems and the factors that supported or impeded their success prior to and after two-week instruction on probability. Fourteen-question Pre- and Post-Instructional Task-Based Questionnaires provided verbal data of nine participants enrolled in a college finite mathematics course while solving problems containing simple, compound, independent, and dependent probabilistic events.

Overall, the general method modeled by the more successful students consisted of the student reading the entire problem, including the question; breaking down the problem into sections, analyzing each section separately; using the context of the question to reason a solution; and checking the final answer. However, this ideal method was not always successful. While some less successful students tried to use this approach when solving their problems, their inability to work with percents and fractions, to organize and analyze data within their own representation (Venn diagram, tree diagram, table, or formula), and to relate the process of solving word problems to the context of the

problem hindered their success solving the problem. In addition, the more successful student exhibited the discipline to attend the class, to try their homework problems throughout the section on probability, and to seek outside help when they did not understand a problem.

However, students did try alternate unsuccessful methods when attempting to solve probability problems. While one student provided answers to the problems based on his personal experience with the situation, other students sought key words within the problem to prompt them to use a correct representation or formula, without evidence of the student trying to interpret the problem. While most students recognized dependent events, they encountered difficulty stating the probability of a dependent event due to their weakness in basic counting principles to find the size of the sample space. For those students who had not encountered probability problems before the first questionnaire, some students were able to make connections between probability and percent. Finally, other inexperienced students encountered difficulty interpreting the terminology associated with the problems, solving the problem based on their own interpretations.

Methods College Students Use to Solve Probability Problems and the Factors that Support or Impede their Success

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

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Mary E. Bamberger, Author

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DEDICATION

This thesis is dedicated to my loving and supportive parents, Matthew and Bernadette Skoda. For it was they who taught me the words, "I can".

Methods College Students Use to Solve Probability Problems
and the Factors that Support or Impede their Success

CHAPTER I

THE PROBLEM

Introduction

Numeracy expectations in American society have changed rapidly in the past two centuries. More specifically, since the introduction of the computer leading to the information age, the emphasis of understanding, interpreting, and applying numbers has grown from the need to deal with basic arithmetic applications to the need to deal with data; number and operational sense; various forms of measurement; variables and their relationships; geometric shapes and spatial visualization; and chance (Dossey, 1997). Even these broader six aspects defined by the College Board's Mathematical Sciences Advisory Committee are insufficient to meet the challenges of today's data-drenched society. Steen (1997) suggested that in addition to these six aspects of numeracy, today's expectations also include assessing claims, detecting fallacies, evaluating risks, weighing evidence, and solving multi-step problems. Although Americans today are, on the whole, better educated and more literate and numerate than any who proceeded them, many employers say they are still unable to find enough workers with the reading, writing, mathematical, and other competencies required in the workplace. Changing economic, demographic, and labor market forces may worsen the problem in the future. Results from a recent National Adult Literacy Survey showed that approximately 50% of

American students graduating from high school have difficulty achieving the standard within the College Board's six aspects of numeracy (Kirsch, Jungeblut, Jenkins, & Kolstad, 1993).

From these six aspects of mathematical behavior encompassing the concept of numeracy, the aspect of chance (probability) is one of the most recent mathematical concepts recommended for the American mathematics curricula. Probability has never been an integral component of the early American mathematics curriculum, despite the fact that as early as the 1920's, mathematics educators were calling for students to be able to read and interpret various forms of probabilistic situations. Although more concerns were expressed in the 1940s and 1950s, little had changed the direction of probabilistic education until 1967. At that time, the National Council of Teacher of Mathematics [NCTM] and American Statistical Association [ASA] formed the Joint Committee on the Curriculum in Statistics and Probability. The goal of this Joint Committee was to promote probability and statistics as important and useful concepts that should be integrated into the mathematics curricula. Even after these recommendations, probability slowly infiltrated the mathematics curricula and was either not considered a required topic within most curricula or considered a topic at the end of the text to be covered if there was time. It was not until the publication of *Agenda for Action* (NCTM, 1980) and *A Nation at Risk* (National Commission on Excellence in Education [NCEE], 1983) that there was a greater awareness for the need of probability not only in the mathematics classroom, but also the physical, biological and social sciences (Garfield & Ahlgren, 1988, Shaughnessy, 1992).

Since the publications of *Agenda for Action* (NCTM, 1980) and *A Nation at Risk* (NCEE, 1983) emphasizing the need for probability to be integrated into the mathematics and science curriculum, professional organizations now provide guidelines and suggestions on the various probability topics to be taught across the curriculum. The NCTM (1989), in its publication of *Curriculum and Evaluation Standards*, called for introducing a number of probabilistic concepts throughout the K-12 school curricula. The NCTM (2000a) continued its support for teaching probability concepts in its most recent publication of *Principles and Standards for School Mathematics*. Similarly, at the post-secondary level, the American Mathematical Association of Two-Year Colleges [AMATYC] (1995), in its publication of *Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus*, also recognized the need to introduce non-mathematics and science major students to the basic probability laws. The Mathematical Association of America [MAA] (Sons, 1996) founded the Subcommittee on Quantitative Literacy Requirements to establish guidelines for quantitative literacy (numeracy) requirements for all students who received a bachelor's degree. Within their quantitative literacy requirements was the need for college graduates to have a deeper and broader experience with probability, extending the probabilistic concepts suggested by NCTM (1989).

Professional science and science education organizations have recommended the integration of probability in both mathematics and science curricula. The American Association for the Advancement of Science [AAAS] (1993), in its publication of *Benchmarks for Science Literacy,* discussed the need for probability in the K-12 curriculum to enhance students' understanding of real-world events. Similarly, the National Research Council [NRC] (1996) guided the development of the *National Science Education Standards*, including the integration of probability in the K-12 curriculum. These proposals called for an increased emphasis on probability in the mathematics and science curricula at the K-12 and collegiate level curricula.

One only has to glance through today's newspapers to see the extent to which the language of probability has become an integral part of our lives. As these examples illustrate, probability permeates day-to-day life:

- *There is a 20% chance of rain tomorrow.*
- *The chance of a patient having cancer, given a positive x-ray is 75%.*
- *Seven out of 10 Australians believe tobacco sponsorship of sports should be banned if alternate sponsorship is available, says a Drug Offence Survey*

Yet, researchers have found professionals, college students, and secondary students using their intuitions, not mathematical reasoning, to interpret probabilistic statements. Successful mathematical interpretation of these kinds of problems require an understanding of the terminology and procedures (i.e. equations, formulas, rules, and their interrelationships) generally used to represent these concepts. For example, Eddy (1982) noticed a tendency among physicians to confuse the predictive accuracy of a mammography report with its retrospective accuracy, using their intuition to interpret the results. These physicians incorrectly interpreted the probability of a patient having cancer, given the patient tested positively for cancer, as the probability of a person testing positively for cancer, given the patient had cancer. Such incorrect interpretations can lead to misdiagnosing a disease. Both professionals and college students with probabilistic background have also misinterpreted probabilistic situations. For instance, Falk (1986, 1989) observed statistics instructors, researchers trained in the use of statistics, and undergraduate students enrolled in a statistics course often misinterpreted what it means to reject the null hypothesis, by illustrating a confusion with the inverse of the conditional statement. The probability that one will reject the null hypothesis, given that the null hypothesis is true is the definition of a Type I error. However, due to some linguistic ambiguities, Falk observed the instructors and students familiar with hypothesis testing had a tendency to interpret the inverse - the probability that the null hypothesis is true, given that the null hypothesis was rejected - as the definition of a Type I error. The misinterpretation of the physicians or the linguistic ambiguities encountered by "experts" in statistics could be improved through a clearer understanding of probability concepts and principles.

Additional studies on college students and secondary students suggested these students also used their bias and subjectiveness, not mathematical reasoning, to interpret probabilistic situations (Cohen, 1957; Falk, 1988, 1989; Kahneman & Tversky, 1972, 1973; Tversky & Kahneman, 1973, 1974, 1980, 1982, 1983). For example, 68% of a group of Stanford University undergraduates stated that it is more likely for a female Stanford student to weigh between 124 and 125 pounds than to weigh more than 135 pounds (Tversky & Kahneman, 1983). These college students reasoned that the narrow specific interval (124-125 pounds) was a better judgement of the female students than the broader interval (above 135 pounds). Further studies on this phenomenon lead Tversky and Kahneman (1983) to hypothesize that people reason that an instance of a specific category can be easier to imagine or to retrieve than an instance of more inclusive categories, even if the inclusive category was the correct response. Studies have also suggested that secondary students who have had formal instruction in probability still misinterpret probabilistic situations found in daily life. Fischbein and Gazit (1984) noticed that secondary students who lost weekly lotteries over a course of two months continued to play, reasoning incorrectly that their chance of winning increases the more they play. Unfortunately, lottery drawings are independent events, and the chance of winning one drawing has no influence on the chance of winning the following drawing.

These previous probabilistic concepts in which professionals, college students, and secondary students used their intuitions to interpret the probabilistic statements are similar to those found in mathematics and science textbooks. In addition, these same probabilistic concepts are also found in professional and everyday conversations. Students eventually become consumers and citizens. They will enter a society relying more on probability to communicate information and influence their decision making. If today's society does not understand, interpret, and apply simple probabilistic situations found on randomizing devices such as dice, cards, and roulette wheels, they are unlikely to correctly interpret diet and medical claims, sports statistics, pre-election results, insurance claims, lotteries, and promotional ads. In addition, without a solid grasp of probability, these misunderstandings of probabilistic situations may limit a person's opportunity for growth within their personal and professional lives.

Based on these results, research identified four main areas relating to a person's misunderstanding of a probabilistic situation. First, most of the research conducted on probabilistic thinking was conducted by psychologists, not by educators, where the psychologists' intent was to identify various judgmental heuristics and difficulties adults encountered solving probability problems; refer to Chapter II and Appendix A for definitions and examples of judgmental heuristics (Cohen, 1957; Falk, 1988, 1989; Kahneman & Tversky, 1973, 1972; Tversky & Kahneman, 1973, 1974, 1980, 1982). These results formed a framework for identifying misinterpretations and difficulties adults encountered in everyday probabilistic events. However, the studies did not investigate factors that might help adults overcome their difficulties. As probability theory became more prominent in the science and mathematics curricula at all levels, more educators became interested in finding the connection between current instructional techniques and student understanding of the theory. However, there is a deficiency of research contributions by North American mathematics educators on student probabilistic thinking. The results of studies conducted on student probabilistic thinking in mathematics curricula in Spain, Israel, and Italy formed a base of knowledge to build studies in this country of how new standards, described above, should be implemented (Castro, 1998; Fischbein & Gazit, 1984; Fischbein, Nello, & Marino, 1991; Fischbein & Schnarch, 1997). Those studies conducted on North American mathematics curricula consisted of the results of national and international assessment and evaluations in a variety of curriculum areas at the K-12 levels (Beaton et al., 1996; NCTM, 2000b). These results indicted that elementary and secondary school students had difficulty with sample space recognition, proportional reasoning skills' applicability to probability, and compound probability statements.

Second, researchers still debate the appropriate educational level at which to introduce probability theory into the curriculum (Fischbein & Schnarch, 1997; Jones, Langrall, Thornton, & Mogill, 1997; Jones, Langrall, Thornton, & Mogill, 1999; Ojemann, Maxey, & Snider, 1965a, 1965b; Piaget & Inhelder, 1951/1975; Watson & Moritz, 1998). By introducing probability too early, students with weak conceptual thought processes could develop judgmental heuristics that might be more difficult to overcome in later years. In addition, if a student did not have the number sense needed to understand the underlying theories of probability, they most likely would rely on intuition rather than mathematical reasoning to solve probability problems.

Third, there is a lack of research on using research-based knowledge of students probabilistic thinking to create a probabilistic thinking framework to inform instruction (Jones, Langrall, Thornton, & Mogill, 1999; Jones, Langrall, Thornton, & Mogill, 1997; Tarr & Jones, 1997). Without a probabilistic thinking framework, it would be difficult to design effective curricula outlining an effective approach to developing probabilistic thinking in students. For example, it is not known whether a student could understand conditional probability without a strong understanding of compound probability problems. Similarly, if a student develops informal qualitative thinking skills in estimating the probability of an event, it is not known if that same student can be successfully introduced to probability comparisons.

Finally, studies looking specifically at the experimental effect of various instructional methods on student probabilistic thinking were derived from a wide range of student ages, curriculum objectives, and international curricula (Austin, 1974; Castro, 1998; Fischbein & Gazit, 1984; Fischbein et al., 1991; Jones et al., 1999; Ojemann, Maxey, & Snider, 1965a, 1965b; Shaughnessy, 1977). A college student with abstract reasoning abilities could learn probability theory better with a manipulative / pictorial approach than a symbolic approach. Additionally, the impact of small group / activity - based instruction might also improve college students' understanding of probability problem solving.

The introduction of probability in the mathematics and science curricula corresponds to the increasing number of studies on the learning and teaching of probability. Previous studies explored students' use of judgmental heuristics and difficulties they encountered, the appropriate education level to introduce probability into the curriculum, the use of research-based knowledge of students' thinking to create probabilistic frameworks, and experimental studies on the effects of various instructional methods on student probabilistic thinking. Probability is a branch of mathematics with wide ramifications in science research, business and industry, politics, and practical daily life. Future studies need to explore how the teaching of theoretical probability can influence students' understanding of probability and their ability to apply their knowledge in appropriate applications.

Statement of the Problem

In reviews of the research on the learning and teaching of probability, Shaughnessy (1992), Shaughnessy and Bergman (1993), Garfield and Ahlgren (1988) identified several issues that they recommended be the focus of future investigations.

After considering these reviews and observing the trends and directions of the research on the learning and teaching of probability, two issues serve as a basis for conceptualizing the current study.

The first issue relates to the methods college students use solving probability problems. Prior research by cognitive psychologists paved a path for mathematics educators by identifying judgmental heuristics adults use as a method to solve probability problems (Cohen, 1957; Falk, 1988, 1989; Kahneman & Tversky, 1973, 1972; Tversky & Kahneman, 1973, 1974, 1980, 1982). Judgmental heuristics, discussed in more details in Chapter II and in Appendix A, has been defined by cognitive psychologists as the reduction of complex tasks of assessing probabilities and predicting their values to simpler unique judgmental operations. Using the results based on the work of the cognitive psychologists, mathematics educators sought the frequency and magnitude at which students used judgmental heuristics to solve probability problems (Fischbein & Gazit, 1984; Fischbein & Schnarch, 1997; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993; Pollatsek, Well, Konold, Hardiman, & Cobb, 1987; Shaughnessy, 1977). Yet, the results of these studies indicated that some of the judgmental heuristics were not observed (Fischbein & Schnarch, 1997; Konold, et al., 1993; Pollatsek, et al., 1987), the frequency of the judgmental heuristics changed with age (Fischbein & Schnarch, 1997), and instruction might have a weak influence on overcoming some judgmental heuristics (Fischbein & Gazit, 1984; Shaughnessy, 1977). One study questioned the explicit use of judgmental heuristics and concluded that some students who correctly gave the judgmental heuristic responses were not using the judgmental heuristic reasoning to solve the problem (Konold, et al., 1993).

Instead of research continuing to verify whether the adult uses a judgmental heuristic or the correct method to solve a probability problem, the direction of research needs to revisit the question "What methods do adults use to solve probability problems?" without assuming that the alternate methods can be explained purely by judgmental heuristics. By not examining the alternate methods college students use to solve theoretical probabilistic situations, it is not known if college students reason chance, random events, and decisions under uncertainty while solving a problem, or if they resort to non-statistical or deterministic explanations of chance phenomenon to arrive at a solution.

The second issue relates to the factors that might support or impede college students' success for solving probability problems. The results from prior research conducted by mathematics educators identified three main categories identifying factors that support or impede students' success at all ages. First, the age of the student and the influence of instruction could be considered two main factors determining student success (Austin, 1974, Castro, 1998; Fischbein & Gazit, 1984; Fischbein, et al., 1991; Shaughnessy, 1977). Second, similar difficulties could be found across various students' ages such as incorrect problem solving approaches and students inventing their own probabilistic language (Beaton, et al., 1996; Konold, et al., 1993; NCTM, 2000b; Pollatsek, et al., 1987). Third, college students might not use judgmental heuristics as frequently as cognitive psychologists previously suggested (Konold, et al., 1993;

Pollatsek, et al., 1987). These studies identified factors to consider when designing curriculum to fit the needs of the students at various ages. In addition, the results from these studies suggested that judgmental heuristics might not be a factor influencing student's ability to solve a probability problem. Yet, these studies observed pre-determined factors and did not search for other possible factors supporting or impeding student success. In addition, the studies observing the factor of instruction were experimental studies also observing predetermined factors (Austin, 1974, Castro, 1998; Shaughnessy, 1977). Designing studies using pre-determined factors might have biased the results of the study by not observing the important factors that might actually be tangential to students' understanding of probability. Instead of research continuing to verify additional predetermined factors, the direction of research needs to examine specific student populations and observe and discover factors supporting or impeding their success which are characteristic of that one specific student population.

Based on the current trends and directions of these two issues of the research on the teaching and learning of probability, it is necessary to start shifting the direction towards observing specific populations of students, searching for their unique methods of solving probability problems, and noting the factors which supported or impeded their success. All college level students, including non-mathematics and science majors, enroll in courses integrating probabilistic concepts into their curriculum. However, non-mathematics and science majors are not required to develop as strong a mathematical background as their mathematics and science major peers. Instead, these non-mathematics and science majors enrolled in a foundational finite mathematics course designed to fit the mathematical needs of their degree requirements, including an introduction to probability theory. An extension of the research on the learning and teaching of probability as it relates to finite mathematics students provided the opportunity to re-examine and learn about the specific factors affecting this student population as it learns about probability theory. Therefore, this study examined the methods used and the factors that supported or impeded college finite mathematics students' success as they solved probability problems. In particular, this study sought the following two research questions:

1. What methods do college finite mathematics students use to solve probability problems?
2. What are factors that support or impede college finite mathematics students' success for solving probability problems?

Significance of the Study

Despite the recent inclusion of probability in the science and mathematics curricula, teachers cannot teach probability as if "starting from scratch." Up through the 1960s, it was generally assumed that when people reasoned in everyday probabilistic situations, they used methods similar to those a statistician would use, but less carefully (Peterson & Beach, 1967). In addition, Piaget and Inhelder (1951 / 1975) claimed that by the age of 12 most children could provide probabilistic reasoning for randomizing devices, such as dice and spinners, and had developed sound probabilistic notions including an intuitive understanding of the Law of Large Numbers. More recently, cognitive psychologists have found that people rely more on experience and subjective interpretation, not on the mathematical reasoning or probability theory, to solve probabilistic situations (Cohen, 1957; Falk, 1988, 1989; Kahneman & Tversky, 1973, 1972; Tversky & Kahneman, 1973, 1974, 1980, 1982). Thus, based on their life experiences, students have already encountered probabilistic situations and used some form of reasoning, either correctly or incorrectly, to obtain solutions. Faced with similar probabilistic situations in the classroom, students will depend on their own prior conceptions, intuition, interpretations, and mathematical abilities to solve problems. This fact poses a challenge for the teaching of probability and statistics. Most teachers wish to give their students opportunities that would allow them to build a working mathematical model of chance. However, teachers have found that their students' probabilistic thought processes are already preoccupied with potentially misleading misconceptions, incorrect mathematical reasoning, and misinterpretations.

This study sought to identify the methods used and the factors that support or impede college finite mathematics students' success as they solved probability problems in order to provide some guidelines for improving instruction and learning in this domain. Once these common methods and influential factors are recognized, probabilistic research and instruction can start to build upon them to develop better conceptual and reasoning processes leading to growth in students' competence in probabilistic problem solving.

This information is important for a number of reasons. First, it can be used in the development and organization of curricula containing probability instruction. Studies on students understanding of probability have not taken into account the possible mathematical skills students need to master prior to introducing probability in the classroom. If a student has not gained a strong background of mathematical concepts such as ratios and proportions, fractions, percents, or decimals, they might not be able to provide strong reasoning skills in a probabilistic situation. Second, this study provides useful information and ideas to current and future mathematics instructors. Although the focus of this study is on the students, the examples that come out of the study may enrich teachers' repertoire of knowledge about student's methods of solving probability problems and factors causing students to succeed or struggle. These results might offer instructors the opportunity to expand their understanding of what intuitive notions students bring to the classroom and what misconceptions need to be addressed as the instructor guides the learning process.

CHAPTER II

REVIEW OF THE LITERATURE

Introduction

This chapter provides an overall picture of the research conducted on the methods used and the factors that supported or impeded college students' success as they solved probability problems. The studies reviewed for this paper were organized into three areas:

- History and Nature of Probability
- Research on Judgmental Heuristics
- Research on Methods used and the Factors that Supported or Impeded College Students Success for Solving Probability Problems

The first section, History and Nature of Probability, provides a historical account of the discovery of probability and its mathematical development, leading to the progression of the various interpretations of probability. These historical accounts and interpretations of probability illustrate the existing gaps between probabilistic intuition and mathematical theory. In addition, these historical interpretations still influence current curriculum development and continual research on adults' assessment of probability of uncertain events.

The second section, Research on Judgmental Heuristics, provides summaries of the results conducted by cognitive psychologists on adults' interpretation of probability of uncertain events. More specifically, the cognitive psychologists conducted the research on subjective probability, one interpretation of probability. Based on the results of these studies, the cognitive psychologists created a theoretical framework for guiding mathematics and science educators by providing some misconceptions and judgmental operations held by adults when interpreting the probability of uncertain events. Therefore, the purpose of the second section is to provide examples of the judgmental heuristics defined by cognitive psychologists and later used by mathematics and science educators in their studies on the learning and teaching of probability.

The intent of including the history and nature of probability along with the results of the judgmental heuristic summaries in this chapter was to provide the historical and theoretical background for the studies conducted the methods used and the factors that supported or impeded college students' success as they solve probability problems, found in the final section of the chapter.

History and Nature of Probability

Unusual features are found in the history, conceptual and mathematical development, and interpretations of probability in comparison to other mathematical fields. These unusual features found in the historical development of probability also illustrate possible reasons for why mathematicians, cognitive psychologists, and philosophers cannot provide an adequate and consistent interpretation of the mathematical field of probability. In addition, these unusual features might also help explain why people today still have difficulty understanding and assessing probabilistic events.

Probability is a mathematical field that is actually quite young, only about 400 years old. In contrast, evidence of games depending on chance is traced to the ancient cultures of Indians, Babylonians, and Egyptians. Archeological digs have uncovered board games from 3500 BC, perfectly balanced fired-pottery die from 3000 BC, and evidence of Egyptians playing the game odd-or-even in 2000 BC (Bennett, 1998; Borovcnik, Bentz, & Kapadia, 1991; Lightner, 1991). Random events also played a vital role for these ancient people in their daily lives: settling disputes among neighbors, selecting a course of military strategy, dividing property, and delegating civic responsibilities or privileges (Bennett, 1998). Despite the everyday presence of chance and random events, the recognition of the mathematical interpretations regarding chance and certain random events is relatively recent. This lack of early knowledge of mathematical interpretation in rolling dice or playing cards has puzzled present day mathematical historians and philosophers. Some theories have been developed in an attempt to explain the slow emergence of associating mathematics to random events. Among the many explanations, historians and philosophers generally accept three main theories.

The first theory asserts that the cultures and beliefs of the past may have had an influence on the inability to recognize the link between random events and mathematical interpretations. Evidence of a belief that God or gods directed earthly events in a predetermined plan in which randomness was not considered was demonstrated in the early use of lotteries and dice for consulting gods (Hacking, 1975; Lightner, 1991). Thus, what is referred to in modern times as random event generators - dice, coins, and cards -

was previously accepted as divine judgments or authoritative decisions, with no definite predictive ability (Borovcnik et al., 1991).

The second theory asserts a lack of appropriate mathematical notations, symbols, and numerate people, which was exemplified by the origin of the pips (or dots) on dice, may have had an influence on the inability to recognize the link between random events and mathematical interpretations. Prior to the 14th century, people recognized the relationships of greater than and less than, but were not proficient in recognizing the relationships between the concepts of numbers and numeracy and the concepts of decimals and fractions. It was not until the Renaissance Period (14th to 17th centuries) and the development of algebra that the ability to write and calculate with Hindu-Arabic numerals was developed by scholars (Lightner, 1991; Hacking, 1975). Once mathematical notations and symbols were invented, and the church was more open to scientific inquiry, mathematicians started to recognize number sense, number patterns, and empirical frequencies associated with certain random events.

In the third and final theory, Hacking (1975) dismisses all previous theories as irrelevant or insufficient explanations for the late development of mathematical interpretations behind probability. Instead, Hacking claimed the slow emergence of the recognition between random events and mathematical interpretations of probability was due more to the dual meaning that has historically been attached to the word probability: scientia (knowledge) and opinio (belief). In the past, there were "high sciences" and "low sciences." High science, which sought absolute truth, included the study of mathematics, mechanics, astronomy, and philosophy. Low science, which rendered opinions based on empirical evidence, included the study of medicine and astrology. The old interpretation of probability was based more upon the association of opinion and aspects of low science. Thus, prior to the 17th century, probability was viewed more as "approval" from authority rather than a mathematical calculation (Hacking, 1975). It was not until the 17th century when Cardano, Galileo, Pascal, Huygens, Leibniz, and Fermat independently linked random events and mathematical interpretations to such diverse phenomena as games of chance; tax rates; and census records of weddings, christenings, and burials (Maistrov, 1974). Thus, by the late 17th century, a true mathematical interpretation of random events and the study of chance eventually turned into the field of mathematics called probability.

Despite the recognition of the mathematics behind probability in the early 17th century, it is still a surprise to realize that early probabilitists did not recognize the concept of equally-likely events until nearly two centuries later (Barnett, 1999; David, 1962). This might be more difficult to imagine since dice and other random event generators created as early as 3000 BC indicated the recognition of the need to create fair and unbiased generators. However, historians believe that previous subjective interpretations of probability influenced mathematicians inability to recognize equally likely events as an important foundation to the mathematical interpretation of probability. Thus, the inability to recognize equally-likely events could have been influenced by subjective interpretations of divine interference, inability to calculate empirical probabilities to support the theory, and the paradox between the probability of an event occurring and the ability to predict an event occurring (David, 1962).

A large number of other paradoxes accompanied the emergence of various probabilistic interpretations over time. These paradoxes indicate the disparity between intuitions and formal mathematical approaches within the difficult conceptual development of the mathematical interpretation of probability. In 1933, Kolmogorov published the first sound mathematical interpretation of probability. In his book, Kolmogorov devised a system of axioms for probability and derived the theories that were immediately acknowledged by mathematicians. However, Kolmogorov's interpretation did not truly clarify "probability." He only elaborated on the structural properties of probability and left the interpretation of the concept of probability as an open question. Therefore, there are still quite a number of distinctive philosophical interpretations that arouse curiosity and intermingle with the ability to understand and accept the mathematical theory behind probability.

These interpretations of probability illustrate the existing gaps between probabilistic intuition and mathematical theory. Four main interpretations of probability still influence the research conducted on the teaching and learning of probability and the development of mathematics and science curricula: subjective, classical, frequentist, and structural.

Subjective Interpretation

The old understanding of probability based upon the association of opinion and an aspect of low science still holds true in the 21st century. Words indicating the "probability", "likelihood", or "chance" of something occurring saturate everyday lives:

"I will probably stay home tonight", "I am unlikely to continue this job", "There is little chance that Kitzhaber will raise the speed limit." Such statements are made with little (if any) regard to a conscious interpretation, let alone any measurement, of the uncertainty concept. Instead, people use their individual experiences, personal degree of belief, and individual assessment of the situation, without regards to mathematical theory. Hence, "a probability concept is developed that is *specific to the individual* in the sense that it relates to the accumulated *personal* experiences *that person* brings to bear in assessing any situation" (Barnett, 1999, p. 84). This interpretation of subjective probability also can carry into situations by which mathematical reasoning could strengthen a decision. Some cognitive psychologists have investigated the understanding of probability and the role of intuition and subjective reasoning on decision making. More specifically, the work of Kahneman and Tversky attempted to categorize certain misconceptions of probability based on subjective interpretations, which they believed were systematic and predictable. The results from these studies on decision theory and judgmental heuristics in probabilistic situations can be found in the next section of this chapter.

Classical Interpretation

Since the recognition between random events and mathematical interpretations of probability occurred in the late 1600s, there was no attempt to define probability. With the recognition of mathematics behind random events, the later assumptions of assigning a probability to an event were that each event was equally likely to occur. Thus, Laplace first attempted to define a mathematical interpretation of probability in the beginning of the 19th century. Laplace's classical interpretation of probability did not include the possibility of unequally likely events. Instead, he defined probability: "as the ratio of the number of outcomes favourable (sic) to the event to the total number of possible outcomes, each assumed to be equally likely" (Barnett, 1999, p. 74). This classical interpretation of probability, that all events are equally likely, restricted its applications to a small range of artificial experiments with objects such as coins, dice, and spinners. This adopted interpretation did not include the possible applications of probability to social, biological, and environmental sciences. Nor did this definition provide the ability to examine probabilities without symmetry. Studies conducted by Piaget and Inhelder (1951 / 1975) and Fischbein (1975) indicated that elementary students had the fundamental ability to work within the classical interpretation of probability, with hopes to become building blocks to the frequentist interpretation of probability.

Frequentist Interpretation

Entering the 19th century, and the growing awareness of statistical applications, mathematicians were asked to estimate the numerical evaluations of probability in everyday life. Predicting insurance rates, taxes on imported goods, and estimating the mortality rate of humans could not be calculated with accuracy using the classical interpretation of probability. Mathematicians started to use their subjective interpretations of an event to provide an approximate solution. Searching for a more objective interpretation to calculating statistical frequency, and using the behavior of limits found in calculus, Venn formalized the idea of probability to include the behavior of limiting values of relative frequencies into indefinitely long sequences of repeatable (and identical) situations. For example, the probability of rolling a 3 on a particular die is the relative number of occurrences if the die were rolled infinitely number of times. In the late 19th century, Venn further refined this interpretation of probability to include non-equally likely events. Hence, the frequentist interpretation of probability, the ability to estimate probabilities on repeatable situations with frequencies, provided the needed conditions to consider independent, mutually exclusive, and conditional events (Barnett, 1999; Borovcnik et al., 1991). Still, this frequentist interpretation of probability does not fill in all the gaps in mathematical probabilistic understanding. Links between these concepts and some theorems still needed to be stronger.

Structural Interpretation

Based on the assumptions made by the classical and frequentist interpretations to probability, Kolmogorov published the first formal and sound mathematical interpretation of probability in 1933. What distinguished this work from the others was the ability to apply a body of definitions and theorems and their corresponding system of axioms to all forms of probabilistic situations requiring mathematical reasoning, filling in the gaps left behind by the frequentist interpretation. As mentioned earlier, this structural interpretation to probability does not define the nature of probability, but it does create the ability to precisely calculate probabilistic situations using the mathematical laws of probability (Borovcnik et al., 1991).

History and Nature of Probability: Concluding Remarks

Though theorists, psychologists, and philosophers may disagree whether or not some event ought to be assigned a probabilistic value and argue over the interpretation of probability, these various interpretations of probability can derive identical probabilistic results for certain random events. Though the end result of finding the probability of an event might be the same, the somewhat different interpretations of probability are more than just a philosophical argument. These different interpretations have important implications when considering the research on learning and teaching probability. Mathematical textbooks usually begin with the classical interpretations and soon move into frequentist and structural interpretations. Some books do introduce experimental situations by building upon the frequentist interpretations using the underlying assumptions of equally likely events. In addition, the placement of these probabilistic interpretations within the curriculum might also help explain many of the discrepancies and inconsistencies in research findings on the learning and teaching of probability. Finally, these different interpretations may also significantly affect the development of probabilistic ideas in the classroom.

Research on Judgmental Heuristics

Subjective probability, one of the four main interpretations of probability, plays an important role in decision making tasks. When making a decision, reaching a conclusion, or providing an explanation, the outcomes are usually based on judgements of the likelihood of uncertain events such as success in a new job, the outcome of an election, or the state of the market. Action on decisions based upon private interpretations of chance has people rely upon their past experiences and maturity in reasoning skills rather than mathematical interpretations. This reliance on experience and personal interpretations, not on mathematical interpretation or theory, is known as subjective interpretation of probability (Barnett, 1999; Cohen, 1957; Hacking, 1975). The reduction of complex tasks of assessing probabilities and predicting values to simpler judgmental operations has been defined by cognitive psychologists as judgmental heuristics (Tversky & Kahneman, 1974). These judgmental heuristics, which sometimes yield reasonable judgements, can also lead to severe and systematic errors (Kahneman & Tversky, 1972, 1973; Tversky & Kahneman, 1973).

A series of studies conducted by cognitive psychologists with both naïve and educated subjects has supported this hypothesis. This section provides an overview of four judgmental heuristics and two conditional probability judgmental heuristics cognitive psychologists associate with subjective interpretation of probability:

- Representativeness
- Availability
- Positive and Negative Recency Effects (Gambler's Fallacy)
- Conjunction Fallacy
- Conditional Probability Judgmental Heuristics

Representativeness

One judgmental heuristic Kahneman and Tversky (1972) associate with subjective probability is representativeness. A person who follows this judgmental heuristic estimates that the probability of an uncertain event is based on how well an outcome represents some aspect of its parent population, or how the event reflects the prominent features of the process by which it is generated. For example, the representativeness of misconceptions of chance states that people expect a sequence of events generated by a random process will represent the essential characteristics of that process even when the sequence is short. People believing that the sequence of flipping a coin five times and obtaining H-T-H-T-H is more likely than H-H-H-T-T, or even the sequence H-H-H-T-H, easily illustrate the judgmental heuristic of representativeness.

Availability

Availability is another judgmental heuristic associated with subjective probability. Tversky and Kahneman (1973) described a person who uses availability as one who evaluates the probability of an event by the ease with which relevant instances come to mind. For example, suppose a word is randomly picked from an English Dictionary. Is it more likely that the word begins with the letter K, or that K is its third letter? Availability tells the naïve person, one who has not had formal education in probability, that since it is much easier to think of words starting with K than of words in which K is the third letter, they would believe the word is more likely to start with K. Unfortunately, in the English language, there are about twice as many words with K in the third position than in the first (Tversky & Kahneman, 1973).

Positive and Negative Recency Effects (Gambler's Fallacy)

Some recent researchers of probabilistic judgmental heuristics classify positive and negative recency effects as subcategories of representativeness. However, the research conducted on recency effects took place before the recognition of the representativeness judgmental heuristic (Cohen, 1957, 1960). Recency occurs when a person is uncertain how to calculate the outcome of the next event, given the results of the previous independent trials. For example, a person using the positive recency judgmental heuristic when predicting a head or tail on a flip of a fair coin tends to believe that after a run of heads, a head is more likely to occur in the next toss. Thus, the positive recency effect causes the person to assume incorrectly that the conditions were not fair. A person using a negative recency judgmental heuristic when predicting a head or tail on a flip of a fair coin tends to believe that after a run of heads, a tail is more likely to occur in the next toss. Thus, the negative recency effect causes the person to believe intuitively that the alternating outcomes seem to better represent a random sequence. The idea of negative recency effect has also been known as "Gambler's Fallacy", in which the gambler believes the events will balance at the end.

Conjunction Fallacy

The conjunction fallacy stems from the extension rule of the Law of Probability: If $A \supset B$, then $P(A) \geq P(B)$. Since the set of possibilities associated with the conjunction (A and B) is included in the set of possibilities associated with B, the same principle can also be expressed by the conjunction rule: P (A and B) $\leq P(B)$. However, Tversky and Kahneman (1983) found in their study that when a person was given an uncertain event involving conjunctions, people tended to use the representativeness and availability heuristics to make a conjunction appear more probable. One of their studies showed that 85-90% of their subjects violated the conjunction rule of probability. This rule was illustrated when after people were given a description of a fictitious female character, who is "bright, single, 31 years old, outspoken, and concerned with issues of social justice", the subjects were more likely to believe that the person was a bank teller and was active in the feminist movement, than that the person was just a bank teller.

Conditional Probability Judgmental Heuristics

The probability of an event varies depending upon the occurrence or nonoccurrence of one or more related events. For example, Oregon sport fishermen are vitally interested in the probability of rain. The probability of rain on a given day, ignoring the daily atmospheric conditions or any other events, is the fraction of days in which rain occurs over a long period. This would be called "unconditional probability." Consider the chance of it raining tomorrow. It has rained almost continuously for two days and a storm is heading up the coast. This probability is conditional on the occurrence of several events, and any Oregonian would tell you that it is much larger than the unconditional probability of rain. Thus, the "conditional probability" of an event is the probability of the event given the fact that one or more events have already occurred. Problem I and II illustrate other conditional probability problems found in textbooks and life experiences:

Problem I (Falk 1988, p. 292):
An urn contains two white balls and two black balls. We blindly draw two balls, one after the other *without* replacement from that urn.

a. What is the probability that the second ball was white, given the first was also white? (correct solution: 1/3)
b. What is the probability that the first ball was white, given that the second ball was also white? (correct solution: 1/3)

Problem II (Kahneman, Slovic, & Tversky, 1982, p. 120):
Which of the following is more probable:

a. That an athlete won the decathlon, if he won the first event in the decathlon
b. That an athlete won the first event in the decathlon, if he won the decathlon
c. The two events are equally probable (correct solution)

Researchers identified common judgmental heuristics people use solving conditional probability problems. Time-axis (Falk, 1983, 1988) and causal bias (Kahneman, et al., 1982; Tversky & Kahneman, 1980) are two conditional probability judgmental heuristics used by people who are unsure how to solve the complex tasks of assessing conditional probabilities.

The time-axis fallacy is a prominent judgmental heuristic of conditional probability. Falk (1983, 1988) recognized that when a person is given a conditional probability situation, and asked about the probability of the first event happening, given the second has occurred, they have a difficult time going "back in time" to comprehend the question correctly. This judgmental heuristic is illustrated in Problem I, above, Falk (1983, 1989) found that the subjects of her study were able to answer part (a) correctly - one third - however, some subjects did not believe part (b) had an answer. The subjects argued that the probability of an outcome of a draw on an event that occurs later is not permissible. Others argued that since the first ball does not care whether the second is black or white, the answer will be one half. Hence, those who use the time-axis heuristic compute the probability of an event occurring at the immediate point of time at which the event takes place. To solve this problem, the conditional statement implies one white ball was removed out of the possible outcomes for the first draw. Therefore, the probability of selecting a white ball on the first draw is 1/3: one white ball to a total of three balls.

Causal bias is another prominent judgmental heuristic of conditional probability. A causal scheme follows a course of cause to consequence. However, when people are faced with finding the probability of an uncertain causal event, they may find it easier to invert this sequence and reason from consequence to cause, the incorrect reasoning. Research conducted by Kahneman, et al. (1982) tested this hypothesis by asking people which is more probable: $P(X|Y)$ or $P(Y|X)$ when X is the natural cause of Y and $P(X) = P(Y)$. In this study, the majority of the subjects answered $P(Y|X) > P(X|Y)$, when these probabilities were equal. An example used in their study was Problem II, from above. Let X be the event of winning the decathlon, let Y be the event of winning the first event. Thus, $P(X) = P(Y) = 1/N$, where N is the number of competitors. In addition, X appears to be the natural cause of Y, but the event X is independent of the event Y. Therefore, the correct answer is (c) since winning the decathlon does not imply the athlete won the first event, nor does winning the first event imply the athlete won the decathlon.

Judgmental Heuristics: Concluding Remarks

By investigating these subjective interpretations, intuitions of probability, misconceptions, fallacies in thinking, and judgmental biases, cognitive psychologists suggested a framework of how and when people use these judgmental heuristics. Some of the following studies reviewed in this chapter used the results of the work by the cognitive psychologists on judgmental heuristics as a theoretical framework for their own studies on the teaching and learning of probability. However, when using the results from these studies as the theoretical framework for a study on the learning and teaching of probability, the researcher must be careful. First, the main goals of the cognitive psychologists were to search for and define possible judgmental heuristics. These studies did not indicate the proportion of people who use these judgmental heuristics. If a small amount of the population appears to use one or more of the judgmental heuristics, the researcher must take into consideration if it is worth the time and effort to explore a specific judgmental heuristic in a classroom setting.

Second, the intent of these studies was not to examine the possibility of outside interaction influencing a person's judgmental heuristic. Before conducting an experiment on instructional strategies influencing a person's judgmental heuristic, it may be feasible for a researcher to first see if instruction on probability theory has an influence on a person's judgmental heuristic before devising experimental instructional strategies to change their reasoning skills.

Finally, the majority of these studies were conducted on probabilistic naïve adults, in which the results were gathered using a forced-choice multiple choice test. The data analysis of the multiple choice test did not support the possibility that either the person chose the "judgmental" answer due to frustration, if the person chose the judgmental answer due to the lack of understanding of the problem, or if the person chose the judgmental answer due to different reasoning other than the defined judgmental heuristic associated with that response. In addition, a person may have selected the "correct" answer, but the reasoning behind this selection might have been incorrect. It may be true that the results of these studies did support that judgmental heuristics are apparent in probabilistic naïve people, but further studies on the strength of these judgmental heuristics, on the influence of outside interaction on a person's judgmental heuristics, and on the possibility of other reasoning behind a solution needs to be examined before these results can help develop a strong framework for the research on the teaching and learning of probability.

Research on the Methods Used and the Factors that Supported or Impeded College Students' Success for Solving Probability Problems

Previous studies conducted on college students ability to develop probabilistic thinking and problem solving abilities suggested methods and factors which supported or impeded college students' success as they solve probability. These studies specifically suggested some factors such as the formal wording or algebraic notation associated with probability problems (Konold, 1989; O'Connell, 1999; Pollatsek, et al., 1987). Other factors included students' preconceived judgmental heuristics, intuitively based misconceptions, or weaknesses in their conceptual and procedural knowledge of the various concepts (Fischbein & Schnarch, 1997; Konold, et al., 1993; Pollatsek, et al., 1987). Finally, two studies explored the influence of the instruction factor on college students' success (Austin, 1974; Shaughnessy, 1977). The intent of this section is to provide summaries of studies identifying the methods used and the factors that supported or impeded college students' success as they solved probability problems. The following studies considered factors of college students' probabilistic reasoning skills, interpretations, intuitively based misconceptions, the relationships of different types of errors while trying to solve probability problems, and the influence of instruction on students' success.

Probabilistic Reasoning Skills

Konold et al. (1993) agreed with previous research conducted by cognitive psychologists on defining judgmental heuristics. However, Konold, et al. found the results limiting. They observed that once a judgmental heuristic had been defined, the cognitive psychologist would then focus their studies on just that one heuristic, while not looking at other possible student difficulties. Konold et al. did not refute the validity of past research conducted on these judgmental heuristics; but instead claimed other inconsistencies students encountered solving probability problems outside these judgmental heuristics. In order to explore this, Konold et al. developed two studies designed to investigate students' probabilistic reasoning skills. The purpose of the overall study was to explore other methods students used to solve probability problems that were designed to illustrate the representative judgmental heuristic through two different exploratory studies. More specifically, this study was designed to determine whether the representative judgmental heuristic was used exclusively, or whether other methods for solving representative problems existed.

The first study consisted of investigating two representative probability problems (Konold et al., p. 397):

> *Four-heads problem:* A fair coin is flipped four times, each time landing with heads up. What is the most likely outcome if the coin is flipped a fifth time?
>
> a. Another heads is more likely than tails
> b. A tails is more likely than another heads
> c. The outcomes (heads and tails) are equally likely (correct answer)
>
> *HT-Sequence problem*:
> Part I: Which of the following is the most likely result of five flips of a fair coin?
>
> a. HHHTT
> b. THHTH
> c. THTTT
> d. HTHTH
> e. All four sequences are equally likely (correct answer)
>
> Part II: Which of the above sequences would be least likely to occur?

Each problem was written on a separate page, with Part I and Part II of the second question on the same page. Once a student replied to a question, during the administration of the questionnaire, they could not change their reply after seeing the other questions.

The sample for the first study consisted of volunteers from three distinct clusters of students. The first cluster consisted of 16 female high school students attending the Summermath program at Mount Holyoke College, a summer program for high school girls interested in studying mathematics. The second cluster of students consisted of 25 undergraduate remedial mathematics students enrolled in a mathematics course, not studying probability, at the University of Massachusetts. The final cluster consisted of 47 students taking a pre-course survey for a statistical methods course in the College of Education, a course required for all advanced-degree candidates in psychology and education at the University of Minnesota.

The data collection for the first study included students' multiple choice responses to the two representative probability problems, stated above, with brief explanations for their choices. Each cluster of students was administered the two problems under different circumstances, different goals, and different times during their instructional period.

The data analysis of the first study consisted of two parts: students' responses to the questions and their brief explanation for their selection. The data analysis pertaining to the students' responses to the questions consisted of visual comparison of the overall percents of each response within each cluster. Thus, each cluster was the unit of analysis for the first study.

The results on the first question of the first study indicated that the majority of the students - 86% overall- correctly chose option *c* as their response, with the remedial students scoring the lowest (70% correct), and the advanced degree candidates the highest (96% correct). The most popular alternative answer - option *b* - was the one consistent with the judgmental heuristic Gambler's Fallacy. This option *b* was selected by 22% of the remedial students, 19% of the Summermath students, and 4% of the advanced degree candidates.

The results of the second question of the first study indicated that the majority of the students - 72% overall - correctly chose option *e*, with the remedial students scoring the lowest (61% correct), the Summermath class in the middle (69% correct), and the advanced degree candidates the highest (79% correct). None of the other four responses was favored over another for those not selecting *e* as the correct answer. However, performance on the follow-up question suggested that the majority of the students were not using correct reasoning on the second question. When responding to the follow-up question - which of the following is *least* likely - overall, only 38% of the students responded that all four sequences were equally likely and unlikely. This means, those who answered Part I correctly, responded again that the sequences were equally likely in Part II. The others who responded correctly on Part I selected one of the sequences as least likely in Part II. Konold et al. (1993) defined this pattern of response as the "Most-Least Switch." Thus, it appeared that many subjects who chose the correct answer in Part I nevertheless did not believe that all sequences were equally likely, as indicated by the assigned probabilities.

The final set of results from the first study consisted of a summary of brief explanations that the students gave for their responses. Claiming that the justifications offered little insight into the underlying rationales the study looked closer at three students and their responses. First, the researchers claimed that there was little evidence in the overall brief explanations to support their hypothesis for a possible Most-Least Switch inconsistency. However, by looking at three student responses, which illustrated Most-Least Switch inconsistencies, it might have been possible to give a stronger case for the existence of a Most-Least Switch heuristic by looking at all the students. In their rationale for the most likely sequence, all three students mentioned that any of the sequences could occur. The answers and accompanying justifications for the question of the least likely sequence seemed consistent with the representativeness heuristic. Based on the results from this study, and the brief explanations given by students, the researchers faced a dilemma. They thought that if the students who gave inconsistent answers were reasoning according to the outcome approach on the first part of the problem, why did they switch to answering the second part of the problem on the basis of the representative heuristic?

The second exploratory study had similar objectives as the first study. The question of interest for the second study was to explore the reasoning of students who committed the Most-Least Switch through interviews. Using the same two representative probability problems from the first study, the second study conducted formal interviews on undergraduate psychology students on their problem solving reasoning skills. The sample consisted of 20 volunteers enrolled in a psychology class. Eleven of the students had taken, or were currently enrolled in, a course in which probability had been taught.

The data collection consisted of a one-hour videotaped structured interview. The taped session included several questions concerning various aspects of probability, including the two questions used in the analysis for this study. However, the wording of Part I and Part II of the second question was modified (Konold et al., 1993, p. 401):

> *HT-Sequence problem*:
> Part II: Which of the following sequences is <u>least</u> likely to result from flipping a fair coin five times?
>
> a. HHHTT
> b. THHTH
> c. THTTT
> d. HTHTH
> e. All four sequences are equally unlikely (correct answer)

The modification included the words "most" and "least" to be underlined, using the word "result" in the question, and changing option *e* to "un"likely. The second modification consisted of Part I and Part II of the second question being written as two different questions, as opposed to Part II being an extension of Part I, as in the first study.

The Interview Procedures included the list of specific questions the interviewer asked the students. The study included an interview-coding guide in which if the student responded to the question correctly, the interviewer put a "+" after the question. If the student responded to the question incorrectly, the interviewer wrote a "-" after the question.

The data analysis consisted of comparing the percents of correct and incorrect answers to the structured questions. For the second study, the unit of analysis was the individual student. First, the students were classified into four categories: the four students who answered all questions correctly; the three students who showed no obvious inconsistencies in their responses, but protocols were incomplete; the 12 students who gave inconsistent responses; and one student who gave consistent but non-normative responses. After grouping the students, the data was further analyzed for possible patterns and inconsistencies among and between the groups.

The results of the study indicated no significant difference in the mean number of correct responses was found based on either gender or prior statistical instruction. In addition, the study did find three inconsistencies among the students probabilistic reasoning skills: qualitative answers versus probabilities, student responses, and constraints on the sum. The inconsistencies of qualitative answers versus probabilities refers to when students responded correctly to both versions of the Heads-Tails Sequencing Problem, but did not believe the sequences had equal probabilities of occurrence. The inconsistency of student responses referred to students responding correctly to the Four-Heads Problem, but did not exhibit an understanding of independence of successive trials. The inconsistency of constraints on the sum referred to students stating probability values whose sum equaled or exceeded one. Pertaining to the first question, Four Heads Problem, 85% of the students answered it correctly and were able to state the correct probability associated with their response. Ironically, two of the three students who answered the problem incorrectly were able to state the correct probability. This inconsistency of qualitative answers versus probability also carried into the second question. Seventy percent of the students answered Part I of the Heads-Tails Sequencing Problem correctly. Three students who answered Part I and Part II correctly were not able to give correct probabilities; hence, those students did not believe the sequences had equal probabilities. It was also found that six of the students gave probability values whose sums were greater than or equal to one. Finally, the results also indicated that four of the students did commit the Most-Least Switch, as defined in the first study.

Overall, the study suggested that if a student correctly answered a representative probabilistic problem, they might be using incorrect reasoning skills to reach that conclusion. Unfortunately, the study did not have the capability of exploring the alternative incorrect problem solving techniques.

Probabilistic Interpretation and Reasoning Skills

Konold (1989) observed that college students recognized basic probability law, such as $P(A) + P(A') = 1$. In addition, college students were comfortable with the classical interpretation of probability. However, cognitive psychologists observed that people tend to use judgmental heuristics when computing the probability of more complex events, as noted in the works of Tversky and Kahneman (1983). Working with college students, Konold (1989) additionally observed that regardless of whether a student used judgmental heuristics or formal probability theory to assess an event, the college students had difficulty interpreting the event and the probabilistic outcome. Therefore, the purpose of this study was to explore the errors college students made interpreting and reasoning probabilistic events.

The population for the study consisted of 16 undergraduate students from the University of Massachusetts at Amherst. The 16 students volunteered to participate in return for extra credit in a psychology course. The study required the students to participate in two interviews, with a five-month lapse between the two interviews.

However, only 12 of the original 16 students participated in the second interview. From this population, two students had previously enrolled in a high school statistics course, and three students had enrolled and completed a statistics course during the lapse between interviews.

The data for the study consisted of two videotaped interviews. The first interview consisted of presenting the students with three questions covering various aspects of probability. The goal of this first interview was to observe students' interpretations, reasoning abilities, consistency within their interpretations and reasoning ability, and to develop a model of non-standard reasoning abilities. The model, referred to as the outcome approach, explored the students' non-standard reasoning abilities in order to obtain a solution of the probabilistic event. The second interview consisted of presenting the students four additional probabilistic situations, similar to the original three. The goal of the second interview was to test the model on another set of questions and to search for responses which had not been observed in the first interview.

The data collection instrument for both interviews consisted of two structured interviews, which were videotaped. The problems were selected to vary along several dimensions of simple probabilistic events, searching for student response types that persisted across different situations, while also containing a variety of tasks. For example, in the first interview, the students were given the Bone Problem (Konold, p. 64):

> *Bone Problem*: I have here a bone that has six surfaces. I've written the letters *A* through *F*, one on each surface. If you were to roll that, which side do you think would most likely land upright? How likely is it that *x* will land upright? [Student is asked to roll the bone to see what happens.] What do you conclude about your prediction? What do you conclude having rolled the bone once? Would rolling the bone more times help you conclude which side is most likely to land upright?

The Bone Problem involved a relatively clear sample space and evident repeatability of trials, easily identifiable chance factors, the ability for the student to view the phenomena statistically, and provided a variety of tasks with the same event. The second problem in the first interview consisted of one probabilistic situation that gave the student the numerical probability of a weather prediction, asking the student to interpret the numeric probability. The final problem provided the student with probabilistic events that could not be assigned numeric probability, asking the student to interpret the event. The questions for the second interview were based on the data analysis of the first interview.

The data collection procedure for both interviews consisted of a one-hour videotaped interview in which the students were orally presented their problems. Students were instructed that they would be given several problems requiring interpretation about probabilistic situations. They were also instructed that the particular answers they gave were of less interest than the reasoning process leading to the answer. Therefore, the students were instructed to think aloud as they attempted to interpret each problem, verbalizing their thoughts as they occurred. The interviewers did not probe the students, but rather coaxed the students to further explain their reasoning. The order of the questions for each interview alternated on each successive interview.

The results from the first interview suggested the students interpretations and reasoning abilities illustrated two main approaches: frequentist approach and outcome approach. The frequentist approach, the ability to estimate probabilities on repeatable situations with frequencies, was observed as the main approach for some students. However, the focus on the results of the study was more concerned with the students illustrating the outcome approach. Thus, the results of the frequentist approach were not reported. The outcome approach was a more nonstandard approach to solving probabilistic situations, in which the students exhibited a lack of probabilistic theory to solve the problem, yet solved the problem in a coherent manner. Those students illustrating the outcome approach could be characterized as involving three general features: predicting outcomes of single trials, interpreting probability as a prediction, and basing probability estimates on causal features rather than on information pertaining to the distribution of the sample space. In the outcome approach, predictions of single trials took the form of "yes" or "no" decisions of whether a particular outcome will occur. For example, four students translated the statement "70% chance of rain" into the more definitive statement, "It's going to rain" (p. 68). In addition, after seeing the occurrence of a single trial, several outcome approach students indicated that a probability value was either right or wrong. For example, the students were given a situation in which the forecast claimed a 70% chance of rain, but it did not rain on that day. These students were then asked what they would conclude about the accuracy of the statement that there was a 70% chance of rain. Six outcome approach students concluded that the forecast was wrong. Finally, several students made statements indicating that they generated or interpreted probability estimates based on a causal analysis of the problem. Returning to the Bone Problem, three outcome approach students suggested that more reliable information to determine which side was more likely to land upright could be obtained from careful inspection of the bone rather than conducting trials.

The results from the first interview suggested that some college students did hold an outcome approach belief to interpreting probability problems. However, the outcome approach is not a belief that individuals either do or do not hold. Out of the 16 students, only two did not give at least one outcome-oriented approach response. Some of the outcome approach responses were made by the majority of students, whereas others were made by only five of the students. Therefore, instead of defining the outcome approach as a discrete category, the study defined outcome approach to be a set of beliefs that the students held to differing degrees. Summarizing the "strength" of an outcome-oriented student, the study counted the number of outcome-oriented statements made by each of the 16 students. Based on 15 probabilistic situations contained within the three questions, each student rated a score of 0 to 15 based on the number of outcome-oriented statements they made. The median for the 16 students was 4.17, with actual scores ranging from 0 to 13.

As stated earlier, the goal of the second interview was to test the outcome approach model on another set of questions and to search for responses which had not been observed in the first interview. Based on the results of the first interview, the outcome approach model consisted of three general features: predicting outcomes of single trials, interpreting probability as a prediction, and basing probability estimates on causal features rather than on information pertaining to the distribution of the sample space. The second interview comprised of four questions, with only one problem referring to a question from the first interview, the Bone Problem. The remaining three questions illustrated three other areas of probabilistic situations: conditional probability, compound probability, and modeling similar probabilistic situations with different random generators. The data collection procedures for the second interview were the same as the first interview. The students were asked to think out-loud and use paper and pencil to illustrate their solutions, if necessary.

The results from the second interview suggested that the outcome approach model defined in the first interview correlated positively with the outcome-oriented responses on the second interview ($r = .797$; $p < .005$, one-tailed). These correlations suggested that the students outcome-oriented responses were fairly consistent, both across different problems and across a five-month time interval. These statistical results are also supported by the statements generated by the students on the four problems. On the conditional probability problem, those responses illustrating the outcome-oriented approach believed they were being asked to form a definitive statement of the event, not to determine the probability of the event. In addition, on the Bone Problem, in which the students were to decide which side of the bone was most likely to land upright, the physical properties of the bones were used to predict the results of ten trials, not the frequency of the ten trials. On the compound probability problem, those responses illustrating the outcome-oriented approach arrived at a prediction of an event based on the results of a single trial and deciding if the event could or could not occur (yes / no decisional approach). Finally, when the students were given two probabilistic situations with same probabilistic outcomes, selecting a marble from an urn or rolling a die, the outcome-oriented approach responses suggested that the urn was an inappropriate model to substitute for the rolling of the die. This reasoning demonstrated that the causal features of one random generator could not be illustrated using another random generator, thus, not representing similar probabilistic values. This result matches the last feature of the outcome approach model, where the students based their probably estimates on causal features of a random generator rather than on information pertaining to the distribution of the sample space.

Overall, the results from this study suggested that college students had difficulty interpreting and reasoning probabilistic events. More specifically, when the students were confronted with an uncertain event, most students would use the formal, probabilistic knowledge when reasoning about situations that are clearly probabilistic and have a simple sample space. However, for situations that were less obviously probabilistic, or that the sample space was not as clearly defined, the students fell back on the use of judgmental heuristics. When the student fell back on a judgmental heuristic, their interpretation and reasoning skills would illustrate an outcome approach to solving the probabilistic problem. Thus, these students based their decisions of a probabilistic event by predicting outcomes of single trials, interpreting probability as a prediction, and basing probability estimates on causal features rather than on information pertaining to the distribution of the sample space.

Interpretations of Conditional Probability

Past research and position papers considered several possible misconceptions regarding the difficulty of students' understanding of conditional probability. Einhorn and Hogarth (1986) hypothesized that the syntax of a conditional problem may confuse students, making them believe it was a conjunction problem. Studies conducted by cognitive psychologists suggested a judgmental heuristic might be a cause of confusion between causality and conditional conditions (Tversky & Kahneman, 1980). Hence, Pollatsek et al. (1987) tried to address these two misconceptions and several others that they felt were important in their study addressing three issues of college students' understanding of conditional probabilities. Defining a "naïve student" as an undergraduate who had not taken a college level statistics course, Pollatsek et al. (1987) first issue investigated naïve students intuitions about conditional probability. Their second issue considered the misconception of the notation P(B|A): probability of the event B occurring, given that event A has occurred. Pollatsek et al. wanted to determine if the problem arose from the confusion between the conditional statement P(B|A) and the conjunction statement P(A and B). Their third issue concerned causal bias. Tversky and Kahneman (1980) had argued that a causal bias existed in judging conditional probabilities. Pollatsek et al. (1987) wanted to test whether this causal bias was truly powerful and persuasive, as previously assumed. In order to investigate these three issues, the study consisted of two different investigations.

The first investigation explored naïve students' fundamental inability to deal with conditional probability and the effect of causal bias. The first experiment consisted of testing 86 undergraduates enrolled in a lower division psychology course at the University of Massachusetts. The students voluntarily participated in this study, and their participation did not have any affect on their course grade.

On the first day of class, 86 undergraduate students completed a six-question, forced-choice questionnaire. Each question consisted of an event for which the students were asked to select one of the following three options that stated the correct relationship between the two conditional probabilities $P(A|B)$ and $P(B|A)$: $P(A|B) < P(B|A)$, $P(A|B) > P(B|A)$, or $P(A|B) = P(B|A)$. Of the six questions, the first three questions contained probabilistic events in which the two events were related, but one event was not necessarily the cause of the other event. These first three questions consisted of scenarios in which Event A could have been thought as necessary but not an adequate cause of Event B, while Event A was strongly implied by Event B; however, the two events were related. For example, if Event A was "being sick" and Event B was "having a fever", Event A both causes and is implied by Event B; however, the conditional probabilities indicated that $P(A|B) > P(B|A)$.

The remaining three questions were chosen to be sensitive to any causal bias that might have existed. The first question considered a scenario in which Event A did not cause Event B, and the two events were not related. The last two questions did not have correct answers for the given information. The six questions did not supply the student with numerical probabilities, only hypothetical situations indicating the possibility of causality. Thus, students were forced to pick a relationship based on their bias.

The data analysis consisted of computing the percent of correct responses to each of the questions for the entire population. The results of the study suggested most of the students gave the correct answer (range of 72% - 87% correct on the three questions). These three questions explored probabilistic events in which the two events were related, but one event was not necessarily the cause of the other event, In all three cases, the students strongly preferred the alternative consistent with P(A|B) > P (B|A). The last three questions referring to the sensitivity of any causal bias, suggested the students had a tendency to choose the alternative consistent with P (effect|cause) > P(cause|effect) ($\chi^2(1)$=11.52, 7.05 and 7.54 for each of the three questions, with $p < .01$). This result suggested the students were not sensitive to the causal bias judgmental heuristic. In addition, 50% of the students were able to choose P(A|B) = P (B|A), as the correct answer being sought. Hence, the researchers concluded from the results of the first experiment that these naïve students did have a fundamental ability to deal with conditional probabilities. In addition, based on the results of this first study, the role of causal bias played a minor role.

Their second investigation explored misconceptions about the notation P(B|A) and possible hindrances naïve students might have calculating these probabilities. The two main hindrances explored in this investigation were the possibility that the wording of the problem may affect their judgement and the confusion between conjunction - P(A and B) - and conditional probability - P(A|B). For this investigation, 120 students were recruited from various sections of an introductory psychology course designed for psychology majors and received course credit for participation. These students were also considered naïve in probability since they had not taken a college level statistics course.

The data collected for the second investigation consisted of the results of four questionnaires. The first set of two questionnaires consisted of seven forced-choice questions judging whether the conditional event, $P(A|B)$, was greater, less than, or equal to the other conditional event, $P(B|A)$. The first set of questionnaires was presented to the students in two different formats, but with the same seven events. Half the students answered questions posed in a probability format, while the other half answered questions posed in a percent format, given with written explanations. The distinction between questions posed in the probability and percent questions was to see if the wording caused the student difficulty in understanding of the problem. The second set of questionnaires, administered to a second group of students, consisted of the same seven questions which the first group of students answered. However, the second group of students was asked to estimate the conditional probability in percents with justifications for their answers. To discourage the students from answering hastily, they were given fixed amounts of time to complete each section of the questionnaire. The seven estimation questions were used to judge the students' confusion between conjunction and conditional probability questions.

Data analysis consisted of the average percent of correct responses to each of the seven questions from each of the two groups. The data analysis consisted of visually comparing the percents for each of the seven questions and supporting their findings with a Chi-Square Analysis. The Chi-Square Analysis considered the two dimensions of wording - probability format and percent format - and the three responses for each question to see if there was a relationship between the variables of wording and student responses. The results from the forced-choice questionnaire set suggested that the performance of judging conditional probability varied widely across the problems - from 30% to 80% correct. Student performance was similar on the probability format of the test and the percent format, with an average of 57.0% and 56.7% correct respectively ($\chi^2(2)>6$; $p < .05$ for all seven questions). These results suggested that no difference was found between student responses with respect to the wording of the questions.

The results from the estimation questionnaire set indicated an 80% agreement between the forced choice responses and the estimation responses, with slightly better performance on the estimation section. Finally, the estimation data were analyzed for their "reasonableness." For this study, reasonableness was defined as answers that met certain criteria for each of the seven questions. The critical scores used to determine the reasonableness of a problem were stated in the study. Based on the critical scores, it was not evident that students reversed the two conditions, i. e. interpreted $P(A|B)$ as $P(B|A)$ or visa versa. With the possible exception of one problem, there was little indication based on the students estimations that the students reversed $P(A|B)$ and $P(B|A)$.

The last data analysis on the second set of questionnaires consisted of the students' justification of their answers. Claiming the students only gave brief justifications, in the form of a phrase or short sentence, the researchers claimed the written data were less useful than anticipated. However, some patterns were identified based on the brief responses. In addition, the majority of the students who had correct answers provided correct justifications.

Overall, the study suggested that naïve students were able to correctly solve conditional probability problems. However, when solving conditional probability problems, certain factors such as wording and unfamiliarity with the problem interfered with their problem solving ability. In addition, the study also found no difference in students' problem solving ability when asked to solve a problem asking for the results stated as a percent or probability. However, there may be confusion translating conjunction and conditional probability statements. Finally, the study did not find evidence that the use of the causal bias judgmental heuristic was as powerful and persuasive as previous research had claimed.

Intuitively Based Misconceptions

Fischbein and Schnarch (1997) investigated probabilistic misconceptions across various ages of students. Their interest in studying the misconceptions of probability across various ages stemmed from a previous study they conducted on the notion of infinity. In this previous study, Fischbein, Tirosh, and Hess (1979) found that various intuitively based misconceptions related to the notion of infinity were stable across student ages. This study was the first stage of a larger study on the evolution of probabilistic misconceptions. Therefore, the secondary purpose of this study was to gather preliminary empirical data, allowing the researchers to obtain a global picture of the evolution of probabilistic misconceptions as an effect of age. Unfortunately, the results from the larger study have not been published yet. Hence, the purpose of this study was to investigate the possibility that intuitively based misconceptions remained the same across various ages of students.

For this study, Fischbein and Schnarch (1997) defined intuition as "a cognition that appears *subjectively* as self-evident, directly acceptable, holistic, coercive, and extrapolative" (p. 96). Intuitive cognition was differentiated from an analytically and logically based non-intuitive cognition by intuitive cognition producing a feeling of obviousness and of intrinsic certainty when the student solved a problem.

The sample for the study consisted of five groups of Israeli students without previous instruction on probability: 20 fifth grade students, 20 seventh grade students, 20 ninth grade students, 20 eleventh grade students, and 19 college students. The researchers claimed the sample represented a range of students with respect to socioeconomic level and cultural background.

The data collecting instrument consisted of a seven multiple choice problem questionnaire. Each problem represented one of seven different judgmental heuristics: representativeness, negative and positive recency effects, simple and compound events, conjunction fallacy, effects of sample size, availability and the time-axis fallacy. Each question consisted of a description of an event with three forced-choice possible answers: the correct response, the common incorrect misconception response, and a distracter. The questionnaire was administered to the students during a regularly scheduled class, allowing them one hour to complete the questionnaire.

Data analysis consisted of finding the average score for each of the five groups for each question. An average score for each question was compiled by computing the percent of students in each group who chose each of the three responses. The analysis of the results consisted of visually comparing the average percents for each misconception, across all age levels.

By comparing the average percents of students correctly answering the questions with the main misconception, the study suggested that the misconceptions of representativeness (from 75% in the youngest group to 22% in the oldest group), negative recency effect (from 35% in the youngest group to 0% in the oldest group), and the conjunction fallacy (from 85% in the youngest group to 44% in the oldest group) decreased with age. The results also suggested that availability (from 10% in the youngest group to 72% in the oldest group) and the effect of the time-axis (from 5% in the youngest group to 44% in the oldest group) increased with age; and positive recency effects (from 0% in the youngest group to 6% in the oldest group), and compound and simple events (from 70% in the youngest group to 78% in the oldest group) remained stable with age. However, the question pertaining to the misconception of the effect of sample size remained a strong misconception across the ages, with only one student from the entire sample answering the question correctly.

Overall, the results from this study suggested that while some probabilistic misconceptions decreased with age, others either increased or remained the same.

Relationships Among Different Types of Errors

Recognizing the existence of cognitive biases in student's reasoning about probability and probabilistic events and the existence of conceptual difficulties learning elementary probability theory, O'Connell (1999) looked specifically at what other types of errors students made solving probability problems. Specifically, O'Connell investigated the relationship between different types of errors which occurred solving probability problems typically found in graduate level introductory statistics textbooks aimed at students in social or behavioral science. This small study was part of a larger study on probability problem solving, not yet published.

The only information known about the sample for the study was that it consisted of 50 education and graduate students, enrolled in a one-semester course in probability and statistics at a large urban university.

At the completion of the probability section of the course, the students received 12 probability problems to complete over the course of one week. While solving the 12 problems, the students were asked to show all the work done to solve the problems. The solutions and explanations on the 12 problems became the data collected for the study. The 12 problems encompassed the five main topics covered in the course: equally likely, non-equally likely, mutually exclusive, independent and conditional events, and assessing their associated probabilities. Several of the problems contained sequential questions, amounting to 50 individual items over all 12 problems.

The data analysis consisted of inspecting each of the 50 items for errors and assigning a code based on the error made on each question, according to the Error Coding

Scheme discussed below. Once each incorrect problem was coded, the categories were tabulated and analyzed using an Additive Tree, discussed below.

The development of the Error Coding Scheme consisted of an analysis of the written work of 180 students from another class solving 93 different probability problems. The sample from the present study did not participate in the data collection for developing the Error Coding Scheme. The first stage of development consisted of creating broad categories that corresponded to the text comprehension errors, conceptual errors, procedural errors, and arithmetic errors. Once the four general categories were identified, each error was then further classified into a sub-category. At the end, there were a total of nine specific text comprehension errors, 18 specific conceptual errors, 71 specific procedural errors, and 12 specific arithmetic errors. A fifth category was used to identify an error that could not be clearly determined under the general or specific categories. Finally, the assignment of codes to the identified errors was arranged hierarchically, according to the level of conceptual error, to facilitate classifications of similar specific errors. Overall, 30 higher-order levels or types of errors were determined through the analysis. This allowed for classification of eight types of text comprehension errors, 11 types of conceptual errors, 10 types of procedural errors, and all arithmetic errors were considered one type. For further clarification of the Error Coding Scheme process, the study did provide some examples of student's work and the assignment of error codes.

The assessment of the Error Coding Scheme's reliability consisted of two different methods. First, using a subset of the work by 30 students not participating in the study, two independent judges were asked to assess each instance of an error as either text comprehension, procedural, conceptual, or arithmetic. Inter-rater assessments produced the following percent agreements and Cohen's kappa estimates for identification of errors from each of the four categories: text comprehension (84%, kappa = .61, $p < .01$), conceptual (93%, kappa = 0.83, $p < .01$); procedural (82%, kappa = 0.82, $p< .01$) and arithmetic (89%, kappa = 0.68, $p < .01$). In addition to reliability assessments for the four broad categories, inter-rater agreement was also used to investigate reliability of classification of errors into one of the 30 types. The researcher and a third rater used the coding scheme to identify errors occurring in the work of 14 additional students solving four probability problems in which 22 of the solutions contained errors. The analysis of this data produced an 84% inter-rater agreement.

The study did indicate some limitations of the Error Coding Scheme. It was possible that one problem may consist of various errors belonging in more than one category. Another problem may be that an error detected in the beginning of the problem may have an affect on an error produced later in the problem. For consistency, error analysis guidelines were developed for the classification.

In order to investigate possible relationships among observed errors, frequency scores corresponding to each error type were calculated for all 50 students participating in the study. Using this technique, it was possible to distinguish the tendency for a student to make a particular type of error. Three of the 50 students made no errors on any of the 12 problems. Thus the relationships among different kinds of errors was assessed using the frequencies of error types for the remaining 47 students.

After assigning an error code, hierarchical clustering was used to help identify a natural structure to the set of text comprehension, conceptual, procedural and arithmetic errors. Using the Pascal program ADDTREE/P, based on Sattath and Tversky's (1997) algorithm for fitting Additive Trees, it was possible to investigate the hierarchical cluster of the proximity matrix. The measure of proximity consisted of the regular Pearson correlation between all pairs of variables. The data analysis provided two different relational observations: conceptual/procedural relationships and relationships among errors.

The results of the data analysis were reported in two areas: error analysis and hierarchical clustering. The error analysis data consisted of stating the frequencies of each error type. The results indicated that the most common errors consisted of procedural errors (total of 271), followed by errors in text comprehension (total of 138) and concepts (total of 110). Relatively speaking, there were few arithmetic errors and unclassified errors, total of 54 and 34 errors respectively. Some procedural errors consisted of the students forgetting the outcomes when defining a sample space, not checking preconditions of a formula, using incorrect versions of a formula, substituting wrong values, and not completing a strategy. The text comprehension errors consisted of assigning probability values to the wrong event, incorrectly identifying the goal, and misinterpreting "at least" and "at most." The concept errors consisted of reporting negative probability; assuming events were equally likely; equating frequency with probability; and misinterpreting independent, mutually exclusive, and complementary events.

The hierarchical clustering used the Additive Tree for comparison of the two relational observations: conceptual / procedural relationships and relationships among all errors. The conceptual / procedural relationship consisted of the correlations between the ten types of procedural errors and the six types of conceptual errors which had an overall frequency greater than or equal to four. The cluster analysis revealed a correlation between actual and estimated proximities for these data as 0.84, accounting for nearly 70% of the variance ($R^2 = 0.6988$). The Additive Tree identified three main clusters. The first cluster suggested that conceptual difficulties in working with the formal language of probability was related to the misconceptions and procedures involving mutually exclusive events, in which formal language was the most prominent in the relationship. The second cluster suggested several combinations of conceptual and procedural errors. The eight items in the cluster indicated a very general relationship between misconceptions of independence; conceptual difficulty in distinguishing between independent and mutually exclusive events; and procedural difficulties in solving probability problems which require some knowledge of independent versus non-independent events. The third cluster suggested there was difficulty in working with formulas in general. The errors in this cluster suggested a relationship between unfinished solution attempts, inventing procedures or rules to "fit" one's understanding of a problem, and difficulty working with formulas for complementary events.

The relationship among errors consisted of the correlations between the five text comprehension errors, the ten procedural and six conceptual error types, and one class representing errors in arithmetic. The cluster analysis revealed a correlation between actual and estimated proximities for these data as 0.79, accounting for nearly 62% of the variance ($R^2 = 0.6169$). The Additive Tree identified six main clusters. The first cluster suggested a relationship between combining unfinished solution attempts with misconceptions concerning the validity of a probability value greater than one. The second cluster suggested a general relationship between five error types. Accuracy of identifying the goals of a problem and the ability to work with inequalities were the two predominant errors related to misconceptions about the formal language of probability and mutually exclusive events, and procedural errors concerning mutually exclusive events. The third cluster suggested a relationship between the difficulty of correctly representing a situation as described in the text of a problem with three procedural errors: identifying independence, analyzing data in tabular form, and working with sequential experiments. The fourth cluster suggested a relationship between the predominant category that information given in a previous problem may be a factor in the tendency to invent procedures or rules and the tendency to assume that events were equally likely. The fifth cluster suggested a relationship between difficulty in assigning a given probability value to the correct event as given in the text with those problems involving conditional probability and errors involving complementary probability. The final cluster suggested a general relationship between arithmetic errors with procedural errors in the general use of formulas and difficulty distinguishing conceptually between independent and mutually exclusive events.

Overall, the researcher reached her goal to seek clarification of the nature of relationships occurring among different kinds of errors in order to provide some guidelines for improving instruction and learning in probability classrooms. The results of this study suggested several relationships among conceptual and procedural errors; associations between conceptual and procedural errors regarding mutually exclusive events; associations between conceptual and procedural errors regarding independent events and related formulas; and procedural difficulty when working with formulas in general. In addition, the results from the cluster analysis suggested test comprehension difficulties were associated with conceptual and procedural errors during problem solving.

Influence of Instruction on College Students

The two studies in this section explored the factor of instruction influencing college students ability to solve probability problems. In the first study, Austin (1974) investigated the effectiveness of various types of manipulatives in the teaching of probability and statistics to university-level students. The second study by Shaughnessy (1977) evaluated the effects of an experimental instructional method of teaching probability in an elementary probability and statistics class on students' use of judgmental heuristics.

After investigating previous studies on the use of manipulatives in the mathematics classroom, Austin (1974) noted a discrepancy in their effectiveness in the higher-level mathematics classroom. Hence, Austin (1974) conducted an experimental investigation of the effectiveness of various types of manipulatives in the teaching of probability and statistics to university-level students. Therefore, the purpose of the study was to examine the effects of various instructional methods using manipulatives on college student probabilistic thinking.

The study consisted of 80 non-mathematics and science students at Purdue University enrolled in two different sections of a sophomore level probability course. With an attrition rate of 71 students, the final population of this study consisted of 19 freshmen, 13 sophomores, 12 juniors, and nine seniors. The students were business, humanities, agriculture, or undeclared majors. Only 40 students had no previous probability instruction, 25 students had high school experience, six students had college experience, and five students had less than three weeks of probability coursework. Before the experiment began, the students were ranked based on their previous mathematics grades for each section. The first three students were randomly assigned individually to one of the three treatment groups. This procedure continued until all students were assigned to a treatment group; thus, each section had three treatment groups for a total of six experimental groups.

The three treatment groups for each section met in separate tape laboratories during the regular class hours, instead of attending a lecture. The students did not have any contact with the instructor, and the laboratory assistants had no knowledge of the purpose of the study. Neither the students nor the assistants were told they were involved in an experiment. Instead, they were told they were part of a study on the feasibility of video taped instruction. During each class meeting, every student received a written lesson and a tape of the lecture. Thus, students could listen to the lecture at their own pace. Daily homework was assigned and returned, graded by the instructor.

Each of the two sections was divided into the following three treatment groups: Manipulative-Pictorial, Pictorial, and Symbolic. Each of the three groups had the same written lessons with the same objectives; however, the treatment differed in their lecture portion. The Manipulative-Pictorial groups performed experiments on random processes found in discrete probability. The students conducted random experiments using coins, dice, random number tables, and marble selection devices. The students also used graphs, diagrams, and figures to motivate their learning. The Pictorial groups were similar to the Manipulative-Pictorial groups; however, the Pictorial groups did not perform the experiments. Instead, the Pictorial groups used data generated by the instructor. The students did not see the experiments or the use of the physical objects being used. The pictorial aspect of the instruction was not changed from the Manipulative-Pictorial mode of instruction. The Symbolic groups were similar to the Pictorial groups; however, the Symbolic groups did not have any pictorial aids. The written material was altered and only words and mathematical symbols were used. Otherwise, the behavioral objectives and problems were identical in the Manipulative-Pictorial, Pictorial, and Symbolic groups.

The experiment took place the first four weeks of the term with the class meeting three times a week. Each taped lecture was approximately 30 minutes long. At the end of the experiment, an examination covering the twelve lessons was given during an evening meeting, so that all students took the same examination at the same time. Of the original 80 students, only 71 completed the experiment.

The data for the experiment were collected through the students' previous mathematics grades and the final examination for the course. The final exam consisted of 40 questions, stratified into cognitive levels, based on the taxonomy used in the National Longitudinal Study of Mathematical Abilities: comprehension, computation, application, and analysis. The percents of items on the exam approximated the percents of the behavioral objectives used in the lessons. The results of the exam were broken into five dependent variables: comprehension score, computation score, application score, analysis score and total score. To check for content validity, the exam items were based on the lesson behavioral objectives. To estimate test-retest reliability, the coefficient alpha reliability was computed for the five dependent variables for each treatment group. The alpha values were between .59 and .93. The study claimed that a coefficient alpha reliability below a .60 was an indication of a test reliability problem; therefore, the study claimed, that since the test had only one score of .59, the test was considered to be reliable enough to draw valid inferences from the data.

The analysis and results for the test were stated in two categories: students' previous mathematics grades and the five dependent variables collected from the final exam. Using the student's previous mathematics grades, a two-way analysis of variance was conducted that compared the two factors of class (2 levels) and treatment (3 levels). With $\alpha = .05$, analysis of variance indicates there were no statistically significant differences between the two sections and the three treatment groups in their previous mathematics grades ($p > .24$ for the three comparisons).

The results from the final exam were also computed using the two-way analysis of variance on each of the five dependent variables. Homogenity of cell variance for each variable was tested using Bartlett's χ^2 method. With $\alpha = .05$, only the computational sub-score test rejected the homogenity hypothesis ($p < .05$ for computational, $p > .05$ for the other four variables). The rejection of the homogenity hypothesis indicates the assumption of equal population variance was not met; thus, the data did not meet the assumptions of a two-way ANOVA (Huck & Cormier, 1996).

When the hypothesis of no difference in the examination score means among the three treatment groups was rejected for a particular variable, Scheffé's method was used to make the pairwise comparisons of treatment means. For the total examination scores, equality of treatment means was rejected. The class and interaction effects were pooled, and the Scheffé's test conducted. The results from the Scheffé's test indicated the Symbolic treatment mean was less than the Manipulative-Pictorial and the Pictorial means. However, the hypothesis of equality of the Manipulative-Pictorial mean and the Pictorial mean was not rejected. Thus the ordering indicated by Scheffé's method indicated the Symbolic mean was less than the Pictorial mean, the Symbolic mean was less than the Manipulative-Pictorial mean, and the Pictorial mean was equal to the Manipulative-Pictorial mean. The Scheffé's method also found the same ordering for the application, analysis, and examination scores.

For the comprehension sub-score, the two-way analysis permitted the pooling of class and interaction effects. The resulting analysis rejected the equality of treatment means. Scheffé's test indicated that the Symbolic treatment mean was less than the

Pictorial treatment mean. However, neither the hypothesis that the Manipulative-Pictorial means was equal to the Symbolic mean nor the hypothesis that the Manipulative-Pictorial means was equal to the Pictorial mean was rejected. Thus the order was the Symbolic mean was less than the Pictorial mean, the Symbolic mean was less than the Manipulative-Pictorial mean, and the Pictorial mean was equal to the Manipulative-Pictorial mean. For the computation sub-score, homogenity of cell variance was rejected with $\chi^2_{obs}(5) = 16.5$. The two-way analysis was not made. Rather, two one-way analyses were made on the scores from each of the two classes. With the separate classes, homogenity of variance was not rejected in either. Neither of the two one-way analyses rejected the equality of treatment means ($p > .05$, pairwise analysis of treatment averages).

The researchers did recognize one limitation to this study: the use of taped lectures. The students had minimal, if any, interaction with the instructor, had no influence on the teaching methods, and did not have the human contact found in most traditional style classrooms. The methods used for this experiment were similar to laboratory experiments.

Overall, the results from the analysis suggested four trends. First, no difference was found in student computational achievement among the three instructional methods. Second, the use of graphs, figures, and diagrams suggested an improvement in student's application and analysis, and total examination scores. Third, if graphs, figures, and diagrams were used, then students' application, analysis, and total scores did not suggest any significant difference between the Manipulative-Pictorial students and the Pictorial students. Finally, if students did use graphs, figures and diagrams, the comprehension score suggested that students who generated the outcome of the experiments did not perform as well as those who were only told the outcome of the experiments.

Shaughnessy (1977) conducted a study on an experimental model of mathematics instruction in elementary probability and statistics and how the instruction maximized the student's chances of overcoming their misconceptions of probability and statistics. By using small-group, activity based, model building approaches to teaching elementary probability and statistics to undergraduates, Shaughnessy conducted an instructional experiment on students' misconceptions about probability and the reduction of reliance upon availability and representativeness judgmental heuristics. Therefore, the purpose of the study was to present an experimental model of mathematics instruction in elementary probability and statistics that would maximize the student's chances of overcoming representativeness and availability judgmental heuristics.

The population for this study consisted of 80 college undergraduate students enrolled in four sections of a finite mathematics course during the Spring 1976 term at Michigan State University. The four sections were randomly selected for this experiment from seven sections being offered that term. The remaining three sections were defined to be the control group. It was unclear from the study, and the summary of results, how many students were enrolled in either the experimental or the control groups. The demographics of the experimental population consisted of primarily freshmen business and accounting majors. The prerequisite for the course was one term of college algebra, which all of the subjects had successfully completed. In addition, 51 of the students claimed they had taken at least one of two secondary-school level remedial algebra courses prior to enrolling in the college algebra. Only seven of the students indicated that they had any previous work in probability. Hence, the students in the study did not have a strong mathematical background.

The experimental groups participated in an activity-based course constructed as an alternative to the lecture method. The experimental group participated in a series of nine researcher-designed activities covering five probability concepts: probability, combinatorics, game theory, expected value, and elementary statistics. The nine activities were carried out in small groups of four or five students, with the students rotating among groups for each activity. The study did provide a detailed description of each of the nine activities. Each activity required the groups to perform experiments, gather data, organize and analyze the data, and reach a conclusion that could be stated in the form of a mathematical principle or model. The role of the instructor during these activities was similar to a facilitator, clarifying students' questions, assisting groups stalled on a particular problem, or answering a question with a question. However, the instructor did not provide the students with answers.

The control group participated in a traditional lecture. The experiment took place over the course of four weeks in which the experimental and control group had similar course content, but the order of the topics was different. The study did mention that the lessons conducted for the experimental group were derived from three textbooks, while the control group used the traditional finite mathematics book assigned to the course. Finally, the study did note that the researcher taught the four experimental classes.

The data for this study were collected in three areas: researcher observations, pre-test, and post-test. The researcher observations were mainly collected to record student-student interactions, instructor-student interactions, and unique occurrences noted by the researcher, in only the experimental groups. The notes taken by the researcher were not used for data analysis, but instead, to augment the study and to provide insight to the reader about the activities and interactions that occurred in the experimental classroom. The 20-question forced-choice pre- and post-tests used for the study were developed based on the questions used by Kahneman and Tversky in their research on judgmental heuristics. The tests measured the students' knowledge of some probability concepts and for their reliance upon the judgmental heuristics of representativeness and availability in estimating the likelihood of events. Besides being a forced-choice test, the test also asked the student to supply a reason for each response. The two tests were identical and were given to both the experimental and control groups.

The data analysis consisted of compiling the pre- and post-test results of the experimental and control groups. To check for possible change in students' use of representativeness and availability, some specific questions on the test were analyzed separately. The study did not indicate which statistical analysis was conducted to reach the conclusions, only the p-values. The analysis suggested that the experimental group was more successful at overcoming reliance upon representativeness ($p < .05$, d. f. = 2), and tend to be more successful at overcoming reliance upon availability ($p < .19$, d. f. = 2). This last claim made by the researcher did not appear to have strong statistical support. The study did provide further analysis of the data by providing the raw data of the students responses to specific questions. However, the researcher did not use this added information in the results, but rather claimed the purpose of the extra analysis was to provide the reader with more insight to the students' responses.

Overall, the results of this study supported the hypothesis of Kahneman and Tversky, which claimed that combinatorially naïve college students rely upon availability and representativeness to estimate the likelihood of events. The results on the post-test suggested that the manner in which college student learned probability may have made a difference in their ability to overcome misconceptions that arise from availability and representativeness. This experiment suggested that the course methodology and teaching model used in an elementary probability course might help develop students' intuition for probabilistic thinking.

Research on the Methods Used and the Factors that Supported or Impeded College Students' Success for Solving Probability Problems: Concluding Remarks

The previous seven studies identified the methods used and the factors that supported or impeded college students' success as they solved probability problems covered a wide range of probabilistic concepts. From simple probability problems, to compound and conditional probability problems, each study sought to either confirm previously defined judgmental heuristics, understand the relationship among other types of errors, or confirm the influence of instruction on students ability to solve probability problems. Unfortunately, the majority of the studies lacked validity and reliability of their testing instruments, lacked clear descriptions of the instructional methods used in the classroom, lacked a description of the mathematical and probabilistic background of the observed students, and the descriptive studies depended on the results of their analytical analysis which takes away the credibility of the study. However, each study did suggest that across the ages, and especially at the college level, students did encounter a wide array of difficulties solving probability problems. In addition, instruction and the type of instructional methods could influence secondary and college level students' probabilistic thinking. With the lack of strong studies on the methods used and the factors that supported or impeded college students' success as they solve probability problems, the results only provided a few suggestions as to the effects of various instructional methods.

Summary

The purpose of the literature review for this study was to provide a summary of the research conducted on the methods used and the factors that supported or impeded college students' success as they solve probability problems. In order to gain knowledge about the research conducted in this area, this chapter summarized the history and nature of probability, the theoretical frameworks of subjective probability defined by cognitive psychologists, and empirical studies appropriate for the purpose of this study. The various developments and interpretations of probability may have important implications on the development of probabilistic ideas in the classroom and conducting the research on the learning and teaching of probability (Barnett, 1999; Hacking, 1975). In addition, the contributions by cognitive psychologists on their attempt to observe and describe peoples decision making abilities when confronted with probabilistic situations helped identify and define probabilistic judgmental heuristics (Cohen, 1957; Falk, 1988, 1989; Kahneman & Tversky, 1973, 1972; Tversky & Kahneman, 1973, 1974, 1980, 1982). These results from cognitive psychologists formed a framework for identifying factors which supported or impeded college students ability to solve probability problems; however, these studies did not investigate factors which might help adults overcome their difficulties. With the blending of these results, educators were able to conduct studies on how students might be able to overcome some of their difficulties through instruction. Therefore, the purpose of the final section of this chapter was to provide an overall summary of the research results conducted on methods used and the factors that supported or impeded college students' success as they solve probability problems.

The reviewed research contained one study specifically observing the methods college students used to interpret probability problems. Konold (1989) observed in his study that college students would approach a problem in one of two different methods: frequentist approach and outcome approach. The frequentist approach consisted of students interpreting probabilistic situations as the outcomes of repeatable situations with frequencies. The outcome approach consisted of students interpreting the probabilistic situation in a nonstandard method, in which the student exhibited a lack of probabilistic theory to solve the problem, yet solved the problem in a coherent manner. In addition, O'Connell (1999) was not directly observing the methods college students use to solve probability problems, but did observe in her study that students used incorrect problem solving approaches to obtain a correct answer. Since the goal of her study was to investigate the relationships among different types of error, O'Connell did not elaborate on the incorrect methods the students did use to solve the problems.

The reviewed research revealed modest results with regards to the factors which supported or impeded college mathematics students success for solving probability problems. This section consisted of three parts: factors that supported, factors that impeded, and evidence of judgmental heuristics in college students ability to solve probability problems.

Unfortunately, the majority of the studies did not focus on factors that supported college students success on solving probability problems. Only one study provided insight to factors that supported their success. Pollatsek et al. (1987) suggested that statements dealing with conditional probability did not confuse naïve college students. Based on implication problems and real-world knowledge problems, naïve college students grasped the conditional probability concept and provided the correct answer. In addition, Pollatsek et al. suggested college students did not encounter difficulties solving probability problems when either stated as a percent or as a probability. Overall, the results of this study suggested that naïve college students had the ability to interpret conditional probability problems and work with percents in probability problems.

The reviewed research revealed some factors that impeded college students ability to solve probability problems. Pollatsek et al. (1987) found that college level students invented their own probabilistic language. In addition, college level students exhibited difficulty with problems if they did not understand the syntax of the question. In particular, students confused conjunction and conditional probability statements. Konold et al. (1993) observed additional difficulties with college level students. They noticed that even if a student was capable of recognizing that two events were equally likely, this student might not assign equal probabilistic values. Similarly, they found that if a student recognized two events as not equally likely, the student might assign equal probabilistic values. Hence, Konold et al. suggested that despite student' correct answer of a probability question, they might not have used the correct problem solving approach. In another study, while observing the methods college students used, Konold (1989) suggested another factor that might impede college students success was students' difficulty interpreting probabilistic events. More specifically, when the students were confronted with an uncertain event, most students would use the formal, probabilistic interpretation when reasoning about situations that were clearly probabilistic and had a simple sample space. However, for situations that were less obviously probabilistic, these students based their decisions of a probabilistic event by predicting outcomes of single trials, interpreting probability as a prediction, and basing probability estimates on causal features rather than on information pertaining to the distribution of the sample space. O'Connell (1999) conducted an exploratory study on the relationship among different types of errors that occurred during college students' probability problem solving. Within 110 specific errors, O'Connell described relationships between certain conceptual, procedural, and arithmetic errors. More specifically, O'Connell suggested the most common errors overall were procedural in nature, followed by errors in text comprehension. Some procedural errors consisted of students forgetting the outcomes when defining a sample space, not checking preconditions of a formula, using incorrect versions of a formula, substituting wrong values, and not completing a strategy. The text comprehension errors consisted of assigning probability values to the wrong event, incorrectly identifying the goal, and misinterpreting "at least" and "at most." Overall, the results from these studies suggested factors that impede college students ability to solve probability problems include misunderstanding the syntax of the problem, unfamiliarity with the problem stated, incorrect interpretation and reasoning of the problem, and inventing personalized probabilistic language.

Some of the reviewed literature explored the possibility of students drawing on judgmental heuristics. Fischbein and Schnarch (1997) suggested that the use of judgmental heuristics could be found in secondary and collegiate level students. They further concluded that the use of some judgmental heuristics changed with the age level of the students. More specifically, Fischbein and Schnarch found that across student ages, there was a decrease in the use of representative, negative recency, and conjunction fallacy judgmental heuristics and an increase in use of availability and the time axis judgmental heuristics. Fischbein and Schnarch also found that the positive recency judgmental heuristic was not evident in any of the students. Konold et al. (1993) was not able to find evidence of the use of the Gambler's Fallacy or the representative judgmental heuristic in secondary and collegiate level students, while Pollatsek et al. (1987) found no evidence of causal bias in collegiate level students. Overall, these studies suggested that the previously defined judgmental heuristics either might not be evident in collegiate students, or that they were evident, but their use might change with age.

The reviewed research considered the influence of instruction on college students' ability to solve probability problems. Two studies explored the influence of instruction on college level students ability to solve probability problems. Austin (1974) explored three college classrooms with video taped lectures employing three types of instructional strategies: Manipulative-Pictorial, Pictorial, and Symbolic. Using this type of instructional method in the classroom, the researcher observed which of the three experimental methods improved students' probability computational achievement, probability application, probability analysis, and overall final exam score. The results from this study suggested that those college students who used graphs, figures and diagrams in their classroom scored higher on application, analysis, and exam scores than those who did not have access to graphs, figures, and diagrams. The college study also found no difference on the application, analysis, and exam scores between those who conducted experiments and generated data, from those who just looked at the data. Finally, the three types of instructional strategies did not have an effect on college student computational achievement.

In addition, Shaughnessy (1977) suggested that small group, activity based model building approach helped college students overcome their reliance upon the representative judgmental heuristic better than those in a lecture class. This study also found inconclusive evidence that either the experimental class or the lecture class helped students overcome reliance upon the availability judgmental heuristic. Overall, the results from these two studies suggested that the use of graphs, figures, and diagrams could improve college students probabilistic thinking and problem solving ability, while a small group, activity based model building approach classroom might help college students overcome the representative judgmental heuristic.

With only two studies exploring the influence of instruction on college level students ability to solve probability problems, clear patterns did not emerge. Instead, the results of these studies did indicate that instruction did have an influence on certain aspects of student probabilistic thinking.

CHAPTER III

DESIGN AND METHOD

Introduction

Based on the current trends and directions of the research on the teaching and learning of probability, it is necessary to shift the direction from observing and verifying predefined factors towards observing specific populations of students, search for their unique methods of solving probability problems, and note the factors which support or impede their success. All college level students, including non-mathematics and science majors, enroll in courses integrating probabilistic concepts into their curriculum. However, non-mathematics and science majors are not required to develop as strong a mathematical background as their mathematics and science major peers. Instead, these non-mathematics and science majors enrolled in a foundational finite mathematics course designed to fit the mathematical needs of their degree requirements, includes an introduction to probability theory. An extension of the research on the learning and teaching of probability as it relates to finite mathematics students provided the opportunity to re-examine and explore factors affecting this student population as it learned about probability theory. Therefore, this study examined the methods used and the factors that supported or impeded college finite mathematics students' success as they solve probability problems. In particular, this study sought the following two research questions:

1. What methods do college finite mathematics students use to solve probability problems?
2. What are factors that support or impede college finite mathematics students' success for solving probability problems?

This chapter presents the methodology used to conduct this study of the students enrolled in a college finite mathematics course. This study used a case study analysis of the verbal data collected through two Task-Based Questionnaires conducted prior to and after the students participated in two-week instruction on probability.

The Participants

The participants for the present investigation consisted of nine volunteers from a class of 24 students and one instructor of a 10-week finite mathematics course offered at a local community college during the Winter 2002 Term. The finite mathematics course, established as a terminal mathematics course to satisfy the baccalaureate core requirement for students pursuing a business degree, paralleled the sequence of topics present in *Finite Mathematics: For Business, Economics, Life Sciences, and Social Sciences* (Barnett, Ziegler, & Byleen, 2002). Prior to enrolling in the finite mathematics course, the students had to successfully complete a college algebra course. As part of their degree requirements, the majority of the students would also enroll in a business calculus course, which was not a prerequisite for the finite mathematics course. The material covered in the finite mathematics course was designed to give the students a fundamental background in the techniques of counting, probability and statistics, an introduction to matrix operations, and elements of linear programming. The section on probability included topics on basic counting techniques, combinatorics, sample space, simple and compound events, conditional probability, independent events, and Baye's Rule. The topics covered in the finite mathematics course were not intended to provide the student with the mathematical background needed if they intended to pursue additional college level mathematics courses, but rather provide the students with the mathematical background needed for their careers and private lives.

The Students

Nine students from the 24 finite mathematics students volunteered to participate in the interviews. With cooperation of the instructor, the researcher introduced herself to the class at the beginning of the term. During the introduction, the researcher presented a short description of the study, described the data she planned to collect, and asked for student participation. The students were presented two ways to participate: providing background information about themselves or participating in two structured interviews. After introductions questions regarding the study and their possible participation, the researcher passed out the two Informed Consent Forms to gather the background information and to gather contact information for those students interested in volunteering for the interviews (see Appendix B). The Informed Consent Forms were then collected, providing the researcher with background information for the 24 students in the class and contact information for the seven males and two females interested in participating in the two interviews.

The Instructor

The instructor for the course brought into the class a strong background in college mathematics teaching and administration. Earning a Bachelors of Science Degree in Mathematics in 1959, he originally planned to teach high school level mathematics and coach baseball. However, the launching of Sputnik in 1963 and the National Defense Education Act changed his career objectives. Accepted into a yearlong institute at a university for high school and science teachers, he found himself enjoying higher level mathematics and decided to stay and pursue a doctorate degree in abstract algebra. Working on his dissertation, he discovered the original research was flawed and had to find a new problem. During the same year, 1968, two local counties recently had built a community college and were searching for a full-time mathematics instructor. He applied for the position and became the first full-time mathematics instructor, evolving into becoming the director for the Math Science Division the following year. Enjoying teaching part-time with his administrative duties, he did miss teaching higher level mathematics. However, he decided to leave his teaching career in 1972 for a position as the Director for Learning Services at the same community college. In 1977, he changed offices and became the Director for Development. This new position consisted of developing and writing grants and contracts for the college. However, in 1984 he decided to return to the classroom, accepting a part-time position as a mathematics and business statistics instructor. By 1988, he became a full-time instructor, until his retirement in 1993, returning as a part-time instructor. Along with his college mathematics teaching and administration background, he also co-authored a contemporary college level mathematics book, covering similar concepts found in a finite mathematics course.

As an undergraduate student, he did not recall enrolling in any courses requiring the study of probability. His first exposure to theoretical probability was while teaching high school early in his career. Reading an article in *Scientific America* in the early 1960s, he learned about a self-learning kit on probability. So he bought the kit and studied probability on his own. Enjoying the activities and applications in the kit, he adapted the self-learning kit to a classroom setting and first tried teaching probability to advanced-level high school freshmen. Returning to college the following year, he enrolled in his first formal college level course in probability. He claimed his first time teaching probability at the college level was not until 1985, when he was teaching business statistics. Since then, he has only taught probability while teaching his finite or contemporary mathematics students.

The Researcher

Due to the qualitative nature of the study, the researcher is the primary instrument for gathering and analyzing data. This method of data collection can bring subjectivity into the data analysis since the data were filtered through her particular theoretical position and biases. Since the data was collected and analyzed from the perspective of the researcher, it is valuable to explain the perspective of the researcher. The researcher's first encounter with formal probability was in high school. Interested in mathematics, she enrolled in a probability and statistics course. Continuing her interest in mathematics, she pursued a Bachelors of Science Degree in Mathematics and obtained a secondary teaching certificate. In her course of studies, the researcher enrolled in a junior level probability and statistics course. However, the emphasis was more on statistics than theoretical probability. After earning her bachelors degree, she taught two years of high school mathematics. Introduced to adult education at an Army education center, she changed her career goals to working with adults. For the next four years, the researcher helped develop and taught adult basic mathematics education courses for Army soldiers. The researcher did not have the opportunity to teach probability during those six years.

During the years working with adult basic education, the researcher continued her own education, completing a Masters of Education Degree in Adult and Higher Education. Due to military cutbacks, the adult basic education courses were cancelled. The researcher decided to continue her education by pursuing a doctorate degree in mathematics education. While working on the doctoral program, the researcher worked concurrently on a Master of Science Degree in Mathematics. As part of her course of study, the researcher enrolled in graduate level courses on probability theory, sparking an interest in the learning and teaching of probability. This interest in learning more about probability led to a minor in statistics and conducting a masters thesis study on college students' understanding of conditional probability.

Despite her recent interests in formal probability, the researcher had always enjoyed games containing probabilistic reasoning. She is an avid player of Monopoly, playing since the age of six. Her mathematical interest in Monopoly grew from becoming a competent banker to knowing the probabilistic strategies to win the game. In addition, the researcher could often be found in Atlantic City trying to beat the odds at the slot machines with her Grandmother. The researcher had always noticed the need for the understanding of probability in daily situations, and its incorporation into basic skills. While completing her doctoral work, she was a mathematics instructor, teaching finite mathematics, business calculus, and college algebra at Oregon State University.

Data Collection Instruments

Five main data sources were used for this study: Background Information Sheet, Pre- and Post-Instructional Task-Based Questionnaires, Classroom Observation Notes, Researcher's Journal, and Fieldnotes. The data for the study were collected over a 10-week period during the Winter 2002 Term.

Background Information Sheet

Prior to instruction on probability, the Background Information Sheet was administered to the class in addition to the Informed Consent Forms (see Appendix B). The purpose of the Background Information sheet was to gather self-reported biographical and background data about the students enrolled in the finite mathematics course. These data included gender, age, major, mathematical background, and mathematical achievement. Students' mathematical achievement was measured by students self-reporting of their previous college level mathematics grades.

Pre- and Post-Instructional Task-Based Questionnaires

The goal of the study was to examine the methods used and the factors that supported or impeded college students' success as they solve probability problems. Since instruction could be a possible factor, the participating students were interviewed before and after instruction on probability. The nine participating students were asked to solve 14 probability problems using a think-aloud protocol. The participants were asked to verbalize their thoughts as they solved the problems. The Pre- and Post-Instructional Task-Based Questionnaires used in the study consisted of 14 probability problems to assess the methods used and the factors that supported or impeded college students' success as they solve probability problems prior to and after instruction (see Appendix D and F). These items assessed the participants' methods of solving probability problems pertaining to sample size, simple probability, compound probability, conditional probability, mutually exclusive events, and independent events. These problems also contained concepts related to basic counting principles found in a finite mathematics course.

The probability problems found in the two Task-Based Questionnaires contained similar probabilistic concepts to those found in the finite mathematics course offered at the community colleges and those probabilistic skills recommended by the MAA and AMATYC for college graduates. The MAA (Sons, 1996) published *Quantitative Reasoning for College Graduates: A Complement to the Standards* as a response to NCTM's (1989) *Curriculum and Evaluation Standards for School Mathematics* establishing quantitative literacy requirements for all students earning a bachelor's degree. In addition, the AMATYC (1995) published *Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus* as a response to NCTM (1989) establishing standards for mathematics programs that specifically address the needs of college students who plan to pursue careers that do not depend on knowledge of calculus or upper-division mathematics. Those students enrolled in the finite mathematics course were students who are working towards a Bachelor's degree in their pursuit of careers that did not depend on knowledge of calculus or upper-division mathematics. Thus, these students are those served by the MAA and AMATYC's goals and standards.

To establish content validity and reliability, the interview procedures and a set of prospective questions were piloted with over 30 volunteer students enrolled in a finite mathematics course at a mid-sized university during the Fall 2001 Term. These volunteers were not part of the population used for the study. The first goal of the pilot study was to establish content validity using students from another finite mathematics course. Therefore, the pilot study was conducted to ensure that the students participating in the study would be familiar with mathematical tools needed to solve the problems and to determine the length of the interview. After each volunteer completed a set of problems, he or she was questioned about the problems to eliminate confusing wording and ambiguous questions. This method assisted in avoiding results that were uninterpretable or that were frustrating situations between respondent and interviewer. The second goal of the pilot study was to establish reliability of the questionnaires. In qualitative studies, reliability is viewed as a concern of the researcher as to the accuracy and comprehensiveness of the data. Through exhaustive pilot testing of over 30 problems, the "human instrument" collecting the data can become more reliable through training, practice, and refining the questions (Merriam, 1998; Bogdan & Bilken, 1998). The refinement of the questions and the experience gained through exhaustive interviews with pilot students strengthened the validity of the data collection instrument and data collection methods. Therefore, as each pilot student reacted to the questions, the questions were refined as needed. The refined questions were then used with the next pilot student. This cycle was continued until it was confirmed that each question conveyed its intent to the pilot students and the replies included responses useful for the study. The final set of pilot questionnaires consisted of 16 refined questions given to a panel of experts to check for content and face validity.

To establish content and face validity of the two questionnaires, a panel of five experts examined the suggested problems adapted from the finite mathematics course, from the recommendations from the MAA (Sons, 1996) and AMATYC (1995), and from the results of the pilot testing. This panel consisted of five professors: two mathematics educators, two mathematicians, and one instructor, in which the two mathematicians and the instructor had previously taught the finite mathematics course. The panel was given the two questionnaires, a set of objectives, and a table of specifications for them to construct. The comments and suggestions for improvement and the results from their Table of Specifications helped to modify the two questionnaires (see Appendix C for the objectives and Appendix F for the results of the Table of Specifications).

The questions containing an 80% agreement on content and face validity were then re-piloted with five additional volunteer students. This second testing of the questions consisted of ensuring the questionnaire could be completed within 40 minutes. Based on the results of the pilot testing and the results of the evaluations conducted by the panel of experts, the final Pre- and Post-Instructional Task-Based Questionnaires consisted of 14 questions each covering the probabilistic topics found in a finite mathematics course.

Classroom Observation Notes

The classroom observation notes consisted of collecting information on the sequence and instruction in the course. The goal of the classroom observation notes were to provide the researcher background knowledge of the objectives, resources, and type of instruction the students received. In addition, classroom observations also allowed the instructor to be aware of key phrases or examples used in class that could influence students ability to reason or explain a problem. These classroom observation notes were gathered in three ways: overall classroom observations, examination of course materials, and observations of classroom behavior of participating students.

Overall, classroom observations were conducted by the researcher during the term in which the student interviews took place. Classroom observations took place prior to and during the two-week instruction on probability within the overall course curriculum. To minimize the observer's classroom influence, the researcher attended classes prior to the course of instruction on probability. In addition, observational data was collected as handwritten notes and audiotapes of classroom interaction during the instruction on probability. The handwritten notes of each class utilized the anecdotal record technique with a focus on instructional methods and techniques for introducing various probability problem-solving techniques (Acheson & Gall, 1997). Special focus was given to problem solving methods modeled by the instructor, summarized in Chapter IV. By observing the instructor's problem solving methods, and other methods presented in class, it was possible to gain a clearer understanding of the participating students' approaches to the probability problems following instruction. The classroom observational data was transcribed for subsequent analysis.

Course materials, such as textbooks, activities, and web sites, provided a resource and reference for students enrolled in the course. The course material, just as with the classroom discourse, might have an impact on students' understanding of probability. The use of textbooks and activities might assist students to gain understanding of probability and influence their problem solving approaches. The course materials that were used in the finite mathematics course were examined to determine what role they played in the development of students' understanding of probability.

Observations of classroom behavior of participating students provided the researcher with additional information pertaining to the participants' opportunity to learn probability. The observations consisted of noting participating students' attendance, homework habits, and classroom participation.

Instructor Formal Questionnaire

The researcher conducted a formal interview with the instructor after the instruction on probability. The goal of the formal interview was to gain insight on the mathematical and probabilistic background of the instructor, to construct the instructor's learning expectations of the students, to ask for clarifications pertaining to the format of the class, and to understand the instructors definition of probabilistic reasoning and what he accepts as probabilistic reasoning on part of his students (see Appendix G for interview questions). The interview took place after the two-week instruction on probability to prevent possible instructor influence on changing the method of instruction and view of probabilistic reasoning. In addition, the instructor did not know the questions posed to his students during the Pre- and Post-Instructional Task-Based Questionnaires. Two questions posed to the instructor asked about his understanding of probabilistic reasoning and what he would accept as evidence of probabilistic reasoning on the part of his students. These two questions were crucial to the study since the students would probably be using the instructor's version of probabilistic reasoning. This understanding of the instructor's views of probability reasoning and his acceptance of students understanding of probability also provided guidance for the researcher while analyzing the students' responses to the questionnaires. The results from this interview are summarized in Chapter IV.

Researcher's Fieldnotes and Journal

Fieldnotes provided the researcher with a permanent record of her observations, providing descriptive and reflective information for data analysis. For this study, the fieldnotes consisted of video and audiotapes of the pre- and post-instructional interviews, audiotapes and written anecdotal records of the class meetings, audiotapes of the formal interviews with the instructor, transcriptions of all video and audiotapes, and a written record of those individual activities that might not be fully recorded by the video or audiotapes. In addition to these field notes, a written account of the researcher commentary, feelings, reactions, hunches, initial interpretation and working hypotheses, was contained within a researcher's journal.

The video and audiotaping of the pre- and post-instructional interview sessions provided various aspects of interaction including talking, gestures, eye gaze, and recording of chronological order of students written responses. The use of the video and audiotapes were beneficial in providing guidance for the transcribed interviews, allowing the chronological order of the student's written solutions to complement their verbal response. Since the use of video and audio recordings cannot take the place of notes, the researcher's journal supplemented the recordings.

The use of audiotapes, written anecdotal records, and further notes in the researcher's journal provided permanent data of the course instruction the participants received on probability. Since one of the objectives of the study was to observe the factors that support or impede college finite mathematics students as they solved probability problems, the classroom observations provided data indicating the instructional strategies the teacher used, the instructional methods used for classroom activities, and the possible influence of the course material. The audiotapes and anecdotal records provided instructional methods and problem solving strategies illustrated in the classroom, which was augmented by the researcher's journal.

Bogdan and Bilken (1992) and Gall, Borg, and Gall (1996) advocate the use of a researcher's journal, which contain the descriptive and reflective aspects of the collected data. They recommend that the descriptive aspects of the researcher's journal should include portraits of the participants, reconstruction of the dialogue, description of the physical settings, accounts of particular events, and descriptions of the observer's behavior. In addition, the reflective aspect of the researcher's journal should include the researcher's personal account of the course of inquiry, reflections on the method of data collection and analysis, reflections on ethical dilemmas and conflicts, reflections on the observer's frame of mind, and emerging interpretations.

The fieldnotes for this study consisted of the video and audiotapes of the pre- and post-instructional interviews, audiotapes and written anecdotal records of the class meetings, audiotapes of the formal and informal interviews with the instructor, and transcriptions of all video and audiotapes. Along with the research journal, this study produced an extensive set of written and visual data to serve as a record of the observations.

Timeline for Data Collection

The primary source of data was collected through interviews with participating students enrolled in the finite mathematics course. The overall data collection occurred in four stages, with several parts to each stage, over a course of ten weeks. The initial stage consisted of the administration of the Background Information Sheet to all the students enrolled in the finite mathematics course. The intent of the background information was to learn about the students enrolled in the class. The second stage consisted of a pre-instructional interview. The intent of the pre-instructional interview stage was to conduct the first Task-Based Questionnaire with the participants prior to instruction. The third stage consisted of collecting course information and classroom observations. The intent of the third stage was for the researcher to gain an understanding of the objectives, resources, and instruction available to the students. The final stage consisted of a post-instructional interview and an informal interview with the instructor. The intent of the post-instructional interview was to collect data from the second Task-Based Questionnaire conducted after instruction. The use of the researcher's journal and fieldnotes was an integral part of each data collection stage. Therefore, the following data collection methods were used to collect and triangulate the research data.

Stage I: Background Information

At the beginning of the term, the students were presented a short description of the study, were given a biographic questionnaire, and were asked for their cooperation in the study. At that time, the entire class was given the Class Participation Informed Consent Form (see Appendix B), explaining the researcher's involvement in the classroom; the Background Information Sheet (see Appendix B), to gather biographical and mathematical backgrounds information on the students; and the Interview Participant Informed Consent Form (see Appendix B), for those who agreed to participate in the interviews.

Stage II: Pre-Instructional Interviews

Prior to instruction on probability, the participating students were scheduled for one-hour videotaped pre-instructional interview, consisting of three parts: pre-session, a task-based session, post-session. The main objectives of the pre-session was to gather further biographical data of each participant, to allow the participant to become comfortable with the recording equipment, to encourage them to speak loudly, and to familiarize themselves with the data collection process and the researcher. The participant also had the opportunity to ask the interviewer questions. An outline of the questions asked in the pre-session can be found in Appendix H.

The problem solving session consisted of the participant solution of probability problems on the Pre-Instructional Task-Based Questionnaire (see Appendix D). Participants were asked to read each probability problem aloud before beginning the problem and to verbalize their thinking while attempting to solve the problems. In comparison with conventional, paper-and-pencil data collection methods, this task-based questionnaire made it possible to focus the attention more on the participant's process of addressing the probabilistic task, rather than just on the patterns of correct and incorrect answers in the results they produce. However, it is still an ongoing debate to whether "verbal data" provides an accurate reflection of the methods students would use had they not been asked to work aloud. Nevertheless, the most important events in a problem solving session may be the ones that do not take place (Ericsson & Simon, 1980, 1993; Schoenfeld, 1985). In order to improve the quality of the participant's reports of their thinking, Ericsson and Simon (1993) recommend three methods to limit major distortions in a participant's thought process. First, the interviewer should urge the students to think aloud and not to reflect on their thought processes. Second, since some students use more nonverbal representations than others do, the interviewer should encourage the drawing of diagrams or translations of images to language. Finally, probing should be used as a method to encourage sufficient amount of thinking aloud and clarification, not as a possible interference of the problem solving method chosen by the participant. Goldin (2000) and Clement (2000) also support Ericsson and Simon (1993) that during task-based questionnaires, probing should be held to a minimum. By asking the student self-monitoring and reflection questions, the resulting method might follow a different path from that which it might have taken without any intervention. Thus, the problem solving session for this student consisted of the participant reading the problem aloud and verbalizing their thinking, without reflection, while attempting to solve the problems.

In order to promote students' familiarization with the problem solving protocol and their ability to think aloud, the researcher provided general instructions and a warm-up task to the student. The goal of the general instructions was to provide guidance for the student on the protocol and how to think aloud (see Appendix I for Instructions). The goal of the warm-up task was to allow the student to try one problem using the verbal protocol and receive feedback from the researcher regarding the students ability to think aloud.

In order to minimize the amount of interference due to clarification probing, the interviewer maintained minimal interruptions during the problem solving process. Instead, the post-session of the initial interview provided an opportunity for the interviewer to clarify her understanding of the participants' approaches during the problem solving session. Thus, the goal of the post-session interview was to provide the researcher the opportunity to ask for the participant's clarification or explanation of any steps not sufficiently verbalized or written during the problem solving session. The pre-instructional interview was transcribed for subsequent analysis.

Stage III: Classroom Observations

Classroom observations were conducted by the researcher during the term which the student interviews take place. In order to minimize the observer's classroom influence, the classroom observations took place prior to and during the two-week instruction on probability. During this stage, the researcher also collected data pertaining to the resources available to the students, such as the math learning center and additional resources the students could access.

Stage IV: Post-Instructional Interviews

Following the classroom observations of the lessons on probability, the participating students took part in a one-hour post-instructional interview. The purpose of the post-instructional interview was to explore possible changes in the participating students' approaches to solving similar probability problems posed in the pre-instructional interview stage on a Post-Instructional Task-Based Questionnaire, (see Appendix E). The post-instructional interview stage was conducted similarly to the pre-instructional interview stage, but only consisted of a problem solving session and a post-session. The problem solving session contained probability problems measuring the same probabilistic concepts posed in the pre-instructional interview. The participants were asked to provide their solution in verbal and written form. At the conclusion of the post-session, the students were asked if they could identify factors which might or might not help them learn probability, see Appendix J for questions. The post-instructional interviews were transcribed for subsequent analysis. During this stage, the researcher conducted the formal interview with the instructor.

Data Analysis

Due to the nature of data collection using task-based questionnaires, a qualitative analysis method - case study analysis - was used to address the two questions of interest. One of the main characteristics of qualitative research is its focus on the intensive study of specific instances of a phenomenon.

Case study analysis is a particular approach in qualitative analysis in which the characteristics of a problem in its natural context are used to clarify a problem or phenomenon (Gall et al., 1996; Krathwohl, 1997; Merriam, 1998). One advantage of a case study analysis is its possibility of an in-depth attempt to understand an individual, allowing the researcher to seek and to explain, not merely to record, an individual's behavior (Ary, Jacobs, & Razavieh, 1990). Researchers generally conduct case studies for one of three purposes: to produce detailed description of a phenomenon, to develop possible explanations of it, or to evaluate the phenomenon (Gall et al., 1996; Merriam, 1998). Since the purpose of the study was to gain in-depth understanding and to provide detailed descriptions of the methods used and the factors that support or impede college finite mathematics students as they solve probability problems, this study used the case study analysis on the student verbal data as its data analysis techniques.

Since this study is descriptive and exploratory in nature, the case study analysis of the qualitative data was an ongoing and iterative process. A constant comparison of the data, the continual process of comparing episodes within and across categories, began during the data collection process, and continued after the data were collected. The constant comparison method allowed for constant defining and refining categories, creating sharp distinctions between categories and deciding the importance of the categories to the study. By applying the method of constant comparison, the researcher arrived at a set of well-defined categories with clear coding instructions. Multiple data collection methods, data triangulation, classroom observations, fieldnotes, and instructor interviews were also used to confirm or disconfirm observations and categorical development, seek reasons for contradictions, and to strengthen the validity of the findings (Gall et al., 1996; Krathwohl, 1997; Merriam, 1998).

The final step to the data analysis was to draw conclusions. The case study analysis helped to create categories and refine detailed descriptions of the categories of the methods students use and the factors which support or impede college students' success in solving probability problems. The analysis of verbal data consisted of observing and comparing sequence and patterns of verbal behavior across different mathematical backgrounds and abilities. In addition, after identifying the methods college students use, and the factors that support or impede their ability to solve probability problems, this study also observed and compared sequences and patterns of verbal behavior of the students before and after instruction on probability. Therefore, this data analysis also considered how the factor - instruction - influenced students' ability to solve probability problems.

Summary

Ultimately, the purpose of this descriptive analysis was to provide rich portraits of the methods used and the factors that supported or impeded college finite mathematics students' success as they solve probability problems. The data collection consisted of assessing the participant's ability to solve probability problems by collecting verbal data through the administration of two probability Task-Based Questionnaires, prior to and after two-week instruction on probability. From these data, the case study analysis allowed for defining categories of students' problem solving methods and factors which supported or impeded their success for solving probability problems. In addition, the verbal data was organized for comparison of sequences and patterns of verbal behavior across various students' mathematical background and ability, as well as student's verbal behavior before and after instruction. Together, the two Task-Based questionnaires helped to create a profile of the methods used and the factors that supported or impeded college finite mathematics students' success as they solve probability problems.

CHAPTER IV

RESULTS

Introduction

This chapter presents the results of a study on the methods college finite mathematics students used to solve probability problems and the factors which supported or impeded their success. This study used a case study analysis of the verbal data collected through task-based questionnaires administered prior to and after the students participated in probability instruction. This chapter begins with an overview of the two-week instruction on probability and of the nine participants and their responses to the Pre- and Post-Instructional Task-Based Questionnaires. The chapter reports their responses while solving the two 14-question Task-Based Questionnaires that examined their methods and factors that supported or impeded college finite mathematics students' success while solving probability problems. Finally, the chapter concludes with an overall summary of the main findings of the study.

Overview of the Class and the Participants

The nine participants in the present investigation, from a class of 24 students, were enrolled in a 10-week college finite mathematics course. The material covered in the course was designed to provide students the opportunity to learn counting techniques, probability and statistics, matrix operations, and elements of linear programming. The section on probability included topics on basic counting techniques, combinatorics, sample space, simple and compound events, conditional probability, independent events, and Baye's Rule. The course objectives for the two-week unit of instruction on probability can be found in Appendix C. Prior to enrolling in the finite mathematics course, the students had successfully completed a college algebra course. As part of their degree requirements, the majority of the students also enrolled in a business calculus course, which was not a prerequisite for the finite mathematics course.

Overview of the Two-Week Instruction on Probability

The entire class participated in a two-week unit covering the basic counting principles and probability during the sixth and seventh week of the 10-week course. The instruction for this unit consisted of eight 50-minute classes in a lecture format. The purpose of this section was to provide an overall description of the two-week instruction. The course objectives for the two-week instruction are found in Appendix C.

The instruction in the course was a typical college lecture course. The class opened with a question / answer session regarding the recommended homework assignment and the remainder of the class was devoted to lecture. The instructor provided an open class environment in which the students felt comfortable asking questions, asking for clarification, or joking with the instructor. In turn, the instructor would ask the class, or sometimes specific students, questions related to previous concepts covered in class or calculations for the current problem. In addition, the instructor provided suggested homework assignments for his students, although these assignments were not collected or graded.

The questions posed by the instructor varied according to whether the instructor was introducing a topic, lecturing on a topic, or helping the students solve a homework problem. For example, when introducing a new topic, the instructor motivated the class by presenting a real-life application and asking questions which integrated the previous lesson's objectives with the current lesson's objectives:

Classroom Discourse:	*Instructor's Work*:
I: (writes number on the board) Big number in the news last night - anybody know what this number is?	64,300,000
S: The lottery.	
I: The lottery in California.	
S: Did someone win it?	
I: Yep!	
S: How many people?	
I: Well, as far as I knew there's only one that won it - in the news that I saw. Now, almost every state - more than half the states - I think there are 37, have lotteries now, or they participate in multi-state lotteries. We have an Oregon lottery. In the Oregon lottery, how does it work?	Oregon Lottery
S: Pick a number.	
I: Pick numbers. Pick six numbers, out of how many?	
S: 44.	

I: Out of 44. Okay, then the six numbers are selected at random. And, if you see it on TV, there's a drum that's rolling and there's some sort of air pressure going on, and they come out one at a time. And they're not necessarily ordered, but you might have something like this.... They put them in order, but you don't have to have them in any particular order, do you? Okay, so the question is, what is the probability of winning? (Pause) You win if you match all six. There's one way to win. How many ways possible?	1. Pay $1, pick 6 numbers out of 44 2. 6 numbers are selected at random. 2 - 7 - 12 - 22 - 37 - 40 Probability of winning?
S: Over 7,000,000.	
I: And how did you calculate that?	
S: I used my calculator.	
I: And how did you use your calculator?	
S: I put 44 in it and nCr.	
I: Okay, 44. And then it's the combination of 44 things, taken 6 at a time. And what number is that?	${}_{44}C_6$
S: 7,059,052.	
I: So we say the probability of the event of you winning - if he is winning - is 1/7,059,052. In a particular lottery that I was looking at of like \$13,000,000 -- it keeps building up because there are multiple players and week by week people don't win. Is this the only way that you can win?	$\frac{\text{1 way to win}}{\text{7,059,052 ways possible}}$ $P(E) = \frac{1}{7{,}059{,}052}$

S: No.

I: Here are the results from one of the lotteries last year. (puts up an overhead). There are two other ways to win, and we're going to figure out those....	(overhead of results)

The instructor continued his lesson on simple probability by asking students "How many ways can that be done?", integrating the previous lesson on permutations and combinations with the current concept of simple probability. However, the instructor only elicited prior knowledge questions when introducing a new topic. The instructor did not ask students methodological or procedural questions in regards to how to think about or solve the current problem during a lecture.

While helping students solve their homework problems, the instructor did not ask students for their input on how to approach the problem. In addition, while solving the problem, the instructor did not appear to diagnose the student's difficulty with the problem. For example, Greg, one of the participants in this study, asked the instructor to solve a compound probability event problem, assigned as suggested homework:

<u>*Classroom Discourse*</u>:	<u>*Instructor's Work*</u>:
I: Okay, on this particular one we are rolling two fair dice and adding the dots on the two sides facing up, and we want to know what is the probability that the sum is divisible by 2 or 3.	
So let's define two conditions, A: The sum is divisible by 2; B: the sum is divisible by 3. We want to know the probability of A or D; is it divisible by 2 or is it divisible by 3?	A: the sum is divisible by 2 B: the sum is divisible by 3

Now our general form for looking at a union is that it's the probability of the first, plus the probability of the second, minus the probability of the intersection. And the subtraction of the probability of the intersection insures that something is not counted twice. It takes out the duplications.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Now, if this is divisible by 2 and this means divisible by 3, what would this mean? (Pointing to $P(A \cap B)$)

S: Both

I: Both. And what would that mean? If something is divisible by 2 *and* it's divisible by 3, then it's by the product of 2 or 3, better known as 6.

	1	2	3	4	5	6
1	2		4		6	
2		4		6		
3	4		6			
4		6				
5	6					
6						

Now... now ones that are divisible by 2 - I mean there's a 2 here, there's a 4 here, there's a 4 here, there's a 6, there's a 6, there's a 6 here - what other ones do we have?

We have 4s - 4 along here - you can sort of trace those down, and you get 12 of them that are divisible by...

Of the 36 that all the different combinations of those two dice, there are 12 of them that are divisible by 2.

$$= \frac{12}{36} +$$

Then we add to that the ones that are divisible by three. Now where would we find the 3s? Well, we find a 3 here, and here. We have 5 in 1 we have along here. There's another row in there. Now, I'm not going to draw the whole table, since you've got it, but how many do you count up that are divisible by 3?

By the way, did you guys agree with this?

S: I said 12 because there was 12 divided by like 6. I mean, there were 12 of them.

I: Oh, okay. So maybe everybody needs to take a quick look at that table - that figure with the dice diagram on it - because I had about 12 up there already. It kinds of bothered me when you said 12.

How many in that table are divisible by 2? How many sums of 2 are there? How many sums of 4 are there? How many sums of 6?

The table is really kind of interesting in the way it's set up - we've had this before. There's one way to get a 2; two ways to get 3; three ways to get 4; four ways to get 5; and five ways to get a 6; six ways to get a 7 - that happens to be what we'll call later as the mode, also as the modal - it's the middle value and it's the most frequent value. 5, 4, 3, 2, 1. So those that are divisible by 2... how many do we have?

S: Six.

I: How many? 1 + 3 is 4; + 5 is 9, + another 9 is 18.

Now, how many are divisible by 3? Those ones are divisible by 3. And there are two of them here; five of

2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	5	4	3	2	1

them there; four of them there; one of them there.

$$= \frac{18}{36} + \frac{12}{36} - \frac{6}{36} = \frac{24}{36} = \frac{2}{3}$$

S: Twelve.

I: A total of 12! Then we have to subtract out the ones that are counted twice, and that's the ones that are divisible by six. And there are six of those. And the probability that the sum is divisible by 2 or 3 is two-thirds.

After Greg asked the instructor to solve this homework problem, the instructor proceeded to solve the problem for Greg, without asking where Greg encountered difficulty with the problem. While the instructor solved the problem, Greg verbally acknowledged the point in the process where he himself encountered the difficulty. From this comment, it appeared the instructor sensed Greg did not understand the question, so he tried to provide a second representation to model the problem for Greg, not using Greg's interpretation of the problem. As also observed with the presentation of a new topic, the instructor did not ask the class methodological or procedural questions with regard to how to think about or solve the current problem.

The course objectives for the class were presented in a lecture format. At the beginning of the class, the instructor would write the objectives of the lecture on the board. Often the instructor would motivate a probability concept through real-life examples by integrating applications of probabilistic interpretation needed for recent newspaper articles. For example, while introducing conditional probability, the instructor presented a recent article reporting the results of a false-positive outcome of the current tuberculosis screening of the Mariner's baseball team. The instructor also presented probability puzzles, such as the Monte Hall Dilemma. These motivational problems appeared to elicit the students' interest in learning probability, as illustrated earlier in the Oregon lottery example. In addition, the instructor presented future applications of the current concept in his lecture. He included examples of the association between the symmetry of combinatorics and the binomial coefficients, of the applications of expected value and of the probability distributions during the lecture on simple probability. Overall, while presenting the course objectives, the instructor provided his students with real-life applications of probabilistic events and linked the current probability concepts to future probability concepts.

One feature that was consistent throughout the instruction was the instructor's ability to represent the counting principles and probability problems using Venn diagrams, tree diagrams, and tables. However, the instructor did not use all three representations throughout the two weeks. At the beginning, the instructor illustrated the applications of Venn diagrams when representing the counting principle problems. As he approached the probability problems, the instructor shifted from organizing and analyzing data by using Venn diagrams to using only tree diagrams and tables. As instruction continued with probability concepts, the instructor did not refer back to the Venn diagrams and its probability applications. By the second week of class, the instructor used solely tree diagrams and tables when organizing and analyzing probability problems.

Overall, the instructor provided a lecture format classroom environment, integrating real-life and future applications to motivate the students and various representations of set operations and probabilistic situations using Venn diagrams, tree diagrams, and tables. Within the relaxed atmosphere of the classroom, the students felt comfortable asking questions, responding to low level questions posed by the instructor, and joking with the instructor.

During the formal interview with the instructor, he explained that in the classroom, he did not want his students to think of probability "as a set of rules that you will apply sort of haphazardly. The main thing I want them to think about is what situations are they dealing with? What elements of chance are involved? How do these work together?" He claimed the main difficulty he had observed teaching students probability was their inability to count the elements in the sample space, "They can't conceive of the combinations of circumstances." Not only did he observe difficulties with their ability to distinguish between combinations and permutations, but also other counting principles such as calculating the complement of an event, breaking down a sample space into disjoint subsets, and identifying duplications within non-disjoint subsets. At the end of the course, when the students were still confused about the mathematics behind interpreting a probabilistic situation, he stated that he would like his students to leave the classroom with the ability to "recognize when chance is at work and the chance of something also entails a chance of it not happening. And they seem to see (probability) as black and white. Crossing that barrier of thinking to two things instead of just one." With all that said and done, he did admit "teaching calculus is far less painful than teaching probability."

Overview of the Nine Participants

At the beginning of the term, the 24 students in the finite mathematics class were presented a short description of the study and asked to participate in the administration of the Pre- and Post-Instructional Task-Based Questionnaire (see Appendix D and E). Nine students, seven American male and two African female students volunteered to participate in this study. The nine participants were called Aaron, Bob, Charlie, Dennis, Evan, Freda, Greg, Harriet, and Ian. Pseudonyms were used to assure the anonymity of the participants. Table 1 below reports the number of correct answers calculated by the nine participants. A more detailed table of each of the student's responses with respect to the probability objectives can be found in Appendix K and L.

Participant	Pre-Instructional Results	Post-Instructional Results	Change
Aaron	23	23	0
Bob	7	16	+9
Charlie	16	16	0
Dennis	9	19	+10
Evan	4	7	+3
Freda	5	5	0
Greg	11	17	+6
Harriet	4	3	-1
Ian	14	14	0

Table 1: Summary of Correct Answers (total of 23 questions)

The participants ranged in age from 19 to 30, with Aaron and Harriet being the younger students and Evan being the oldest. While seven of the participants were pursuing a business degree, Evan was pursuing a degree in forestry recreational resources and Bob was pursuing a degree in health care administration. Bob was the only participant who enrolled in finite mathematics as an elective course; whereas, the other eight participants were required to enroll in the course for their degree requirements. Charlie had the strongest mathematics background, completing college level calculus and discrete mathematics courses. Evan and Ian returned to school after 10 years out of the classroom and built up their math skills starting from basic arithmetic courses offered at the college. Both Freda and Harriet attended high school in their native African countries. More than half of their high school classes were taught in English. Overall, the nine participants for this study provide a wide spectrum of adult learners that are found in a community college setting.

Each of the nine participants entered the finite mathematics classroom with various levels of problem solving and probability abilities. Aaron, Charlie, Dennis, and Harriet had encountered set operations and probability theory in previous classes. Both Aaron and Harriet recalled set operations and probability theory from high school classes, while Charlie had enrolled in a college level discrete mathematics course containing set operations and probability. Dennis had previously enrolled in finite mathematics. Only Aaron successfully recalled the concepts of set operations and probability theory on the first questionnaire. On the second questionnaire, while Aaron and Dennis were refining their understanding of probability, Charlie and Harriet still were unable to solve the majority of the problems successfully.

Freda had recalled using set operations and simple probabilistic concepts in high school. Struggling over set operations and probability theory on the first questionnaire, Freda continued encountering difficulties interpreting and solving the problems on the second questionnaire.

Finally, Bob, Evan, Greg, and Ian did not recall learning set operations and probability theory prior to enrolling in finite mathematics. While Bob and Greg tried to use their knowledge of percents to solve the problems on the first questionnaire, Ian used his strong mathematical skills to interpret and solve the majority of the problems correctly, and Evan interpreted the probability problems to be either right or wrong. On the second questionnaire, both Bob and Greg successfully solved the majority of the problems. Ian did not improve in his attempts, and Evan still encountered difficulties interpreting probability situations.

Overall, the nine participants displayed nine different levels of problem solving and probabilistic abilities. Three of the participants - Aaron, Dennis, and Greg - were successful meeting the probability objectives by the second questionnaire. Three participants -Bob, Charlie, and Ian - were moderately successful and three participants - Evan, Freda, and Harriet - were unsuccessful. For a more detailed portrait of each of the nine participants, including a description of their classroom behavior and homework habits in the finite mathematics classroom, refer to Appendix M for their complete case studies.

Answers to the Questions of Interest

This next section discusses the methods used and the factors that supported or impeded these nine participants as they solved probability problems. The purpose of the first section of this chapter was to provide an overview of the two-week instruction on probability and of the nine participants. The purpose of this second section was to take their responses and answer the two questions of interest.

1. What methods do college finite mathematics students use to solve probability problems?
2. What are factors that support or impede college finite mathematics students' success for solving probability problems?

In order to answer the two questions of interest, a cross-case analysis of the verbal and written data from the nine participant portraits resulted in patterns and themes that demonstrated their ability to solve probability problems. In addition to the cross case analysis, the methods and factors of more and less successful participants were compared. For the purpose of this study, a more successful participant was defined as a problem solver who was able to analyze a word problem, develop an appropriate method, organize and analyze data with appropriate representation, and check the reasonableness of their answer. A less successful participant was unable to analyze or interpret the problem, develop or implement an appropriate method, or perform the necessary arithmetic or algebraic operations while working within the various representations. Each of the two questions of interest for this study was addressed separately in the next two subsections. Table 2 provides an overview of the methods the participants used and the factors which supported or impeded their success.

	General Word Problems	Probability Specific
Methods	General methods for solving word problems: • Individualized methods: – Aaron – Charlie – Freda – Greg • Association of numbers within their solutions with the context of the problem • Reliance on a procedure • Observations of short-cuts	Specific methods for solving probability work problems: • Individualized methods: – Evan – Bob – Harriet – Ian • Interpretation of probability as a trick question • Representations to organize and analyze the data
Factors	Prerequisite content and skills: • To rephrase a problem or question • To use arithmetic reasoning skills • To work with percentages in a problem • To recognize their weaknesses in solving a problem • To check the reasonableness of their answer • Instruction	Probability content and skills: • To use set operations • To interpret probability terminology • To understand the experiment • To interpret the probabilistic event • To verify the data in the problem • To be able to use a representation to organize and analyze the data • To compare probabilities when given a probability distribution • To use correct probabilistic terminology • To recognize the combinations of compound events • To recognized the difference between independent and dependent events • To find the probability of an event • Instruction

Table 2: Methods Used and Factors Which Supported or Impeded Success

Methods Students Used to Solve Probability Problems

This first subsection answers the first question of interest: What methods do college finite mathematics students use to solve probability problems? More specifically, this subsection provides an overall organized presentation of the procedures or techniques the participants used for attaining the solution to the problem. The identified methods used by the participants were divided into two subdivisions: methods used for solving word problems and methods used for solving probability problems. In addition, this section discusses successful and unsuccessful methods.

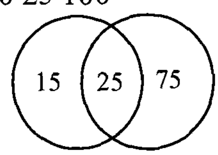
Methods Used for Solving Word Problems

The intent of the first question was to observe the methods college finite mathematics students used to solve probability problems. Embedded in this question was that each student might have their own style or method when approaching a mathematics problem, regardless of the concepts embedded in the problem. The purpose of this first subdivision was to provide an overall summary of the methods used for solving word problems on the Pre- and Post-Instructional Task-Based Questionnaires. Seven word problem methods were used by the participants. These seven methods are not to be interpreted as distinct, disjoint methods. Instead, some of the participants integrated more than one method in one problem or changed their methods according to their comfort level in each problem.

The seven word problem methods consisted of four different individualized methods exhibited by Aaron, Charlie, Freda, and Greg; Aaron, Dennis, and Greg's association of the numbers in their solutions with the context of the problem; Bob, Dennis, Freda, Greg, and Harriet's reliance on a procedure, such as a formula or step-by-step procedure; and Aaron and Ian's observations of shortcuts in a problem.

The first method consisted of Aaron's individualized word problem method. Aaron, the most successful participant, continually used his own method for solving problems: read the entire problem and the question; broke down the problem into pieces; analyzed each piece separately; used an appropriate representation to organize and analyze the data, used the context of the problem to reason a solution, and checked his final answer. An example of this can be observed in Aaron's verbal response to solving Problem 7 on the first questionnaire:

> You have torn a tendon and are considering surgery to repair it. The orthopedic surgeon explains the risks to you. Of the 500 patients who underwent this surgery, infection occurred in 40 of such operations, the repair failed in 100 of such operations, and both infection and failure occur together in 25 of such operations. Find the probability that if you undergo surgery, the operation will succeed and you are free from infection?

Student's Response	*Student's Work*
So, you have 500 total	500
And you got, um, 40 for infection, 100 failed, 25 together	40 25 100 
So that leaves 75, 15...	
Uh, 75 plus 25 plus 15 equals that they either got infection or failed, or both	75 + 25 + 15 = 115 infection or failed or both

So you take 500 minus 115 and you get 385 over 500 $500-115=\frac{385}{500}$

And reduce that and you get 77% chance the operation will succeed and you are free from failure $=77\%$
(Pause while he checks his work) Okay.

Aaron's method for solving these probability problems developed from his strong mathematics background and his awareness of probability and set operations concepts. Aaron continually used this complete cycle for solving each problem posed on both questionnaires, without an observed change.

The second method consisted of Charlie's individualized word problem method. Charlie, the participant with the strongest mathematics background, created his own method for solving problems: read the problem, stated an estimate, solved the problem, and checked his answer against his initial guess and the context of the problem. However, Charlie was not always successful with this approach. After Charlie read a problem and did not know how to solve it, Charlie would rely on his belief that "all you need is the numbers, and then just manipulate it enough and then you'll somehow come out with something." After using this algorithm of manipulating the numbers, Charlie would compare his various results with his initial intuitive answer, selecting the answer that was the closest to his original guess. This approach was successful for some problems. However, Charlie did encounter difficulties when the numbers he obtained through his algorithm did not compare favorably to his original guess. When faced with this dilemma, Charlie would try to convince himself that the number he obtained through his algorithm matched the context of the question, even if the numerical answer was wrong.

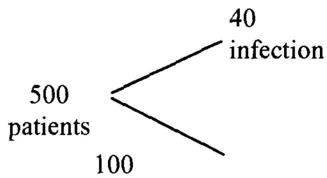
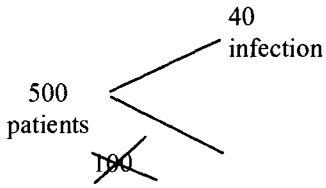
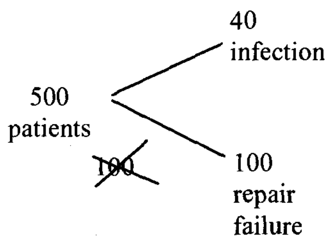
An example of this can be observed in Charlie's verbal response to solving Problem 9 on the second questionnaire:

> A warning system installation consists of two independent alarms. During an emergency, the probability of each alarm operating properly is 0.95 and 0.90 respectively. Find the probability that at least one alarm operates properly in an emergency.

Student's Response		*Student's Work*
At least one engine fails in flight. So, it's going to have to be less than .05% because that is the probability… Oh, never mind. So the probability of one failing has to be the combination of the left one and the right one.		
So, it's hard to say, because if you add them together, you get .15, which is 15% of an engine failing.	.15	15%
And if you multiply them then you get .005, which is half a percent. So, find the probability that at least one engine fails in flight…	.005	.5%
I guess I'd have to go with the addition, because that's a pretty good percent that it's going to fail. I would not want to fly that plane.	15%	

Charlie continued to approach the problems on the second questionnaire with the same method he used on the first questionnaire: estimate an answer, manipulate the numbers, and check the calculated answers against the estimated answer.

The third method consisted of Freda's individualized word problem method of extracting information and organizing the information using a Venn or tree diagram before reading the question. Freda, the participant who attended high school in Botswana, was familiar with Venn and tree diagrams, but not probability. After constructing a diagram, organizing the information, Freda would proceed to use the original Venn diagram or tree diagram throughout the problem. However, Freda would sometimes construct an incorrect representation of the problem because the method she selected was inappropriate, thus causing a conflict in her interpretation of the problem and the interpretation of the diagram. An example of this can be observed in Freda's response in solving Problem 7 on the first questionnaire:

> You have torn a tendon and are considering surgery to repair it. The orthopedic surgeon explains the risks to you. Of the 500 patients who underwent this surgery, infection occurred in 40 such operations (Freda does not finish reading the problem)
>
> *Student's Response*
>
> So, 500 patients, 40 get an infection. Okay. The repair failed in 100 such operations - ooh! 100. And both infection and failure - okay. This is infection
>
> *Student's Work*
>
> 
>
>
> This is 100 failure. I don't know! (crosses out 100)
> Okay. In both the infection and the failure together in 25 of such oper...huh?
>
> 
>
>
> You have torn a tendon and are considering surgery to repair it. The orthopedic surgeon explains the risks to you. Of the 500 patients who underwent this surgery, infection occurred in 40 of such operations, the repair failed in 100 of such operations, and both infection and failure occur together in 25 of such operations. Find the probability that if you undergo surgery, the operation will succeed and you are free from infection?
>
> 
>
>
> Okay, 500 patients who underwent the

> surgery. Infection occurs in 40 and the repair failed - the repair failure in 100, and both infection and failure together in 25 of such operations. And both infection and failure occur in 25.

Freda started to read the problem and immediately drew a tree diagram. She tried to fit the data to the tree diagram, but was confused as to where to put the 100 repair failure. Once she decided where to put 100-repair failure, the next statement confused Freda, "In both infection and the failure occur together in 25 of such operations." She did not know how to represent this statement on her tree. After this confusion, Freda decided to read the entire problem and the question. Knowledge of set operations was needed to solve this problem. It is not always possible to represent set operations using a tree diagram. Venn diagrams and tables are a more suitable representation. Freda consistently used her method on the first questionnaire: read the problem, construct a diagram, read the question, analyze the diagram, and answer the question. After the two weeks of instruction, Freda approached the second questionnaire in a slightly different manner. This time, Freda would read the entire problem before selecting a diagram for organizing and analyzing the problem. On the second questionnaire, Freda was able to select the correct diagram for the question, but still encountered difficulties organizing and analyzing the information in her diagram.

The fourth method consisted of Greg's improved individualized word problem method on the second questionnaire. Greg, a participant whose unfamiliarity with the problems on the first questionnaire caused him to try different methods, refining his approaches to the problems on the second questionnaire. On the first questionnaire, Greg read the entire problem, extracted numbers from the problem, labeled the numbers, and then proceeded with various operations to obtain a solution. This approach caused Greg to unsuccessfully solve the majority of the problems. However, after gaining comfort with set operations and probability within the two weeks of instruction, Greg's overall method for solving the problems changed, successfully solving the majority of the problems. On the second questionnaire, Greg read each problem, determined which representation would best suit the situation; drew a tree diagram or table, including calculations of the cells or branches not necessary for the problem; verified that he constructed his representation correctly by checking his calculations on the rows, columns, or branches of the trees; then proceeded to read the question and answer it. An example of this can be observed in Greg's response to solving Problem 7 on the second questionnaire:

> An auto dealer sold 300 luxury cars in one month. His records indicated that 45 cars had their air conditioners failing before the warranty expired, 60 had their alternator failing before the warranty expired, and both the air conditioner and alternator failure occurred together in 15 cars before the warrantee expired. Mr. Jones purchases a luxury car from the dealer. What is the probability that the air conditioner and alternator on the luxury car will ***not*** fail before the warranty expires?

Student's Response

(Greg's verbal responses while constructing the table)
Well, air conditioning failed; air conditioning not failed. Alternator, no. AC, AC primed. Alternator failed; alternator not fail. Total. Total.

Okay, his records indicate that 300 cars -- his records indicate that 45 cars had their air

Student's Work

	A	A	T
AC			
AC'			
T			

conditioning fail before the warranty expired.
(Greg's verbal responses while filling in the table)
45, 60 had alternators fail before the warranty expired, 45. And then if 15 cars had both air conditioning -15 cars, so finish this out, this is 30. This is 30. No, wait! Expired -- 45 cars had air conditioning expire. 60 had AC. 60. That number's wrong. 60. So that's 45. 45 out of 300 is 255. And 60 out of 300 is 240. And that's 210. Yeah!

	A	A	T
AC	15	30	45
AC'	45	210	255
T	60	240	300

What is the probability that the air conditioning and alternator on the luxury car will ***not*** fail before the warranty expires?
Well, it would be inter -- probability of AC intersection A prime, over 300 which equals 210 out of 300. 210 divided by 300 is .7 or 70% chance total will ***not*** fail before warranty expires.

$$\frac{P(AC' \cap A')}{300} = \frac{210}{300}$$

.7 or 70% chance that it will not fail before warrantee expires

Greg read the entire problem and recognized a table could be used to organize and analyze the data. After constructing the table, Greg proceeded to fill in the table using the information in the problem and to check that the totals of the columns and rows add up to the grand total of the problem. After ensuring his table was constructed correctly, Greg read the question, extracted the required information from the table, and successfully answered the question.

The fifth method consisted of Aaron, Dennis, and Greg's associating the numbers in their solutions with the context of the problem. Aaron continually integrated this method while solving all his problems. In addition, when Dennis or Greg was comfortable with a problem, they would also respond to the problem by using the context of the problem in their word problem method. An example of this method can be observed in Dennis' verbal response to solving Problem 6 on the second questionnaire:

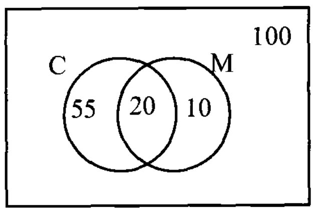
> From a survey at a preschool, a pharmaceutical company found that 75% of the students contracted chicken pox, 30% contracted measles, and 20% contracted both chicken pox and measles. If a child at the preschool is randomly selected, what is the probability that: The student contracts chicken pox, or measles, or both?

Student's Response

Universal set of 100
Okay, 75% of students contracted chickenpox, 30% contracted measles, 20% contracted both chicken pox and measles, so you put the 20 in the middle. Should take the 75; subtract 20, 'cause it's 55 for just chicken pox. And we've got 30 subtract 20 and that gives us 10 for just measles
Okay, the student contracted chicken pox, or the measles, or both

Now we've got to get 100 subtract 55 subtract 20, subtract 10, equals 15 who don't have either. Okay, so a. 100 total subtract 15 who don't have either comes out to 85 who have chicken pox, or the measles, or both.

Student's Work



a) .85

As Dennis solved Problem 6, he was able to verbalize his solution by associating the numbers in his solutions with the context of the problem. This method was not evident in the other six participants verbal responses, where they would extract numbers without labeling them or associating them back to the context of the problem. However, the three participants who did associate the numbers in their solutions with the context of the problem were successful solving those problems.

The sixth method consisted of Bob, Dennis, Freda, Greg, and Harriet's word problem method of reliance on a procedure, such as a formula or systematic procedure to solve the problem. Harriet continually searched for a formula to solve the problems. However, Bob, Dennis, Freda, and Greg occasionally showed evidence they were relying on their own procedures to solve the problem instead of using mathematical reasoning. For example, when Bob tried to solve Problem 7 from the second questionnaire, which involved knowledge of set operations, Bob first ruled out the possibility of needing counting theory to solve the problem. Reading the problem again, Bob recognized the set operations in the problem and proceeded to solve the problem:

> An auto dealer sold 300 luxury cars in one month. His records indicated that 45 cars had their air conditioners failing before the warranty expired, 60 had their alternator failing before the warranty expired, and both the air conditioner and alternator failure occurred together in 15 cars before the warrantee expired. Mr. Jones purchases a luxury car from the dealer. What is the probability that the air conditioner and alternator on the luxury car will *not* fail before the warranty expires.

Student's Response	*Student's Work*
Okay - so there's 45, there's 60 - counting, that has nothing to do with it - wait…	
So, lets see, 60… what will we use… we go 60… 60 minus 15 equals 45, and 45 minus 15 equals 30.	60 - 15 = 45 45 - 15 = $\underline{30}$ 75
So, 75 out of 300. Let me see if that looks right (puts fraction into calculator) .25 What is the probability that the air conditioner and alternator…. ooh, What is the probability that it will not? I 'd say 75% chance it will not fail.	75 / 300 = .25 75% chance it will not fail

Bob knew that the 15 cars which both incurred an air conditioner or alternator failure was counted twice in the sentence, "45 cars had their air conditioners failing before the warranty expired, 60 had their alternator failing before the warranty expired." Knowing he had to subtract to accommodate this overlap of data, Bob proceeded to subtract the 15 cars which both incurred these failures from both sets, as if recalling a procedure. Bob did not notice he subtracted the 15 cars, which had both incurred these failures twice in the problem. This example indicated Bob's reliance on the set operation "and" to prompt him to subtract the overlap of cars, without understanding the overlap occurs only once.

Another example of reliance on a formula was when Dennis, Freda, and Greg each tried to find the probability of a simple event. They knew a useful formula was the number of elements in an event over the total number of elements. However, each one used this formula incorrectly. An example of this can be observed in Greg's attempt at solving Problem 10 on the first questionnaire:

> A market research firm has determined that 40% of the people in a certain area have seen the advertising for a new product. Given that they have seen the advertising, 85% have purchased the product. What is the probability that a person in this area has seen the advertising and purchased the product?

Greg proceeded to divide .40 by .85, noting that answer, and then dividing .85 by .40. This is the formula for finding the probability of a simple event. However, this problem asks the participants to find the probability of a dependent event, or the product of 40% and 85%. Unfortunately, the majority of those students relying solely on choosing a procedure were not successful at solving the problems.

The seventh method consisted of Aaron and Ian's word problem method of observing connections and similarities between problems and used them to find short cuts in the problems. They sought underlying structures of similar problems to solve the current problem. Aaron and Ian used this method when they came across Problems 6 and 11 in which there were multiple questions on the same set of data. Aaron and Ian both noticed that for Problem 6, Parts A and C were complements of each other. Therefore, when solving Problem 6 Part C, they explained in their verbal responses that they were taking the complement of the probability obtained in Part A. In addition, while finding the probabilities of the various events in Problem 11, both of them also recognized that some problems had the same number of elements in the sample space or in the event as a previous problem. Thus, they took those results and used them in another problem, saving effort on re-calculating the event. Finally, Aaron noticed two instances, Problem 9 and Problem 14, where finding the complement of an event was easier than finding the probability of the event. This ability to see the connections and similarities between the problems, allowed Aaron and Ian to find shortcuts to solving their problems. Thus they were able to perform fewer computations to correctly solve the problem.

Overall, these prior seven word problem methods were observed used at least once by the nine participants. In addition, some participants created their own method for approaching the word problems. While Aaron, Charlie, Freda, and Greg used their own individualized word problem method when approaching most problems, Aaron, Dennis, and Greg employed a method of associating numbers in their solutions with the context of the problem. In addition, Bob, Dennis, Freda, Greg, and Harriet's method consisted of reliance on a procedure, such as a formula or systematic procedure to solve the problem. However, some of these methods were not necessarily "the best" approaches to solving the problem. While the more successful participants - Aaron, Dennis, and Greg- used methods such as seeking relationships to prior exposed concepts and using the context of the question to reason a solution, some of the less successful participants also tried similar methods, but were unsuccessful. An example of an unsuccessful attempt at this method was Charlie's algorithm for manipulating numbers. Charlie's method of stating an estimate at the beginning and comparing it to the calculated response appeared to be a strong approach to solving problems. However, Charlie's algorithm for manipulation of numbers to generate calculated responses in the word problem method was not the best method for Charlie to use in all situations. Therefore, the methods observed in this study were not categorized as "right" or "wrong." It was observed that the participants had their own unique method for working on word problems, but what made them more or less successful were the factors contributing to their ability to obtain a solution, such as Charlie's arithmetic skills. The next subdivision considers the specific probabilistic methods the participants used to solve the problems.

Methods for Solving Probability Problems

The previous subdivision provided an overall summary of the general word problem methods the participants used while solving probability problems. As noted in the previous subdivision, Aaron, Charlie, Freda, and Greg used their individualized word problem method for some problems. In addition, for some problems, more than one participant used similar word problem methods. However, when required to apply their knowledge of probability in order to solve the problem, some of the participants used methods intended specifically for solving probability problems. Therefore, this second subdivision of probability methods provided an overall summary of the probability methods the participants used while solving probability problems on the Pre- and Post-

Instructional Task-Based Questionnaires that were unique to solving probability problems. At least six probability methods were observed to be used by the participants. These six probability methods consisted of three different individualized probability methods exhibited by Evan, Bob, Harriet, and Ian; Bob and Freda's difficulty interpreting two problems; and Aaron, Dennis and Greg's use of representations to organize and analyze the data.

The first method consisted of Evan's method of solving unfamiliar probability problems without mathematical reasoning. Evan, the participant who was concurrently enrolled in a statistics class, was uncomfortable interpreting probability and statistics problems. On the first questionnaire, Evan claimed that probability is "cut and dry" and the solution is answered in a "yes / no fashion." Therefore, when approached with the problems on the first questionnaire, Evan would reason that the spinners at a carnival game "are not designed to be won all the time" (Problem 3) or that "I won't be one of the 40" out of 500 patients who will develop an infection after surgery (Problem 7). Unfortunately, on the second questionnaire, Evan perceived the probability problems more as a new concept requiring new set of procedures to learn in order to pass the class. Combined with his original approach to solving problems, Evan continued to exhibit his yes/no interpretation of solving problems, while also trying to identify the correct procedure to solve the problem, still solving the problems unsuccessfully.

The second method consisted of Bob's method of seeking relationships between the new probability concepts and his prior learned concepts. This was evident in Bob's attempt at the problems on the first questionnaire. Bob was a participant who did not recall learning probability. His unfamiliarity with probability mixed with his dislike of word problems caused him to approach each problem using a different method. However, as Bob continued solving the problems, he started to make an association between probability and percents. By the end of Problem 8, he commented, "I'm seeing a whole, so like these are percents, in a way. And one would be 100%." Bob approached the problems on the second questionnaire as percent problems, but encountered difficulties when he tried to interpret his answers. For example, when Bob correctly solved Problem 5, he arrived at the answer .275, then commented, "What do you call that? .275 chance that a customer drinks and smokes?... I don't know if it's a percent or what kind of thing... I think it's (the answer) is a percent because of probability." Bob was confused as to the representation of the final answer as a percent or a decimal.

Bob was not the only participant trying to seek a relationship between probabilistic concepts and his prior learned concepts. On Greg's first attempt at the questionnaire, he also tried to use some reasoning found in percent problems to reason the solutions to some of the problems. When Greg provided answers to the problems on the first questionnaire, he would represent the answer as a percent: "so it's 7% don't drink coffee", "29% females prefer coffee", and "92.5% that one will work." Greg stated answers as if he were finding the percent of the event occurring, not the probability of the event occurring. It was not evident from the data if Greg recognized the difference between probability and percents. Both Bob and Greg tried to develop a scheme to solve probability problems, using what they knew about percents.

The third method consisted of Harriet's overall method of seeking the formulas used to solve probability problems. Harriet, the participant who attended high school in Nigeria, recalled learning about probability in high school and had taken a college level statistics course. When faced with a familiar problem, Harriet tried to recall the formulas associated with the problems. She knew that simple probability consisted of finding the number of elements in the event and dividing it by the total number of elements in the sample space. In addition, Harriet recalled algebraic formulas for finding the probability of an unknown event and for working with set operations. However, when faced with a problem in which she could not recall the formula, she tried to formulate an equation, as opposed to reasoning throughout the problem. An example of this was observed in Harriet's attempt to solve Problem 7 on the first questionnaire:

> You have torn a tendon and are considering surgery to repair it. The orthopedic surgeon explains the risks to you. Of the 500 patients who underwent this surgery, infection occurred in 40 of such operations, the repair failed in 100 of such operations, and both infection and failure occur together in 25 of such operations. Find the probability that if you undergo surgery, the operation will succeed and you are free from infection?

<u>*Student's Response*</u>	<u>*Student's Work*</u>
Okay, Okay, let's try x plus 40 plus 125... No, ... If 40 were infected and 100 out of... Okay, minus 25.	$x + 40 + 100 - 25$ $= 115$
So, I subtract 500	500
That is 380... 385	385
I know this is not going to come out right, but... it will be .77	$\frac{385}{500} = 0.77$

Harriet recalled a formula for working with set operations. Using this formula, Harriet correctly solved the problem. On the first questionnaire, Harriet recalled an incorrect formula for Problem 8, but on the second questionnaire, she recalled the correct formula.

The fourth method consisted of Ian's overall method of using his strong reasoning skills to solve the probability problems he had not encountered before. Ian, the participant who returned to college 10 years after earning a GED and owned his own business, exhibited strong mathematical reasoning skills. These strong skills enabled him to interpret and solve the majority of the probability problems on the first questionnaire correctly. When coming across unfamiliar terminology, Ian would state his definition of the term and used his definition to solve the problem. In addition, while solving the problems, Ian would try to integrate the newly found words into his verbal responses. Through his reasoning, he was able to correctly solve problems requiring him to find the probability of simple events and apply the properties of probability, including the complement of the event. However, Ian did exhibit difficulty interpreting the problems requiring knowledge of set operations and the recognition of dependent events. Overall, Ian was comfortable solving the probability problems due to his strength in his mathematical reasoning ability, but did not indicate improvement in his ability to solve the probability problems on the second questionnaire.

The fifth method consisted of Bob and Freda's views that Problems 10 and 12 on the first questionnaire were trick questions and could not be solved using mathematics, in which the answer was subject to the interpretation of the reader. They both felt that the problem provided them with the answer, but they were to interpret the problem and pick the correct number from the data given. Based on this reasoning, they did not attempt to solve the problem. They both did not know how to interpret the conditional probability term "given" in the problems. An example of this can be observed in Bob's attempt at solving Problem 12 on the first questionnaire:

> Two boxes each contain red marbles and blue marbles. One marble is drawn at random from a box (each marble has an equal chance to be drawn). If the marble is red, you win $1. If the marble is blue, you win nothing. You can chose between two boxes:
> - Box A contains 1 blue marble and 4 red ones
> - Box B contains 3 blue marbles and 4 red ones
>
> Which box offers a better chance of winning, or are they the same? Explain.
>
> *Student's Response:*
> This is a trick question. Better chance of winning. If you're looking in a different perspective... see, if... Is this considered math? I could say, you know, this is like a geometry - there's no right or wrong, I feel. But I feel like - I could say two things, like here - better chance. I could see it where there's four (red marbles) in both of them, so that means I only have four chances to win, so that would make it the same amount of chance (for each box). Or, I could see it as this one (Box A) only contains 1 blue marble, so the chance of me picking up that blue marble when there's 3 more of the red, I would have a better chance of winning that than if there's just 1 less blue marble than the reds (Box B). I could go both ways. I feel like this is a trick question. So, I will go with, I would choose the box - I would choose Box A. So I would put a big "I" would choose Box A, because there are 3 less blue marbles than there are red.

Bob was trying to understand the problem, but stated it was a "trick question" depending on the interpretation of the problem: there were 4 red marbles in both of them, so there was an equal chance to win from each box; or Box A has three more red than Box B, so Box A had the better chance of winning. By the second questionnaire, Bob no longer exhibited difficulty interpreting these problems as trick questions. Instead, Bob used probabilistic reasoning skills to solve Problem 12. Unfortunately, Freda continued believing the two questions were trick questions when she encountered them again on the second questionnaire.

The sixth method consisted of Aaron, Dennis, and Greg's method of using the appropriate representation for a probability problem - Venn diagram, tree diagram, or table - to organize and analyze the data in the probability problem successfully. While Charlie and Freda unsuccessfully organized and interpreted Venn diagrams; Bob, Evan, Harriet, and Ian did not attempt to use alternate representations, such as Venn diagrams, to organize or analyze probability problems on the second questionnaire.

Overall, these six probability methods were observed to be used at least once by the nine participants. While Evan, Bob, Harriet, and Ian devised their own probability method for some of the problems, Bob and Freda encountered problems which they thought were trick questions, requiring them to select an answer. Finally, Aaron, Dennis, and Greg used representations to organize and analyze the data. However, some of these methods were not necessarily "the most successful" approach to solving these problems. In fact, the less successful participants - Bob, Evan, Harriet, and Freda - used the six probability methods while trying to solve the probability problems, while the more successful participants - Aaron, Dennis, and Greg - only used the last probability method of organizing and analyzing the data correctly. As stated earlier, it was observed that each participant had their own unique word problem method, but what might make them more or less successful at their method could be the factors contributing to their ability to obtain a solution. The next subsection of the cross-case analysis considered these factors, which supported or impeded their success while solving probability problems.

Factors that Supported or Impeded Student Success Solving Probability Problems

This second subsection of the cross-case analysis considered the second question of interest: What are factors that support or impede college finite mathematics students' success for solving probability problems? More specifically, this subsection provided an overall organized presentation of the causes or components contributing to the participants' ability to obtain a correct solution. For example, Charlie exhibited an overall method of estimating an answer, solving a problem, and checking his calculated answer against his initial guess and the context of the problem. While Charlie continued to use this method, his weak arithmetic skills, his inability to recognize set operation terminology, and his inability to use the correct probabilistic terminology when solving the problem contributed to his inability to solve the problems. At the same time, Charlie exhibited the ability to read tables and diagrams, rephrase the problem, and recognize the properties of probability. These causes or components contributing to Charlie's ability (or inability) to obtain a correct solution was defined as the "factors" which support or impede the students success solving probability problems. As in the previous section, these identified factors were divided into three subdivisions: factors in solving word problems, factors in solving probability problems, and factors within instruction.

Factors in Solving Word Problems

The intent of the second question was to observe the factors that supported or impeded college finite mathematics student's success for solving probability problems. Embedded in this question was students' own personal factors contributing to their ability to obtain a correct solution, regardless of the concept embedded in the problem. The purpose of this first subdivision of word problem factors, therefore, was to provide an overall summary of these "common" factors contributing to the participant's ability to obtain a correct solution. At least five word problem factors were observed to be used by the participants and were presented in order of their placement in the problem solving process: the ability to rephrase a problem or question, to use arithmetic reasoning skills, to work with percents in a problem, to recognize their weakness solving a problem, and to check the reasonableness of their answers. These five factors are not to be classified as successful or unsuccessful factors. In fact, each factor could support or impede a participant's success at solving the probability problems. For example, a participant might display a weak or strong understanding of percents, a useful skill for interpreting probabilistic events. Therefore, the intent of this section was to identify the factors, in which each factor could either support or impede the participants' success as they solved probability problems. In addition, these five factors are not to be interpreted as distinct, disjoint factors. Instead, some of the participants integrated more than one factor in one problem or interchanged their factors according to their comfort level in each problem.

The first factor consisted of Aaron, Bob, Charlie, and Dennis' ability to rephrase the problem or question either while solving the problem or after stating an answer. This was differentiated from the participants who continually re-stated exact phrases from the problem. When Aaron, Bob, and Dennis incorporated this factor into their overall method, they were successful at solving the problems. However, Charlie was not as successful. Instead, Charlie would rephrase the problem to try to make it fit the answer he calculated. While Aaron would try to rephrase the problem while solving the problem, Bob and Charlie would rephrase the problem in order to check whether their response was correct. Aaron, Bob, Charlie, and Dennis would rephrase the problem or question on both questionnaires.

The second factor consisted of Charlie, Freda, and Harriet's inability to use arithmetic reasoning skills. Charlie's word problem method was unsuccessful due to his weak arithmetic reasoning skills. Charlie claimed "all you need is the numbers, and then just manipulate it enough and then you'll somehow come out with something." An example of this can be observed in Charlie's verbal response to solving Problem 6 on the first questionnaire:

> From a survey at a large university, a market research company found that 75% of the students owned stereos, 45% owned cars, and 35% owned both cars and stereos. If a student at the university is selected at random, what is the probability that the student owns either a car, or a stereo, or both?
>
> *Student's Response*
> So, hmm, my first intuition is just to add them up, but then ... you get over 100%, which isn't good. So then, obviously that can't be the answer, so then I multiply. I go .75 times .45 times .35 equals (uses calculator). No, jeez... 11.8% Oh, that can't be right. Oh, umm ... Well, this is weird. I guess I will have to say 100%. Because, hmm... well, if 75% of the students own stereos...45% own cars, and 35% own both cars and stereos ... maybe I could go 75% minus 35% would equal 40%, so ... umm... okay, never mind. Okay, I don't know the question.

Charlie added and multiplied the numbers found in Problem 6. After pondering the reasonableness of the two quantities, Charlie decided that neither of them was correct.

Freda and Harriet also encountered difficulties with their weak arithmetic reasoning skills. Their weakness with fractions was exhibited while solving Problem 4 on the first questionnaire.

> In a deck of cards, 1/6 are green, 1/12 are yellow, 1/2 are white, and 1/4 are blue. If someone takes a card from the bag without looking, which color is it most likely to be? Why?

Trying to compare the four fractions, Freda decided to add the denominators of the fractions to find the common denominator. Harriet proceeded to also find the common denominator for each fraction. However, Harriet was successful finding a common denominator, but due to a miscalculation, Harriet calculated there was a total of 15/12 cards in the deck. Observing her rather large number, Harriet did not double-check her work, but instead, continued to the next problem.

The third factor consisted of Charlie, Evan, Freda, Greg, and Harriet's inability to work with percents. Charlie stated that he "always thought it was easier to understand percents, than, say ratios or fractions"; however, when solving a problem involving percent, Charlie continued to admit, "I think whenever I see percents, I multiply... Which is obviously wrong. But, I mean 'cause you don't add up percents." Therefore, whenever Charlie encountered a problem containing percents he would multiply the percents together. This view of percents caused Charlie to incorrectly solve two problems requiring the addition of percents.

Freda, Greg, Harriet, and Ian each encountered difficulties with percents when confronted with Problem 6 on the first questionnaire:

> From a survey at a large university, a market research company found that 75% of the students owned stereos, 45% owned cars, and 35% owned both cars and stereos. If a student at the university is selected at random, what is the probability that the student owns either a car, or a stereo, or both?

While solving this problem, neither Freda nor Greg recognized when they violated the properties of percents. They both proceeded to add the three percents given to them in the problem to calculate the total of 155. This rather large number did not appear to bother them as they continued solving the problem. In addition, Harriet also tried to add the three percents together for Problem 6, calculating a total of 1.55. Recognizing this number was too large, Harriet proceeded to check her knowledge of percents by proceeding to divide 75 by 100 to ensure that it produces the decimal equivalent of .75. Then, Harriet proceeded to conduct random calculations to obtain a smaller, more reasonable number she felt more comfortable reporting as her answer:

Student's Response	*Student's Work*
(After calculating 1.55)	
1.55 and that's the total…	1.55
75 divided by 100	.75
Okay, the probability that they student owned either a car or a stereo is… 75 plus 45 - and just the total number of - the total percent of students that owned both cars and -- total percent of students that on cars is 45% and out of stereos is 75%, which adds up to 120%	45% +75% = 120%
That's no good…. Student owns either a car or a stereo	
1 over 75 plus 1 over 45	1/75 + 1/45
That adds up to 1.35	1.33+0.02 = 1.35

> So, the probability that the student owns
> either a car or a stereo - I came up with
> 1.35

After looking back at her answer, Harriet responded, "But, I know that is not right, because it's in percent and I kind of converted it to 100, but now I am not really sure if I was suppose to solve the problem in percent or not." It was evident that Harriet was frustrated at the set operations terminology solving Problem 6. However, she tried to produce an answer in which the decimal was reasonable by trying random calculations. Unfortunately, Harriet did not know the syntax of her calculator and that 1/75 is equal to 1.33×10^{-2} or .0133 instead of her written answer of 1.33.

Proceeding to Problem 6 the second questionnaire, Harriet, Freda, and Evan continued to encounter difficulties working with percents:

> From a survey at a preschool, a pharmaceutical company found that 75% of the students contracted chicken pox, 30 % contracted measles, and 20% contracted both the chicken pox and measles. If a child at the preschool is randomly selected, what is the probability that the student contracted the chicken pox, or the measles, or both?

Evan felt uncomfortable working with the percents contained in the problem. When he first approached this problem, Evan responded, "So... 75... I'm thinking there's something else I need here. I don't know why. Maybe like a number of how many students there were total? That would be nice." After the interview, Evan was asked why he wanted to know the number of students there were total. Evan responded, "Because I guess I'm looking for something that - I think I'm looking for something to take these percents from." Similarly, when Harriet first approached Problem 6, she asked, "And I have to use percents here?... They didn't give me the total ... The problem is like I need to get a total...and I don't have to change the percents? As into a regular number divided by 100?" When approaching this problem containing percents, both Evan and Harriet wanted to know the total number of students there were in the survey. However, when Freda approached this problem on the second questionnaire, she treated the problem as if the percents were the total number of students, and gave an answer representing the number of students, as opposed to the probability or percent answer.

Ian encountered difficulty with percents on his attempt at solving Problem 10 on the first questionnaire. Problem 10 required Ian to find the probability of an event, knowing the percent of elements, which contain certain characteristics. Ian did not encounter difficulties interpreting Problem 10, but instead encountered difficulties working with the percents in the problem. Ian wanted to find the number of elements in the event, instead of the probability of the event. Ian's conflict with percents was not similar to Evan or Freda' conflict. However, Ian also wanted to know the number of elements in the event. While in the previous subsection identifying methods the participants used, Bob and Greg sought a relationship between probabilistic concepts and percent concepts. While this was their overall approach to solving the problems, the difficulty Charlie, Evan, Freda, Greg, and Harriet encountered was working with percents in the problem. This current factor, the inability to work with percents, caused them to encounter difficulties in their overall method.

The fourth factor consisted of Bob and Greg's ability to recognize their own weakness in a problem. For example, while solving Problem 5 on the second questionnaire, Bob correctly calculated the answer .275 but was unsure how to represent his final answer: as a percent, decimal, or fraction. Bob recognized that he did not know how to present his final answer, claiming "see, this is something where I might bring to it to class the next day... like I might put a star (next to it) because I don't know if it is a percent or what kind of thing." In addition, while Bob was solving Problem 6 on the second questionnaire, requiring knowledge of set operations, Bob was confused by the association of the numbers in the problem. Thus, Bob commented again "before I get confused, I would put a star next to this one too", referring to his previous comment that he would bring it to class the next day to ask a question. In addition, on the first questionnaire, Greg kept reading the various questions, stating, "I don't believe I have the tools to solve these." Both Bob and Greg exhibited strong reasoning skills, but were also able to recognize where their weaknesses were while solving the problem.

The fifth factor consisted of Aaron, Bob, Charlie, Greg, and Ian's ability to check the reasonableness of their answer. This factor was differentiated from those participants who stated an answer to the current problem, then continued to the next problem. Since Aaron felt comfortable with the problems on both questionnaires, he was able to check the reasonableness of all his answers. On the second questionnaire, Bob preferred to finish the problem, and then check his answer against his rephrased problem improving his success on those problems. On both questionnaires, Charlie preferred to provide an estimated first guess to the problem and checked if his final answer was similar. On the first questionnaire, Ian tried to check the reasonableness of his answers, but unfamiliarity of the problems caused him not to check as often as he did on the second questionnaire. When Aaron, Bob, Greg, and Ian checked the reasonableness of their answer, they were successful at solving that particular problem. However, Charlie encountered difficulty when his initial guess did not match his calculated response and would try to match his calculated answer to the problem. Bob, Greg, and Ian did observe incorrect responses when checking their final answer. This caused them to return to the problem and find their mistake.

Overall, these prior five word problem factors were observed to be used at least once by the nine participants. Each factor had the ability to help or hinder the participant's success while solving the problems. For example, Aaron, Dennis, and Greg integrated the ability to rephrase the problem or question in their overall approach successfully, while Charlie also integrated this factor in his overall approach, but was not as successful. In addition, Charlie, Freda, and Harriet encountered difficulties with their arithmetic reasoning skills, while Charlie, Evan, Freda, and Harriet encountered difficulties working with percents contained in the problem. The next subdivision of the factors that supported or impeded college finite mathematics students' success for solving probability problems considered specific probability factors.

Factors in Solving Probability Problems

The previous subdivision provided an overall summary of the word problem factors that supported or impeded college finite mathematics students' success for solving probability problems. Some of the factors were unique to some participants, while more than one participant were impacted by other factors. However, when approached with problems requiring their knowledge of probability, some of the participants used factors intended for solving probability problems. Therefore, this second subdivision of probability factors provided an overall summary of the factors that supported or impeded college finite mathematics students' success for solving probability problems. More specifically, this subdivision provided an organized presentation of the causes or components contributing to the participants' abilities to obtain a correct solution to a probability problem, in which the factors were characteristic of specific content knowledge of probability. At least 11 probability factors were observed to be used by the participants. These 11 probability factors were presented in order of their placement in the solution process: ability to use set operations, to interpret probability terminology, to understand the experiment, to interpret a probabilistic event, to verify the data in the problem, to use a representation (Venn diagram, tree diagram, or table) to organize and analyze the problem, to compare probabilities when given a probability distribution, to use correct probabilistic terminology, to recognize the combinations of compound events, to recognize the difference between independent and dependent events, and to find the probability of a dependent event. As stated in the previous subdivision of factors, each factor cannot be defined as successful or unsuccessful. For example, some participants were successful while others were unsuccessful choosing a representation (Venn diagram, tree diagram, or table) to organize and analyze the problem. In addition, these 11 factors are not to be interpreted as distinct, disjoint factors. Instead, some of the participants integrated more than one factor in one problem or interchanged their factors in each problem.

The first factor consisted of the each participant's ability and inability to use set operations. On the first questionnaire, Evan stated that he had difficulty with the vocabulary associated with probability, "Because the vocabulary they use, a lot of the words that I would use every day but the meanings have been *completely* switched around to something where you're, 'What? My entire life that word had not meant that', and when I read (it now) I interpret it a totally different way." Evan's opinion was evident in the other participant's interpretations of the problems requiring set operation knowledge. On the first questionnaire, Bob, Greg, and Ian admitted they had not been exposed to set operations before. When Bob read Problems 6 and 7, requiring knowledge of set operations, Bob stated, "if *this* is probability, then that means I have no clue on this" and did not approach the problems. When Greg and Ian read Problems 6 and 7, they proceeded to solve the problems believing that each of the three sets - "student owns either a car, or a stereo, or both" - were disjoint. While solving Problems 6 and 7 on the second questionnaire, Ian and Bob recognized the set operations embedded in the problem, but did not know how to interpret the probability of the event. However, Greg was somewhat successful, using a tree diagram and a table to solve the problems on the second questionnaire.

Those participants who had prior exposure to set operations were also unable to recognize or interpret the terminology associated with set operations. On the first questionnaire, both Charlie and Dennis recognized the key words "and" and "or", prompting them to think that they must either multiply or add the numbers. However on the second questionnaire, while Charlie continued using these key words, Dennis recognized their association with set operations, and successfully solved the problems. On both questionnaires, Freda did recognize the key words "and" and "or" and their association with Venn diagrams. Unfortunately, while drawing her Venn diagrams, Freda then proceeded to treat each set disjointedly, as Greg and Ian first believed. Overall, only Aaron and Dennis were proficient at recognizing the set operation terminology embedded in the problems and were able to solve these problems successfully by the second interview.

The second factor consisted of Evan, Harriet, and Ian's inability to correctly interpret probability terminology. Evan encountered difficulty on both questionnaires interpreting the phrases "would be expected", "equally likely", and "equal chance." When interpreting equally likely and equal chance, Evan indicated that he interpreted these phrases as each event having the same chance of being selected, regardless of the size of each event in comparison to one other. While solving problems on both questionnaires, Harriet would repeatedly verbalize key words such as "without looking", "likely to be", and "random." When asked about her continual repetition of these words, Harriet claimed she knew the definition of the words, but they were confusing her ability to interpret the problem. Without prior knowledge of the probabilistic terminology, Ian did encounter difficulty trying to interpret the meaning of the phrases "fair spinner" on Problem 3 of the first questionnaire. After thinking more about the problem, Ian stated that "fair spinner" must mean it was an equal chance for the arrows of the spinner to land on either color. However, on the second questionnaire, Ian did not exhibit difficulties interpreting the terminology in the problem. While solving the problem, the inability to correctly interpret some of the probability phrases prevented some of the participants from obtaining the correct solution.

The third factor consisted of Harriet's inability to understand the experiment. It was unclear if the other participants understood the experiment described in the problems; however, Harriet encountered the most difficulty interpreting the experiment for Problem 3 on both questionnaires. Requiring her to compare the probability of an event to the experiment of spinning two spinners or flipping two coins, Harriet thought the experiment was for the continual spinning of the two spinners or continually flipping the two coins until the event of interest appeared. An example of this can be observed in Harriet's attempt to solve Problem 3 on the second questionnaire:

> The two fair coins shown below are part of a carnival game. One side of a coin is black, the other side white. A player wins a prize only when both coins land on black after each coin has been tossed once.
>
> 
>
> Caroline thinks she has a 50-50 chance of winning.
> Do you agree? Explain why or why not.

Harriet responds, "Yeah, I agree because it's not like she's limited to playing it once, tossing it around once... Yea, I agree because she - I mean, there's a chance that after tossing it around for so many times she's going to get both coins landing on black." Harriet reasoned on Problem 3 from the first questionnaire that if the person spins the two spinners, they would eventually get two black. With this interpretation of the experiment, Harriet did not know how to interpret the experiment on either of the two questionnaires. The inability to understand the experiment caused Harriet to define her own experiment, thus preventing her from obtaining a correct solution.

The fourth factor consisted of Evan and Freda's interpretation of a probabilistic event. Freda interpreted Problems 3 and 12 in which she had to compare two probabilistic events, as a "win or lose" situation. Evan interpreted probability as a "yes / no fashion", in which the answer can be determined using non-mathematical reasoning. An example of this could be observed in Freda's verbal response to Problem 4 on the second questionnaire:

> The two fair coins shown below are part of a carnival game. One side of a coin is black, the other side white. A player wins a prize only when <u>both</u> coins land on black after each coin has been tossed once.
>
> 
>
> Caroline thinks she has a 50-50 chance of winning.
> Do you agree? Explain why or why not.

Freda's response to this question was similar to her response on Problem 4 on the first questionnaire: "Yea, I agree... Because there are only two sides. It's either you win or lose, so chances are 50 - 50." It is possible due to the language barrier, that Freda did not understand the experiment. However, she used the same reasoning for Problem 4 on the first questionnaire and Problem 12 on the second questionnaire. This misinterpretation of a probabilistic event prevented Evan and Harriet from obtaining a correct solution.

The fifth factor consisted of Aaron, Charlie, and Greg's ability to verify the data given in a problem and to verify the data in their representation of the problem. When given a problem with a frequency table, like Problems 5 and 11, Aaron, Charlie, and Greg would double-check the numbers in the table, verifying that the rows and columns added up to the stated number in the sample space. In addition, when asked to compute the probability of an event using these tables, all three participants circled the cells of interest and continually verified that their calculations from the tables were correct. In addition, whenever Greg constructed an alternate representation of the problem, such as a tree diagram or table, he would completely draw the tree or table, including the calculations of the branches or cells not necessary for the problem. Once the tree or table were constructed, Greg would double-check his numbers, verifying that he constructed his representation correctly by checking the branches, rows, and columns of his representation, then proceeded to answer the question. However, when Aaron constructed his representations, he only drew the branches, cells, or circles required for solving the specific problem. Charlie never used an alternate representation for any of the problems. All three participants proficiently double-checked the numbers in given tables and constructed tree diagrams and tables. Nevertheless, this verification process did not prevent Greg from using the wrong representation for Problem 6 on the second questionnaire, thus calculating an incorrect answer. In their method for obtaining a solution, Aaron, Charlie, and Greg would verify that the numbers in a problem were correct. While this factor appeared to help students with their solution process, it did not help them to obtain a correct solution if they were using the incorrect representation for the problem.

The sixth factor consisted of Aaron, Dennis, Greg, and Freda's ability to use a representation - Venn diagram, tree diagram, table, or formula - in order to organize and analyze the data: Venn diagrams, tree diagrams, tables, and formulas. While Aaron and Dennis relied more on using a Venn diagram, Greg used the tree diagram and table interchangeably, and Freda used the Venn diagram, tree diagram, and table. Problems 6 and 7 required knowledge of set operations and could be solved using one of the four representations. Aaron, Dennis, Greg, and Freda were the only participants attempting to construct a representation for Problems 6 and 7, while Aaron and Dennis were the only successful participants to also construct the correct representation for the problem. In addition, Aaron and Dennis were the only two participants able to solve both Problems 6 and 7 correctly. When observing how the participants solved Problem 9, Aaron and Greg were the only participants not only to solve it, but to also use a representation to solve it. The ability to use a representation could contribute to a participant's successful method while solving a probability problem. However, the participant must also be familiar with the process of organizing and analyzing data in the representation in order to be successful at solving the problem.

The seventh factor consisted of Charlie, Freda, and Harriet's inability to compare probabilities when given a probability distribution. On the first questionnaire, Problem 4 caused some difficulties for Charlie, Freda, and Harriet:

> In a deck of cards, 1/6 are green, 1/12 are yellow, 1/2 are white, and 1/4 are blue. If someone takes a card from the bag without looking, which color is it most likely to be? Why?

Charlie and Freda wanted to calculate the frequency of each event and compare their frequencies. Charlie incorrectly assumed there were 52 cards in any deck, thus producing 26 white, 13 blue, about 8 green, and about 4 yellow cards in this deck. However, the sample he had created had only 51 elements, since he was not concerned about loosing the .67 of a green card when dividing 52 by 6, or the .33 of a yellow card when dividing

52 by 12. Similarly, Freda also incorrectly assumed there were 24 cards in the deck, obtaining 24 by adding the numbers in the denominator to find a common denominator for all the fractions. Once assuming there were 24 cards in the deck, Freda proceeded to find the frequency of each event. Both Charlie and Freda arrived at the correct answer of white being the color most likely to be selected. On the second questionnaire, when given a similar problem for Problem 4, Freda then assumed there were 52 cards in that deck, and proceeded to find the frequency of each event, truncating the number of cards as Charlie did for the first problem and producing a sample size of 51 cards. Charlie proceeded to find the decimal equivalent of each fraction and compare the results for the second questionnaire.

Harriet also encountered difficulty comparing probabilities when given a probability distribution, as in Problem 4, stated above. Harriet wanted to find the number of cards in the deck, but proceeded to find the number of cards by adding the fractions together. When incorrectly calculating the number 15/ 12, she stated, "How am I going to know what color a person gets?" then turns to the next problem. On the second questionnaire, when given a similar problem, Harriet proceeded to add up the fractions again. Upon realizing the fractions add up to 1, Harriet stated she did not know, and proceeded to the next problem. While Charlie and Freda wanted to know the frequency of an event in order to compare their probabilities, Harriet did not know how to approach the problem. While the participants had their own method for solving the problem, their inability to work with a probability distribution prevented the participants from successfully comparing probabilities of an event.

The eighth factor consisted of Aaron, Bob, Charlie, Dennis, and Ian's ability to use the correct probabilistic terminology when solving the problem and stating the answer on the second questionnaire. Aaron was proficient at integrating probabilistic terminology in his verbal responses. However, Aaron encountered difficulty determining which terminology to use to represent the complement of an event, when he said, "And you take the opposite, the complement, inverse, or something like that." Bob encountered difficulty providing a label with his final answer. After correctly calculating the answer .275 for Problem 5 on the second questionnaire, Bob was unsure how to represent his final answer: as a percent, decimal, or fraction. Bob recognized that he did not know how to present his final answer, claiming "see, this is something where I might bring it to class the next day… like I might put a star (next to it) because I don't know if it is a percent or what kind of thing." For the rest of the problems, Bob would not provide a label with his numerical answers.

Charlie encountered similar difficulties both using the correct probabilistic terminology when solving a problem and stating an answer. While taking the complement of an event for Problem 7, he responded, "So, the chance of not failure is the opposite of that." In addition, when trying to compare the probabilities of an event given its probability distribution, Charlie incorrectly responded, "So, I guess I 'd have to go with the yellow since it has the highest fraction." Dennis encountered difficulties trying to use the correct terminology when comparing probabilistic events. For Problem 4 on the second questionnaire, Dennis responded, "Well, they're more likely to get yellow, because there's a higher percent of yellow in the bag. As far as odds go anyway. Or probability, or whatever you call it. I'm not sure." Ian encountered difficulties on how to state the final answer. For Problem 10 on the second questionnaire, he responds, "So about 18% or - I guess it's not stated as a percent... it's just... that's my answer." In addition, Ian responds, "Whatever 50 represents. I guess percents, I don't know", for Problem 13 on the second questionnaire. Trying to integrate probability terminology into their problem solving, Aaron, Bob, Charlie, Dennis, and Ian encountered difficulties searching for the correct word.

The ninth factor consisted of Charlie, Dennis, and Evan's inability to recognize the combinations of two compound events. Charlie needed to find the probability of two fair spinners landing on black - with a 50 / 50 chance of each spinner landing on black, after each spinner has been spun once. Charlie considered the event of obtaining a black and white as obtaining the same event white and black. This same factor occurred again when Charlie tried to solve Problem 14. Charlie responded that the event of selecting a red sock followed by a blue sock was the same event as selecting a blue sock followed by a red sock. For Problem 3, Evan was to find the probability of flipping two coins - black on one side, white on the other side - such that both coins land on black after each coin had been tossed once. While solving the problem, Evan considered the event of black and white as the same event as white and black. Dennis indicated similar reasoning as Charlie and Evan when solving Problem 3 on the first questionnaire.

The tenth factor consisted of Ian's inability to recognize the difference between independent and dependent events. While solving the problems, Ian was successful at computing independent and dependent events on the second questionnaire; however, his interpretation of dependency was that the outcome of a compound experiment - independent or dependent - was a "dependent" event when the outcome of one of the events was not known. An example of Ian's confusion of independent and dependent events was illustrated in Problem 13. For Problem 13, Ian needed to find the probability of a complex experiment of two successive events without replacement:

> A gumball machine contains 5 red, 7 white and 9 green gumballs. The gumballs are well mixed inside the machine. John gets two gumballs from this machine in succession, without replacement. What is the probability that both gumballs are red?

Ian correctly reasoned that "In the first one, its 5 out of 21... let's just say he gets a red one on the first time, I mean there's only 4 more in there, and there's also only 20 now." This time Ian knew that the first ball was red, however to find the probability that both balls were red, Ian then proceeded to add the fractions 5/21 and 4/20. When asked at the end of the interview why he added the two numbers, he responded, "Because it's similar to..." and refers to Problem 3:

> The two fair coins shown below are part of a carnival game. One side of a coin is black, the other side white. A player wins a prize only when <u>both</u> coins land on black after each coin has been tossed once.

> Caroline thinks she has a 50-50 chance of winning.
> Do you agree? Explain why or why not.

After looking at his work on Problem 3, he added,

> No, I guess I did multiply those (referring to Problem 3), because (the first coin) was dependent on the other (coin). This (coin) was dependent on the other (coin). And in this case (Problem 13), it wasn't. Because we knew that the first (gumball) was a specific one. And this one (Problem 3) we didn't. This one (Problem 3) was dependent on the roll of the first one. There was a 50-50 chance here. But see, in this one (Problem 13), we knew that she go the first gumball.

Ian's interpretation of dependency was that the outcome of a compound experiment - independent or dependent - was "dependent" when the outcome of the one of the events was not known, such as the color of the first marble selected from the bag.

The eleventh factor consisted of the nine participants' abilities to find the probability of a dependent event. There were four problems on each questionnaire requiring knowledge of dependent events. Due to the variety of responses related to solving a dependent event, this factor was broken into three parts, Problem 10, 11, and 13.

Of the nine participants, eight attempted problem 10. All eight participants encountered difficulty interpreting and solving Problem 10 on the first questionnaire. Problem 10 consisted of finding the probability of an event (probability that a person in the area has seen the advertising and purchased the product) while being provided an intuitively prescribed dependent event (85% of the 40% who have seen the advertising purchased the product):

> A market research firm has determined that 40% of the people in a certain area have seen the advertising for a new product. Given that they have seen the advertising, 85% have purchased the product. What is the probability that a person in this area has seen the advertising and purchased the product?

Aaron, Dennis, and Charlie each recognized the key words "and" and "of" after rephrasing the problem to read "seen the advertising *and* purchased the product" and "85% of the 40%." Upon recognition of the key words, Aaron, Dennis, and Charlie proceeded to multiply the two numbers. All three participants knew they had obtained the correct solution, but none of the three explained their reasoning as one event depending on the other event. They admitted the key words told them to multiply the numbers. As Bob and Freda approached Problem 10, they both thought this was a trick question, in which either 85% or 40% was the correct response, based on a person's interpretation of the problem. As Greg approached this problem, he was able to rephrase the problem, knowing he had to find 85% of the 40%, but thought this interpretation meant to divide the two numbers, proceeding to calculate .40/.85. As mentioned previously, Ian wanted to find the number of people in the event, not the percent. Ian then proceeded to calculate $(0.85)(0.4)x$ where x was equal to the number of people who lived in that area. While Aaron, Dennis, Charlie, Greg, and Ian recognized key words such as "and" and "of" to solve the problem, Bob and Freda thought they were to select the correct answer. None of the participants indicated recognition of the dependent event occurring in the problem.

When approaching Problem 10 again on the second questionnaire, Aaron and Greg recognized the dependency in the event, providing verbal and written responses indicating their recognition and understanding of the dependent events. Both Aaron and Greg successfully integrated the use of a tree diagram to represent the dependency of the two events. However, Charlie, Dennis, and Ian continued to multiply the two answers after recognizing the two key words "and" and "of." In addition, Freda still thought the answer was provided, and she just had to pick the correct answer. Finally, Bob had no idea how to approach the problem on the second questionnaire and divided the two percents just to supply an answer.

The participants indicated a better understanding of dependent events as they approached Problem 11 on the first questionnaire. Only six of the participants tried Problem 11. Problem 11 required them to find the probability of seven events: the first set of three were independent events, the second set of three were dependent events, and the last problem was an independent event:

A university cafeteria surveyed 500 students for their coffee preferences. The findings are summarized as follows:

	Does Not Drink Coffee	**Prefer Regular Coffee**	**Prefer Decaffeinated Coffee**
Female	25	145	70
Male	15	200	45

A student is selected at random from this group. Find the probability that:

a. the student does not drink coffee
b. the student is male
c. the student is a female who prefers regular coffee
d. given that the student is male, the student prefers decaffeinated coffee
e. given that the student prefers decaffeinated coffee, the student is male
f. given that the student prefers regular or decaffeinated coffee, the student is female
g. the student is a male student who prefers regular or decaffeinated coffee

For the first questionnaire, Aaron was the only completely successful participant. Surprisingly, Bob and Ian, two participants who had not worked with probability before, were able to correctly answer the first six parts, but interpreted Part G as a dependent event, the concept found in the three prior parts. Unfortunately, both Charlie and Greg were not affected by the wording of the problems, responding to each part as if they were calculating the probability of an independent event. When approaching Problem 11 on the second questionnaire, Aaron and Greg were the only two completely successful participants. However, Bob, Charlie, Dennis, and Ian were able to correctly answer the first six parts, while still interpreting Part G as a dependent event. Surprisingly, while the participants were solving Problem 11, the only factor preventing a participant from being successful was their inability to recognize when the problem was asking for an independent or dependent event. The participants did not indicate difficulty reading the table, setting up the fractions, or misinterpreting the overall problem.

All six of the participants attempting to solve Problem 13 recognized the dependent event; however, did not know how to calculate the probability of the event. Problem 13 required the participants to calculate the probability of a complex experiment, in which they needed to find the probability of two successive events without replacement:

> A box contains 2 red, 3 white and 4 green balls. Two balls are drawn out of the box in succession without replacement. What is the probability that both balls are red?

On the first questionnaire, only Aaron, Charlie, Dennis, and Ian recognized the need to break down the problem into two different probabilistic events: probability of selecting a red ball first and probability of selecting a red ball second. In addition, while calculating the probability of each event, Aaron and Charlie both recognized that both the number of elements in the event and in the sample decreases as a ball was selected without replacement. Both Aaron and Charlie were able to correctly compute the probability of the event to be (2/9)(1/8) = 2/72. Similarly, Dennis and Ian recognized that the sample size decreased as the balls were being selected. Stating that the probability a red ball would first be selected was 2/9, Ian could not decide if the second probability would be 2/8 or 1/8, responding, "Your chance is either going to be 2 out of 8 or 1 out of 8, depending on whether or not you actually drew the red one. But since it is such a small chance, I'm going to say that you -probably didn't get it." Based on this reasoning, Ian concluded the probability of the second ball being red was 2/8. However, Dennis tried to write the ratios of the two events occurring, recognizing the reduction in the sample space, but did not know how to state the final answer.

Ian Freda, Greg, and Ian also recognized that after a red ball was selected, that the size of the sample and the event decreased. However, they did not know to separate the problem into two different probabilistic events. Freda concluded the probability was 2/6 where 2 represented the two balls taken out, and 6 represented the other 6 balls. While Greg concluded that the probability will be 2/9 where 2 represents the two red balls taken out, and 9 represents all the balls in the sample space.

When approaching Problem 13 again on the second questionnaire, Aaron, Charlie, and Greg correctly broke down the problem into two events and calculated the probability of each event before combining the results. Ian conducted a similar procedure, but when he found the probability of each event, he proceeded to add the two probabilities. Dennis also knew to break down the problem into two events, but instead of calculating the probability of each event, Dennis found the number of ways two red balls could be selected from the sample space.

The eleventh factor, the ability to find the probability of a dependent event, provided different results depending on the type of problem posed, and if the probability was provided or if the participant had to calculate the probability of each event. When given Problem 10, most participants for both questionnaires, easily perceived the "correct" operation to be multiplication, because of the key words "and" and "of." Except Aaron and Greg, the participants did not indicate they recognized a dependent event in the problem. However, when given Problem 11, in which the participants were to read the size of the event and sample space from a table, the majority of the participants were successful calculating the probabilities. They did recognize the problem contained independent and dependent events. However, some participants encountered difficulty distinguishing between the two. Finally, when given Problem 13, some participants recognized the need to break down the problem into two probabilistic events. In addition, most of the participants recognized for Problem 13 that the size of the sample and the event decreased when the ball was not replaced. However, the difficulty the participants encountered solving Problem 13 was deciding how to represent the two probabilities, how to combine the two probabilities, and if they needed to know the outcome of the first event to be able to calculate the probability of the second event. Overall, the participants recognized the need to find the probability of a dependent event, but encountered different difficulties depending on the context of the problem.

Overall, these prior 11 probability factors were observed to be used at least once by the nine participants. Each probability factor had the ability to help or hinder the participant's success while solving the probability problem. For example, Aaron, Dennis, Greg, and Freda tried to use a representation - Venn diagram, tree diagram, or table - to organize and analyze the data. However, while Aaron and Dennis successfully organized and analyzed the data to obtain a correct answer, Greg and Freda were not as successful, encountering difficulties with using the correct format of a diagram or incorrectly organizing the data in the diagram. In addition, Harriet was unable to understand the experiment, while Evan and Freda were unable to interpret a probabilistic event.

Factors in Instruction

The two goals of this study consisted of an examination of the methods used and the factors that supported or impeded their success as they solved probability problems. After identifying possible factors that support or impede college students' success in solving probability problems prior to instruction, an embedded goal from the second question was to also observe one potentially important factor - instruction - and its influence on college students' ability to solve probability problems. Therefore, the purpose of this third subdivision was to provide an overall summary of the observed changes in student's methods and factors from the administration of the first questionnaire, to the administration of the second questionnaire. Instruction is defined to be the two-week exposure to probability within the finite mathematics classroom, their homework and classroom habits, and the help students sought outside the classroom. Due to the design of the study, these observed changes do not imply causality of instruction. Instead, the goal of this section was to report the main observed changes.

The two previous subsections outlined word problem methods, probability methods, word problem factors, and probability factors. Within these subsections, as each method and factor was defined, the method or factor included the changes which occurred from the administration of the Pre-Instructional Task-Based Questionnaire, to the administration of the Post-Instructional Task-Based Questionnaire. Observed changes included the participants' use of Venn diagrams, tree diagrams, and tables to organize and analyze the data and recognition of the difference between independent and dependent events. While each method and factor was discussed individually, the purpose of this last subdivision of factors which support or impede student success while solving probability problems was to provide an overview of the observed changes in the participants' content knowledge, novice participants' ability to solve probability problems, and experienced participant's ability to solve probability problems from the administration of the first questionnaire to the second questionnaire. Table 3 provides an overview of the observed changes.

	Content Knowledge	Novice Participants (Bob, Evan, Freda, Greg, and Ian)	Experienced Participants (Aaron, Charlie, Dennis, and Harriet)
Pre-Instructional Questionnaire	• Simple events: Aaron, Charlie, Dennis, Ian • Comparison of Two Events: Aaron, Charlie, Dennis, Ian • Properties of Probability: Aaron, Bob, Charlie, Greg, Ian • Compound Events: Aaron • Independent Events: Aaron • Dependent Events: Aaron • Unable to finish a problem: Bob, Charlie, Dennis, Evan, Harriet, Freda	Methods Used: • Percent and arithmetic reasoning: Bob, Greg, Ian • Non-Mathematical reasoning: Evan • Incorrect knowledge of set operations: Freda	Methods Used: • Familiarity with problems: Aaron • Procedural: Dennis, Harriet • Number manipulation: Charlie
During Instruction	The instructor: • Motivated probability topics with real-life applications • Posed content and calculation questions • Provided suggested homework problems • Used various representations to model problem • Did not pose methodological questions • Did not diagnose student difficulty	• Attended all class and tried homework: Bob, Greg: • Sought help from tutors: Greg • Attended half of classes, did not attempt homework: Evan, Freda, Ian • Did not participant in class: Ian	• Tried homework and sought help from tutors: Dennis • Missed one class: Dennis, Harriet • Missed three classes: Aaron, Charlie • Did not attempt homework: Aaron, Charlie, Harriet • Did not participate in class: Harriet
Post-Instructional Questionnaire	• Simple Events: Same as before plus Bob, Greg, • Comparison of Two Events: Same as before plus Bob and Greg • Properties of Probability: Same as before plus Dennis, Evan • Compound Events: Same as before plus Dennis • Independent events: Same as before • Dependent events: Same as before • Unable to finish a problem: Same as before minus Charlie, Dennis	• Improved skills in Set Operations, Independent and Dependent Events, Properties of Probability: Bob, Greg • Incorporated various representations: Greg, Freda • Used same method of arithmetic reasoning: Ian • Incorporated personal interpretation and procedure: Evan	• Refined problem solving techniques: Aaron, Dennis • Improved skills in Properties of Probability: Harriet • Used same method of manipulating numbers: Charlie

Table 3: Factor of Instruction

The first change consisted of the participants' probabilistic content knowledge. The two questionnaires contained 14 questions on sample space, simple events, compound events, independent events, dependent events, comparison of two events, and properties of probability. Appendix K and L summarize the content responses of each participant on both questionnaires. Overall, only one student, Aaron, was able to correctly answer all 14 questions and their corresponding sub-questions correctly on both questionnaires.

On the Pre-Instructional Task-Based Questionnaire, Aaron, Charlie, Dennis, and Ian were the only participants to correctly answer all problems on simple events and comparison of two events. In addition, Aaron, Bob, Charlie, Greg, and Ian were able to correctly answer the problems on the properties of probability. None of the eight participants, outside of Aaron, was able to solve all the problems associated with compound, independent, and dependent events. Finally, several participants were unable to finish problems associated with compound or dependent events. Bob, Charlie, Dennis, Evan, and Harriet were unable to complete the problems associated with compound events, while Bob, Evan, and Freda were unable to finish the problems associated with dependent events.

During instruction, the instructor introduced various the probability topics with real-life applications. While presenting the content to his students, the instructor posed content and computational related questions. While solving a problem, the instructor modeled various representations – Venn diagram, tree diagram, table, or formula – to organize and analyze the problem without asking the class methodological or procedural questions in regards to how to think about or solve the current problem. At the beginning of the section on probability, the instructor provided the students with suggested homework problems.

On the Post-Instructional Task-Based Questionnaire, the same participants plus Bob and Greg correctly answered all problems on simple events. In addition, the same participants plus Bob and Greg correctly solved the problems associated with comparison of two events, while the same participants plus Dennis and Evan correctly solved the problem on properties of probability. Aaron and Dennis was the only participant to correctly answer the problems associated with compound events. Finally, several participants were unable to finish problems associated with comparison of two events, compound events, and independent events. Harriet was unable to finish the problems associated with comparison of two events. Bob, Evan, Freda, and Harriet were unable to finish the problems associated with compound events. Dennis was unable to finish the problems on dependent events.

Overall, from the administration of the Pre-Instructional Task-Based Questionnaire to the administration of the Post-Instructional Task-Based Questionnaire, only one participant was able to solve all the problems correctly. However, there was an increase in the number of participants correctly solving simple, independent, and dependent events, while many of the participants encountered difficulty completing or correctly solving problems associated with compound events on both questionnaires.

The second change consisted of the "novice participants' ability to solve probability problems. Bob, Evan, Freda, Greg, and Ian did not recall formal instruction on probability prior to enrolling in the finite mathematics course. While Freda did recall simulations of rolling of a die or flipping of a coin in a previous class, she did not recall learning content outside of simple probability concepts. In addition, Evan was simultaneously enrolled in a statistics course while taking finite mathematics. During the administration of the Pre-Instructional Task-Based Questionnaire, Bob, Evan, Freda, Greg, and Ian each encountered difficulty solving the problems. Bob, Greg, and Ian appeared to use their prior knowledge of percent and arithmetic reasoning when attempting the problems. Evan approached each problem using non-mathematical reasoning and explanations to the problems. In addition, Evan viewed each answer in a yes / no fashion. Finally, Freda recalled Venn and tree diagrams from high school. Using her knowledge of set operations, Freda tried to solve the problems, but encountered difficulty selecting, organizing, and analyzing the data in the diagram.

During the two-week instructional period, only Bob and Greg were the only two novice participants to attempt the assigned homework problems. While Greg sought help with the tutors at the mathematics resource center at the college, Bob would ask either Greg or his roommates for assistance on the homework problems. Evan, Freda, and Ian each indicated they did not attempt the homework problems during the two-week instructional period. In addition, Bob and Greg were the only two participants attending all eight classes. While Freda missed the first four classes, Evan and Ian missed two classes each. Finally, while Greg continually asked the instructor questions and participated the most in class, Evan and Freda would only ask the instructor questions if they did not understand the lecture. Bob and Ian preferred to take notes and not ask questions or participate during class.

During the administration of the Post-Instructional Task-Based Questionnaire, all five novice participants - Bob, Evan, Freda, Greg, and Ian - indicated some growth in their ability to solve probability problems. Bob and Greg exhibited the largest growth overall. On the first questionnaire, both Bob and Greg used their prior knowledge of percents in their attempts to solve the unfamiliar problems. However, on the second questionnaire, both Bob and Greg improved on their ability to recognize set operations, to notice the differences between independent and dependent events, and to recognize the properties of probability. While Greg gained a proficiency of using representations (Venn diagram, tree diagram, and table) to solve the majority of the problems, Bob did not attempt to use any representations to organize and analyze the data on any of the problems, causing him to unsuccessfully solve the problems requiring set operations. Freda and Ian's attempts on the second questionnaire remained the same. However, while Freda was more successful organizing and interpreting problems using Venn and tree diagrams, Ian continued using his strong arithmetic reasoning skills to unsuccessfully complete the second questionnaire. Finally, Evan combined his original approach to solving problems using his personal interpretation while also trying to select the correct procedure for solving the problem. Evan did acknowledge that his difficulty solving the problems was his inability to interpret the terminology associated with probability.

Overall, from the administration of the Pre-Instructional Task-Based Questionnaire to the administration of the Post-Instructional Task-Based Questionnaire, only two novice participants, Bob and Greg, indicated a strong growth in their ability to solve the problems. In addition, Bob and Greg were the only novice participants who sought outside resources to help them with their homework. However, Evan and Freda did indicate some growth in their ability to solve probability problems. Both Evan and Freda sought the correct procedures to solve the problems, but did encounter difficulty organizing and analyzing the data. Finally, Ian did not indicate a change in his ability to solve probability problems. Ian used the same reasoning skills on the second questionnaire, while also solving more problems incorrectly on the second questionnaire than on the first.

The third change consisted of the "experienced" participants' ability to solve probability problems. Aaron, Charlie, Dennis, and Harriet each recalled enrolling in prior courses containing probability theory. Aaron and Harriet recalled learning probability in their high school mathematics courses, while Charlie recalled learning probability in his college level discrete mathematics course and Dennis recalled learning probability in his first attempt at finite mathematics. During the administration of the Pre-Instructional Task-Based Questionnaire, Aaron was the only participant to correctly answer all problems. However, Charlie, Dennis, and Harriet each encountered difficulty solving the problems. Charlie, the participant with the strongest mathematics background, created his individualized algorithm of estimating an answer, then manipulating the numbers to arrive at a solution similar to his initial guess. This algorithm caused him to encounter difficulty solving problems with set operations and dependent events. On the first questionnaire, Dennis tried to recall the procedures for solving probability problems, but easily was frustrated when he could not recall how to use the diagrams and formulas to obtain an answer. Finally, Harriet recalled formulas associated with each problem. However, her weak recollections of the properties of probability caused her to incorrectly produce the correct formula.

During the two-week instructional period, only one experienced participant, Dennis, attempted the assigned homework problems. In addition, Dennis sought help with his homework by visiting the tutors at the college's mathematics resource center. None of the four experienced participants attended all eight classes on probability. While Dennis and Harriet missed only one class, both Aaron and Charlie missed three classes. Finally, while Dennis continually asked the instructor questions and participated the most in class, Aaron and Charlie would only participate in class if the instructor called on them. Harriet appeared distracted during the lecture, preferring to organize her notebook than taking notes and participating in class.

During the administration of the Post-Instructional Task-Based Questionnaire, only two experienced participants, Aaron and Dennis, exhibited growth in their ability to solve probability problems. Both Aaron and Dennis exhibited refined problem solving techniques from their prerequisite knowledge. While Aaron was able to correctly solve all the problems on the first questionnaire, Aaron was able to articulate more of his problem solving process, incorporating more probabilistic terminology while solving the problems on the second questionnaire. Dennis exhibited the ability to use the Venn diagram correctly on the second questionnaire and was more competent using the procedures and formulas presented in class to solve the dependent events. However, Charlie and Harriet did not exhibit a growth in the ability to solve the probability problems. Charlie continued to use his algorithm of estimating an answer, then manipulating the numbers to arrive at a solution similar to his initial guess. Harriet exhibited a stronger understanding of the properties of probability, which allowed her to recall the correct formulas for solving the problem, while still encountering difficulty using the formulas.

Overall, these three main changes - content knowledge, novice participants' ability to solve probability problems, and experienced participant's ability to solve probability problems - were observed in all the participants from the administration of the Pre-Instructional Questionnaire, to the administration of the Post-Instructional Questionnaire. While Bob and Greg exhibited the largest growth in their ability to solve probability problems, Charlie continued using his algorithm of manipulating numbers to solve the problems. In addition, these results indicated that the participant with the strongest mathematical background - Charlie - was not the most successful participant. In fact, Bob, Dennis, and Greg indicated that the two-week exposure to probability, trying their assigned homework problems, and seeking outside help did change their approaches to solving probability problems. Unfortunately, some participants, such as Charlie, Evan, Harriet, and Ian, acknowledged they did not try their homework, nor seek outside help to understand the problems, thus were unsuccessful solving the problems on both the first and second questionnaire.

As stated earlier, the nine participants displayed nine levels of problem solving and probabilistic abilities. However, four of the participants - Aaron, Bob, Dennis, and

Greg - could be considered the more successful participants overall. Aaron was familiar with the probability problems posed on the two questionnaires and successfully solved both questionnaires. While Aaron sporadically attended class and did not try the homework problems, Bob, Dennis, and Greg attended class, tried homework problems, sought help outside the classroom, and asked questions during class. While these three more successful participants were not the most successful at solving the problems on the first questionnaire, their ability to solve the problems improved by the administration of the second questionnaire. Bob, Dennis, and Greg had the ability to solve the majority of the problems on the second questionnaire successfully.

One of these participants - Ian - could be considered a moderately successful participant. Ian entered the finite mathematics classroom with strong arithmetic reasoning skills, allowing him to be moderately successful at solving the probability problems on the first questionnaire. While Ian sporadically attended class, did not try the homework problems, and did not ask questions in class, he exhibited a stronger understanding of probability properties and terminology. On the administration of the second questionnaire, Ian still used his same arithmetic reasoning skills, but was able to integrate some of the newly learned concepts in his problem solving process.

Finally, four of these participants - Charlie, Evan, Freda, and Harriet - could be considered unsuccessful participants at solving the probability problems. These four unsuccessful participants entered the finite mathematics classroom with weak arithmetic reasoning and computational skills. These weaknesses caused them to be unsuccessful at solving the problems on the first questionnaire. In addition, these unsuccessful participants sporadically attended class and did not try the homework problems. However, Freda and Evan would ask questions during class and interrupt the instructor when they did not understand the lesson. On the administration of the second questionnaire, Charlie, Evan, Freda, and Harriet still tried to use their previous methods to solve the problems, still encountering difficulties with their arithmetic reasoning and computational skills.

Summary

The two goals of this study consisted of examining the methods used and the factors that supported or impeded their success as they solved probability problems. This chapter presented the results of a two-month study on college students enrolled in a college finite mathematics course. In response to these two goals, the two previous sections of this chapter provided an overview of the class and the participants and an overall summary of the methods and factors that supported or impeded their success as they solved probability problems. Therefore, the purpose of the final section of this chapter was to provide a summary of the results on the methods used and the factors that supported or impeded college student's success as they solving probability problems.

By observing the procedures and techniques the students used for attaining the solutions to the probability problems, not only did the students have their own style or method for solving probability problems, but it was also observed that the students had their own style or method when approaching a mathematics problem, regardless of the concepts embedded in the problem. While some methods were unique to some students, other methods were shared by more than one student. In addition, while the more successful student used similar methods, the less successful students used a variety of methods.

Overall, the methods modeled by the more successful students consisted of the student reading the entire problem, including the question; breaking down the problem into sections, analyzing each section separately; using the context of the question to reason a solution; and checking the final answer. While breaking down the problem, the successful students would also extract the numbers from the problem, select a reasonable representation for the problem (Venn diagram, tree diagram, table, or formula), and correctly label the numbers and diagram in the problem solving process.

However, this ideal method was not always successful. What made the method more or less successful were the factors contributing to their ability to obtain a solution. While some less successful students tried to use this approach when solving their problems, their inability to work with fractions and percents, to organize and analyze data in their own representation, and to relate the problem solving process to the context of the problem hindered their success solving the problem. The more successful students using this method to solve their problems also exhibited the ability to work with set operation terminology; to represent a probability as a decimal, fraction, or percent; to recognize the properties of probability; and to rephrase the problem or question to ensure they understood the question. In addition, the more successful student exhibited the discipline to attend the class, to participate in class discussions and ask questions, to try their homework problems throughout the section on probability, and to seek outside help when they did not understand a problem. Finally, some of the more successful students did encounter difficulties in their overall method. While some of the more successful students relied on their own procedures to determine which representation to use, other successful students relied on the wording of the problem to provide them a hint to the mathematical operation to be conducted on the numbers in the problem.

The previous successful word problem method could be used on all mathematics problems, regardless of the concepts embedded in the problem. In addition, it was not the method that determined if a student was successful solving the problem, what made the method more or less successful were the factors contributing to their ability to obtain a solution. However, the results from this study suggested students did try alternate methods when attempting to solve probability problems. One student interpreted probability in a yes/no fashion, providing answers to the problems based on his personal experience with the situation. Other students sought key words in the problem to prompt them to use a correct representation or formula, without evidence of the student trying to interpret the overall problem. While most students recognized dependent events, they encountered difficulty stating the probability of a dependent event due to their weakness in basic counting principles. For those students who had not encountered probability problems before the first questionnaire, some students were able to make connections between probability and percent, thus using the correct method, but not the correct interpretation. In addition, other inexperienced students thought the problems on the first questionnaire were trick questions that could not be solved mathematically. Finally, those students who had not seen probability problems before encountered difficulty interpreting the terminology associated with the problems. When the students encountered difficulties interpreting the problem or knowing the definitions of the terminology in the problem, the students would attempt various methods to solve the problem based on their interpretations or definitions.

Due to the design of the study, it cannot be assumed that instruction alone influenced the observed changes in the student's methods and factors over the two-week instructional period. While some students sought help outside the classroom environment, other students were already familiar with the material, and, yet, other students recalled the concepts on the two questionnaires. Nonetheless, the students approaches had changed from the administration of the first questionnaire to the administration of the second questionnaire. Some of these changes included clarification of students probabilistic reasoning and the mathematics behind it, increase confidence in organizing and representing data in a representation (Venn diagram, tree diagram, table, or formula), recognition of the properties of probability, and awareness of the difference between independent and dependent events. However, the students also exhibited different difficulties on the second questionnaire that were not evident on the first questionnaire. Some additional difficulties included students' confusion on the representation of the final answer: a percent, a decimal, or a fraction, inability to organize and analyze the data in a Venn or tree diagram, and viewing probability problems as a new concept requiring new set of procedures to learn in order to pass the class.

CHAPTER V

DISCUSSION

Introduction

Based on the current trends and directions of the research on the learning and teaching of probability, research should start shifting direction from observing and verifying predefined factors towards observing specific populations of students, searching for their unique methods of solving probability problems, and noting the factors which support or impede their success (Garfield & Ahlgren, 1988; Shaughnessy, 1992; Shaughnessy & Bergman, 1993). An extension of the research on the learning and teaching of probability as it relates to finite mathematics students provides the opportunity to re-examine and explore methods and factors affecting this student population as they learn about probability. Therefore, this study examined the methods used and the factors that supported or impeded college finite mathematics students' success as they solve probability problems.

The purpose of this chapter is to provide a discussion of the main findings, including a comparison to previous research, followed by the limitations of the study, implications for instruction, and recommendations for future research.

Discussion of the Main Findings

Problems involving probability or probabilistic reasoning, such as those typically encountered by finite mathematics students, clearly demand an appreciation of probability concepts and principles (Shaughnessy, 1992). More importantly, students' ability to successfully solve these kinds of problems requires an understanding of the terminology and procedures (i.e., equations, formulas, rules, and their interrelationships) used to represent these concepts, the processes necessary to solve the problem, and the ability to use probability as a way of thinking. This study sought to identify the methods and factors that supported or impeded college students' successes as they solved probability problems to provide understanding about improving instruction and learning in this domain. Once these common methods and influential factors are recognized, probabilistic research and instruction can begin to build upon this knowledge with the hope to develop better conceptual knowledge and reasoning processes leading to growth in students' competence in probabilistic problem solving.

From an educator's prospective, a student's understanding of probability problem solving is recognized by the ability to successfully work in the formal system of concepts and procedures which define this domain (Schoenfeld, 1985, 1992). Diagnosis of student difficulties in this area is a complicated task, as many factors are related to each other and attempts to remediate a single factor many not result in an improved ability in the student's overall method to solve different, or even similar types of problems. As a preliminary step towards diagnosing and remediation of students' difficulties in probability problem solving, this study has revealed some of the difficulties the nine participants encountered solving probability problems. The first set of difficulties the participants encountered were the methods they used for attaining a solution to the problem. These methods included the reliance on a procedure, the poor interpretations of probabilistic situations, and the incorrect use of a representation to organize and analyze the data. However, what made these and other methods more or less successful were the factors contributing to their ability to obtain a solution. These hindering factors included their arithmetic reasoning skills and the ability to work with percentages, their probabilistic linguistic skills, their misunderstanding of independent and dependent events, and their struggles to state a final answer.

First, the results of this study suggested that some participants relied on a procedure to solve the problem as their main method for solving the problem. Some of the students used their own unique procedure, while other students used procedures associated with other concepts. For example, some students recognized words implying the union or intersection of two events. Once recognizing the words and phrases, they immediately drew a Venn diagram and proceeded to fill in the diagram before reading the question. Other students recognized a concept being tested in the problem and proceeded to think of the formula associated with that concept. Finally, one student noticed that in Part A of a problem, the question was asking for the product of two numbers, thus, assuming Part B of the same problem was asking for the sum of the two numbers. In all three instances, the students incorrectly interpreted the diagram, wrote down the wrong formula, or assumed the wrong operation. Thus, they were relying on a procedure to follow in order to solve the problems. Prior research literature has acknowledged the possibility that students use procedural methods to obtain a solution rather than conceptual methods (Skemp, 1987; Greeno, 1987). In addition, while using these procedural methods, it is not clear how much a student understands the concepts behind the problem. The results of this study suggested some of the participants still relied on applying procedures without indicating a conceptual understanding of the problem, even after exposure to instruction on probability.

Second, the results from this study suggested that some participants encountered difficulty with interpreting probabilistic situations, causing them to rely on an incorrect interpretation. While some students interpreted probability as a right / wrong or win / lose situation, other students interpreted probability as the need to find the percent of the elements exhibiting the given condition. In a previous study, Konold (1989) explored college students' interpretation of probability problems. Konold conducted an informal interview presenting the students with three questions covering various aspects of probability. Konold concluded that students would approach a problem in one of two different methods: frequentist approach or outcome approach. The frequentist approach consisted of students interpreting probabilistic situations as the outcomes of repeatable situations with frequencies. In the outcome approach, some students' predictions of single trials took the form of "yes" or "no" decisions of whether a particular outcome will occur. This outcome approach for interpreting probability situations suggested by Konold was evident in some of the participants in this study.

Third, the results from this study suggested that some participants tried to use alternative representations to organize and analyze the data as their method for solving the problems. Some of the problems required the students to work with set operations. When presented a problem requiring set operations, various representations allow the student to organize and analyze the data: Venn diagram, tree diagrams, tables, and formulas. However, most students chose not to organize and analyze data in a set operation problem, causing them to count a set twice or to delete a set altogether. Ballard (2000) observed similar difficulties with finite mathematics students solving set operation problems. Ballard observed the more successful student appeared competent organizing and analyzing set operation problems using Venn diagrams, while the less successful student encountered difficulty translating English statements into Venn diagrams. Surprisingly, Ballard also observed that the less successful student did not encounter difficulty translating a Venn diagram into an English statement. In this present study, it was also observed that most of the students were successful reading and interpreting the tables provided to them. Therefore, the results of these studies suggested that while students have the ability to read data from a diagram, they still encountered difficulty organizing and analyzing data when they needed to provide their own representations.

However, some of these methods were not necessarily "the most successful" approach to solving these problems. In fact, the less successful participants used certain methods to solve the probability problems that were sometimes different from the methods the more successful participants used while solving the problems. As stated earlier, it was observed that each participant had their own unique word problem method, but what might make them more or less successful at their method could be the factors contributing to their ability to obtain a solution.

First, the results from this study suggested that some participants encountered difficulty with basic arithmetic skills and ability to work with percentages, factors contributing to their inability to solve the problem. Some students exhibited difficulty adding, subtracting, and comparing fractions, and working with percents in a problem. While one student added up the denominators of a fraction in order to find a common denominator, another student was unable to distinguish the largest fraction in a set. In addition, when working with percents in a problem, many students were unable to solve a problem because they wanted to know the number of elements in the problem, not the percent of elements. Oddly, one student's method for solving problems appeared ideal: read the entire problem, estimate an answer, compute an answer, and compare the two results. Unfortunately, due to his poor arithmetic skills, he was unable to recognize his arithmetic mistakes and to correctly solve problems. O'Connell (1999) found similar results in her study. O'Connell sought the interrelationships among different types of mathematical errors occurring while college students solved probability problems. In her study, O'Connell analyzed the college students' written solutions, observing the relationship between the presence and type of errors observed in their work. O'Connell concluded there was a relationship between students' arithmetic errors and the ability to use and interpret a formula. In addition, those students committing these arithmetic errors also encountered difficulty with their problem solving strategy and use of an appropriate method to solve the problem. However, the results from Pollatsek, et al. (1987) suggested college students did not encounter difficulty solving probability problems when stated either as a percent or as a probability. In this study, Pollatsek et al. administered two force-answer multiple choice tests to two different groups, one test was written using decimals to represent the probability of an event, the other test was written using percents to represent the probability of an event. Pollatsek et al. observed the two groups - decimal representation and percentage representation - indicated an equal performance in their ability to solve probability problems. The results from these studies suggested college students still encounter difficulty with basic arithmetic skills, preventing them from interpreting new situations and learning new concepts. However, the results of this study indicated some participants encountered difficulty recognizing the difference between decimal and percentage representation of a probability, which conflicts with findings of Pollatsek, et al.

Second, the results from this study suggested some participants encountered difficulty with the language of probability terminology, a factor contributing to their inability to solve the problem. Students encountered difficulty interpreting words associated with probability: random, equally likely, fair. One student interpreted "equally likely" to mean each event had an equal chance to be selected, regardless of the proportion of elements in each event. As discussed in Chapter II, the linguistic ambiguities of the word "probability" stemmed from associations with divine judgments or authoritative decisions, with no definite predictive ability (Borovcnik et al., 1991). In addition, the word probability was connected with the old interpretation of probability based upon the association of opinion and aspects of low science (Hacking, 1975). Finally, in everyday language, the words "probably," "possibly," and "surely" are associated with potentially quantifiable terms when in a sentence indicating the event will happen, such as, "I will probably go to the meeting tonight" (Cohen, 1957). Based on these linguistic ambiguities with the word "probability," students might be encountering similar difficulties with the words associated with probability. A study conducted by Fischbein et al. (1991) on elementary and secondary Italian students suggested the students did not understand the words "possible," "impossible," and "certain." In particular, some of the students associated the word "impossible" with "rare." The terminology used in probabilistic situations is also found in everyday language. However, the results from these studies suggested the students might not have a clear definition of the terminology, especially when associated specifically with probabilistic situations.

Third, the results from this study suggested some participants encountered difficulty understanding independent and dependent events, a factor contributing to their inability to solve the problem. This study explored various representations if independent and dependent events. Ranging from students' abilities to read and select the data from a table, to students solving dependent events requiring sequential selection with or without replacement, to students' ability to recognize the dependency in the wording of a problem. While solving problems requiring them to select the data from a table, the majority of the participants had the ability to read the table correctly to find the size of the event, but encountered difficulty deciding the size of the sample space. When solving problems requiring sequential selection with or without replacement, some participants wanted to know the outcome of the first selection before attempting to find the probability of the second outcome. Ironically, these same students knew that the sample size decreased on the second outcome, when there was no replacement. Finally, some participants encountered difficulty recognizing dependency in the wording of the problem. They were unable to recognize that one group was a sub-group of the original sample.

Results from previous studies suggested that elementary students (Jones, et al., 1997, 1998), secondary students (Castro, 1998; Fischbein & Gazit, 1984; Tarr and Jones, 1997), and college students (Pollatsek, et al., 1987; O'Connell, 1999) all have the fundamental ability to work with dependent events. However, these studies also suggested areas in which college students encountered difficulty understanding independent and dependent events. O'Connell (1999) sought the interrelationships among different types of mathematical errors in college students' written solutions to probability problems. Analyzing the college students' written work, O'Connell concluded there was a relationship in students' misconceptions and incorrect procedural knowledge while solving independent events, mutually exclusive events, and dependent events requiring sequential selection with or without replacement. Pollatsek, et al. (1987) found that when college students solved dependent events, they encountered difficulty with certain factors such as wording and unfamiliarity with the problem. Therefore, the results from these studies suggest college students have the fundamental ability to work with independent and dependent probability situations, but do encounter difficulty interpreting and solving the problems mainly due to their unfamiliarity.

Finally, the results from this study suggested some participants encountered difficulty stating a final answer, a factor contributing to their inability to solve the problem. Due to the methodology of the study, the verbal responses to the questions did not provide clear enough data to distinguish when a student was using probabilistic reasoning or non-probabilistic reasoning to reach the final solution. However, the students did encounter difficulty trying to state the final answer in probabilistic terminology, indicating difficulty understanding the problem. The students would state their final answer with "percent chance," "percent," or "chance" after their numerical result; state their answer as ratio; or state their answer as just a number. It was not clear whether the students understood the process to reach the final answer, or if they did not understand the problem. It also was not always apparent whether the students were merely using a procedure to calculate a number or were correctly using probability reasoning to determine a solution and then stating a number.

What has emerged from the results of this study and the results of prior studies was that there are no simple explanations as to how students approach and solve probability problems. Solving probability problems does not consist of reading a problem and following a procedure leading to a solution. As observed in this study, the participants had their own unique word problem methods, but what made them more or less successful at their methods were the factors contributing to their ability to obtain a solution. The purpose of this discussion was to begin groundwork towards diagnosing and remediation of student difficulties while solving probability problems. These three methods - reliance on a procedure, poor interpretation of a probabilistic situation, and the incorrect use of a representation - and four factors - basic arithmetic skills, language of probability terminology, recognition between independent and dependent events, and the statement of the final answer - presented above did hinder the participants ability to solve the probability problems. However, it was observed that these factors, when used successfully in their individualized word problem method, could help students solve probability problems.

Limitations of the Study

The purpose of this section was to discuss the limitations of this study. Some limitations included the selection of participants, the data collection instruments and procedures, the design and analysis of the data, and the performance of the researcher.

One limitation to the study was the selection of participants for the study. The population for the study consisted of 24 students and one instructor of a 10-week finite mathematics course offered at a local community college. From this population, nine students volunteered to participate in the interviews. The sample size was acceptable for the research method and design of individual case studies. However, since the nine participants volunteered for the study without conducting purposeful sampling, they do not necessarily represent the class. The nine participants consisted of seven American male students and two African female students. In addition, the selection of the class and the instructor was due to the availability of a finite mathematics class and the convenience of the researcher. Relying on the instruction and instructional philosophy of one instructor kept the case study confined to one finite mathematics classroom. However, the goal of a case study is to observe a "bounded system" (Merriam, 1998). Therefore, this case study consisted of observing the nine participants and one instructor bounded in this one finite mathematics classroom.

Another limitation to the study was the data collection instruments acting as a treatment and procedures for collecting the data. As mentioned previously, the development of the two Task-Based Questionnaires may have been influenced by the bias of the researcher. In addition, the implementing of a pre-test / post-test design with structured interviews created several implications to the design of the study. The exposure to a pre-test may affect the participant's performance in the class and on the post-test. This administration of a pre-test may sensitize students to respond to the treatment a different way than they would if they had not been pre-tested. This is referred to as pre-test sensitization, which is a potential external-validity problem.

In addition, the administrative procedures and the length of the two questionnaires caused anxiety and frustration for some of the participants. Each of the 14 problems was presented to the students on a separate piece of paper. The first question provided them a chance to practice their verbal protocol procedures. Therefore, the students completed the remaining 13 without intervention from the researcher. However, some participants were apprehensive over the size of the pile of questions, while other participants wanted "human contact" and reassurance on their problem solving protocols. Three participants were unable to complete the first questionnaire, due to their frustration with the problems. One participant was unable to complete the second questionnaire due to a time conflict. Although interviews provided valuable data, just as all data, this data collection instrument could be susceptible to bias, to pre-test sensitization, and to the failure to meet the mathematical abilities of the participants.

A third limitation to the study was the design and analysis of the study. The framework for the use of case study analysis does have some limitations in its design (Gall, et al., 1996; Krathwohl, 1997; Merriam, 1998). The goal behind the framework was to conduct a within-case analysis of the methods the nine participants used and the factors that supported or impeded their success, then to conduct a cross-case analysis. One limitation to the design and analysis of the study was the selection of the information to be coded. Transcripts of the verbal data provided manageable distinct data, but the information about the behavior and environment was just as important. The physical environment, the participant or interviewer's behavior, and the overall tone of the interviewing process must also become part of the distinct episodes. A second limitation to the design and analysis of the study was defining and refining the categories, considering the breadth and width of the categorical definitions. The more complex or implicit the categorical definitions, the more difficult it is to gather evidence on it. In addition, with not enough data collected, the categories might not be well established. A third limitation to the design and analysis of the study was the singular episodes or participants that do not fit into any of the defined categories, i.e. outliers. Extreme cases might provide unique insights to the question of interest, but need to be verified and ensured there is no disconfirming evidence before analyzing the outlier. By analyzing behavior purely on transcribed verbal data, it was possible to misinterpret what actually took place in the problem sessions.

A fourth limitation of the study was the performance of the researcher. The researcher developed the data collection instruments, collected the data, and analyzed the data. Due to the background and experience of the researcher, the researcher could have led to unintentional bias in the development of the data collection instrument, and in the collection and analysis of the data overall. Precautions were taken to minimize any introduction to bias. However, using various data collection techniques, continual triangulation of the data, and awareness of the pitfalls as was done throughout this study, the results of the data analysis can be strengthened.

Implications for Instruction

The results of this study have implications for mathematics and science education at all levels, but specifically to finite mathematics level courses including the study of probability. It was observed that the nine participants in this study had their own unique word problem methods, but what made them more or less successful at their overall method were the factors contributing to their ability to obtain a solution. This study highlighted some of the factors for which students encountered difficulty solving probability problems. Therefore, this section provides suggestions for instructors who teach probability and want to improve both their students ability to interpret probabilistic situations and to solve probability problems.

First, several participants encountered difficulty interpreting the probabilistic statements in the problem: equally likely, equal chance, fair, random, and without replacement. Using their own interpretation of these terms, some participants assumed that "equally likely" meant that each event in the experiment had an equal chance of being selected. Similarly, students struggled with the effects to the sample space and the probability of an event when an outcome was not replaced. Finally, some students interpreted the results as a "yes / no" outcome. Perhaps instructors could be more attentive towards presenting more statistical definitions and interpretations of the concepts used within probability, while also providing the opportunity for students to interpret various situations using the new terminology.

Second, the participants in this study approached the problems using their own unique word problem methods. However, instead of trying to interpret problems, most of the participants were searching for key words to trigger an association with mathematical operations: of, and, is, both, or, etc. Unfortunately, these key words do not have a unique mathematical operation associated with them. For example, the word "and" could be a trigger word to imply the multiplication of two numbers or it could indicate the intersection of two sets. Perhaps instructors could demonstrate alternate examples of the times these main key words trigger more than one mathematical operation and to discuss how to interpret the problem, leading to stronger and sounder methods to solve the problem.

Third, the participants encountered difficulty determining the number of elements in the event or the sample space. This difficulty ranged from their inability to work with basic concepts of combinatorics, to recognize the impact of dependent events on the sample space, and to interpret set operation terminology. Without an ability to work with the basic counting principles, students will encounter more difficulty computing the probability of an event. Some finite mathematics curriculum introduces the basic counting principles in the proceeding chapter. In addition, this might be the first time the students were exposed to basic counting principles. Students might not see the connection between the basic counting principles and probability. Perhaps instructors could emphasize the implications of the basic counting principles in the probability problems.

Fourth, for those participants who employed alternate representations for various problems, about half of them encountered difficulty organizing and analyzing the data in the representation. In addition, one participant thought Venn diagrams were to be used solely for set operations problems, tables for frequency distributions, and tree diagrams for dependent events. A possible solution to this confusion is for instructors to stress that each problem can be modeled by at least four representations: a Venn diagram, a tree diagram, a table, and / or a formula. In addition, instructors could incorporate the inclusion of all four representations in their instruction, allowing the students opportunity to organize and analyze the problem using each representation, to decide which representations can be used to solve the problem, and to select the most effective representation.

Fifth, half of the participants encountered difficulty interpreting a problem that provided the percents of the elements in an event, as opposed to the frequency or probability of the event. While some of the participants wanted to know the number of elements in the entire set before attempting the problem, others did not recognize the properties associated with percents. In addition, one participant associated probabilities with the mathematical operation of multiplication. Students bring their misconceptions of percents into the classroom. This does not mean that instructors need to "re-teach" their students percents before proceeding to the chapter on probability. However, instructors could remind their students of the three ways to represent the proportion of elements in an event: percents, decimals, and fractions. In addition, instructors could interchange the representations in a problem, allowing students to understand the connection and to gain more skills working with various representations.

Finally, several participants encountered difficulty trying to select the proper representation of the probability of an event: as a percent, as a decimal, or as a fraction. This difficulty on recognizing the differences in the proper representation of the probability of the event could create problems when the students try to interpret the difference between "90% chance of rain today" and "out of 10 days in which the weather patterns were similar to today, it rained 90% of the time". Perhaps the instructor could discuss with their students the interpretation of all three representations when stated in a probability problem.

Overall, when teaching probability, the students need to be able to recognize when chance is at work, and that the chance of something happening also entails the chance of it not happening. In order to reach this objective, perhaps more time is needed on discussing or interpreting the concepts of probability. However, it is not uncommon to see a teacher pushing the students all too quickly into the traditional route of manipulation of probabilities then churning out "right answers", perhaps in the hope that understand would follow later.

Recommendations for Future Studies

This study highlighted possible methods that college students use and the factors which support or impede their success solving probability problems. The findings from this study also suggested other areas for further research.

First, the small sample for this study was appropriate, but the findings from a larger group of students could produce trends that were not evident with this group. In addition, a larger group could give more insight to the prevalence of the resulting trends. Additionally, testing a larger, random sample of students on the methods and factors that emerged from this study could confirm the findings of this study.

Second, two of the participants exhibited a two-sided interpretation of probability. These two participants interpreted the answers to the theoretical probability problems a "yes / no "; "win / lose" response. Similar findings by Konold (1989) indicated some college students interpreted probabilistic situations in this "outcome approach". However, in Konold's study, the students were interpreting probabilistic statement "70% chance of rain" into the more definitive statement, "It's going to rain" (p. 68). However, due to the design of this study, it was not possible to question the two participant's views and interpretations on probability. Further research may consider further exploration pertaining to the similarities and differences in college students interpretations of probability situations and theoretical probability situations.

Third, the majority of the students in this study exhibited difficulty working with fractions and percents. While some students did not know how to find a common denominator when adding fractions, other students did not recognize the properties of percents. Similar results from the Seventh Mathematical Assessment of the National Assessment of Educational Progress [NAEP] (NCTM, 2000b) and the Third International Mathematics and Science Study [TIMSS] (Beaton, et al., 1996) suggested that elementary and secondary level students were generally weak in rational number concepts and had difficulties with basic concepts involving fractions, decimals and percents. A low percent of students had correct responses to exercises involving complex concepts and skills requiring undersigning of underlying mathematical principles. As these secondary students enter college level mathematics, they bring with them the same difficulties with fractions, decimals, and percents. Surprisingly, the students in this study had attained the skills to pass a college algebra course before entering the finite mathematics classroom. Future studies could focus on the reasonableness and need of incorporating strategies to improve students rational number concepts either before or within the finite mathematics classroom prior to instruction on probability.

Fourth, as mentioned above, the results from this study suggested students encountered difficulty determining the number of elements in the event or sample space. This difficulty ranged from the inability to work with basic concepts of combinatorics, to recognizing the influence of dependent events on the event and sample size, to interpreting set operational terminology. Without an ability to work with the basic counting principles, students will encounter more difficulty computing the probability of an event. Further investigation may consider the factors that support or impede student's success working with basic counting principles.

Fifth, the NCTM, AMATYC, MAA recommend introducing probabilistic topics through activities and simulations, not abstractions. In addition, these groups stress real-life applications requiring probabilistic reasoning, while also asking for student's reactions and interpretations of the application. These are all suggestions from professional organizations. The literature pertaining to the teaching and learning or probability makes it clear that far more research has been done on the psychology of probability than on students' interpretation and understanding of probabilistic situations. It is not certain that these recommendations help students overcome their misinterpretation of probabilistic situations, their difficulty with the language ambiguities embedded in the language of probability, and their difficulty interpreting the end product of their calculations. Those studies conducted on the effects of various instructional methods on college students were conducted in an experimental research design (Austin, 1974; Shaughnessy, 1977). Without a baseline study on the affects of current instructional methods on student learning, it was difficult to conclude that the experimental instructional methods improved student probabilistic thinking and problem solving abilities more than those students attending classes using current probability instructional methods. Future studies are needed on classroom observations of current instructional methods and their effect on student probabilistic thinking.

Sixth, the design of the study did not have the capabilities to observe if the students conceptually understood the problem or the problem solving method they chose to solve the problem. To most of the students, this was their first exposure to set and probability theory. It was not clear whether they understood these new concepts or were just mimicking the procedures presented in class. This study was not designed to observe if there was evidence of the student understanding in the process. The data only provided the student's verbal response as they solved the problems. From this data, it was evident that some students were searching for key words associated with mathematical operations. Other students knew they could use Venn diagrams, tree diagrams, and tables to help organize the data. However, it was not evident from the data if the students understood the probabilistic situation or if the student used probabilistic reasoning to solve the problem. Future studies on students understanding of probabilistic situations could focus more in-depth on student's conceptualization of probabilistic thinking.

Finally, some of the participants were successful solving probability problems. These participants also exhibited an understanding of probabilistic events, strong mathematical reasoning skills, and the ability to solve probability problems. Future studies are needed on focusing on these more successful students and what factors made them successful at solving probability problems.

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APPENDICES

APPENDIX A

Probability Terminology

Availability: Judgmental heuristic associated with subjective probability. Tversky and Kahneman (1973) described a person who uses availability as one who evaluates the probability of an event by the ease with which relevant instances come to mind. For example, suppose a word is randomly picked from an English Dictionary. Is it more likely that the word begins with the letter K, or that K is its third letter? Availability tells the naïve person, one who has not had formal education in probability, that since it is much easier to think of words starting with K than of words in which K is the third letter, they would believe the word is more likely to start with K. Unfortunately, in the English language, there are about twice as many words with K in the third position than in the first (Tversky & Kahneman, 1973).

Causal Bias: Judgmental heuristic associated with subjective probability, specifically, conditional probability. A causal scheme follows a course of cause to consequence. However, when people are faced with finding the probability of an uncertain causal event, they may find it easier to invert this sequence and reason from consequence to cause. Research conducted by Tversky and Kahneman (1980) tested this hypothesis by asking people which is more probable: $P(X|Y)$ or $P(Y|X)$ when X is the natural cause of Y and $P(X) = P(Y)$. In this study, the majority of the subjects answered $P(Y|X) > P(X|Y)$. An example used in their study was Problem II (Tversky & Kahneman, 1980, p. 51):

Which of the following is more probable:

a. That a girl has blue eyes if her mother has blue eyes
b. That the mother has blue eyes, if her daughter has blue eyes
c. The two events are equally probable

The correct answer being (c) - P(Girl has blue eyes) = P(Mother has blue eyes), since the distribution of eye color is essentially the same in successive generations. Those who illustrate causal bias would select option (a) which is P(Girl has blue eyes | Mother has blue eyes) or showing probability of consequence to cause.

Classical Theory: Interpretation of Probability. The probability of an outcome is simply the ratio of the number of favorable possibilities to the total number of basic alternatives. For example, finding the probability of rolling a 2 on a six-sided fair die is 1 / 6. This interpretation of probability was prevalent until the time of Bernoulli, due to the assumption, and major limitation, that all outcome alternatives must be equally likely. (Barnett, 1999)

Combinatorics: The branch of mathematics devoted to the study of permutations and combinations, counting techniques determining the number of elements in a set (Barnett, et al., 2002).

Complement of a Probabilistic Event: See Compound Probabilistic Events

Compound Probabilistic Events: Since events are subsets of a sample space, it is possible to use set operations to form other events. The use of the set operations to find the probability of that event is called the compound probabilistic event. Suppose there were two events, E and F, in a sample space S, then it is possible to compute the probability of the union of the two events, noted by P(E or F); the intersection of the two events, noted by P(E and F); or the complement of one of the events, noted as P(E'). Example: In a remote jungle village the probability of a child contracting malaria is .45, the probability of contracting measles is .65, and the probability of contracting both is .20. Hence P(malaria and measles) = .20; P(malaria or measles) = .90; P(not malaria) = .55 (Rolf, 2001).

Conditional Probabilistic Events (Dependent Probabilistic Events): The probability of an event that varies depending upon the occurrence or nonoccurrence of one or more related events. In mathematical notation, the translation of "the probability of event A occurring given that event B has occurred" is $P(A|B) = P(A \text{ and } B) / P(B)$. For example, Oregon sport fishermen are vitally interested in the probability of rain. The probability of rain on a given day, ignoring the daily atmospheric conditions or any other events, is the fraction of days in which rain occurs over a long period. This would be called "unconditional probability." Consider the chance of it raining tomorrow. It has rained almost continuously for two days and a storm is heading up the coast. This probability is conditional on the occurrence of several events, and an Oregonian would tell you that it is much larger than the unconditional probability of rain. Thus, the "conditional probability" of an event is the probability of the event given the fact that one or more events have already occurred (Rolf, 2001).

Conjunction: A compound probabilistic event in which the probability is based on the union of two events. This is denoted by P(E or F),where E and F are two events from the same sample space. Note: "E or F" is the event consisting of those outcomes that are in E or in F or both. See Compound Probabilistic Events for an example (Rolf, 2001).

Conjunction Fallacy: Judgmental heuristic associated with subjective probability. The conjunction fallacy stems from the extension rule of the Law of Probability: If $A \supset B$, then $P(A) \geq P(B)$. Since the set of possibilities associated with the conjunction (A and B) is included in the set of possibilities associated with B, the same principle can also be expressed by the conjunction rule: $P(A \text{ and } B) \leq P(B)$. However, Tversky and Kahneman (1983) found that when a person was given an uncertain event involving conjunctions, people tend to use the representativeness and availability heuristics to make a conjunction appear more probable. One of their studies showed that 85-90% of their subjects violated the conjunction rule of probability. This was illustrated when after people were given a description of a fictitious female character, who is "bright, single, 31 years old, outspoken, and concerned with issues of social justice", the subjects were more likely to believe that the person was a bank teller *and* was active in the feminist movement, than that the person was just a bank teller.

Experiment (in Probability): A procedure that has the same possible outcomes every time it is repeated but for which no single outcome is predictable. For example, rolling a die, selecting a card from a deck of cards, or flipping a coin are all possible experiments (Rolf, 2001).

Dependent Probabilistic Events: See Conditional Probabilistic Events

Event (in Probability): An event is a subset of the sample space of an experiment. We say that an *event occurs* when any of the outcomes of the event occurs. For example, consider the experiment of rolling a die. The event of rolling an odd number is the subset $\{1, 3, 5\}$, the event of rolling an even number is the subset $\{2, 4, 6\}$; and the event of rolling a prime number is the subset $\{2, 3, 5\}$ (Rolf, 2001).

Falk Phenomenon: See Time-Axis Fallacy.

Frequentist Theory: Interpretation of probability. The probability of an event is its limiting frequency of occurrence in an infinite series of trials. Thus, the probability of rolling a 3 on a particular die is the relative number of occurrences if the die were rolled an infinite number of times. Venn formalized and further refined this interpretation of probability to include non-equally likely events. Hence, the frequentist interpretation of probability, the ability to estimate probabilities on repeatable situations with frequencies, provided the needed conditions to consider independent, mutually exclusive, and conditional events (Barnett, 1999; Borovcnik et al., 1991).

Gambler's Fallacy: See Positive and Negative Recency Effects

Independent Probabilistic Events: Two events that can both occur at the same time; but the occurrence of one event does not affect the chance of the occurrence or nonoccurrence of the other event. For example, the first toss of a coin has no influence on the outcome of the second toss. However, if a person selects a card from a deck of cards, without replacement, this does have an effect on the chance of the selection of the second card; hence, selecting a card from a deck of cards without replacement is not an independent event, while the tossing of a coin is an independent event (Rolf, 2001).

Intersection of Two Probabilistic Events: See Compound Probabilistic Events

Intuitive Cognition: See Misconceptions: Intuitive vs. Non-Intuitive Cognition

Judgmental Heuristics: Suppose one is faced with determining the outcome of an election, the guilt of a defendant, or the chance of winning at the roulette table. If this person has no exposure to knowledge of chance, or the statistical theory of prediction, they will try to reduce the complex tasks of assessing probabilities and predicting values to simpler judgmental operations. The reduction of complex tasks of assessing probabilities and predicting values to simpler judgmental operations has been defined in research on the understanding of probability as probabilistic judgmental heuristics (Tversky & Kahneman, 1974). These heuristics, which sometimes yield reasonable judgements, can also lead to severe and systematic errors (Kahneman & Tversky, 1972, 1973; Tversky & Kahneman, 1973). Six main judgmental heuristics are found in the research literature on the teaching and learning of probability: Availability, Positive and Negative Recency Effects (Gambler's Fallacy), Conjunction Fallacy, Representativeness, Time-Axis Fallacy (Falk Phenomenon), and Causal Bias.

Misconceptions: Intuitive vs. Non-Intuitive Cognition: The articles reviewed for this paper appear to interchange the words "conceptions" and "intuitions." At first glance, it may appear that these two words are the same; however, as observed in the review of literature on the difficulties encountered solving probability problems, a concept can fall under one of two categories: intuitive and non-intuitive concepts (Fischbein, 1975). Thus, when a student is asked, "What is the shortest distance between two points", intuitively, they may reply "a straight line." This intuitive response indicates a feeling of obviousness, of intrinsic certainty, and a subjective answer, which appears to be absolutely true without the need of a formal proof. As opposed to the elicited question, "Given the coordinates of two points on the Cartesian Plane, what is the numerical distance between the two points", the teacher may be looking for a more non-intuitive response of the distance formula. This non-intuitive response indicates the student required some background knowledge to derive a solution, which is analytically and logically based. In either case, a student can have "mis-intutions" or "mis-non-intuitions" which may be also regarded as "misconceptions" in the research on difficulties students encounter solving probability problems.

Mutually Exclusive Events: Two events that cannot both occur at the same time.

Negative Recency Effects: See Positive and Negative Recency Effects.

Odds: The ratio of the probability of an event's happening to the probability of its not happening. The odds can be calculated as either the odds *for* or the odds *against* the event occurring. For example, If you roll a fair die once, the probability of rolling a 4 is 1/6, whereas the odds in favor of rolling a 4 are 1 to 5, and the odds against rolling a 4 are 5 to 1 (Barnett, et al., 2002).

Positive and Negative Recency Effects (Gambler's Fallacy): Judgmental heuristic associated with subjective probability. Some researchers of probabilistic heuristics classify positive and negative recency effects as subcategories of representativeness. However, the research conducted on recency effects took place before the recognition of the representativeness heuristic (Cohen, 1957, 1960). Recency occurs when a person is uncertain how to calculate the outcome of the next event, given the results of the previous independent trials. For example, a person using the positive recency heuristic when predicting a head or tail on a flip of a coin tends to believe that after a run of heads, a head is more likely to occur in the next toss. Thus, the positive recency effect causes the person to assume incorrectly that the conditions were not fair. A person using a negative recency heuristic when predicting a head or tail on a flip of a coin tends to believe that after a run of heads, a tail is more likely to occur in the next toss. Thus, the negative recency effect causes the person to believe intuitively that the alternating outcomes seem to better represent a random sequence. The idea of negative recency effect has also been known as "Gambler's Fallacy", in which the gambler believes the events will balance at the end.

Probability: A ratio expressing the chance or likelihood that a certain event will occur, given the number of possible outcomes of an experiment. In mathematical notation, the probability of event A occurring is written P(A), where $0 \leq P(A) \leq 1$. For the purpose of this paper, *probability* refers to the introductory mathematical concepts found in an elementary probability course recommended by NCTM (2000a), AMATYC (1995) and MAA (Sons, 1996): sample space, simple probabilistic events, compound probabilistic events, dependent probabilistic events, and independent probabilistic events. Probability can be expressed as a common fraction, decimal fraction, or a percent.

The Properties of Probability:

a. The probability of an event is a number between 0 and 1, inclusive
b. The sum of the probabilities of all events in the sample space is 1
c. The probability that an event does not occur is 1 minus the probability that the event does occur

Representativeness: Judgmental heuristic associated with subjective probability. A person who follows this judgmental heuristic estimates that the probability of an uncertain event is based on how well an outcome represents some aspect of its parent population, or how the event reflects the prominent features of the process by which it is generated. Tversky and Kahneman (1982) pursued their interest in representativeness to define six subcategories: insensitivity to prior probability of outcomes, insensitively to sample space, misconceptions of chance, insensitivity to predictability, illusion of validity, and misconceptions of regression. For example, the representativeness of misconceptions of chance state that people expect that a sequence of events generated by a random process will represent the essential characteristics of that process even when the sequence is short. People believing that the sequence of flipping a coin five times and obtaining H-T-H-T-H is more likely than H-H-H-T-T, or even the sequence H-H-H-T-H, easily illustrate misconception of chance (Kahneman and Tversky, 1972).

Sample Space: The set of all possible outcomes in an experiment. For example, when flipping a coin, the sample space consists of heads and tails; whereas, rolling a die, the sample space consists of the numbers one through six (Rolf, 2001).

Simple Probabilistic Events: Computing the probability of an event, given the outcomes in the sample space S are equally likely to occur. Let E be the event. Then the probability of an event E, denoted $P(E)$, is

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}$$

Structural Theory: Interpretation of probability. All probabilistic situations have a formal and sound mathematical foundation by which one can precisely calculate the situation by using the mathematical laws of probability. Structural theory is the ability to apply a body of definitions and theories and their corresponding system of axioms to all forms of probabilistic situations. Kolmogorov is credited with providing the Structural Theory of Probability in 1933 (Borovcnik et al., 1991).

Subjective Probability: Acting on a probabilistic event with incomplete or unsure knowledge of the event. The actions are based upon private assessment of chances, not mathematical assessment, which in turn depends upon past experiences and maturity in reasoning skills (Cohen, 1957). Availability, Conjunction Fallacy, Positive and Negative Recency Effects, and Representativeness are all examples of Subjective Probability.

Subjective Theory: Interpretation of probability. All probability is a matter of personal judgement, and is thus subjective (Barnett, 1999).

Time-Axis Fallacy (Falk Phenomenon): The time-axis fallacy is the most prominent judgmental heuristic of conditional probability. Falk (1986, 1988) recognized that when a person is given a conditional probability situation, and asked about the probability of the first event happening, given the second has occurred, they have a difficult time going "back in time" to comprehend the question correctly. This heuristic is illustrated in Problem I (Falk 1988, p. 292):

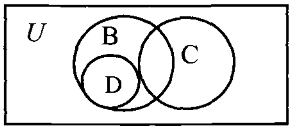
> An urn contains two white balls and two black balls. We blindly draw two balls, one after the other *without* replacement from that urn.
>
> a. What is the probability that the second ball was white, given the first was also white?
> b. What is the probability that the first ball was white, given that the second ball was also white?

Falk (1983, 1989b) found that the subjects of her study were able to answer part (a) correctly - one third - however, some subjects did not believe part (b) had an answer. The subjects argued that the probability of an outcome of a draw on an event that occurs later is not permissible. Others argued that since the first ball does not care whether the second is black or white, the answer will be one half. Hence, those who use the time-axis heuristic want to compute the probability of an event occurring at the immediate point of time at which the event takes place.

Union of Two Probabilistic Events: See Compound Probabilistic Events

Venn Diagrams: Geometric figures used to picture sets, set relations, and provide a useful method of counting techniques (Barnett, et al., 2002)

Example: Let U = {students at Lincoln High School}
B = {band members at Lincoln High School}
D = {band members at Lincoln High School who play drums}
C = {chorus members at Lincoln High School}



APPENDIX B

Informed Consent Forms and Background Information Sheet

Probabilistic Thinking in College Students
Research Consent Form
(Class Participation)

By signing this form below, I attest to the following:

1. I understand that I am a participating in a research study. The purpose of the research is to examine the methods college students use to solve probability problems and factors which influence the ability to solve the problems. My participation will consist of providing the researcher with biographical information about myself and allowing the researcher to observe instruction in my course.
2. I understand that my participation in this study is voluntary, and that I may withdraw my participation at any time with no penalty.
3. The researcher has explained the purpose and procedures of this research study, and I have been given an opportunity to receive answers to my questions.
4. I understand that the researcher will keep my responses confidential and will destroy all records at the completion of the research.
5. I understand that I will not receive any compensation for my participation in this study.
6. I understand that the results of the inventory tests will not affect my grade.

Questions concerning this research should be directed to either:
Mary Bamberger at (541) XXX-XXXX or bamberg@orst.edu
Dianne Erickson at: (541) XXX-XXXX or erickso@orst.edu

Questions concerning your rights as a human subject should be directed to the IRB Coordinator, OSU Research Office, (541) 737-3437

My signature below indicates that I have read and that I understand the procedures described above and give my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

______________________________ __________________
Name (printed) date

______________________________ __________________
Signature Signature of Researcher

Biographical Information Sheet

Age ___________

Gender __________

Major (s) _______________________

Please list all previous and current mathematics courses that you have taken (include high school and other college courses). If possible, please include the grade which you earned for that course:

Please list all previous and current courses that you have taken (including mathematics courses) in which you have learned about probability (include high school and other college courses).

Please list as many probability and statistics concepts that you remember learning in your previous and current courses (you do not necessarily need to remember how to use them, I am interested in learning what you have been exposed to in the past)

Circle the phrase that best describes you:

How would you rate yourself as a mathematics learner? Math is easy / OK / difficult to learn.

How do you like math? I love / tolerate / hate math.

I take math because I like it / it is required

Probabilistic Thinking in College Students
Research Consent Form
(Interview Participation)

By signing this form below, I attest to the following:

1. I understand that I am a participating in a research study. The purpose of the research is to examine the methods college students use to solve probability problems and factors which influence their ability to solve the problems. My participation will consist of participation in two (2) one-hour video and audiotaped interviews, which will consist of solving probability problems, and providing biographical information about myself. I will also provide contact information to the researcher for scheduling the interviews.
2. I understand that my participation in this study is voluntary, and that I may withdraw my participation at any time with no penalty.
3. During the two (2) one-hour video and audiotaped interviews, I have the right to refuse to answer any question(s), if I choose.
4. The researcher has explained the purpose and procedures of this research study, and I have been given an opportunity to receive answers to my questions.
5. I understand that the researcher will keep my responses confidential and will destroy all records at the completion of the research.
6. I understand that I will not receive any compensation for my participation in this study. My participation, or lack of participation, will not affect my grade or standing in the class in any way.
7. I understand that the results of the inventory test and the video and audiotaped interviews will not be shared with my instructor, nor affect my grade.

Questions concerning this research should be directed to either:
Mary Bamberger at (541) XXX-XXXX or bamberg@orst.edu
Dianne Erickson at: (541) XXX-XXXX or erickso@orst.edu

Questions concerning your rights as a human subject should be directed to the IRB Coordinator, OSU Research Office, (541) 737-3437

My signature below indicates that I have read and that I understand the procedures described above and give my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

______________________________ ______________________

Name (printed) date

______________________________ ______________________

Signature Signature of Researcher

Contact Information

Name: ______________________________________

Telephone number: ____________________________

E-mail Address: _______________________________

The best way to contact me is by: __________________________________

The best times to contact me are: __________________________________

Probabilistic Thinking in College Students
Research Consent Form
(Instructor)

By signing this form below, I attest to the following:

1. I understand that I am participating in a research study. The purpose of the research is to examine the methods college students use to solve probability problems and factors which influence their ability to solve the problems. My participation will consist of taking part in two audiotaped interview with the researcher and allowing the researcher to observe instruction in my course.
2. I understand that my participation in this study is voluntary, and that I may withdraw participation at any time with no penalty.
3. I understand that I may refuse to answer any question(s), if I choose.
4. The researcher has explained the purpose and procedures of this research study, and I have been given an opportunity to receive answers to my questions.
5. I understand that the researcher will keep my responses and the students' responses confidential and will destroy all records at the completion of the research. The only person who will have access to this information will be the researcher and no names will be used in any data summaries.
6. I understand that I will not receive any compensation for my participation in this study.

Questions concerning this research should be directed to either:
Mary Bamberger at (541) XXX-XXXX or bamberg@orst.edu
Dianne Erickson at: (541) XXX-XXXX or erickso@orst.edu

Questions concerning your rights as a human subject should be directed to the IRB Coordinator, OSU Research Office, (541) 737-3437

My signature below indicates that I have read and that I understand the procedures described above and give my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

______________________________	____________________
Name (printed)	date
______________________________	____________________
Signature	Signature of Researcher

APPENDIX C

Finite Mathematics Course Objectives

Objectives used for instruction and to construct Pre- and Post-Instruction Task-Based Questionnaire

Finite Mathematics students will be able to:

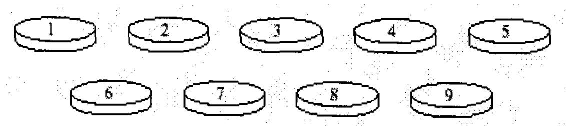
1. Sample space
   Determine the distribution of items in a sample space or list all possible outcomes of an experiment
2. Simple probabilistic events (of equally likely events)
   Assign a probability to a simple event, given the sample space or relative frequency of the event
3. Compound probabilistic events
   Assign a probability to the union, intersection, or complement of two events, given the sample space or the probabilities of each event
4. Independent probabilistic events
   Assign a probability to an event not effected by a related event or additional conditions imposed on the event (independent)
5. Dependent probabilistic events
   Assign a probability to an event effected by a related event or effected by additional conditions imposed on the event (dependent / conditional probability)
6. Mutually Exclusive probabilistic events
   Assign probability to events which have no outcomes in common (mutually exclusive)
7. Comparison of probabilistic events
   Compare probabilistic results, given the sample space or the probabilities of each event
8. Probability Properties
   A. Recognize that any probability is a number between 0 and 1
   B. Recognize that all possible outcomes together must have probability 1
   C. Recognize that the probability that an event does not occur is 1 minus the probability that the event does occur

APPENDIX D

Pre-Instructional Task-Based Questionnaire

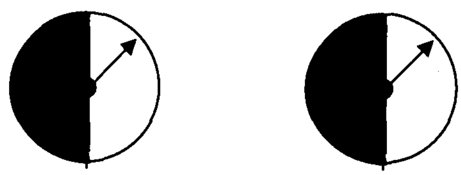
1. (warm-up task) From a batch of 3000 light bulbs, 100 were selected at random and tested. If 5 of the light bulbs in the sample were found to be defective, how many defective light bulbs would be expected in the entire batch?

2. The nine chips shown are placed in a sack and then mixed up:



   Madeline draws one chip from this sack. What is the probability that Madeline draws a chip with an even number?

3. The two fair spinners shown below are part of a carnival game. A player wins a prize only when <u>both</u> arrows land on black after each spinner has been spun once.



   James thinks he has a 50-50 chance of winning.
   Do you agree? Explain why or why not.

4. In a deck of cards, $\frac{1}{6}$ are green, $\frac{1}{12}$ are yellow, $\frac{1}{2}$ are white, and $\frac{1}{4}$ are blue. If someone takes a card from the deck without looking, which color is it most likely to be? Why?

5. An auto dealer sold 120 minivans in one month. His records show that the following repairs were required to these minivans during the first year:

Repair	*Frequency*
Minor	70
Major	30
No Repairs	20

   Mrs. Sleigh purchases a minivan from the dealer. Find the probability that she will return during the first year for minor repairs.

6. From a survey at a large university, a market research company found that 75% of the students owned stereos, 45% owned cars, and 35% owned both cars and stereos. If a student at the university is selected at random, what is the probability that:
    a. The student owns either a car, or a stereo, or both?
    b. The student does not own a car?
    c. The student has neither a car nor stereo?

7. You have torn a tendon and are considering surgery to repair it. The orthopedic surgeon explains the risks to you. Of the 500 patients who underwent this surgery, infection occurred in 40 of such operations, the repair failed in 100 of such operations, and both infection and failure occur together in 25 of such operations. Find the probability that if you undergo surgery, the operation will succeed and you are free from infection?

8. All human blood can be typed as one of O, A, B, or AB, but the distribution of the types varies a bit with the race. Here is the distribution of the blood type of a randomly chosen black American:

Blood Types of Black Americans

Blood Type	*O*	*A*	*B*	*AB*
Probability	0.49	0.27	0.20	?

    a. What is the probability of type AB blood? Why?
    b. Maria has type B blood. She can safely receive blood transfusions from people with blood types O or B. What is the probability that a randomly chosen black American can donate blood to Maria?

9. A warning system installation consists of two independent alarms. During an emergency, the probability of each alarm operating properly is 0.95 and 0.90 respectively. Find the probability that at least one alarm operates properly in an emergency.

10. A market research firm has determined that 40% of the people in a certain area have seen the advertising for a new product. Given that they have seen the advertising, 85% have purchased the product. What is the probability that a person in this area has seen the advertising and purchased the product?

11. A university cafeteria surveyed 500 students for their coffee preferences. The findings are summarized as follows:

	Does Not Drink Coffee	**Prefer Regular Coffee**	**Prefer Decaffeinated Coffee**
Female	25	145	70
Male	15	200	45

A student is selected at random from this group. Find the probability that:
the student does not drink coffee

a. the student is male
b. the student is a female who prefers regular coffee
c. given that the student is male, the student prefers decaffeinated coffee
d. given that the student prefers decaffeinated coffee, the student is male
e. given that the student prefers regular or decaffeinated coffee, the student is female
f. the student is a male student who prefers regular or decaffeinated coffee

12. Two boxes each contain red marbles and blue marbles. One marble is drawn at random from a box (each marble has an equal chance to be drawn). If the marble is red, you win \$1. If the marble is blue, you win nothing. You can chose between two boxes:
    - Box A contains 1 blue marble and 4 red ones
    - Box B contains 3 blue marbles and 4 red ones

    Which box offers a better chance of winning, or are they the same? Explain.

13. A box contains 2 red, 3 white and 4 green balls. Two balls are drawn out of the box in succession without replacement. What is the probability that both balls are red?

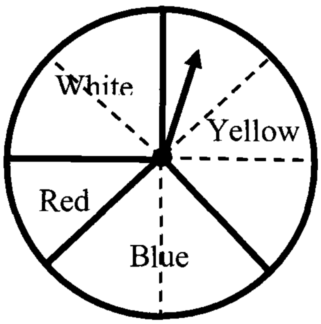
14. Imagine your bedroom is dark and you cannot see a thing. There are three socks in your drawer: two are blue and one is red (they are identical except color). If you withdrew two socks, are you more likely to get two blues (a match), or a blue and red (no match) or are the outcomes equally likely? Explain.

APPENDIX E

Post-Instructional Task-Based Questionnaire

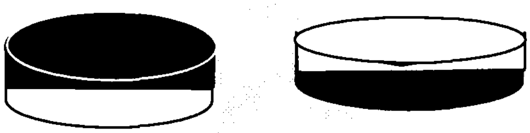
1. (warm-up task) From a shipment of 500 batteries, a sample of 25 was selected at random and tested. If 2 batteries in the sample were found to be dead, how many dead batteries would be expected in the entire shipment?

2. The following spinner is a fair spinner.



Angie spins the fair spinner once. What is the probability that the arrow will land on yellow?

3. The two fair coins shown below are part of a carnival game. One side of a coin is black, the other side white. A player wins a prize only when <u>both</u> coins land on black after each coin has been tossed once.



Caroline thinks she has a 50-50 chance of winning.
Do you agree? Explain why or why not.

4. In a deck of cards, $\frac{1}{6}$ are green, $\frac{1}{3}$ are yellow, $\frac{5}{18}$ are white, and $\frac{2}{9}$ are blue. If someone takes a card from the bag without looking, which color is it most likely to be? Why?

5. A local supermarket randomly surveyed 800 customers about their smoking and drinking habits. The results of this survey are summarized in the table below.

	Smokers	**Non-Smokers**
Drinkers	220	480
Non-Drinkers	20	80

Calculate the probability that a randomly selected customer drinks and smokes.

6. From a survey at a preschool, a pharmaceutical company found that 75% of the students contracted chicken pox, 30 % contracted measles, and 20% contracted both the chicken pox and measles. If a child at the preschool is randomly selected, what is the probability that:
   a. The student contracted the chicken pox, or the measles, or both?
   b. The student did not contract the chicken pox?
   c. The student contracted neither the chicken pox nor measles?

7. An auto dealer sold 300 luxury cars in one month. His records indicated that 45 cars had their air conditioners failing before the warranty expired, 60 had their alternator failing before the warranty expired, and both the air conditioner and alternator failure occurred together in 15 cars before the warrantee expired. Mr. Jones purchases a luxury car from the dealer. What is the probability that the air conditioner and alternator on the luxury car will ***not*** fail before the warranty expires?

8. Customers at a local coffee shop participated in a coffee taste test. The participants were asked to state which of the following four brands of coffee they prefer. The brands were labeled *A, B, C,* and *D*. The results are partially summarized as follows:

Coffee Preferences

Brand	*A*	*B*	*C*	*D*
Probability	0.25	0.43	0.05	?

   a. What was the probability that the participants preferred Brand *D*? Why?
   b. You walk up to the taste test and sampled the four varieties. What is the probability that you will prefer either Brand *A* or *B*?

9. A private plane consists of two independent engines. The probability that the left one fails in flight is 0.10, and the probability that the right one fails is 0.05. Find the probability that at least one engine fails in flight.

10. A bank observes that 60% of its customers prefer to visit a teller than an ATM when making a transaction. Given that a customer has visited the teller, 30% deposited money into their accounts. What is the probability that a customer at this bank had visited a teller and deposited money into their account?

11. An ice chest at a family reunion picnic contains three brands of soft drinks: Pepsi, Coke, and 7 - UP. In addition, there are some regular and some diet drinks of each brand. Out of 500 cans, the number of cans of each type of soft drink is summarized as follows:

	Pepsi	**Coke**	**7-UP**
Regular	25	145	70
Diet	15	200	45

You reach into the chest and select one drink at random. Find the probability that you:

a. selected a Pepsi product
b. selected a diet soft drink
c. selected a Diet Coke
d. given it was a diet soft drink, selected a Pepsi
e. given it was a Pepsi product, selected a diet soft drink
f. given it was a Pepsi or Coke product, selected a regular soft drink
g. selected a regular soft drink that was either Coke or 7 - UP

12. Two boxes contain red marbles and blue marbles. One marble is drawn at random from a box (each marble has an equal chance to be drawn). If the marble is red, you win $1. If the marble is blue, you win nothing. You can chose between two boxes:
    - Box A contains 1 blue marbles and 2 red ones
    - Box B contains 6 blue marbles and 9 red ones

    Which box offers a better chance of winning, or are they the same? Explain

13. A gumball machine contains 5 red, 7 white and 9 green gumballs. The gumballs are well mixed inside the machine. John gets two gumballs from this machine in succession, without replacement. What is the probability that both gumballs are red?

14. Imagine your bedroom is dark and you cannot see a thing. There are three socks in your drawer: two are blue and one is red (they are identical except color). If you withdrew two socks, are you more likely to get two blues (a match), or a blue and red (no match) or are the outcomes equally likely?

APPENDIX F

Table of Specifications

Pre-Instructional Task-Based Questionnaire

Objective	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
1. Sample Space	60%													80%	2
2. Simple Event	60%	100%			100%									80%	4
3. Compound Event			100%			100%	80%		80%						4
4. Independent									100%		100%				2
5. Dependent										100%	100%		100%		3
6. Mutually Exclusive											100%				1
7. Comparison				80%								80%			2
8. Probabilistic Laws						80%		100%							2

Post-Instructional Task-Based Questionnaire

Objective	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
1. Sample Space	60%													80%	2
2. Simple Event	60%	100%			100%									80%	4
3. Compound Event			80%			100%	80%		80%						4
4. Independent									100%		100%				2
5. Dependent										100%	100%		100%		3
6. Mutually Exclusive											100%				1
7. Comparison				100%								80%			2
8. Probabilistic Laws						80%		100%							2

APPENDIX G

Instructor Formal Interview Questionnaire

Formal Classroom Observation Interview (conducted after instruction on probability)

1. Can you tell me about your teaching background?
2. Can you tell me your background in learning probability?
3. Can you tell me about your background in teaching probability?
4. What are your expectations of your students understanding of probability?
5. What are some of the successes and difficulties you think your students will encounter trying to solve probability problems?
6. What is your understanding of mathematical reasoning, or probabilistic reasoning?
7. What will you accept as evidence of probabilistic reasoning on the part of your students?
8. Do you find my intrusion on the class is interfering with the students ability to learn in the classroom?
9. Did my presence in the classroom cause you to change your course curriculum in any way?
10. Is there anything else that you think could be helpful for me to know?

APPENDIX H

Pre-Instructional Interview

Tell me about yourself.

1. What year of college are you in?

2. What is your major?

3. Tell me about the mathematics courses you had in high school.

4. What other college mathematics courses have you taken, including any you are taking this term?

5. Do you feel confident about the coursework in your Finite Mathematics class?

6. Have you had a course in probability before?

7. Is there any thing else I should know about you and mathematics? Probability? (hard, easy, like, don't like, etc.)

8. Are there any questions you would like to ask me?

APPENDIX I

Talk-Aloud Script

In this experiment, I am interested in what you say to yourself a you solve mathematics problems. In order to do this, I am asking you to *talk aloud* as you work on the following problems. What I mean by talk aloud is that I want you to say out loud *everything* that you say to yourself silently why solving the problem. Just act as if you are at home doing homework, speaking to yourself. I would like you to talk aloud *constantly* from the time you are presented the problem until you have given your final answer to the question. If you are silent for any length of time, I will remind you to keep talking aloud. I don't want you to try to plan out what you say or try to explain to me what you are saying. Just act as if you are in the room speaking to yourself. Do you have any questions?

Good, before you solve the problems for the study, I want you to practice talking aloud on this practice problem (Problem 1). First, read the question as it is written, then proceed to solve the problem as if it was a homework problem, and talk aloud. When you are finished, we can discuss your ability to talk aloud.

Now that you finished the first problem, I would like you to continue solving the other problems while talking aloud. As you finish a problem, turn the page and proceed to the next problem. Continue this until you are finished.

APPENDIX J

Post-Instructional Interview

Questions Before Post Instructional Task-Based Questionnaire:

The purpose of this component of the interview is to find out more about what aspects of this class you found beneficial in learning how to solve probability problems.

1. Tell me about your homework habits. For this class, you are expected to read the section on probability prior to the lecture. In addition, the instructor has a time-line as to when homework is to be completed. Do you happen to follow these suggestions?
2. Do you use your textbook or lecture notes for studying for this class?
3. Did you use any outside resources when studying for this section on probability?
4. When you do not understand a concept in class, do you ask for help?
5. What did you find most helpful in understanding probability?
   - Lecture
   - Lecture notes
   - Textbook
   - Supplemental material
   - Solutions manual
   - Working with others
   - Homework problems
   - Computer assignments
6. What did you find the least helpful in understanding probability?

Questions After Post Instructional Task-Based Questionnaire:

1. How comfortable do you feel solving the probability problems found in class?
2. How do you perceive your probability knowledge and understanding?
3. Do you think you learned anything about probability in this class? Can you tell me any specifics?
4. What helped you learn about probability the most?
5. Do you like word problems?
6. These problems I presented to you might be problems you interpret in everyday events. Did you find the context of the problem helpful in solving the problem? How?
7. Is there anything you would like to tell me regarding learning probability?

APPENDIX K

Interview 1: Summary of Responses

Student	1. Sample Space	2. Simple Event	3. Compound Events	4. Comparison of Two Events	5 Simple Event	6a. Compound Events	6b. Compound Event	6c. Compound Events	C, I, DNF
Aaron	Correct	Correct	Correct	Correct	Correct	Correct	Correct	Correct	(8, 0, 0)
Bob	Correct	Correct	Incorrect	Correct	D. N. F.	D. N. F.	---------	---------	(3, 1, 2)
Charlie	Correct	*Correct*	Correct	Correct	Correct	D. N. F.	Correct	Incorrect	(6, 1, 1)
Dennis	D. N. F.	Correct	Correct	Correct	Correct	D. N. F.	Correct	---------	(5, 0, 2)
Evan	Correct	D. N. F.	*Correct*	Correct	D. N. F.	D. N. F.	---------	---------	(3, 0, 3)
Freda	Incorrect	Correct	Incorrect	Correct	Incorrect	Incorrect	*Correct*	Incorrect	(3, 5, 0)
Greg	Correct	Incorrect	Correct	Correct	Correct	Incorrect	Incorrect	---------	(4, 3, 0)
Harriet	D. N. F.	Correct	D. N. F.	Incorrect	Correct	D. N. F.	D. N. F.	---------	(2, 1, 4)
Ian	Correct	Correct	Correct	Correct	Correct	Incorrect	Correct	D. N. F.	(6, 1, 1)
TOTALS	(6, 1, 2)	(7, 1, 1)	(6, 2, 1)	(8, 1, 0)	(6, 1, 2)	(1, 3, 5)	(5, 1, 1)	(1, 2, 1)	(40,10,13)

Student	7. Compound Events	8a. Properties of Probability	8b. Properties of Probability	9. Compound Events	10. Dependent Event	11a. Independent Event	11b. Independent Event	11c. Independent Event	C, I, DNF
Aaron	Correct	Correct	Correct	Correct	Correct	Correct	Correct	Correct	(8, 0, 0)
Bob	---------	Correct	Correct	D. N. F.	D. N. F.	Incorrect	Correct	D. N. F.	(3, 1, 3)
Charlie	Incorrect	Correct	Correct	Incorrect	Correct	Correct	Correct	Correct	(6, 2, 0)
Dennis	D. N. F.	Incorrect	*Correct*	D. N. F.	Correct	---------	---------	---------	(2, 1, 2)
Evan	---------	Incorrect	*Correct*	Incorrect	D. N. F.	---------	---------	---------	(1, 2, 1)
Freda	Incorrect	Incorrect	*Correct*	D. N. F.	D. N. F.	---------	---------	---------	(1, 2, 2)
Greg	Incorrect	Correct	Correct	Incorrect	Incorrect	Correct	Correct	Correct	(5, 3, 0)
Harriet	Correct	---------.	D. N. F.	---------	---------	---------	---------	---------	(1, 1, 1)
Ian	Incorrect	Correct	Correct	Incorrect	Incorrect	Correct	Correct	Correct	(5, 3, 0)
TOTALS	(2, 4, 1)	(5, 4, 1)	(8, 0, 1)	(1, 4, 3)	(3, 2, 3)	(4, 1, 0)	(5, 0, 0)	(4, 0, 1)	(32,15,10)

Student	11d. Dependent Event	11e. Dependent Event	11f. Dependent Event	11g. Independent Event	12. Comparison of Two Events	13. Dependent Event	14. Comparison of Two Events		C, I, DNF
Aaron	Correct	Correct	Correct	Correct	Correct	Correct	Correct		(7, 0, 0)
Bob	D. N. F.	D. N. F.	Incorrect	Incorrect	Correct	---------	---------		(1, 2, 2)
Charlie	Incorrect	Incorrect	Incorrect	Correct	Correct	Correct	Correct		(4, 3, 0)
Dennis	---------	---------	---------	---------	Correct	D. N. F.	Correct		(2, 0, 1)
Evan	---------	---------	---------	---------	Incorrect	D. N. F.	---------		(0, 1, 1)
Freda	---------	---------	---------	---------	Correct	---------	---------		(1, 0, 0)
Greg	Incorrect	Incorrect	Incorrect	Correct	Correct	Incorrect	Incorrect		(2, 5, 0)
Harriet	---------	---------	---------	---------	Correct	---------	---------		(1, 0, 0)
Ian	Incorrect	Correct	Correct	Incorrect	Correct	Incorrect	Incorrect		(3, 4, 0)
TOTALS	(1, 3, 1)	(2, 2, 1)	(2, 3, 0)	(3, 2, 0)	(8, 1, 0)	(2, 2, 2)	(3, 2, 0)		(21, 15, 4)

Note: (n = 9)
D. N. F. indicates the student did not finish the problem
Dashes indicate data were not recorded
Amounts in parenthesis record numbers of problems correct, incorrect, and not finished, respectively
Correct indicates the student stated the correct solution to the problem, but their reasoning was not correct

APPENDIX L

Interview 2: Summary of Responses

Student	1. Sample Space	2. Simple Event	3. Compound Events	4. Comparison of Two Events	5 Simple Event	6a. Compound Events	6b. Compound Event	6c. Compound Events	C, I, DNF
Aaron	Correct	Correct	Correct	Correct	Correct	Correct	Correct	Correct	(8, 0, 0)
Bob	Correct	Correct	Correct	Correct	Correct	D. N. F.	Incorrect	Correct	(6, 1, 1)
Charlie	Correct	Correct	Correct	Correct	Correct	Incorrect	Incorrect	Incorrect	(5, 3, 0)
Dennis	Correct	Correct	Correct	Correct	Correct	Correct	Correct	Correct	(8, 0, 0)
Evan	Incorrect	Correct	Incorrect	Correct	Incorrect	D. N. F.	---------	---------	(2, 3, 1)
Freda	Correct	Correct	Incorrect	Correct	Correct	Incorrect	Incorrect	Incorrect	(4, 4, 0)
Greg	Correct	Correct	Correct	Correct	Correct	Incorrect	Incorrect	Incorrect	(5, 3, 0)
Harriet	Incorrect	Correct	D. N. F.	D. N. F.	Correct	D. N. F.	D. N. F.	D. N. F.	(2, 1, 5)
Ian	Correct	Correct	Correct	Correct	Correct	Incorrect	Correct	Incorrect	(6, 2, 0)
TOTALS	(7, 2, 0)	(9, 0, 0)	(6, 2, 1)	(8, 0, 1)	(8, 1, 0)	(2, 4, 3)	(3, 4, 1)	(3, 4, 1)	(46, 17, 7)

Student	7. Compound Events	8a. Properties of Probability	8b. Properties of Probability	9. Compound Events	10. Dependent Event	11a. Independent Event	11b. Independent Event	11c. Independent Event	C, I, DNF
Aaron	Correct	Correct	Correct	Correct	Correct	Correct	Correct	Correct	(8, 0, 0)
Bob	Incorrect	Correct	Correct	Incorrect	Incorrect	Correct	Correct	Correct	(5, 3, 0)
Charlie	Incorrect	Correct	Correct	Incorrect	Correct	Correct	Correct	Correct	(6, 2, 0)
Dennis	Correct	Correct	Correct	Incorrect	Correct	Correct	Correct	Correct	(7, 1, 0)
Evan	Incorrect	Correct	Correct	D. N. F.	---------	Correct	Correct	Correct	(5, 1, 1)
Freda	Incorrect	Incorrect	*Correct*	Incorrect	D. N. F.	Incorrect	Incorrect	Incorrect	(1, 6, 1)
Greg	Correct	Correct	Correct	Correct	Correct	Correct	Correct	Correct	(8, 0, 0)
Harriet	---------	Correct	---------	---------	---------	---------	---------	---------	(1, 0, 0)
Ian	Incorrect	Correct	Correct	---------	Correct	Correct	Correct	Correct	(6, 1, 0)
TOTALS	(3, 5, 0)	(8, 1, 0)	(8, 0, 0)	(2, 4, 1)	(5, 1, 1)	(7, 1, 0)	(7, 1, 0)	(7, 1, 0)	(47, 14, 2)

Student	11d. Dependent Event	11e. Dependent Event	11f. Dependent Event	11g.Independent Event	12. Comparison of Two Events	13. Dependent Event	14. Comparison of Two Events		C, I, DNF
Aaron	Correct	Correct	Correct	Correct	Correct	Correct	Correct		(7, 0, 0)
Bob	Correct	Correct	Correct	Incorrect	Correct	Incorrect	Correct		(5, 2, 0)
Charlie	Correct	Correct	Correct	Incorrect	Correct	Correct	Incorrect		(5, 2, 0)
Dennis	Correct	Correct	D. N. F.	D. N. F.	Correct	Incorrect	Correct		(4, 1, 2)
Evan	Incorrect	Incorrect	Incorrect	Incorrect	Incorrect	---------	---------		(0, 5, 0)
Freda	Incorrect	Incorrect	Incorrect	Incorrect	Incorrect	---------	---------		(0, 5, 0)
Greg	Correct	Correct	Correct	Incorrect	Incorrect	Correct	---------		(4, 2, 0)
Harriet	---------	---------	---------	---------	D. N. F.	---------	---------		(0, 0, 1)
Ian	Incorrect	Correct	Correct	Incorrect	Incorrect	Incorrect	Incorrect		(2, 5, 0)
TOTALS	(5, 3, 0)	(6, 2, 0)	(5, 2, 1)	(1, 6, 1)	(4, 4, 1)	(3, 3, 0)	(3, 2, 0)		(27, 22, 3)

Note: (n = 9)
D. N. F. indicates the student did not finish the problem
Dashes indicate data were not recorded
Amounts in parenthesis record numbers of problems correct, incorrect, and not finished, respectively
Correct indicates the student stated the correct solution to the problem, but their reasoning was not correct

APPENDIX M

Case Studies: Portraits of The Nine Participants

The purpose of this Appendix is to present portraits of each of the nine participants. The student selection and data gathering methodology are described in Chapter III. The following portraits included information about the student's mathematics and probability background, their attitudes towards mathematics, the methods they used to solve the problems, and the factors that supported or impeded their success during the first and second administrations of the Task-Based Questionnaires. In addition, the portraits included a description of their classroom behavior and homework habits in their finite mathematics classroom. An overview of the answers given by the nine participants attempting the two questionnaires are found in Appendices K and L. The goal of presenting detailed portraits of the nine students was to acknowledge the individuality of the student's performance while also observing the similarities among the group as a whole. The nine students who volunteered to participate in this study were called Aaron, Bob, Charlie, Dennis, Evan, Freda, Greg, Harriet, and Ian. Pseudonyms were used to assure the anonymity of the participants.

Aaron

Aaron was a 19-year old sophomore debating whether to pursue a degree in either mathematics or in business administration and economics. Having a strong mathematics and physics background from high school, Aaron enjoyed enrolling in various mathematics courses in college required for both of his educational goals. He was fortunate to have a probability and statistics course in high school, and encountered more probability in his college level business statistics classes prior to enrolling in finite mathematics. When asked which probability concepts he recalled from these previous classes, Aaron exhibited a wide knowledge of mathematical and probabilistic concepts as he described his familiarity with counting theory, probability, and inferential statistics.

Aaron demonstrated his strong mathematics and reasoning skills on the first Task-Based Questionnaire by successfully solving all the problems. His comfort with mathematics was apparent in his arithmetic and reasoning ability. Aaron preferred to conduct his operations by hand, work with fractional representations; and estimate fractions without a common denominator. In addition, the problems did not appear to challenge Aaron's reasoning ability. Throughout his problem solving session, Aaron did use the context of the problem and proper mathematics terminology while verbalizing his procedures. In addition, when working on a problem, Aaron would sometimes reword the problem or question to ensure he was interpreting the problem correctly. However, after solving the problems, it did not appear that Aaron checked his answer for its reasonableness. When asked about this factor, Aaron said he recognized the problems and felt confident in his problem solving ability. Normally, he would check his answer but thought that these were basic problems and did not take the time to check his answers.

Aaron's approach to solving the first set of problems consisted of examining each phrase of the problem, then looking at the problem as a whole before deciding the method he would use to solve the problem. In addition, his method incorporated the use of the context of the problem to reason through his solution, using appropriate representations to organize and analyze the data, and going back to the problem to check the reasonableness of the answer.

While solving the problems, Aaron demonstrated an understanding of the relationships between basic counting principles and probabilistic properties. Together, with his mathematical reasoning and knowledge of probability properties, Aaron was able to create shortcuts for solving problems, demonstrated in his verbal response to Problem 9:

> A warning system installation consists of two independent alarms. During an emergency, the probability of each alarm operating properly is 95% and 90% respectively. (That's a good system). Find the probability that at least one alarm operates properly in an emergency.
>
> *Student's Response:*
> Umm... you take 100, or 1 minus .95 and you get .05 that that would fail (pointing to the first alarm), and you get 1 minus .90 equals .1 that that one would fail (pointing to the second alarm).
>
> You multiply them and you get point .005 that both fail. Then you take 1 minus that they will both fail, and you get .995 (that at least one alarm operates properly in an alarm)

Aaron exhibited an understanding of using the complement of the probability of "at least one event occurring" to find the probability of "none of the events occurring", while incorporating the multiplication principle to find the probability of neither event will occur. From this Aaron then used the complement of "none will occur" again to find the probability that "at least one will occur."

Finally, Aaron exhibited a strong understanding of set operations, choosing Venn diagrams to help him solve compound probabilistic events. In addition, Aaron demonstrated the ability to interpret the various events represented in a Venn diagram, as shown in Problem 7:

> You have torn a tendon and are considering surgery to repair it. The orthopedic surgeon explains the risks to you. Of the 500 patients who underwent this surgery, infection occurred in 40 of such operations, the repair failed in 100 of such operations, and both infection and failure occur together in 25 of such operation Find the probability that if you undergo surgery, the operation will succeed and you will be free from infection.

<u>*Student's Response*</u>

So you have 500 total,

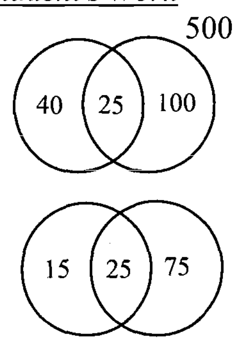
And you got, um, 40 for infection, 100 failed, 25 together.

So, that leaves 75, 15...

Uh, 75 plus 25 plus 15 equals 115 that they either got infection or failed, or both.

So you take 500 minus 115 and you get 385 over 500 and reduce that and you get 77%.

<u>*Student's Work*</u>



Aaron recognized the word "both" as an indicator of the union of two events, which are inclusive of the probability of each individual event. In addition, Aaron recognized the need to find the complement of the number of patients who incurred infection or repair failure to find the number of patients who will have successful operations and will be free from infection.

Overall, due to familiarity with these types of problems, Aaron's method for solving these probability problems developed from his awareness of probability and set operation concepts. Aaron would read each question, break down the problem into different groupings, look at each grouping separately, and use the context of the problem to reason a solution. Factors, which made him successful, were his ability to work with Venn diagrams to solve the set operation problems and his comfort with probability properties, which allowed him to find shortcuts in his procedures. In addition, Aaron was able to use his arithmetic skills, familiarity of working with these problems, and the context of the problem throughout the first questionnaire.

Aaron only attended five of the eight days in class covering the lessons on probability. When he attended class, Aaron sat in the back of the classroom with only his calculator on the desk. If the instructor were presenting a problem requiring calculations, Aaron would try the problems on his calculator. When the other students could not provide the correct answer to a problem posed by the instructor, Aaron would volunteer his correct answer to the class. Aaron did not ask any questions during the lessons and interacted with only two other students: Charlie and Dennis. On the second questionnaire, Aaron admitted that he did not try any of the homework problems and learned the most from observing the instruction. Having seen the concepts before, Aaron treated the lectures as a time to enhance his understanding of probability.

On the second Task-Based Questionnaire, Aaron successfully solved all the problems and showed evidence of refinement of his probabilistic reasoning and problem solving ability. For example, on the first questionnaire, Aaron was asked why he multiplied two probabilities on Problem 10 to get the correct answer:

> A market research firm has determined that 40% of the people in a certain area have seen the advertising for a new product. Given that they have seen the advertising, 85% have purchased the product. What is the probability that a person in this area has seen the advertising and purchased the product?

Aaron replied, "It says, 'What's the probability they've seen advertising *and* purchased the product'. So when it says *and* you multiply." This reasoning indicated he recognized the key word "and" implying multiplication.

However, on the second questionnaire, Aaron solved Problem 10 by also multiplying the two probabilities:

> A bank observes that 60% of its customers prefer to visit a teller than an ATM when making a transaction. Given that a customer has visited the teller, 30% deposited money into their accounts. What is the probability that a customer at this bank has visited a teller *and* deposited money into their account?

Aaron's final reasoning for the solution was, "You take .6 times .3 and you get .18. Yeah, 18%." This time, when asked why he multiplied, he replied

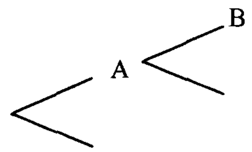
Student's Response

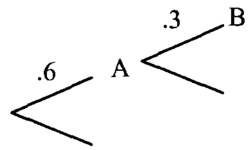
Um, like what we did in class, we had the one choice, A, and then the next choice B

This one would be the .6 and this one would be the .3

And then there are four different ones (branches) but you only look at this branch. And you multiply these (pointing to .6 and .3)

Student's Work





However, this time Aaron preferred to use the tree diagram to represent the associations among the probabilities of dependent events as compared with his previous problem solving reasoning of multiplying the two numbers because of the key word "and."

Aaron exhibited additional refinement while using the Venn diagram. On the first questionnaire, Aaron exhibited solid understanding for using Venn diagrams (refer to Problem 7 above). However, when asked to solve Problem 7 on the second questionnaire, he still used the Venn diagram, but his verbal response indicated a clearer understanding of the problem:

> An auto dealer sold 300 luxury cars in one month. His records indicated that 45 cars had their air conditioners failing before the warranty expired, 60 had their alternator failing before the warranty expired, and both the air conditioner and the alternator failure occurred together in 15 cars before the warrantee expired. Mr. Jones purchases a luxury car from the dealer. What is the probability that the air conditioner and alternator on the luxury car will *not* fail before the warranty expires?

<u>*Student's Response*</u>

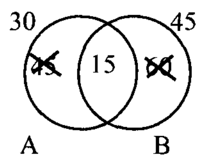
Okay, you've got 45, 60,

And in the middle you've got 15.

So, just A, not B, is 30; just B and not A is 45.

Add them all together and you get 90 out of 300, and you take the opposite, the complement, inverse or something like that and you get 210 over 300, which is 7/10ths. Yeah, is the probability that neither one will fail.

<u>*Student's Work*</u>



In this solution, Aaron decided to find the number of events representing the three sets "A and not B", "A and B", and "not B and A." This verbal response indicated a stronger grasp of set operations.

After observing Aaron's mathematical reasoning skills that incorporated his knowledge of set operations and probability properties, it was surprising to hear him solve Problem 12, the comparison of two probabilistic events, using an alternate strategy:

> Two boxes contain red marbles and blue marbles. One marble is drawn at random from a box (each marble has an equal chance to be drawn). If the marble is red, you win $1. If the marble is blue, you win nothing. You can chose between two boxes:
>
> - Box A contains 1 blue marbles and 2 red ones
> - Box B contains 6 blue marbles and 9 red ones
>
> Which box offers a better chance of winning, or are they the same? Explain.

Instead of comparing the probabilities or odds of the event, Aaron multiplied the ratio of marbles in Box A by 6, making 6 blue marbles and 12 red marbles in Box A. Comparing the proportion of marbles in Box B (with 6 blue marbles and 9 red marbles) to the "new" Box A (with 6 blue marbles and 12 red marbles), Aaron claimed Box A had the higher number of red marbles overall, when each box had 6 blue marbles. Therefore, Box A had the better chance "because if you chose out of it 6 times, there will be the same amount of blue, but 12 red ones (marbles) instead of 9."

On the second questionnaire, two additional factors emerged. First, as observed in his verbal explanations in Problems 7 and 10, Aaron used the notation A and B to label each the set, as opposed to the context words used in the problems. This could cause confusion when given more than two sets. In addition, as observed in his solving of Problem 10, Aaron also tried to incorporate more probability terminology into his verbal reasoning. However, he appeared confused as to which terminology to use to represent the complement of an event, when he said, " and you take the opposite, the complement, inverse or something like that."

On the second questionnaire, Aaron appeared as if he had clarified some of his probabilistic reasoning and certain mathematics operations embedded in the problems. His method remained the same: read the question, broke down the problem while analyzing each piece separately, used the context of the problem to reason a solution, and checked his answer for reasonableness. Factors that supported his success were his ability to recognize set operation terminology, to organize and analyze the problems with Venn and tree diagrams, to use probability properties to create shortcuts, to rephrase the question, and strong arithmetic skills.

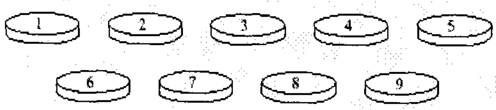
Bob

After graduating from high school, Bob was unable to find a job and decided to return to school. Now a freshman at the age of 20, Bob was pursuing a degree in health care administration in which finite mathematics was an elective mathematics course for him. He chose to enroll in finite mathematics because he observed it had a mixture of mathematical concepts. He enjoyed solving mathematical problems, but claimed "word problems are my downfall in math… Because if you look at the whole thing, it's just going to fully screw you up because of the of the stuff in there - a word problem - has nothing to do with what you really got to do." Bob would rather "just go over it (the problems) on my own… and I'll get it down if I keep doing it over and over." He had successfully completed intermediate and college algebra, but did not recall enrolling in a course containing probability.

On the first Task-Based Questionnaire, it was evident that Bob had not seen probability before. His unfamiliarity with probability mixed with his dislike of word problems caused him to approach each problem using a different method. Even when Bob did feel comfortable with his solution, he did not use the same approach for a similar problem. Bob was successful solving problems requiring knowledge of the properties of probability and comparing probabilities of two events, but encountered difficulties solving simple, compound, independent, and dependent events.

While solving a simple probabilistic event problem, such as Problem 2, Bob tried different strategies until he settled on one that he felt comfortable using:

> The nine chips shown are placed in a sack and then mixed up:
> Madeline draws one chip from this sack. What is the probability that



> Madeline draws a chip with an even number?

Student's Response:	*Student's Work:*
Okay, so I see nine chips - so there's nine chips,	
That means there's 1, 2, 3, 4 even numbers,	
So I'll -- how do I -- I mean, I would just say there's four even numbers, so I would say she has a probability of one out of four ... one out of nine chances.	$\frac{1}{4}$
Actually, four out of nine.	$\frac{4}{9}$
Four out of nine chances. That's what I would say. I mean, I'm just hoping... 'cause I never really -- yeah - four-ninths.	

For this problem, Bob was given the sample space, and was able to find the probability of the event. However, he could not apply this reasoning when given the frequency of an event. As indicted by his solution to Problem 5, he did not stick with his initial method to compute the probability of a simple event:

> An auto dealer sold 120 minivans in one month. His records show that the following repairs were required to these minivans during the first year:
>
> | *Repair* | *Frequency* |
> |---|---|
> | Minor | 70 |
> | Major | 30 |
> | No Repairs | 20 |
>
> Mrs. Sleigh purchases a minivan from the dealer. Find the probability that she will return during the first year for minor repairs.
>
> <u>*Student's Response:*</u>
>
> Okay. How do you figure this out? Purchases a van from the dealer - find the probability that she will return. Okay. How would I do this? This is the kind of questions I just leave alone and come to class the next day. Okay. The probability that she will return the first year - whoa! - To the auto dealer, so how would I figure this out? How would I figure this out? Jeez. I would figure this out by looking in the back of the book. I don't know how I would do this.

Requiring familiarity with set operations, the sixth problem, was frustrating Bob. However, he was still willing to attempt each problem.

At the end of Problem 7, Bob commented "Maybe that is my problem, is I don't know how to find probability. And I bet you if I knew how to find probability, I would know how to do this." As Bob continued onto the other problems, he started to make an association between probability and percents. By the end of Problem 8, he comments "I'm seeing a whole, so like these are percents, in a way. And 1 would be 100%." But under the same breath claimed, "This is confusing." Thus, by Problem 11, when he is asked to compute the probability of a simple event, given the frequency of each event, Bob stated, "I have to find a percent." By the end of the interview, he stated, "Yeah, I think I need to do this probability stuff. I have to figure out what probability is."

In addition, Bob began to believe some of the problems were trick questions possessing two ways to solve it depending on how the problem was interpreted. Problem 10 allowed him to demonstrate his displeasure with word problems and his inability to interpret word problems, even after receiving clarification:

> A market research firm has determined that 40% of the people in a certain area have seen the advertising for a new product. A market research firm... given that they have seen the advertising, 85% have purchased the product. What is the probability that a person in this area has seen the advertising and purchased the product?

<u>*Student's Response:*</u>

S: Okay. What area are they talking about? Are they talking about this area?

R: What is the question?

S: It says, what is the probability that a person in *this* area…

R: So say they surveyed all the people here at this college and found 40 % of the people at this college have seen the advertising for a new product.

S: Okay. What is the probability that a person in this area has seen the advertising and purchased the product? Don't they just give me the answers then, that means? That would be a trick question. That means 40 % ... saw... the advertisement ... and 80 %, oh 85 % purchased the product.

Bob reasoned that the answer was either 85% or 40%, depending on how he was to interpret the question. Bob thought he was given yet another trick problem when he turned to Problem 12:

> Two boxes each contain red marbles and blue marbles. One marble is drawn at random from a box (each marble has an equal chance to be drawn). If the marble is red, you win $1. If the marble is blue, you win nothing. You can chose between two boxes:
>
> • Box A contains 1 blue marble and 4 red ones
>
> • Box B contains 3 blue marbles and 4 red ones
>
> Which box offers a better chance of winning, or are they the same? Explain.
>
> <u>*Student's Response:*</u>
>
> This is a trick question. Better chance of winning. If you're looking in a different perspective... see, if... Is this considered math? I could say, you know, this is like a geometry - there's no right or wrong, I feel. But I feel like - I could say two things, like here - better chance. I could see it where there's four (red marbles) in both of them, so that means I only have four chances to win, so that would make it the same amount of chance (for each box). Or, I could see it as this one (Box A) only contains 1 blue marble, so the chance of me picking up that blue marble when there's 3 more of the red, I would have a better chance of winning that than if there's just 1 less blue marble than the reds (Box B). I could go both ways. I feel like this is a trick question. So, I will go with, I would choose the box - I would choose Box A. So I would put a big "I" would choose Box A, because there are 3 less blue marbles than there are red.

From Problem 10 and 12, Bob was trying to understand the problems, but stated they were "trick questions" depending on the interpretation of the problem.

Overall, Bob did not have a specific method for solving these problems. The lack of the definition of probability, and the lack of not knowing the formulas and procedures, he felt he was at a disadvantage. As Bob continued through the questionnaire, he started to approach the problems as if they were percent problems. Given two word problems not requiring mathematical knowledge of probability, his inability to interpret the problem hindered his ability to solve them. In his own words, "Well, I am just stunned. I just feel like it's the worst I ever did in math."

In class, Bob was an ideal student. He would show up to class early, pull out his notebook and calculator before the class began and ask Greg, the student sitting in front of him, for some help on the homework problems. During the lecture, Bob took notes, and tried to follow along in the textbook and on the calculator. During the second interview, he admitted he did not ask questions during the lecture because he was too busy taking notes and digesting the information. When asked what he writes in his notes, he says he "writes, like, to me, what's making sense." However, he did not find his lecture notes useful to study from because he felt he was just "writing a whole bunch of numbers." When asked about his homework habits, Bob admitted, "I don't really do it... I don't really have the motivation to do it because you don't have to turn it in... but I think it would be a whole lot better if I could, like if I had more motivation and I did more of the problems, just to memorize it." When he did try some of the homework problems and did not understand the problem, he either would ask one of his roommates or waited to ask Greg the next day in class. To study for the exams, Bob waited until the instructor "gives out the study sheets, that's when I really go over the chapter. Like I look at the type of problems that's going to be on the test, and then I look at it and then, like, I try to figure it out from there."

By the second Task-Based Questionnaire, it was apparent that Bob had gained an understanding of probability and its properties and was familiar with set operations. Bob was able to solve the simple and independent probabilistic events with ease. He also gained the ability to compare probabilistic events and recognize complement and dependent events. However, Bob encountered difficulties with the application of set operations, the computation of dependent probabilistic events, and the correct format in which to state the final answer.

Bob admitted that this questionnaire was his first time trying to use set operations in the context of probability. While solving the problem, he was trying to recall what he learned from class while solving Problem 6:

> From a survey at a preschool, a pharmaceutical company found that 75% of the students contracted chicken pox, 30 % contracted measles, and 20% contracted both the chicken pox and measles. If a child at the preschool is randomly selected, what is the probability that:
> a. The student contracted the chicken pox, or the measles, or both?
> b. The student did not contract the chicken pox?
> c. The student contracted neither the chicken pox nor measles?

Bob was able to separate the three sets correctly into 55% contracted just the chicken pox, 10% contracted just the measles, and 20% contracted both the chickenpox and measles. He did not draw any diagrams to represent these three sets. Once at this point, he did not know how to proceed and answer Part A. Oddly, he was able to answer Part C correctly - 15% - by adding all three sets together and subtracting from 100%. Bob reasoned Part C correctly, but did not notice the association with Part A.

On the first questionnaire, Bob did not recognize dependent events and its effect on the resultant size of the sample and the event. On the second questionnaire, Bob recognized dependent events, but exhibited some difficulty interpreting them. For Problem 11, Bob was required to find the probability of seven events: the first set of three was independent events, the second set of three was dependent events, and the last problem was an independent event. On the second questionnaire, Bob successfully calculated the first six problems, but did not recognize the need to compute a simple event on the last problem.

On Problem 5, Bob was able to calculate the value of a simple probabilistic event, but was confused with the representation of his answer. He arrived at the answer .275 then commented, "What do you call that? .275 chance that a customer drinks and smokes? See this is something where I might bring to class the next day… because I don't know if it's percent or what kind of think… I think it's (the answer) is a percent, because of probability." Bob was confused as to the representation of the final answer, as a percent or a decimal.

Still, Bob's reasoning on Problem 10 did not improve. On the second questionnaire, he tried to solve the problem:

> A bank observes that 60% of its customers prefer to visit a teller than an ATM when making a transaction. Given that a customer has visited the teller, 30% deposited money into their accounts. What is the probability that a customer at this bank has visited a teller *and* deposited money into their account?

This time Bob reasoned that "half of its customers visited and deposited into their account", because, he explained later, 30% is half of 60%.

After the second questionnaire, Bob's method for solving probability problems consisted of reasoning the majority of the problems as percents, searching for a procedure, working out a large number of homework problems "to memorize" the procedure, and staying away from word problems. On the first questionnaire, it was evident that while Bob wanted the "formula" for probability, he was able to make a connection by using percents. However, with that recognition, he still had difficulty reasoning through some problems not requiring an understanding of probability. On the second questionnaire, Bob had the procedures he was looking for on the first questionnaire. He had the ability to use the procedures on simple probabilistic events. He did incur problems using set operations and dependent events. His problem solving methods also did not include drawing a diagram to assist with his reasoning strategies. Finally, Bob appeared pleased at his ability to solve the problems on the second questionnaire and noticed his own weakness of not doing enough homework problems to learn the general procedures. In addition, Bob did encounter some difficulties on the second questionnaire. First, Bob was able to recognize set operations but did not know how to organize and analyze the numbers in the problem. Second, Bob was able to recognize dependent events, but still was unable to find the probability of a dependent event. Third, Bob was uncertainty of how to state the final answer, as a decimal, a percent, or a fraction. However, Bob was able to see a connection between probability and percents.

Charlie

Charlie, a 20-year-old sophomore, recently changed his major from computer engineering to business. While pursuing his engineering degree, he completed differential, integral, and vector calculus and discrete mathematics. While enrolled in finite mathematics, Charlie realized he preferred the theoretical side of mathematics found in calculus, to the practical, real-life application approach he was experiencing in finite mathematics. However, Charlie claimed:

> I never really liked math... Well, I mean, it's weird, because my whole life all my teachers have taught me that, you know, like you can do it this way, but then they teach you the easier way, and then they teach you other ways. So, I mean, figured from that like all you need is the numbers, and then just manipulate it enough and then you'll somehow come out with something. And, surprisingly, it's worked!... But math is also one of the things where like you can sit down with it and then there's always more than one way to figure it out, so I kind of try and figure it out the way that they ask us not to do (it).

Charlie's experience with probability included a probability class in high school and the set theory and probability theory he had in his discrete mathematics course.

Charlie's primary method for solving probability problems during the first Task-Based Questionnaire consisted of reading the problem, stating an estimate, solving the problem, and check the answer against his initial guess and the context of the problem. This appeared to be a solid approach to solving the problems overall. However, after Charlie would read the problem and not know how to solve it, he would rely on his belief that "all you need is the numbers, and then just manipulate it enough and then you'll somehow come out with something." He created an algorithm of performing various operations to the numbers in the problems then checking the reasonableness of the result. Using this method, Charlie was successful solving problems on simple events, comparison of two events, and properties of probability. However, Charlie encountered difficulty solving compound, independent, and dependent events.

An example of Charlie's algorithm for manipulating numbers was exhibited in Problem 6. For Problem 6, Charlie did not recognize the use of set operations on the compound event, instead, he decided to add, multiply, and subtract the numbers until he achieved a reasonable answer. In addition, not feeling comfortable with the context of the problem, Charlie did not state an estimate at the beginning:

> From a survey at a large university, a market research company found that 75% of the students owned stereos, 45% owned cars, and 35% owned both cars and stereos. If a student at the university is selected at random, what is the probability that:
>
> a. The student owns either a car, or a stereo, or both?
> b. The student does not own a car?
> c. The student has neither a car nor stereo?
>
> *Student's Response to Part A:*
> So, hmm, my first intuition is just to add them up, but then ... you get over 100%, which isn't good. So then, obviously that can't be the answer, so then I multiply. I go .75 times .45 times .35 equals (uses calculator). No, jeez... 11.8% Oh, that can't be right. Oh, umm ... Well, this is weird. I guess I will have to say 100%. Because, hmm... well, if 75% of the students own stereos...45% own cars, and 35% own both cars and stereos ... maybe I could go 75% minus 35% would equal 40%, so ... umm... okay, never mind. Okay, I don't know the question.
>
> *Student's Solution to Part B:*
> Uh, 55%? Because it would be 100% of the students minus 45% that own cars equals 55%? I believe.
>
> *Student's Solution to Part C:*
> And c, the student has neither a car nor a stereo. Oh baby! I would go with 11.8%, just out of intuition (the answer from multiplying the three numbers).

Using his algorithm, Charlie multiplied, added, and subtracted the numbers found in Problem 6. After pondering on the reasonableness of the three quantities, Charlie decided that none of the quantities was correct. Using this algorithm, Charlie only calculated Part B correctly.

Charlie continued "manipulating" numbers on Problems 7, 9 and 13 seeking a reasonable answer, believing his "first intuition is to multiply them (the numbers in the problems)." However, for Problems 3 and 13, Charlie solved the problems by only multiplying the fractions without trying other arithmetic operations. When asked at the end of the interview why he only multiplied for each of these problems, he replied " I think it is just because I've always been taught that when you add fractions it's bad, like you just don't add fractions. So, it - I mean, it would always make more sense to me to multiply them."

Continuing on his manipulation of numbers algorithm, Charlie was surprised when it did work for Problem 10:

> A market research firm has determined that 40% of the people in a certain area have seen the advertising for a new product. Given that they have seen the advertising, 85% have purchased the product. What is the probability that a person in this area has seen the advertising and purchased the product?

Charlie started the problem by saying, "Once again, I think I have to go with the - trying multiplication first." After producing .34 on his calculator, he tries to reword the problem to fit his answer:

> So - this is going to sound weird - with these kind of problems what I do, like, 85% have purchased the product of the people that have seen advertising 85% have purchased the product, so that means that the advertising has got to be pretty good. So, if 40% of the people have seen the advertising, then 85% of that -Oh! 85% of that 40% will have bought the product.

After rewording the problem, he realized he hit the correct operation on the first try, thus stating the correct answer of 34%.

Besides his continual use of his own problem solving algorithm - estimate, calculate, check - Charlie continually checked the reasonableness of some solutions by trying to find alternate methods of calculation, by rewording the problem to see if the answer made sense with the context of the problem, or by checking his calculations.

Surprisingly, for Problems 3, 4 and 9, he was able to convince himself that the number he obtained through his algorithm matched the context of the problem, even if the numerical answer was wrong. After Charlie multiplied the two numbers together in Problem 9, he tried to convince himself he found the correct numerical solution:

> A warning system installation consists of two independent alarms. During an emergency, the probability of each alarm operating properly is 0.95 and 0.90 respectively. Find the probability that at least one alarm operates properly in an emergency.

Charlie multiplied the probabilities of the two events and calculated the probability to be .855. He reasoned this answer was correct because:

> which from my classes I found that when you have - yeah, when you have two things that are different probabilities of happening, that the chance that at least one of them happens is going to be less than both of them. Yeah, it's going to be less than both of them. So, if my lowest number is point 90, so this fits - point 855.

However, with independent systems, the probability of at least one of them operating should be larger than the probability of only one of them working, thus, should be greater than .90 (for the correct solution to this problem, see Aaron's solution).

Charlie also encountered difficulties with his reasoning skills. When given a probability distribution, he wanted to know the number of items in the original sample; when calculating the probability of two dependent events, he wanted to know what happened the first time; and used non-mathematical reasoning in his final response.

Charlie's method included reading the problem and estimating an answer. However, when he tried this with Problem 4, he did not trust his first or second intuition and had to back it up with mathematical reasoning:

> In a deck of cards, 1/6 are green, 1/12 are yellow, 1/2 are white, and 1/4 are blue. If someone takes a card from the deck without looking, which color is it most likely to be? Why?

When starting to solve this problem, Charlie claimed his first intuition was that the answer was yellow, "my first intuition was to go with the big number (largest denominator)." After reading the problem again, Charlie was debating on white, "because it says that half of the deck are white." Using an alternative approach from his intuition, Charlie proceeded to find the size of each event. Assuming there were 52 cards in the deck, Charlie computed there were 26 white, 13 blue, about 8 green, and about 4 yellow. With these numbers, he concluded, "obviously you have the biggest chance of drawing the white." However, the sample he had just created had only 51 elements, because he was not concerned about loosing .67 of a green card when dividing 52 by 6, or the .33 of a yellow card when dividing 52 by 12. Charlie's method for solving this problem was to find the number of items in each event and compare the size of each event.

Charlie did not appear comfortable calculating the probability of a complex experiment in which two of the objects were drawn without replacement. When trying to find the probability of the complex experiment in Problem 13, Charlie wanted to know what had happened on the first event:

> A box contains 2 red, 3 white and 4 green balls. Two balls are drawn out of the box in succession without replacement. What is the probability that both balls are red?

Charlie correctly stated, "you have a 2/9ths chance that the (first) ball you draw will be red." However, when Charlie wanted to find the probability of the second ball being red, Charlie reasoned, "so, you draw one out and then on your next time you put your hand in there, there are only 8 balls, but the chance of you drawing a red one kind of depends on whether you drew a red one the first time or not. Charlie encountered the same dilemma when he tried to solve Problem 14, where he had to find the probability of the complex experiment of drawing two socks.

On the first questionnaire, Charlie tried to maintain the same method for solving all the problems: read the problem, state an estimate, solve the problem, and check the answer against his initial estimate and the context of the problem. However, he did not always succeed at all the steps. If Charlie did not understand the question, he did not venture an initial estimate. In addition, when Charlie did not know how to solve the problem, he would use his algorithm of manipulating the numbers. Finally, if Charlie felt he found a reasonable solution, he would try to convince himself it was reasonable.

Factors that helped Charlie approach these problems included his arithmetic skills, his ability to recognize probability properties, and his ability to recognize the complement of an event. The factors that hindered Charlie's ability to solve the problems included his poor interpretation skills when trying to reword the problem, his inability to recognize set operations, and his inability to work with dependent events.

Charlie was one of the entertainers in class. Notorious for being late, Charlie would often slither into the classroom during a lecture while trying not to be detected. Once in his seat, he was easily distracted. Charlie would fidget in his seat, twirl his pen, or play with a rubber band. He did not distract other students, but after arriving late, would ask Aaron the section being covered in lecture that day. Charlie often had a notebook and calculator on his desk. During the lecture, he would interact with the instructor, ask questions regarding the lecture, and use his calculator to compute answers to problems posed during the lecture. When asked if he took notes during lecture, he claimed, "No, I don't think I've ever taken notes. Actually, I'm kind of just there to catch the main points, and what he's (the instructor) going over and when the tests is going to be, and to hand stuff in." On the second questionnaire, he was asked what he used to study for the course. Charlie claimed homework was what helped him understand the concepts the most, "homework is excellent, like that's how I learn basically everything." However, Charlie found this class tedious stating "on the last two tests I didn't read it (the book) at all, and then I read the whole chapter the night before the test, and I did that on both tests and I got As on both of them. So I think... like it's (the course) is too slow for me." Having taken discrete mathematics before, he believed this course was more of a refresher course for him. Prior to the second questionnaire, Charlie did not look over the chapter on set notation and probability, nor did he try his homework.

On the second Task-Based Questionnaire, Charlie appeared tired and did not venture an initial estimate to his problems, as he did during the first questionnaire. However, his overall method for approaching the problems remained the same. He still encountered difficulty solving compound, independent, and dependent events, while successfully solving simple events, comparison of two events, and properties of probability.

Returning to Problem 6, Charlie recognized that the problem was requiring set operations to interpret and solve the problem. He chose to use the Venn diagram to represent the data; however, after drawing the two circles, Charlie realized he did not know how to use the Venn diagram to help him solve the problem, and he resorted to his algorithm of manipulating numbers:

> From a survey at a preschool, a pharmaceutical company found that 75% of the students contracted chicken pox, 30 % contracted measles, and 20% contracted both the chicken pox and measles. If a child at the preschool is randomly selected, what is the probability that:
> a. The student contracted the chicken pox, or the measles, or both?
> b. The student did not contract the chicken pox?
> c. The student contracted neither the chicken pox nor measles?

Student's Solution for Part A:

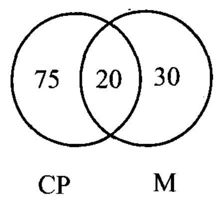
So, since you have one, or the other, or both, with the Venn diagram -

So we'll say this one is chicken pox, over here, and this one is measles, and this one is both.

So 75% contracted the chicken pox; 30 contracted measles, and 20% contracted both.

So, if a child at the preschool is selected randomly, what is the probability that: The student contracted the chicken pox, or the measles, or both? Huh... oh no! Uh, well, I guess I'd just go with a times b times c, which is I'd just multiply all these across and see what I get. 4 1/2%. So, ... Dang it! That seems like a really low number for that. Well I guess I'll just go with that, because I didn't pay attention.

Student's Work



Frustrated that he could not recall how to use a Venn diagram to represent the problem, Charlie continued multiplying numbers to obtain the incorrect answers to Part B and C of Problem 6.

Charlie continued his algorithm of manipulating the numbers to solve the remaining problems. However, this time when asked why he only multiplied the numbers in Problem 6 and 7, without considering the sum of the numbers as a possible solution, he stated,

> Those have percents that threw me off. I think whenever I see percents like that later problem (Problem 10) where I was talking about it was 30% of the 60%; I think I need to multiply. So, I kind of see like all percent problems in that way, I think. Which is obviously wrong. But, I mean, 'cause you don't add up percents… I think I am stuck on that error.

The same factors found on the first questionnaire were evident on the second questionnaire. For Problem 9, he multiplied and added the two numbers, deciding to take the product of the two numbers as his answer. Again, this process was incorrect, and he proceeded to try to reason a possible solution.

Charlie did not have the same problems comparing probabilistic events as he did on the first questionnaire. Regarding Problem 4 in which Charlie had to compare probabilistic values, Charlie did not try to find the size of each event. Instead, Charlie found the common denominator of the fractions and compared the numerators to find the event with the largest chance to occur.

However, a new factor did emerge on the second questionnaire. On the first questionnaire, Charlie had difficulty calculating the probability of a complex experiment in which he had to calculate the probability of two items being selected without replacement. Charlie wanted to know the outcome of the first event. This requirement to know the outcome of the first event was still evident while he solved Problem 13 on the second questionnaire. However, on the second questionnaire, Charlie changed his approach to solving Problem 14 to listing the sample space instead.

> Imagine your bedroom is dark and you cannot see a thing. There are three socks in your drawer: two are blue and one is red (they are identical except color). If you withdrew two socks, are you more likely to get two blues (a match), or a blue and red (no match) or are the outcomes equally likely?

Charlie demonstrated an understanding that he was finding the probability of two dependent events. Breaking down the problem, Charlie wanted to find the probability of withdrawing a blue and a red from the drawer. He was stuck trying to figure out the probability of pulling a blue one first against the probability of pulling a red one first. He recognized the dependency, but needed to know which sock was pulled out first and while listing the sample space, decided that the event of selecting a red sock first followed by the blue sock was the same event as selecting a blue sock first followed by the red sock. Unfortunately, these are two different events; however, they do have the same probability of being selected.

Charlie's inability to recognize the possible combinations of two events also occurred when he tried to calculate the possible combinations of two independent events, as in Problem 3:

> The two fair coins shown below are part of a carnival game. One side of a coin is black, the other side white. A player wins a prize only when both coins land on black after each coin has been tossed once.

> Caroline thinks she has a 50-50 chance of winning.
> Do you agree? Explain why or why not.

Charlie's original response to the problem was "When you want both of them, you want to multiply, so that's a quarter chance of having them both be black." At the end of the interview, he was asked how he knew to multiply, he explained, "Because there are four possibilities, like you can have two blacks, a black and a white, a white and a black, or a white and a white. But, I think these were kind of the same (referring to the combination of a black and a white), so maybe it's a third. Because I don't -- no, I'm just going to stop thinking about that." Charlie could not provide an explanation as to why he accepted his original response of 1/4. While writing out the sample space, he started to question if the event " a black and a white" was the same as the event "a white and a black."

Overall, Charlie used the same problem solving method for both questionnaires: read the problem, state an estimate, solve the problem, and check the answer against his initial guess and the context of the problem. However, on the second questionnaire, a new factor emerged: Charlie's understanding of combinations. This lack of understanding combinations could also influence his inability to find the probability of two complex events without replacement.

Dennis

Dennis was a 22-year-old freshman who attended college sporadically since graduating from high school. Coming out of Army Basic training, Dennis was pursuing a degree in business administration. He preferred to avoid mathematics classes and only enrolled in the courses required by his degree. Mathematics was not his favorite topic. He had taken college algebra twice and was enrolled in finite mathematics the prior year. Earning a C in his first attempt at finite mathematics, he returned to the class hoping to earn a better grade. He believed he was the perfect student for the study because "this is the second time taking it and I still find it just as hard… I think probability is what threw me for a loop (the last time)... it seems a little bit over my head." This term he was concurrently taking business calculus with the finite mathematics course.

On the first Task-Based Questionnaire, Dennis demonstrated his strong arithmetic skills, by preferring to reason with fractions rather than converting to the decimal equivalent. Dennis successfully solved the simple events and comparison of two events, while encountering difficulty with compound events, independent events, dependent events, and properties of probability. He was intimidated by the multi-stepped problems and skipped over ones he knew he could not solve. He entered the problem solving session with a carefree attitude, stating, "I don't think I will be able to solve any of these anyway." His attitude changed slightly when he was able to solve the first five problems.

Once he began reading Problem 6 aloud, Dennis recognized his weakness in solving certain probability problems. While reading the problem for the first time, Dennis would interject his comments:

> From a survey at a large university, a market research company found that 75% of the students owned stereos, 45% owned cars, and 35% owned both cars and stereos. *Yeah, this is where I go bad.* If a student at the university is selected at random, what is the probability that: *(laughs) oh boy! I know I'm going to have a lot of these in class, too.* The student owns either a car, or a stereo, or both? *Oh, that's where the old Venn diagram comes into place.*

Dennis attempted to use a Venn diagram to represent the problem, but could not recall the procedure to fill in the various sets. He recalled "that there is some kind of subtraction, but I'm not sure which one does what." However, skipping Problem 6 and continuing onto Problem 7 that required similar reasoning, Dennis recalled how to use a Venn diagram and successfully set up the Venn diagram to represent the set operations implied in Problem 7. Once the Venn diagram for Problem 7 was set up, Dennis then realized he did not know how to interpret the numbers, stating, "Well, I think that part's right, but I'm not sure how to do the rest...all right" and proceeded to the next problem.

Confused with the set operations and their representation on a Venn diagram, Dennis recognized two key words associated with set operations: "and" and "or." Moreover, Dennis knew the algorithm associated with word problems in which a word problem with the word "and" means multiply and the word "or" means addition. This is the reasoning he encountered when solving Problem 8:

> All human blood can be typed as one of O, A, B, or AB, but the distribution of the types varies a bit with the race. Here is the distribution of the blood type of a randomly chosen black American:
>
> Blood Types of Black Americans
>
> | *Blood Type* | *O* | *A* | *B* | *AB* |
> |---|---|---|---|---|
> | Probability | 0.49 | 0.27 | 0.20 | ? |
>
> a. What is the probability of type AB blood? Why?
> b. Maria has type B blood. She can safely receive blood transfusions from people with blood types O or B. What is the probability that a randomly chosen black American can donate blood to Maria?
>
> <u>*Student's Response Part A:*</u>
> Well I would think it would be A and B together.
> Times them together.
> I don't know, I always do that, I always times them together.
> .27 times .2 equals .054. That sounds good to me. All right.
>
> <u>*Student's Response Part B:*</u>
> All right, O or B, instead of O and B. That's great, I love that! O or B...
> Hmm... I'll just the opposite. I'll just add them together: .49 plus .2 is .69. Great. There you go.

Asked at the end of the interview if he knew blood type AB was a separate type than either type A or B, Dennis claimed, "Yes, I do! But I don't know how else to solve that problem."

Dennis appeared to have difficulty finding the probability of a complex experiment without replacement. Proceeding to Problem 13, Dennis recognized the dependency of the complex events, but he did not know how to set up the probability of each individual event:

> A box contains 2 red, 3 white and 4 green balls. Two balls are drawn out of the box in succession without replacement. What is the probability that both balls are red?

Student's Response	*Student's Work*
There's 7, 8, 9 - 9 balls	
1 out of 9	1 : 9
And then 1 out of 8	1 : 8
Then I am thinking I've got to do something with those 2 (pointing to the white and green balls) but I am not sure what. What is the probability that both are red? Yeah. Well, okay.	

Dennis did not finish the problem; however, based on his verbal explanation, Dennis had difficulties stating the probability of selecting a red ball from the box on the first draw. Dennis did recognize that once the first ball was removed, then the box had 8 remaining balls in the bag. He then stated the probability of selecting a red ball from the box on the second draw as 1 out of 8.

Observing how Dennis solved Problem 13 clarified some of the possible thought processes observed on Problem 14. Dennis solved Problem 13 in a similar manner to the method that Charlie used:

> Imagine your bedroom is dark and you cannot see a thing. There are three socks in your drawer: two are blue and one is red. If you withdrew two socks, are you more likely to get two blues, or a blue and a red, or are the outcomes equally likely?

Dennis was able to look at each event separately and successfully calculated the probability of withdrawing two blue socks. While calculating the probability of obtaining two blue socks, Dennis did not use the same procedure from Problem 13 when he wanted to obtain two red marbles. Instead, Dennis used the correct procedure to calculate the dependent event: (2/3)(1/2). However, when Dennis tried to calculate the probability of withdrawing a blue and a red, he found the probability of drawing a red and blue, in that order: (1/3)(2/2). After comparing these two probabilities, Dennis then realized that the probability of obtaining a blue and a red depended on which color was chosen first:

> I don't know, maybe that's some messed up logic, but... Anyway, so when you've got three in there at random, you've got two chances of getting a blue one. And you've got one chance of getting a red one. And then after that one's gone and if you did have a blue, then you'd have a one out of two chance of getting the red. And if you did get the red, well then you have a 2 in 2 chance of getting the blue

Dennis' logic was correct, but he could not decide which had the larger probability.

Overall, Dennis did recall some of the procedures associated with solving probability problems. He recognized the set operations in Problems 6 and 7 and wanted to represent these problems using Venn diagrams. However, Dennis encountered difficulty setting up the Venn diagram in Problem 6. Once he felt comfortable with setting up the Venn diagram in Problem 7, Dennis encountered more difficulty interpreting the various sets. In addition, Dennis displayed strong arithmetic skills, demonstrating the ability to work with fractions and percents. Additional factors which hindered Dennis' ability to solve the problems included his need to find the procedure, his difficulty interpreting the probability of a complex experiment of two successive events without replacement, and his lack of effort in checking the reasonableness of his answer.

While solving the problems on the first questionnaire, Dennis did not appear to be motivated to solve the problems. He would approach the problems by reading the question, and if he could not devise an immediate answer or method to solve the problem, Dennis would skip the problem and try the next problem. If he felt the problem was too difficult, he would not try the problem. However, when asked about certain problems and his approaches to solving the problems after he finished the questionnaire, Dennis did put some effort into explaining his reasoning to reach the solution.

Dennis was another ideal student who arrived early to class, sat in the middle of the classroom, and continually took notes. Students who sat in Dennis' vicinity, including Aaron and Charlie, depended on Dennis' record keeping skills to keep them up to date on assignments and exams. The instructor often called on Dennis to answer questions, and Dennis was not afraid to ask questions during lecture. During the second interview Dennis mentioned that he did write the lecture notes down into his notebook. He also brought a calculator and followed along with the problems on the board. When asked about what helped him understand probability the most, Dennis claimed it was the lecture notes and textbook examples. Fortunate to have mornings available to study, Dennis explained he sat in the college's mathematics learning center at least nine hours a week to work on his homework problems. When he was stuck on a problem, he sought help from a tutor at the learning center. However, Dennis had to split the time studying his two mathematics courses, finite mathematics and business calculus.

By the second Task-Based Questionnaire, Dennis had completed the majority of the suggested homework problems, but he still acknowledged he had difficulty with the probability problems. Dennis' enthusiasm while completing the second questionnaire made it apparent that he felt as if he learned something in class. On the second questionnaire, Dennis successfully solved simple events, comparison of two events, compound events, and properties of probability. In addition, Dennis encountered difficulty distinguishing between independent and dependent events. Just as in the first questionnaire, he was able to solve the first five problems with ease.

Dennis' method for searching for a correct procedure was apparent while solving Problem 5:

> A local supermarket randomly surveyed 800 customers about their smoking and drinking habits. The results of this survey are summarized in the table below.

	Smokers	**Non-Smokers**
Drinkers	220	480
Non-Drinkers	20	80

> Calculate the probability that a randomly selected customer drinks and smokes.

After reading the problem, Dennis reworded the question, "He's got to drink and he's got to smoke." Realizing that in both the original question and his reworded question, Dennis said the word "and" making him think, "something like A and B has to occur." With that in his mind, he responded, "And then I don't know if you're supposed to do all that sample space divided by whatever space, and then you've got the space and all that space stuff, I am not sure. So I'll just do what I think is the easiest on and hopefully that will be correct." This approach suggested Dennis was searching for the correct procedure to solve the problem, and this was the first procedure he could recall.

As Dennis approached Problem 6, requiring knowledge of set operations and the properties of probability, Dennis demonstrated the ability to represent the problem using a Venn diagram, double checking his arithmetic, and correcting himself when he tried to interpret the different pieces. His confidence and ability to work with Venn diagrams appropriately carried into his ability to solve Problem 7. However, while solving Problem 6 and 7, Dennis' verbal reasoning sounded mechanical, as if he was following a procedure.

On the first questionnaire, Dennis recognized the key words associated with set operations, found in Problems 6 and 7. This recognition of key words carried into his approaches of solving Problem 8 and 11 on the second questionnaire, also requiring him to recognize the key words. While solving Problem 8, he stated "A or B? Well that's outstanding. I've probability got to times them together." However when he multiplied the two numbers, he realized the resulting number was too small, stating, "Oh, it's 'or', oh, you just have to add them." While solving Problem 11, he exclaimed, "Pepsi or Coke. Well, that's just a nice little curve ball there. Pepsi or Coke. Well, if it was Pepsi *and* Coke, that would be easier to do, but since it's *or*, I'm really not sure." Searching for key words apparently did not mean he knew how to use them. For both Problems 8 and 11, his initial response was to multiply the two numbers, when the correct procedure was to add the two numbers.

Dennis' ability to solve the first twelve problems dramatically improved after his exposure to probability again in class. His strong arithmetic ability, his ability to use the Venn diagram to represent the problem, and his ability to break down problems into smaller episodes allowed him to reason through the problems.

Returning to Problems 13 and 14, where he previously encountered difficulty finding the probability of two successive events without replacement, Dennis' original approach changed completely. For Problem 13, Dennis resorted to finding how many ways two red gumballs could be gotten from the machine:

> A gumball machine contains 5 red, 7 white and 9 green gumballs. The gumballs are well mixed inside the machine. John gets two gumballs from this machine in succession, without replacement. What is the probability that both gumballs are red?

Instead of finding the probability of this situation, Dennis stated, "Then whatever the permutation of 21 items chosen 5 at a time multiplied by the permutation of 20 items chosen 4 at a time, whatever that equals, I am not sure…." Dennis did not proceed to find a numerical or probabilistic answer to this problem, but rather left the combinatorics representation of how many ways two red gumballs can be selected from the machine. Unfortunately, his reasoning of the correct number of ways two red gumballs can be selected from the machine was still wrong. The correct answer to the number of ways the two red balls can be selected is the combination of 5 items taken 2 at a time, or (5)(4)/2 since order did not matter. However, Dennis recognized the dependency of a ball being taken without replacement.

Dennis exhibited comfort with using a tree diagram to represent a problem. Dennis tried to use a tree diagram to represent the ways two socks can be withdrawn from the drawer for Problem 15:

> Imagine your bedroom is dark and you cannot see a thing. There are three socks in your drawer: two are blue and one is red (they are identical except color). If you withdrew two socks, are you more likely to get two blues (a match), or a blue and red (no match) or are the outcomes equally likely?

Using the tree diagrams correctly, Dennis stated the six possible combinations of selecting two socks from the draw, clarifying in his head the difference between selecting a blue and a red as opposed to a red and a blue, the combinatorics problem he encountered on the first questionnaire.

Overall, Dennis approached the first questionnaire knowing there were certain procedures involved in solving probability problems. Without a reasonable grasp of those concepts, Dennis would skip over the problems he knew he could not do without knowing the procedure. For the remaining problems, Dennis used his strong arithmetic skills to reason a possible solution or skip the problem completely. Approaching the second questionnaire, Dennis knew he had the tools at hand to solve the problems. He was able to use the Venn diagram on those problems requiring set operations and a tree diagram to list all possible outcomes on another problem. He still relied on key words, such as "and" and "or" to guide him to the correct mathematical operation, but still confused the words and application.

Some factors that helped Dennis solve the problems included his strong arithmetic skills, his ability to catch his arithmetic mistakes, his ability to use a diagram to represent a problem allowing him to organize and analyze the problem, and his recognition of the probability properties. Some factors that hindered Dennis' ability to solve the problems included his inability to work with dependent events, his difficulty with reading a table, and his need to search for a procedure by noting the key words.

Evan

After 10 years out of school, Evan, a prior medical corpsman for the Marine Corps and a current ski patroller, decided to go back to college full-time and earn a degree in recreational resources. As a 29-year-old sophomore, Evan believed mathematics was easier to learn now than when he was in high school, but Evan considered "math is simply a weeder class for me. It's 'can I get through this, am I capable of being taught'." Evan believed he had matured and "realized in order to get to where I want to go I have to do these math classes." The term he was enrolled in finite mathematics, Evan also helped teach a basic mathematics class at the college. He constantly talked about his experience teaching younger students arithmetic skills and was amazed at his inability to work with basic arithmetic operations when he first began teaching. Evan was concurrently enrolled in a statistics course while taking finite mathematics. When asked about his statistics course, he replied,

> It's hard for me to get used to because, you know, I'm - my pattern of thinking is a whole lot different, I think, than someone who would be really into statistics. So, it's hard for me to learn. You know, I'm sometimes very cut and dry, like there was a question today having to do with the probability of life on Mars. I don't think of it as being a probability, I think of it as being - you know, that, to me, would be an opinion. There is. It's cut and dry, *there has been life* here before. He

> doesn't want me to look at it in such a yes/no fashion. So, he looks at me and he goes, "Ah, you just have to open your mind a little more", and I go, "my mind is really open, that's why I believe that there's life in outer space", you know, but the probability that it exists - I don't worry about it. The probability that it exists has nothing to do with me because I believe that it's out there. So, that's where I kind of conflict with it sometimes, but, eh!

In addition, Evan stated that he had difficulty with the vocabulary associated with probability and statistics, "Because the vocabulary they use, a lot of the words are words that I would use every day but the meanings have been *completely* switched around to something where you're, 'What? My entire life that word has not meant that', and when I read (it now) I interpret it a totally different way."

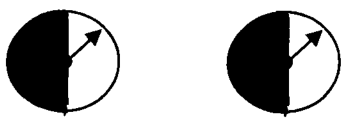
On the first Task-Based Questionnaire, Evan exhibited difficulty applying mathematical reasoning to solve the problems. Instead, Evan opted to use non-mathematical reasoning to solve the majority of the problems. He attempted to find an explanation that made sense to him, instead of using mathematics to solve it. He also encountered difficulty interpreting the phrases associated with probability. Overall, Evan was unable to successfully solve any of the problems on the first questionnaire.

Evan claimed he could not solve Problem 1 because he did not understand the vocabulary:

> From a batch of 3000 light bulbs, 100 were selected at random and tested. If 5 of the light bulbs in the sample were found to be defective, how many defective light bulbs would be expected in the entire batch?

After reading the problem, Evan stated, "See, there you go again, there's the vocabulary - 'would be expected in the entire batch'." He was stuck on this phrase and could not continue.

Evan also encountered difficulty deciphering between the empirical probabilistic concepts on the questionnaire and the probabilistic concepts associated with statistics. On Problem 2, Evan wanted to "calculate the probability off a chart" to find the probability of a simple event when given the sample space. He was confused between the applications of probability distribution tables found in his statistics course and the simple probabilistic concepts found on the questionnaire. Finally, for Problem 3, Evan gave up on finding a mathematical answer and started using a non-mathematical explanation for solving the problem:

> The two fair spinners shown below are part of a carnival game. A player wins a prize only when both arrows land on black after each spinner has been spun once.
>
> 
>
> James thinks he has a 50-50 chance of winning.
> Do you agree? Explain why or why not
>
> *Student's Response*
> No, I don't agree. There we go again, my favorite fan of Statistics - explain... I would probability say - you know, in a carnival, I would say 20 or 30% chance that you'll win. Because those games are not designed to be won all the time.

Evan did not search for a numerical answer, however, was able to state a reasonable range for the probability of the event, based on his experiences with carnival games.

While solving the other problems, Evan would continue using his non-mathematical reasoning to provide an answer. For example, after reading Problem 7 requiring knowledge of set operations, Evan stated, "With stuff like this, I don't think we've gotten far enough in class to give me the tools to work with." However, before approaching the next problem, Evan did provide an answer to Problem 7:

> You have torn a tendon and are considering surgery to repair it. The orthopedic surgeon explains the risks to you. Of the 500 patients who underwent this surgery, infection occurred in 40 of such operations, the repair failed in 100 of such operations, and both infection and failure occur together in 25 of such operations. Find the probability that if you undergo surgery, the operation will succeed and you are free from infection?
>
> *Student's Response*
> Well, see, there, go you again. You're asking me, who has a fairly decent medical background, and I would know - I would say *I* would get the operation just because, I mean, only 40 of them are occurring in infection out of 500. You know, and the part that failed, that was almost probably due to human error on their part for people who just won't sit around and actually let it rehab. And, you know, the 25 out of the 500 who did both, who knows? Human beings.
> I just know that I'm fully capable of taking care of myself, and, you know, a lot of the reasons some of these things fail isn't necessarily doctor error...So yeah, if only 40 - if infection was the only thing I would actually be worried about - if only 40 out of 500 were getting an infection, yeah, I'm fine. I won't be one of the 40

Evans reasoning for Problem 7 was that the chance of an operation being successful was more dependent on the individual's ability to take care of themselves than the data the doctor collected regarding the operation. Evans answer to Problem 7 was not incorrect; it was how Evan would have interpreted the situation if he was faced with such a problem.

Evan claimed he had a strong medical background, both as a Marine medic and as a ski patroller. He used this background information to supply an answer for Problem 7. However, for Problem 8, Evan decided to look at the trend of the numbers in the table than to rely on his medical knowledge:

All human blood can be typed as one of O, A, B, or AB, but the distribution of the types varies a bit with the race. Here is the distribution of the blood type of a randomly chosen black American:

Blood Types of Black Americans

Blood Type	*O*	*A*	*B*	*AB*
Probability	0.49	0.27	0.20	?

What is the probability of type AB blood? Why?

Student's Response
Well, I wouldn't be able to tell you the exact probability of AB blood, but I can say it's probably going to be less than .20. I mean, I'm seeing a falling trend of numbers here

With this response, Evan was noticing the pattern of numbers decreasing in the table without taking into consideration of the four disjoint, independent events.

Evan's frustration with probability and his weakness to interpret word problems caused a greater confusion when he encountered unknown terminology found in Problem 12. For Problem 12, Evan was unfamiliar with the phrase, "each marble has an equal chance of being drawn":

Two boxes each contain red marbles and blue marbles. One marble is drawn at random from a box (each marble has an equal chance to be drawn). If the marble is red, you win $1. If the marble is blue, you win nothing. You can chose between two boxes:

- Box A contains 1 blue marble and 4 red ones
- Box B contains 3 blue marbles and 4 red ones

Which box offers a better chance of winning, or are they the same. Explain.

Student's Response
Huh. Well it says each marble has an equal chance to be drawn. So is that just telling me it's half and half, either you're going to get a blue one or you're going to get a red one, right? Hmm. I would say they're both the same, because if each marble has an equal chance of being drawn, what's the difference of how many blue ones or red ones there are? That's what I would say.

Also observed in Problem 12, Evan explained probability in terms of his "yes / no fashion" stated during the interview. By Problem 12, Evan was clearly frustrated and did not attempt the last two problems.

While solving the problems on the first questionnaire, Evan displayed the inability to interpret some problems, poor arithmetic skills, and the need for wanting a procedure to answer the question. Of the four problems he did get correct, two of them he provided the wrong reasoning, however stated the correct answer. Evan did not display an overall method to solving these problems. After reading the first three problems and trying to solve them, he became frustrated and thought he was wasting time because he did not know how to approach them. Not concerned about time, he tried to solve the remaining problems, but when he could not understand the problem, Evan would start going off track and talk about related subjects or what he learned in statistics class. It was apparent that Evan did not like word problems, was not comfortable with the terminology, and viewed probability as a yes / no interpretation not subject to interpretation. In addition, Evan indicated more of a reliance on his book, searching for the correct tools to solve the problem, or a formula to put all the numbers into rather than trying to reason the correct procedure to solve the problem.

Evan attended class for half of the lectures pertaining to set operations and probability. When he did attend class, Evan sat up front, listening to the lesson without taking notes. He always had his calculator on his desk and would assist the instructor with computations. During the second interview, Evan claimed he was "not afraid to ask questions at all", but during the lectures, Evan did not appear to ask the instructor for many clarification questions. Instead, Evan enjoyed injecting jokes and sly remarks in the lecture, providing humor in the classroom. During the second interview, Evan was asked about his study habits for this section on probability. Evan explained he had not had the chance to look over the textbook or start the homework from the lesson. Asked about his view on solving word problems, Evan remarked:

> It's funny you ask that question. I used to hate them. I used to absolutely hate them, but like I've found out with any other mathematics that I've been doing, the more you just try and keep a positive attitude about it, and just sit down and do them and make the effort to learn how to read them, and how to break them down - I mean, you know I used to stare at word problems for hours and not be able to figure it out, but once you start breaking the information down into pieces that I can digest, you know, it makes it a lot easier and I actually have fun now trying to solve them because it literally is a challenge every single time I get a word problem. None of them come easy to me.

Evan's sporadic appearances in lecture provided him with some guidance for approaching the problems on the second Task-Based Questionnaire. While solving the first two problems, Evan felt more confident in his abilities, but was still searching for a formula. Evan still encountered difficulty successfully solving each problem on the second questionnaire; however, Evan was successful solving simple events, comparison of two events, and properties of probability. Evan still encountered difficulty solving compound, independent, and dependent events.

On the second problem, Evan needed to calculate the probability of a simple event, given the sample space. He solved the problem by finding the reciprocal of the answer: the number in the entire sample space over the number in the event. When he calculated an answer of 2.67, he appeared satisfied with the response, and continued to the next problem. However, once on Problem 5, where Evan had to calculate the probability of a simple event given a frequency table, Evan used the same formula used on the second problem, and calculated the answer 3.63. This rather large number did cause some conflict. Evan recognized this number was larger than 1, not possible for a probability. In order to correct the result, Evan calculated the reciprocal of this response - the correct procedure - and calculated .275. After the interview, when asked if the answer to the second problem was correct, Evan just laughed, and corrected it with no response.

On the first questionnaire, Evan used non-mathematical reasoning to solve Problem 3. However, on the second questionnaire, Evan's response to Problem 3 exhibited more mathematical reasoning:

> The two fair coins shown below are part of a carnival game. One side of a coin is black, the other side white. A player wins a prize only when both coins land on black after each coin has been tossed once.
>
> 
>
> Caroline thinks she has a 50-50 chance of winning.
> Do you agree? Explain why or why not.
>
> *Student's Response*
> See, it's that 50-50 chance again. No, I don't think she has a 50-50 chance, because if it were 1 of the coins I might start to think that she had a 50-50 chance, but with 2 I don't think it's equally likely that both of them are going to land face up. Just because there's, what, 4 different -- what is that, black/white, black/black, white/white - so there's 3 different ways it can land? Three different ways it could come up. So I don't think there's a 50-50 chance. I think it's less than half the time that'll happen.

However, Evan was still unable to provide a numeric solution to the problem. In addition, stemming from this problem, Evan did not associate the event of flipping a black and a white as a different event of flipping a white and a black coin, similar observation from Dennis and Charlie's responses.

Approaching Problem 6, Evan read the question. Not knowing how to solve it, he skipped to the next problem.

> From a survey at a preschool, a pharmaceutical company found that 75% of the students contracted chicken pox, 30 % contracted measles, and 20% contracted both the chicken pox and measles. If a child at the preschool is randomly selected, what is the probability that the student contracted the chicken pox, or the measles, or both?

When asked at the end of the session why he could not start solving the problem, Evan stated, "I was thinking there's something else I needed here. I don't know why. Maybe like a number of how many students there were total... that would be nice." When asked how would he change his approach if he knew the total number of students surveyed, Evan explained, "Because I guess I'm looking for something that - I think I am looking for something to take these percents from." After the interview, he was asked if there were 100 students in this survey, could he solve it? Evan replied, "well, then this is beyond 100%", referring to the sum of the three numbers in the problem.

When he approached Problem 7, requiring similar reasoning as Problem 6, however, Problem 7 provided the frequency elements in each event, instead of the percent of items. Evan tried some random calculations on this problem, and without stating an answer, moved onto the next problem. When asked if Problem 7 in which the frequency of each event was given was easier to solve, Evan claimed it was "more comforting" but still had difficulty interpreting the set operations. It was apparent for Problems 6 and 7 that Evan had difficulty with the set operations, but it was not apparent if his inability to attempt Problem 7 was due to his weakness working with percents or with set operations.

Turning to Problem 8, Evan recognized the probability distribution embedded in the problem. This time, instead of noticing the trends of the numbers in the table, he recognized the properties of probability and that the probabilities in the table should add up to 1. Recognizing this property of probabilities, Evan was able to solve Problem 8 correctly.

Finally, returning to Problem 12, Evan still had difficulty interpreting and solving the problem:

> Two boxes contain red marbles and blue marbles. One marble is drawn at random from a box (each marble has an equal chance to be drawn). If the marble is red, you win $1. If the marble is blue, you win nothing. You can chose between two boxes:
>
> - Box A contains 1 blue marbles and 2 red ones
> - Box B contains 6 blue marbles and 9 red ones
>
> Which box offers a better chance of winning, or are they the same? Explain

Evan first started searching for a formula and chose the formula he used in Problem 5: the number in the event over the number in the entire sample space. As in Problems 2 and 5, he knew the correct formula to use for the problem, but Evan did not know how to use it to solve the problem. For Problem 12, instead of Evan calculating the probabilities of each event, he calculated the odds in favor for the event occurring:

> <u>*Student's Response*</u>
> Just 6 divided by 9; 1 divided by 2 (calculating the odds for an event). Well, I would say that Box B has -- no, that's not right. Let me stop and think here... Point 5, point 6... so what is it about this question that I don't like? I would rather pick out of Box B. Because when you're thinking about probability, my *luck* would be better with Box B. Because with just 3 marbles in one box I would definitely pick the blue one and not win.

As he solved the problem, he started to doubt his answer, so he decided to provide an answer based on his non-mathematical reasoning.

Throughout the two questionnaires, Evan would comment about the terminology in the problems. Specifically, he appeared to struggle when he saw the phrases "equally likely" and "equal chance." When asked why he struggled with these phrases, Evan replied:

> Because they're -- sometimes they can be very misleading to me. "Equal" chance to be drawn, then you know that if it's an equal chance to be drawn, if you're saying that, then I should assume that it's a 50-50 chance. See, that's where these things -- you say "equal", I automatically start thinking they both have an equal chance; they're both equally likely, right?

Continuing on this conversation, Evan indicated that he interpreted the phrase "equal chance" as each event had the same chance of being selected, regardless of the size of each event in comparison to another.

Overall, Evan perceived mathematical probability as a procedure, preferring to represent answers as a decimal, while still using his non-mathematical reasoning to provide an alternate answer. Evan left the second interview still solving some problems with non-mathematical reasoning, but was starting to apply the mathematics to the problems. Evan encountered difficulty interpreting phrases such as "equally likely" and "equal chance" because they make him think an event has a 50% chance of occurring. Evan's ability to recognize and apply set operations to the problems did not improve on the second questionnaire; however, he did recall the properties of probability while recognizing that one of his problems was incorrect since the answer was not a number between 0 and 1. Evan still had a reliance on procedures, looking for a formula. In addition, when given a problem with percents, Evan wanted to know the number of children in that sample in order to solve the problem.

Freda

Freda, a 24-year-old sophomore from Botswana, was on a fellowship to study business administration in the United States. Not learning English until she was 13 years old, Freda explained that in her country half of her secondary education courses were taught in English and the other half taught in the language of her country, depending on the nationality of her teacher. Freda was accustomed to learning mathematics in English, but still felt the language barrier hindered her ability to interpret word problems. Freda believed the mathematics courses she had taken through American colleges were extremely easy. She explained,

> The math that I took in high school, I think it was very hard, than the math that I'm taking here. Because back home, you know, they're not allowed to use calculators and things like that, and we say final exam we mean something like - when you are in high school - high school is three years and junior high school is two years - so when they say final exam it's the material that you've covered within three years. For instance, for (the math final exam) it's the math that you've covered for three years. So, that's what we call final exams. And a lot of people fail. So when I got here, and the first time I was (taking a math exam), okay, I write like a passing grade may be when you have 90 you pass, you have 80, so I was like, "What!" I get an A? Because in my country 90 and above is like rare. I mean people don't get it, you know. Like 60, you've passed, and I was like here, 60, you've failed? I mean it's like very low. I was like, "Oh my, I'm not going to do it." So then I started taking exams here and realized that it is really easy. That's what I think. I think the education system around here is easier than back home. You know, like when you graph you don't use graphing calculators, you just use your head and pencil and pen.

Freda commented that since she has been attending American schools, she was "so used to the calculator that I cannot even calculate in my own head. You see, the calculators are spoiling us." Freda enjoys enrolling in a mathematics class and stated, "I really like math, but I don't know math, but I like it." She did not recall learning theoretical probability: she only "knows a little bit of probability, like when you toss a die and things like that, and when you use cards." While solving the problems, Freda indicated that she had learned set operations, Venn diagrams, and tree diagrams in high school.

Freda was curious to see what type of problems would be on the first Task-Based Questionnaire. While reading each problem for the first time, Freda extracted the numbers from the problem, label the numbers in context of the problem, and then read the question. Sometimes, Freda would try to organize the information with a Venn or tree diagram before reading the question on most questions, regardless of the problem. Once she started using a diagram, she would proceed using the diagram throughout the problem. Freda had the impression that she must use up all the space provided, so she drew rather large diagrams. Freda successfully solved simple events and comparisons of two events; however, encountered difficulty solving compound events, independent events, dependent events, and properties of probability.

Freda felt comfortable solving the first two problems, but when asked to interpret or compare the probability of an experiment, found in Problems 3 and 4, Freda encountered difficulty explaining the her reasoning to the answer. With Problem 3, Freda had difficulty stating an answer:

> The two fair spinners shown below are part of a carnival game. A player wins a prize only when <u>both</u> arrows land on black after each spinner has been spun once.
>
> 
>
> James thinks he has a 50-50 chance of winning.
> Do you agree? Explain why or why not

<u>Student's Response</u>	<u>*Student's Work*</u>
I think I agree. Yeah. Because ... because there are only two chances There are only two chances. It's either you win or lose. So, it's 50-50. (laughs) I think I got this one right, but I don't know	Yes Because there are only two chances. It's either you win or lose, so it's 50 - 50

Freda concluded that when given a probabilistic situation, there are only two chances: you win or you lose. Surprisingly, Freda did not indicate this type of reasoning on any of the other problems.

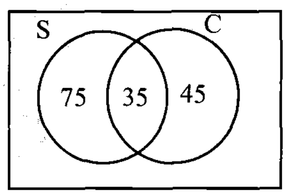
Freda also encountered difficulty comparing the probabilities of various events. For Problem 4, when given the probability of each event and asked to determine the event which is most likely to occur, Freda wanted to know the total number of elements in the original sample space to compare the number in each event, not their probabilities:

Student's Response:	*Student's Work:*
In a deck of cards, 1/6 are green. Okay, 1/6 are green.	1/6 g
1/12 yellow,	1/12 y
1, 2 (sic) are white,	1/2 w
and one quarter are blue.	1/4 b
If someone takes a card from the deck without looking, which color is it most likely to be, and why? In the deck 1 out of 6 are green, 1 out of 12 are yellow. Okay. I'm going to try to see the total of how many of this, how many of this. So 6 plus 12 is 18, 20, 24. (adding up the denominators)	24
So there are 24 cards in a deck.	
So...huh. I'm going to say 1/6 of 24.	1/6 x 24
One divided by six. Okay. One divided by six times 24.	
So, 4 for the yellow, oh the green.	= 4 green
And then 1/12 of 24 - this is 2 and then 2 for the yellow.	= 2 yellow
And then 1/2 of 24, because 1/2; 12 for the white.	=12 white
Six blue.	= 6 blue
So, without looking, which is most likely to be, which color is most likely to be picked, I guess.	
Okay. I think it's white. White, because...because... I think it's white because there are more white cards. There are more white cards in the deck (laughs)	White, because there are more white cards in the deck

Not only did Freda want to know the number of elements in each event, Freda had difficulty working with fractions. To find the total number of cards in the deck, Freda added the denominators to determine the deck had 24 cards. This approach of finding the number of elements in each event to determine the event most likely to occur was also observed in Charlie's first attempt at solving this problem.

Freda recalled using Venn and tree diagrams to represent set operations and the basic counting principles from high school. However, Freda encountered difficulty trying to use the Venn diagram to represent the information in a problem. In addition, while trying to solve Problem 6 requiring knowledge of set operation and percents, Freda did not recognize that her resulting probabilities were larger than 100%. While continuing to solve Problem 6, Freda was trying to use a formula to calculate the probability of each event, while misinterpreting the Venn diagram she constructed:

> From a survey at a large university, a market research company found that 75% of the students owned stereos, 45% owned cars, and 35% owned both cars and stereos. If a student at a university is selected at random, what is the probability that a student owns a car?

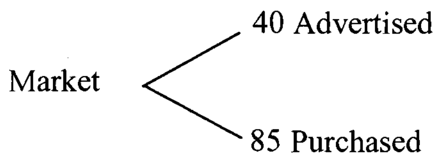
<u>Student's Response</u>	*<u>Student's Work</u>*:
...(Freda had drawn her Venn diagram and not is trying to answer the question) If a student at the university is selected at random, what is the probability that: The student owns either a car, or a stereo, or both? So, the probability that a student owns a car, a stereo (points to 75 and 45 respectively) Huh. I guess this is going to be ... the probability that a student owns either a car or a stereo or both... Huh. I guess I'm going to have to add on this and see what they add up to 75 plus 35 plus 45, they all add up to 155.	 = 155
The probability that he will own a car is 45, 50 - this plus this.	80 / 155 Car
And stereo, I'm going to add this and this. I don't know, I'm just 110 stereo, and 35.	110 / 155 Stereos 35 / 155 Both

Freda recognized on Problem 6 that there was a procedure to find the probability of an event, but while trying to fulfill this procedure. Freda was not able to organize and interpret the Venn diagram she had drawn, nor recognize the properties of percents.

Finally, while attempting the problems on the first questionnaire, Freda encountered difficulty trying to interpret Problem 10:

> A market research firm has determined that 40% of the people in a certain area have seen the advertising for a new product. Given that they have seen the advertising, 85% have purchased the product. What is the probability that a person in this area has seen the advertising and purchased the product?

Freda's original approach to solving the problem was to extract the numbers from the problem while trying to read the problem. Before reading the question, Freda constructed a tree diagram to represent the two numbers in the problem:



After reading the question, Freda had difficulty interpreting the word "given" in the problem. She responded "Given that they have seen the advertising, which ones? These ones? (pointing to the 40%) or these ones (pointing to the 85%)?" Freda was confused as to which group the problem was referring to. Bob had a similar interpretation with this problem. Both Bob and Freda thought the problem was asking about two distinct groups and they were to decide which group was involved in the question.

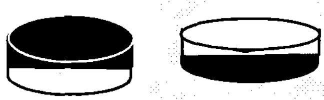
Overall, Freda's method for solving the problems consisted of extracting the numbers from the problems while reading the problem, drawing diagrams to represent the problem, then reading the question and try to answer it. Factors hindering Freda's ability to solve the problem included the inability to correctly organize and interpret data in a Venn or tree diagram, the need to find the number of elements in a sample space to compare their probabilities, the inability to work with fractions, and the weak arithmetic skills, especially with fractions. Freda indicated she had not had a formal class on probability, yet she knew she needed to find the probability by setting up a fraction. This dependency on a procedure did not allow her to observe that a problem gave her the probabilities of each event.

Due to an illness, Freda missed a week of lecture during which the instructor went over the section on probability. She was present for the lessons on the set operations and an introduction to probability. When she did attend class, Freda arrived late, sat in the front row next to Evan, and continually took notes. During class, Freda rarely asked questions during the lecture or answered questions the instructor posed to the class. She only interacted with two students in class: Evan and Harriet. She was continually laughing at Evans jokes and would rely on him to provide her with missed information pertaining to class. Before starting the second questionnaire, Freda admitted to only completing homework on the sections containing basic counting principles and simple probability.

On the second Task-Based Questionnaire, Freda slightly changed her approach to solving the problems. Instead of reading parts of the problem and extracting data from the problem, Freda would read the entire problem and the question before extracting numbers from the problem. Once selecting the numbers from the problem, she tried a variety of diagrams to represent the problem: Venn diagram, tree diagram, or a table. However, after selecting a diagram to represent the problem, Freda was not able to organize or analyze the information in the diagram. Overall, Freda successfully solved simple events and comparison of two events, while still encountering difficulty with compound events, independent events, dependent events, and properties of probability.

By the time Freda reached Problem 12, she was tired and reverted back to her original method of solving the problem, which was read a phrase, extract information, and construct a diagram to represent the information, before reading the question. Freda did not use a tabular representation of the data on the first questionnaire, which she did use on the second questionnaire. This could have been due to Freda missing a week of class. When she did return to class after her illness, the instructor had only used tables to solve the problems posed in class, no longer using the Venn or tree diagram to represent problems.

Some of the same difficulties Freda encountered during the first questionnaire continued on her second questionnaire. Freda still encountered difficulty interpreting and comparing the probabilities of various events. Returning to Problem 4, Freda still interpreted probability as a win or lose situation:

> The two fair coins shown below are part of a carnival game. One side of a coin is black, the other side white. A player wins a prize only when <u>both</u> coins land on black after each coin has been tossed once.
>
> 
>
> Caroline thinks she has a 50-50 chance of winning.
> Do you agree? Explain why or why not.

Freda's response to this question was similar to her response on Problem 4 on the first questionnaire: "Yea, I agree... Because there are only two sides. It's either you win or lose, so chances are 50 - 50." It is possible due to the language barrier, that Freda did not understand the experiment. Unfortunately, on the first questionnaire, Freda did not have the opportunity to solve Problem 12, a similar concept. However, on the second questionnaire, Freda did have the time to solve Problem 12, and surprisingly, still used the same approach to solving Problem 12 as she did Problem 2:

> Two boxes contain red marbles and blue marbles. One marble is drawn at random from a box (each marble has an equal chance to be drawn). If the marble is red, you win $1. If the marble is blue, you win nothing. You can chose between two boxes:
>
> - Box A contains 1 blue marbles and 2 red ones
> - Box B contains 6 blue marbles and 9 red ones
>
> Which box offers a better chance of winning, or are they the same? Explain

When first starting to read the problem, Freda explained while pointing to Box A, "Okay, here the probability is 50-50... whether you win or lose." However, she entered a conflict in her explanation when she approached Box B. Looking at the number of marbles in Box B, she decided to look at the ratio of the marbles in each box and compare the ratios:

> And then here (referring to Box B)... the ratio is 2 to 3... 2 to the 3 red ones. I think Box B. Because the ratio is high. It's higher than this one (Box A). I mean, here (Box A) the chances are 1 is to 2. And here (Box B) the chances are 2 is to 3. Or, are they the same? Let me see. ...They look like they're the same. Because like the ratio is 2 to 3, so the chances of not winning is 2 and the chances of winning is 3. It's just like here, you know, I mean the difference is just 1, even here the difference is just 1.

Freda reasoned that the chances of selecting a red marble from either box would be the same probability because the numbers in each of the ratios were different by a difference of one. After the interview, Freda was asked about her initial statement of "a 50 - 50 chance because you can only win or lose" regarding both problems. Freda explained she still believed it was a 50-50 chance of winning for Problem 4 because "it depends on how much you are given. Like here (Problem 4) it's white and black, you know, it's just 1 white and 1 black... so here (Problem 12) we are talking about marbles, and here it's going to depend on how many marbles you are given... yeah, it's going to depend on like how much red marbles we're given... yes there are two colors, but it depends on the number (of each color)." Freda was starting to confuse herself as to what 50-50 chance of winning meant to her.

Freda encountered similar difficulties when trying to compare probabilistic events. On the second questionnaire, Freda was given a probability distribution of the colors in a deck of cards. Instead of comparing the probabilities, Freda wanted to know the frequency of each color in the deck. Freda used this same method when solving Problem 4 on the second questionnaire:

> In a deck of cards, 1/6 are green, 1/3 yellow, 5/18 white, 2/9 are blue. If someone takes a card from the bag without looking, which color it is most likely to be? And why?

Instead of adding the denominators of the fractions together to find the number of cards in a deck as she did on the first questionnaire, on the second questionnaire Freda assumed there were 52 cards in this deck and proceeded to find the frequency of each color in the deck. However, 52 is not divisible by 6, 3, 18 or 9. So, when Freda tried to find the frequency of each color, she rounded the number she computed on her calculator for each color, and found there were 9 green, 17 yellow, 14 white, and 11 blue cards for a grand total of 51 cards in the deck. The truncating of the remainder of cards did not affect

Freda's reasoning skills. This same dilemma of loosing a card from the deck due to truncation was also observed in Charlie's first attempt to solve Problem 4.

Freda did try to construct tree diagrams to represent the Problems 11 and 12, Venn diagrams for Problems 6 and 7, and a table for Problems 5, 9, 10. Unfortunately, on all these representations, Freda did not correctly set up a diagram or table, nor correctly interpret them. After setting up (incorrect) diagrams for problems 6, 7, and 11 in which she was given the frequency of an event, instead of stating the probability of an event occurring, Freda's final answer consisted of stating the frequency of the event in question occurring.

Finally, returning to Problem 10, Freda still had difficulty interpreting the problem. On the first questionnaire, Freda thought the problem was describing two distinct groups of people. While attempting to solve Problem 10 on the second questionnaire, Freda still claimed, "that's kind of a tricky question":

> A bank observes that 60% of its customers prefer to visit a teller than an ATM machine -- an ATM when making a transaction. Given that a customer has visited the teller, 30% deposit money in their accounts. What is the probability that a customer in the bank has visited a teller and deposited money in their account?

Freda recognized that "60% of the customers preferred to visit the teller than the ATM" and constructed a table to represent this information, trying to figure out which cell would contain the 30%. She realized that wherever she tried to put the 30%, that row or column did not add up to 100%:

Teller	60%	
ATM		

Freda did not understand how the 30% could be represented in the table and decided to proceed to the next problem. Bob had also used similar reasoning to solve this problem. Bob claimed that it was a trick question, depending on how it was interpreted, that was how you knew which number to select as the answer.

Overall, Freda's approach for solving the problems was similar between the two questionnaires: Read the problem, extract numbers and label or place into a diagram or table, read the question, and try to use the constructed diagram to answer the question. Freda was familiar with tree and Venn diagrams from previous mathematics courses, but she tried to learn how to integrate the use of a table when solving the problems. Either way, Freda was not successful on selecting, organizing, or analyzing a diagram while solving the problems on either questionnaires.

Freda was hindered in her ability to solve the problems by her weak arithmetic skills, including ratios, proportions, and percents; lack of effort to check if she found the probability of an event or the frequency of the event; misunderstanding of probability as a 50 - 50 chance of winning; inability to recognize the sample space, and weakness in interpreting word problems. On the first questionnaire, Freda was aware that there was a procedure to find the probability of an event by constructing a fraction, but she did not use this approach while solving the problems on the second questionnaire.

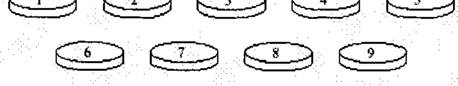
Greg

Greg, a 21-year-old sophomore, was fortunate to have the time and resources to continue his college education. Greg had taken the required mathematics classes in high school, but believing his mathematics skills were not strong enough to pursue a degree in business and accounting, Greg began his college career enrolling in pre-algebra. After two years of continually taking math, Greg was taking his final mathematics course, finite mathematics, required for his degree. Greg remembered that his high school teachers tried to introduce probability into the curriculum, but nothing beyond experiments of flipping a coin or rolling a die. He did not recall learning probability in any of his college classes.

Approaching the problems on the first Task-Based Questionnaire, Greg exhibited strong arithmetic skills and reasoning ability. Not having seen many of these types of problems before, Greg did not maintain one method for approaching the problems. Greg tried various methods on each problem, and when one method worked, he did not necessarily apply it to another problem with a similar concept. Overall, Greg was successful solving properties of probability, but did encounter difficulty solving simple, compound, independent, dependent, and comparisons of two events.

Greg encountered some difficulty interpreting probability. For example, on Problem 2, Greg was given the sample space, but could not determine the probability of a stated event without knowing which event was to occur:

The nine chips shown are placed in a sack and then mixed up:

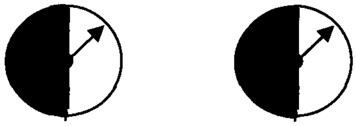


Madeline draws one chip from this sack. What is the probability that Madeline draws a chip with an even number?

Greg could not determine the probability of the event without knowing which chip was selected,

> It doesn't state which chip it is... because it didn't say if it was an odd chip she pulled out or an even chip... oh, wait, 9 chips are drawn yeah, it doesn't state which chip, which number the chips is... My understanding is it didn't say what the number was on the chip, so that leaves a - I feel it leaves a huge gap.

Greg explained that he needed to know which chip was drawn. Surprisingly, when Greg approached Problem 3, he proceeded to list the entire sample space using a tree diagram and deduced the probability of this experiment:

> The two fair spinners shown below are part of a carnival game. A player wins a prize only when <u>both</u> arrows land on black after each spinner has been spun once.
>
> 
>
> James thinks he has a 50-50 chance of winning.
> Do you agree? Explain why or why not.

After reading the problem, Greg proceeded to respond to the question by explaining, " I don't agree... Yeah, he has a 1 in 4 chance because... " in which Greg proceeds to draw a tree diagram representing the four possible outcomes for this experiment, then concluding, "so, it's 1/4 possibilities." When asked after the interview why he solved this problem using a tree diagram, he claimed he had used them before in high school and remembered a similar experiment. However, when asked again about Problem 2 after the interview, Greg did not see the connection between the two problems, still claiming, "Really, it doesn't say which one she drew, so there's inaccurate information here. There's not an abundant supply of information" to solve the problem.

Throughout Greg's attempts on the first questionnaire, he demonstrated strong arithmetic skills and reasoning skills. He was able to check the reasonableness of his answers by comparing the answers to an estimate. However, Greg did encounter difficulties calculating decimal equivalents of fractions with his calculator. He continually would enter fractions incorrectly into his calculator, and when he noted the decimal answer was too large for the question, he would then try to enter the reciprocal of his answer into the calculator. He felt confident that he was performing the correct operation, but was not proficient in using his calculator to find the decimal approximation.

When Greg approached Problem 6 and 7, he had difficulty working with the set operations embedded in the problems. He tried to solve them with the methods he was familiar with, but kept explaining, "these answers are wrong, the answer cannot be greater than 100%", recognizing that his attempts did not produce reasonable solutions. His inability to interpret set operation problems, exemplified in Problems 6 and 7, and his ability to solve Problem 8 correctly illustrated his comfort with the properties of probability, especially his understanding the concept of the complement of an event. The majority of the answers Greg provided were stated in percent format, so Greg might have approached these problems using percents.

However, despite Greg's strong mathematics and reasoning abilities, he was not able to work with Problem 10:

> A market research firm has determined that 40% of the people in a certain area have seen the advertising for a new product. Given that they have seen the advertising, 85% have purchased the product. What is the probability that a person in this area has seen the advertising and purchased the product?

For Problem 10, Greg was confused on how to find the probability of the event. He observed that on the previous problems that to find the probability of an event, he needed to divide a number by a number. Greg proceeded to divide .40 by .85 and compare it to .85 divided by .40. Realizing these results were not reasonable for the question, he then stated, "It doesn't state how many people were there… needs how many people total." This same dilemma of needing to know the number of objects that are in an event in order to find the probability of an event has also been found in the interviews conducted on other students.

Without having exposure to probability problems before, Greg did not recognize the concept of dependent events, nor how to calculate the probability of dependent events. For Problem 11, three of the parts required the calculation of the probability of a dependent event. Greg assumed the problem was testing his ability to read a table and find the probability of compound events. However, when solving Problem 13, Greg needed to find the probability of a complex experiment of two successive events for which there was no replacement:

> A box contains 2 red, 3 white and 4 green balls. Two balls are drawn out of the box in succession without replacement. What is the probability that both balls are red?

Greg concluded there was a 2 out of 9 chance of both balls being red. Greg did not take into consideration that the problem was to find the probability of two dependent events.

Overall, Greg did not exhibit one specific method while approaching the problems on the first questionnaire. Not having seen probability before, Greg would read the entire problem, extract the numbers from the problem providing labels for all his numbers, and then try to solve the problem. Greg used some of his knowledge of percents to find an answer. In addition, Greg recognized his weakness for not knowing how to approach the problems containing set operations. His strong arithmetic and reasoning skills helped him correct his arithmetic errors and check the reasonableness of the solution. However, he illustrated difficulty when faced with a percent problem. Greg wanted to know how many people were surveyed in Problem 10. In addition, Greg was unable to recognize dependent events, but tried to produce an answer he thought was reasonable.

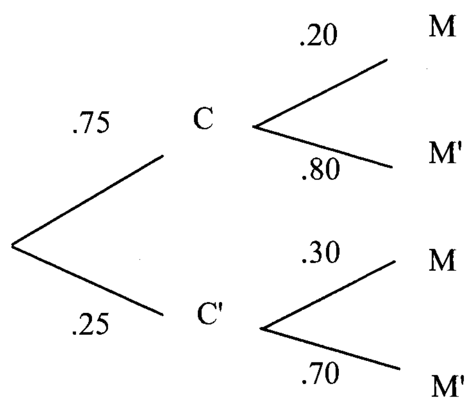
Greg was an ideal student in class. Greg knew he had a weakness in mathematics and took advantage of visiting the mathematics learning center at the college for free help on his homework. Greg was able to keep up with his assigned homework on a daily basis and was the center of an ongoing joke in class regarding his long list of homework questions to be solved in class. His peers joked with him regarding his homework habits, but they admitted they were jealous of his motivation to keep on task. Greg never missed a class and would sit next to the wall. He would continually answer questions for Bob throughout class, who sat behind Greg. Greg appeared to have strong organizational skills, arriving with a neat notebook and his calculator at his side. Whenever he did not understand a certain aspect of the lecture, Greg would not be afraid to ask for an explanation. During the second interview, Greg admitted that he completed all his homework for the section on probability. The combined effort of diligent work on his homework; the lectures providing him with the understanding of the technical terms used in the textbook and a visual presentation of how to solve the problems; and him seeking help both inside and outside the class made Greg feel more confident with the concept of probability.

Greg's study habits were evident when he approached the problems on the second Task-Based Questionnaire. Greg improved his ability to work with set operations, compound events, and dependent events. With various tools available to Greg, he was able to approach each problem with a similar method. Greg read each problem; determined which representation would best suit the situation; drew a complete tree diagram or table (including the calculations of cells or branches not necessary for the problem), verified the constructed representation by checking the rows, columns, or branches of the tree, then proceed to read the question and answer it. Greg appeared confident in recognizing the correct representation for each problem. However, on the second questionnaire, Greg incorrectly solved two problems, Problem 6 and Problem 12, because of his search for a procedure and the inability to interpret the question. Overall, Greg was able to successfully solve simple events, comparison of two events, properties of probability, independent, and dependent events. However, Greg still encountered difficulty with compound events.

Despite Greg's improved performance in solving probability problems, evidence on Problems 6 and 8 indicated that Greg still searched for a procedure to solve the problems instead of reasoning from the question. While solving the problems on the second questionnaire, Greg would draw representations of those problems containing the frequency of an event with a table and those problems containing the probability of the events with a tree diagram. When Greg approached Problem 6, he recognized the set operation terms embedded in the problem and initially intended to solve the problem using a table. However, once he re-read the problem, he observed that the numbers were given in percents. This prompted him to incorrectly solve the problem using a probability tree:

> From a survey at a preschool, a pharmaceutical company found that 75% of the students contracted chicken pox, 30 % contracted measles, and 20% contracted both the chicken pox and measles. If a child at the preschool is randomly selected, what is the probability that:
>
> a. The student contracted the chicken pox, or the measles, or both?
> b. The student did not contract the chicken pox?
> c. The student contracted neither the chicken pox nor measles?

Greg proceeded to create the following probability tree, assuming the two events were dependent events rather than independent events.



As Greg completed his probability tree, he included the probability of each branch. Continuing his method of solving probability problems, Greg did verify that the sum of the four branches of the tree was one. However, the problem did not specify whether the two events were dependent on each other. In addition, Greg did not compare the numbers in the problem to the percents he calculated for each branch. For example, the problem stated, "20% contracted both chicken pox and measles." The top branch of Greg's tree indicated the event of both chicken pox and measles. When using a probability tree, the probabilities of each successive branch would be multiplied, yielding the result of the probability of both chicken pox and measles as .15. Greg did not observe that his calculated response of 15% contradicted the reported 20% in the problem. Greg instead continued correctly answering the three questions using the tree he had created from this problem. Greg was dependent on using one method for problems stating the probability of an event and using another method for problems stating the frequency of an event. When Greg approached Problem 6, he was unable to use the correct representation for a problem containing set operations and the percent of the elements in a set.

Greg appeared to have knowledge of the tools he needed to solve the probability problems. Nevertheless, he knew that in order to find a probability of an event, the procedure was to find the number of elements in an event and divide it by the number of elements in the entire sample space. However, Greg continued having difficulty with problems that presented him the probabilities of various events and he had to calculate the probability of a compound event. For Problem 8, Greg wanted to find a fraction to represent the probability of an event occurring, given the probabilities of the event:

> Customers at a local coffee shop participated in a coffee taste test. The participants were asked to state which of the following four brands of coffee they prefer. The brands were labeled *A, B, C,* and *D*. The results are partially summarized as follows:
>
> Coffee Preferences
>
> | *Brand* | *A* | *B* | *C* | *D* |
> |---|---|---|---|---|
> | Probability | 0.25 | 0.43 | 0.05 | ? |
>
> What was the probability that the participants preferred Brand *D*? Why?

Greg correctly calculated the answer to the question as one minus the sum of the three given probabilities. However, in his explanation, Greg responded, "And the reason for that is because probability equals 1, so add total numbers, divide by 1.0, (the answer) would be 1 minus .73." Greg wanted to divide the sum of the probabilities of the three events (.73) by the probability of all the events occurring (1) rather than adding the two probabilities.

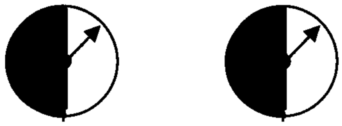
One difficulty that Greg encountered was finding the probability of a dependent event. When faced with Problem 11, Greg was able to correctly state the probability of the first three parts that required him to find the simple probability of an event. However, when Greg encountered a question requiring him to find the probability of a dependent event, he did not recognize the dependency in the problem, and continued to solve the problem as if the two events were independent. Surprisingly, when Greg approached Problem 13, which required him to find the probability of receiving two red gumballs from a gumball machine in succession without replacement, Greg recognized the dependent event. For Problem 13, he proceeded to solve the dependent event using a probability tree correctly. Greg's probabilistic reasoning skills dramatically improved on the second questionnaire. Greg was also proficient at using tree diagrams and tables to organize and analyze the problems.

Harriet

Harriet, a 19-year-old sophomore from Nigeria studying in the United States, was pursuing a degree in business administration. Harriet had spoken English for as long as she could remember. In her country, each community has their own language in which to communicate with each other. However, inter-communal communication is conducted in English. Like Freda, Harriet's schooling consisted of a mixture of classes taught in English and in her native language, depending on the nationality of the teacher. Harriet claimed she understands English better than she speaks English. Harriet recalled probability was integrated into the curriculum while in high school, but she did not recall which concepts she learned. Since enrolling in college, Harriet took college algebra and a statistics course.

On the first Task-Based Questionnaire, Harriet's methods for solving probability problems were sporadic. She recalled formulas, algebraic expressions, and tried to express her final answer in a fraction. When she had to compare probabilities, she continually ensured the probabilities added up to one, but then did not know how to proceed to answer the question. In addition, Harriet encountered some difficulty interpreting words and phrases in the problems, and she encountered some additional difficulty with her mathematical reasoning skills. Explaining that she wanted to complete the entire questionnaire during the interview, Harriet rushed through the problems. If she did not understand the wording of the problem or the question of interest, Harriet moved onto the next problem without providing a solution to the current problem. Overall, on the 14 problems, Harriet attempted nine, completed five, and solved four correctly. The four she did solve correctly pertained to calculating simple and compound events.

Harriet approached the second and fifth problem with ease, calculating the simple probability of an event. She read the questions and provided detailed explanations to their solutions. Approaching Problem 3, Harriet encountered difficulty with the wording of the problem.

> The two fair spinners shown below are part of a carnival game. A player wins a prize only when both arrows land on black after each spinner has been spun once.
>
> 
>
> James thinks he has a 50-50 chance of winning.
> Do you agree? Explain why or why not.

Trying to interpret the problem, Harriet continually repeated the phrase, "After each spinner has been spun once." After contemplation on its interpretation, she asked, "Does it actually tell how many times it should be spun around?" Receiving clarification on the experiment, Harriet stated, "I can't tell. Is that okay?" and proceeded to the next problem.

Harriet encountered additional difficulty comparing probabilistic events. When trying to solve Problem 4, Harriet wanted to know the total number of cards and which color the person selected:

> In a deck of cards, 1/6 are green, 1/12 are yellow, 1/2 are white, and 1/4 are blue. If someone takes a card from the deck without looking, which color is it most likely to be?

After extracting the information from the problem, she reasons, "Okay, total number of cards should be 1 over 6 plus 1 over 12 plus 1 over 2 plus 1 over 4." Harriet proceeded to find common denominators for each fraction and due to a miscalculation, found there was a total of 15/12 cards. She then stated, "How am I going to know what color a person gets?" Then turns to the next problem. At the end of the interview, her arithmetic mistake was shown to her, and she was asked again if she could solve the problem. Harriet claimed then, "It's likely to be yellow... or green." Asked why she selected those two colors, she said, "I don't know why, but I think it's green or yellow" and could not provide an explanation. Yellow and green had the largest denominators.

Harriet encountered difficulty with Problem 6, finding the probability of a compound event. Having studied set theory before, Harriet recognized the key words "and", "both" and "or" in the problem. However, the problem stated the percent of each set, and Harriet had difficulty finding a suitable representation for the numbers. For Part A, Harriet originally left the numbers in percents, then she changed the numbers into decimals, and finally, she tried to write them as fractions. Giving up on how to represent the numbers, Harriet proceeded to Part B, and decided to try a Venn diagram. Finally, after trying different representations, Harriet claimed, "It does not tell us how many students... do they want the answer in percent, or probability does not come in percents." After reading the problem again, Harriet claimed her answer to the problem was, "I don't think probability comes in a percent", and says the problem cannot be done.

However, Harriet did recall set operations and was completely successful solving Problem 7, requiring her to find the probability of a compound event:

> You have torn a tendon and are considering surgery to repair it. The orthopedic surgeon explains the risks to you. Of the 500 patients who underwent this surgery, infection occurred in 40 of such operations, the repair failed in 100 of such operations, and both infection and failure occur together in 25 of such operations. Find the probability that if you undergo surgery, the operation will succeed and you are free from infection?

Similar to Problem 6, Harriet recognized the key words "and", "both" and "or" in the problem. This time, Harriet was comfortable with the problem giving her the frequency of the events and recalled a formula she learned in school:

Student's Response	*Student's Work*
Okay, lets try x plus 125	
If 40 were infected and 100 out...	
Okay, minus 25	$x+40+100-25$
Giving me 115 so I subtract 500	
I subtract my outcome from the total number of patients who are undergoing surgery	115 500
That is 385	385
I know this is not going to come out right... on the probability that 385 people undergo this operation successfully. So the probability of me undergoing this operation successfully will be 385...	$\frac{385}{500}$

From the formula Harriet recalled, she was able to find the number of people who did not undergo the operation successfully. From this number, Harriet subtracted it from the number of total people who had undergone the surgery to find those who underwent the surgery successfully. Surprisingly, Problem 7 required the same reasoning as Problem 6, but Harriet was confused by the percents in Problem 6 and therefore, could not interpret and solve Problem 6.

Harriet's remembrance of a formula for solving probability problems also arose while she tried to solve Problem 8. For Problem 8, Harriet was given the probability distribution of the blood types of black Americans. However, one probability was missing. Harriet was asked to calculate the probability of the missing category:

All human blood can be typed as one of O, A, B, or AB, but the distribution of the types varies a bit with the race. Here is the distribution of the blood type of a randomly chosen black American:

Blood Types of Black Americans

Blood Type	*O*	*A*	*B*	*AB*
Probability	0.49	0.27	0.20	?

What is the probability of type AB blood? Why?

As in Problem 7, Harriet recalled a formula, but could not figure out how to work with the formula to answer the problem:

Student's Response	*Student's Work*
0.49 plus 0.27 plus 0.20 plus x equals 0	$0.49+0.27+0.20+x=0$
I don't know... I'm going to add the probability of type O, type A, and type B. Okay, that comes out with 0.96 plus x and my x is representing type AB.	$0.96+x=0$
Oh... that is not right. The probability of type AB is... 1 is 1. I came up with 1. I don't know if that is right.	1

After the interview, she was asked how she arrive at the final answer "1" for the problem. Harriet explained that she found the value of x to be 0.96, so she divided 1 by 0.96 and she got 1. One divided by 0.96 is equal to 1.041667 and Harriet rounded the answer to the nearest whole number. When asked about the formula she was using, she said she recalled this formula for finding the probability of an event, but knew it was not the right formula when she was trying to solve for x. Asked if she knew what the correct formula was, Harriet smiled and said, "I'll get back to you."

Overall, Harriet encountered various difficulties solving probability problems on the first questionnaire. However, she approached the familiar problems with formulas she recalled from her high school classes. For the problems containing simple probability, Harriet remembered to find the fraction of the number of items in the event over the total number of items in the sample space. For the problems requiring more skill, Harriet tried to recall formulas and their applications, as opposed to determining a good method for reasoning the problem. For those problems she tried to answer, Harriet had difficulty understanding the experiment, recalling the proper formula, using percents to represent the amount of items, and recognizing arithmetic mistakes.

In the classroom, Harriet was often late and missed half the classes pertaining to probability. While in class, Harriet sat in the back of the room and often was distracted by her notebook. It was not uncommon to find Harriet organizing her notebook, reading papers not related to the class, or working on other homework. Harriet was not responsive in class, did not ask instructor any questions during the lecture, and did not try the problems posed in class. When asked during the second interview about her homework, Harriet claimed she had a difficult term and did not have a chance to look over the section on probability. Instead, she started completing her homework in the next section because it was easier. In addition, she claimed she prefers to learn from the book than lectures for this course.

Unfortunately, Harriet had a scheduling conflict for the second Task-Based Questionnaire. Arriving at the second interview five minutes late, she claimed that she had another appointment in 30 minutes. Not able to find anther convenient time to reschedule the interview, she agreed to solve as many problems as the 30 minutes allowed. While working on the problems, she was getting frustrated at her inability to solve the problems.

Harriet used the same methods and encountered the same difficulty on the second questionnaire as she did in the first. She easily solved Problem 2 and 5, finding the probability of a simple event, still encountered difficulty interpreting the experiment for Problem 3, still could not compare probabilities of various events, and used the same formulas to solve Problems 6 and 8 on compound events and properties of probability. Unfortunately, due to time, she did not try Problem 7, also on compound events.

Harriet's inability to understand the experiment for Problem 3 from the first questionnaire carried into the second questionnaire:

> The two fair coins shown below are part of a carnival game. One side of a coin is black, the other side white. A player wins a prize only when both coins land on black after each coin has been tossed once.
>
> 
>
> Caroline thinks she has a 50-50 chance of winning.
> Do you agree? Explain why or why not.

Harriet responds, "Yeah, I agree because it's not like she's limited to playing it once, tossing it around once... Yea, I agree because she - I mean, there's a chance that after tossing it around for so many times she's going to get both coins landing on black." Harriet reasoned on Problem 3 from the first questionnaire that if the person spins the two spinners, they would eventually get two black.

Harriet exhibited difficulty comparing probabilistic events on the first questionnaire. On the second questionnaire, Harriet encountered the same difficulties when asked to solve Problem 4:

> In a deck of cards, 1/6 are green, 1/3 are yellow, 5/18 are white, and 2/9 are blue. If someone takes a card from the bag without looking, which color is it most likely to be? Why?

Approaching the problem in the same manner as the first questionnaire, Harriet read the question, stating, "so the total deck of cards are - I don't know" and proceeded to add up the fractions. Upon realizing the fractions add up to 1, Harriet stated she did not know, and proceeded to the next problem.

When Harriet got to Problem 8, she used the correct formula to find the missing probability. This time, she found the missing probability by using the equation $0.25 + 0.43 + 0.05 + x = 1$ to find the missing probability. This time she realized that the probabilities added up to 1.

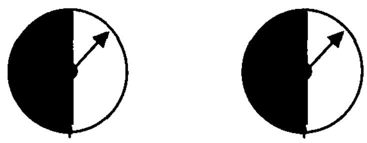
Harriet only improved on one factor from the first questionnaire to the second: recognizing that probabilities of disjoint events add up to one. Other than that, Harriet used the same methods to solve the problems on the second questionnaire. Harriet knew the formulas to use when calculating a simple probability of an event and the formula for finding the probability of an event, given the probability distribution. However, she still was not able to compare probabilities or work with problems in which the number of elements in a sample or event was presented as a percent. Unfortunately, she did not attempt the problems requiring her to find the probability of dependent events.

Ian

Ian, currently a 27-year-old sophomore, earned his GED at the age of 16 and decided to go to college at the age of 25. He saw a need to earn a degree in business administration after trying to start his own business in landscape maintenance. Not having the mathematics skills required for his degree, he began college by enrolling in a pre-algebra course and improved his skills by completing college algebra and business calculus before enrolling in finite mathematics. Although Ian kept his own records for his business, Ian claimed his job did not require many calculations. However, he enjoyed learning mathematics and would like to take more mathematics courses. He did not recall taking a course containing probability and was curious to see how it was different from other mathematics topics he had learned in college.

On the first Task-Based Questionnaire, Ian's strong arithmetic reasoning skills helped him interpret and solve the majority of the problems correctly. Without a knowledge base of probabilistic terminology, Ian did encounter some difficulties trying to interpret the meaning of phrases associated with probability. However, as he continued to solve the problems, he tried to integrate the newly found words into his verbal responses. Ian was able to reason correctly the more basic problems requiring him to find the simple probability of an event or apply the properties of probability. However, Ian did not recognize the terminology associated with set operations, nor did he observe that some of the problems required him to find the probability of a dependent event. Overall, Ian enjoyed solving the problems and was looking forward to learning probability and how to solve the problems on the questionnaire correctly. Ian did encounter difficulty solving compound events, and distinguishing between independent and dependent events. However, Ian successfully solved simple events, properties of probability, and comparison of two events.

Ian solved the first two problems with ease. Proceeding to Problem 3, Ian sought clarification of the term "fair spinner" before proceeding to answer the question:

> The two fair spinners shown below are part of a carnival game. A player wins a prize only when both arrows land on black after each spinner has been spun once.
>
> 
>
> James thinks he has a 50-50 chance of winning.
> Do you agree? Explain why or why not

Stating his interpretation of "fair spinner" as the equal chance of the each spinner landing on each color, Ian proceeded to answer the question. After reading the question again, Ian answered, "I would say no, because, *each* of these has a 50% chance of landing on black, and since both of them need to land on the black, it seems like a 25% chance, So, I don't agree" After the interview, when asked why he thought it should be a 25% chance, Ian explained, "Well, if each one is 50%, depending on- if they both have to be over there (pointing to the black sides) - then it would be 50% of that one because both have to be black." Ian reasoned that the chance was 50% of 50% or a chance of 25%.

However, Ian had difficulty providing an answer for Problem 10, requiring him to find the probability of an event, knowing the percent of the elements that contain certain characteristics. Ian was given the percents of two dependent events in which the probability of the two events could be calculated without an understanding of probability. However, Ian did not encounter difficulties interpreting Problem 10, but instead encountered difficulties working with percents in the problem and stating a final answer:

> A market research firm has determined that 40% of the people in a certain area have seen the advertising for a new product. Given that they have seen the advertising, 85% have purchased the product. What is the probability that a person in this area has seen the advertising and purchased the product?

Ian wanted to know the number of people in the area that were sampled for the survey. After reading the problem, Ian responded, "So, it's 85% of this 40%... 40% of 85% of x being equal to the number of people that live in the area." His final written answer was "0.4(0.8x)", where x was equal to the number of people. Ian tried to find the number of people who had seen the advertising and had seen the product, instead of the probability of this event.

Ian easily solved Problems 2 and 5 requiring him to find the probability of a simple event. Ian continued the same reasoning for solving simple probabilities into his approach for solving Problem 11. However, in Problem 11, there were seven embedded questions pertaining to calculating the probability of independent and dependent events. Ian proceeded solving all the parts in Problem 11 as if all the questions were asking him to solve a simple probabilistic event.

Although Ian did not recognize the words in Problem 11 to prompt him to solve a dependent event, Ian did recognize the depended event occurring in Problem 13. While solving Problem 13, Ian recognized the sample sized changed after removing a ball, but he did not know how to represent this mathematically:

> A box contains 2 red, 3 white and 4 green balls. Two balls are drawn out of the box in succession without replacement. What is the probability that both balls are red?

Ian responded to this question by observing,

> "So, take *one* out. Okay. The first one is going to be 1 out of 9 chance that it is red, and then there's going to be - actually it's 2, it's 2 our of 9 chance that it's going to be red, and then after you take that out, depending on whether or not you picked a red one, your chance is either going to be 2 out of 8 or 1 out of 8 depending on whether or not you actually drew the red one. But since it's such a small chance (of selecting a red ball) I'm

> going to say that you probably didn't get it. So, all right, I am going to say 4 out of 72 chance.

Ian calculated a 4 out of 72 chance by multiplying the probability of getting a red ball the first time - 2 out of 9 chance - by the probability of getting a red ball the second time, assuming a red ball was not selected the first time because "it's such a small chance" of selecting a red ball the first time - 2 out of 8. In this problem, Ian recognized that the overall sample size did change as a ball was extracted, however, Ian was not sure how to determine which ball was drawn first.

Overall, Ian used his strong mathematical reasoning skills to help him solve the unfamiliar problems found on the first questionnaire. His reasoning skills allowed him to correctly solve problems requiring him to find the probability of independent events and apply the properties of probability, including the complement of an event. However, Ian did exhibit difficulty recognizing the problems requiring knowledge of set operations and recognition of dependent events.

Ian missed the first week of class covering the lessons on probability. However, Ian was present for the remaining week. When Ian did show up for class, he was normally five minutes late for class, sat up front, and prepared his notebook for the day's lesson. He preferred to just listen to the lecture, not ask questions, and used his notebook only to write down key information. Later, he explained that he learned more from the book than the lecture, so Ian used lecture time to just observe the various methods used to solve a problem. When asked about his homework habits, he explained that he tried to keep up with the lectures.

After describing his homework habits, Ian smiled and said, "Well, I have a girlfriend now, and it changes everything. I used to study a lot more!"

When Ian proceeded to the second Task-Based Questionnaire, Ian admitted that he had not finished the homework for the section on probability. On the second questionnaire, Ian appeared less confident solving the problems by not interpreting and rewording the problems as he did on the first questionnaire. Ian encountered similar difficulties recognizing set operations and dependent events. Overall, he successfully solved simple events, and properties of probability, while encountering difficulty with comparisons of two events, compound events, and distinguishing between independent and dependent events.

Ian was able to solve the first five problems with ease, finding the probability of a simple event and comparing probabilistic events. However, when he approached Problem 6, requiring knowledge regarding set operations, Ian recognized the terminology, but was confused by their interpretations:

> From a survey at a preschool, a pharmaceutical company found that 75% of the students contracted chicken pox, 30 % contracted measles, and 20% contracted both the chicken pox and measles. If a child at the preschool is randomly selected, what is the probability that:
> a. The student contracted the chicken pox, or the measles, or both?
> b. The student did not contract the chicken pox?
> c. The student contracted neither the chicken pox nor measles?

Student's Response	*Student's Work*
Part A:	
Okay, well chicken pox is 75%	75%
The measles is 30%	30%
And both is 20%	20%

Ian assumed the word "or" meant that each number was a separate answer. Stating the three quantities as his final answer, Ian was asked at the end of the questionnaire how he determined this answer. Ian explained, "Well, since it says 'or' instead of 'and', I'm going to assume that the individual is one of these, as opposed to chicken pox, and the measles, *and* both." Ian's confusion between the interpretation of "and" and "or" continued to Part C of Problem 6. To find the probability that a student contracted neither the chicken pox, nor the measles, Ian responded:

> I think that's going to be 5%. Because, 75% contracted the chicken pox. But 20% contracted both chicken pox and measles, so 5% didn't contract chicken pox

Ian calculated the 5% by subtracting both the 75% of those who contracted the chicken pox and the 20 % who contracted both from 100%. Still indicating confusion between "and" and "or", Ian encountered the same interpretation dilemma while solving Problem 7, also requiring interpretations of set operations.

Surprisingly, Ian did not construct alternate representations of problems, such as Venn diagrams, tree diagrams, or tables, to help him organize and analyze the data. However, Ian was more comfortable working with percents on Problem 10:

> A bank observes that 60% of its customers prefer to visit a teller than an ATM when making a transaction. Given that a customer has visited the teller, 30% deposited money into their accounts. What is the probability that a customer at this bank had visited a teller and deposited money into their account?

This time, Ian recognized the needed to find 30% of the 60%, multiplying 0.3 by 0.6 to obtain 0.18, instead of trying to find the total number of people, as he tried to do on the first questionnaire. Unfortunately, there was not enough time at the end of the interview to ask how he knew to multiply the two numbers.

Finally, Ian was feeling more confident about dependent events, but still was confused on the associated operation with compound events. He struggled with the independent and dependent statements in Problem 11, solving half of them correctly. However, Ian returned to Problem 13, in which he needed to find the probability of a complex experiment of two successive events without replacement:

> A gumball machine contains 5 red, 7 white and 9 green gumballs. The gumballs are well mixed inside the machine. John gets two gumballs from this machine in succession, without replacement. What is the probability that both gumballs are red?

Ian correctly reasoned that "In the first one, its 5 out of 21… let's just say he gets a red one on the first time, I mean there's only 4 more in there, and there's also only 20 now." This time Ian knew that the first ball was red, however to find the probability that both balls were red, Ian then proceeded to add the fractions 5/21 and 4/20. When asked at the end of the interview why he added the two numbers, he responded, "Because it's similar to…" and refers to Problem 3:

> The two fair coins shown below are part of a carnival game. One side of a coin is black, the other side white. A player wins a prize only when <u>both</u> coins land on black after each coin has been tossed once.
>
> 
>
> Caroline thinks she has a 50-50 chance of winning.
> Do you agree? Explain why or why not.

After looking at his work on Problem 3, he added,

> No, I guess I did multiply those (referring to Problem 3), because (the first coin) was dependent on the other (coin). This (coin) was dependent on the other (coin). And in this case (Problem 13), it wasn't. Because we knew that the first (gumball) was a specific one. And this one (Problem 3) we didn't. This one (Problem 3) was dependent on the roll of the first one. There was a 50-50 chance here. But see, in this one (Problem 13), we knew that she go the first gumball.

Ian's interpretation of dependency was that the outcome of a compound experiment - independent or dependent - was "dependent" when the outcome of the one of the events was not known, such as the color of the first marble selected from the bag.

Overall, Ian's method for solving the problems on the second questionnaire changed slightly from the administration of the first questionnaire. On the first questionnaire, Ian used his strong mathematical reasoning skills to try to interpret the problems. On the second set of problems, Ian was aware of the words associated with probability and proceeded to solve the problems in a more mechanical fashion, searching for a procedure, than trying to interpret the problem. However, Ian still encountered the same difficulties of not understanding set operational terminology and misinterpreting dependent and independent events. However, Ian was successful solving simple probability problems and recognizing the properties of probability.