AN ABSTRACT OF THE DISSERTATION OF

Charuwan Singmuang for the degree of Doctor of Philosophy in Mathematics Education presented on June 3, 2002.
Title: Thai Preservice Middle School Mathematics Teachers' Subject Matter Knowledge and Knowledge of Students' Conceptions of Division of Rational Numbers with Respect to Their Classroom Practices.

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The study investigated the impact of Thai preservice middle school mathematics teachers' knowledge of subject matter and of students' conceptions of division of rational numbers with respect to their classroom practices in a teaching environment controlled by a required national curriculum. Four preservice teachers were selected with different knowledge of subject matter and of students' conceptions of division of rational numbers: high knowledge of subject matter and high knowledge of students' conceptions, high knowledge of subject matter and low knowledge of students' conceptions, low knowledge of subject matter and high knowledge of students' conceptions, and low knowledge of subject matter and low knowledge of students' conceptions.

Each preservice teacher was observed three weeks, each class day during the teaching of units on division of decimals, representing fractions as decimals, and division of fractions. Formal interviews were conducted with each of the four preservice teachers prior to and after teaching each unit. Informal interviews were conducted prior to and after teaching each lesson. Materials used in the normal teaching of the class were collected. Interviews with the preservice teachers' mentors were conducted before and after each unit. The mentors were interviewed daily before or after the instruction. Interviews with supervisors were conducted each time they supervised the preservice teachers.

Results showed that all preservice teachers planned and taught division of rational numbers procedurally following an algorithmically-based national curriculum. The preservice teachers with higher subject matter knowledge used multiple examples. They could make up examples when the students asked questions. In contrast, the lower subject matter knowledge preservice teachers rarely created new examples while they were teaching. The high knowledge of students' conceptions preservice teachers used their knowledge of students' conceptions throughout the lessons more often than the low knowledge of students' conceptions preservice teachers. After teaching the lessons, they all gained knowledge of subject matter and of students' conceptions of division of rational numbers. The depth of knowledge of subject matter and of students' conceptions of division of rational numbers is as essential for preservice middle school mathematics teachers' teaching in a nonvoluntary curriculum as it is in a voluntary curriculum.

THAI PRESERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' SUBJECT MATTER KNOWLEDGE AND KNOWLEDGE OF STUDENTS' CONCEPTIONS OF DIVISION OF RATIONAL NUMBERS WITH RESPECT TO THEIR CLASSROOM PRACTICES

by
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I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

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Charuwan Singmuang, Author

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This dissertation is dedicated to the loving memory of my father,
Capt. Sa-Nga Singmuang

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CHAPTER 1
THE PROBLEM

Introduction

Teaching mathematics well is a complex endeavor, and there are no easy recipes for success (National Council of Teachers of Mathematics [NCTM], 2000). Mathematics teachers must know the subject, their students, and teaching pedagogy. They also must have ability to apply this knowledge in a variety of field-based settings. In addition, mathematics instruction requires an understanding of the impact that socioeconomic background, cultural heritage, attitudes and beliefs, and political climate have on the learning environment. Above all, teaching mathematics entails developing a store of knowledge that combines personal understanding, sensitivity and responsiveness to learners with the education, skills, understanding, and disposition to teach (NCTM, 1991).

To be effective, teachers must know and thoroughly understand the mathematics they are teaching, and they must flexibly draw on that knowledge. Additionally, teachers need to understand the gap between what their students know and what they need to learn while challenging and supporting them. Effective teachers also need to know the concepts with which students often have difficulty and understand ways to clarify common misunderstandings (NCTM, 2000).

Today, mathematics educators have voiced a concern with the way mathematics is taught. They have repeatedly called for changes that emphasize teaching for understanding and meaningful learning (Davis, 1986; Educational Technology Center, 1988; Lampert, 1988; NCTM, 2000). If a teacher's role is to help the learner achieve understanding of the subject matter, teachers must obviously receive training and instruction that prepare them to teach for understanding. However, teachers are also learners, and their understanding of content and pedagogy is powerfully influenced by their own experiences as students. Unfortunately, teaching and learning in high schools and colleges is limited to a traditional didactic pattern: faculty treat students as passive recipients of knowledge presented primarily through lecture, textbooks, and demonstrations with a focus on procedural fluency (Boyer, 1987; McDiarmid, 1989).

Teacher educators are no exception. They tend to take preservice teachers' subject matter knowledge for granted and merely provide techniques and materials for presenting lessons (Ball & Feiman-Nemser, 1988; Floden, McDiarmid, & Wiemers, 1989). Even when teachers are taught cognitive approaches to instruction, this knowledge is presented in the form of lectures about children's naive scientific theories or the difference between procedural and conceptual knowledge in mathematics (Ball, 1988). Rarely are teachers treated as learners who actively construct understanding: rarely are teachers encouraged to actively construct meaning. As a consequence, many high school and college students including preservice teachers graduate with limited conceptual understanding of the subject matter taught (Ameh & Gunstone, 1988; Ball, 1990a; Neale & Smith, 1989; Peck & Connell, 1991; Smith, 1987; Stofflett & Stoddart, 1991).

Unfortunately, teachers who do not conceptually understand the content are unlikely to teach it conceptually (Stoddart, Connell, Stofflett, & Peck, 1993). And as mentioned by McDiarmid, Ball, and Anderson (1989), teachers tend to teach mathematics as they were taught. Therefore, the resulting inadequate mathematical competency of both the students and the teachers causes students to revert to rote-learned procedural knowledge when under pressure to complete tasks (Tall, 1995). Even mathematics majors who plan to teach and are able to produce correct answers for division involving fractions, zero, and algebraic equations, frequently struggle to make sense of division with fractions, have difficulty connecting mathematics to the real world, and are unable to develop explanations that go beyond a restatement of the rules (Ball, 1988). Mathematics teachers who have deficiencies in subject matter knowledge are likely to pass their misconceptions and misunderstandings on to those they teach (Babbitt & Van Vactor, 1993). In contrast, a teacher who has solid mathematical knowledge for teaching is more capable of helping his/her students achieve a meaningful understanding of the subject matter (Even, 1990). That is, students gain understanding that goes beyond procedures; they comprehend concepts.

To this end, several reform efforts recognizing the importance of the preparation of mathematics teachers (Carnegie Task Force, 1986; Holmes Groups, 1986; NCTM, 1991) have focused on preservice programs. These reforms are designed to improve professional teacher education. For example, the mathematics education community recommends that preservice teachers examine their conceptions of mathematics and how it should be taught (NCTM, 1991; Mathematical Association of America [MAA], 1991).

Statement of the Problem

Understanding division is a basic conceptual knowledge that is vital for students to understand a variety of ideas in mathematics (McDiarmid & Wilson, 1991). When studying division, students learn about rational and irrational numbers, place value, the connections among the four basic operations, as well as the limits and power of relating mathematics to the real world (Ball, 1990b). Children's and teachers' understanding of multiplicative concepts, such as division, and rational numbers is important to their ability to gain mathematical understanding (Behr, Knoury, Harel, Post, & Lesh, 1997). Therefore, elementary and middle school teachers' understanding of division is essential if their students are to develop ideas about division that are likely to help them understand other mathematical concepts, such as fractions, decimals, and ratios (McDiarmid & Wilson, 1991). Thus, because it is a central concept to mathematics at all levels and figures prominently throughout the K-12 curriculum (Ball, 1990b), division of rational numbers is an important content area for investigation.

And although division of rational numbers is a key topic in the elementary and middle grades of several countries, it is the last of the four fundamental operations learned and is often considered the most mechanical (Fendal, 1987; Payne, 1976). In the United States, division of rational numbers is part of the mathematics curricula in the fourth, fifth and sixth grades (NCTM, 1989). In Thailand, it is part of the mathematics curricula in fifth, sixth, seventh and eighth grades (Institute for the Promotion of Teaching Science and Technology [IPST], 1990). Division is often perceived by teachers and students alike as being the most difficult topic confronting them in the classroom (Burns, 1991; Burton & Knifong, 1983; Leinhardt & Smith, 1985). Therefore, since students often have difficulty learning division with rational numbers, teachers should understand it completely in order to teach it (Ball, 1990b).

Recognition of the importance of teachers' knowledge has encouraged researchers to examine preservice and inservice teachers' subject matter knowledge of division of rational numbers (Ball, 1990a, 1990b; Graeber, Tirosh, & Glover, 1989; McDiarmid & Wilson, 1991; Tirosh & Graeber, 1989; Simon, 1993). The results of this research have indicated that preservice and inservice teachers' subject matter knowledge of division is fragmented and largely procedural. Explanations tend to be disconnected from an underlying concept of division (Ball, 1990a; McDiarmid & Wilson, 1991), and conceptual knowledge is weak, with no connection between procedure and concept (Simon, 1993). Borko et al. (1992) also indicate that such a limited understanding negatively influences the teaching experiences of a student teacher.

Educators generally agree that learning occurs when a student's existing conceptions are challenged (or built upon) (Tirosh, 2000). Consequently, those calling for reform in mathematics education have emphasized that teachers gain knowledge of students' common conceptions and misconceptions (Australian

Education Council, 1991; NCTM, 1989, 1991, 2000). However, only a few researchers have examined preservice teachers' knowledge of students' conceptions of division of rational numbers (Even & Tirosh, 1995; Tirosh, 2000). The results have indicated that preservice teachers have a poor ability to analyze the reasoning behind students' responses (Even & Tirosh, 1995). Prior to the preservice program's methods course in which emphasis is placed on knowledge of students' conceptions, preservice teachers' ideas are not directed toward major sources of students' incorrect responses in dividing fractions. By the end of the methods course, however, most preservice teachers are familiar with the various sources of incorrect responses (Tirosh, 2000).

Based on results from previous studies, it is important to ascertain specific areas in which preservice teachers' subject matter knowledge and knowledge of students' conceptions fall short in improving the quality of teaching mathematics, in general, and of division of rational numbers in particular. Since many of tomorrow's teachers are today's preservice teachers, their subject matter knowledge and knowledge of students' conceptions of fundamental operations must be of concern to teacher educators and researchers. As mentioned by Simon (1993), a research base with respect to prospective teachers is essential in developing instructional interventions to help prospective teachers extend, amend, and modify their knowledge.

Previous studies have examined elementary preservice teachers' knowledge of mathematics while enrolled in the methods courses. Nevertheless, few studies have investigated preservice middle school mathematics teachers after they have finished their mathematics courses or as they begin their practice teaching. Although these preservice teachers may have conceptions about mathematics teaching, they have yet to develop a strong foundation for applying those connections in the classroom.

In addition, while several research studies concerning preservice mathematics teachers' knowledge have been done across several countries, including Israel and the United States, none have been conducted in Thailand. Although some researchers have investigated the connections among preservice teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers as reflected in classroom practice, their investigations were within the context of a voluntary/unrestrictive curriculum. For example, the United States offers more than one mathematics curriculum for middle school. Several mathematics textbooks have been developed by different authors based on different philosophies and learning theories. Therefore, American teachers have some latitude in choosing mathematics textbooks for their classrooms. They also have freedom to create and arrange their curriculum and design their lessons. Research has supported the concept that teachers with strong subject matter knowledge and knowledge of students' conceptions are needed in this kind of context.

In contrast, Thailand has one national curriculum. The Institute for the Promotion of Teaching Science and Technology developed the curriculum along with the textbooks, teachers' guidebook, teaching materials series, and supplementary documents. The mathematics textbooks developed by this organization serve as the main textbooks in schools in Grades 7-12. Thai mathematics teachers are required to teach from these textbooks. They have less freedom in designing and arranging their curriculum and lessons than teachers in a voluntary/unrestrictive curriculum. Given this situation, one question can be raised: Does the research conducted in voluntary situations necessarily hold for nonvoluntary/restricted classroom environments? To investigate this question, a study is needed of Thai preservice middle school mathematics teachers' subject matter knowledge, their knowledge of students' conceptions of division of rational numbers, and their classroom practices.

The guiding framework for this research is two fold. First, this research is to describe Thai preservice middle school mathematics teachers' subject matter knowledge and their knowledge of students' conceptions of division of rational numbers as reflected in classroom practices in a teaching environment controlled by a required national curriculum. Second, this research will address a question: Is the depth of mathematical knowledge and the knowledge of students' conceptions as essential for preservice middle school mathematics teachers teaching in a controlled national curriculum as that needed in a voluntary curriculum? This study is designed to provide a rich description of preservice teachers' teaching with various levels of subject matter knowledge and knowledge of students' conceptions. Based on this research the future studies can make the comparative analysis.

Definition of Terms

The research question involves several terms that need to be defined:

Thai preservice middle school mathematics teachers are senior students majoring in mathematics education. For this study, they are enrolled in one teachers' college in Thailand. The students are in the fourth year of a four-year undergraduate teacher education program. They have already finished their mathematics content and mathematics methods courses. In Thailand, division of rational numbers is part of the mathematics curricula in the seventh grade. At the time of the study, the preservice teachers had been teaching seventh grade mathematics classes in the public schools for three months.

A **rational number** refers to a number that can be written as the ratio p/q where p and q are integers, and q does not equal zero.

Subject matter knowledge refers to a preservice teacher's personal knowledge of division of rational numbers, including key facts, fundamental and crucial concepts, principles, and explanatory frameworks concerning division of rational numbers.

Knowledge of students' conceptions refers to a preservice teacher's personal knowledge of common conceptions, misconceptions, and difficulties of students and the possible sources for these conceptions, misconceptions, and difficulties.

Significance of the Study

This study examines Thai preservice middle school mathematics teachers' subject matter knowledge and their knowledge of students' conceptions of division of rational numbers as reflected in their classroom practices. The study provides a description of the role these two domains play in classroom practices within a teaching environment controlled by a required national curriculum. The resulting information can be used to design appropriate preservice teachers' programs to support them in becoming teachers in a country with a national curriculum.

As mentioned earlier, few studies have investigated preservice middle school mathematics teachers' subject matter knowledge and their knowledge of students' conceptions of division of rational numbers. All of the previous research studies in this area has been conducted in classroom situations of voluntary/unrestrictive curriculum. With the lack of investigation in involuntary/restrictive curriculum contexts, the literature is incomplete. The results of this study provide a more complete picture of the knowledge structures that preservice middle school mathematics teachers possess in a nonvoluntary/restrictive curriculum environment and furnish information for a more complete understanding of the importance of multiple knowledge bases (subject matter and students' conceptions of mathematics) in teaching mathematics and will provide information for improving the preservice teacher preparation program to prepare mathematics teachers for Thailand.

This study also provides useful information and ideas for current and future middle school mathematics teachers. Although this study is not measuring the effectiveness of division instruction, the picture of current instructional practice may motivate novice middle school mathematics teachers to reflect on their own teaching of division of rational numbers.

CHAPTER 2
REVIEW OF THE LITERATURE

Introduction

This study investigates Thai preservice middle school mathematics teachers' subject matter knowledge and their knowledge of students' conceptions of division of rational numbers as reflected in classroom practices in a teaching environment controlled by a required national curriculum. Recent educational reform efforts have emphasized teaching for understanding (Hiebert, 1992; NCTM, 2000). Hiebert (1992) defined understanding in terms of the way an individual represents and structures information. A mathematical idea, procedure, or fact is understood if its mental representation is part of an internal network of representations. The degree or extent of understanding is determined by the number and the strength of the connections. A fundamental assumption of cognitive psychology is that these knowledge structures and mental representations of the world play a central role in an individual's perceptions, thoughts, and actions (Brown & Borko, 1992). In the case of teaching, teachers' thinking is directly influenced by their knowledge, both mathematical and pedagogical. Their thinking, in turn, contributes to their instructional behavior. If the goal of a mathematics teacher education program is to help teachers implement programs of instruction that develop student understanding of mathematics, it is reasonable to expect that teachers have, and are continuing to develop, a well connected and extensive knowledge base to support their mathematics teaching (Howald, 1998).

The assertion that knowledge related to subject matter is an essential component of teachers' professional knowledge is neither new nor controversial (Ball & McDiarmid, 1990). Shulman (1987) proposed that knowledge of subject matter for teaching consists of two overlapping knowledge domains: subject matter knowledge and pedagogical content knowledge. Shulman's (1987) conceptualization has served as a framework for much of the current research on teacher knowledge (Ball, 1990a, 1990b; Ball & McDiarmid, 1990; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992). This theoretical model allows the examination of teacher knowledge, the relationships among the domain of knowledge identified in the theory, and effects on classroom practices and outcomes (Howald, 1998). Preservice teachers also need to possess a great deal of knowledge so that they can find solutions to many pedagogical problems as they practice teach: how to represent mathematical ideas, how to deal with problems of misunderstanding, how to manage the classroom, how to plan lessons, and how to choose materials to teach (Cha, 1999).

In order to better understand the framework for this study, the review of the literature focuses on specific areas of investigation: preservice teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers; and the research on preservice teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers in classroom practice. Although the study focuses on preservice teachers, studies for this review include results about both inservice and preservice mathematics teachers.

Types of Teachers' Knowledge

Shulman (1987) outlined seven categories of knowledge that underlie the teacher understanding needed to promote comprehension among students: subject matter knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners and their characteristics, knowledge of educational contexts, and finally, knowledge of educational ends, purposes, and values, along with their philosophical and historical grounds. Among these categories, Shulman emphasized pedagogical content knowledge as of special interest.

Subject matter knowledge consists of two types of understandings, substantive knowledge and syntactic knowledge. Substantive knowledge includes key facts, concepts, principles, and explanatory frameworks in a discipline;

whereas, syntactic knowledge consists of the rules of evidence and proof within a discipline. Borko and Putnam (1995) mentioned, "Syntactic structures are the cannons of evidence and proof that guide inquiry in a discipline—the ways of establishing new knowledge and determining the validity of claims" (p.45). Shulman (1986) emphasized, "Teachers need not only understand that something is so; the teachers must further understand why it is so. They need to understand why a given topic is particularly central to a discipline whereas another may be somewhat peripheral" (p.9).

Shulman (1987) added, "Pedagogical content knowledge is of special interest because it identifies a distinctive body of knowledge for teaching. Pedagogical content knowledge represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners and presented for instruction" (p. 8). Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of a teacher (Shulman, 1987).

Carpenter, Fennema, Peterson, and Carey (1988) defined that pedagogical content knowledge includes knowledge of the conceptual and procedural knowledge that students bring to the learning of a topic, the misconceptions about the topic that they may have developed, and the stages of understanding that they are likely to pass through in moving from a state of having little understanding of the topic to mastery of it. It also includes knowledge of techniques for assessing students' understanding and diagnosing their misconceptions, knowledge of instructional strategies that can be used to enable students to connect what they are learning to the knowledge they already possess, and knowledge of instructional strategies to eliminate the misconceptions that may have developed.

Grossman (1989) expanded on Shulman's definition of pedagogical content knowledge by identifying four components: (1) teachers' overarching conception of the purposes for teaching particular subject matter; (2) teachers' knowledge of pupils' understandings and potential misunderstandings of a subject area; (3) teachers' knowledge of curriculum and curriculum materials; and (4) teachers' knowledge of strategies and representations for teaching particular topics.

Cochran, DeRuiter, and King (1993) emphasized that pedagogical content knowledge differentiates expert teachers in a subject area from subject area experts. Pedagogical content knowledge is concerned with the manner in which teachers relate their subject matter knowledge (what they know about what they teach) to their pedagogical knowledge (what they know about teaching) and how subject matter knowledge is a part of the process of pedagogical reasoning.

One important component of pedagogical content knowledge is knowledge of students' conceptions of subject matter. It includes knowledge of students' understanding, conceptions, and misconceptions of particular topics in a subject matter (Grossman, 1990). This component differs from general knowledge of learners by virtue of its focus on specific content (Borko & Putnam, 1996). Knowing the understandings and misunderstandings about particular topics that students bring to class are important for teachers, especially when their emphasis is on teaching for understanding rather than on mechanical or rote leaning (Borko & Putnam, 1996). Teachers must have some knowledge about what students already know about a topic and what they are likely to find puzzling in order to generate appropriate explanations and representations (Grossman, 1990). Borko and Putnam (1995) concluded that teachers with well-developed pedagogical content knowledge understand how students typically learn a particular subject. They are aware of the topics in a field that students are likely to find difficult, know what the common difficulties are, and have strategies for addressing those difficulties in their representational and adaptational repertoires.

Preservice Teachers' Knowledge of Subject Matter and of Students' Conceptions

In recent years, preservice teachers' subject matter knowledge and their knowledge of students' conceptions of mathematics have been the focus of much research attention. This portion of the review provides 11 studies that explored and described preservice teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers. Nine studies (Ball, 1990a, 1990b; Graeber, Tirosh, & Glover, 1989; McDiarmid & Wilson, 1991; Simon, 1993; Tirosh & Graeber, 1989; Tirosh & Graeber, 1990; Tirosh & Graeber, 1991; Wheeler & Feghali, 1983) investigated preservice teachers' subject matter knowledge. One study (Even & Tirosh, 1995) explored preservice teachers' knowledge of students' conceptions and one study (Tirosh, 2000) determined preservice teachers' knowledge of subject matter and of students' conceptions of division of rational numbers.

Wheeler and Feghali (1983) conducted a study to investigate preservice teachers' understanding of zero. Understanding included the formal and informal uses of zero: arithmetic computations involving zero, awareness of zero and the null set as conceptualized by Piaget, mathematical uses of zero in set cardinality and counting situations, and nominal uses of zero in social contexts. The subjects were 52 preservice teachers (47 females and five males) enrolled in one of three sections of an elementary school mathematics methods course at a state university in the summer of 1981.

Two sets of tasks were pilot-tested and used: a group-administered written test and an individually administered interview. Each set consisted of four tasks. The group test consisted of a division test and three elaboration tasks. The interview tasks considered (a) nominal uses of zero, (b) mathematical uses of zero, (c) classification, and (d) partitioning. Each interview task consisted of five or fewer subtasks. Tasks (a) and (b) were presented first and second; tasks (c) and (d) followed in random order.

The instructor of the methods course—one of the investigators—administered the written test to each section. The subjects were then individually interviewed by one of the two investigators using a common protocol; written verbatim records were kept by the interviewer. The study was conducted during the second quarter of the term, prior to any specific instruction regarding zero. The group tasks were administered before the interview in one section and after the interview in the other two. The subjects completed the written tasks within 12 minutes and the interview within 30 minutes. For all subjects, both test and interview were completed within 36 hours.

The results from the division test indicated that the division items not involving zero were the easiest and those involving zero as the divisor, the hardest. The most frequent error pattern occurring in groups of items with zero as the dividend or divisor was to identify the quotient as identical to the dividend or divisor. Of the 11 subjects who made multiple errors in the zero dividend subtask, eight (73%) consistently erred in this manner; on the zero divisor subtask, 30 (79%) of 38 subjects with multiple errors showed this pattern.

Each of the 52 written responses to "What is zero?" was categorized in one or more of eight categories: number, empty set (including null set), nothing (including absence of things), symbol (including numeral, representation, and digit), placeholder, identity for addition, other, and unclassifiable. Approximately 40% (21) of the subjects gave responses that were classified in a single category; the other responses required multiple categorization. When asked if zero was a number, eight (15%) subjects answered no. A representative response was: "Zero is not a number because it has no value; it is a digit signified by 0."

Each of the 52 responses to "What is zero divided by zero?" was categorized correct ("cannot divide by zero," "undefined," "no answer exists"), zero, one, or other. More responses were incorrect (77%) than correct. Most frequently, the subjects said that zero divided by zero was zero (67%). Of the 27 subjects given, the most frequent justification (55%) referred to division; "How can one divide nothing into nothing and receive anything but nothing" was a typical justification.

The researchers found that the preservice teachers did not possess an adequate understanding of zero. These future teachers exhibited a reluctance to identify zero as a number. They also found division items with zero as a dividend or a divisor to be difficult; their error patterns were stable. In quantitative settings, these preservice teachers preferred to use "zero" rather than "none." In nominal social contexts, when vocalizing zero, they preferred the use of "oh" to that of "zero." Consequently, they were not sufficiently prepared to teach concepts of zero.

Graeber, Tirosh, and Glover (1989) conducted a study to explore whether preservice elementary teachers have misconceptions such as "the divisor must be a whole number" or "multiplication always makes bigger and division always makes smaller," by determining whether preservice teachers select the correct operation when they are presented with problems having data that conflict with the implicit rules of the primitive behavioral models of multiplication and division. The researchers were also interested in noting whether the preservice teachers would exhibit other misconceptions and the extent to which such misconceptions were similar to those previously noted among children.

The subjects included 129 female college students enrolled in one of the mathematics content or mathematics methods courses for early elementary education major in a large university in the southern United States. Both the content and the methods courses included attention to the multiplication and division of whole and rational numbers.

A test with 12 multiplication and 14 division problems was administered to the 129 students early in the quarter. The multiplication and division problems were divided between two forms, and problems requiring addition or subtraction were interspersed with the multiplication and division problems to reduce the likelihood that correct answers would result from guessing. The 18-item forms were distributed alternately so that half the preservice teachers in each class received each form. In an effort to obtain the preservice teachers' intuitive responses, the test asked them not to perform the needed operation but simply to write an expression in the form of "a number, an operation, and a number" that would lead to the solution of the problem.

Thirty-three preservice teachers were then selected for interviews. All these students had given incorrect answers to one or more of the eight most commonly missed problem. The interview schedule was devised to obtain more information about the conceptions the preservice teachers held and the reasoning they used. During an audiotaped interview, the interviewee was given a problem similar to each of the problems she had missed on the written test and was asked to write an expression that could be used to solve it. If the response was incorrect, she was asked to explain why she wrote the expression she did and to explain and show how she would check her work. After discovering or being shown that the written expression led to an incorrect answer, she was asked to verbalize what she believed led her to write the wrong expression.

The results from the written test indicated that 39% of the preservice teachers answered four or more of the 13 multiplication or division problems incorrectly. Moreover, every interviewee gave evidence of holding some misconceptions.

With regard to multiplication, the data indicated that violation of the primitive model was a source of difficulty. Some interviewees reported that the presence of decimals was a source of difficulty. However, the preservice teachers did very well on some problems containing decimal factors, such as $.75 \times 15$, 2.25×15, and 3.25×15. In these problems the decimals played the role of operand. Thus, the decimal numerals' role seemed more influential than its presence. The written work also provided that the common misconception "multiplication always makes bigger and division always makes smaller" was held by preservice teachers. More than 25% of the sample incorrectly wrote a division expression as appropriate to the solution for multiplication problems. Eight of the 33 interviewees claimed that division was the appropriate operation for solving such problems. Six argued that division always makes smaller and multiplication always makes larger. The interviews also showed that some preservice teachers' reasoning about multiplication problems with decimal operands involved an overgeneralization of procedures used with unit fractions. The researchers further concluded that neither the size of a whole-number operator nor the familiarity of a decimal operator influenced the difficulty of the multiplication problems.

With regard to division, the preservice teachers had high scores on the problems conforming to the primitive partitive model of division and on three of the problems conforming to the primitive quotative model. However, they were less successful on the other two quotative division problems. The interview data indicated that the context of these two problems (the number of rows of tiles of a given height needed to cover a wall of a given height) was exceedingly difficult for the preservice teachers. Once the problem was clarified for the interviewees, they had no difficulty writing an appropriate expression to solve either of them. The researchers mentioned that in these two cases the difficulty of the context far exceeded the difficulty caused by the characteristics of the numbers.

The data further indicated that the problems that violated constraints of the primitive division models proved more difficult. The majority of the incorrect responses to problems in which the whole-number divisor was larger than the dividend ($5 \div 15$, $5 \div 12$) were expressions that revealed the roles of the divisor and the dividend. The interviews enabled the researchers to shed some light on whether the difficulty was strictly one of improper notation, a misconception about commutativity, or adherence to the stereotyped notion that the divisor must be smaller than the dividend. During the interviews, 22 of the 33 interviewees reversed the role of the divisor and dividend in this type of problem. All 22 claimed that in division the larger number should be divided by the smaller number. Four of the interviewees claimed that, "it is impossible to divide a smaller number by a bigger number."

Three problems ($3.25 \div 5$, $.75 \div 5$, $1.25 \div 5$) involved the division of a decimal by a larger whole number. These problems were easier than those with whole-number divisors greater than the whole-number dividend. Four interviewees who had reversed the roles of the dividends and divisors in problems with whole numbers were asked about their correct response to $3.25 \div 5$. One interviewee reported that she first wrote an expression like $5 \div 3.25$. However, when she was confronted with the decimal divisor $(3 \div 0.15)$, she interchanged divisor and dividend. This evidence showed that some preservice teachers answered these problems correctly only because they sought to avoid a decimal divisor. In this sense, the faulty rule that "the divisor must be a whole number" seemed to have greater strength than the faulty rule about the relative sizes of divisor and dividend.

The interview data also suggested that for some preservice teachers the belief that "the divisor must be a whole number" was supported by the conventional algorithms for division by decimals and by fractions in the a/b form. In the first case the decimal is always "changed to a whole number," and in the second, multiplication, not division, was used.

The researchers concluded that preservice teachers were influenced by the same primitive behavioral models for multiplication and division that influenced the 10- and 15-year-old students in another study. They suggested that because today's preservice teachers were tomorrow's teachers, the learning/teaching cycle might perpetuate misconceptions and misunderstandings about multiplication and division. Thus, efficient strategies were needed for training teachers to monitor and control the impact that misconceptions and primitive models have on their thinking and their student's thinking.

In the study just discussed, Graeber, Tirosh, and Glover (1989) found that a substantial number of preservice teachers had difficulty selecting the correct operation to solve multiplication and division word problem involving positive decimal factors less than one. Interviews indicated that some of the preservice teachers held explicit misbeliefs about the operations. Other preservice teachers apparently were influenced by implicit, unconscious, and primitive intuitive models for the operations. Based on this result, Tirosh and Graeber (1989) conducted a study to assess the extent to which the beliefs "multiplication always makes bigger" and "division always makes smaller" are explicitly held by preservice elementary teachers. It was also designed to provide insight into the sources of the preservice teachers' beliefs about multiplication and division.

The subjects were 135 female and one male preservice teachers enrolled in the mathematics content or mathematics methods courses for early elementary (kindergarten to grade five) education majors in a large university in the United States. The paper and pencil instruments were administered to these preservice elementary teachers. They were asked to respond to the following six statements related to the misbeliefs, "multiplication always makes larger" and "division always makes smaller" by labeling each statement "True" or "False" and justifying their response.

A. In a multiplication problem, the product is greater than either factor.
B. The product of $.45 \times 90$ is less than 90.
C. In a division problem, the quotient must be less than the dividend.
D. In a division problem, the divisor must be a whole number.
E. The quotient for the problem 60/.65 is greater than 60.
F. The quotient for the problem $70 \div \frac{1}{2}$ is less than 70.

The preservice teachers were reminded of the relationship among three terms: divisor, dividend, and quotient.

Data were also collected on the preservice teachers' computational skills and on their performance in writing expressions to solve word problems. Two of the computational exercises, $.38 \times 5.14$, and $.75\overline{)3.75}$, provided counterexamples to the beliefs under discussion. Preservice teachers responded to either 16 or 21 word problems (including 13 multiplication and division problems) depending on whether they were enrolled in the course in the winter or spring. The problems used in the winter and spring were similar but not always identical.

Of the 136 preservice teachers, 33 preservice teachers from one of the winter classes and 38 from one of the spring classes were interviewed by the researchers in order to obtain more information about their conceptions of multiplication and division. The preservice teachers were asked to write expressions that would solve multiplication or division word problems similar to those they had missed on the written word problem instrument. They were also asked to explain the logic they used in selecting the operation. The preservice teachers' justifications for each of the statements of belief were compiled, and similar responses were grouped into categories.

With regard to beliefs about multiplication, the results indicated that 87% of the 130 preservice teachers who responded to both of the multiplication statements related to the misbelief "multiplication always makes bigger" responded correctly to both; only 3% of them responded incorrectly to both of the statements. Although only about 10% of the preservice teachers held the misbelief explicitly, the data on preservice teachers' performance in writing expressions for word problems suggested that many of the preservice teachers were implicitly influenced by this misbelief. When the operator in a word problem was a decimal less than 1, about 50% of the preservice teachers responded with a division expression. However, when the operator was a whole number, 90-95% of the preservice teachers wrote correct expressions. This high rate of correct responses for whole number operators held for both whole number and decimal operands. The researchers further mentioned that the influence of the misbelief was also evident in interviews.

With regard to beliefs about division, of the 129 preservice teachers who responded to all four statements related to the misbelief "division always makes smaller," 28% responded correctly to all four of the statements and 3% responded incorrectly to all four. The majority of the preservice teachers responded incorrectly to the statement "in a division problem, the quotient must be less than the dividend." The justification given by more than half of the preservice teachers who responded incorrectly were valid if restricted to the domain of whole numbers. The majority of those who responded correctly either cited a specific example, or appealed to special cases within the domain of whole numbers.

The pervasive nature of this misbelief was also evidenced by the fact that about 45% of the preservice teachers wrote multiplication expressions for the division word problems with decimal divisors less than one. A word problem of this type was, "Girls club cookies are packed .65 pounds to a box. How many boxes can be filled with 5 pounds of cookies?" Eighteen of the 40 preservice teachers who responded to this problem did so incorrectly; 14 of those 18 wrote .65 × 5 or 5 × .65. During the interviews, a number of individuals explained that they rejected division as the appropriate operation for the problem because they knew that the answer ought to be larger than 5. Since division makes smaller, they chose multiplication.

Although the responses to the statement "in a division problem, the quotient must be less than the dividend" suggested that a great number of the preservice teachers tacitly assumed that divisors were whole numbers, the responses to the statement "in a division problem, the divisor must be a whole number" indicated that 61% of the preservice teachers explicitly acknowledged the possibility of decimal or fraction divisors. This discrepancy indicated the preservice teachers acknowledged the existence of forms with decimal divisors but did not link such forms with a resulting quotient greater than the divisor.

Another apparent contradiction arose when the data from the computation exercises were compared with the written justifications. All of the preservice teachers attempted a computational exercise with a decimal divisor, $.75\overline{)3.75}$, and 87% completed it correctly. However, 19% of these preservice teachers argued that a divisor must be a whole number. Most of these preservice teachers' written justifications referred to algorithmic procedures. Their reliance on procedure was also evidenced in the interviews. Some interviewees felt division by a decimal was clearly impossible.

The justification of many of the preservice teachers who responded incorrectly to the statement "the quotient for the problem 60/.65 is greater than 60," and "the quotient for the problem 70 ÷ ½ is less than 70," included a statement to the effect that division makes smaller. These preservice teachers were defining division in terms of the primitive partitive model. A preservice teachers' inability to access the measurement model of division also hampered efforts to determine a reasonable answer.

The researchers mentioned that all of the division beliefs statements discussed were logically connected with the notion that division always makes smaller. The preservice teachers successfully used procedural knowledge when justifying specific examples, and they used the primitive models of the operations or their generalizations about procedures when responding to general statements. The researchers concluded that a substantial percent of the preservice teachers involved were influenced by misconceptions about multiplication and division. For many preservice students the misbeliefs about multiplication were likely to be implicit, while the misbeliefs about division were more apt to be explicit.

From the study by Tirosh and Graeber (1989), one misconception that a majority of preservice teachers appeared to hold explicitly was that in division the quotient must be less than the dividend. Based on this finding, Tirosh and Graeber (1990) conducted a study to investigate conflict teaching as a means of probing the misconceptions held by preservice elementary teachers that in a division problem the quotient must be less than the dividend.

The study included a pretest, individual interviews of some of the subjects, and a posttest—all given to one male and 57 female preservice elementary teachers in a large university in the southeastern United States during their regularly scheduled class sessions. Three distinct paper-and-pencil pretest instruments were administered. The preservice teachers were asked to calculate four multiplication and division examples involving decimals. The test included the examples 3.25×5.14, 0.38×5.14, $3.75 \div .75$, and 5.00×15. After handing in their solutions, the preservice teachers then responded to seven statements about multiplication and division. Among the statements were the following:

1. In a division problem, the quotient must be less than the dividend.
2. The quotient for the problem $10 \div .65$ is greater than 10.
3. The quotient for the problem $70 \div \frac{1}{2}$ is less than 70.

For each statement the preservice teachers were to write "true" or "false" and then write a defense for their answer. They were asked to respond to items 2 and 3 without computing the quotients (on the posttest, item 1 was changed to "In a division problem, the dividend must be greater than the quotient." Posttest items 2 and 3 had the same form but involved different numbers).

Next, the preservice teachers were given 21 word problems and asked to write an expression that would lead to the solution of each of them, that the subjects were not likely to select only division or multiplication. The word problems included problems to be solved by each of the four operations. The problems included four items for which multiplication or division by a decimal less than one was appropriate:

1. A box of sand has a volume of .80 cubic meters. What is the volume of .25 of the box?

2. A motorcycle goes 40 miles per gallon. How far will it travel on .75 gallons?

3. You prepared 5 liters of punch. You have punch cups that hold .2 liters. How many cups can you fill with the prepared punch?

4. Girl-club cookies are packed .8 pounds to a box. How many boxes can be filled with 7.2 pounds of cookies?

During the six weeks that intervened between the pretest and the posttest, neither of the courses covered material related to multiplication or division with decimals or fractions. The content course covered the topics in problem-solving strategies, geometry, and measurement. The methods course topics covered during this interval included strategies for teaching numeration and whole-number operations.

Twenty-one female preservice teachers who had correctly computed $3.75 \div .75$ but who had agreed, incorrectly, with the written statement "The quotient must be less than the dividend" were selected as the subjects for individual interviews. The researchers conducted 35- to 50-minute individual interviews which were designed to help the preservice teachers verbalize their conceptions of division, explicate their conceptions about the relative size of the dividend and the quotient, recognize the inconsistency between their expressed conviction and the results of a computation with a decimal divisor less than one, and reflect on sources of their original misconceptions. The interviews, conducted over a 3-week period, were audiotaped and transcribed. A minimum of three weeks elapsed between the interview and the administration of the posttest instruments. The posttest instruments were similar but not identical to the pretest instruments.

Responses to the initial interview question about the meaning of division fell into four categories: partitive interpretation of division (12 subjects), partitive and measurement interpretation (three subjects), inverse operation of multiplication (three subjects), and none of these categories (three subjects).

When asked about restrictions on the divisor, dividend, or quotient, three subjects argued that the quotient can be either less than or greater than the dividend. Three subjects first expressed the misconception, then almost immediately verbalized some uncertainty and expressed a need to check their conviction. The remaining 15 subjects argued that the quotient is always less than the dividend.

During the course of the interviews, 14 of the subjects identified causes of their misconception as the requirement for only whole numbers (seven subjects), the assumption that with decimals division works in the same way as with natural numbers (one subject), an identification that decimals are confusing and misleading (three subjects), and conclusions from the standard algorithm (three subjects).

From written instruments, the majority of the justifications written on the pretest suggested that the subjects used the partitive model for division. Other responses were based on the argument that division is the inverse of multiplication—multiplication makes bigger, and therefore division makes smaller.

The subjects did considerably better with all three-division statements on the posttest. Moreover, the subjects' performance in writing expressions for word problems improved from pretest to posttest. Fourteen of the 21 subjects improved their performance. The remaining seven, including the one who did not reach a conflict in the interviews, wrote exactly the same number of correct expressions on the posttest as they had on the pretest. Furthermore, a comparison of the responses on the two tests showed that the improvement was consistent with progress in overcoming the misconception that division always makes smaller.

The researchers found that many of the subjects relied heavily on the domain of whole numbers. They expressed the conviction that characteristics of division with whole numbers should also apply in the domain of rational numbers. Moreover, although all of the subjects correctly completed the algorithm for division by a decimal less than one, some of them apparently believed that "division makes smaller" more firmly than they believed in the result of their calculation. A number of the subjects either did not understand or overlooked the equivalence-preserving nature of the "move the decimal points" part of the standard algorithm.

The researchers also mentioned that good performance on computational tasks masked inadequate relational knowledge. It appeared that the inconsistency between the "division makes smaller" conception and the conflicting results of a specific computation were easily detected, and the majority of the subjects realized the conflict shortly after arriving at a correct computation. However, eight subjects' inadequate understanding of the algorithm made it easy for them to resolve the apparent conflict in an inappropriate manner. Many of the subjects were relatively unfamiliar with the measurement interpretation of division and relied heavily on the primitive partitive model. Therefore, they had no convenient means of interpreting a division expression with a decimal divisor. The researchers further mentioned that the primitive model, the initial and relatively long school experience with the operations in the domain of whole numbers and the standard long division algorithm were bases of the misconceptions.

Based on the findings from the previous studies (Graeber, Tirosh, & Glover, 1989; Tirosh & Graeber, 1990), Tirosh and Graeber (1991) conducted a study to determine whether preservice teachers' success in solving division word problems was affected by (a) the type of division problem or (b) the common misconceptions related to primitive models of division. Of particular interest was the relative effect of problem type and misconceptions on preservice teachers' ways of thinking about division.

The subjects were 80 female college students enrolled in one of the mathematics content or methods courses for early elementary education majors in a large university in the southeastern United States. Each of 80 preservice teachers was given two paper and pencil instruments: Writing an Expression for Word Problem and Writing Division Word Problems.

With respect to the Writing an Expression for Word Problem instrument, the preservice teachers were asked to write an expression that would lead to the solution of each given word problem. Problems included addition, subtraction, and multiplication along with 16 division problems, eight of which were measurement type problems and eight of which were partitive type problems. Four of the division problems for each type included data that conformed to the corresponding primitive model; the remaining four problems included data that violated the "the dividend must be greater than the divisor" constraint common to both primitive models. In constructing the division word problems, an attempt was made to use problems with similar contexts and similar numerical data. The division word problems were interspersed with the other problems to reduce the likelihood that the correct answers would result from guessing. The total of 32 problems was divided between two test forms in order that each subject would have a reasonable number of questions to complete. One-half of the preservice teachers in each of the classes received one form and the other half received the other form. The sample word problems are as follows:

Partitive Problems

- It takes 5.25 meters of ribbon to wrap 3 packages of the same size. How many meters of ribbon are required to wrap one of these packages?

- It takes 3.25 meters of ribbon to wrap 5 packages of the same size. How many meters of ribbon are required to wrap one of these packages?

Measurement Problems

- You prepared 5.25 liters of punch. You have punch bowls that hold 3 liters. How many punch bowls can you fill with the prepared punch?
- You prepared 3.25 liters of punch. You have a punch bowl that holds 5 liters. How much of the punch bowl can you fill with the prepared punch?

With respect to the Writing Division Word Problem instrument, four division expressions ($6 \div 3$, $2 \div 6$, $4 \div .5$, and $.5 \div 4$) were presented to the preservice teachers, and they were asked to write a word problem for each. The types of problems written were used to gain insight into the preservice teachers' preference for and access to the types of division word problems. The responses on this instrument were also used to assess the effect of the common misconceptions. Correct word problems for the expressions $2 \div 6$ and $.5 \div 4$ must counter the misconception "the divisor must be less than the dividend" common to both primitive models.

Each of the 33 preservice teachers enrolled in one of the four mathematics methods classes was interviewed to obtain more information about the conceptions they held and their reasoning. Typically, the preservice teacher would first be given problems similar to those missed on the written instrument and asked to write expressions that could be used to solve them. Then, they were asked to explain why they responded the way they had and how they could check their work.

The results indicated that preservice teachers were more successful in writing expressions for partitive type word problems than for measurement problems. Only 44% of the preservice teachers wrote correct measurement type problems for the expression, $4 \div .5$; 18% of the respondents incorrectly wrote reversed ($.5 \div 4$) partitive word problems for this expression.

The interviews strengthened the belief that the majority of the preservice teachers have access only to the partitive interpretation of division. When asked to interpret the expression $6 \div 2$, almost always, they offered a partitive explanation. When asked to offer another interpretation, most of them were unable to do so without many cues and prompts from the interviewers.

Many preservice teachers presented with simple expressions that easily lent themselves to a measurement interpretation were unable to find the quotient without resorting to the standard division algorithm. They were also unable to translate $4 \div .5$ into the question, "How many five-tenths are there in four?" Presented with a measurement interpretation of division, they reacted as if the idea was completely new.

The researchers mentioned that the preservice teachers were more successful with word problems that did not challenge common misconceptions than with word problems that challenged the misconceptions. The expressions the preservice teachers' wrote indicated that many of them had been affected by the constraint of both primitive models, "the divisor must be smaller than the dividend." Preservice teachers' statements during the interviews indicated that the constraints of the primitive partitive division model dominated their thinking even when they solved measurement type problems.

Data on the preservice teachers' success in writing expressions for word problems was used to compare the relative affect of common misconceptions with the affect of problem type on preservice teachers' performance in writing expressions for word problems. A 2×2 analysis of variance was performed using percent of correct responses for each item as the unit of analysis. There were two problem types (partitive, measurement) and two states related to common misconceptions (do not challenge misconceptions, challenge misconceptions). The analysis showed that both the type of problem and a problem's status with respect to misconceptions had significant effects on the preservice teachers' performance in writing expressions for word problems. The interaction for problem type by conformity to primitive models was not significant. The F values indicated that the status of conformity to the misconceptions had a larger effect on preservice teachers' performance on this task than did problem type.

The researchers summarized that when preservice teachers were asked to write expressions for division word problems, they were more successful with word problems that did not challenge common misbeliefs than they were with problems that challenged the misbeliefs. The preservice teachers were more successful in writing expressions for partitive type problems than for measurement type problems. Furthermore, when asked to write word problems for given expressions they almost unanimously wrote partitive type problems. Moreover, the effect of two factors—problem type and conformity to misconceptions—appeared to vary with the type of task being attempted. For the task of writing expressions for a given word problem, the preservice teachers' success was more affected by conformity to the misconceptions than by the problem type. For the task of writing word problems for given expressions, the preservice teachers' success seemed more affected by the compatability of the expressions with the partitive interpretation of division than by their adherence to the constraints imposed by the primitive models. This result suggested that when the preservice teachers were placed in the position of having to attach meaning to a division expression, they tended to give it a partitive interpretation.

As part of a large study, Ball (1990a) examined prospective teachers' substantive knowledge of division and their beliefs about mathematical justifications and the reason for making something true or reasonable in mathematics. The participants of the study were 10 elementary and nine secondary education students. The prospective elementary teachers were majoring in elementary education or child development and had no disciplinary specialization. The secondary teacher education students were mathematics majors or minors. They were systematically selected to vary on several key criteria: gender, academic history in college mathematics, and self-acknowledged dispositions toward mathematics. Of the 19 students, six were men and 13 were women. One student was black, one was Asian, the others were Caucasian. On a 4-point scale, the average overall grade-point average of the elementary majors was 3.18; the secondary majors averaged 3.09. The mean grade-point average for mathematics courses was 3.12 and 3.13, respectively. The elementary teacher candidates had taken, on average, fewer than two mathematics sources since high school, and these courses were often remedial. The secondary teacher candidates had taken an average of 32 quarter credits, or more than nine college-level mathematics courses. Although the group was not selected to be representative, the demographic data indicated that the sample was similar to the population of teacher education students in terms of characteristics such as ethnicity, gender, social class, and age.

Three different mathematical contexts were employed to examine prospective teachers' knowledge of division: division with fractions, division by zero, and division with algebraic equations. In each case, the teacher candidates were asked to explain or to generate representations. Probes were developed to examine the teacher candidates' ideas about what it means to justify or to explain something in mathematics.

These students were interviewed when they were about to enroll for their first education course. The interviews were tape-recorded and transcribed. Careful substantive analyses of the interview questions led to the creation of a set of response categories for each. The responses were coded on two dimensions: (1) their correctness and (2) the nature of the justification provided by the teacher candidates.

With regard to division by fractions, the participants were asked the following:

> People have different approaches to solving problems involving division with fractions. How would you solve this one: $1\frac{3}{4} \div \frac{1}{2}$? Sometimes teachers attempt to find real-world situations or story problems to show the meaning or application of some particular piece of content. This can be pretty challenging to do. What would

> you say would be a good situation or story for $1\frac{3}{4} \div \frac{1}{2}$–something real for which $1\frac{3}{4} \div \frac{1}{2}$ is the appropriate mathematical formulation? (Ball, 1990a, p. 134)

The results indicated that 17 out of 19 participants were able to calculate the division correctly. Five teacher candidates, all mathematics majors, generated an appropriate representation of $1\frac{3}{4} \div \frac{1}{2}$, however, this representation did not come easily to any of them. Five teacher candidates generated representations that did not correspond to $1\frac{3}{4} \div \frac{1}{2}$. The most frequent error was to represent division by two instead of division by one-half. Eight teacher candidates could not generate any representation, correct or incorrect, for $1\frac{3}{4} \div \frac{1}{2}$.

Next, the participants were asked the following questions: "Suppose that a student asks you what 7 divided by 0 is. How would you respond? Why is that what you'd want to say?" The results indicated that, of the 19 teacher candidates, five explained the meaning of division by zero in response to this question. Four of them gave answers that focused on what division by zero means. Two approaches were used: (1) showing that division by zero was undefined and (2) showing that the quotient "explodes" as the divisor decreases. Twelve of the prospective teachers responded by stating rules, seven of them explained division by zero in terms of a correct rule such as "you can't divide by zero," five other teacher candidates responded in terms of an incorrect rule such as "anything divided by zero is zero." Two prospective elementary teachers said they could not remember the answer to $7 \div 0$.

Finally, the participants were asked: "Suppose that one of your students asks you for help with the following: If $\frac{x}{0.2} = 5$, then $x = \ldots$How would you respond? Why is that what you'd do?" In response to this question, the teacher candidates focused on the mechanics of solving algebraic equations. Only one prospective teacher—an elementary major—talked about the meaning of the equation. Fourteen of the prospective teachers, including all of the mathematics majors, focused on the mechanics of manipulating algebraic equations. Four elementary teacher candidates did not know how to solve the equation themselves.

The researcher concluded that the prospective teachers' understanding of division with fractions consisted of remembering a particular rule; that understanding was unattached to other ideas about division. Most of the teacher candidates did not seem to refer to the more general concept of division or to multiplicative ideas and relationships to provide their explanations. Instead they recognized division by zero as a particular case for which there was a rule. Their explanations were simply statements of what they thought to be true for this specific case. Furthermore, half the elementary candidates had the rule wrong. They also mentioned that relying on what prospective teachers have learned in their precollege mathematics classes was unlikely to provide adequate subject matter preparation for teaching mathematics for understanding.

Ball (1990b) conducted a study to examine subject matter knowledge of preservice elementary and secondary teacher education students as they entered formal teacher education in five preservice program sites around the United States. The sample included 217 elementary education majors and 35 mathematics majors who planned to teach high school. The sample was not selected to be representative of the population of prospective teachers. Still, the demographic data indicated that the subjects were similar to the general population of teacher education students in terms of characteristics such as ethnicity, gender, social class, and age.

The study design was longitudinal. At repeated intervals, the researcher administered a questionnaire to all college students in the sample and then interviewed and observed a smaller "intensive" sample of students whom the researcher followed more closely throughout their preservice program and into their first year of teaching. The questionnaire and interview were designed to explore participants' ideas, feelings, and understandings about mathematics and writing, about the teaching and learning of mathematics and writing, and about students as learners of these subjects. Many of the questions were grounded in scenarios of classroom teaching and woven with particular subject matter topics. Mathematical topics included rectangles and squares, perimeter and area, place value, subtraction with regrouping, multiplication, division, fraction, zero and infinity, proportion, variables and solving equations, theory and proof, slope and graphing.

Frequencies were calculated for relevant questionnaire responses. Careful substantive analyses of the interview questions led to the creation of a set of response categories for each question. These categories were modified in the course of data analysis to better accommodate teacher candidates' responses. Most questions were cross-analyzed on several dimensions: subject matter understanding, ideas about teaching, learning and the teacher's role, and feelings or attitudes about mathematics, pupils, or self.

A questionnaire item asked respondents to select from among a set of four story problems those that represented a given division statement, for example:

> Which of the following is a good story problem to illustrate what $4\frac{1}{4} \div \frac{1}{2}$ means? Choose all that apply.
> a) A recipe calls for $4\frac{1}{4}$ cups of milk. How much milk is needed for half a batch?
> b) It takes $4\frac{1}{4}$ hours to drive 200 miles. How far will we have gone in half an hour?
> c) Jim needs $4\frac{1}{4}$ pounds of lentils. How many half-pound bags should he buy?
> d) None of these. Instead:______.
> e) I'm not sure. (Ball, 1990b, p. 453)

From the interview, the researcher first asked the prospective teachers how they were taught to divide fractions and asked them to show by doing $1\frac{3}{4} \div \frac{1}{2}$. Then the researcher told them that many teachers try to identify picture models, stories, or real world representations to help pupils understand mathematical ideas and asked them whether they could think of something to represent the statement $1\frac{3}{4} \div \frac{1}{2}$. If they were able to describe a representation, the researcher probed their understanding of its correspondence to their paper-and-pencil calculations. If they were unable to come up with anything, the researcher told them that many people find this difficult and asked them why they thought it was hard.

The results indicated that the elementary school candidates as well as the secondary school candidates (who were majoring in mathematics) had significant difficulty "unpacking" the meaning of division with fractions. These results fit with evidence from other parts of the interviews and questionnaires that suggested that the teacher education students' substantive understanding of mathematics was both rule bound and compartmentalized. Only 30% of the preservice teachers chose the story problem that represented 4¼ ÷ ½. In addition, even when presented with four choices, 10% of the elementary and 6% of the secondary teacher candidates selected the "I don't know" opinion.

The interviews data indicated that all of the teacher candidates were able to calculate 1¾ ÷ ½. They remembered to invert and multiply—the rule they had been taught—and although a few were momentarily unsure about which number to invert, most were able to carry out the procedure. However, few secondary teacher candidates and no elementary candidates were able to generate a mathematically appropriate representation of the division. Only four out of 35 teacher candidates in the intensive sample (11%) were able to describe a completely appropriate representation of 1¾ ÷ ½. Correct responses did not come easily, however. Twelve out of 35 teacher candidates (34%) generated representations during interviews that did not correspond to the problem. The most frequent error was to represent division by 2 instead of division by ½. The teacher candidates' comments showed that they saw the question as being about fractions instead of division. Nineteen (15 elementary and four secondary) of 35 teacher candidates (54%) were unable to generate a representation for 1¾ ÷ ½. Most of them tried to use the numbers (1¾ and ½), but they did not represent division.

The researcher concluded that although few of the prospective teachers even mentioned division explicitly, the difficulties all of them experienced suggested a narrow understanding of division. Although they worried about the fractions in the problem, they only considered division in partitive terms: forming a certain number of equal parts. This meaning corresponded less easily to division with fractions than did the measurement model. Some prospective teachers lacked explicit understanding of concepts and principles even when they could perform the calculations involved. Few of them explicitly interpreted the division with fraction tasks as cases of division. Furthermore, some predominant assumptions were revealed: doing mathematics means to follow set procedures step-by-step to arrive at answers; knowing mathematics means knowing "how to do it"; and mathematics is a largely arbitrary collection of facts and rules.

In the same line as Ball's (1990b) study, McDiarmid and Wilson (1991) conducted a study to explore the mathematics knowledge of teachers in two alternate route certification programs. The research question was: What do practicing and prospective teachers seem to learn about teaching mathematics and writing to diverse learners during their participation in the two certification programs?

For each program, the researchers collected self-administered questionnaire data from all participants (N = 700) three different times. Then the researchers selected a random subsample (n = 55) to follow intensively over the four years of the study, interviewing them at the beginning of their program, immediately after the program, and then at the end of their first year of full-time teaching. In addition, the researchers observed eight intensive subsample teachers in their classrooms at least twice. Finally, the researchers collected data on the learning opportunities the teachers had in their programs, including interviews with faculty and administrators and observations of key courses and other experiences. The researchers drew from the questionnaire and interview data collected from the teachers, rather than the data from observations of or interviews with program staff.

The subsample (n = 55) consisted of individuals with undergraduate degrees in mathematics who were participating in the two alternate route certification programs in the sample. Participants in these programs received two to four weeks of training the summer prior to assuming full teaching responsibilities. During their first year of teaching, they also regularly attended seminars and inservice workshops designed for them, and in one program they were assigned mentors.

Among these 55 teacher-respondents, eight were in the intensive subsample and were, therefore, interviewed and observed as well. All eight of these were secondary teachers of mathematics. To supplement the interview data used in the analyses reported here, the researchers also included data from eight alternate route teachers whose bachelor's degrees were in fields other than mathematics but who, as elementary teachers, were responsible for mathematics instruction in their classrooms.

The researchers administered a 306-item questionnaire to all participants at 11 sites. The questionnaire consisted of conventional Likert-scale and forced-choice items, some of the latter involving scenarios of teaching situations. The questionnaire was designed to tap respondents' belief about and knowledge of diverse learners, mathematics and writing, teaching and learning, learning to teach, and the role of social context. Because the study participants responded to the questionnaire at least three times (before, during, and after their teacher education programs), the researchers also used their responses to track changes in their beliefs and knowledge.

The second instrument was a highly structured interview used with teachers in the subsample. The interview, designed to complement the questionnaire and requiring approximately two hours to conduct, consisted of a series of teaching scenarios. Questions about each scenario were developed to elicit more elaborate and detailed information about teachers' beliefs and knowledge of, among other things, diverse learners, learning, the role of the teacher, mathematics and writing, and teaching. Responses to interview questions allowed the researchers to explore alternative explanations for patterns apparent in questionnaire results.

In this article, the researchers used responses to a subset of the mathematics items on the questionnaire as well as responses to three of the mathematics scenarios on the interview. These items represented the ideas of division, the nature of zero, operations with integers, proportion and slope.

With regard to knowledge of rules of thumb, algorithms and representations of mathematical ideas, secondary mathematics teachers in the subsample performed considerably better on items of algorithmic knowledge of mathematics (rules of thumb) than they did on items that tapped their knowledge of the logical foundations of mathematical ideas. Most recognized that the statement that "the greater number always goes inside the bracket" in setting up a division problem is false. Most also recognized as false the statement that "you can't subtract a larger number from a smaller one." Some participants (86%) recognized that "invert and multiply" is the correct algorithm for dividing fractions. When they were presented a rule of thumb that had an exception, only 40% of the teachers with degrees in mathematics understood that "any number divided by itself is 1" is not true, since 0 divided by 0 is undefined.

When asked to judge whether a series of common mathematical statements could be explained or was "just one of those things in mathematics that you have to remember," 29% of the teachers with degrees in mathematics incorrectly believed that there was no explanation for why the result is positive when multiplying two negatives. Seven percent thought the fact that the slope of a vertical line is undefined could not be explained. Finally, 27% of the subsample thought, incorrectly, that there was no explanation for why any nonzero number to the zero power is 1. This series suggested that a significant proportion of these teachers were likely to perpetuate the view that much of mathematics is more or less arbitrary and must be memorized.

When asked to choose a story problem represented by division of fractions, 1¼ divided by ½, only 33% of the subsample chose a story that correctly represented a division by fractions problem, "You are making homemade taffy and the recipe calls for 1¼ cups of butter. How many sticks of butter (each stick = ½ cup) will you need?" Although teachers with undergraduate degrees in mathematics performed adequately on some of the questions requiring algorithmic knowledge, a significant number had difficulty with items requiring an understanding of the mathematical meaning of those algorithms or how to represent them.

In the interviews, the participants were asked about the division by fractions. Responses indicated that participants in both groups generally could get the right answer to the problem, "What is 1¾ divided by ½?" Recall that 33% of the teachers with mathematics degrees chose, on the questionnaire, an appropriate representation for 1¼ divided by ½. When the researchers asked teachers in the intensive subsample to identify a story that would help their students understand how the answer was derived, many participants either could not generate a representation at all or told a story that represented division by 2 and not by ½. Of the eight teachers with mathematics degrees who planned to teach secondary mathematics that the researchers interviewed at the beginning of their alternate route program, six offered examples that illustrated either dividing 1¾ into halves or by 2. When the researchers asked the same question to eight elementary teachers in the alternate route programs, none of them could, when they entered their program, generate an appropriate story or example. All the interviewees found the request to represent problems requiring division by fractions difficult.

The researchers also found that although some teachers did apparently learn between their initial and final interviews, the responses of other teachers showed no change. Four of the secondary teachers who initially offered examples that represented either division by 2 or into halves subsequently provided appropriate examples in the interviews the researchers conducted at the end of their first year of teaching, but only one of the eight elementary teachers seemingly had learned to represent this division problem.

On other interview questions, the evidence on learning mathematics from teaching was less clear. When the researchers asked eight secondary teachers, at the beginning of their alternate route programs, to explain 7 divided by 0, only four of them offered explanations or examples that indicated they understood the nature of division and of zero. At the end of their first year of teaching, only one of the four secondary teachers whose initial explanation was mathematically inappropriate offered a correct explanation.

Simon (1993) conducted a study focused on two aspects of prospective elementary teachers' mathematical knowledge central to understanding division—the connectedness of their knowledge and their understanding of units. Two research questions grounded the study. The first research question was "How connected is the prospective elementary teacher's knowledge of division?" Three types of connections were examined: (a) between procedural and conceptual knowledge, (b) among concepts (e.g., between division and subtraction), and (c) between arithmetic operations and the real-world situations on which they were based. The second research question was "To what extent do prospective elementary teachers know the units that correspond to the quantities they work with in contexts involving division?"

The study included two phases of data collection, written responses to problems and interview data. The subjects for the written (n = 33) and the interview (n = 8) phases of data collection were prospective elementary teachers from a large state university. They were randomly selected from the list of volunteer students enrolled in the methods course, "Teaching Mathematics in the Elementary Schools." No students participated in both phases of the study.

In the first phase, five problems were administered to 33 prospective elementary teachers who were directed to show all work and to write full explanations in response to the problems. For each problem, the responses were sorted into groups and the groups were modified until they were judged to characterize the range of responses. In the second phase, eight prospective elementary teachers were interviewed as they worked on Problem 3, 4, and 5 from the original problem set. The interviews were used to obtain an in-depth view of the subjects' thought processes and understanding of the overlapping areas of connectedness and units. The problems were always administered in this order so that work with Problems 4 and 5 would not improve performance on Problem 3. For each problem, the subjects were asked to think aloud as they solved the problem. The interview transcript analysis was interpretive according to the taxonomy developed by Konold and Well (1981).

Five problems provide a discussion of the results from the written data.

1. Story Problems: "Write three different story problems that would be solved by dividing 51 by 4 and for which the answers would be, respectively:

a) 12¾ b) 13 c) 12

You should have three realistic problems." The results on this problem indicated that subjects had the most success on part (a) of story problems and the least success on part (b). All the problems judged to be correct for Problem 1(b) were quotitive; those judged to be incorrect were partitive. Overall, 74% of the problems generated were partitive and 17% were quotitive.

2. Division by a Fraction: "Write a story problem for which ¾ divided by ¼ would represent the operation used to solve the problem." Seventy percent (23 students) were not able to create an appropriate problem for ¾ divided by ¼. Twelve of these students (36%) created problems that would be represented by a number expression other than the one that was given. Problems for ¾ × ¼ were most common. Four students (12%) gave other types of incorrect problems and seven students (21%) gave incomplete or no responses.

3. Calculator Remainder: "How could you find the remainder of 598,473,947 divided by 98,762 by using a calculator?" None of the subjects were able to generate two strategies: subtracting the product of the whole number part of the quotient, and the divisor from the dividend and multiplying the decimal part of the quotient by the divisor. Twenty-four percent were able to offer one valid method of finding the remainder, 6% by method one and 18% by method two. Thirty-three percent said that a remainder could be obtained by putting the decimal over a power of ten, 12% indicated that it could be read from the decimal, and 12%

said to divide the decimal part of the quotient by the divisor. Fifteen percent gave no clear answer to the question, and 3% gave an answer that could not be classified.

4. Cookies: "Serge has 35 cups of flour. He makes cookies that require 3/8 of a cup each. If he makes as many such cookies as he has flour for, how much flour will be left over?" Thirty percent claimed that there was 1/3 of a cup of flour left over. Thirty-three percent had other incorrect solutions, and 21% had no solution.

5. Long Division: "In long division carried out as in the example below, the sequence divide, multiply, subtract, bring down is repeated.

```
     59
 12)715
   -600
    115
   -108
      7
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Explain what information the multiply step and the subtract step provide and how the contribute to arriving at the answer." None of the subject provided a conceptual explanation. Their responses were devoid of references to the meaning of division and were generally of two types. The first type indicated that the operation was necessary to get the next number in order to continue the long division algorithm. The second type was based not on the meaning of division but rather on the local goal of the procedure at the given point. Subjects would say that the multiplication step checks to see that the number selected for the quotient is not too large. They further would explain that the subtraction step checks to see that the number that they had chosen is not too small. The interview data also provided a more detailed picture of the mathematical knowledge of these prospective teachers. The interviews confirmed the results found in the written responses.

The researcher further mentioned that the prospective elementary teachers were unable to think flexibly and consciously about division as partitive and quotitive. Moreover, they seemed to be unable to connect the meaning of division with the symbolic division carried out by long division. Some subjects were able to make sense of division in the context of a word problem, but were unable to connect what they knew in these contexts to an abstract level of thinking about division.

Even and Tirosh (1995) conducted a study to explore one component of pedagogical content knowledge: teachers' planned presentations of the subject matter knowledge of undefined mathematical operations. Special attention was to teachers' planned reactions to students' questions and hypotheses.

The participants were 33 Israeli secondary mathematics teachers. The participants were first asked to answer a questionnaire that included defined and undefined mathematical expressions by providing numerical solutions if possible, and if not, by explaining why not. All the subjects were then individually interviewed and were asked to describe their in-class reactions to a list of suggested definitions of 4/0, 0/0, 0^0, and $(-8)^{1/3}$ which were presented as if they were made by students.

The results indicated that most of the subjects knew that 4 divided by 0 is undefined. However, when asked to explain why, most could not supply any appropriate explanation. Some provided a rule-based argument—both to themselves and to students. For example, one teacher argued that "in mathematics there is such a rule that one cannot divide by 0." She also advocated this rule-based approach as adequate to students' inquiries. Her reaction to a student's suggestion that 4 divided by 0 is 0 was:

> I'll tell them that it is forbidden to divide by zero. I'll explain that mathematics and physics are different. In physics I can explain everything in terms of nature, of reality. In mathematics we have rules, and we operate according to them. These rules often do not seem reasonable. When studying mathematics, one has to adopt these rules and to operate accordingly. There is no reason and there is no point at all in looking for explanations. One just has to accept them. (Even & Tirosh, 1995, p. 10)

These sentences revealed the teacher's own limited conceptions of the reasoning behind the decision not to define 4/0. She had memorized the statement that a/0 is undefined, and was unwilling to attempt to question the logic behind this decision or to challenge it. This teacher viewed mathematics as a bag of unexplainable rules that students should accept, memorize, and use. She considered the teaching of mathematics as a process in which students absorb what they are told. As a result she could not seriously consider students' responses. She expected her students to unquestioningly memorize the rules, much like she had done.

Furthermore, in the case of 4/0, teachers were provided with two common students' responses. One suggested that "4/0 = 4. When you divide by zero, you cannot actually perform the division, and thus you're left with the entire quantity." The other said that "4/0 = 0, dividing by 0 is impossible and thus the answer is 0." The teachers were then asked to describe their in-class reactions to each of these students. They were not explicitly asked to provide explanations for the sources of the students' mistakes. Obviously, these mistakes could evolve from conceptualizing zero as nothing, from viewing division only as sharing, from both these conceptions, or from others yet to be explored. Further probing to better understand their line of reasoning was needed.

The analysis of the teachers' reactions to students' definitions however revealed that most of the teachers did not attempt to make this initial inquiry to better understand their reasoning. In fact, the vast majority of the teachers judged the students' answers only in terms of being right or wrong, and provided them with their own explanations for the right answer. Few teachers tried to carefully examine the students' way of thinking. A rather unique reaction was that of Maya who made careful assumptions about the students' reasoning, tried to find procedures for validating her assumptions, and to define means by which she could discuss these cases not only with one specific student, but with the entire class. She tried to identify both strengths as well as weaknesses that should be addressed in the discussion.

The researchers concluded that many of the teachers made no attempt at understanding the sources of students' responses. When asked directly, they found it difficult to explain why students reacted the way they did. Sensitivity to students' thinking becomes even more difficult under the pressure of real teaching instead of an interview setting.

Tirosh (2000) conducted a study to provide a comprehensive description of prospective elementary school teachers' subject matter knowledge and knowledge of students' conceptions of division of fractions, before and after instruction. A class of 30 prospective elementary teachers participated in the study. All were women in their second year of a four-year teacher education program in an Israeli State Teachers' College. At the beginning of the academic year these prospective teachers completed a questionnaire designed to assess their subject matter knowledge and knowledge of students' conceptions of rational numbers. Each participant was interviewed. During the entire academic year these 30 prospective teachers participated in a mathematics methods course designed to develop their subject matter knowledge and knowledge of students' conceptions of rational numbers.

A diagnostic questionnaire on subject matter knowledge and knowledge of students' conceptions of rational numbers was administered to the prospective teachers at the beginning of the course. The questionnaire included two division-of-fraction items. For item 1, the participants were requested to (a) calculate the following expressions: $1/4 \div 4$; $1/4 \div 3/5$; $4 \div 1/4$; $320 \div 1/3$, (b) list common mistakes students in seventh grade may make after finishing their studies of fractions, and (c) describe possible sources for each of these mistakes. For item 2, the participants were asked to (a) write an expression to solve the problem without calculating the expression, (b) write common incorrect responses, and (c) describe possible sources of these incorrect responses to the following word problems:

1. A five-meter-long stick was divided into 15 equal sticks. What is the length of each stick?

2. Four friends bought $\frac{1}{4}$ kilogram of chocolate and shared it equally. How much chocolate did each person get?

3. Four kilograms of cheese were packed in packages of $\frac{1}{4}$ kilogram each. How many packages contained this amount of cheese?

The results from the pretest indicated that 5 of the 30 prospective teachers gave incorrect responses to some of the four division-by-fraction expressions in item 1. Two incorrectly gave a quotient of 1 instead of 1/16 on the first example. Another participant argued that 1/4 ÷ 3/5 = 62, and two others incorrectly argued that 320 ÷ 1/3 = 106.666....

Parts (b) and (c) of item 1 probed participants' knowledge (knowing that and knowing why) of children's ways of thinking about division of fractions. Regarding knowing that, 27 of the 30 participants gave one typical incorrect response for each expression. The other three participants listed two common incorrect responses for at least one expression. Regarding knowing why, 26 of the 30 participants said that one possible source of incorrect responses was an incorrect application of the standard division-of-fractions algorithm. The other four participants also related common incorrect responses they attributed to students for the first two expressions (1/4 ÷ 4 and 1/4 ÷ 3/5) to incorrect applications of the division algorithm, but these four participants referred to intuitive but incorrect beliefs about the operation of division as another possible source of incorrect responses to the third and fourth expressions (4 ÷ 1/4 and 320 ÷ 1/3).

With respect to knowing that (listing common incorrect responses to division expressions involving fractions), prospective teachers' responses to item 1 indicated that the vast majority of them had this knowledge. With respect to knowing why (understanding the possible sources of specific students' reactions to this item), the data indicated that most participants attributed these mistakes to algorithmic errors, only a few suggested both algorithmic and intuitive sources of students' incorrect responses. The participants believed that the steps of the algorithm are memorized and that if a step is forgotten, students will be unable to reconstruct it through mathematical inquiry. The possibility of performing the division without using the standard algorithm was not considered.

Analysis of responses to the word problems (part (a) of item 2) revealed that all but one participant provided correct responses. Regarding knowing that, 22 participants knew that the expression $15 \div 5$ is a common error for the first word problem. The expressions $4 \div ¼$ and $¼ \times 4$ were listed as common errors for the second word problem by 11 and eight prospective teachers, respectively; and the expressions $4 \times ¼$ and $¼ \div 4$ were chosen as the most common errors for the third word problem by 12 and three prospective teachers, respectively. For each of the three word problems, about a third of the participants' responses to the request to list common mistakes (part (b) of item 2) were of the type "I do not know" (8, 11, and 15 prospective teachers provided such responses to word problems 1, 2, and 3, respectively).

Of the prospective teachers who listed common incorrect responses for the word problems, four referred to intuitive beliefs about the same prospective teachers who had mentioned this source for errors in their responses to item 1. In their descriptions of possible sources of incorrect responses to two of the three word problems (problems 1 and 3), these participants noted children's tendencies to attribute properties of operations with natural numbers to fractions. For the first word problem, they argued that a common incorrect response is $15 \div 5$. One of them explained that "children have learned, for many years, about natural numbers only and they are used to dividing the big number by the small one." The other three participants simply stated that students are used to having the big number first. For word problem 3, these four prospective teachers mentioned $1/4 \times 4$ as a common erroneous response, stating that students tend to think that multiplication makes bigger and division makes smaller, so knowing that the answer should be greater than 4, they chose multiplication. In response to word problem 2, these four prospective teachers noted that they could not think of any specific mistakes. Three of them mentioned that reading comprehension problems might affect students' responses to this and other word problems. Other prospective teachers who listed prevalent mistakes for the three word problems either referred to algorithmically-based error sources of such responses or mentioned the general "reading comprehension difficulties" source of incorrect responses. Among participants whose responses to the request to list common mistakes were of the type "I do not know," five expressed strong, naive beliefs in the ultimate power of good mathematics instruction during interviews. One argued, "All depends on the quality of the instruction. If the teacher was good, his students will not make mistakes after studying the topic." Three participants related the errors to students' mathematical capabilities; for example, one said, "Good students will not make mistakes. Weak students will make all sorts of mistakes."

At the midpoint of the course (after 11 lessons), participants received a home assignment involving division of fractions. The final assignment was given as an in-class test at the end of the course. The home assignment and the final assignment provided indications of the participants' subject matter knowledge and knowledge of students' conceptions halfway through the course and at the end of the course, respectively.

The researcher concluded that, before the prospective teachers entered the course, most mentioned only algorithmically-based mistakes or reading-comprehension difficulties as possible sources of incorrect responses to tasks involving division of fractions. Most participants knew how to divide fractions but could not explain the procedure. By the end of the course, however, most participants were familiar with various sources of incorrect responses. In particular, they were aware of students' tendencies to attribute properties of division of natural numbers to division of fractions; of the constraints that the primitive, partitive intuitive model of division imposed on the operation of division, and of related intuitively based mistakes. Moreover, they pursued and examined various explanations of the standard division-of-fractions algorithm in light of both their accessibility by children and their possible long-term effects on students' mathematical conceptions and ways of thinking.

Preservice Teachers' Knowledge Structures in Classroom Practice

Although much research exists concerning preservice teachers and inservice teachers' subject matter knowledge of mathematics, only a few researchers (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Buckreis, 2000; Lehrer & Franke, 1992; Leinhardt & Smith, 1985) investigated the relationship between subject matter and pedagogical knowledge with classroom teaching.

Leinhardt and Smith (1985) conducted a study to explore the organization and content of subject matter knowledge of expert arithmetic teachers. The samples of the study were eight fourth-grade mathematics teachers, four experts and four novices. These teachers were a subsample of 12 expert and four novice teachers who participated in a three-year study of expertise. The expert teachers were selected because of the unusual and consistent growth scores of their students in mathematics over a five-year period. The novices were student teachers in their last year of a teacher-training program and were considered the best student teachers by their supervisors. The subgroup of four expert teachers was chosen from the set of 12 because they taught fourth grade. Two of the experts seemed to have high knowledge of subject matter, one had moderate knowledge, and one had low knowledge. This estimation was determined by their in-class discussions over three years and by considering their presentations and explanations as well as their errors. The four novices had moderate to low subject matter knowledge.

In the first two years of the study, extensive data were collected on the teachers. They were observed for approximately three months each year while teaching math. They were videotaped for 10 hours. Finally, they were interviewed on several topics: the taped lesson (stimulated recall), how they planned and evaluated their lessons, and their fraction knowledge. The teachers and student teachers were also given card sorts consisting of 40 math problems randomly selected from the computational sections of fourth-grade texts. Transcriptions of the protocols became one database, and transcriptions of videotapes and observations of in-class performance became another.

The analyses were of two types. First, the fraction interview and mathematics card sorts were analyzed to determine any consistent patterns of knowledge and understanding as well as confusion and misunderstanding. Second, three of the teachers, two with high knowledge and one with middle-level knowledge, were examined more closely. Videotapes of each of the three teaching a lesson on reducing fractions that lasted one or two periods were examined in detail.

The outcome of the card sorts and fraction knowledge interviewed indicated that two teachers had relatively high math knowledge, one had relatively middle-level knowledge, and one had barely sufficient math knowledge for classroom instruction. Novices generally had low knowledge. There were observable differences between the high knowledge experts and novices. High knowledge experts sorted 45 math topic cards into 10 categories and ordered the topic by difficulty to teach or perform. They also grouped addition and subtraction together and then ordered problems by difficulty through to decimals. Novices made categories for every one or two problems and noted little differentiation in problem difficulty. They also indicated almost no internal connections.

With regard to teaching fractions, similar distinctions appeared in the tapes and interviews. Four items seemed to discriminate interestingly between groups of teachers. In response to the request to define a fraction, seven of the teachers referred to equal parts of a whole, thus retaining the notion of equal segments and their relationship to the whole. One teacher defined fractions as the points between zero and one or zero and any other whole number, including the whole numbers. This teacher was the only one who consistently used the number line as a frame for the lessons and the only one who saw fractions as having a measurement property.

When asked for a definition of equivalence, all teachers defined it correctly, emphasizing the regional equality. When asked "Are 3/7 and 243/567 equivalent?" The two high- and one middle-knowledge expert saw quickly that 81 was a common factor and that the fractions were equivalent, but the low-knowledge expert and two novices found 81 but did not know what to do with it and eventually stated that the fractions were not equivalent.

The third item involved the concept of unit. When asked to draw pictures representing 3/4, 5/5, and 5/4, all but one of the teachers did so successfully. They were also asked to indicate the unit for each of the fractions, 3/4, 5/5, 4/4, respectively.

The fourth item asked the teachers if there were any differences between the ratio of a set and a fraction of a set. The item first asked how ratios were similar to or different from fractions. Then the teachers were shown a figure and asked to specify what fraction was shaded and the ratio of unshaded to shaded. All of the teachers either said that a fraction and ratio were identical or similar or said they did not know.

The results of the interviews indicated the disparity between the teachers' ability to express an algorithm and their lack of understanding of the underlying mathematical concepts and relationships. The researchers then analyzed the mathematical content of classroom lessons given by three of the experts who demonstrated adequate subject matter knowledge. These lessons focused on the topic of reducing fractions. The lessons were examined by constructing semantic nets that described the content of the subject matter presentation.

The researchers concluded that experts had more elaborate and deeper categories for problems; novices had more horizontal, separate-category systems. However, among the experts there were differences in levels of subject matter knowledge. In-depth analysis of the explanation behavior revealed substantial differences in the details of the three experts' presentations to students. There was considerable difference in the level of conceptual information presented as well as differences in the degree to which procedural algorithmic information was presented. Teachers had decidedly different emphases in their presentations, and they entered the topic differently. They used different representational systems: number line, regional, and numerical.

Lehrer and Franke (1992) used personal construct psychology and the logic of fuzzy sets to elucidate the content and organization of the teachers' knowledge of fractions. They also explored associations between teachers' personal constructions and their classroom teaching. The researcher questions were as follows:

1. Does personal construct psychology provide a means to elicit the various components of teacher knowledge found in the other research?

2. Are there conditional relationships among the components of a teacher's knowledge?

3. Is there any relationship between the portrait of teacher knowledge obtained within the personal construct framework and teaching actions in the classroom?

Two teachers (Ms. Hunter and Ms. Gardner) participated in the study. These teachers had clear differences in their teaching practices prior to the beginning of the research. Ms. Hunter, a second-grade teacher with 17 years teaching experience, often posed problems to students, listened to their solutions with an eye toward understanding student thinking, and generally displayed a highly improvisational form of teaching. Ms. Gardner, a fifth-grade teacher with five years teaching experience, generally followed the order of the textbook when posing problems and largely confined her attempts to resolve student impasses by recourse to a few standard examples. In summary, Ms. Hunter and Ms. Gardner defined more- and less-skilled practices in the teaching of mathematics. The participants were presented three fraction problems and asked to identify which two problems were more similar to each other yet different from the third in terms of (a) the content, (b) how students think about the problems, and (c) the pedagogical actions they associate with the particular problems. Ten triads, which included 12 fraction problems, were presented to each teacher. The triads focused on delineating the teacher's notions of fractions related to (a) identification and representation, (b) order, (c) equivalence, and (d) operations.

With the presentation of each triad the teacher was initially probed for content knowledge followed by probing for pedagogical knowledge. In all cases teachers were probed until they responded that they had nothing else to say about the similarities and differences among the problems. Following the presentation of all 10 triads, the teachers were shown a list of their elicited constructs. Teachers then rated, on a 10-point scale, the degree to which each elicited construct was important to each of the 12 fraction problems. A rating of one indicated that the construct was not important or relevant to the problem, whereas a rating of 10 indicated that the construct was very important or relevant to the problem. The resulting grid of ratings was analyzed to determine relationships among problem types and constructs.

Following the elicitation of constructs, one of the researchers observed the two participants teaching fractions. The observation consisted of verbatim written notes including what was written by both the teachers and the students whenever possible and audiotaped classroom dialogue to substantiate the written notes.

Grids were created from the constructs of each teacher. The constructs were classified according to whether the construct related to the content knowledge of fractions (e.g., 8/8 = 1); general pedagogical knowledge (e.g., concrete materials needed); the pedagogy associated with the teaching of fractions (e.g., use a related fraction); or other content knowledge (e.g., understanding negative numbers). Independent rater agreement for two raters was 94%. Disagreements were resolved by consensus.

The results of the study indicated that wide variability existed between the two teachers' responses to the fraction problem triads. Ms. Hunter provided 33 constructs. Her content-related constructs included some general notion of fractions but did not include notions of equivalence except in terms of needing a common denominator. Most constructs focused on pedagogy both generally and specifically as related to the teaching of fractions. About 30% of Ms. Hunter's constructs were classified as cognitional knowledge. None of Ms. Hunter's elicited constructs related to the teaching of algorithms or procedures. Instead, her constructs focused on the processes necessary for solving fraction problems and the need to build relationships among elements of students' thinking about fractions. On the other hand, Ms. Gardner provided 18 constructs. She provided constructs about the concepts underlying fractions, as well as the procedures used in solving fraction problems. But, she did not discuss ordering or operations with fractions except in terms of reducing fractions to their lowest terms and teaching a procedure. The nine general pedagogical constructs included constructs that were applied to all of the problems discussed, whereas the pedagogical content constructs related to a specific problem. None of her elicited constructs were classified as cognitional knowledge.

In the classroom observation, Ms. Gardner introduced subtraction of fractions with like denominators. Throughout this lesson, her actions were consistent with the constructs elicited. Her focus was on fractions as parts of a whole as she indicated. She drew pictures and had the students draw pictures, focusing on the fact that the denominator determines the number of parts and the same amount of parts as given makes a whole. She focused on reducing fractions to their lowest terms and chose problems for initial presentation of the new concept that were familiar to the children. Whether Ms. Gardner worked in a large group or with individual children her responses to children having difficulty was to attempt to provide an explanation. Hence, generally her explanations did not build on earlier explanations and were not tuned to the students' level of understanding of the mathematical ideas.

Ms. Hunter's class was on dividing a number of objects into different fractions parts. In Ms. Hunter's classroom both the context in which the children learn fractions and how the children thinking played a significant role. Furthermore, much of the lesson was student driven. Ms. Hunter sequenced the questions and the fractions used so that she built from the fractions more familiar to the students to those that were unfamiliar. She also sequenced the fractions so that the students themselves could build relationships between the fractions. These actions were consistent with her constructs about student knowledge.

One major difference between the two teachers was the degree to which they accounted for student thinking. Ms. Gardner not only provided fewer constructs than Ms. Hunter in each of four categories, she also did not provide any constructs about cognitional knowledge. Ms. Gardner's knowledge was less well tuned than that of Ms. Hunter. Moreover, there was no connection between pedagogy, pedagogical content, and content knowledge of fractions in Ms. Gardner's constructs. The lack of connection between these constructs was echoed in her practice.

The researchers confirmed that personal construct psychology offered a coherent and consistent framework for interpreting the interactive roles played by the multiple constructions teachers placed on classroom events. They also concluded that there were relationships between the structure of teachers' knowledge and the practice of teaching.

Borko, Eisenhart, Brown, Underhill, Jones, and Agard (1992) conducted a study to examine the process of becoming a middle school mathematics teacher by following a small number of novice teachers throughout their final year of teacher preparation and first year of teaching. The researchers investigated novice teachers' knowledge and beliefs related to the following domains: mathematics, general pedagogy, mathematics-specific pedagogy, mathematics curriculum, learners and learning, elementary schools, middle schools, learning to teach, teachers as professionals, and self as teacher. They also examined the nature of participants' thinking during preactive, interactive, and postactive teaching. Moreover, one student teacher's knowledge related to a single topic in the elementary and middle school mathematics curriculum—division of fractions was also examined. The analysis focused on a classroom lesson in which that student teacher was unsuccessful in responding to a student's request for a conceptually-based justification for the standard division-of-fraction algorithm.

Eight seniors in an elementary teacher education program at a large southern university participated in the first year of the project. All eight were members of a cohort of 38 students in a year-long senior year experience or model that included professional course work and student teaching. The model was specifically intended for preservice teachers interested in middle school teaching.

All eight participants had selected mathematics as an area of concentration and indicated an intention to teach middle school mathematics on graduation. They were selected purposefully to represent diverse educational backgrounds and range of competencies in mathematics. All eight were average or above average in their academic performance, compared to other students in the model. Ms. Daniels, the participant whose teaching was analyzed in detail, had the most extensive mathematic background of any of the student teachers in the program, having completed her first three years at the university as a mathematics major. Ms. Daniels maintained a C average through two years of calculus, an introductory course in mathematical proof, a first course in modern algebra, and four computer science courses. Moreover, the researchers also attempted to select participants consistent with the ethnic and gender makeup of the cohort. The participants included seven white females and one black female.

The design of the teacher education program called for each cohort member to have four different student teaching placements (seven weeks each; two each semester) in a city unified school district of approximately 15,000 students. During the first three placements, the cohort taught for half of the school day and took courses taught by university faculty; during the final placement, they taught the full school day. During the first 12 weeks of the academic year, mathematics, language arts, and reading methods courses were taught; during the second 12 weeks, courses in science and social studies methods and diagnosis were taught. Ms. Daniels' first student teaching placement was in a self-contained sixth-grade classroom in an elementary school. Her second placement was in a second-grade classroom. Her third assignment was with a mathematics teacher in a junior high school. For her fourth placement, she returned to the sixth grade, but this time to another classroom in a different elementary school. The teaching episode reported in the study occurred in her fourth placement.

The primary source of information about the participants' knowledge and beliefs was a baseline interview administered at the beginning, middle, and end of the school year. Open-ended questions, many of which were based on vignettes describing hypothetical classroom situations involving mathematics, were intended to elicit the participants' knowledge and beliefs about mathematics, pedagogy, mathematics pedagogy, learning to teach, and other domains of teachers' professional knowledge and beliefs.

Division of fractions was one topic that received special attention in the interview. Participants were asked to compare the presentation of this topic in two sixth-grade textbook sections (provided by the interviewer), to describe how they would teach the topic to a sixth-grade class and how they would evaluate student learning, and to react to a hypothetical student's homework assignment on the topic.

To gather information about the university experience, the researchers observed each session of the mathematics methods course, interviewed the instructor after each class session about his goals and objectives for the session and his reactions to it, and interviewed participants about their reactions to the course. The researchers also interviewed the participants, their methods instructors, the university supervisors, and the teacher education program director about their overall impressions of the university's teacher education program. To supplement these data, the researchers collected documents pertaining to the teacher education program and to the students' progress in it.

To gather information about the novice teachers' thinking and actions in the classroom, the researchers conducted week-long visits to each participant's class near the end of her first, third, and fourth student teaching placements. Primary data sources were daily observations of the participants' mathematics instruction, interviews about their planning for that instruction and interviews asking for their reactions to the lessons and to specific lesson components. These data were supplemented by copies of written lesson plans, worksheets, and other handouts.

The researchers organized the data analysis into six strands, representing the six components of the conceptual framework: individual participant's knowledge and beliefs, individual's classroom thinking and actions, university experience, public school experience, personal history, and research project. Analysis for the classroom thinking and actions strand consisted of coding and examining all data related to the teaching episode in order to construct a description of Ms. Daniels' actions in that episode, as well as her planning, interactive thinking, and reflections about the lesson. For the knowledge and beliefs strand, the researchers sorted the coded baseline data and methods course artifacts to identify data specific to division of fractions and closely related topics. Analysis of the university experience consisted of coding all the university data, identifying major themes in these data, and then reconsidering all occasions when multiplication or division of fractions came up in the methods class.

The results of the analyses indicated Ms. Daniels believed that good mathematics teaching included primarily (a) making mathematics relevant for students and (b) making mathematics meaningful to students. Making mathematics relevant for students required teachers to incorporate into their lessons (a) applications of mathematics that students can use in their everyday lives, (b) applications of mathematics that students believe might be useful someday or to someone, and (c) mathematics-related activities that students enjoy. Making mathematics meaningful meant that students should be encouraged to "understand the math, not just know the process, but to understand the reasoning behind it and the logic of it." Over the course of her student teaching year, Ms. Daniels' knowledge and beliefs related to division of fractions changed somewhat. However, there was evidence that her understanding of division of fractions remained quite superficial and that her repertoire of ways to apply or represent division of fractions was limited, even by the end of the student teaching year.

The researchers concluded that Ms. Daniels' conception of good mathematics teaching included several components that were compatible with current views of effective mathematics teaching. However, these beliefs are difficult to achieve without a stronger conceptual knowledgebase and a greater commitment to use available resources and to engage in hard thinking than she possessed.

Buckreis (2000) conducted a study to explore how differences in an elementary mathematics teacher's subject matter knowledge structure impacted classroom teaching and student learning. In particular the investigation attempted to answer the following questions:

1. What is the appearance of an elementary mathematics teacher's subject matter knowledge structure of addition, subtraction, multiplication, and division?

2. How do differences in this knowledge structure relate to classroom teaching and student learning?

The study included two phases. Phase 1 focused on the selection of a single case. An open-ended questionnaire and interview were used to identify the subject matter knowledge structure for addition, subtraction, multiplication, and division of three elementary teachers. One teacher was selected who demonstrated clearly different levels of knowledge for multiplication and division. An additional interview provided information on the teacher's specific climate for teaching mathematics and details about the unit on multiplication and division to be observed.

Phase 2 included daily classroom observations for approximately one hour each day of a seven-week unit on multiplication and division. Informal interviews were conducted with the teacher throughout the unit to better understand the lessons and allow the teacher an opportunity to clarify statements and actions. A final teacher interview occurred after the last classroom observation. At the conclusion of the observations, the students were assessed to determine their knowledge of multiplication and division based on the teacher's unit objectives. Six students, representing the range of class performance, were interviewed to provide additional insights into the students' learning.

The results indicated that the teacher's subject matter knowledge of multiplication was strong but her knowledge of division was faulty and incomplete on several topics including the different meanings of division, the conceptual underpinnings of division procedures, the relationships between symbolic division and real life problems, and the idea of divisibility. Although the translation of the teacher's subject matter knowledge was complex, it seemed to be directly related to classroom teaching and students' learning. The teacher's narrow understandings were associated with an incomplete development of the full range of division situations. The researcher had commented that although the students had significantly more success on the post assessment problems involving multiplication than on those involving division (understandable since the teacher spent more time teaching multiplication than division), a more worrisome concern was that the students in this study exhibited serious misconceptions associated with the meanings of division, division computation, and notions of divisibility. It is possible that the teacher passed these misconceptions to her students.

Discussion and Conclusions

From the review of research of preservice teachers' knowledge of subject matter and of students' conceptions of division of rational numbers, there is evidence that the subject matter content knowledge of preservice teachers, especially elementary, is not strong (Ball, 1990a, 1990b; Even & Tirosh, 1995; Graeber, Tirosh, & Glover, 1989; McDiarmid & Wilson, 1991; Simon, 1993;

Tirosh, 2000; Tirosh & Graeber, 1989; Tirosh & Graeber, 1990; Tirosh & Graeber, 1991). Most preservice teachers possessed the idea that zero divided by zero was zero. Divisions with zero as a dividend or a divisor were difficult for these preservice teachers (Wheeler & Feghali, 1983). Although many of the teacher candidates could produce correct answers for division, several could not, and few were able to give mathematical explanations for the underlying principles and meanings (Ball, 1990a; Even & Tirosh, 1995; McDiarmid & Wilson, 1991; Tirosh, 2000). Prospective teachers exhibited serious shortcomings in their understanding of division. They seemed to have appropriate knowledge of the symbols and algorithms associated with division, but many important connections seemed to be missing, leaving a very sparse "web of knowledge" (Ball, 1990a; 1990b; Simon, 1993; Tirosh & Graeber, 1989). The results also indicated that preservice teachers' abilities to analyze the reasoning behind students' responses were poor (Even & Tirosh, 1995). Before the methods course emphasized knowledge of students' conceptions, the preservice teachers were unaware of major sources of students' incorrect responses in dividing fractions. By the end of the course, however, most preservice teachers were familiar with various sources of incorrect responses (Tirosh, 2000).

The literature reviewed highlighted several aspects about preservice teachers' subject matter knowledge, knowledge of students' conceptions as reflected in classroom practices. Unfortunately, researchers studying preservice teachers' subject matter knowledge (e.g., Graeber, Tirosh, & Glover, 1989; Simon, 1993; Tirosh & Graeber, 1990; Tirosh & Graeber, 1991; Wheeler & Feghali, 1983) and knowledge of students' conceptions of division (Tirosh, 2000) often have not studied the preservice middle school mathematics teacher. Although they may have conceptions about mathematics teaching, they have yet to develop a strong foundation for applying those connections in the classroom. Further research is needed to get a more complete picture of the knowledge structures that preservice middle school mathematics teachers possess in the area of division of rational numbers. In several studies, the researchers did not provide evidence of validity for measures they used in their research (e.g., Ball, 1990a, 1990b; Graeber, Tirosh, & Glover, 1989; Tirosh & Graeber, 1989; Tirosh & Graeber, 1990; Tirosh, 2000; Wheeler & Feghali, 1983). Little or no confidence in the results of these studies that used such measures can be claimed. Moreover, several researchers did not discuss the evidence of reliability for measures they used in their research (e.g., Graeber, Tirosh, & Glover, 1989; Tirosh & Graeber, 1989; Tirosh & Graeber, 1990; Tirosh, 2000; Wheeler & Feghali, 1983). Given these limitations, a more careful and thorough investigation of the nature of such knowledge structures is needed. Some studies (e.g., Graeber, Tirosh, & Glover, 1989; Simon, 1993; Wheeler & Feghali, 1983) did focus on knowledge of individuals as they participated in research contexts (written problem solving and individual interviews). Further research is needed to examine preservice teachers as they engage with their peers and in contexts in which they are developing and using their knowledge of division.

From studies of preservice teachers' knowledge of subject matter and of students' conceptions of division of rational numbers as reflected in classroom practice, relationships were identified between the structure of teachers' knowledge and the practice of teaching (Lehrer & Franke, 1992). Although the translation of the teacher's subject matter knowledge was complex, it seemed to be directly related to classroom teaching and students' learning (Buckreis, 2000). Experts had more elaborate and deeper categories for problems; novices had more horizontal, separate-category systems. There was a difference in the level of conceptual information presented as well as differences in the degree to which procedural algorithmic information was presented. Teachers had decidedly different emphases in their presentations, and they entered the topic differently. They used different representational systems (Leinhardt & Smith, 1985). The teacher's narrow understandings were associated with an incomplete development of the full range of division situations (Buckreis, 2000). Although the student teacher's conception of good mathematics teaching included components compatible with current views of effective mathematics teaching, these beliefs were difficult to achieve without a stronger conceptual knowledgebase and a greater commitment to use available resources and to engage in hard thinking that she possessed (Borko et al., 1992).

Unfortunately, those research studies have been conducted within the context of a voluntary/unrestrictive curriculum. With the lack of investigation of involuntary/restrictive curriculum contexts, the literature is not complete. An investigation of preservice teachers' knowledge and classroom practice within an involuntary/restrictive curriculum is needed in order to address the gap in the research literature.

Thailand has one national curriculum. The national curriculum and mathematics textbooks have been developed by the Institute for the Promotion of Teaching Science and Technology (IPST). Thai mathematics teachers teach in a system where a national curriculum is clearly defined and provided to the teacher. They have little, if any, choice in the flow of the curriculum. Given such educational systems, many important questions can be raised: What is the nature of preservice teachers' subject matter knowledge of division of rational numbers? What is the nature of preservice teachers' knowledge of students' conceptions of division of rational numbers? Is it important to have depth in these knowledge structures where the curriculum is provided? If so, why? How does preservice teacher understanding of division of rational numbers influence the manner in which they teach?

Furthermore, the study of division of rational numbers (e.g., a divisor is zero or decimal numbers) is delayed for seventh and eighth grades in Thailand. Does the delay and the defined curriculum affect the impact of the teachers' knowledge structures on how and what students are able to learn? Are students better prepared to understand the study of rational number division with the instruction provided by teachers who may or may not have in-depth subject matter knowledge and knowledge of students' conceptions? Do classroom practices lead to student misinterpretation? The answers to questions previously mentioned are useful. Consequently, derived information can be used to answer the final question: Does the research conducted in voluntary situation necessarily hold for nonvoluntary/restricted classroom environments? Thus, examining Thai preservice middle school mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers and classroom practices within involuntary/restrictive curriculum context is of interest.

CHAPTER 3
DESIGN AND METHOD

Introduction

The investigation of Thai preservice middle school mathematics teachers' subject matter knowledge and their knowledge of students' conceptions of division of rational numbers as reflected in their instructional practices while teaching division of rational numbers in a required national curriculum utilized a case study research design. The design was selected as a suitable methodology for dealing with critical problems of practice and extending the knowledgebase of various aspects of education.

Thai preservice middle school mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers as well as their instructional practices while teaching division of rational numbers in a required curriculum is largely unknown. Therefore, the exploratory nature of the case study design makes it an appropriate approach for describing Thai preservice middle school mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers with respect to their classroom practices in a teaching environment controlled by a required national curriculum. In addition, a question may be answered: Is the depth of mathematical knowledge and the knowledge of students' conceptions as essential for preservice middle school mathematics teachers' teaching in a controlled national curriculum as that needed in a voluntary curriculum? To gain a richer description of the preservice teachers' knowledge and a deeper understanding of any possible connection between preservice teachers' knowledge of subject matter and of students' conceptions with their classroom instructional practices in a non-voluntary/restricted classroom environment, four cases were explored.

Subjects

The subjects for this study were selected using purposeful sampling. In an effort to identify four preservice teachers who had varied subject matter knowledge and varied knowledge of students' conceptions of division of rational numbers, 25 preservice teachers were initially sought for the first phase of the study. Nineteen preservice teachers willing to participate in the study were administered the questionnaire designed to assess their knowledge of subject matter and of students' conceptions of division of rational numbers. Once the questionnaire data were analyzed, nine preservice teachers were selected to participate in the interviews in order to clarify their responses on the questionnaire. After analyzing the interview data, the researcher was able to identify the four preservice teachers representing different knowledge of subject matter and of students' conceptions of division of rational numbers categories. They had not yet addressed division topics in their seventh grade classroom. Four preservice teachers then were observed while teaching division of rational number units.

Method

A case study design utilized qualitative and quantitative techniques of data collection and analysis. The use of a case study design was purposefully intended for the development of an in-depth description of classroom teaching of four preservice teachers who had different subject matter knowledge and knowledge of students' conceptions on division of rational numbers. Multiple sources of data were collected with each type and phase of data being analyzed separately in order to derive any patterns or themes of information.

Phase 1: Selection of Four Preservice Teachers

The primary purpose of the first phase of the study was to identify the subject matter knowledge and knowledge of students' conceptions of division of rational numbers and to ultimately select the cases for further study. Initial contact with 25 preservice teachers at a teachers' college in Thailand was made before student teaching began. A meeting provided a brief explanation of the purpose of the study and requested their participation in the study. The preservice teachers were told that they would be administered the questionnaire, interviewed, and observed as part of a study exploring the teaching of division of rational numbers at the middle school level. They were also told that the observations would not affect their student teaching grade. It was expected that such an explanation would help reduce the preservice teachers' concerns about critical evaluations and minimize the impact of the observations on classroom instruction. In addition to explaining the general intent of the study, the preservice teachers were given a letter (Appendix A) describing the types of data to be collected and the time commitments involved in being part of the study. The letter also assured the preservice teachers that all data collected as part of the study would remain anonymous. A written response form allowed the preservice teachers to confirm their willingness to participate in the study. The participants were asked to sign and return the letter within one week.

Nineteen preservice teachers willing to participate in the study were administered the questionnaire designed to assess their knowledge of subject matter and of students' conceptions of division of rational numbers. The English version of the questionnaire is provided in Appendix G and the Thai version of this questionnaire is provided in Appendix H. The preservice teachers were ranked according to their subject matter knowledge (SMK) and knowledge of students' conceptions (KSC) determined by the scores from the questionnaire. The top five ranks represented a high level (H) of subject matter knowledge or a high level (H) of knowledge of students' conceptions and the bottom five ranks represented a low level (L) of subject matter knowledge or a low level (L) of knowledge of students' conceptions. The middle scores represented a medium level (M) of subject matter knowledge or a medium level (M) of knowledge of students' conceptions. In using the subject matter knowledge questionnaire, scores from 15 to 26 were classed as low; scores from 28 to 33 were classed as medium; and scores from 34 to 40 were classed as high. In using the knowledge of students' conceptions questionnaire, scores from 12 to 17 were classed as low; scores from 18 to 24 were classes as medium; and scores from 28 to 30 were classed as high. These determinations were based on the scores of 19 preservice teachers.

The knowledge levels of each preservice teacher are presented in Figure 1 along with their raw scores, within parentheses, and the grade level of their student teaching experience.

Participant	SMK Level	KSC Level	Grade Level Taught
1	L (22)	L (12)	7
2	L (25)	M (22)	7
3	H (34)	L (15)	7
4	H (36)	H (28)	7
5	M (29)	M (22)	7
6	M (33)	L (17)	8
7	M (32)	H (28)	8
8	M (29)	M (22)	8
9	H (36)	H (28)	8
10	H (37)	H (30)	9
11	M (32)	L (15)	6
12	H (40)	M (21)	9
13	L (26)	M (19)	2
14	M (33)	M (18)	4
15	M (33)	H (28)	9
16	M (32)	M (24)	9
17	M (28)	M (23)	9
18	L (15)	L (16)	9
19	L (24)	M (24)	5, 6

Figure 1. Knowledge levels of 19 preservice teachers based on their raw scores in the questionnaire.

Participants 1-9 were selected to be interviewed on their subject matter knowledge and knowledge of students' conceptions of division of rational numbers.

They had different knowledge of division of rational numbers based on the questionnaire results. They taught at seventh or eighth grades. They had not yet addressed division topics in their classroom. They were willing to be observed. Participant 2 was initially placed in a medium knowledge of students' conceptions category on the basis of the questionnaire results. However, the interview results suggested that participant 2 had a slightly high knowledge of students' conceptions of division of rational numbers. Thus, he was recategorized as having high knowledge of students' conceptions of division of rational numbers category on the basis of his interview results.

Following the analysis of the interview data, four preservice teachers were selected with as diverse knowledge structures as possible. They met the following criteria:

- The preservice teachers had knowledge of division of rational numbers that satisfied one of the following categories:
    - High knowledge of subject matter-high knowledge of students' conceptions (H/H): participant 4
    - High knowledge of subject matter-low knowledge of students' conceptions (H/L): participant 3
    - Low knowledge of subject matter-high knowledge of students' conceptions (L/H): participant 2
    - Low knowledge of subject matter-low knowledge of students' conceptions (L/L): participant 1
- They taught at seventh grade,
- They had not yet addressed the topic of division in their classroom,
- They were willing to be observed.

The four preservice teachers identified then were contacted to reconfirm their interest. The preservice teachers' schedules for teaching the units were requested. Letters (Appendices B, C, and D) were sent to the preservice teachers' mentors, their supervisors, and students in the preservice teacher class and their parents requesting participation in the study. Following permission by all parties involved, four preservice teachers were selected as subjects for the study.

A questionnaire supplemented by an interview was used to develop a description of the preservice teachers' subject matter knowledge and their knowledge of students' conception structures with regard to division of rational numbers.

Phase 2: Classroom Observations

Four preservice teachers who had clearly different knowledge structures were selected to participate in the second phase of the study designed to generate an in-depth description of classroom teaching of division of rational numbers, an area in which the four preservice teachers had different structures of subject matter knowledge and knowledge of students' conceptions. This phase included classroom observations, formal and informal classroom observation interviews with the four preservice teachers, formal and informal interviews with mentors and supervisors, the collection of classroom documents, and the development of the researcher's journal.

A synthesis of the data from the questionnaire instruments and the follow-up interviews for the four selected preservice teachers provided a detailed characterization of their subject matter knowledge and their knowledge of students' conceptions of division of rational numbers prior to teaching. This characterization formed the basis for the observation of how preservice teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers were applied in classroom situations.

A formal interview of each of the four cases was conducted prior to their teaching of each unit involving division of rational numbers. The interview focused on obtaining information about the unit to be observed. Following the interview, extensive classroom observations were conducted. The class was observed for every lesson taught of the units on division of decimals and division of fractions.

An informal interview of each of the four preservice teachers was conducted prior to the teaching of each lesson. Moreover, interviews were completed after each lesson and each unit. Furthermore, interviews with the preservice teachers' mentors were conducted before and after each unit. The mentors also were interviewed daily before or following the instruction. Interviews with supervisors were conducted every time they supervised the preservice teacher. These interview data were used as additional sources of data to confirm (or disprove) the researcher's interpretation of the preservice teachers' conceptions of division of rational numbers and their teaching in classroom practices. The details of these interviews are discussed with the description of the data sources.

Data Sources

This study was designed to examine Thai preservice middle school mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers with respect to classroom practices. To gather and triangulate information, a variety of instruments was used: 1) a questionnaire on subject matter knowledge and knowledge of students' conceptions of division of rational numbers; 2) interviews following the questionnaire on subject matter knowledge and knowledge of students' conceptions of division of rational numbers; 3) classroom observation on division of rational numbers; 4) classroom observation interviews with four preservice teachers; 5) interviews with the mentors and supervisors; 6) additional documents; and 7) the researcher's journal.

Questionnaire

Before the preservice teachers' student teaching, a questionnaire, developed by the researcher, based on instruments used by several researchers (e.g., Ball, 1990; Chalardkid, 1994; Chang, 1997; Nowlin, 1991; Simon, 1993; Tirosh, 2000) was administered to 19 preservice teachers. The questionnaire focused on assessing preservice teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers in the middle grade. The questions were limited to this mathematics level because: 1) these preservice teachers would have had this mathematics background, and 2) as teachers, these preservice teachers would not be teaching beyond Grade 9 in the first two or three years of their teaching. While it is desirable that teachers have knowledge of mathematics that extends beyond the mathematics they are required to teach (Speedy, 1989), this study was interested primarily in investigating whether the preservice teachers had the basic knowledge of the mathematical concept (division of rational numbers) and processes they were required to teach as well as knowledge of students' conceptions of division of rational numbers. The questionnaire items are provided in Appendix G.

In designing the questionnaire, several aspects of division of rational numbers and students' conceptions of division of rational numbers were considered. The instrument addressed the aspects of the meaning of division, division of integers, division of fractions, and division of decimals, knowledge of students' common conceptions, misconceptions, and difficulties of division of rational numbers and possible sources for these conceptions, misconceptions and difficulties.

The objectives for the questionnaire were pooled from the context of Thai mathematics curriculum developed by the Institute for the Promotion of Teaching Science and Technology (IPST) and grouped into four sections accordingly to levels of Wilson's (1971) taxonomy: computation, comprehension, application, and analysis. They were also based on the Curriculum and Evaluation Standards for School Mathematics' (NCTM, 1989) guidelines on assessment of conceptual and procedural knowledge. The objectives are presented in Appendix E.

The questionnaire was divided into two parts. The questions in Part 1 measured the preservice teachers' subject matter knowledge of division of rational numbers. There were 20 items consisting of multiple-choice and open-ended questions. The questions in Part 2 measured the preservice teachers' knowledge of students' conceptions of division of rational numbers. There were 11 items in Part 2 consisting of open-ended questions. The open-ended structure of the questions precluded a distortion of the results due to such chance factors as guessing during multiple choice items. Scoring guides were used to help score the responses, in order to reduce the possibility of scoring one preservice teacher higher than others for essentially the same answer. A table of specifications for the final assessment is provided in Appendix F. Content validity for the questionnaire was established by having the questionnaire reviewed by five mathematics or mathematics education professionals, prior to administration, to determine if the items were consistent with the stated objectives. Eighty percent agreement by the reviewers was reached in the first round.

The Thai version of the questionnaire was developed by the researcher through a series of steps to ensure equivalency of meaning and freedom from cultural bias. The English version was translated into Thai by the researcher. Another professional translator who had no knowledge of the original English version then translated the Thai version back into English. The two English questionnaires were compared and corrected for discrepancies in vocabulary, phrasing, and syntax and cast into a single, modified version. Afterwards, the modified Thai version was sent to two Thai professors of mathematics education to review for format, correctness, etc. Based on this feedback, the researcher corrected the Thai version. The Thai version questionnaire is provided in Appendix H.

The questionnaire was pretested before use in the study. The pretest included a sample of individuals from the population from which the researcher drew respondents. The pretest form of the questionnaire provided space for respondents to make criticisms and recommendations for improving the questionnaire. The researcher asked respondents to state in their own words what they thought each question meant. The questions were revised and retested until 80% of the members of the pretest sample understood them accurately. Reliability of the questionnaire was established by calculating a spilt-half correlation coefficient. The reliability for the subject matter knowledge questionnaire was .82 and the reliability for the knowledge of students' conceptions questionnaire was .72.

Interview of Preservice Teachers

The questionnaire lacked the ability to probe deeply into the respondents' opinions and understandings. To obtain a more complete picture of the preservice teachers' knowledge, interviews were used to investigate the reasoning of the preservice teachers. Data from the interviews provided explanations for preservice teachers' responses on the questionnaire. The interviews allowed the researcher to follow up on the respondent's answers to obtain more information and clarify vague statements.

A semi-structured interview was conducted with nine selected preservice teachers. The interview consisted of two parts: (1) related questions which did not appear on the questionnaire, and (2) a review of responses to the questionnaire. These two parts addressed both subject matter knowledge and knowledge of students' conceptions.

Interview questions (Appendix I) focused on background information of preservice teachers and probed the responses to the questionnaire items on division of rational numbers. The interview also provided a means of clarifying thinking processes that were not evident from the written responses. The primary goals of this interview were to clarify understandings of subject matter knowledge and knowledge of students' conceptions on division of rational numbers and gain additional insight into how preservice teachers organized their knowledge of division of rational numbers. The data from these interviews provided the background for Phase 2 in the selection of four preservice teachers' classroom observations.

A panel of five Thai mathematics education experts was asked to determine the face validity of the interview questions. These experts were asked to determine whether the questions matched the goals of the interviews and whether the questions were leading or biased. An 80% agreement among the mathematics education experts was a determination of validity. One-hundred percent agreement was reached on the first round. Because one of the purposes of the interviews was to verify preservice teachers' responses on the questionnaire, piloting the interview was not possible before the study began. The interview was piloted during the study. After the administration of the questionnaire, one preservice teacher who was not selected for the interview served as the pilot interview. Information from this interview guided the protocol for the actual interviews.

Classroom Observations

The context where teaching takes place influences teachers' use of their knowledgebase (Fennema & Franke, 1992). For that reason classroom observations were necessary to provide a detailed description of the role the four preservice teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers played in their instructional behavior as they taught units focused on division of rational numbers. For each case study, classroom observations and videotaping began one week before the day division of rational numbers was introduced and continued until it was no longer the focus of instruction. The class was observed every day these units were taught. During the observations, the researcher focused on all elements of instruction such as the setting, the teacher, the learners, and the activities and interactions happening in the classroom. The focus included such instructional elements as presentations, discussions, problem solving activities, hands-on activities, assessment, questions, and interactions with students.

All transactions between the teacher and students were videotaped and fieldnotes were taken. A special microphone was attached to the teacher in order to record the teacher-student interactions. The fieldnotes included information concerning the teacher's movements, student behavior, teacher and student actions and interactions, and general classroom tone. All board and overhead work was also recorded as part of the fieldnotes as well as any materials used during the class. Fieldnotes and videotapes were reviewed each evening for tentative trends and hypotheses to provide an ongoing focus to the observations.

The primary purpose of this study was to analyze preservice teachers' conceptions of division and teaching division of rational numbers. Classroom observations were used to determine the types of conceptions of mathematics and its teaching that were displayed in classroom practices. Further, classroom observations allowed for an investigation into the connection of preservice teachers' conceptions of division to classroom practices.

The researcher observed each preservice teacher to provide a more detailed and comprehensive picture of classroom practices. When observing in classrooms, the researcher was as unobtrusive as possible. The observation process was the same for each lesson. The researcher arrived before class began and asked the preservice teacher for the plan for the lesson. The researcher asked how the students were arranged and what was expected for the lesson. The researcher set up the video camera in the back of the classroom and began taping. As students entered, the researcher was seated to take notes. Throughout the lesson, the researcher noted subject matter knowledge being discussed, students' misconceptions and difficulties, time, activities, preservice teacher actions, student behavior and any reflections or interpretations made by the researcher. Observations lasted the entire class period. An example of a classroom observation form is presented in Appendix J.

Classroom Observation Interviews

A semi-structured interview was conducted with the preservice teachers prior to the teaching of each of the following units: division of decimals and division of fractions. The interview focused on obtaining information about the units to be observed. The researcher requested a daily teaching schedule. After the teaching of each unit, a semi-structured interview was conducted with the preservice teachers. Each formal interview lasted at least half an hour. The interview was audiotaped and transcribed.

Informal interviews were conducted prior to the lesson of each unit. The purpose of the interviews was to obtain information about the objectives, topic, main concepts, activities, sequence of the day's lesson. Additional informal interviews were also conducted after the class or the school day about the day's lesson in order to understand the events the researcher had observed and heard in the lesson. Moreover, each preservice teacher had a chance to reflect upon the instruction related to the topic and to discuss what changes, if any, were planned for future teaching about that topic. The interview was audiotaped for later analysis. Each informal interview lasted approximately 10 minutes. Examples of questions of the various classroom observation interviews are presented in Appendix K.

Interviews with Mentors and Supervisors

A semi-structured interview was conducted with both the mentor and supervisor prior to and following the teaching of preservice teachers on each of the following units: division of decimals and division of fractions. Informal interviews were conducted with mentors daily before or after each lesson. Formal interviews also were conducted with the supervisors following every observation of the preservice teachers' teaching. The interviews were audiotaped for later analysis. Each formal interview lasted approximately half an hour and each informal interview lasted approximately 10 minutes. Examples of interview questions are provided in Appendix L.

Additional Classroom Documents

Additional classroom documents were obtained to gather a complete view of the preservice teachers' actions and classroom activities during the observation phase of the study. All documents used in the normal course of teaching during the observation phase of the study such as worksheets, textbook activities, hands-on activities, homework assignments, assessments, and lesson plans were collected.

Researcher Journal

Since the researcher was the principal data collection instrument for the classroom observations, and as such, could be a major threat to the reliability of the data analysis, it was important to establish possible sources of biases or misinterpretations. Thus, a daily journal was kept containing the researcher's reflections on the classroom observations. The journal included thoughts, questions, reactions, interpretations, and insights made in the course of the observations. These deliberations were used to guide the interviews with preservice teachers by providing the preservice teachers an opportunity to clarify observed actions. The process of allowing the preservice teacher an opportunity to clarify actions discouraged the researcher from relying on personal interpretations of the behaviors of the preservice teacher and students. By acknowledging personal preconceptions, values, and beliefs, the researcher had an opportunity to challenge the developing notions about the preservice teachers' subject matter knowledge and knowledge of students' conceptions with respect to classroom instruction.

Data Analysis

In an attempt to produce an in-depth description of preservice mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers with respect to classroom practices, six types of data were analyzed: the questionnaire of 19 preservice teachers, the semi-structured interviews of nine preservice teachers, classroom observations of four preservice teachers, interviews of mentors and supervisors, classroom documents, and the researcher journal.

The first stage of the data analysis was to assess the responses on the questionnaire. After the questionnaires were completed, the data collected for each preservice teacher were read separately and analyzed. The questionnaire was divided into two parts: part one measured the preservice teachers' subject matter knowledge of division of rational numbers; part two measured the preservice teachers' knowledge of students' conceptions of division of rational numbers. Each preservice teacher's score on each part of the questionnaire were computed and ranked from high to low.

After the analysis of the questionnaire responses was complete, nine preservice teachers representing different knowledge categories were individually interviewed in order to validate their responses to the questionnaire and to provide them with an opportunity to clarify and elaborate on their written responses. All interviews were audiotaped and transcribed. After the audiotapes of the preservice teacher interviews had been transcribed, the researcher analyzed the data in a manner similar to that of the questionnaires.

Four preservice teachers representing H/H, H/L, L/H, and L/L knowledge categories ultimately participated in the classroom observation phase of the study. The classroom observation data were analyzed during the process of collecting the data to uncover patterns, develop working hypotheses, and guide further data collection with respect to each preservice teacher. This process involved reviewing data from fieldnotes of classroom observations and transcripts of informal and formal interviews with the four preservice teachers, from the preservice teachers' mentors and supervisors, and from the researcher's journal. Ideas and patterns were jotted down, key words and phrases used by the preservice teacher or students circled, and particularly important sections highlighted.

A more intensive analysis of the classroom observation data took place at the conclusion of the classroom observations. First classroom transcripts and documents were placed in chronological order by date observed and each piece of data was numbered sequentially with similar kinds of materials being kept together in order to facilitate locating data. After the data were numerically ordered, they were read several times in order to develop an initial "picture" of classroom teaching. The next step of the analysis was the preparation of the case record for each preservice teacher with the goal of bringing the data together in an organized fashion. Transcription of videotapes provided narrative comments to complete the record of what had occurred in the classroom. Information that had been written on the overhead or the board was inserted at the corresponding points of the transcripts. The final transcripts thus provided a synthesis of the videotapes and fieldnotes. These transcripts of the division of rational number lessons combined with the interview transcripts with preservice teacher, mentor, and supervisor and supporting documents constituted the case record for each preservice teacher. Ultimately the analysis process involved searching through the data and recording words and phrases for regularities and patterns in the teaching of division of rational numbers. The words and phrases generated from this search then were used as coding categories to sort the data.

To answer the research question, each individual preservice teacher's questionnaire and interview results were reviewed, and each of the preservice teacher's lessons was examined for any links or connections between preservice teacher's knowledge of division of rational numbers and of students' conceptions of division of rational numbers with the instruction. Because pairs of preservice teachers who participated in the teaching phase shared knowledge categories, it was possible for the researcher to investigate and discuss similarities and differences in lessons across preservice teachers. Patterns in preservice teachers' lessons were compared and contrasted in light of the different types of knowledge held by the preservice teachers.

In general, the descriptions of all four of these categories were of interest: what classroom practices were exhibited by members of each category, how this information differed between each pair of neighboring classes, and what interaction effects existed between the two dimensions in a teaching environment controlled by a required national curriculum.

In detail, four comparisons were made:

- H/H and H/L with L/H and L/L in order to consider the connection of subject matter knowledge on classroom practice;
- H/H and L/H with H/L and L/L in order to consider the connection of knowledge of students' conceptions on classroom practice;
- Both (1) H/H with H/L and L/H with L/L and (2) H/H with L/H and H/L with L/L in order to consider the interaction of subject matter knowledge and knowledge of students' conceptions on classroom practice.

Based on these analyses, the description of Thai preservice middle school mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers with respect to classroom practices in a teaching environment controlled by a required national curriculum emerged.

Researcher

Since data were collected and analyzed by the researcher, efforts were made to address possible biases of the researcher. A brief description of the researcher is provided to assist the reader in determining the perspective from which data collection and analysis were processed.

The researcher is currently a doctoral student in Mathematics Education at Oregon State University. She received a Bachelor of Education degree majoring in mathematics from Udonthani Teachers College, Thailand. The researcher obtained a Masters of Education degree majoring in mathematics education from Chulalongkorn University, Thailand. The researcher taught mathematics at the secondary school level for 10 years and taught mathematics and mathematics education at the college level for seven years. In the first two years of teaching, the researcher taught mathematics to students in the seventh grade. At this grade level, division of rational numbers was addressed in Chapter 3 and 5 of the textbook developed by the Institute for the Promotion of Teaching Science and Technology (IPST), Ministry of Education. The researcher taught division of rational numbers based on this textbook. From her experience, the researcher perceived that division of rational numbers was the most mechanical and least understood topic in elementary and middle schools. From her teaching experiences, the researcher found that children made algorithmically-based mistakes when dividing rational numbers. The most common errors included inverting the dividend instead of the divisor or inverting both the dividend and the divisor before multiplying numerators and denominators.

At Oregon State University, the researcher took several courses: assessment and evaluation, teaching methods, research designs, and teaching practicum. As a result of her experience from these courses and her teaching, the researcher formed strong personal views about teaching and learning. She also developed skills in interviewing and obtaining classroom observations.

CHAPTER 4
RESULTS

Introduction

The purpose of this study was to describe Thai preservice middle school mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers as reflected in their classroom practices in a teaching environment controlled by a required national curriculum. In this description of the research results, four preservice teachers studied are described in depth; their backgrounds, subject matter knowledge, knowledge of students' conceptions of division of rational numbers, and interview responses are reported. The preservice teachers have been assigned pseudonyms to protect their anonymity. Classroom observations are reported for each preservice teacher, and the four preservice teachers are compared. Finally, the similarities and differences of their classroom practices discussed.

Four preservice teachers, Chai, Lada, Sak, and Nisa, were identified for the teaching phase of the study. Each preservice teacher was categorized as having high or low subject matter knowledge of division of rational numbers, as well as high or low knowledge of students' conceptions of division of rational numbers. The intent was to identify a preservice teacher representative of high knowledge of subject matter and high knowledge of students' conceptions, high knowledge of subject matter and low knowledge of students' conceptions, low knowledge of subject matter and high knowledge of students' conceptions, and low knowledge of subject matter and low knowledge of students' conceptions. These four preservice teachers were the best representatives of their groups, because there were overlaps in their categories, offering an opportunity for comparisons of preservice teachers within, as well as between, categories. The information for each preservice teacher is presented in Figure 2.

		Knowledge of Students' Conceptions	
		High	Low
Subject Matter Knowledge	High	Chai	Lada
	Low	Sak	Nisa

Figure 2. Knowledge structures categorizations of preservice teachers.

Thai Mathematics Curriculum

Several countries offer more than one mathematics curriculum for middle school, and a number of authors have developed mathematics textbooks based on different philosophies and learning theories. Within the context of a more voluntary/unrestrictive curriculum, teachers have some latitude in choosing the mathematics textbooks for their classrooms. They also have freedom to create and arrange their curriculum and design their lessons.

In contrast, Thailand has one mathematics curriculum. This curriculum is developed by the Institute for the Promotion of Teaching Science and Technology (IPST). The textbook for Grade 7 developed by the IPST serves as the main textbook in this grade level. Within the context of this nonvoluntary/restricted curriculum, Thai middle school mathematics teachers are required to teach. They have less freedom in designing and arranging their curriculum and lessons than teachers in a voluntary/unrestrictive curriculum. Preservice teachers in Thailand are also expected to follow this curriculum.

The Thai mathematics textbook for Grade 7 consists of seven chapters:

Chapter 1: Representation of Numbers by Numerals

Chapter 2: Properties of Natural Numbers

Chapter 3: Decimals

Chapter 4: Measurement and Estimation

Chapter 5: Straight Lines and Angles

Chapter 6: Fractions

Chapter 7: Length, Area, and Volume

During the course of this study, Chai, Lada, Sak, and Nisa taught seventh grade mathematics classes in the public schools in Bangkok, Thailand and used the required textbook. Their mentors suggested they build their lesson plans from this curriculum. Chai, Sak, and Lada taught in the same school district. Chai and Sak taught in the same school. Even though they were at the same school, they did not discuss their lesson plans or how to teach division of rational numbers.

The school district committee arranged the order of the chapters. They recommended that the teachers in the district teach Chapter 6 before Chapter 5. Chai's and Sak's mentors suggested they follow the sequence suggested by the school district committee which was different from the sequence in the textbook. Although Lada taught in the same school district as Chai and Sak, she did not follow the suggestion given by the school district committee. Lada's mentor suggested she follow the sequence of the chapters in the textbook. Nisa taught in another school district. Her mentor suggested she teach the topics in the same order as in the textbook.

The four preservice teachers' teaching covered all of the chapters in the textbook. However, based on the advice of their mentors, they covered them in the order as shown in Figure 3.

Textbook	Chai	Lada	Sak	Nisa
Chapter 1	1	1	1	1
Chapter 2	2	2	2	2
Chapter 3	3	3	3	3
Chapter 4	4	4	4	4
Chapter 5	6	5	6	5
Chapter 6	5	6	5	6
Chapter 7	7	7	7	7

Figure 3. Four preservice teachers' chapter sequencing.

Division of rational numbers was included in Chapters 3 and 6 of this textbook. The units observed were division of decimals (Chapter 3), representing fractions as decimals (Chapter 3), and division of fractions (Chapter 6). Examples of those units are presented in Appendix M.

Students had previously learned division of decimals in their sixth grade. Thus, in this seventh grade level, division of decimals emphasized division of decimals where the dividend or the divisor has many decimal places and division of decimals with a remainder. The sequencing and topics of division of decimals included in the preservice teachers' lesson plans and their classrooms are presented in Figure 4.

Topics in the Text	Chai		Lada		Sak		Nisa	
	Plan	Teach	Plan	Teach	Plan	Teach	Plan	Teach
A	1	-	1	-	1	1	1	1
B	2	1	2	1	2	2	2	2
C	3	2	3	2	3	3	3	3
D	4	3	4	3	4	4	4	4

Topics:
- A: Dividing a natural number by a natural number.
- B: Dividing a decimal by a natural number.
- C: Dividing a decimal by a decimal (Multiply both the dividend and the divisor by the power of 10).
- D: Division word problems with decimals

A number (1, 2, 3, or 4) represents the sequence the preservice teachers planned and taught the topic.
" - " means the preservice teachers did not plan or teach that topic.

Figure 4. Four preservice teachers' instructional emphasis and sequencing of topics in the division of decimals unit.

All preservice teachers' lesson plans on division of decimals covered all topics that appeared in the textbook and in the same order as in the textbook. During the lessons, Sak and Nisa taught their students all of the topics. Chai did not teach division of natural numbers because there was a special activity before his class. The students came to class 30 minutes late, and the actual time spent on division of decimals was about 20 minutes. Chai adjusted the lesson by skipping division of natural numbers. Since Lada was sick for a week, her mentor taught the lesson on division of decimals algorithm for her. The researcher observed Lada's lesson on division word problems. She reviewed division of decimals algorithm before teaching division word problems. Chai and Nisa taught the "moving the decimal points" strategy in their lesson. This strategy was not presented in the textbook.

Representing fractions as decimals was part of Chapter 3. The primary concept in this unit was that every fraction has a repeating decimal representation. The sequencing and topics of representing fractions as decimals unit included in the preservice teachers' lesson plans and their classrooms are presented in Figure 5.

Topics in the Text	Chai		Lada		Sak		Nisa	
	Plan	Teach	Plan	Teach	Plan	Teach	Plan	Teach
A	1	1	1	1	1	1	1	1
B	2	2	2	2	2	2	2	2
C	3	-	-	-	3	3	3	-

Topics:
A: Converting fractions to terminating decimals
B: Converting fractions to repeating decimals
C: Representing terminating decimals as repeating decimals

A number (1, 2, or 3) represents the sequence the preservice teachers planned and taught the topic.
" - " means the preservice teachers did not plan or teach that topic.

Figure 5. Four preservice teacher's instructional emphasis and sequencing of topics in the representing fractions as decimals unit.

Sak planned and taught every topic presented in the textbook. Although Chai and Nisa planned to teach a topic on representing terminating decimals as repeating decimals, they did not teach this topic in class. Lada did not plan and teach a topic on representing terminating decimals as repeating decimals; instead, she planned and taught a topic on converting repeating decimals to fractions. This topic was presented in the textbook for Grades 8 and 9.

Division of fractions was part of Chapter 6 in the textbook. The sequencing and topics of division of fractions included in the preservice teachers' lesson plans and their classrooms are presented in Figure 6.

Topics in the Text	Chai		Lada		Sak		Nisa	
	Plan	Teach	Plan	Teach	Plan	Teach	Plan	Teach
A	-	-	1	1	1	1	1	1
B	-	-	2	2	2	2	2	2
C	1	1	3	3	3	3	3	3
D	2	2	4	4	4	4	4	4

Topics:

A: Representing $\frac{a}{b} \div \frac{c}{d}$ as a complex fraction

B: Dividing fractions by multiplying both the numerator and denominator by the reciprocal of the denominator

C: Dividing fractions using the "invert the divisor and then multiply" strategy

D: Division word problems with fractions

A number (1, 2, 3, or 4) represents the sequence the preservice teachers planned and taught the topic.
" - " means the preservice teachers did not plan or teach that topic.

Figure 6. Four preservice teachers' instructional emphasis and sequencing of topics in the division of fractions unit.

Lada, Sak, and Nisa planned and taught all of the topics that appeared in the textbook. Chai planned and taught only two topics: dividing fractions using the "invert the divisor and then multiply" strategy and division word problems with fractions. During the lessons, Chai introduced the symbolic form, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ where a, b, c, and d are natural numbers, to the students. When Chai taught the other section, he also taught the students writing division of decimals problem in the form $\frac{a}{b} \div \frac{c}{d}$ as a complex fraction $\frac{\frac{a}{b}}{\frac{c}{d}}$.

Chai

Chai was categorized as having high knowledge of subject matter and high knowledge of students' conceptions (H/H) on the basis of the questionnaire and interview results. He majored in mathematics education and minored in education measurement at a teachers college in Thailand. The requirements of this major included 33 semester hours of general education, 66 semester hours of specialized education, and 10 semester hours of free electives. In addition, he took 42 semester hours of mathematics, including principles of mathematics, calculus and analytic geometry, number systems, linear algebra, abstract algebra, set theory, graph theory, and foundations of geometry. He had taken 10 semester hours of practicum and field experience in teaching mathematics, including three semester hours of a mathematics methods class, one semester hour of participation and observational study, one semester hour of practicum, and five semester hours of student teaching.

Chai found mathematics to be a subject of rules. He explained, "Mathematics is about the rules and principles. They are never changed." When asked to define what a rational number was, Chai responded that rational numbers are numbers written in fraction form, including positive integers, zero, negative integers, decimals, and fractions. Chai's knowledge structure of division had two meanings, repeated subtraction and equal sharing. For equal sharing, he explained, "He has a cake. He wants to share it equally among his five brothers. How much cake will each brother receive?" For a repeated subtraction example, he said, "A jar contains ten ping-pong balls. If we draw two balls from the jar, one after the other, how many times are all the pig-pong balls drawn?"

Subject Matter Knowledge of Division of Rational Numbers

Chai understood division of rational numbers well. During the interview, he declared that two notational forms, "$a \div b$," and "$\frac{a}{b}$," could be written in place of the phrase "a divided by b." He correctly identified the dividend and the divisor from this phrase. This result was consistent with his questionnaire result. He correctly identified that $1\frac{3}{4}$ was the dividend, $-\frac{1}{4}$ was the divisor, and -7 was the quotient of the statement, "$1\frac{3}{4} \div (-\frac{1}{4}) = -7$." On the questionnaire, Chai correctly identified that $0 \div (-2497)$ was 0. He wrote that $0 \div a = 0$, when a was any nonzero real number. When asked to explain why $0 \div (-2497)$ was 0 in the interview, he explained using the missing-factor approach. He stated that any number multiplied by 0 was 0. In other words, $0 \div (-2497) = 0$, since $(-2497) \times 0 = 0$.

Chai identified that $\frac{3}{0}$ and $\frac{2}{3} \div 0$ were undefined. His explanation was that there was no number such that 0 times the number was equal to the dividend. During the interview, when asked to explain why $\frac{3}{0}$ was undefined, he provided this explanation, "Well, we cannot partition 3 into groups, 0 for each group." Chai indicated that the statement, "Any rational number divided by itself is 1" was not true, as 0 divided by 0 was undefined.

With respect to division of fraction problems, Chai correctly identified the multiplication sentence that represented a given division of fractions sentence. Chai's knowledge of the division of fractions was sufficient to respond correctly to the questions involving computation of division of fractions. He solved each of them using the "invert divisor and multiply" algorithm. During the interview, Chai solved all division of fractions problems presented in symbolic forms using the same strategy used on the questionnaire. Unfortunately, he was not able to show that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, where a, b, c, and d are integers and b, c, and d $\neq 0$.

Chai correctly solved two division of decimal problems. He also used the "moving the decimal points" strategy to make the divisor a whole number and then divided as in whole number division. Moreover, when the quotient resulted in a repeating decimal, he correctly wrote the decimal in the short form by writing a dot over the repeating digit. When asked to explain his work in the interview, Chai mentioned replacing the original problem with an equivalent problem where the divisor was a natural number. Chai explained further that the "moving the decimal points" strategy was derived from multiplying both the dividend and the divisor by the same power of 10 to make the divisor an integer.

An inconsistency became apparent, however. Although Chai correctly computed 5 as the quotient of $0.75\overline{)3.75}$ from the questionnaire question, he agreed with the statement, "In a division problem, the dividend must be greater than the quotient." He explained, "The dividend is divided by the divisor. The dividend is equally partitioned. Thus, the quotient must not be greater than the dividend."

On the questionnaire, Chai was able to write expressions for the given word problems, "$6\frac{3}{4}$ kilograms of cheese were packed in boxes, each box containing $\frac{3}{4}$ kilogram. How many boxes were needed to pack all the cheese?" and "A rope $4\frac{1}{2}$ feet long is to be partitioned into 30 shorter pieces. How many inches long will each of the shorter pieces be?" When asked to reverse the process, by giving the expression and requested a word problem, Chai was unable to construct a word problem that satisfied the given expression. During the interview, he was able to write the story problem that was solved by dividing 51 by 4 and for which the answer was $12\frac{3}{4}$. He wrote, "A 51-meter-long log was divided into 4 smaller pieces with equal length. How many meters long will each of the small pieces be?" However, when asked to write a story problem for which the answer was 13, he was unsuccessful.

When given the problem designed to assess his understanding of the connection between a real-world context and the interpretation of a remainder or fractional part of the quotient, "Somsak must deliver 20 tons of rambutans. If his truck can carry 3 tons at a time, how many trips must he make to finish delivery?", Chai was able to provide a correct solution. Although he wrote the expression $20 \div 3$, he did not solve this problem by carrying out the long division algorithm. He solved it using the repeated-subtraction approach.

Chai had no trouble solving other word problems. When asked to write a story problem for which $\frac{3}{4}$ divided by $\frac{1}{4}$ represented the operation used to solve the problem, Chai correctly constructed the story problem. He wrote that, "Tim had $\frac{3}{4}$ ton of rice. He sold $\frac{1}{4}$ ton each to dealers. How much rice did each dealer get?" Chai seemed to have a broad enough concept of division to make sense of the division of fractions less than 1.

When asked to order four fractions, $-\frac{8}{9}, -\frac{7}{8}, \frac{13}{17}, \frac{11}{15}$, from smallest to largest, Chai was able to provide a correct order. His strategies were finding the least common multiple of the denominators, finding the equal fractions, and then comparing the numerators. Chai incorrectly identified that the set $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\}$ was closed with respect to the operation of division. He showed the work as follows:

$$\{1 \div 2, \frac{1}{2} \div 2, \frac{1}{4} \div 2, \frac{1}{8} \div 2, \ldots\}.$$

Despite some mathematical errors, on the basis of his questionnaire and interview responses, Chai was placed in a high knowledge of subject matter of division of rational numbers category.

Knowledge of Students' Conceptions of Division of Rational Numbers

Chai was able to identify conceptions, difficulties or misconceptions that children experience with division of rational numbers. He was able to describe possible sources of difficulties or misconceptions. Chai correctly expressed $0 \div 7$ as 0. On the questionnaire, he identified that one typical incorrect response students might make in computing $0 \div 7$ was 7, thinking that zero was nothing. Thus, zero divided by any number was equal to that number (the divisor). During the interview, Chai added that another possible erroneous response students might make was 0.7 where the students were confused by the decimal.

Concerning division of zero by zero, Chai identified that 1 and 0 were two common erroneous responses for this problem. He indicated the students observed that $\frac{4}{4} = 1$, $\frac{8}{8} = 1$. Thus, they concluded that $0 \div 0 = 1$. For a second possible error, he indicated that the students thought that any number divided by 0 was 0.

On the questionnaire, Chai indicated that 1 and $\frac{1}{16}$ were incorrect responses students might make when they solved $\frac{1}{4} \div 4$. Chai stated that the students might divide 4 by 4 to get the answer of 1. The source for other errors was that the students did not use the knowledge of $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. During the interview, he correctly indicated that $\frac{1}{4} \div 4 = \frac{1}{16}$.

Chai listed two incorrect responses students might make in solving the problem $\frac{1}{4} \div \frac{3}{7}$. He wrote that students changed $\frac{1}{4} \div \frac{3}{7}$ to $\frac{1}{4} \times \frac{3}{7}$ or $\frac{4}{1} \times \frac{7}{3}$. During the interview, Chai concluded that the common errors students might make in dividing fractions were that they inverted the dividend instead of the divisor or inverted both the dividend and the divisor before multiplying numerators and the denominators.

When asked to determine the student's work, $\frac{4}{6} \div \frac{2}{6} = \frac{2}{6}, \frac{6}{10} \div \frac{2}{10} = \frac{3}{10}$, Chai was able to identify the student's error and use her procedure to answer the question. Chai mentioned that the student performed the division algorithm in the same manner as addition and subtraction algorithms. She divided the numerator of the dividend by the numerator of the divisor, and kept the denominator.

From the questionnaire, Chai indicated that $5 \div \frac{1}{5}$ and $\frac{1}{5} \times 5$ were incorrect expressions students might make when they solved the problem, "Five friends bought $\frac{1}{5}$ kilogram of chocolate and shared it equally. How much chocolate did each person get?" The sources of those errors were that the students did not look at the unit and they did not consider division as a possible operation for solving this word problem. During the interview, he added that reading comprehension problems might affect students' responses to this problem. He stated that they did not know whether they should do division or multiplication.

When asked to determine a student's work on the long division algorithm where the student incorrectly placed the digits in the quotient from right to left, Chai correctly indicated that this student was confused about place value. When asked to determine a student's work on a division problem that required the student to add a zero as a place holder in the quotient, Chai correctly indicated that the student did not add a zero as a place holder in the quotient when a number brought down was not great enough to be divided. Chai was able to use this student's pattern to answer the question. When asked to determine a student's work on a long division with a remainder, Chai correctly indicated that the student wrote the remainder as an extension of the quotient.

When given a situation, "A student solves the problem, 'If cheese is 75.60 baht (Thai currency) per pound, how much is 0.78 pounds?' By 75.60 ÷ 0.78," and asked to determine whether or not the student's method was correct, Chai responded that the operation used by this student was not correct. When asked why the student might make this error, Chai responded:

> One possible way might be the student thought that the larger number might be the numerator. Another possible way might be the student thought about partitioning into pound....Thus, the student chose division.

Chai indicated that he would correct the student's misconception by giving the explanation, "1 pound of cheese is 75.60 baht. Thus, 0.78 pounds of cheese is 75.60 × 0.78 baht." Chai was placed in a high knowledge of students' conceptions of division of rational numbers category based on his understanding of students' conceptions.

Classroom Observations of Lessons on Division of Rational Numbers

Chai did his student teaching at the large-size high school in Bangkok. The school had approximately 1,600 students in the seventh through twelfth grades. Chai taught three seventh grade mathematics classes. The class sizes ranged from 41 to 43 students. The class periods lasted approximately 50 minutes. Chai taught from Monday through Friday. He taught, at most, two periods a day and nine periods a week. His schedule was not consistent from day to day. For example, on Monday he taught classes the second and the third periods of the day and had the rest of the day to grade the homework and prepare for another class. Whereas on

Friday, he taught a class the first period and had two hours before he had to teach the next period. His class weekly schedule is presented in Appendix N.

Chai's mathematics classroom was on the fifth floor of the building. The portrait of the King and the Queen of Thailand, the pictures of the Buddha image and the Thai flag were hung on the front wall over the chalkboard. There were two doorways, several windows, and two ceiling fans in this room. The teacher's desk was in the right front corner of the classroom which was next to the door. The students' desks were arranged in pairs facing the chalkboard. There was enough space for the teacher to move around the room. Two bulletin boards in front of the room were decorated with seasonal items. Chai hung several mathematics posters, including the multiplication table, table of measures, and formulas at the back wall of the room.

The researcher observed one of Chai's classes for many lessons on division of rational numbers. This class was taught three days a week, Monday afternoon, Tuesday afternoon, and Thursday morning. The class included 41 students, with 25 girls and 16 boys. On average, the lessons lasted about 40 minutes with five minutes to put away materials. At the beginning of the class, the students used the first few minutes to enter the room, take their seats, and organize their materials. At the end of the class, they used a few minutes to put away their materials before leaving.

Chai usually went to the classroom early in order to prepare for the lesson; he typically divided the chalkboard into sections and wrote or posted the rules or the examples for the topic of the day. When students entered, they wrote the rules or problems in their notebooks. Chai often encouraged the students to solve the problems before he began the lesson. He sometimes moved around the room to see what the students were doing.

When all the students were in the class, a student who was the head of the class led the students in greeting the teacher. In the beginning of the class, Chai always told the students what they were going to study. He often reviewed previously covered topics or addressed prior knowledge relevant to the day's lesson. Chai provided additional practice and reinforcement of previously learned materials. The review also allowed him to correct and reteach difficult topics.

After the review, Chai presented a new topic by asking them to read the problems or the rules chorally. He also used a question-answer technique to go over the examples. Most of the questions were facts. Mostly Chai expected choral responses rather than calling on individual students for responses, although a few students called out responses individually. Chai rarely called on specific students to answer his questions. Only a few questions were addressed to specific students.

Chai presented the content of the lessons in a deductive manner. He introduced division of rational numbers algorithms first and then offered word problems as applications of computational skills to make a link between the computation and every day life. Chai sometimes assigned the students work to complete in class as he circulated around the room to see what they were doing and answer their questions. He interrupted their work frequently to address the whole class with hints, insights, and questions. He kept his students on task during the lessons not allowing wasted time. He sometimes had a volunteer student put the work on the chalkboard. Chai presented the information in his lessons both verbally and in writing but also asked the students to take notes. Near the end of the class, Chai wrote the homework assignment on the chalkboard or distributed a worksheet assignment. Most of the problems were selected from the textbook. Some problems were selected from other resources as well. At the end of class, the head of the class led the students to bow to the teacher. They often said, "Thank you, teacher."

Chai's mentor observed every lesson he taught. She was in the class from the beginning until the end of the class. She sometimes sat among the students, stood in front of the class, or moved around the room helping the students. Chai's mentor also helped him manage the class and distribute the homework assignments and other materials. In class, whenever Chai had finished instruction and if she felt that the students were confused, she often summarized the topics. During the lesson, Chai did what his mentor suggested.

Chai's mathematics supervisor observed his lessons twice although the observations were not on division of rational numbers. Chai and his mathematics supervisor discussed his written lesson plans, instructional materials, question-answer techniques, and classroom management and discipline.

Chai created three daily lesson plans on the division of rational numbers. Examples of Chai's lesson plans are presented in Appendix O. He followed the sequencing and topics presented in the textbook developed by the IPST. During the classroom observations, Chai spent four days teaching division of rational numbers. The sequencing and topics of division of rational numbers included in Chai's lesson plans and his classrooms are presented in Figure 7.

<table>
<tr><th colspan="2">Lesson Plan</th><th colspan="2">Classroom</th></tr>
<tr><th>No.</th><th>Topic</th><th>Day</th><th>Topic</th></tr>
<tr><td rowspan="2">1</td><td rowspan="2">Division of Decimals Algorithm and Division Word Problems for Decimals</td><td>1 (Tuesday)</td><td>Division of Decimals Algorithm</td></tr>
<tr><td>2 (Monday)</td><td>Division Word Problems for Decimals</td></tr>
<tr><td>2</td><td>Representing Fractions as Decimals</td><td>3 (Tuesday)</td><td>Representing Fractions as Decimals</td></tr>
<tr><td>3</td><td>Division of Fractions Algorithm and Word Problems</td><td>4 (Thursday)</td><td>Division of Fractions Algorithm and Word Problems</td></tr>
</table>

Figure 7. Chai's instructional emphasis and sequencing of topics in the division of rational numbers units.

During the classroom observations, Chai did not follow his lesson plans exactly in that he did not use all examples in the plans. He used some division problems that he created before the class that were in the same line as the examples in the textbook. Sometimes he created some problems as he was teaching. He did exactly what his mentor suggested. Basically, Chai followed his mentor suggestions in teaching the sections following the first lesson.

Due to a school special activity, the first day was only 20 minutes. Chai had to adjust his teaching and, therefore, his teaching did not exactly follow his plans. The examples presented in the classroom evoked more students' errors than those presented in the lesson plan.

Chai worked closely with his mentor who taught other sections of seventh grade mathematics. Chai provided in his mentor the lesson plans ahead of time. He said that his mentor had suggested he teach by writing lesson plans (learning outcomes, instructional objectives, and so on), using the chalkboard, managing the classroom, and so on. His mentor told Chai that he did not need to follow his lesson plans. The mentor instructed Chai to use whatever strategies the students understood providing examples along the same line as the textbook and explanations of the content. Following the lesson, the students needed to achieve as mentioned in the objectives. Before teaching the lesson, Chai and his mentor discussed the plan for the day. After the lesson, they reviewed the lesson he taught, the flaws of his teaching, and how to adjust his teaching.

Day 1: Division of Decimals Algorithm

Before teaching the division of decimals unit, Chai planned to spend two lessons on division of decimals and follow the order of the topics in the textbook developed by the Institute for the Promotion of Teaching Science and Technology (IPST). Chai felt that some students might not be able to perform the division of decimals algorithm because they were unable to carry out whole number division. Therefore, he planned to review whole number division. He also stated that some students had not memorized the multiplication and division facts. Thus, these students would have difficulties in learning the division algorithm. To remediate this error, Chai had the students commit these facts to memory. He had the students recite during the recess or after school in his office. The students would receive extra credits for this activity.

On the first day, Chai presented the rules for dividing decimals and then he gave the students three examples, two involving dividing a decimal by a natural number and one requiring division of a decimal by a decimal. The students learned to make the divisor a natural number by multiplying it by a power of 10. The students were also taught a shortcut—the "moving the decimal points" strategy. The relationship between the two strategies was explained to the students.

Before class, Chai divided the chalkboard into five sections and posted the rules for dividing decimals on the first section of the chalkboard:

Division of Decimals

Rules for dividing decimals

1. To divide a decimal by a natural number, place the decimal point in the quotient directly above the decimal point in the dividend. Then divide as you would with natural numbers. If there is a remainder, add zero in the dividend.
2. To divide a decimal by a decimal, multiply both the dividend and the divisor by 10, or by 100, or by 1,000, or by 10,000, and so on according to the number of decimals places of the divisor to make the divisor a natural number, then divide as you would with natural numbers.

In addition to the rules, Chai wrote four division problems on each section of the chalkboard for the students to record and solve in their notebooks at the beginning class.

When almost all of the students were in the room, Chai captured the class's attention by saying, "Today's lesson is division of decimals. Are you ready?" Several students answered chorally, "We are ready." Chai then said, "There are two rules in dividing decimals. The first rule is used when the divisor is a natural number." Chai had the students read the first rule chorally. Several students read the first rule. Then he explained before the class:

> If the divisor is a natural number, divide as you would with natural numbers without considering the decimal point. You will consider it when you will place it in the quotient. The decimal point in the quotient must be right above the decimal point in the dividend.

Chai reminded the students how to place the decimal point in the quotient. While he was talking, Chai looked around the classroom. He presented a division problem $8.056 \div 5$ that required students to add zeros as place holders in the dividend. Chai reminded the students that a division problem could be written as $5\overline{)8.056}$.

Chai guided the students through the example using a question-answer technique. At the chalkboard, he recorded the answers given by the students and the students copied the work in their notes. When the final remainder was 1, Chai asked the whole class what they would do. Several students responded, "Add zero at the dividend." Chai wrote the zero after the digit "6" in the dividend. Chai did not include any discussion of whether the values of 8.056 and 8.0560 were equal and the students did not question the result. The class continued the division, getting the result of 16112. A student called out, "Place the decimal point." Chai asked the class where to place the decimal point. A few students responded and gave a correct answer. Chai wrote on the chalkboard, "$8.056 \div 5 = 1.6112$." He asked the class whether they got the same answer. Some students raised their hands to show their agreement.

Chai presented the students with another division problem, $12.516 \div 12$. This problem required students to add a zero as a place holder in the quotient. This problem was neither in the lesson plan nor in the textbook. Chai did not write this problem in a long division form at the outset. Instead, he asked the class to identify the dividend and the divisor of the problem. Several students gave correct answers. Chai accepted their answer and wrote $12\overline{)12.516}$ on the chalkboard. The whole class then carried out the division together. Chai used the same strategy as he used in the first problem. He asked the students questions to guide their work recording their responses. When the dividend was less than the divisor, several students told Chai to add a zero at the quotient. The class continued the division, getting the result of 1043. Chai asked the students where to place the decimal point in the quotient. Several students correctly identified the placement of the decimal point in the quotient. Chai wrote $12.516 \div 12 = 1.043$ on the chalkboard. Chai asked the class whether they got the same answer. A girl replied, "Yes, I did." Chai responded to that girl, "Great!"

The class did not check the accuracy of the quotient by using other ways such as the relationship between multiplication and division. Chai did not place the decimal point in the quotient right away while he was performing the long division algorithm. He placed it after he had carried out the algorithm by asking the students to identify the correct placement. Several students correctly identified the placement of the decimal point in the quotient. Chai reminded the students to place the decimal point in the quotient directly above the decimal point in the dividend. This reminder helped them, preventing some from incorrectly placing the decimal point in the quotient.

After the class had finished the second problem, Chai's mentor walked to the chalkboard and suggested that they place the decimal point immediately in the quotient. His mentor convinced the students that if they placed the decimal point in the quotient after they had finished carrying out the algorithm, some students might forget to place the decimal point in the quotient. While his mentor explained to the whole class, Chai stood in front of the class and listened to his mentor and the students listened to the mentor copying the work in their notebooks as well.

Chai then had the students read the second rule chorally. He then explained that if the divisor was a decimal, they had to make the divisor a natural number by multiplying it by 10, or by 100, or by 1000, or by 10000. Chai reminded the students to change a decimal to a natural number, "If the divisor is a one-place decimal, then by what number must it be multiplied?" A girl replied, "10." Chai accepted her answer as a correct answer. Chai asked, "If the divisor is a two-place decimal, then by what number must it be multiplied?" Another girl replied, "100." Chai then asked, "If the divisor is a three-place decimal, by what number must it be multiplied?" Another girl answered, "1,000." Next Chai asked, "What if it is a four-place decimal?" A few students responded, "10,000." Chai asked, "What if it is a five-place decimal?" A few students replied, "100,000." Chai then responded that multiplication by 100,000 caused them to remove the decimal point and divide as with natural numbers. The students had no difficulty in finding those numbers. While this conversation was taking place, Chai summarized the ideas on the first section of the chalkboard.

> If the divisor is a one-place decimal, then multiply it by 10.
> If the divisor is a two-place decimal, then multiply it by 100.
> If the divisor is a three-place decimal, then multiply it by 1,000.

Chai followed by presenting a division problem with a one-place decimal divisor, $8.056 \div 5.3$ which was on the board. This problem was presented neither in his lesson plan nor in the textbook. He reminded the students that if the divisor was a decimal, they had to make the divisor a natural number by multiplying it by 10, or by 100, or by 1000, or 10000. Chai then discussed making the divisor of the problem a natural number. Chai asked the whole class, "In this problem, how many decimal places are there in the divisor?" A few girls replied, "One decimal place." Chai asked further, "By what number must it be multiplied?" A few girls responded, "10." Chai then asked, "If we multiply the divisor by 10, then we have to multiply what number by 10?" A few students answered, "The dividend." Chai emphasized that they had to multiply both the divisor and the dividend by 10. Chai also reminded the students that 8.056 was the dividend, 5.3 was the divisor and $8.056 \div 5.3$ could be written in a fraction form as $\frac{8.056}{5.3}$. Chai and the students carried out the multiplication and the division algorithms, getting an answer of 1.52. Chai always asked the students to guide the work. He put the work suggested by the students on the chalkboard. No attempt was made to verify the accuracy of the quotient.

At this time, Chai immediately placed the decimal point in the quotient directly above the decimal point in the dividend with student's help. Time ran short during the third problem, so Chai did not work the fourth problem, 2.575 ÷0.25. Five minutes before the class period ended, Chai then checked the students' understanding and summarized the lesson. Chai said, "If the divisor is a natural number, divide and place the decimal point directly above the decimal point in the dividend. If the divisor is a decimal, then make the divisor a natural number by multiplying it by 10, or by 100, or by 1000. We have to determine the number of decimal places in the divisor. If the divisor is a one-place decimal, by what number must it be multiplied?" A few students replied, "10." Chai asked further, "If it is two-place decimal, multiply it by?" A few students responded, "100." Chai asked, "If it is three-place decimal, multiply it by?" A few students replied, "1,000." A boy raised his hand and asked whether he had to copy the work on the chalkboard or not. Chai told to the whole class that they had to copy the two examples.

In the first day's lesson, Chai presented the information not only verbally and in writing form but also asked the students to take notes. Near the end of the class period, Chai assigned the students six homework problems on division of decimals algorithms selected from the textbook. Five problems were like the ones worked in class, two with natural number divisors and three with decimal divisors. One problem with a decimal divisor required the quotient to nearest hundredths. At the end of the class, Chai allowed students to leave the class. There were a few students who asked Chai about the homework at his desk. Other students put away their materials and left the class.

After class, Chai mentioned that the lesson was too rushed because he ran out of time. He said that in the next lesson he would review the topic on division of decimals algorithms before teaching division word problems. Chai also mentioned that since the division algorithm employed all four operations, addition, subtraction, multiplication, and division, the students could analyze the errors in other operations and if the students were able to divide decimals correctly, they would not have difficulties in other operations.

Day 2: Division Word Problems for Decimals

Chai told the researcher before class on the second day that he would review division of a decimal by a decimal. He made this decision after grading the homework from the previous lesson. Chai found that some students had no trouble in making the divisor a natural number. However, they made an error in multiplying the dividend by the power of 10. Chai said that he would review multiplication. Chai mentioned that his mentor suggested that he emphasize "moving the decimal points." He further mentioned that in solving word problems some students might have difficulties in identifying what question was being asked and what operation was required in order to answer the question.

Chai arrived at the classroom early and wrote a word problem on the chalkboard, "A train can travel 6.25 kilometers in 2.5 minutes. How long does it take the train to travel 82.5 kilometers?" Once class began, Chai reviewed the long division algorithm of decimals using the problem $3.456 \div 1.2$. Chai and the students proceeded to multiply both 3.456 and 1.2 by 10. They carried out the division of $12\overline{)3.456}$ together long hand, getting the quotient of 2.88. Chai reminded the students to place the decimal in the quotient directly above the decimal point in the dividend. However, no attempt was made to check the accuracy of the answer. Chai had the students read the word problem chorally. He then asked the students to identify the given information. Several students answered the questions. He told the students that to solve this problem they had to use the rule of three and explained each step of this rule. While he talked, Chai wrote on the chalkboard as follows:

A train travels 6.25 kilometers in 2.5 minutes

A train travels 1 kilometer in $\frac{2.5}{6.25}$ minutes

A train travels 82.5 kilometers in $\frac{2.5}{6.25} \times 82.5 = 33$

Chai and the students proceeded to multiply 2.5 by 82.5, getting the product of 206.25. They multiplied both the dividend and the divisor by 100, getting a fraction of $\frac{20625}{625}$. Chai mentioned that if the dividend and the divisor of a fraction were large numbers, they could reduce that fraction to lowest terms before performing the long division algorithm.

Chai reminded the students that to reduce a fraction, they needed to divide the numerator and the denominator by the number that would divide both evenly. Chai and the students then reduced the fraction to the lowest terms together, getting a fraction of $\frac{33}{1}$. With the help of the students, Chai concluded that $\frac{33}{1}$ was 33. Thus, the students saw that a whole number could be written as a fraction with a denominator of 1. The class summarized that a train travels 82.5 kilometers in 33 minutes. Chai did not draw a diagram while solving this problem.

While the students were taking notes, Chai wrote the second word problem on the chalkboard.

> A rectangular land is 34 meters and 50 centimeters wide and 45 meters long. If this land was sold for 3,881,250 baht, what was the cost of the land per square meter?

Chai had the students read the problem chorally and discussed with the class:

> Chai: What is the width of this land?.
> Several students: 34 meters and 50 centimeters.
> Chai: How many meters? Write it as a decimal.

A few students: 34.50.
Chai: What is its unit?
Several students: Meters.
Chai: Thus, after we wrote it as a decimal, we got 34.50 meters. What is the length of this land?"
Several students: 45 meters.
Chai: The length is 45 meters. Thus the area of this land is? Do you know how to find the area?"
A few students: Yes.
Chai: The area of a rectangle. This land has a rectangular shape. What is the formula for finding rectangular area?"
A student: Length times width.
Several others: Length times width times height.
Chai: Pardon me.
A few students: Width times length.
Chai: Width times length. What is the width of the land?
Several students: 34.50 meters.
Chai: What is the length of the land?
Several students: 45.

Chai and the students carried out the multiplication of 34.5 times 45 together, getting an answer of 1552.50. Chai asked the students to identify the unit of the area. Several students gave the correct answer: square meters. Chai then said, "How much money did this land cost?" Several students answered, "3,881,250 baht." Chai concluded, "We want to know the price of this land per square meter. What operation should we use?" Several students replied, "Division." Chai had the students identify the dividend and the divisor. Several students replied and gave the correct responses. Chai and the students multiplied the divisor and the dividend by 10 to make the divisor a natural number. They proceeded to divide 38,812,500 by 15,525, getting the quotient of 2,550. Chai also reminded the students that they could reduce a fraction to its lowest terms before performing the long division algorithm. Chai asked the students to identify the unit and the correct answer was offered. Chai and the students then summarized that this land cost 2550 bath per square meter. Chai did not draw any diagram while he and the students were solving the problem. Near the end of the class period, Chai pointed out the relationship between multiplication by a power of 10 and the "moving decimal points" strategy using three examples. At the end of the lesson, Chai assigned three word problems as the homework that were selected from the textbook.

After class, Chai mentioned that some students still had a problem with reducing fractions. He said he would talk individually with the students who had this difficulty. Some students did not understand the "moving the decimal point" strategy. His mentor had suggested he review this strategy and the relationship with multiplication by the power of 10. After Chai had taught all sections on the division of decimals, he told the researcher that some students performed the division incorrectly. The error was due to their lack of knowledge of basic multiplication or division facts. Some students had difficulty in moving the decimal points in the numerator. Chai further mentioned that the students did not like learning division of decimals. They said it was difficult and complex.

Day 3: Representing Fractions as Decimals

Before the class began Chai told the researcher that he would do a review of division of decimals when the divisor was a decimal. His mentor had also suggested that he concentrate on the "moving the decimal points" strategy. When asked to identify the difficulty that students might have in learning this topic, Chai recognized that they could be confused when the dividend was less than the divisor. From elementary school grades, the students were accustomed to performing division with the dividend greater than the divisor. Thus, they might think that they could not perform division when the dividend was less than the divisor. His mentor had suggested he use simple examples and small numbers.

Chai went to the classroom early and wrote on the chalkboard the rule for converting fractions to decimals.

> To convert fractions to decimals, divide the numerator by the denominator. The quotient will be a natural number or a repeating decimal.

He wrote three fractions on the chalkboard, $\frac{2}{5}, \frac{4}{8}, \frac{6}{8}$. Chai had the students write the rule in their notebooks. He also moved around the room as the students took notes.

Chai began the lesson with a review of dividing a decimal by a decimal. He wrote three division problems, $25.429 \div 18.326$, $394.23 \div 4.8$, $1200.4 \div 12.65$, on the chalkboard. He converted the first division problem into a fraction form and asked the students to identify each part of the problem. He then reminded the students that the numerator was the dividend, the fraction bar represented the division symbol and the bottom number was the divisor. Chai asked the students what they would do next. A girl replied, "Change the decimal divisor to a natural number." Chai accepted her answer and confirmed that they had to make the decimal divisor an integer or a natural number. Chai then asked the class to identify the number of the decimal place of the divisor. Several students gave a correct answer to three decimal places. Chai asked further, "What number must it be multiplied by? A few students replied, "1,000." Chai multiplied both 25.429 and 18.326 by 1,000. Chai emphasized that they needed to multiply both the numerator (dividend) and the denominator (divisor) by the same number and reminded the students that if the divisor was a two-place decimal, it had to be multiplied by 100 and if the divisor was a one-place decimal, it had to be multiplied by 10. Chai then mentioned moving the decimal point back (to the right) in making the divisor a natural number. He pointed out the relationship between the "multiplying by the power of 10" and the "movement of the decimal points" strategies. While Chai talked, he wrote the following on the chalkboard:

$$25.429 \div 18.326 = \frac{25.429}{18.326} = \frac{25.429 \times 1{,}000}{18.326 \times 1{,}000} = \frac{25{,}429}{18{,}326.000}$$

Chai and the students did not perform the long division in class. Chai told the students to perform the division by themselves at home. Chai and the students then went over another two problems in the same manner. Chai put all the work on the board.

Chai's mentor observed his class. She sat in the class next to the front door. After Chai had finished a review, his mentor summarized for the class that if the divisor was a decimal, the students had to make the divisor an integer. She reviewed the "moving the decimal points" strategy again and pointed out its relationship to the "multiplying by the power of 10" strategy.

Next, Chai presented the topic of representing fractions by decimals to the students by having the class read the rule written on the chalkboard chorally. Chai explained that a fraction could be expressed in a decimal form by dividing the numerator by the denominator. The result could be a natural number or a decimal. The decimal could be a nonrepeating or repeating decimal. Chai presented the students with three fractions, $\frac{2}{5}, \frac{4}{8}, \frac{6}{8}$, where the divisors were greater than the dividends. These particular fractions could be expressed as terminating decimals.

Chai had the students convert the first fraction to a decimal. The conversation that followed illustrates some students' misconception and how Chai corrected their misconception.

Chai: How can $\frac{2}{5}$ be expressed as a decimal?
A few students: 2.5.
Chai: We have to use the rule. What is the rule?
A few students: Divide the numerator by the denominator.
Chai: Divide the numerator by the denominator. What is the denominator?
Several students: 5.
Chai: What is the numerator?
A few students: 2.
Chai: 2 is the numerator. In this example what number is the dividend?

Several students: 2.
Chai: What is the top number called?
Several students: The dividend.
Chai: The dividend. What does this symbol "__" represent?
A few students: The division symbol.
Chai: The division symbol. Thus, what is the bottom number called?
A few students: The divisor.
Chai: We will perform the long division algorithm. Tell me again, what number is the dividend?
Several students: 2.
Chai: What number is the divisor?
Several students: 5.

Chai wrote "$5\overline{)2}$" on the chalkboard. Chai and the students then carried out the division of $5\overline{)2}$, getting an answer of 0.4. Chai quickly commented that the quotient was not equal to 2.5. The students had to use the rule for converting a fraction to a decimal.

Chai had the students identify the divisor and the dividend of the next problem. The correct responses were given by several students. Chai and the students proceeded to divide 4 by 8, getting the result of 0.5. He put all the work on the chalkboard. Chai asked the class to identify the dividend of a fraction $\frac{6}{8}$. Several students gave two responses, including, "8," and "6."At this time, some students showed a misconception. They thought that the larger number might be the dividend. Chai immediately said that the dividend was not necessarily greater than the divisor. He further emphasized that the top number was the 'dividend' and the bottom number was the 'divisor.' Chai and the students then carried out the division of $8\overline{)6}$, getting a decimal 0.75. Chai put all the work on the chalkboard himself.

After the class completed three examples, Chai reminded the students one more time that $\frac{2}{5}$ was not 2.5, $\frac{4}{8}$ was not 4.8, and $\frac{6}{8}$ was not 6.8. He also reminded the students to add a zero in the quotient when the dividend was not great enough to be divided. In each problem, the dividend was less than the divisor, thus when they performed the long division algorithm, they had to add zero as a place holder in the dividend.

Unfortunately, Chai and the students simply added a zero after the dividend without placing the decimal point between the dividend and 0. Chai's mentor recognized this shortcoming. She suggested that instead of adding a zero at the dividend, they had to add a zero after the remainder to continue the division. Chai and his mentor did not point out that 2 and 2.0 were the same numbers.

Chai presented another fraction, $\frac{17}{8}$ where the numerator was greater than the denominator. He had the students identify the dividend and the divisor. At this time, several students gave correct responses. Chai and the students performed the long division algorithm suggested by his mentor, getting an answer of 2.125. Chai added that what happens if the quotient has a remainder, the students had to add a zero after the remainder and divide until the quotient had no remainder. A boy asked what happens if the quotient has a remainder. Chai's mentor responded that the problem would identify the number of decimal places needed. Chai continued by referring to one division problem from the homework, asking the students to find the quotient to the nearest hundredths. Chai asked a boy whether he could do this problem or not. This boy did not respond to his question. Chai explained that the students had to carry the division to thousandths and determine whether the digit was greater than or equal to five. If it was greater than or equal to five, they needed to round up. If it was less than five, they needed to round down.

Chai pointed to $\frac{17}{8}$ on the chalkboard and said, "Look at this problem. We need the quotient to the nearest hundredths. Now we have the digit at the thousandths place, right?" A boy replied, "Yes." Chai asked, "What will the answer be?" A boy responded, "2.13." Chai pointed to the digit 5 of 2.125 on the chalkboard and asked, “Because this digit is 5, how do we round it?” A few students answered, “Up.” Chai asked, “Round it up from 2 to what number? A few students replied, “Three.” Chai told the students that estimation and rounding topics were in the next chapter.

Chai had the students take notes. While they were taking notes, Chai wrote three fractions, $\frac{1}{9}, \frac{5}{11}, \frac{1}{6}$ on the chalkboard. These fractions could be represented as repeating decimals. Chai asked the students to identify the dividend and the divisor of $\frac{1}{9}$ before they performed the division. A few students gave a correct answer. While Chai wrote “$9\overline{)1}$” on the chalkboard, he reminded the students that the dividend was not necessarily greater than the divisor. The top number was the dividend and the bottom number was the divisor.

With the help of the students, Chai used the long division algorithm to express this fraction as a decimal. He put all the work on the chalkboard. When they proceeded to divide 1 by 9, getting a remainder of 1’s, Chai asked the students whether there were any trends that the quotient had no remainder. “No” was the response given by a few students. Chai explained that the division would never end and the quotient would always be 1’s. The decimal continued without end. The quotient for $\frac{1}{9}$ was 0.111.... Chai explained to the students how to read and write the repeating decimal in a short form, getting the result of $\frac{1}{9} = 0.111... = 0.\dot{1}$.

Chai and the students then proceeded to change other fractions to decimals in the same manner as they did with the first fraction. The quotient for $\frac{5}{11}$ was $0.4545... = 0.\dot{4}\dot{5}$ and the quotient for $\frac{1}{6}$ was $0.166... = 0.1\dot{6}$. While Chai and the students were performing the division in each problem, Chai asked the students to look for a pattern. He reminded the students to divide to get enough digits to be quite sure that the pattern would be clear. Chai explained that these decimals were called repeating decimals and wrote each repeating decimal in two ways, writing three dots after the decimal and writing a dot over the repeating digit or writing two dots, one over the first digit in the repeated sequence and the last digit.

To check students' understanding, Chai wrote four fractions, $\frac{2}{9}$, $\frac{6}{11}$, $\frac{52}{11}$, $\frac{8}{30}$, on the chalkboard and had the students convert them to repeating decimals. The students worked quietly. While the students worked, Chai wrote on the chalkboard the assignment of 16 problems selected from the textbook. The problems were similar to those worked in class. As the students worked on the problems, Chai moved around the room answering their questions. He reminded them to follow the examples. Five minutes later, a boy handed in the work at Chai's desk. Four girls handed in their work. Chai and these girls discussed the dividend and the divisor of the problem. Then several students handed in their work. Chai corrected their work at his desk. Some students did the work incorrectly. Chai allowed the remaining students to complete their work until the end of the class period. At the end of the class, Chai reminded the students about the homework assignment and allowed the students to leave the classroom. A girl led the class bow to the teacher.

After the class, Chai told the researcher that he gave the students a few examples. He did not present an example of a repeating decimal with three digits repeating. Thus, for the next section, he would give the students a variety of examples.

On the next day, the researcher observed Chai's lesson on representing fractions as decimals again but a different class. At this time, he began the lesson with converting decimals to fractions. Chai encouraged the students to see the relationship between the number of the decimal place and the divisor.

Chai: One-place decimal is the result of dividing by what number?
Several students: 10.
Chai: Divide by 10. For example, 0.2 has one decimal place, it can be written as?
Several students: 2 over 10.
Chai: Two-place decimal is the result of dividing by what number?
Several students: 100.
Chai: 100 or 10^2. For example, 0.15 can be written as?
Several students: 15 over 100.
Chai: 15 over 100. Three-place decimal is the result of dividing by what number?
Several students: 1,000.
Chai: 1,000. For example, 0.123 can be written as?
Several students: 123 over 1000.

While Chai was explaining, he also wrote on the chalkboard. The students took notes. Chai then had the students convert three decimals to fractions. For converting fractions to decimals, Chai gave the students the same examples as the previous class section. The researcher noticed that the students in this section could see the relationship between the terminating decimals and fractions. After Chai had taught all classes on the unit on representing fractions by decimals, he told the researcher that the students had not previously studied repeating decimals. They enjoyed learning this unit.

Day 4: Division of Fractions

Before teaching this unit, Chai told the researcher that the mistakes the students would make when dividing fractions were algorithmically-based mistakes. He indicated that the common errors included inverting the dividend instead of the divisor or inverting both the dividend and the divisor before multiplying numerators and denominators. Chai said that he would emphasize the rules for division of fractions and ask the students to take notes. He thought that this would reduce errors.

Chai came to the classroom early and wrote the rules for dividing fractions on the chalkboard.

<u>The rules for dividing fractions</u>

1. Convert mixed numbers, if any, to improper fractions.
2. Change the division sign into a multiplication sign and then switch the numerator and the denominator of the divisor. This can be represented by

   $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ where a, b, c, and d are any counting numbers.

He wrote three problems, $3\frac{5}{24} \div 1\frac{1}{6}$, $\frac{5}{14} \div \left(\frac{15}{8} \div \frac{12}{7}\right)$, $7\frac{1}{2} \times \left(\frac{24}{5} \div 3\right)$, on the chalkboard. The students wrote the rules in their notebooks, while waiting for class to begin.

Chai's mentor observed his class. She sat at the back of the room. A girl told Chai that she could not do the homework. Chai responded to the girl that he would explain it to her after class. A boy told Chai that he did not understand the multiplication of fractions. This boy did not attend in the class when Chai presented the lesson on multiplication of fractions. Chai responded that he planned to review that lesson.

Chai began the lesson by writing a multiplication problem $\frac{11}{25} \times \frac{3}{4}$ on the chalkboard. A boy said, "Cross multiply." Chai quickly corrected the student, saying that it was not a cross-multiplication problem. The students had to multiply the numerators together and the denominator together. Thus the product was $\frac{33}{100}$. While he was explaining, Chai wrote all the work on the chalkboard.

Chai then asked the students, "If they are mixed numbers, what should we do? Several students replied, "Change to improper fractions." Chai then had the students change $2\frac{1}{2}$ and $3\frac{3}{5}$ to improper fractions. The students correctly provided the improper fractions. Chai reminded the students to multiply the numerators together and the denominators together. He added, “If a fraction can be reduced to its lowest terms, reduce it.” He reminded the students that a fraction could be reduced by dividing both the numerator and the denominator by a common factor. Chai and the students proceeded to cancel the common factors, 5 and 2, getting the result of 9. While Chai was explaining, he wrote the procedures on the chalkboard.

After reviewing multiplication of fractions, Chai presented division of fractions to the students by having the students read the first rule on the chalkboard chorally. Chai asked the students whether it was the same as the rule for multiplication. Students answered, as a group, “Yes, it is the same.” Chai emphasized that if the numbers were mixed numbers, they needed to change the numbers to improper fractions first. This rule could be applied to addition, subtraction, multiplication, and division.

Chai then had the students read the second rule chorally. Chai emphasized that after changing mixed numbers to improper fractions, the students had to change the division sign into a multiplication sign and then switch the numerator and the denominator of the divisor. Chai also reviewed the position of the dividend, divisor, and quotient in a division problem written in horizontal form, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. He explained,

> If you want to divide $\frac{a}{b}$ by $\frac{c}{d}$, you have to keep the dividend, $\frac{a}{b}$, the same and change the division sign into multiplication sign. Then switch the divisor $\frac{c}{d}$ to $\frac{d}{c}$. Next, you multiply a by d and multiply b by c. If you can reduce it, reduce it.

Chai had the students read the problem, $3\frac{5}{24} \div 1\frac{1}{6}$, chorally. He asked them what they would do if the problem had mixed numbers. A few students answered, "Change them to improper fractions." Chai and the students changed the mixed numbers to improper fractions together. They then changed the division sign to the multiplication sign, switched the divisor, and multiplied the two fractions altogether, getting the result of $\frac{11}{4}$. The class then changed this fraction to a mixed number. Chai put all the work on the chalkboard. He reminded the students how to change an improper fraction to a mixed number and reminded the students that the denominator was 4.

The next example was a division problem with parentheses. Chai had the students read the problem as a group. Chai asked the students what they would do first. A girl replied, "In the parentheses." Chai reminded the students that if parentheses were used, they needed to simplify within the parentheses first. Chai and the students then changed the division problem $\frac{15}{8} \div \frac{12}{7}$ to the multiplication problem $\frac{15}{8} \times \frac{7}{12}$. Chai wrote on the chalkboard as follows:

$$\frac{5}{14} \div (\frac{15}{8} \div \frac{12}{7}) = \frac{5}{14} \div (\frac{15}{8} \times \frac{7}{12})$$

Chai asked the students whether the fractions in the parentheses could be reduced. A student replied, "Yes." Chai then asked the students to identify the common factor. A girl replied, "3." Chai and the students proceeded to cancel the common factor of 15 and 12, getting the result of $\frac{5}{8} \times \frac{7}{4}$. A girl said that she would cancel the common factor of 8 and 4. Chai quickly reminded the students that they could cancel when they multiplied fractions or when they had changed a division problem into a multiplication problem. They had to reduce any numerator with any denominator. The class then proceeded to multiply $\frac{5}{8} \times \frac{7}{4}$, getting the result of $\frac{35}{32}$.

The class proceeded to compute the division problem $\frac{5}{14} \div \frac{35}{32}$ as a multiplication problem, $\frac{5}{14} \times \frac{32}{35}$. Chai and the students cancelled the common factors of 5 and 35 and of 14 and 32, getting the result of $\frac{1}{7} \times \frac{16}{7}$. Chai and the students then carried out the multiplication, getting the answer of $\frac{16}{49}$.

The last problem was a multiplication-division problem with a parentheses and a divisor was a natural number, $7\frac{1}{2} \times \left(\frac{24}{5} \div 3\right)$. Chai had the students do the problem by themselves. While the students worked, Chai circulated to see how they were doing and answered their questions. He asked a few students whether they understood or not.

A boy asked Chai whether the fraction form of 3 had a denominator of 1. Chai then reminded the class that 3 was a natural number. Every natural number had the denominator of 1. While Chai was moving around the room, he found that a few students had difficulty in writing 3 as $\frac{3}{1}$. He also found that some students simply multiplied $\frac{24}{5}$ by 3. To remediate this error, Chai reminded the students to write a whole number as a fraction with denominator of 1 before switching the divisor. A student volunteered to put the correct work on the chalkboard. The class agreed that his answer was right. They applauded. Chai asked the students, "Who got the answer of 12. Several students raised their hands showing their agreement. Chai then reviewed this problem again. Chai gave students some time to take notes. While the students were taking notes, Chai moved around the room answering students' questions. He found that some students still had difficulty in writing 3 as a fraction with a denominator of 1. He reminded the students that the denominator of 3 was 1. They had to multiply $\frac{24}{5}$ by $\frac{1}{3}$.

While the students were taking notes, Chai wrote on the chalkboard a division word problem, "A train traveled $281\frac{1}{4}$ kilometers in $3\frac{1}{8}$ hours. What was the average speed of the train?" Chai had the students read the problem chorally. Chai then explained the information in the problem one more time and asked the students what operation was required to answer the question. A few students replied that division was needed. Chai then asked the students to change mixed numbers to improper fractions. Chai explained that a train traveled $\frac{1125}{4}$ kilometers in $\frac{25}{8}$ hours. The question was how many kilometers a train can travel in an hour. Chai then asked the students to identify the dividend and the divisor. A student identified $\frac{25}{8}$ as the divisor. Chai and the students changed the division problem to the multiplication problem, reduced fractions, and carried out the multiplication. Chai reminded the students how to reduce fractions. While Chai was explaining, he wrote on the chalkboard as follows:

In $\frac{25}{8}$ hours, the train traveled $\frac{1125}{4}$ kilometers

In 1 hour, the train traveled $= \frac{1125}{4} \div \frac{25}{8}$

$= \frac{1125}{4} \times \frac{8}{25}$

= 90 kilometers

Thus, the average speed of the train was 90 kilometers per hour.

Chai did not draw any diagrams while they were solving this problem. Five minutes before the class ended, Chai wrote on the chalkboard another word problem,

> Aom weighs 60 kilograms and Dum weighs 90 kilograms. How many times does Aom weigh more than Dum? How many times does Dum weigh more than Aom?"

Chai had the students read the problem chorally. The following illustrates the conversations that took place between Chai and the students.

> Chai: How many kilograms does Aom weigh?
> A few students: 60.
> Chai: 60. How many kilograms does Dum weigh?
> Several students: 90.
> Chai: How many questions do we have to answer?
> A few students: Two questions.
> Chai: What is the first question?
> A few students: How many times does Aom weigh more than Dum?
> Chai: To answer the first question, we will divide 60 by Dum's weight.

Chai wrote $\frac{60}{90}$ on the chalkboard. Chai and the students reduced it to the lowest terms, getting the result of $\frac{2}{3}$. They concluded that Aom weighed $\frac{2}{3}$ of Dum's weight. The conversation continued as Chai mentioned that the second question was how many times Aom weighed more than Dum. Chai then asked, "How many kilograms does Dum weigh?" A student replied, "90." Chai then asked, "How many kilograms does Aom weigh?" A student responded, "60." Chai wrote $\frac{90}{60}$ on the chalkboard. Chai then asked, "How many times?" A student replied, "$\frac{3}{2}$." Chai asked further, "$\frac{3}{2}$ of Aom's weight. Or?" A student replied, "$1\frac{1}{2}$."

Chai summarized for them, saying that if the question asked how many times Aom weighed more than Dum, where Aom was mentioned first, then Aom's weight is the numerator and Dum's weight is the denominator. If the question asked how many times Dum weighed more than Aom, then Dum's weight was the numerator and Aom's weight was the denominator. At the end of the class, Chai passed out an extra assignment for Chapter 6. He reminded the students about the homework assignment. He then allowed the students to leave the class.

After class, Chai told the researcher that some students forgot to write whole numbers with a denominator of 1. Some students had computation errors because they did not memorize multiplication and division facts. He said that he did not present the students with problems written in complex fraction forms. For the other sections, he planned to present the students with this type of problem. The researcher observed the other two sections. Chai presented the students with problems written in complex fraction forms,

$$\frac{\frac{2}{5}\times\frac{15}{12}}{\frac{1}{2}}, \quad \frac{6\frac{1}{2}\times 2\frac{3}{4}}{2\frac{1}{6}\times 2\frac{1}{5}}$$

He also reminded the students to write whole numbers with a denominator of 1.

After Chai had taught all sections this unit, Chai told the researcher that a few students still had errors in dividing fractions. They inverted the dividend instead of the divisor or inverted both the dividend and the divisor before multiplying numerators and denominators.

Impact of Chai's Knowledge of Subject Matter and of Students' Conceptions on His Instruction

Chai appeared to have the necessary content knowledge to support the instruction of a topic based on the division of rational numbers. He made few content errors in teaching these lessons. Prior to teaching he was not able to identify a generalized statement such as $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ where a, b, c, and d are integers and b, c, and d $\neq 0$. During his lesson, he did not show the students where the "invert the divisor and multiply" algorithm originated. Chai introduced division of fractions by writing the rules of division of fractions on the chalkboard: to divide one fraction by another, invert the divisor (the second fraction if written with division sign) and proceeded as in multiplication. He introduced the students to the equation, "$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, but he did not show or give any reason why the students had to invert the divisor and then multiply. Thus, Chai's method for using the algorithm was reflected in his lessons.

During his lesson on division of word problems, Chai encouraged the students to solve the problems using the rule of three. This was the same approach he had used in the questionnaire and interview items. The researcher concluded that Chai's procedure to divide fractions was reflected in his lessons.

Although Chai was able to identify the relationship between multiplication and division, he did not apply this knowledge in his lesson. He did not encourage his students to check the accuracy of the quotient using the relationship between these two operations.

Prior to teaching, Chai was able to identify the important terminologies of division of rational numbers such as dividend, divisor, and quotient. He knew that students were often confused between the dividend and the divisor. They mixed up the order of the divisor and the dividend in the $a \div b$ form and the $b\overline{)a}$. Thus, Chai often asked the students to identify the dividend and the divisor of the problem during the lessons. He also wrote the division problem on the chalkboard.

Chai recognized potential students' errors in the long division algorithm acknowledging that division of decimals tended to be difficult for students, especially when the division problem required them to add zeros as place holders in either the dividend or the quotient. During the lesson, Chai presented division problems that violated these errors. He often reminded the students to add zeros as place holders in the divided or the quotient. Moreover, if the students revealed these errors in the class, he corrected them right away. Chai also recognized that if the division did not come out even when taken as far as the digits given in the dividend, some students wrote the remainder as an extension of the quotient. During the lesson, Chai reminded the students that when the division did not result in a remainder of zero, they needed to divide to one extra place value position in order to know how to round the quotient.

Chai knew that the students often had algorithmically-based mistakes in dividing fractions. They inverted the dividend instead of the divisor or inverted both the dividend and the divisor. Thus, in his lessons, Chai emphasized this idea by writing on the chalkboard, "Change a division sign to a multiplication sign and then switch the numerator and the denominator of the divisor." When the class performed the division of fractions, Chai often reminded the students to invert the divisor before multiplying.

Finally, Chai indicated that students might make the errors when dividing a fraction by a whole number, failing to write a whole number with a denominator of 1. They simply multiplied the fraction by that whole number. To avoid this error, during the lesson, Chai presented the students with a division problem with the divisor as a whole number. He reminded the class to write a whole number as a fraction with the denominator of 1 before inverting the divisor. Moreover, when a fraction with the denominator of 1 appeared in any lesson, Chai often reminded the students that it was a whole number and they were able to write it as a fraction with a denominator of 1.

In the interview before class, Chai indicated that some students had trouble in dividing a decimal by a decimal. Thus, during the lesson, Chai reminded students to make the divisor a natural number by multiplying it by the power of 10. He also reminded them to multiply both the divisor and the dividend by the same number. Chai not only told the students, but also proceeded to multiply the divisor and the dividend by the same number with the whole class.

In the interview before the class on representing fractions by decimals, Chai indicated that students might be confused when the dividend was less than the divisor and might think that the division could not be performed. During the lesson, Chai pointed out that a fraction could be expressed in a decimal form by dividing the numerator by the denominator. The dividend was not necessarily greater than the divisor. The top number was the dividend and the bottom number was the divisor. Chai had used the knowledge about students' conceptions throughout the lessons.

In planning for his lessons, Chai focused on an algorithmic presentation following an algorithmically-based curriculum developed by the IPST. Although he had a strong conceptual understanding, Chai followed the curriculum that did not emphasize conceptual development, and he did not deviate from the curriculum.

During the lessons, Chai taught division of rational numbers procedurally. He thought of examples that guided the students towards understanding of algorithm that he was teaching. He exhibited incomplete knowledge only when working with the representation of fractions by decimals. He often prevented students' errors and corrected their misconceptions, reflecting his high knowledge of students' conceptions.

Chai's mentor directly impacted his teaching. Before teaching each lesson, Chai and his mentor discussed the lesson and how to teach it. She observed every lesson Chai taught. Whenever Chai had finished instruction and she felt that the students were confused, she often summarized the topics. After the lesson, Chai and his mentor discussed the lesson he taught, the flaws of his teaching and how to adjust his teaching. Chai followed his mentor's suggestions.

Lada

Lada was categorized as having high knowledge of subject matter and low knowledge of students' conceptions (H/L). She majored in mathematics education and minored in English. She had taken 42 semester hours of mathematics. Her mathematics courses included principles of mathematics, calculus and analytic geometry, differential equations, number systems, foundations of geometry, linear algebra, abstract algebra, set theory, statistics and probability and graph theory. She had taken 10 semester hours of practicum and field experience in teaching mathematics, including three semester hours of a mathematics methods class, one semester hour of participation and observational study, one semester hour of practicum, and five semester hours of student teaching. She had taken three semester hours of English methods class.

Lada explained that mathematics was the process of finding answers using principles and theories. Mathematics was used in several disciplines, including, physics, chemistry, and biology. When asked to define what a rational number was, Lada was unable to provide a definition. She gave examples of rational numbers, including, fractions and decimals. Lada incorrectly identified that integers and repeating decimals were not rational numbers.

Lada's knowledge structure of division had two meanings, partition and measurement divisions. She gave a partition problem as, "She has six oranges. She wants to share them among her three sisters equally, how many oranges will each sister receive?" She also gave a measurement problem as, "She has six oranges. She wants to group them, three for each group. How many groups can she make?"

Subject Matter Knowledge of Division of Rational Numbers

Lada understood division of rational numbers well. When asked to express the phrase "a divided by b" in notational forms, Lada was able to express the given phrase in four forms, "$a \div b$," the fraction forms "$\frac{a}{b}$," "a/b," and the algorithmic form used in a long division "$b\overline{)a}$." She was able to identify the terms of division such as dividend, divisor, and quotient from the given statement, $1\frac{3}{4} \div (-\frac{1}{4}) = -7$. When asked about restrictions on the divisor, the dividend, or the quotient, Lada had no trouble. She indicated that the statement, "In a division problem, the dividend must be greater than the quotient" was not true, as $\frac{3}{5} \div \frac{1}{2} = \frac{6}{5}$. She disagreed with the statements, "In a division problem, the dividend must be greater than the divisor," and "In a division problem, the divisor must be the integer." She correctly identified that the statement, "If both p and q are negative, then $p \div q$ is negative" as untrue.

On the questionnaire, Lada correctly carried out the operation of $0 \div (-2497) = 0$. She was able to identify that $\frac{3}{0}$ as undefined. When asked to explain her response in the interview, she provided a rule-based argument. Lada argued that, "Well, I can only remember this rule." Lada's knowledge of the division of rational numbers when the dividend or the divisor was 0 was unclear. When asked to give the explanation of the quotient $\frac{2}{3} \div 0$, she explained that $\frac{2}{3} \div 0 = 0$ because all division with 0 results in 0. Moreover, she incorrectly indicated that the statement, "Any rational number divided by itself is 1" was true. She did not consider the case that 0 divided by 0 was undefined.

With respect to division of fractions, Lada was able to identify the multiplication sentence related to a given division sentence. Based on this knowledge, she correctly responded to the items involving computation. When asked to explain her work during the interview, her explanations were not simply invert the divisor and then multiply. The example below illustrates how Lada solved one division problem.

$$(-\frac{1}{4}) \div (-\frac{3}{5}) = \frac{(-\frac{1}{4}) \times (-\frac{5}{3})}{(-\frac{3}{5}) \times (-\frac{5}{3})}$$

$$= (-\frac{1}{4}) \times (-\frac{5}{3})$$

$$= \frac{5}{12}$$

Additionally, she was able to show that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ where a, b, c, and d are integers and b, c, and d $\neq 0$. Her work is shown as follows:

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}}$$

$$= \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}}$$

Multiply both the numerator and the denominator by $\frac{d}{c}$ to make the divisor equal 1.

$$= \frac{a}{b} \times \frac{d}{c}$$

$$\therefore \quad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

In this proof, she used a natural consequence of various notations of fractions, the definition of multiplication of fractions, and several mathematical principles (e.g., $a \times \frac{1}{a} = 1, a \neq 0$; $\frac{a}{1} = a$).

When given a division of decimals problem where the divisor was a decimal and the quotient was a natural number. Lada correctly identified 5 as the answer for $0.75\overline{)3.75}$. She replaced the original problem with an equivalent one having a natural number divisor and performed the long division algorithm. When asked to explain her method, Lada mentioned that she moved the decimal points until both the numerator and the denominator were integers. She added that this rule came from multiplying both numbers by 1000.

When Lada was confronted with the division problem, 1.33 divided by -2.1 (which required her to add zeros as place holders in either the dividend or the quotient), she incorrectly carried out the division.

$$
\begin{array}{r l}
 & \underline{\quad -0.60303} \\
-210 \,\big) & 1330 \\
 & \underline{1260} \\
 & \quad 700\ \_ \\
 & \quad \underline{630} \\
 & \qquad 700\ \_ \\
 & \qquad \underline{630} \\
 & \qquad\quad 7
\end{array}
$$

During the interview, Lada, however, was able to correctly complete the long division algorithm.

Lada seemed to have the ability to transform problem elements from one mode to another. She correctly wrote the expressions represented by two story problems. Lada had an understanding of the connection between the real-world context and the interpretation of the remainder or fractional part of the quotient. She wrote the following quotative division story problem, “Fifty-one liters of syrup is put into bottles holding 4 liters each. How many bottles are filled?” Lada claimed that this story problem would be solved by dividing 51 by 4, for which the answer would be 13. She correctly solved another word problem.

Lada had a broad enough concept of division to make sense of division of fractions less than one. She constructed the quotative division problem, "She has $\frac{3}{4}$ of dessert. She needs to put it on trays with $\frac{1}{4}$ each. How many trays contain this amount of dessert?" and claimed that this story problem could be solved by using $\frac{3}{4}$ divided by $\frac{1}{4}$. However, Lada had a problem in solving this word problem about division, "Somchai drove 665 kilometers and used gas $\frac{3}{4}$ of tank. If he has a full tank of gas, how far he can drive?" She was not able to solve this problem, but during the interview, she correctly solved the problem by setting up the proportion and then cross-multiplying.

On the questionnaire, Lada incorrectly identified that the set $\{1, 2, 4, 8\}$ was closed with respect to the operation of division. When asked to define the Closure Law in the interview, Lada was not able to state what the closure law meant. Lada, however, was placed in a high knowledge of subject matter of division of rational numbers category on the basis of her questionnaire and interview results.

Knowledge of Students' Conceptions of Division of Rational Numbers

Lada's knowledge of students' conceptions of division of rational number was classified as low level. Lada correctly identified that $0 \div 7$ was equal to 0 but when asked to identify two possible errors students might make in evaluating it, she was only able to identify one error. She wrote that 7 was the possible error students might make. An explanation of the source of this response was, "Several students think that any number divided by zero is that number." Although Lada could give the common mistake, her reason about the possible source was wrong. She confused the meaning of these two phrases "zero divided by any number," and "zero divided into any number."

An inconsistency became apparent. Lada herself still had a misconception about dividing by zero. On the questionnaire, Lada indicated that a quotient of 1 was the common error students might make in the computing of $0 \div 0$. She claimed that the students might think that any number divided by itself was 1. She was not able to identify other possible errors. During the interview, Lada indicated that $0 \div 0$ was equal to 0. She confirmed that there was only one possible error students might make. The error was $0 \div 0 = 1$.

When asked to determine the common error students might make about division of fractions problem, $\frac{1}{4} \div 4$. Lada indicated that 4 was the possible incorrect answer. She wrote that students divided both numerator and denominator by 4. However, using her logic, the answer was $\frac{1}{4}$. In addition, she made an error herself. Lada incorrectly gave a quotient of 1 instead of $\frac{1}{16}$ in computing $\frac{1}{4} \div 4$. When asked to determine the common error students might make about division of fractions, $\frac{1}{4} \div \frac{3}{7}$, Lada gave $\frac{1 \div 3}{4 \div 7}$ as the error students might make. The source of this error was that the students divided fractions in a way similar to multiplication.

On the questionnaire, Lada identified that $5 \times \frac{1}{5}$ and $5 - \frac{1}{5}$ were incorrect expressions students might make when they solved the word problem "Five friends bought $\frac{1}{5}$ kilogram of chocolate and shared it equally. How much chocolate did each person get?" Lada, however, was not able to provide possible sources of those errors. She simply wrote that the students did not understand the questions. During the interview, she mentioned that the students thought that the word problem was about sharing. Thus, they calculated $5 - \frac{1}{5}$ and the result would be the answer. When asked to determine the student's error patterns on three long division problems, Lada was able to identify two error patterns.

The following conversation between the researcher (R) and Lada illustrates Lada's lack of knowledge of students' misconception.

R: A student solves this problem, "If cheese is 75.60 baht per pound, how much is 0.78 pounds? By 75.60 ÷ 0.78. Is it correct?

Lada: No it is not correct. It has to be 0.78 divided by 75.60. The question asks us to find the price of 0.78 pounds butter which is less than a pound. Thus 0.78 is the dividend and 75.60 is the divisor. This student may think that the larger number is the dividend and the smaller number is the divisor.

Based on Lada's responses to the questions, she was placed in the low knowledge category for knowledge of students' conceptions of division of rational numbers.

Classroom Observations of Lessons on Division of Rational Numbers

Lada did her student teaching at the medium-size high school in Bangkok. The school had approximately 900 students in the seventh through twelfth grades. Lada taught three seventh grade mathematics classes. The class sizes ranged from 34 to 38 students, all boys. The class periods lasted approximately 50 minutes. Lada taught from Monday through Friday teaching, at most, two periods a day and nine periods a week. She taught at varying times of the day, sometimes with extended breaks, sometimes without. For instance on Wednesday she taught a class the second period of the day and had two hours before she had to teach the next period whereas on Thursday she taught two classes the second and the third periods of the day and then she had the rest of the day to grade the homework and prepare for another class. Her class weekly schedule is presented in Appendix O.

Lada's mathematics classroom was on the third floor of the building. Two portraits of the King and the Queen of Thailand were hung on the front wall of the room over the chalkboard. Two bulletin boards in front of the room were decorated with seasonal items. There were two doorways and several windows in the classroom. The teacher's desk was in the right front corner of the classroom which was next to the first door. The students' desks were arranged in pairs facing the chalkboard for the entire school day. There was enough space for the teacher to move around the room.

One of Lada's classes was observed by the researcher for many lessons on division of rational numbers. This class was taught three days a week, Monday afternoon, Thursday morning, and Friday afternoon. The class included 34 students. On average, the class lasted about 40 minutes with five minutes to put away materials. At the beginning of the class, the students used the first few minutes to enter the room, take their seats, and organize their materials. At the end of the class, they used a few minutes to put away their materials before leaving.

Lada usually went to class on time and stood outside the room. When almost all of the students were in class, she walked into the room. A student, the head of the class, led the students to salute the teacher. All the students stood up. The teacher then allowed them to sit down. When the teacher was in class, the students knew that they had to be ready to learn.

After Lada had the students' attention, she told the students what they were going to study in each lesson. She often reviewed previously covered topics, addressed prior knowledge relevant to the day's lesson, or asked about the homework. A typical approach Lada used in reviewing was to write several problems on the chalkboard, one at a time and then call on a student to put the work on the chalkboard. She sometimes called on a student to answer the question verbally. The students always stood up to answer the questions.

After the review, Lada presented a new topic to the students by using a question-answer technique to go over the examples. Most of the questions were facts. Lada always asked the students to guide the work. She stood in front of the room putting all work on the chalkboard. Lada sometimes drew diagrams when she presented word problems. Lada often called on the students by their identification numbers or their names to answer the questions or put up the work on the chalkboard. The students always stood up to answer the questions. When Lada asked the questions, the students sometimes raised their hands for volunteering and wait to be called upon. Sometimes some students called out.

Lada's lessons were teacher-centered. After presenting the students with some examples, Lada sometimes assigned work to be completed in class. While the students worked, she often circulated around the room to monitor and assist students. Sometimes, the students were very noisy and talked to each other. Lada often walked towards those students to see what they were doing. Lada presented the information in her lessons not only verbally and in writing but also asked the students to take notes.

Near the end of the class, Lada sometimes wrote the homework assignment on the chalkboard. All of the problems were selected from the textbook. At the end of the class, when Lada allowed students to leave the class, the head of the class led the students to salute the teacher. All the students stood up and said chorally, "Thank you."

Lada rarely discussed lessons with her mentor. Lada's mentor rarely observed the lessons she taught. When he observed her class, he sat outside the classroom and monitored through the door. When the students saw him, they were very quiet.

Lada's mathematics supervisor observed her lessons twice. Although the observations were not on division of rational numbers, Lada and her mathematics supervisor discussed her written lesson plans on division of decimals and division of fractions.

Lada created four lesson plans on the division of rational numbers, one lesson plan per day. Examples of Lada's lesson plans are presented in Appendix P. The sequencing and topics of division of rational numbers included in Lada's lesson plans and her classrooms are presented in Figure 8.

<table>
<tr><th></th><th>Lesson Plan</th><th></th><th>Classroom</th></tr>
<tr><th>No.</th><th>Topic</th><th>Day</th><th>Topic</th></tr>
<tr><td>1</td><td>Division of a Decimal by a natural number Algorithm and Word Problems</td><td rowspan="2">1
(Monday)</td><td rowspan="2">Division Word Problems For Decimals</td></tr>
<tr><td>2</td><td>Division of a Decimal by a decimal and Word Problems</td></tr>
<tr><td>3</td><td>Representing Fractions as Decimals</td><td>2
(Wednesday)</td><td>Representing Fractions as Decimals</td></tr>
<tr><td rowspan="2">4</td><td rowspan="2">Division of Fractions Algorithm and Division Word Problems for Fractions</td><td>3
(Thursday)</td><td>Division of Fractions Algorithm</td></tr>
<tr><td>4
(Friday)</td><td>Division Word Problems for Fractions</td></tr>
</table>

Figure 8. Lada's instructional emphasis and sequencing of topics in the division of rational numbers units.

In the lesson plan on representing fractions as decimals, Lada added one topic that was not presented in the textbook, converting repeating decimals to fractions. This topic was included in the mathematics textbooks for Grades 8 and 9. Because Lada was sick for a week, her mentor taught some lessons on division of decimals. He covered the topics on division of a decimal by a natural number algorithm and division of a decimal by a decimal algorithm on the same day. Thus, Lada's first lesson observed by the researcher was on division word problem for decimals.

The examples presented in the observed lessons were not the same as those in the lesson plans, but they were similar. The homework problems were selected from the textbook. During the classroom observations, Lada did not exactly follow her lesson plans. She did not use all her examples either. She used some division problems that she created before the class. Sometimes she created some problems as she was teaching.

Lada did not work closely with her mentor who taught other sections of seventh grade mathematics. The mentor allowed Lada to design her lessons and activities by herself. Lada sometimes handed in her mentor lesson plans ahead of time. However, her mentor did not write any comments in the lesson plans. Thus, there was no evidence of any of the mentor's ideas on her lesson plans.

Lada's mathematics supervisor observed her class twice but did not discuss the plans with her before the class. Thus, Lada's mathematics supervisor did not have any impact on the lesson observed. After the class, Lada and her mathematics supervisor discussed the correctness of the mathematics content and the activities she should use in teaching the lesson. Thus, Lada was able to use her mathematics supervisor's advice for improving her teaching in later sections.

Day 1: Division Word Problems for Decimals

Lada began the lesson with a review of division of decimals algorithm. She wrote a division problem, $125 \div 0.5$, on the chalkboard and called on a student to write a solution on the chalkboard. However, he was not able to perform the algorithm. A student called out, "The answer is 10." Lada asked, "10?." No one responded. Lada then asked another student to write a solution on the chalkboard. On the chalkboard, he multiplied both the dividend and the divisor by 10. However, he incorrectly computed a basic multiplication fact, $125 \times 10 = 125$. So when he carried out the division of $5\overline{)125}$, he got the wrong answer, 25, to the original problem.

Lada asked the whole class whether the answer was correct or not. A few students replied that it was correct. Some said, "No." Lada then guided the class see the error. She pointed to the work on the chalkboard and asked the class, "What is the product of 25×5?" A student answered, "125." Lada then asked, "What error did he make?" A student answered, "He forgot to add a zero when he multiplied 125 by 10." Lada commented that it was not correct because he had forgotten to add a zero at the dividend.

Lada then turned to the chalkboard and converted $125 \div 0.5$ to an equivalent problem in fraction notation, $\frac{125}{0.5}$ and asked the students whether these two forms were alike or not. Lada's language confused the students. Thus, some students thought they were alike, other did not. Lada told the students that they were alike. She then asked the class, "Do you know how to make the divisor a natural number? By what number must it be multiplied? A few students replied, "Multiply it by 10." Lada repeated the students' answer and asked, "Can we multiply only the divisor by 10?" No responses were given. Lada then asked the students what they would do next. A few students replied, "Multiply the top number by 10." Lada accepted their answer as correct and concluded, "Multiply both the numerator and the denominator." While she talked, Lada also put the work on the chalkboard. Lada then asked the students to identify the product of 125×10. A few students responded, "1250." Lada and the students then proceeded to divide 1250 by 5, getting the quotient of 250.

Lada called on a student to solve $14.56 \div 0.12$ on the chalkboard. Lada created this problem while she was teaching. It required the student to add zeros as place holders in the dividend and the quotient was a repeating decimal. When the student said that he could not solve it, Lada then called on another student to solve the problem on the chalkboard. He wrote a division problem in a fraction form, $\frac{14.56}{0.12}$. He multiplied both the divisor and the dividend by 100, getting the result of $\frac{1456}{12}$. While he was carrying out the division of $12\overline{)1456}$, he observed that the remainder 4 kept repeating.

> A student: The decimal in the quotient will be 3 infinitely. Can I round the quotient to 121.34?"
> Lada: Are you sure that the next digit would be greater than 5?"
> A student: No, it is not.

Lada then said to the class, "The quotient is 121.333... but the digit 3 keeps repeating. What is the meaning of repeating?" A few students responded, "Put the dot above." Lada accepted their answer and said, "Some students have learned about it before this class. Thus, the answer is $121.\dot{3}$." While she talked, Lada wrote "$121.\dot{3}$" on the chalkboard. A student called out, "Round it up." Lada quickly responded, "We cannot round it up. We have to round it down." While Lada had pointed out the ways of writing a repeating decimal, writing dots over the repeating digits and rounding a decimal up or down to a given decimal places, she did not explain how to round a decimal. Lada might have thought that the students would learn about rounding in the next chapter. Lada then asked the class, "Do you understand how to divide decimals? Do you need another review?" A few students responded, "No. We understood." The class spent about nine minutes on this review. While the student put the work on the chalkboard, Lada walked towards two students and checked their homework.

Next, Lada wrote on the chalkboard a division word problem, "There is 10.2 liters of fish sauce. How many 0.85 liters bottle can be filled?" This problem was neither in the textbook nor in the lesson plan. It was a measurement division problem with a decimal divisor. Lada told the students to take notes. Lada then had the class read the problem chorally. Lada and the students discussed the main idea of the problem. She said, "What are you being asked to find?" Several answers were given including, "Bottle," "The number of bottles," and "Fish sauce." Lada asked, "How much fish sauce do we have?" A few students responded, "10.2 liters." Lada repeated their answer and asked, "We want to fill fish sauce into the bottle. Each bottle can be filled with 0.58 liters. What should we do?" A few students replied, "Do the division." A few students said, "Divide 10.2 by 0.85." Lada accepted their answer without asking them to explain why they would do the division operation.

Lada drew a picture of 0.85 liters bottle and asked the students to identify the capacity of a bottle. A few students responded, "0.85 liters." Lada then asked, "How many bottles can be filled with all fish sauce?" A few students said, "Divide." Lada then asked the students to estimate how many bottles they could draw. Several answers were given, including, "Five bottles," "One and a half bottles," "Two bottles," and "three bottles." Lada asked the students to identify the dividend and the divisor. The correct responses were given by a few students. Lada then wrote on the chalkboard, "There is 10.2 liters of fish sauce. We want to fill all fish sauce in 0.85 liter bottles." A few students called out, "12 bottles." Lada responded, "How many bottles?" A few students replied, "12 bottles." Lada then asked the students to identify the dividend. A few students answered, "10.2." Lada repeated students' answer and asked the class to identify the divisor. A few students replied, "0.85." Lada then asked, "What is the quotient? A few students responded, "12 bottles." While she talked, Lada wrote "10.2 ÷ 0.85 = 12 bottles" on the chalkboard. She asked, "Do you want to prove whether it is true or not. A few students said, "No." Thus, Lada accepted the answer as correct without verifying it.

Lada wrote another problem on the chalkboard, "A man runs 100 meters in 12.4 seconds. How many meters can he run in 7.5 seconds? (Round your answer to two decimal places)." This problem was neither selected from the textbook nor in the lesson plan. While Lada wrote on the chalkboard, the students took notes. Lada and the students discussed the problem. Lada asked, "What question is being asked? What information is given?" A student replied, "A man runs 100 meters in 12.4 seconds." Lada repeated the student' answer and asked, "How many meters can he run in 7.5 seconds?" No one responded. Lada asked, "What information is given?" No responses were given. Lada said, "This man can run 100 meter within? A few students answered, "12.4 seconds." Lada drew a line representing a distance of 100 meters and wrote "12.4 seconds" above it. She explained, "If this man uses 7.5 seconds, we do not know where the end point is from this diagram." Lada then wrote "7.5 seconds" below the line as follows:

← 12.4 seconds →

100 meters

← 7.5 seconds →

Lada asked, "Do you know the distance?" No responses were given. Lada then asked, "What should we do?" A student replied, "Divide 12.4 by 100." Lada quickly responded, "Divide 12.4 by 100?" No one responded. Lada asked, "Do you know what numbers multiplied by what number or what number divided by what number? A student replied, "No." Other students did not respond. Lada noticed that the students did not know how to solve this problem. She explained using a proportion approach. She said, "A man runs 100 meters in 12.4 seconds." She wrote the first ratio on the chalkboard as $\frac{100}{12.4}$. Lada further explained that they did not know how many meters this man could run. Thus, she indicated that they should let x represent the distance this man could run in 7.5 seconds. Lada wrote the second ratio on the chalkboard as $\frac{x}{7.5}$. She asked the students how they could find the value of x. No one responded. Lada explained that cross-multiplication could be used to find the value of x. She wrote on the chalkboard,

$$\frac{100}{12.4} \times \frac{x}{7.5}.$$

Lada then wrote on the chalkboard:

> In 12.4 seconds he can run 100 meters.
> In 7.5 seconds he can run $\frac{7.5 \times 100}{12.4}$.

She reminded the students to make the divisor a natural number by asking the class, "The divisor is a decimal. By what number must the divisor be multiplied to make it a natural number?" A few students replied, "10." Lada repeated their response and asked the class, "What is the result?" A few students answered, "124." Lada then asked, "What about the top number? A student replied, "75." Lada asked, "What is the product of 7.5 and 100?" A student responded, "750." Lada said, "Then multiply it by 10. Thus, the result is?" A few students said, "7500." Lada responded, "Thus, 7500 divided by 124." Lada gave the students some time to perform the division long hand. This problem required the students to add zeros as place holders in the dividend.

While the students worked, Lada stood in front of the class but looked around the room. Lada then asked the students to identify the answer. The students gave several answers. Lada and the students then proceeded to divide 7500 by 124 to verify the answer. She did all the work on the chalkboard. Lada incorrectly computed a basic multiplication fact, $124 \times 6 = 704$. Thus, when they carried out the division of $124\overline{)7500}$, they got the answer of 63.7. Lada did not notice where she made the error. A few students found the error. They told Lada that the product of 124 and 6 was 744. With the students' help, Lada performed the long division once on the chalkboard, getting an answer of 60.483. Lada did not explain why they could write 7,500 as 7,500.00. Moreover, when they had to add the decimal point in the quotient, Lada did not remind them that they had to place the decimal point in the quotient directly above the decimal point in the dividend.

When they had carried out the division to thousandths, Lada mentioned, "From the problem, how many decimal places must be in the answer?" A few students replied, "Two decimal places." Lada repeated the students' response and explained, "3 is less than 5, omit it. Thus, the answer is 60.48 meters." While she talked, Lada wrote "60.48 meters" on the chalkboard. However, she did not remind the students that to find a quotient to a given decimal place, they had to carry the division to one additional decimal place.

Ten minutes before the class ended, Lada assigned the students five word problems to work in class. These problems were selected from the textbook. While students worked, Lada circulated around the class to see how they were doing and answer their questions. One student handed in the work in class but the others did not finish their work. Lada's mentor observed her teaching in the middle of the class time and sat outside the room. The students knew that he observed the class. They were quiet.

After the class, Lada told the researcher that some students were not able to perform long division. She also said that the students did not understand word problems. From the researcher's perspective, Lada's observation was correct. The students did not understand.

Day 2: Representing Fractions as Decimals

Lada began the lesson by saying that today's lesson was on representing fractions as decimals. The students had to study a worksheet in pairs. This worksheet had eight pages where the students were required to read together and learn the material.

Lada created this worksheet herself, reflecting her own understanding. The instructional sheet included the following content, sequenced in this order: converting fractions to terminating decimals, converting fractions to repeating decimals and converting repeating decimals to fractions. Example of the worksheet is presented in Appendix P.

Lada arranged the students into pairs, and handed out the worksheet, one set for each pair. Lada explained the instructions for doing the worksheet to the class. While the students were studying the material, Lada walked around the room answering the students' questions.

After the class, Lada told the researcher that some students made errors. They wrote $2\overline{)3}$ for $2 \div 3$ and identified the quotient as 1. They did not consider a fractional part. Some students thought that they were not able to divide 2 by 3. Some students incorrectly indicated that "0.6060..." was the quotient of $2\overline{)3}$. Lada said that she did not know whether the student mixed up the order of the divisor and the dividend in the a ÷ b form. Thus, in other class sections she thought she should mention the direction of the "a ÷ b" symbolism. The researcher observed another section of Lada's class on representing fractions as decimals. Lada knew from the previous section that one error students made was the direction of the "a ÷ b" symbolism, which was confused with "$a\overline{)b}$." Thus, Lada reminded the whole class that $\frac{1}{2} = 1 \div 2$ and it meant $2\overline{)1}$. She wrote another example on the chalkboard, $\frac{5}{6} = 5 \div 6$, $6\overline{)5}$. She also reminded the students that $5\overline{)6}$ was not a correct form for $\frac{5}{6}$ or 5 ÷ 6.

After teaching all sections, Lada mentioned that the instructional sheet was not complete and it had typing errors. She also had to add the $a\overline{)b}$ form in the sheet. In addition, there were few examples. She needed to add more examples of repeating decimals with three or four repeating digits in the sheet. The researcher observed that the students did not understand the lesson. They made errors in writing a repeating decimal as a short form. They wrote an expression like, $0.\dot{2}3\dot{5}$ = 235235.... When asked after class, the students said that they did not understand. The researcher observed that Lada could not control the class. The students did not concentrate on the task. They talked to each other and the classroom was noisy.

Day 3: Division of Fractions Algorithm

Before teaching a division of fractions unit, the researcher asked Lada to identify the misconceptions that the students might make in dividing fractions. Lada, however, was unable to identify the common errors of inverting the dividend instead of the divisor or inverting both the dividend and the divisor before multiplying numerators and denominators

Lada began the third day lesson by asking the students whether they understood multiplication of fractions or not. Several students responded that they did not understand. Lada then discussed the rules of multiplication of fractions. She asked the students what the rules for multiplication of fractions were. A few students replied that they would multiply the numerators together and multiply the denominators together. Lada repeated their response and asked what to do first with a mixed number. A few students responded that they would change it to an improper fraction. Lada then summarized that to multiply fractions; one must convert mixed numbers, if any, to improper fractions and then multiply the numerators together and multiply the denominators together.

Next, Lada wrote a multiplication problem, $\frac{2}{3} \times \frac{5}{7}$, on the chalkboard and asked the students what they would do. This multiplication problem was similar to the example presented in her lesson plan. A few students replied, "Multiply the numerators and multiply the denominators." Lada accepted their response and asked the class what the product was. A few students answered, "10 over 21."

Lada then wrote another multiplication problem, $1\frac{1}{2} \times \frac{3}{4}$, on the chalkboard. This problem was not presented in the lesson plan. Lada asked the students what they would do. A few students replied that they would change the mixed number to an improper fraction. A student called out, "It is 3 over 2." A few students said, "3 over 2 multiplied by 3 over 4." Lada asked the students what the product was. A few students replied, "9 over 8." Lada asked them to change it to a mixed number. A few students quickly replied, "$1\frac{1}{8}$." Lada asked the class whether they understood or not. Several students answered chorally that they understood.

After the above review, Lada introduced the division of fractions by asking the students to express $6 \div 2$ in fraction form. A few students gave the correct answer, $\frac{6}{2}$. When Lada asked the students to change $11 \div \frac{1}{2}$ to a fraction form, they were not able to write it. A few students said, "11 over 1." Lada explained, "11 over $\frac{1}{2}$. The denominator is $\frac{1}{2}$ and 11 is the numerator." A student said, "I am confused." From the researcher's perspective, several students were confused as well.

Lada then wrote another problem, $\frac{3}{7} \div \frac{4}{9}$, on the chalkboard and asked the students to express it as a fraction form. A few students said, "$\frac{3}{7}$ over $\frac{4}{9}$" Lada wrote $\frac{\frac{3}{7}}{\frac{4}{9}}$ on the chalkboard and asked for agreement from the class. A few students replied, "Yes." A few students asked, "How can we divide them?" Lada responded that they will know it in a few minutes.

Lada then asked the students to create a fraction division problem. A few students said, "$\frac{3}{7} \div \frac{11}{3}$." Lada asked the class to express it in a fraction form. A few students replied, "$\frac{3}{7}$ over $\frac{11}{3}$." Lada wrote $\frac{\frac{3}{7}}{\frac{11}{3}}$ on the chalkboard and asked the students what they would do next. A few students called out, "Divide." Lada explained that to divide, the students had to make the denominator equal to 1. While she talked, Lada pointed to $\frac{11}{3}$ on the chalkboard and asked, "What will you do to make the divisor equal to 1? By what number must it be multiplied?" She gave the students some time to think. One minute later, Lada asked a student, "What number must $\frac{11}{3}$ be multiplied by to get the product of 1?" The student did not give any response. Lada then asked the class, "What number must $\frac{11}{3}$ be multiplied to get the product of 1?". A student said, "I think it must be the numbers from 1 through 10." Another student said, "I don't know." Another student said,

"$\frac{3}{11}$." Lada accepted the last student's response as a correct answer. To verify that the product of $\frac{11}{3}$ and $\frac{3}{11}$ equal 1 Lada said to the class, "$\frac{11}{3}$ multiplied by $\frac{3}{11}$. Can we cancel them?" Several students responded, "Yes." Lada explained, "Cancel 3 over 3 and 11 over 11. Thus, the result is?" Several students replied, "1." Lada explained, "After we multiplied the denominator by $\frac{3}{11}$, we have to multiply the numerator by $\frac{3}{11}$. That is, we have to multiply both the numerator and the denominator by $\frac{3}{11}$. Lada then asked, "The dividend is?" Lada and the students said chorally, "$\frac{3}{7}$ multiplied by $\frac{3}{11}$." Lada asked further, "Divided by?" Several students answered, "1." Lada explained, "Any number divided by 1 is that number, what do we get?" No one gave any responses. Lada had to summarize that $\frac{\frac{3}{7}\times\frac{3}{11}}{1}=\frac{3}{7}\times\frac{3}{11}$. Lada then asked the students what the product of $\frac{3}{7}\times\frac{3}{11}$ is. A few students replied, "9 over 77." Lada summarized, "You can see that $\frac{3}{7}\div\frac{11}{3}=\frac{3}{7}\times\frac{3}{11}$." A student called out, "We don't have to show these steps." Lada further asked this student, "What would you do?" A students responded, "Switch them or cross multiply." Lada agreed with his answer and asked the class to summarize what they had seen. No one responded. Lada pointed to the last line on the chalkboard and said, "From the last line, $\frac{3}{7}$ divided by $\frac{11}{3}$ is equal to $\frac{3}{7}$ multiplied by $\frac{11}{3}$. What can you conclude about the rules of division of fractions?" No one responded. Lada said further, "In division, what should we do to get an answer?" A few students said, "Keep the dividend." Lada responded, "Yes. Then we have to change the division sign into?" Several students answered, "Multiplication sign." Lada and the students said chorally, "Switch the numerator to the denominator and the denominator to the numerator." Lada then asked the class whether they understand or not. Several students responded chorally that they understood. Lada then wrote the rules on the chalkboard and had the students write it in their notebooks.

> The Rules for Dividing Fractions.
> To divide fractions, change the division sign into a multiplication sign then switch the numerator and the denominator. If it has any mixed number, convert to an improper fraction before performing division.

The researcher observed that Lada did not emphasize switching the numerator and the denominator of the divisor. While the students were taking notes, Lada walked around the room to see what they were doing. A student said that he did not want to change a mixed number to an improper fraction. Lada replied that he had to change a mixed number to an improper fraction; however, she did not give any reason.

Lada then wrote a division problem, $\frac{4}{5} \div \frac{3}{15}$, on the chalkboard. She called on a student to solve the problem. This student responded that he would find the common denominator. Lada responded that finding the common denominator was used in addition and subtraction. Lada then asked for volunteers. Several students raised their hand. Lada then called on a student who had hand raised to solve the problem.

A student:	Change the numerator to the denominator.
Lada:	How?
A student:	4 over 5 multiplied by 15 over 3.
Lada:	Yes. Sit down.

A few students called out, “The product is 60 over 15.” Lada responded that, “Can you reduce it?” A few students answered, “Yes, the common factor is 5.” With the students’ help, Lada reduced $\frac{60}{15}$ to $\frac{12}{3}$. Lada then asked, “Can you reduce it one more time?” A few students replied, “Yes.” With the students’ help, Lada reduced $\frac{12}{3}$ to $\frac{4}{1}$. Lada then asked the students, “Can we write $\frac{4}{1}$ as 4?” A few students answered, “Yes.”

Lada wrote another problem on the chalkboard, “$\frac{11}{10}\times\frac{2}{3}\div\frac{7}{12}$.” A student asked whether he had to cancel first then multiply or multiply first then cancel. Lada ignored his question; instead, she had two students put the work on the chalkboard. However, they were not able to evaluate this problem. Lada called on several students to solve this problem on the chalkboard. When they were not able to evaluate this problem, Lada guided them by asking what the rules of division were. A few students answered, “Change the numerator to the denominator.” Lada further probed what they would do next. A few students replied that they would change the division sign into the multiplication sign. Lada concluded that, “Change the division sign into the multiplication sign and then change the numerator to the denominator. Thus, the result is?” Lada answered her question, “12 over 7.” While she talked, Lada wrote on the chalkboard, $\frac{11}{10}\times\frac{2}{3}\times\frac{12}{7}$. Lada did not emphasize the order of operations to the students. Lada then asked, “Can we reduce them?” With students’ help, Lada reduced the multiplication problem to $\frac{44}{35}$. She then asked the students to change an improper fraction to a mixed number. A few students replied, “$1\frac{9}{35}$.”

To check the students' understanding, Lada wrote on the chalkboard division problems, $\frac{3}{5} \div \frac{4}{2}$, $\frac{4}{9} \div \frac{3}{8}$, $\frac{9}{13} \div \frac{7}{2}$, one at a time and called on the students to solve these problems individually. If the first student was not able to give the correct answer, she called on another student. She kept calling on the students until the correct answers were given. The students always stood up when they answered the questions. Lada always asked the class for agreement.

Lada then wrote a division problem, $1\frac{1}{2} \div \frac{2}{3}$, on the chalkboard and called on several students to solve this problem. They gave incorrect answers, including, "$\frac{2}{10}$," "$\frac{2}{4}$," "$\frac{5}{2}$," and "$\frac{6}{4}$." At this time, Lada made an error. When a few students answered that the quotient was 1. Lada accepted that answer as correct without performing any division to verify the answer. However, a smart student argued that the answer was $\frac{9}{4}$. Lada then proceeded to multiply $\frac{2}{3}$ by $\frac{2}{3}$, getting the answer of $\frac{9}{4}$. Lada asked the students who got the answer of $\frac{9}{4}$. Several students raised their hand. This type of error occurred again when Lada asked the students to divide $\frac{9}{12}$ by $\frac{12}{9}$. Several students raised their hands and called out that the quotient was 1. Lada accepted the response as correct without comment. Both Lada and the students still had an error in dividing fractions.

Lada then wrote a division problem, $(\frac{1}{2} + \frac{1}{2}) \div \frac{1}{4}$, on the chalkboard. A student called out, "4." Lada worked on the chalkboard to verify his answer. Next, she wrote a division problem on the chalkboard, $\frac{2}{9} \div \frac{16}{9}$, and said, "Raise you hand first and then answer." A few students raised their hands and answered, "8." Lada called on a volunteer solve the problem. However, his answer was not correct. Lada asked the class whether his answer was correct or not. Several students indicated that his answer was not correct. Several students raised their hand volunteer to answer the question. Some of them called out that the quotient was 8. Lada worked on the chalkboard and responded that the quotient was $\frac{1}{8}$.

Near the end of the class, Lada wrote the homework assignment on the chalkboard. The homework problems were selected from the textbook. Fifteen problems involved computation algorithms. Five problems involved division word problems for fractions.

Lada told the researcher after the class that the students were able to draw conclusion about the rules for dividing fractions. However, some students demonstrated a misconception of inverting both the dividend and the divisor before multiplying.

Day 4: Division Word Problems for Fractions

At the beginning of the class period, Lada asked for students who had problems in doing the homework. Several students raised their hands and said that they were not able to do the homework. Lada selected the first problem from the textbook, "The product of two numbers is $\frac{2}{3}$. One number is $\frac{5}{4}$. Find the second number." She had the students read the problem. Then Lada had a student read the problem again. Lada asked the class what information they knew from the problem. She said, "We know the product of two numbers is equal to?" A few students replied, 2 over 3." Lada responded, "The product of two number is $\frac{2}{3}$. What else do we know?" A few students responded that one number was 5 over 4. Lada further said, "This means that 5 over 4 multiplied by one number is equal to what number?" A few students answered, "2 over 3."

While she talked, Lada wrote "$\frac{5}{4} \times a = \frac{2}{3}$" on the chalkboard. Lada asked the students how they could find the value of a. A few students replied, "5 over 4 multiplied by 2 over 3." Lada then asked, "5 over 4 multiplied by a. If we move it to the other side, then the multiplication will change to?" Several students called out, "Division." Lada did not explain why the multiplication problem was changed to the division problem when the number was moved to the other side. Lada wrote on the chalkboard, "$a = \frac{2}{3} \div \frac{5}{4}$" and asked, "What should we do next? What are the rules for division?" Several students responded that they would change the division sign into the multiplication sign. Lada summarized, "Change the division sign to the multiplication sign and change the numerator to the denominator and the denominator to the numerator." However, Lada did not remind the students verbally that they had to invert the denominator and the numerator of the divisor. Lada then asked the students what the product was. Several students responded that the product was $\frac{8}{15}$.

Lada selected another word problem from the textbook, "Dang has 57 baht. Dum has 38 baht. How many times as much money does Dang have than Dum has? How many times as much money does Dum have than Dang has? Lada had the students read the problem as a group. She and the students discussed the problem. Lada asked the students how much money Dang had. A few students responded that Dang had 57 baht. Lada further asked the students how much money Dum had. A few students replied that Dum had 38 baht. Lada further said, "The problem asked us to find how many times Dang has as much money as Dum has and?" A few students answered chorally, "How many times as much money does Dum have than Dang has?" Lada asked, "To find how many times Dang has as much money as Dum has, which operation will be used, addition, subtraction, multiplication, or division?" A few students answered, "Division." Lada then asked the students, "What number is the dividend?" A few students responded, "57 divided by 38." Lada responded, "57 divided by 38. Can we reduce it to the lowest terms?" Lada gave the students some time to work. A few students answered, "$1\frac{19}{38}$." Lada further asked, "Can you reduce it?" Some students thought they could reduce it, others did not. Lada then cancelled 19 over 38, getting the result of $\frac{1}{2}$. Lada and the students summarized that Dang has $1\frac{1}{2}$ of Dum's money.

Lada further asked, "What should we do if we want to know how many times as much money Dum has than Dang?" A few students responded, "38 divided by 57." A student called out, "3 over 2." Lada said, "38 divided by 57. Can we reduce it?" Some students thought this fraction could be reduced to its lowest terms, other did not. Lada then told the students to use 19 as a common factor. Lada and the students then divided 19 out, getting the lowest term of $\frac{2}{3}$. They concluded that Dum's money was $\frac{2}{3}$ of Dang's money.

Lada then raised another problem, "Father is 35 years old. Son is 3 years and 6 months old. How many times is the father older than his son? How many times is the son older than his father?" Lada and the students solved this problem together using the same strategy as the previous problem. They carried out the division of $35 \div 3\frac{1}{2}$, getting the answer of 10. They concluded that the father was 10 times older than his son. Lada and the students proceeded to divide $3\frac{1}{2}$ by 35, getting the quotient of $\frac{1}{10}$. They concluded that the son was $\frac{1}{10}$ older than his father. Lada did not remind the students that 35 and $\frac{35}{1}$ named the same numbers.

Twenty minutes before the class ended, Lada assigned the students seven word problems to do in class. While the students worked, Lada moved around the room answering students' questions. However, the students did not finish their work in class. Lada allowed them to hand it in as homework on the following day.

After finishing the unit, Lada told the researcher that the students had difficulties in dividing fractions. They did not invert the denominator and the numerator of the divisor. Sometimes they inverted both the divisor and the dividend. Moreover, the students were unable to multiply fractions. They were not able to reduce fractions, change a mixed number to improper fractions and vice versa.

Impact of Lada's Knowledge of Subject Matter and of Students' Conceptions on Her Instruction

Prior to teaching, Lada appeared to have the necessary content knowledge to support the instruction of a topic based on division of rational numbers. During the lesson, the students made some errors; Lada was able to correct them. On the questionnaire, Lada was not able to solve the division of decimals. During the instruction, she continued to have some difficulty with the content of division of decimals and incorrectly multiplied two numbers. For the most part she was able to explain the mathematics correctly.

Lada's knowledge of students' conceptions was low prior to teaching. With her weakness, when she wrote the rules of division of fractions on the chalkboard, she wrote "invert and multiply" instead of writing "invert the divisor and multiply." She did not emphasize that "invert" meant "invert the divisor." She did not realize that the missing one word might create students' errors. Also, during the lessons, Lada rarely prevented students' errors and infrequently corrected their misconceptions. However, after she taught some classes, she gained more knowledge about students' conceptions. She was able to identify the common errors students made. She intended to prevent these errors when teaching other sections.

In the planning for her lessons, Lada focused on an algorithmic presentation following an algorithmically-based curriculum developed by the IPST. Although she had a strong conceptual understanding, Lada followed a curriculum that did not emphasize conceptual development.

During the lessons, Lada taught division of rational numbers procedurally. She created a worksheet that did not adequately guide the students towards understanding of the algorithm that she was teaching. Her low knowledge of students' conceptions was observed in the lesson as well. However, after teaching some lessons, Lada was able to identify more common errors students might make. She made some changes in her lessons.

Lada's mentor rarely observed the lessons she taught. He allowed Lada to design her lessons and activities by herself. Lada sometimes handed her mentor her lesson plans ahead of time. However, he did not write any comments in the lesson plans. Thus, there was no evidence of her mentor's influence on Lada's lesson plans and teaching.

Sak

Sak was categorized as having low knowledge of subject matter and high knowledge of students' conceptions category (L/H). He majored in mathematics education and minored in education measurement. He had taken several mathematics courses, including principles of mathematics, calculus and geometry series, number systems, introduction to geometry, foundations of geometry, linear algebra, abstract algebra, set theory, statistics and probability, and graph theory. He had taken 10 semester hours of practicum and field experience in teaching mathematics, including three semester hours of a mathematics methods class, one semester hour of participation and observational study, one semester hour of practicum, and five semester hours of student teaching.

Sak explained that mathematics was about operations, including, addition, subtraction, multiplication and division and things used in everyday life. Although Sak had taken several mathematics classes, he was unable to define a rational number. He simply gave examples of rational numbers, including, positive integers, zero, negative integers, fractions, decimals. Sak's knowledge structure of division had one meaning, equal sharing. When asked to give an example of a division word problem, he gave a measurement problem such as, "He has nine oranges. He wants to group them, three for each group. How many groups can he make?" Sak was not able to think of other meanings of division.

Subject Matter Knowledge of Division of Rational Numbers

Sak's understanding of division of rational numbers was poor. When asked to express "a divided by b" in notational forms in the interview, Sak was able to express it in two notational forms, "$\frac{a}{b}$," and "$a \div b$." Additionally, he had trouble identifying the dividend and the divisor from this statement, although he successfully identified the dividend, the divisor and the quotient of the division statement in the questionnaire.

Given problems in division of rational numbers when the dividend or the divisor is 0, he correctly identified that $0 \div (-2497)$ is 0. Sak's reason was that any number multiplied by zero was zero. His reason based on the fact that division was the inverse of the multiplication. Sak identified that $\frac{3}{0}$ and $\frac{2}{3} \div 0$ were undefined. $\frac{2}{3} \div 0$ was undefined because there was no number such that 0 times the number was $\frac{2}{3}$. However, Sak incorrectly indicated that the statement, "Any rational number divided by itself is 1" was true. His reason was that -10 divided by -10 was 1. He did not consider the special case that 0 divided by 0 is undefined.

With respect to division of fraction problems, Sak correctly identified the multiplication sentence that represented a given division of fractions sentence. Sak's knowledge of the division of fractions was sufficient to respond correctly to the items involving computation of division of fractions. He solved each of them using the "invert divisor and multiply" algorithm. Sak was unable to show that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ where a, b, c, and d are integers and b, c, and d $\neq 0$.

Another inconsistency became apparent. Although Sak correctly computed $\frac{5}{12}$ as the quotient of $(-\frac{1}{4}) \div (-\frac{3}{5})$, he agreed with the statement, "In a division problem, the dividend must be greater than the quotient." He said that 9 divided by 3 was 3. The dividend, 9, was greater than the quotient, 3. However, 4 was not divided by 5. Thus, the dividend had to be greater than the quotient. This explanation was based on his belief that the division was equal sharing.

Solving division of decimal problems was troublesome for Sak. He incorrectly gave 3.75 as the quotient of $0.75\overline{)0.75}$. He was not able to perform the division of 1.33 divided by (-2.1). During the interview the researcher asked him to solve the problems again. At this time, Sak was able to solve the problems correctly. He multiplied the divisor by the power of 10 to make the divisor a natural number first then carried out the long division algorithm.

Sak could see the connection between the real-word context and the interpretation of the remainder or fractional part of the quotient. He correctly solved the problem, "Somsak must deliver 20 tons of rambutans. If his truck can carry 3 tons at a time, how many trips must he make to finish delivery?" When asked to reverse the problem by giving the expression and requested a word problem, Sak was not able to construct it. He was not able to write a story problem that would be solved by dividing 51 by 4 and for which the answer would be 13. Furthermore, he did not have a broad enough concept of division to make sense of the division of fractions less than one. The lack of this concept led him to be unable to write a story problem to represent "$\frac{3}{4}$ divided by $\frac{1}{4}$." Sak failed to solve several division word problems. When asked to order the given fractions from smallest to largest, Sak incorrectly ordered them as $-\frac{8}{9}, -\frac{7}{8}, \frac{13}{17}, \frac{11}{15}$.

Like Chai and Lada, Sak was not able to demonstrate understandings of mathematical structure. He incorrectly identified a set as closed with the operation of division. From the questionnaire, Sak indicated that the set $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ was closed with respect to the operation of division. During the interview, the researcher asked him to explain why he selected this set. Sak said that actually he did not understand the question. He did not remember about the Closure Law. He selected this solution by guessing. The questionnaire and interview results both suggested that Sak had a low knowledge of division of rational numbers.

Knowledge of Students' Conceptions of Division of Rational Numbers

Sak was not able to identify the common errors students might make with computing division when the dividend or divisor was zero. He wrote that 0 was the typical incorrect response for $0 \div 7$. He explained that the students did not understand the difference between $0 \div 7$ and $7 \div 0$. They thought that any number divided by zero was zero. This explanation showed that Sak was confused between the dividend and the divisor. However, he correctly mentioned that $0 \div 7$ was 0 during the interview. He indicated that the students might have confused the meaning of $7 \div 0$ with $0 \div 7$. They might think that $7 \div 0$ was also equal to 0.

On the questionnaire, Sak incorrectly stated that "undefined" was the common error response for $0 \div 0$ students might make. The source of that error was that the students thought zero was nothing. From the researcher's perspective, Sak may have held this belief as well. Sak, however, correctly stated that $0 \div 0$ was undefined during the interview. He added that another possible error students might make was $0 \div 0 = 0$.

Sak did well in determining the common errors students might make with division of fractions. Sak wrote that $\frac{1}{4} \times \frac{4}{1} = \frac{4}{4} = 1$ was the common error students might make in computing $\frac{1}{4} \div 4$. The source of this error was that students changed the division sign to multiplication sign but they did not flip the divisor. Sak indicated that $\frac{3}{28}$ was the common error students might make when they carried out the division of $\frac{1}{4} \div \frac{3}{7}$. He wrote that the students changed the division sign to multiplication sign. However, they did not invert the divisor. They multiplied the numerators together and the denominators together.

Sak mentioned that $\frac{1}{5} \div 5$ was a correct expression representing the problem, "Five friends bought $\frac{1}{5}$ kilogram of chocolate and shared it equally. How much chocolate did each person get?" He indicated that $\frac{1}{5} - 5$ was an incorrect expression students might make when they solved this problem. The source of this error was that the students thought that the word "shared" meant "took away."

When asked to determine a student's work in a long division algorithm when the student incorrectly placed the digits in the quotient from right to left, Sak correctly identified her error and used her strategy to solve the given problem. When asked to determine a student's work on a division problem that required the student to add a zero as a place holder in the quotient, Sak correctly indicated the student's error pattern. He indicated that the student did not add a zero as a place holder in the quotient. Sak was able to use this student's pattern to answer the question. When asked to determine a student's work on a long division with a remainder, Sak correctly identified the student's error. He indicated that the student wrote the remainder as an extension of the quotient.

During the interview, Sak was able to correct some problems that he missed in the questionnaire. The interview results suggested that Sak had a slightly higher knowledge of students' conceptions of division of rational numbers. On the basis of interview results, Sak was placed in a high knowledge of students' conceptions category.

Classroom Observations of Lessons on Division of Rational Numbers

Sak did his student teaching at a large high school in Bangkok. The school had approximately 1,600 students in the seventh through twelfth grades. Sak taught three seventh grade mathematics classes. The class sizes ranged from 42 to 44 students. The class periods lasted approximately 50 minutes. Sak taught four days a week. He did not teach mathematics on Tuesday. He taught, at most, three periods a day and nine periods a week. His schedule was not consistent from day to day. For example, on Monday he taught one class and had the rest of the day to grade the homework and prepare for another class whereas on Friday he taught three classes the first, the fourth, and the sixth periods of the day. He did not teach three periods in a row. His class weekly schedule is presented in Appendix O.

Sak's mathematics classroom was on the fifth floor of the building. The portrait of the King and the Queen of Thailand, the pictures of the Buddha image and the Thai flag were hung on the front wall over the chalkboard. There were two doorways, several windows, and two ceiling fans in this room. The teacher's desk was in the right front corner of the classroom which was next to the door. The students' desks were arranged in pairs facing the chalkboard. Sak allowed his students to select their seats most times. There was enough space for the teacher to move around the room. Two bulletin boards in front of the room were decorated with seasonal items. Sak hung several mathematics posters at the right wall of the room.

One of Sak's classes was observed by the researcher for many lessons on division of rational numbers. This class was taught three days a week, Wednesday morning, Thursday afternoon, and Friday morning. The class included 42 students, with 19 boys and 23 girls. On average, the class lasted about 40 minutes with five minutes to put away materials. At the beginning of the class, the students used the first few minutes to enter the room, take seats, and organize their materials. At the end of the class, they used a few minutes to put away their materials before leaving.

Sak usually went to the classroom early. Sometimes Sak and the students had to wait outside the room because there was another class using the room. While waiting for the students to file into the room, Sak sometimes discussed homework, exams, or other topics with the students.

When all the students were in class, a student who was the head of the class led the students in greeting the teacher. This action was another way to get the students' attention. The students knew that they had to be ready to study. After Sak had the students' attention, he always told the students what they were going to study in each lesson. He wrote the topic of the day on the chalkboard and had the students write it in their notebooks.

Sak sometimes reviewed previously covered topics, addressed prior knowledge relevant to the day's lesson, or discussed the homework. A typical approach Sak used in reviewing was to write one or two problems on the chalkboard and then ask students to solve the problem on the chalkboard.

After the review, Sak presented a new topic to the students by providing one or two examples. Sak used a question-answer technique. Most of the questions were facts. Usually, the students answered the questions as a group. A few questions were addressed to specific students. While he had the students do the practice problems, Sak often had other students write their work on the chalkboard. He often walked around the room to monitor and assist students. Sak presented the information in his lessons not only verbally and in writing form but also asked the students to take notes.

Sak's mentor often observed his class. She sometimes sat in the class and sometimes she stood outside the class. Thus, the students did not know if she was observing the class. When Sak taught wrong content, his mentor told Sak privately and had him correct it. She never interrupted the class or summarized any topics herself. When Sak's mentor presented in class, the students were quiet. Sak mentioned that his mentor suggested he teach the students without using technical terms; he should use simple language that the students understood easily.

Near the end of the class, Sak always wrote the homework assignment on the chalkboard. The homework problems were selected from the textbook. At the end of the class, Sak allowed the students to leave the class. The head of the class led all of the students to salute the teachers. They often said, "Thank you."

Sak created five lesson plans on the division of rational numbers, one lesson plan per day. Examples of Sak's lesson plans are presented in Appendix Q. In planning the lessons, Sak used a textbook developed by the IPST, the teacher's manual and other books. He followed the sequencing and topics presented in the textbook. Sak mentioned that his examples were mostly selected from the textbook. He also selected some interesting examples from other resources. Sak stated that he selected the examples which were not too difficult for his students. From the researcher's perspective, Sak's lesson plans were detailed and followed the scope and sequence of the textbook.

During the interview, Sak mentioned that in class he would combine two lessons and teach them in one period. He said that the school often had activities in the morning before the first period. The first period sometimes was skipped, and sometimes class period was reduced to 45 minutes. Thus, he might not be able to complete a lesson as he expected. Sak also said that his mentor suggested he not rush because several students did not understand basic concepts. Some students had not memorized the multiplication and division basic facts, and some students could not read and understand the word problems.

Although Sak created five lesson plans, he taught them in six days. During classroom observation, Sak taught multiplication word problems for decimals and division of decimals in the same period. Sak taught representation of fractions by decimals and measurement in the same period. The sequencing and topics of division of rational numbers included in Sak's lesson plans and his classrooms are presented in Figure 9.

Lesson Plan		**Classroom**	
No.	*Topic*	*Day*	*Topic*
1	Division of a Decimal by a Natural Number	1 (Wednesday)	Division of a Decimal by a Natural Number
2	Division of a Decimal by a Decimal	2 (Thursday)	Division of a Decimal by a Decimal and Division Word Problems for Decimals
3	Division Word Problems for Decimals		
4	Representing Fractions as Decimals	3 (Friday)	Representing Fractions as Decimals
5	Division of Fractions and Division Word Problems	4 (Thursday)	Division of Fractions Algorithm
		5 (Friday)	Division of Fractions (Complex Fractions)
		6 (Wednesday)	Division Word Problems for Fractions

Figure 9. Sak's instructional emphasis and sequencing of topics in the division of rational numbers units.

Sak worked very closely with his mentor who had taught for more than 10 years. Sak discussed writing his lesson plans with his mentor showing her the lesson plans before he taught each lesson. His mentor made some comments about the correctness of the content, the sequence, examples, and activities. Before teaching each lesson, Sak and his mentor discussed the plan. His mentor often suggested an activity for a single lesson. Sak's mentor observed his classes often. If Sak made any errors or skipped some steps in the lesson, his mentor usually told him to correct those errors or added some steps. After class, Sak and his mentor often discussed the lesson taught. From the researcher's perspective, Sak followed his mentor's suggestions.

Sak's mathematics supervisor observed his lessons twice. No observation was on division of rational numbers. Thus, the researcher was not able to ascertain how much influence the mathematics supervisor had on Sak's teaching of division of rational numbers. In general, the mathematics supervisor commented on Sak's classroom management. However, Sak was able to ask his mathematics supervisor's advice for improving his teaching in other sections or other topics.

Day 1: Division of a Decimal by a Natural Number

Before teaching a division of decimals unit, Sak told the researcher that he would follow the order of the topics in the textbook. He mentioned that students learned a division of decimals topic from their sixth grade mathematics classes; some students, however, did not memorize the multiplication table. Without memorizing the basic facts, they were not able to perform multiplication. Finally, they were unable to carry out the division algorithm because they were not able to estimate their answers. Sak mentioned that one of the errors students might make when they performed the division algorithm was carelessness. They did not check the accuracy of the quotient. They could check the answer by multiplying the divisor and the quotient or by performing the division one more time.

Sak spent the first day on multiplication word problems and dividing a decimal by a natural number. The lesson took about 40 minutes. Sak reviewed multiplication word problems for about 25 minutes. There was 15 minutes left for division of decimals. To introduce the students to the division of decimals, Sak told the students that they are going to learn division of decimals and asked the class who could not perform the division algorithm. A boy responded, "I can perform division."

Sak then wrote the problem $15 \div 3$ on the chalkboard. This problem was neither in the lesson plan nor in the textbook. Sak asked the class, "Can you perform the division of this problem?" A few students responded that they could perform the division. Sak then asked them to identify the quotient. A boy quickly replied that 5 was the quotient of the problem. Sak then asked the class how to find the quotient. A boy at the back of the room responded, "15 divided by 3." Sak then asked the students to identify the dividend. A few students responded chorally, "15 is the dividend." Sak had the students identify the divisor. A few students answered chorally that 5 was the divisor. Sak wrote $15 \div 3$ as $\frac{15}{3}$. With the help of the students, Sak cancelled the common factor, 5, getting the quotient of 5.

Sak asked the students if any of them could not perform the long division algorithm. No one responded. He then asked the students what the dividend was. A few girls who sat in the front section identified that 5 was the dividend. Sak then asked them to identify the divisor. A few girls responded, "3 is the divisor." While he had the students identify the dividend and the divisor, Sak wrote the problem as $3\overline{)15}$ on the chalkboard. With the help of a few girls, Sak performed a long division algorithm, getting the same answer. Sak asked the students whether the answer was correct. A few students replied that it was correct. Sak had pointed out that a division problem could be written in several ways, $15 \div 3$, $\frac{15}{3}$, $3\overline{)15}$.

After the review, Sak told the students, "The next topic is the division of a decimal by a natural number." He wrote a division problem, $369.45 \div 15$, on the chalkboard. This problem was in the lesson plan and the textbook. Sak told the students that dividing a decimal by a natural number could be performed by using the same rules as dividing a natural number by a natural number. Sak asked the students to identify the dividend. A few students answered chorally that 369.45 was the dividend. Sak then asked the class to identify the divisor. A few students replied as a group that 15 was the divisor. Sak wrote the problem as $15\overline{)369.45}$. The class then performed a long division algorithm together long hand. Sak stood in front of the class putting the work on the chalkboard.

When they had to place the decimal point in the quotient, the discussion between Sak and the students took place.

> Sak: The remainder is 9. What do you see?
> A girl: The decimal point. Put the decimal point above.
> Sak: Do we have to put anything?
> A few students: Put the decimal point above.
> Sak: To divide a decimal by a natural number, we usually place a decimal point in the quotient directly above the decimal point in the dividend, then divide as if it is natural number division."

Sak and a few students went on to divide 369.45 by 15, getting the quotient of 24.63. Sak then again reminded the students to place the decimal in the quotient straight up from the decimal point in the dividend. Sak and the students discussed checking the accuracy of the quotient. Sak asked, "If we want to check whether our answer is correct or not, what will we do?" A few students replied, "Multiply the quotient by the divisor." Sak further asked, "What numbers do we have to multiply together?" A boy responded that 24.63 multiplied by 15. A few students answered, "The quotient multiplied by the divisor." Sak asked, "What number is the quotient?" A few students responded, "24.63." Sak further asked, "Multiplied by?" A few students answered, "15." Sak asked, "What is the product?" A few girls quickly answered, "369.45." Sak put all the work on the chalkboard and wrote, "Check: 24.63 × 15 = 369.45." However, Sak and the students did not perform the multiplication algorithm to verify the accuracy of the quotient.

While students were taking notes, Sak wrote another division problem, 31.8 ÷ 25, on the chalkboard. This example required students to add two zeros as place holders in the dividend. It was in the lesson plan and the textbook. Sak had the class identify the dividend. A few students answered chorally, "31.8." Sak then had the students identify the divisor of the problem. A few students replied chorally, "25." Sak also wrote the problem in the form $25\overline{)31.8}$. The class performed the long division algorithm together. When they had to place the decimal point in the quotient, Sak told the students that they had to place the decimal point in the quotient directly above the decimal point in the dividend.

When they had to add zeros in the dividend, a conversation between Sak and the students took place. Sak said, "If the division has a remainder, what should we do?" A few students answered, "Insert zero." Sak then asked them where they would insert a zero." Two responses were given including, "After eight," "In the dividend." Sak concluded, "Insert a zero in the dividend." Sak had reminded the students to add zeros in the dividend. However, he did not discuss why they were able to insert a zero to the right of the decimal point.

While they were performing a long division algorithm, Sak incorrectly computed a basic subtraction fact, 180 – 175 = 50. Sak did not notice that he made an error. However, several students found this error and corrected it. This error reflected Sak's low subject matter knowledge. Sak and the students then continued performing the division, getting the answer of 1.272. Sak and the students continued the conversation about checking the accuracy of the quotient by multiplying the divisor and the quotient to see if the product was equal to the number divided. However, they did not proceed to multiply 24.63 by 15 to verify that the answer was 369.45. Sak and the students again discussed adding zeros at the dividend. Sak mentioned, "If there is a remainder, where should we add zero? A few students answered, "In the dividend." Sak then added, "Add zero in the dividend and then bring it down. Next what will we do?" A few students replied, "Keep dividing." Sak summarized, "Keep dividing it until the final remainder is zero or carry the division until we get the required decimal places."

To check students' understanding, Sak wrote four division problems on the chalkboard, 0.755 ÷ 5, 11.98 ÷ 4, 35.01 ÷ 18, and 0.108 ÷ 9. These problems were selected from the textbook. However, they were not present in the lesson plan. The second and third problems required the students to add zeros as place holders in the dividend. The fourth problem required students to add zeroes as place holders in the quotient. From these problems Sak was able to check whether the students had difficulty with placing zeros in the quotient and the dividend or not. Sak had students go to the chalkboard and perform the long division algorithm. While the students were putting the work on the chalkboard, Sak walked in the front of the class to examine the students' work. Each student was able to compute the quotient correctly. When they had to check the accuracy of the quotient, they simply wrote, "Check: the quotient times the divisor is equal to the dividend." They did not actually check by performing multiplication algorithm. After the students finished their work, Sak called on two students to determine whether the first problem was correct or not. These students said that the quotient was correct. Sak and the class then determined the correctness of each problem together.

At the end of the class period, Sak assigned the homework. He wrote the instructions on the chalkboard. The problems were selected from the textbook. Three problems were division of decimals with natural number divisors. One of these problems required the students to provide zeros as place holders both in the dividend and the quotient. Two problems were division word problems. These problems were not mentioned in the lesson plan.

After class period, Sak told the researcher that most of the students understood division of a decimal by a natural number. His mentor suggested he review dividing a natural number by a natural number where the quotient was a decimal before teaching dividing a decimal by a natural number. Sak said that he would do as his mentor suggested with other sections.

Day 2: Division of a Decimal by a Decimal and Word Problems

Sak mentioned after the first day's lesson (but before the second day's lesson) that his mentor suggested that he should review dividing a decimal by a natural number where the quotient was a decimal. If there were many decimal places in the quotient, he had to round the quotient to the requested decimal place, for example, to the nearest hundredth. Sak mentioned that some students might have difficulty with placing the decimal point in the quotient. They did not know whether they had to place the decimal point in the quotient directly above the decimal point in the dividend. He said that in this lesson he would also emphasize placing the decimal point.

Sak began the lesson with a review of the division presented in the first lesson. He started by writing one problem selected from the previous lesson on the chalkboard, $369.45 \div 15$. His mentor suggested he use this problem. The students did not actually practice with a new problem. Sak and the students went over that problem, getting the quotient of 24.63. They did not check whether or not the quotient was correct.

Sak wrote a division problem where the divisor was a one-place decimal, $0.299 \div 1.3$, on the chalkboard. This problem was either in the lesson plan or in the textbook. Sak then asked the student to identify the dividend of the problem. A few students responded, "0.299 is the dividend." Sak then asked the students to identify the divisor of the problem. A few students answered chorally, "1.3 is the divisor." Sak rewrote it into a fraction form, $\frac{0.299}{1.3}$. Sak told the students that they had to multiply both the numerator and the denominator by 10 to make the divisor a natural number. Sak and the students proceeded to multiply 0.249 and 1.3 by 10, getting a fraction $\frac{2.99}{13}$. The class then carried out the division of $13\overline{)2.99}$ together long hand, getting the quotient of 0.23. They made no attempt to verify the accuracy of the quotient.

Sak wrote another division problem where the divisor was a one-place decimal, $0.264 \div 2.5$, on the chalkboard. This problem was in the lesson plan, but was not selected from the textbook. Sak converted it to a fraction $\frac{0.264}{2.5}$. Sak and the students then replaced it with an equivalent having a natural number divisor, $\frac{2.64}{25}$, by multiplying both the dividend and the divisor by 10. This problem required adding zeros as place holders both in the dividend and the quotient. In the lesson plan, Sak's mentor had suggested that he find the answer to nearest hundredths. Sak and the students proceeded to divide 2.64 by 25, getting the quotient of 0.1056, a four-place decimal. At this point, Sak made some errors. The conversation between Sak and a student illustrates how Sak made errors in rounding the decimal.

> Sak: The quotient was a four-place decimal. We need the quotient to be a two-place decimal, thus the answer is 0.10. If we need the quotient to be a three-place decimal, then the answer is 0.105 and if we need the quotient to be a four-place decimal, then the answer is 0.1056.
> A student: Why don't you round it up?"
> Sak: No, we will not do that."

Reflecting his low subject matter knowledge, Sak did not give any reason why the student was not able to round it up.

To check students' understanding, Sak wrote four division problems, 0.360 ÷ 0.4, 765.8 ÷ 0.05, 0.0748 ÷ 0.022, and 32.584 ÷ 1.25 on the chalkboard. These problems were presented in the lesson plan. They were not selected from the textbook. Sak asked for volunteers to write their work on the chalkboard. Four students volunteered to solve problems on the chalkboard. Three volunteer students were able to find the quotient correctly with the help of their friends. However, no one checked the accuracy of the quotient using a relationship between multiplication and division. A student who volunteered for the last problem was able to change the problem to an equivalent problem with the divisor of a natural number. While he was carrying out a long division algorithm, a student called out that the answer was 26.0672. Sak responded that he needed the quotient to be a two-place decimal. The student stopped carrying out the algorithm at the hundredths place and gave 26.06 as an answer. Sak accepted the answer as correct. Again, inadequate knowledge on the part of Sak led him to provide the students with a response that was mathematically inadequate. From the researcher's perspective, Sak's misconception was translated to the students.

Sak then presented the following division word problem: "There is 18.75 meters of dress fabric. One dress requires 1.25 meters. How many dresses can be made from the entire piece of fabric?" This problem was selected from the textbook. It was not presented in the lesson plan. Sak and the students translated this problem into the mathematical expression, $\frac{18.75}{1.25}$, multiplied both the numerator and the denominator by 100, and performed a long division algorithm, getting the quotient of 15. They did not check the accuracy of the quotient. At the end of the class, Sak assigned the students homework selected from the textbook. The problems involved the division of decimals algorithm and division word problems for decimals.

In his lesson plan, Sak used examples selected from the textbook and sequenced them in the same order as the textbook. Practice problems were also selected from the textbook. The homework problems were selected from the textbook as well. During the lesson, he followed his lesson plan.

Sak also taught division word problems on this day. In his lesson plan, there were two word problems. One problem was selected from the textbook. In class, Sak did not follow his lesson plan. He gave the students a word problem selected from the textbook. It was not the same problem as in the lesson plan.

After class period, Sak told the researcher that most of the students understood division of decimals. However some problems took time to solve. Thus, he would reduce their difficulty and use them with other sections.

Day 3: Representing Fractions as Decimals

Before teaching the representing fractions as decimals unit, Sak told the researcher that the students had basic knowledge on division of decimals and changing decimals as fractions. Thus, this topic was not difficult for the students.

To begin the lesson, Sak reminded the class that they had learned whether a fraction and a decimal were related. Sak asked the students to change $\frac{1}{2}$ to a decimal. Several responses were given, including, "0.12," "0.5," and "1.2." Sak replied that 0.5 was the correct answer and wrote "$\frac{1}{2} = 0.5$" on the chalkboard. However, he did not give any reason why $\frac{1}{2}$ was 0.5. Sak then asked the students to change $\frac{3}{8}$ to a decimal. A boy responded, "I don't know." There were no responses from other students. Sak wrote $\frac{15}{5}$ on the chalkboard said, "Let's look at an easier one. To what number is $\frac{15}{5}$ equal?" A few girls responded to Sak's question that $\frac{15}{5}$ was equal to 3." Sak then said, "Three, right? What format is it?" A girl replied, "A decimal." Sak then asked again, "What format is '3'? Is it an integer?" A few students said that 3 was an integer. Sak said to the students, "What format is it?", while pointing to " 0.5 " on the chalkboard. A few students responded, "A decimal." Sak repeated, "A decimal. What is the dividend?" and pointed to $\frac{1}{2}$. A few students indicated that 1 was the dividend. Sak responded that 1 was the dividend and asked the students to identify the divisor. A few students indicated that 2 was the divisor. Sak responded that 2 is the dividend and asked what form $\frac{1}{2}$ was. A few students answered, "A fraction form." Sak said, "It is in the fraction form. Thus, how can we write $\frac{3}{8}$ into a decimal form?" A few students replied, "8 divided by 3." Sak then asked the students to identify the dividend of $\frac{3}{8}$. The correct answer was given by a few students. Sak asked the students to identify the divisor of $\frac{3}{8}$. A few students responded that 8 was the divisor. Sak then asked the student what the quotient was. A boy quickly replied, "0.3" However, Sak did not pay attention to his answer. Sak and the students carried out the division of 3 divided by 8, getting the quotient of 0.375. Sak asked students whether 0.375 and $\frac{3}{8}$ were equal to each other or not. A few students responded that they were equal.

A girl requested one more example. Sak then wrote another fraction, $\frac{11}{20}$, on the chalkboard and asked the class to write it into a decimal. A student said, "11 divided by 20. Sak repeated, "11 is the dividend. What is the divisor?" A few students replied that 20 was the divisor. Sak wrote $20\overline{)11}$ on the chalkboard. The class then carried out the long division algorithm, getting a decimal number of 0.55. Sak stood in front of the room putting the work on the chalkboard. While they were performing this algorithm, Sak and the class had to write 11 as 11.00. However, Sak and the students did not discuss adding zeros in the dividend. A boy asked, "Why do we have to add zeros?" Sak did not respond to this question; instead, he asked the students to take notes.

While the students were taking notes, Sak wrote another fraction, $\frac{1}{3}$, on the chalkboard. A girl said, "1 divided by 3." Sak had the students identify the dividend. A few students indicated that 1 was the dividend. Sak then asked, "What is the divisor. A few students responded that 3 was the divisor. Sak asked, "What is the quotient? A student said, "0.333...is the quotient" Sak and the students then performed the long division algorithm. While they were performing the division, a few students said that the remainder was not zero. A few students added that the division would never end. Sak asked, “Will the division end?” A few students confirmed, “The division will never end. The digit 3 repeats.” Sak asked again, “What is the repeating digit?” A few students answered as a group, “3.” Sak further asked, “Can we continue dividing?” A few students replied, “Yes.” Sak asked, “What is the repeating digit?” A few students answered, “The digit 3.” Sak asked, “What is the answer?” Several students responded chorally, “0.33.” Sak said, “Are there any more digits?” Several students replied chorally, “Yes.” Sak then wrote “$\frac{1}{3} = 0.333...$” on the chalkboard. Sak did not explain the meaning of the three dots at the outset. A student asked why he added three dots after the decimal. Sak then asked the students what the three dots meant. A few students responded, “Continue forever.” Sak further probed, “What digit continues forever?” A few students responded, “The digit 3.” Sak then asked again, “The digit 3. Thus, what is the repeating digit?” A few students answered, “The digit 3.” Sak concluded that, “The digit 3 continues to repeat indefinitely. We will write a decimal in a short form by writing a dot over the repeating digit which is 3.” While he gave an explanation, Sak wrote $0.\dot{3}$ on the chalkboard. He told the students how to read the decimal and gave the students some time to take notes.

While the students were taking notes, Sak wrote a fraction, $\frac{7}{15}$, on the chalkboard and asked the students to express it in a decimal form. The fraction was a repeating decimal. The period of this numbers was one. Sak gave the students some time to work. Sak then performed the division of $15\overline{)7}$ on the chalkboard with the help of a few students, getting the quotient of 0.466.... Sak had the students identify the repeating digit. A few students responded that the digit 6 repeated. Sak then had the students identify how to write it in a short form. A few students answered correctly. Sak wrote the decimal in a short form and the way to read it on the chalkboard. While the students were taking notes, Sak walked to right side of the room to see what the students were doing. His mentor who sat on that side suggested he underline the phrase, "the digit 3 repeats" to remind the students which digits repeated. Sak did what his mentor suggested.

Sak wrote another fraction, $\frac{2}{18}$, on the chalkboard. This problem was presented neither in the lesson plan nor in the textbook. Sak asked the students to identify the dividend and the divisor. A few students responded that the dividend was 2 and the divisor was 18. Sak then had the students perform a long division algorithm. While the students worked, Sak stood in front of the class. A few students called out that the quotient was 0.111.... Sak asked them whether it was correct. The students confirmed that it was correct. Sak wrote on the chalkboard, "$\frac{2}{18} = 0.111\ldots$. It can be written in a short form as $0.\dot{1}$." Sak again asked the students to identify the repeating digit. A few students responded that the digit 1 repeated. Other students were busy taking notes.

Sak wrote another fraction, $\frac{2}{11}$, on the chalkboard. A girl said, "2 divided by 11." Sak gave the students some time to perform a long division algorithm. A boy called out that the quotient was 0.555. Sak did not pay attention to his response. Sak asked the class, "What is the answer?" A boy responded, "0.555." Sak did not react to his response. A girl said that the quotient was 0.181818.... Sak accepted her answer and wrote on the chalkboard, "$\frac{2}{11} = 0.181818\ldots$. While he was writing, Sak asked the students what digits repeated. A few students replied that the digits 18 repeated. Sak then asked the students how to write this repeating decimal in a short form. No one responded. Sak explained that they had to place a dot above the first digit and another dot above the last digit. Sak then had the students read this decimal. A boy read it correctly.

Sak explained that instead of writing three dots, two dots might be placed above the repeating digits, wrote one dot above the first digit and wrote another dot above the last digit of the series of the repeating digits. For example, $\frac{2}{11}$ = 0.181818... = $0.\dot{1}\dot{8}$.

Sak wrote a repeating decimal 0.256256256... which had three digits repeating on the chalkboard. This number was presented neither in the lesson plan nor the textbook. Sak asked the students how they wrote it in a short form. No one responded. Sak then asked the students what digits continued indefinitely. A girl replied that the digits 256 continued to repeat indefinitely. Sak again asked the students how to write the dots over the repeating digits. No responses were given. Sak explained that they had to place a dot over the first repeating digit and place another dot over the last repeating digit. While he gave an explanation, Sak wrote the dots over the digits 2 and 6. Sak asked the students whether they would place the dot over the digit 5. A few students responded that they would not do that. Sak then asked the students whether they understood or not. A girl replied that she understood. There were no responses from the rest of the class.

Sak wrote a repeating decimal 0.2345345... which had one nonrepeating digit and three repeating digits on the chalkboard. This number was not presented either in the lesson plan or the textbook. Sak then asked students to identify the repeating digits of this decimal. Several students responded that the digits 345 were the repeating digits. Sak asked whether the digit 2 repeated. Several students responded chorally, "No." Sak then asked the students how to write it in a short form. A few students said that they would place the dots over the digits 3 and 5. Sak asked the students whether they would write the dot over the digit 2 or not. A few students answered chorally, "No." Sak wrote "0.2345345... = $0.2\dot{3}4\dot{5}$" on the chalkboard and had the students read it. Several students read it chorally and correctly. Sak praised, "Great."

Next Sak wrote the last fraction, $\frac{1}{4}$, on the chalkboard. This fraction was a terminating decimal. It was presented both in the lesson plan and in the textbook.

Sak then asked the students to write it in a decimal form. A few students responded that the decimal for $\frac{1}{4}$ was 0.25. Sak then asked, "Can we add a zero at the end of the decimal?" A few students agreed that they could add a zero at the end of the decimal. Sak further probed, "Does the value change?" A few students replied, "No." Sak further asked, "What is the repeating digit?" A few students responded, "Zero." Sak further asked the students how to write it in a short form. No one responded. Sak wrote on the chalkboard "0.250" and asked the students where to place the dot. At this time, a few students answered correctly. Sak wrote on the chalkboard, "$\frac{1}{4} = 0.25000\ldots$ and it can be written in a short form as $0.25\dot{0}$." Sak asked the students whether every fraction could be written as a repeating decimal. Sak summarized the lesson by asking the students how to change a fraction to a decimal. A few students responded that they would do the division and the quotient would be the decimal.

Sak then asked the students to open the textbook. He wrote the assignment on the chalkboard and asked the students to take notes. The problems were the even questions selected from the textbook. The problems asked students to express fractions in repeating decimals. The lesson on representing fractions as decimals took about 25 minutes. The rest of time, Sak spent on a measurement unit.

After the class period, Sak told the researcher that the students still had difficulty in writing a repeating decimal in a short form. Some students wrote "0.235235235…" as "$0.\dot{2}\dot{3}\dot{5}$." They incorrectly put the dot over the digit 3. Sak mentioned that for other sections he would emphasize how to write a repeating decimal with many digits repeated.

Day 4: Division of Fractions

Before teaching this unit, Sak told the researcher that his mentor had suggested he review multiplication of fractions with the product of 1. His mentor suggested he present the lesson by changing the division of fractions problem into a complex fraction and then multiplying the dividend and the divisor by the reciprocal of the divisor to make the divisor equal to 1. She suggested Sak remind the students that if they multiplied the divisor by a number, they had to multiply the dividend by the same number and to remind the students about the fact that any number divided by 1 was equal to that number. When asked to identify the errors students might make in dividing fractions, Sak mentioned that one error students might make was to confuse the division algorithm with an addition and subtraction algorithm. That is, to divide fractions, the students might find equivalent fractions with a common denominator. Then the quotient would represent the quotient of the numerators over the common denominator. He thought another error students might make was that they would confuse the multiplication algorithm with addition and subtraction algorithms. After they had changed the division problem into the multiplication problem, they would find the equivalent fractions with common denominator. Then the product would represent the product of the numerators over the common denominator. If the students had had difficulty with multiplication, certainly the teacher could expect difficulty with division.

On the fourth day, the lesson was on multiplication and division of fractions. Sak went over multiplication problems and one multiplication word problem in about 30 minutes. He began the division of fractions by reviewing multiplication of fractions with the product of 1. Sak asked the students to identify the fraction $\frac{3}{5}$ had to be multiplied by to have a product equal to 1. The students gave the correct answer, $\frac{5}{3}$. Sak had the students identify what fraction $\frac{1}{2}$ had to be multiplied by to have a product equal to 1. A boy gave the correct answer, $\frac{2}{1}$. The second example helped some students who had trouble with finding the reciprocals of whole number or of fractions with the numerator of 1. Sak summarized that if the number was multiplied by its reciprocal, then the product was 1. While he talked, Sak wrote on the chalkboard:

$$\frac{3}{5}\times\frac{5}{3}=1$$
$$\frac{1}{2}\times\frac{2}{1}=1$$

Sak asked the students to take notes.

To introduce the division of fractions, Sak wrote $\frac{10}{5}$ on the chalkboard and had the class read it. Students read, "Ten over five." Sak asked the students what "ten over five" meant. Students gave several responses, including, "Ten divided by five," and "Ten is the dividend and five is the divisor." Sak wrote on the chalkboard, "$\frac{10}{5}$ means $10 \div 5$" and asked the students to identify the dividend. Several students replied correctly. Sak then had the students give the meaning of the fraction bar. A few students replied the fraction bar represented the division symbol. Sak then asked the students to identify the divisor. Several students answered correctly.

Sak asked the students to read and identify the dividend and the divisor of another division problem, $\frac{2}{5}\div\frac{3}{5}$. A few students read it correctly and gave the correct responses. Sak wrote on the chalkboard, "$\frac{2}{5}\div\frac{3}{5}$ can be written in a fraction form as $\frac{\frac{2}{5}}{\frac{3}{5}}$." A student said it was a complex fraction. Sak accepted his response and concluded that it was in a complex form. Sak asked a few students whether they understood or not. The students said that they understood.

Sak wrote another division problem, $\frac{2}{5} \div \frac{3}{10}$, on the chalkboard and asked the students to write it as a complex fraction. A few students said, "Change the division symbol to multiplication symbol and change the numerator to the denominator and the denominator to the numerator." Sak asked those students to give the reasons why they had to change the symbol from division to multiplication. Several answers were offered, including, "It is easier," "It is difficult to divide," and "It is easy. If we change it to multiplication problem, we can reduce it." Sak responded to those students that it was a shortcut. The students had to know where this shortcut came from. Sak said that the first step was to write $\frac{2}{5} \div \frac{3}{10}$ as a complex fraction and the next step was to make the divisor equal to 1. Sak asked the students to identify the divisor and its reciprocal. The students responded correctly. Sak further asked, "By what fraction must $\frac{3}{10}$ be multiplied to have a product equal to 1?" A few students replied, "$\frac{10}{3}$." Sak asked, "Can we multiply only the denominator?" A few students answered, "No." Sak further probed, "We have to multiply?" A few students replied, "We have to multiply both the numerator and the denominator." While the conversation was taking place, Sak wrote on the board as follows:

$$\frac{2}{5} \div \frac{3}{10} = \frac{\frac{2}{5}}{\frac{3}{10}} = \frac{\frac{2}{5} \times \frac{10}{3}}{\frac{3}{10} \times \frac{10}{3}}$$

Several students were confused about changing the value of a fraction. Sak assured the students about the value of a fraction by saying, "We want to make a denominator equal to 1. By what fraction must $\frac{3}{10}$ be multiplied to have a product equal to 1?" A few students responded, "$\frac{10}{3}$." Sak asked further, "Can we multiply only the denominator?" A few students replied, "No." Sak probed, "If we multiplied only the denominator, is the value of a fraction still the same?" A few students responded that the value of the fraction was not the same. Sak concluded that the value would change. They needed to multiply both the numerator and the denominator. Sak then asked the students to identify the new numerator and the denominator. A few students replied that $\frac{2}{5} \times \frac{10}{3}$ was the numerator and 1 was the denominator. A student asked why the denominator was 1. Sak and other students replied that $\frac{3}{10} \times \frac{10}{3} = 1$. Sak then asked what the final result was. A boy replied that the result was $\frac{2}{5} \times \frac{10}{3}$. When asked to give the reason, no one responded.

Both Sak and the researcher observed that several students had difficulty in naming equivalent fractions. He convinced the students that $\frac{2}{5} \times \frac{10}{3}$ divided by 1 was equal to $\frac{2}{5} \times \frac{10}{3}$ by asking, "2 divided by 1 is equal to?" A few students replied that the quotient was 2. Sak asked, "3 divided by 1 is equal to?" A few students replied that the quotient was 3. Sak then asked, "Any number divided by 1 is equal to?" A student replied, "That number." Sak then said, "$\frac{1}{3}$ divided by 1 is equal to?" Two responses were given, including "1" and "$\frac{1}{3}$." Sak said, "1, right?" No one responded. In this conversation Sak made one error. He said that $\frac{1}{3}$ divided by 1 was 1. He also wrote on the chalkboard, "$\frac{1}{3} \div 1 = 1$." Sak did not know that he made a mistake. Thus, when he asked the students what $\frac{2}{5} \times \frac{10}{3}$ divided by 1 was; two responses were given, including "1" and "The same number." It appeared that Sak's error was translated to the students.

Sak then turned the class's attention to $\frac{2}{5} \div \frac{3}{10}$ and pointed out that the students could skip several steps, getting $\frac{2}{5} \div \frac{3}{10} = \frac{2}{5} \times \frac{10}{3}$. Sak reminded that this procedure showed that why they changed the division sign to the multiplication sign and changed the denominator to the numerator and the numerator to the denominator.

However, Sak did not remind the students that they had to invert the divisor. Sak and the students then performed the operation by canceling the common factor, multiplying fractions and changing an improper fraction into a mixed number, getting the answer of $1\frac{1}{3}$. Sak reminded the students that they had to know how to change a division of fraction problem to a multiplication of fraction problem. At the end of the class, Sak handed out a worksheet on fractions.

After the class, Sak told the researcher that some students still had difficulty in reducing fractions, dividing both the numerator and the denominator by a common factor.

Day 5: Complex Fractions

Sak began the lesson by calling five students to the chalkboard to write homework problems. After the students finished their work, Sak asked the whole class whether the solution of each problem was correct. Five students correctly solved the problems. One student forgot to write the denominator of one fraction. Sak did not correct it. Sak asked the class, "Who did all problems correctly?" Several students raised their hands. Sak then had the students identify the form of the problem $\dfrac{\frac{3}{4} - \frac{1}{2}}{\frac{1}{6}}$. The students replied that it was in a complex fraction form.

Sak and the students then proceeded to solve this problem, getting an answer of $1\frac{1}{2}$. The final result is shown as follows:

$$\frac{\frac{3}{4}-\frac{1}{2}}{\frac{1}{6}}=\frac{\frac{3}{4}-\left(\frac{1}{2}\times\frac{2}{2}\right)}{\frac{1}{6}}$$
$$=\frac{\frac{3}{4}-\frac{2}{4}}{\frac{1}{6}}$$
$$=\frac{\frac{1}{4}}{\frac{1}{6}}$$
$$=\frac{1}{4}\div\frac{1}{6}$$
$$=\frac{1}{4}\times\frac{6}{1}$$
$$=\frac{3}{2}$$
$$=1\frac{1}{2}$$

Sak reminded the students about the order of operations. Sak gave the students a few minutes to take notes. As the students were taking notes, Sak's mentor suggested he write a complex fraction in a horizontal form with a division symbol. Sak went back to the first example and pointed out to the students that $\left(\frac{3}{4}-\frac{1}{2}\right)\div\frac{1}{6}$ could be written in place of $\frac{\frac{3}{4}-\frac{1}{2}}{\frac{1}{6}}$. The researcher observed that writing a complex fraction in a division symbol form helped the students solve the problem more easily.

The class solved another complex fraction problem, $\dfrac{\frac{1}{2}+\left(\frac{1}{3}\times\frac{3}{4}\right)}{1\frac{1}{2}-\left(\frac{1}{3}-\frac{1}{4}\right)}$. They wrote the given complex fraction in the form with a division symbol as recommended by Sak's mentor. They carried out the problem together, getting the answer of $\frac{9}{17}$. Sak used only parentheses as grouping symbols. These symbols confused some students. As students were taking notes, Sak's mentor suggested he use several grouping symbols such as the brackets or the braces. Sak went back to the second problem and changed some parentheses to the braces and asked the students to do it in this manner.

Ten minutes before the class ended, Sak wrote four complex fraction problems on the chalkboard as an assignment. These problems were selected from other sources. While the students were taking notes, Sak circulated answering students' questions.

After the class, Sak told the researcher that he would use his mentor's suggestions in other sections, using several types of grouping symbol and writing a complex fraction in a form with a division symbol.

Day 6: Complex Fractions and Division Word Problems

Sak began the lesson by writing one homework problem, $(2\frac{1}{6}+\frac{5}{9})\div(2\frac{5}{7}-1\frac{1}{4})$, on the chalkboard. Sak reminded the students to do the operations within the parentheses first, change mixed numbers to improper fractions before addition and subtraction, and name equivalent fractions before addition and subtraction. Sak and the students performed the addition and subtraction algorithms together, getting the result of $\frac{49}{18}\div\frac{41}{28}$. Sak reminded the students how to divide fractions by asking, "How do you divide fractions?" A few students answered, "Change the division symbol to multiplication symbol." Sak said, "Change the division symbol to multiplication symbol. What would you do next?" A few students replied, "Switch the numerator and the denominator." Sak asked, "We switch the numerator and the denominator of the dividend or the divisor?" A few students responded, "The divisor." Sak probed, "Do we have to switch the dividend?" A few students answered, "No." Sak explained, "We do not have to switch the dividend. We have to keep the dividend the same." Sak then wrote the problem as $\frac{49}{18} \times \frac{28}{41}$. Sak and the students canceled the common factor and proceeded to compute $\frac{49}{9} \times \frac{14}{41}$. Sak reminded the students that they had to multiply the numerators together and the denominators together. It was not a cross-multiplication. After getting the product of $\frac{686}{369}$, Sak reminded the students how to change the improper fraction into a mixed number. Sak and the students then changed the improper fraction to a mixed number $1\frac{317}{319}$. Sak further reminded the students that if there were parentheses within parentheses, they had to do the operations in the inner parentheses first. Sak reminded the students that when the problem was in the division of fractions mode, the students could not cancel the common factor. They had to change the division problem to a multiplication problem and then divide both a numerator and a denominator by a common factor before multiplying fractions.

Sak and the students solved another homework problem, $4\frac{1}{2} \div \{1\frac{5}{6} + (\frac{5}{26} \times 1\frac{1}{2})\}$, together, arriving at the answer of 2. While they were solving this problem, Sak reminded the students about the order of operations, performing the multiplication, addition, and division, respectively. He also reminded the students that they had to convert mixed numbers to improper fractions before multiplication or division. Sak also reminded the students that they could reduce before multiplying fractions. Further, he reminded the students to invert the divisor in order to change the division problem to a multiplication problem.

After the students had reviewed addition, subtraction, multiplication and division of fractions algorithms, Sak wrote a fraction word problem on the chalkboard,

> There was a full tank of water. The water leaked out of $\frac{1}{3}$ of the tank. When 4 liters of water was taken out of the tank, a half of the water was left. How many liters of water can this tank hold?"

This problem required the students to use more than one operation to answer the question. Sak also drew the picture of tank partitioned into three equal sections. While they were working on this problem, Sak made one error. He incorrectly wrote, "A full tank of water can be written as a fraction as $\frac{3}{3}$ liters." His mentor saw this error. She told Sak to delete the word, "liters." Sak and the students continued solving this problem, getting the answer of 24 liters. Sak reminded the students to draw a picture when solving a word problem if they could. At the end of the class period, Sak passed out an assignment on fraction word problems.

After class, Sak told the researcher that the students were confused as to whether or not cancellation could be performed when the fractions were being divided. Sak added that the students still had difficulty in recalling the order of operations. They did not perform the operation in the parentheses first. The students still had difficulty in evaluating complex fractions.

Impact of Sak's Knowledge of Subject Matter and of Students' Conceptions on His Instruction

Prior to teaching, Sak was able to write the phrase, "a divided by b" into two notation forms, "$a \div b$," and "$\frac{a}{b}$". During the lessons, he always wrote a division problem in several forms. During the lesson, Sak made some errors, reflecting his low knowledge of subject matter. He incorrectly computed a basic subtraction fact. He gave incorrect explanations on rounding decimals. When he had to show why $\frac{2}{5}\times\frac{10}{3}$ divided by 1 was equal to $\frac{2}{5}\times\frac{10}{3}$. He incorrectly identified that $\frac{\frac{1}{3}}{1}=1$. Thus when he asked the students to identify the result of $\frac{2}{5}\times\frac{10}{3}$ divided by 1, the students gave two responses, including, "$\frac{2}{5}\times\frac{10}{3}$," and "1." However, Sak did not notice why the students gave two responses. He simply concluded that $\frac{2}{5}\times\frac{10}{3}$ divided by 1 was 1. Moreover, when Sak had the students do some problems on the chalkboard, one student made an error. He did not write the denominator for a fraction. Sak did not see it, so he did not correct it. Moreover, Sak incorrectly wrote the unit of a fraction. Fortunately, his mentor was available to observe this lesson. She told Sak to correct it.

Prior to teaching, Sak correctly identified potential student's error of not placing a zero as a place holder in the quotient. During the lesson, Sak presented the students with several examples that violated this error. The class worked on these examples together. Thus, if any students had difficulty, Sak was able to correct them right away. Sak mentioned that some students might have difficulty with placing the decimal point in the quotient. During the lesson, whenever they performed the long division algorithm, he often reminded the students to place the decimal point in the quotient directly above the decimal in the dividend. Sak realized that one of the errors students might make when they performed the division algorithm was computational error. They did not check the accuracy of the quotient. During the lesson on division of decimals, Sak often reminded the students to check the accuracy of the quotient by using the relationship between multiplication and division. Unfortunately, the class did not actually proceed to multiply the divisor and the quotient to check accuracy.

Sak used his knowledge of students' conceptions throughout the lessons, reflecting his high knowledge of students' conceptions. The strategies used were discussion, presenting examples that evoked the errors, and reminding the students about the errors. Moreover, after teaching the lessons, Sak was able to identify the nature and the likely sources of related common misconceptions held by the students.

Sak planned his lesson according to the prescribed curriculum which was an algorithmically-based curriculum developed by the IPST. He did exactly what the curriculum told him to do. He did not deviate from the curriculum. During the lessons, Sak taught division of rational numbers procedurally. He thought of examples that guided the students towards understanding of algorithm that he was teaching. He rounded decimals incorrectly. He incorrectly identified that the quotient of $\frac{1}{3} \div 1$ as 1. He wrote the unit for a fraction by mistake, reflecting his low subject matter knowledge categorization. However, during the lessons, Sak often prevented students' errors, reflecting his high knowledge of students' conceptions.

Sak's mentor directly impacted his teaching. Before teaching each lesson, Sak and his mentor discussed the lesson and how to teach it. She often observed the lessons Sak taught. When Sak taught incorrect content, his mentor told Sak privately and had him correct it. She never interrupted the class or summarized any topics herself. After the lesson, Sak and his mentor discussed the lesson he taught, the flaws of his teaching, and how to adjust his teaching. He did what his mentor suggested.

Nisa

Nisa was categorized as having low knowledge of subject matter and low knowledge of students' conceptions (L/L) on the basis of the questionnaire and interview results. She majored in mathematics education and minored in English. She had taken 42 semester hours of mathematics, including, principles of mathematics, calculus and analytic geometry, differential equations, number systems, linear algebra, abstract algebra, set theory, graph theory, and foundations of geometry. She had taken 10 semester hours of practicum and field experience in teaching mathematics, including, three semester hours of mathematics methods class, one semester hour of participation and observational study, one semester hour of practicum, and five semester hours of student teaching. She had taken three semester hours of English methods class.

Nisa explained that mathematics is the subject about using thinking to solve problems. Mathematics is used in several ways. Without mathematics, people would not know how much money they have. They would not have any currency systems. Although Nisa has taken several mathematics classes, she was not able to define a rational number. She simply gave examples of rational numbers, including integers, counting numbers, fractions, and terminating decimals. Nisa incorrectly identified real numbers as rational numbers. When asked about the meaning of division, Nisa gave only a partitive interpretation of division. She described that division meant to partition into equal sizes. When asked to give an example, Nisa mentioned, "If we have 20 baht (Thai currency) and want to give it to five persons, how much money will each person receive?" However, Nisa did not mention whether each person would receive the same amount of money. When asked to give the answer for her example, Nisa mentioned that the money 20 baht was divided into five portions. Thus, each person received 4 baht. When asked to give another meaning of division, Nisa said, "Division will mean sharing but not in equal sizes. It will have a remainder."

Subject Matter Knowledge of Division of Rational Numbers

Nisa's understanding of division of rational numbers was poor. When asked to express the phrase, "a divided by b" in notational forms, Nisa was able to express it in two notational forms, "$\frac{a}{b}$," and "$a \div b$." She had no trouble, though, in identifying the dividend and the divisor, she correctly identified that $1\frac{3}{4}$ was the dividend, $-\frac{1}{4}$ was the divisor, and -7 was the quotient of the statement, "$1\frac{3}{4} \div (-\frac{1}{4}) = -7$." When asked about restrictions on the divisor, the dividend, or the quotient, Lada had trouble. She incorrectly indicated that the statement, "In a division problem, the dividend must be greater than the quotient," was true.

On the questionnaire, Nisa was not able to state specific facts about division of rational numbers when the dividend or the divisor was 0. She failed to carry out an algorithm for the division of $0 \div -2497$. She did not know if $\frac{3}{0}$ was undefined. Lack of this knowledge rendered Nisa unable to give the most appropriate explanation of a quotient when the divisor was 0. During the interview, Nisa correctly said that $\frac{3}{0}$ was undefined. When asked to explain her response, Nisa gave the explanation, "It cannot be partitioned into 0 groups, and thus it is undefined." She further explained that $\frac{2}{3}$ could not be divided into groups of zero. Zero was the divisor. If the quotient could be found, then the product of zero and the quotient had to be $\frac{2}{3}$. There was no number such that 0 times the number was $\frac{2}{3}$. Thus, $\frac{2}{3} \div 0$ was undefined. Nisa correctly identified that $0 \div -2497 = 0$. However, Nisa misunderstood the statement, "Any rational number divided by itself is 1" to be true, as 2 divided by 2 is 1. She did not consider the special case of 0 divided by 0 as undefined.

When Nisa was confronted with division of fraction problems, she correctly identified the multiplication sentence that was most closely related to the given division sentence. She correctly carried out 2 of 3 problems concerning division of fractions using the "invert and then multiply" strategy. During the interview, she changed the original division problem to a complex fraction form and then multiplied both the divided and the divisor by the reciprocal of the divisor. She explained that by doing this, the divisor was equal to 1 and the numerator was the product of the dividend and the reciprocal of the divisor. Nisa was able to show that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ where a, b, c, and d are integers and b, c, and d $\neq 0$.

An inconsistency became apparent. Although Nisa correctly calculated $0.75\overline{)3.75}$ to be 5, she agreed with the statement, "In a division problem, the dividend must be greater than the quotient." Division of decimals tended to be difficult for Nisa when the division problem required her to add zeros as place holders in either the dividend or the quotient. She incorrectly carried out the division of 1.33 divided by -2.1:

$$
\begin{array}{r l}
 & \underline{\;0.60032\;} \\
-2.1 & \big)\,1.330 \\
 & \underline{1.26} \\
 & 0.0700 \\
 & \underline{0.63} \\
 & 0.440 \\
 & \underline{0.42} \\
 & 0.02
\end{array}
$$

Nisa also had difficulty in dividing numbers with different signs. She incorrectly identified that a positive decimal was the quotient of the problem. This type of error was also shown in a division of fractions problem. Nisa wrote that the positive number, $3\frac{27}{30}$ was the answer to the problem $4\frac{1}{3} \div (-1\frac{1}{9})$. Her work is shown as follows:

$$\begin{aligned} 4\frac{1}{3} \div (-1\frac{1}{9}) &= \frac{13}{3} \div \frac{10}{9} \\ &= \frac{13}{3} \times \frac{9}{10} \\ &= \frac{117}{30} = 3\frac{27}{30} \end{aligned}$$

Although Nisa successfully computed $\frac{5}{12}$ as the quotient of $(-\frac{1}{4}) \div (-\frac{3}{5})$, she agreed with the statement, "If both p and q are negative, then $p \div q$ is negative."

When asked to write expressions representing two word problems designed to assess the misbelief, "Multiplication always makes bigger and division always makes smaller," and the misbelief, "The dividend is always greater than the divisor," Nisa was able to only write one expression, and her inconsistency became apparent. On the questionnaire, Nisa wrote the expression $4\frac{1}{2} \div 30$ to represent the word problem, "A rope $4\frac{1}{2}$ feet long is to be partitioned into 30 shorter pieces. How many inches long will each of the shorter pieces be?" During the interview, Nisa changed the expression to $30 \div 4\frac{1}{2}$. This problem violated the belief, "The dividend is always greater than the divisor."

Nisa was not able to make a connection between a real-word context and an interpretation of a remainder or fractional part of a quotient. She was unable to write a story problem for "dividing 51 by 4 and for which the answer is 13." Lacking this connection, she was unable to solve the problem, "Somsak must deliver 20 tons of rambutans. If his truck can carry 3 tons at a time, how many trips must he make to finish delivery?" Nisa also had a limited concept of division to make sense of the division of fractions less than one. She was not able to write a story problem to represent "$\frac{3}{4}$ divided by $\frac{1}{4}$."

Like other respondents, Nisa was not able to demonstrate an understanding of the Closure Law by identifying a set as closed with the operation of division. Nisa incorrectly identified that {1, 2, 4, 8} was closed with the operation of division. When asked to define the Closure Law during the interview, Nisa was not able to define it. Instead, she said that the Closure Law did not have the Commutative Law for division. The quotients were not equal to each other. For example, $\frac{5}{4} \neq \frac{4}{5}$. Moreover, Nisa was not able to order the given fractions from smallest to largest. On the basis of her questionnaire and interview responses, Nisa was placed in a low knowledge of subject matter category.

Knowledge of Students' Conceptions of Division of Rational Numbers

Nisa's knowledge of students' conceptions of division of rational numbers was classified into a low level category. On the questionnaire, she was unable to identify common mistakes students might make in computation of the division problems, $0 \div 7, 0 \div 0, \frac{1}{4} \div 4,$ and $\frac{1}{4} \div \frac{3}{7}$. During the interview, she said that $0 \div 7$ was 0. She added that two possible errors students might make were 7 and no answer. Nisa also mentioned that $0 \div 0$ had no answer because zero was nothing. She added two possible errors students might make were 0 and 1.

Nisa incorrectly computed that $\frac{1}{4} \div 4$ was $\frac{1}{4}$. She mentioned that one possible wrong solution that students might make in computing this problem was 1. To get this result, she thought they canceled 4 over 4.

When asked to list two common errors students might make when solving the problem, "Five friends bought $\frac{1}{5}$ kilograms of chocolate and shared it equally.

How much chocolate did each person get?," Nisa was able to identify only one common incorrect expression. She wrote that the students might write $5 \div \frac{1}{5}$. They might think that there were $\frac{1}{5}$ kilograms of chocolate and five children. Thus, they divided five children by the amount of chocolate.

Nisa was able to identify a student's error when she computed a division of fractions problem the same way as she added or subtracted two fractions. Moreover, Nisa was able to identify a student's error in a long division algorithm where he forgot to record a zero as a place holder in the quotient. Unfortunately, when asked to determine a student's work on a long division algorithm when the student incorrectly placed the digits in the quotient from right to left and a student's work on a long division with a remainder when the student wrote the remainder as an extension of the quotient, Nisa was not able to identify their errors.

When given a situation, "A student solves this problem, "If cheese is 75.60 baht per pound, how much is 0.78 pounds? By 75.60 ÷ 0.78," and asked to determine whether or not the student's method was correct; Nisa responded that the operation used by this student was not correct. She incorrectly suggested that this student had to solve the problem by 0.78 ÷ 75.60. The problem in this situation violated her belief, "Multiplication always makes bigger and division always makes smaller." The responses show that Nisa held a misconception.

Classroom Observations of Lessons on Division of Rational Numbers

Nisa did her student teaching at the medium-size high school in Bangkok. The school had approximately 800 students in the seventh through twelfth grades. Nisa taught three seventh grade mathematics classes. The class size ranged from 42 to 45 students, with all boys. Nisa taught from Monday through Friday. She taught, at most, three periods a day and nine periods a week. She taught at varying times of the day, sometimes with extended breaks, sometimes without. She did not teach three periods in a row. For example, on Monday she taught two classes the fifth and the sixth periods of the day and then she had the rest of the day to grade the homework and prepare for another class whereas on Wednesday she taught a class the second period and had three hours before she had to teach the next period. Her class weekly schedule is presented in Appendix O.

Nisa's mathematics classroom was on the third floor of the building with an air conditioner. There were two doorways and several windows in this room. A television set was hung in front of the room. Two bulletin boards in front of the room were decorated with seasonal items. Two science posters were hung on the right wall. The teacher's desk was in the right front corner of the classroom which was next to a door. The students' desks were arranged in pairs facing the chalkboard. They were arranged in this manner for the entire school day. Nisa allowed her students to select their seats. There was enough space for the teacher to move around the room.

The researcher observed one of Nisa's classes for many lessons on division of rational numbers. This class was taught three days a week, Tuesday morning, Wednesday afternoon and Thursday afternoon and was scheduled for 50 minutes a day. The class included 44 students. On average, the class lasted about 35 minutes with five minutes to put away materials. At the beginning of the class, the students used the first few minutes to enter the classroom, take seats, and organize their materials. At the end of the class, they used a few minutes to put away their materials before leaving.

Nisa usually went to the classroom early, and waited for her students to arrive from another building. When all the students were in the class, a student who was the head of the class stood up to lead the students to greet the teacher. This action was another way to get the students' attention. The students knew that they had to be ready to study.

Nisa usually told the students what they were going to study at the beginning of the class. She often reviewed previously covered topics or addressed prior knowledge relevant to the day's lesson. In reviewing, Nisa sometimes wrote two or three problems on the chalkboard.

After the review, Nisa presented a new topic to the students by having the students attempt to solve one or two examples. She sometimes discussed the importance of the topic. Nisa sometimes called on volunteer students to answer the questions, sometimes called on nonvolunteer students, or sometimes had the students put the work on the chalkboard. She usually paid attention to the students who gave responses to her questions. After the student had studied the new topic, Nisa usually wrote practice problems on the chalkboard and had the students work individually or had volunteer/nonvolunteer students work at the chalkboard.

Nisa's lessons were teacher-centered, rather than student-centered. Although Nisa called on students to answer the questions individually and worked on the chalkboard, only the students in the front of the room paid attention to the lessons. Some students at the back of the class talked to each other, and Nisa did not pay attention to them. She rarely circulated around the room to see what they were doing. Nisa summarized the information in the lessons orally without writing them on the chalkboard or asking the students to take notes. When Nisa called on the students, they always stood up to answer the questions.

Near the end of the class, Nisa often assigned homework problems selected from the textbook or distributed a worksheet assignment. During the class time, the students could leave or enter the class with the permission of the teacher. At the end of the class, when Nisa allowed the students to leave the class, the head of the students led the class to salute the teacher. All the students stood up and said, "Thank you."

Nisa created six lesson plans on the division of rational numbers, one lesson plan per day. Examples of Nisa's lesson plans are presented in Appendix R. In planning the lessons, Nisa used the textbook developed by the IPST as the main resource. She followed the sequencing and topics presented in the textbook. From the researcher's perspective, Nisa's lesson plans were greatly detailed including examples, activities, and assessment methods. Although Nisa created six lesson plans, she taught them in five days. The sequencing and topics of division of rational numbers included in Nisa's lesson plans and her classrooms are presented in Figure 10.

	Lesson Plan		**Classroom**
No.	*Topic*	*Day*	*Topic*
1	Division of a Decimal by a Natural Number	1 (Thursday)	Division of a Decimal by a Natural Number
2	Division of a Decimal by a Decimals	2 (Tuesday)	Division of a Decimal by a Decimal
3	Multiplication and Division Word Problems	3 (Wednesday)	Multiplication and Division Word Problems
4	Representing Fractions as Decimals	4 (Thursday)	Representing Fractions as Decimals
5	Division of Fractions Algorithm	5 (Thursday)	Division of Fractions Algorithm and Division Word Problem for Fractions
6	Division Word Problem for Fractions		

Figure 10. Nisa's instructional emphasis and sequencing of topics in the division of rational numbers units.

Nisa followed the first four lesson plans very carefully. However, she taught division of fractions algorithm and division word problems on the same day. For the most part, the examples presented in the observed lessons were the same as those presented in the lesson plans.

Nisa worked very closely with her mentor who was an experienced mathematics teacher for more than 10 years. They discussed writing her lesson plans. Nisa handed in the lesson plans to her mentor before she taught each lesson. Her mentor made some comments about the correctness of the content, sequencing, examples, and activities and gave them back to her before class, sometimes after class. Nisa corrected them and sent them back to him one more time. Nisa mentioned that her mentor suggested she give the students several practice problems besides the problems in the textbook. Each classroom section would receive different problems and less knowledgeable students would obtain easier practice problems than the more adept students.

Nisa's mathematics supervisor observed her lessons twice. The topics were on comparing fractions and representing fractions as decimals. Nisa and her mathematics supervisor discussed her written lesson plans, instructional materials, and question-answer techniques both before and after classes.

Day 1: Division of a Decimal by a Natural Number

Nisa told the researcher before the class that this lesson related to the previous lesson, multiplication of decimals. She mentioned that after getting the quotient, the multiplication operation would be used to check the accuracy of the quotient.

Nisa began the lesson by reviewing of division of a natural number by a natural number where the quotient was a natural number. She asked the students to find the answers to division problems, "24 ÷ 3" and "55 ÷ 5." She did not write the problems on the chalkboard. Instead of using the statement, "24 divided by 3", she said, "24 divided into 3." Her language might have confused some students. However, the correct answers were offered by a few students who sat in the front section of the class.

Nisa wrote a division problem, 27.75 ÷ 25, on the chalkboard and had the students identify the dividend. This problem was presented in her lesson plan. A few students replied that 27.75 was the dividend. Nisa then asked, "What is '25' called?" A few students answered, "The divisor." Nisa wrote on the chalkboard, "$25\overline{)27.75}$." While she was writing, a few students called out, "1.15," "1.11." A student called out, "The answer is 1.11." Nisa did not pay attention to those students. The students at the back of the room did not pay attention to the lesson. Nisa and the class performed the long division algorithm together. She did not remind the students which number went into the bracket before performing a long division.

When Nisa and the students had to place the decimal point in the quotient, Nisa asked, "Where should we place the decimal point?" A student replied that, "Place it directly above the decimal point in the dividend." Nisa accepted his response and reminded the students to place the decimal point in the quotient directly above the decimal point in the dividend. With the help of a few students, Nisa completed the division, getting the quotient of 1.11. She told the students that they had to check the accuracy of the quotient. She said that they could check the answer by finding the product of 25 and 1.11. Without performing multiplication, Nisa wrote on the chalkboard, "$25 \times 1.11 = 27.75$." Nisa summarized that the decimal point in the quotient had to be placed directly above the decimal point in the dividend.

Nisa wrote another example, $35.01 \div 18$, on the chalkboard. This problem required the students to add a zero as a place holder in the dividend. It was selected from her lesson plan. Nisa called on a student to identify the dividend and the divisor of this problem. He gave the correct responses. Nisa wrote $18\overline{)35.01}$ on the chalkboard. Nisa and the students carried out the long division method together. When the division did not end, Nisa asked a student what the remainder was. The students answered correctly. Nisa responded, "The remainder is 9. Can 9 be divided by 18?" A few students answered, "No." Nisa then asked the students what they would do. A few students replied, "Add zero." Nisa further probed, "Add zero. Where will you add zero?" A few students answered chorally, "In the dividend." Nisa further asked, "Does putting a zero at the end of the dividend change its value?" A student replied, "No." Nisa concluded, "The value of the dividend does not change. For example, 35.010 and 35.01 are equal. Thus, putting a zero at the end of the dividend does not change its value." Nisa and the students completed the division, getting the quotient of 1.945. Nisa said that the decimal point in the new dividend could be omitted while performing the division. The students had to leave one space for the decimal point. Nisa did not mention checking the accuracy of the quotient as she did with the previous problem.

Nisa then wrote another division problem, 51.75 ÷ 15, on the chalkboard. This problem was selected from her lesson plan. She asked for a volunteer student to put the work on the chalkboard. Several students had hands raised. Nisa called on a volunteer student to go to the chalkboard to perform the long division algorithm. She wrote 51.75 ÷ 15 as $15\overline{)51.75}$ for the student. As this student was writing his work on the chalkboard, Nisa had other students write the examples in their notebooks. She stood in front of the class observing the student's work reminding him to place the decimal point in the quotient right above the decimal point in the dividend. During this time, a student at the back row of the room asked Nisa why she added a zero in the previous problem. After the student finished his work, Nisa went back to the second example and said to the class, "9 cannot be divided by 18, thus we add zero to the dividend. Then we bring down zero to make 9 as 90. Thus, 18 can divide into 90." Nisa then asked the student whether he understood or not. The student nodded his head.

Nisa then turned the class's attention to the last problem and asked whether or not the problem was correct. Some students thought it was correct, others did not. Without asking the students to explain why they thought so, Nisa simply concluded that it was correct without giving any explanation. Nisa did not have the students check the solution by using the relationship between multiplication and division.

Nisa asked the students to summarize how to divide a decimal by a natural number. She said, "How can we divide a decimal by a natural number?" A student replied, "Perform long division." Nisa responded, "You can see that we divide as if dividing a natural number by a natural number. What is the difference?" A student responded, "The decimal point." Nisa responded, "The difference is a decimal point. Where do we place the decimal point?" A student answered, "Directly above." Nisa then called on this student to answer loudly. The student stood up and answered, "Place directly above." Nisa further probed, "Can you give more explanation?" No response from this student. Nisa said, "You have to explain clearly. We have to place the decimal point in the quotient directly above the decimal point in the dividend." Nisa and the students summarized the rules of dividing a decimal by a natural number verbally. Nisa, however, did not write the rules in any written form.

Nisa then presented the students with a division problem, 0.75 divided by 5. This problem was not in the lesson plan. Nisa and the students then proceeded to divide 0.75 by 5, getting an answer of 0.15. For the last three problems, Nisa did not remind the students to verify their answers by using the relationship between multiplication and division. Moreover, there were no examples of division problems that required the students to add zeros as place holders in the quotient. Nisa then gave the students two minutes to copy four problems in their notebooks. As the students were taking notes, Nisa remained at her desk. She did not circulate to see how they were doing.

Twenty minutes before the end of the class, Nisa wrote six problems on the chalkboard, five division problems with one-digit divisors, and one division problem with a two-digit divisor. Two problems required students to add zeros as place holders in the dividend. One problem required the students to add zero as a place holder in the quotient. These problems were selected from the lesson plan. As students worked, Nisa sometimes walked around the class, answered students' questions. Mostly, she was at her desk. A few students went to ask some questions. At the end of the class, Nisa asked students to summarize the rules of dividing a decimal by a natural number one more time. When students could not summarize the rules, Nisa concluded them herself. Several students handed in the work. But, after the class, Nisa told the researcher that some students were confused about adding zero to the dividend. She mentioned that she had reminded the students that putting a zero at the end of the decimal number did not change its value. From the researcher's perspective, Nisa was correct that some students would not understand. Nisa also stated that they forgot to put the decimal point in the quotient.

Day 2: Division of a Decimal by a Decimal

On the second day Nisa began the lesson by reviewing dividing a decimal by a natural number, $0.063 \div 7$, and $16.95 \div 5$. These problems were presented in the lesson plan. To introduce a division of a decimal by a decimal, Nisa wrote $5.46 \div 2.6$ on the chalkboard and asked the students how to solve this problem. No one responded. Nisa converted the problem to a fraction form, $\frac{5.46}{2.6}$ and asked the students how to make a divisor a natural number. A few students answered, "Multiply the divisor by 10." Nisa asked further, "What is the product of 2.6×10?" Several students responded chorally, "26." Nisa further probed, "Can we multiply only the divisor by 10?" A few students replied, "No." Nisa asked, "What would you do?" A few students replied, "Multiply the dividend by 10." Nisa accepted their response and wrote on the chalkboard, $\frac{5.46 \times 10}{2.6 \times 10}$. Nisa then asked the students to find the product of 5.46×10. A few students quickly replied, "54.6." Nisa asked the students whether they could find the quotient of this problem. A few students responded, "Yes." Nisa summarized that after making the divisor a natural number, the students could do the division as it was a division of natural numbers. Nisa then asked the students to carry out the division of 54.6 by 26. A few students gave 2.1 as the answer. Nisa accepted that answer as correct without giving any explanation or asking the students to check the accuracy of the quotient.

Nisa wrote another division problem on the chalkboard, $34.309 \div 0.22$. This example was included in the lesson plan. Nisa taught this problem as in the first example. A few students identified that the numerator and the denominator had to be multiplied by 100. Nisa and the students then multiplied 34.309 and 0.22 by 100, getting the result of $\frac{3430.9}{22}$. Nisa had the students carry out the long division algorithm. While students worked, Nisa stood in front of the room. She did not walk around the room to see how they were doing. A student asked, "How did you get 22?" Nisa responded to the whole class,

> A decimal can be made a natural number by multiplying it by 10, or 100, or 1,000. We will consider at the number of the decimal places in the divisor. If the divisor has one decimal place, then we will multiply it by 10. If it has two decimal places, we will multiply it by 100. If it has three decimal places, we will multiply it by 1,000. From this example, the divisor has two decimal places, thus we multiplied the divisor by 100. Now the divisor is a natural number, divide as if divided by a natural number.

However, Nisa did not remind the students that they had to multiply both the divisor and the dividend by the power of 10.

A student said, "The quotient is 155.95." Nisa replied, "How about other students, did you get it?" A few students said, "I got it." Nisa further asked, "How much did you get" A few students answered, "155.95." Nisa did not check whether other students got the same answer. She simply accepted the students' response as correct without checking the accuracy. Nisa gave the students some time to take notes. She sometimes walked around to see how they were doing.

As the students were taking notes, Nisa wrote a division problem where the divisor had three decimals places on the chalkboard, $1.36 \div 0.064$, a problem that requires the students to add zeros as place holders in the dividend. It was selected from the lesson plan. Nisa said, "In dividing a decimal by a decimal, we multiply the divisor by 10, 100, or 1,000 to make the divisor a natural number. Are there any rapid ways to make the divisor a natural number?" A student replied, "Move the decimal point." Nisa said to the class that he would use the "moving the decimal points" method and asked him how he moved the decimal point. The student answered, "Move the decimal point in the dividend." Nisa responded that his method was not quite right. Nisa then asked another student how he would do. A student replied that he would move the decimal point in the divisor. Nisa further probed, "How do you move it? What are the rules in moving it?" No one responded.

A student said that he got the answer. Nisa then asked him how he got the answer. The student said that he used the same method as the examples 1 and 2. Nisa responded that she needed them to use the "moving the decimal points" strategy. A student stood up and said, "If the divisor is a three-place decimal, move the decimal point three places. Nisa confirmed, "Move the decimal point three places and also move the decimal point in the dividend." She further mentioned, "In dividing a decimal by a decimal, we move the decimal point to make the divisor a natural number." However, Nisa did not write the rules on the chalkboard.

Nisa then asked the students to identify the direction they had to move the decimal point. A student responded that he would move it to the left. Nisa corrected him by saying, "We have to move it to the right. There is no rule for how many places we have to move. But in moving the decimal point, we will move it until the divisor is a natural number." Nisa reminded the students that they had to move the divisor and the dividend the same number of places. Nisa did not point out the relationship between moving the decimal points and multiplying by the power of 10. Nisa's language was not clear. Nisa sometimes said "move the divisor" instead of using "move the decimal point in the divisor."

Nisa then turned the students' attention to a division problem, $1.36 \div 0.064$. She asked the students to make the divisor a natural number using the "moving the decimal points" step. When they had to move the decimal point in the divisor, 0.064, Nisa said, "We had to move the decimal point three places. Now we have the divisor a natural number, 64." Nisa then asked the students how many places they had to move the dividend. No one responded. Nisa answered the question herself. She said, "Three places. The second move we got 136. We have to add a zero. How much will we get?" A student answered, "1360." However, Nisa did not explain or ask any students to give an explanation why they had to add a zero. Nisa then asked whether or not the dividend was a natural number. Several students agreed that it was a natural number. Nisa further mentioned that when the dividend was a natural number, they needed to divide as if it was a decimal divided by a natural number. She then asked for volunteer students to solve this problem on the chalkboard. A few students' hands went up to volunteer. Nisa then called on two volunteers to show their work on the chalkboard. After the students finished their work, Nisa asked other students whose answer was correct. Some students thought they were both correct, other did not. Nisa commented that both answers were correct, but the first student's work was not complete. However, she did not discuss with the students which part of the work was missing.

Fifteen minutes before the class ended, Nisa wrote four division problems where the divisors were decimals on the chalkboard. The quotients of these problems were terminating decimals. Those problems were selected from the lesson plan. As the students worked, Nisa rarely walked around the class. Mostly, she stood in front of the class. A group of students asked Nisa to explain how to make the divisor a natural number. Nisa spent eight minutes with these students. Nisa found that some students still did not understand why they had to multiply the divisor by 10, 100, and 1,000. She then asked the students to give the reason. A student replied, "To make the divisor a natural number." Nisa then concluded that they had to make the divisor a natural number and then perform the division of a decimal by a natural number. Nisa asked the students whether they understood. A few students responded that they understood. A few students finished the assignment in class. Unfortunately, no division problem that required the students to add zeros as place holders in the quotient was given to the students. Placing zeros in the quotient might be troublesome for some students. Nisa might not have known whether some students would have this type of difficulty.

Nisa followed her lesson plan exactly, using all the examples and practice problems. Her mentor observed her class from the back of the room. He did not comment during the lesson.

After the class Nisa told the researcher that the students still had an error on the multiplication of decimals. She said that they had to use multiplication to make the divisor a natural number by multiplying it with 10 or 100. She further mentioned that some students correctly multiplied the divisor but had an error when they multiplied with the dividend. Thus, for other sections, she would present the examples from easy to hard. She would present the examples where the divisor was a one-place decimal, two-place decimal, and three-place decimal. She would emphasize that the students had to multiply both the dividend and the divisor. Concerning the "moving the decimal points" strategy, Nisa said that she would tell the students to move the decimal points to the right.

Day 3: Multiplication and Division Word Problems

Nisa told the researcher before the class that the students were confused about multiplication and division word problems. They were not able to identify whether the given problem was a multiplication word problem or division word problem. She said that she would point out that if it was a division word problem, it would contain the statement, "The whole thing is partitioned into equally small parts." Thus they would use division operation. She mentioned that multiplication meant increasing.

On the third day, Nisa began the lesson with a review of multiplication of decimals. She asked the students, "If the multiplicand is a 3-place decimal and the multiplier is a 2-place decimal, what is the product? A student answered, "The product is a 5-place decimal. Nisa accepted his answer as correct. Nisa and the students then discussed the benefits of solving multiplication and division word problems for decimals. A student said that one benefit was use in everyday life. Nisa responded, "In our daily life, we pay 10 baht a day for bus fare. How much money do we pay each time?" A student replied, "3.50 baht." Nisa further asked, "Do you know how much money we pay in a month?" Two students gave different answers, including, "More than 100 baht," and "220 baht." Nisa responded, "220 baht. Do you use multiplication or division in finding this answer?" Two responses were given, including, "Multiplication," and "Addition." Nisa responded, "Addition?" A few students answered chorally, "Multiplication." She accepted the answer as correct and concluded, "We will use multiplication. This is the advantage of solving multiplication and division word problems for decimals." However, she did not mentioned why they had to use multiplication instead of addition.

Nisa posted a multiplication word problem on the left side of the chalkboard, "Suchart walks 14.78 kilometers a day. If he walks nine and a half days, how many kilometers can he walk?" Instead of asking the students to read the problem, Nisa read it herself and asked the students whether they would use multiplication or division operation in solving the word problem. A few students answered, "Multiplication." Nisa then asked the students why they would use the multiplication operation. A student answered that it was easy. Nisa called on another student to answer. A student stood up and said, "Multiply 14.78 kilometers by nine and a half days." Nisa added, "We will perform a multiplication operation. Do you notice how many kilometers Suchart can walk in a day?" A few students replied, "14.78 kilometers." Nisa said, "Suchart walks 14.78 kilometer a day. If he walks two days, how many kilometers can he walk? When students did not respond immediately to her question, Nisa answered it herself. She said, "We have to add another 14.78 kilometers. He walks nine and a half days. How can we express nine and a half days in a decimal form?" She asked the students at the back of the room to answer her question. A student replied, "9.5." Nisa accepted his answer and said, "9.5 days. He walks 14.78 kilometers a day. He walks nine and a half days or 9.5 days. What question is being asked?" A few students answered, "How many kilometers can he walk?" Nisa responded, "We want to know how many kilometers he can walk in 9.5 days. What numbers do we have to multiply together?" A student responded, "9.5 multiplied by 14.78." Nisa had the students carry out the multiplication of 14.78 times 9.5. A student called out, "The answer is 19.810." Nisa did not respond to his answer; instead, she called on two students to give the answer. They gave different answers. Nisa showed the answer written on the chart to the students. She reminded the student who gave the wrong answer to check his computation. Nisa then gave the students some time to take notes and said that the students had to hand in their notebooks at the end of the class period. Nisa sometimes walked around the room to see what the students were doing.

While the students were taking notes, Nisa posted a division word problem on the chalkboard, "A 3.6 meter long cloth costs 173.80 baht. How many baht per meter does this cloth cost? Instead of asking the students to read the problem, Nisa read the problem out loud and asked the students whether they would use a multiplication or division operation in solving this problem. A few students responded, "Division." Nisa called on a student to answer her question. A student replied that he would use a division operation. Nisa asked the student why he would use a division operation. The student did not respond. Nisa then asked the students to identify the dividend. A few students replied, "The dividend is 172.80."

A student asked why it was a division problem. Nisa replied that it was a division problem because a 3.6 meter long cloth cost 173.80 baht. "You wanted to know the price of this cloth per meter. Thus, you had to divide the total cost by the amount of cloth." However, Nisa's explanation was not different from the given problem. She simply reread the problem, reflecting her low knowledge of subject matter.

Nisa asked the students to identify the dividend one more time. Two responses were given, including, "36," and "172.80." Nisa responded that they had to divide the total money by the length of the cloth, 3.6, to get the cost of the cloth per meter. Nisa did not draw any picture when she gave the explanation. Nisa reminded the students how to put the dividend and the divisor in a long division form. She said, "If 172.80 is the dividend, where do we put it?" A student responded, "In the bracket." Nisa responded, "In the bracket and then divide it by 3.6." While she talked, Nisa wrote the notation, $3.6\overline{)172.80}$, on the chalkboard. Nisa asked, "Do you remember how to divide a decimal by a decimal? What should we do?" A student replied, "Write as a fraction." Nisa responded, "Divide a decimal by a decimal?" Another student answered, "Make it a natural number." Nisa accepted his answer as correct and said, "We have to make the divisor a natural number. How do you make the divisor as a natural number" A student responded, "Multiply by 10." Nisa did not pay attention to his response. Nisa had shown a limited knowledge of the relationship between decimals and fractions. She did not seem to know whether dividing decimals could be performed by using fractions.

Nisa had the students perform the long division algorithm. A few students answered, "48." A student asked Nisa how to make the divisor a natural number. Nisa then asked the class how to make the divisor a natural number. A few students replied, "Multiply by 10." Nisa further asked why they multiplied the divisor by 10." A student responded, "It is a 1-place decimal." Nisa responded, "It is a 1-place decimal. Thus, we multiple it by 10 to make it a natural number. What is the shortcut?" A few students replied, "Move the decimal points." Nisa accepted their response and concluded that an alternative way was the "moving the decimal points" strategy. The students had to move the decimal points in the dividend and the divisor to the right by the same number of places. A few students called out that the answer was 48. Nisa called on two students to give the answer. They gave no responses. Nisa told the class that the answer was 48. At this time, Nisa reminded the students to check the accuracy of the quotient by asking the students, "If we want to check whether or not the cloth costs 48 baht per meter, what numbers do we have to multiply together? A student replied, "48 multiplied by 3.6." Nisa accepted his answer as correct.

Nisa posted a chart of division word problem on the chalkboard, "64.5 kilograms of rice is put into bags holding 1.5 kilograms each. How many bags are filled?" Nisa gave the students some time to solve this problem. While the students worked, she stood in front of the class. A few students called out that the answer was 43 bags. Nisa did not pay attention to their answers; instead, she had two students solve the problem on the chalkboard. Nisa asked the whole class whether the multiplication or division operation would be useful for solving the problem. Several students replied, "Division." However, Nisa did not ask the students to give any explanation of why the division operation would be used. As the students were putting their work on the chalkboard, they wrote the unit of the answer as "kilograms." Nisa commented that in this problem the unit was "bags" because they wanted to know how many bags were filled. Nisa then summarized that the answer was 43 bags. She simply accepted the answer as correct without asking the students to check the accuracy of the quotient. At the end of the class, Nisa passed out the homework assignment. There were two multiplication word problems and one division word problem. These problems were selected from other sources.

From the researcher's perspective, Nisa followed her lesson plan closely. Her mentor did not observe her teaching. After the lesson Nisa said that the class period was reduced to 45 minutes. The students came to class 10 minutes late. The actual time spent on word problems was only 30 minutes. There was no time for practicing. Thus she had to skip one multiplication word problem. However, Nisa thought that the students understood.

Day 4: Representing Fractions as Decimals

Nisa told the researcher before the class on the fourth day that her mentor had suggested she teach this lesson after the students knew that they could convert a fraction to a decimal by dividing the numerator of a fraction by its denominator. Nisa could show the students that when the denominator had a large value, dividing the numerator by the denominator might be very slow. He suggested that she make the denominator 9, or 99, or 999 and have the students notice that the quotient would be a repeating decimal. If the denominator of a fraction was 9, then that fraction was a repeating decimal with one repeating digit. If the denominator of a fraction was 99, then that fraction was a repeating decimal with two repeating digits. She mentioned that this topic was difficult for the students. They were not able to perform the division.

Nisa began the lesson by writing a fraction, $\frac{1}{2}$, on the chalkboard and asked the students to convert it to a decimal form. A student answered, "0.5." Nisa then asked the students to explain how they got 0.5. A few students answered immediately, "1 divided by 2." Nisa responded, "1 divided by 2, right? A student replied, "Yes." However, Nisa did not remind the students that $\frac{1}{2}$ is a way of writing the quotient of $1 \div 2$. Nisa wrote another fraction, $\frac{1}{4}$, on the chalkboard and had the students convert it to a decimal. A few students gave the correct answer right away, 0.25. Nisa accepted those answers as correct without comment.

Nisa then had the students convert $\frac{1}{8}$ to a decimal. The students took some time to divide 1 by 8. The students gave several responses, including, "0.45," "0.125," "0.13," and "0.128." Nisa told the students to raise their hands before answering the question. A student raised his hand. Nisa called on that student to give the answer. A student replied, "0.128." Nisa then called on another student to give the answer. This student was not able to give any answer. Nisa then called on another student to answer. The student responded that the quotient was 0.45. Nisa called on several students to give the answer. The answers were varied. Nisa then asked the students for the correct answer. A few students responded that the correct answer was 0.125. Nisa accepted their answer as correct. She explained to the students that a fraction could be written as a decimal by dividing the numerator by the denominator. A student asked, "Can we compare?" Nisa responded, "This fraction? Divide the numerator by the denominator." However, Nisa did not ask this student why he wanted to make the comparison.

Several students were confused. They asked Nisa to perform the division algorithm. Nisa then asked the students, "In $\frac{1}{2}$, what number is the dividend?" A student replied, "1 is the dividend." Nisa responded, "1 is the dividend. In division, where do we put the dividend?" A few students answered, "Inside the bracket." Nisa responded, "Inside the bracket, 2 is the divisor" and wrote "$2\overline{)1}$" on the chalkboard. Nisa and the students then went over this example again using the long division algorithm.

To check students' understanding, Nisa wrote six fractions, $\frac{1}{3}, \frac{2}{3}, \frac{4}{33}, \frac{2}{11}, \frac{7}{15}$, and $\frac{3}{22}$ on the chalkboard and had students in each column write these fractions as decimals by using the division algorithms. Volunteer students put their answers on the chalkboard, including, "$0.333333\dot{3}3\dot{3}$," "$0.6666\dot{6}6\dot{6}$," "0.1818181," "0.121212," "$0.4666\dot{6}6\dot{6}$," and "0.1363636---," However, some students incorrectly wrote the dots over the repeating digits. They might have known about writing the dot(s) over the repeating digit(s) from the textbook, but they misunderstood. Some students did not write three dots after the decimals.

Nisa then asked the students, "What do you notice about the quotient?" A student said, "It has a remainder." Another student said, "It is a repeating decimal." Nisa responded that it is a repeating decimal and asked the students, "What do you notice about the digits?" The students gave two responses, including, "They are the same," and "They repeat." Nisa further probed, "Do you know why the digits repeat in the quotient?' No responses were given. Nisa then asked again why the digits repeat in the quotient. A student replied, "They repeat." Nisa did not have the students perform the long division on the chalkboard. From the researcher's perspective, it was difficult for the students to see whether the remainders were the same.

Nisa turned the students' attention to $\frac{1}{3}$. She said, "From this example, $\frac{1}{3}$, what do you notice?" A student replied, "The digits repeat over and over." Nisa responded, "The digits repeat over and over. What do you notice about the remainder?" A student replied, "They are the same." Nisa said, "The remainders are the same, right? However, a few students who carried out the division of 1 by 3 saw this pattern. Nisa and the students had to divide 1 by 3 using a long division algorithm together. While performing the long division algorithm, Nisa did not place the decimal point after the 1 before adding a zero. Thus, the dividend was 10 instead of 1. From the researcher's perspective, some students might think that the dividend was 10.

After getting the quotient, Nisa said, "What do you notice about the remainder?" A student said that they are equal. Nisa explained that, "The remainders will always be the same numbers. That is why the quotient is a repeating decimal." A student pointed to the number "0.121212," the quotient of 4 divided by 33 and said, "This number does not repeat." Nisa replied, "The digits 1 and 2 repeat." However, Nisa did not give more explanation. She further said, "The digits repeat in the quotient. Do you know what this type of number is called?" A few students replied, "A repeating decimal." Nisa did not check whether the students gave the correct answer at the outset. A student indicated that two quotients were not correct. He went to put the answers, "$\frac{4}{33} = 0.121212$" and "$\frac{2}{11} = 0.181818$" on the chalkboard.

Nisa pointed out that $\frac{1}{2}$ could be written as a decimal as 0.5 and asked the students to identify what they had noticed about the quotient of $\frac{1}{3}$. A few students replied that the quotient continued without end (a non-terminating decimal). The division would never end. Nisa then summarized that a decimal with digits that repeat was called a repeating decimal and discussed this idea with the students.

Nisa said, "How can you write $\frac{1}{3}$ as an equivalent fraction with a denominator of 9? By what number must it be multiplied?" A few students replied, "Multiply both the denominator and the numerator by 3." Nisa responded, "Multiple both the denominator and the numerator by 3. Thus, the fraction is?" A few students answered, "3 over 9."

Nisa pointed to $\frac{2}{3}$ on the chalkboard and said, "How about this fraction?" A student replied, "Multiply it by 3." Nisa then asked the students to identify the fraction. A student responded, "6 over 9. Nisa then asked, "What do you notice about the decimal form of $\frac{1}{3}$?" When the students did not respond immediately to her question, Nisa answered it herself. She said, "0.333. If the denominator is 9, what do you notice about the numerator?" A student responded, "Increases." Nisa further probed, "How? What is the relationship between the numerator and the quotient?" A student replied, "They are equal." Nisa said, "They are the same. What about this fraction?" and pointed to $\frac{2}{3}$. A student said, "They are equal." Nisa then called on a student to explain the pattern. However, the student could not conclude it. Nisa then pointed to $\frac{4}{33}$ and asked, "How about this fraction? A student responded, "Multiply it by 3 over 3." Nisa said, "Multiply it by 3 over 3. What is the result?" A few students answered, "12 over 99." Nisa further asked, "Do you see the relationship?" While she talked, Nisa pointed to "$\frac{4}{33} = 0.121212$" and "$\frac{4}{33} \times \frac{3}{3} = \frac{12}{99}$" on the chalkboard. However, no responses were given.

Nisa pointed to "$\frac{3}{9}$" on the chalkboard and asked, "If the denominator is 9, what is the numerator?" A student replied. "3." Nisa pointed to "$\frac{1}{3} = 0.333$" and asked, "What is the repeating digit?" A few students answered, "3." Nisa repeated that, "The repeating digit is 3. If the denominator is 9, the numerator is?" A student responded, "3." Nisa added that this decimal had a 3 as a repeating digit. Nisa then pointed to $\frac{2}{3} = 0.666$ and asked the students, "How about this?" A student replied, "6." Nisa then went back to "$\frac{4}{33} = 0.121212$" and "$\frac{4}{33} \times \frac{3}{3} = \frac{12}{99}$" and asked, "If the denominator is 99, how many digits repeat in the quotient? No one responded. Nisa asked again, "What do you notice when the denominator is 99?" A student responded, "Several digits." Nisa then asked, "How many digits?" Several responses were given, including, "99," "3," "11," and "4 multiplied by 3 is equal to 12."

Nisa encouraged the students to find fractions that were equivalent to repeating decimals that had one repeating digit, two repeating digits, three repeating digits, etc. However, the students were unable to see the relationship. Nisa had to summarize for them. Nisa then went back to $\frac{2}{11}$ and asked the students what they would do. A student said, "Multiply by 9 over 9." Nisa further asked, "When the denominator is 99, how many digits repeat?" Two responses were given, including, "99 digits," and "2 digits." Nisa accepted that there were 2 digits repeated and asked the students to identify those digits. A few students identified that the digits 1 and 8 repeated.

A student asked how to write $\frac{3}{22}$ as a fraction with the denominator of 99. Nisa replied that some fractions could be written as fractions with the denominators of 9, or 99, or 999, but some fractions could not. For this fraction they could not change the denominator to 9, or 99 because there was no number that multiplied 22 and the product was 99 or 9. In this case, you had to perform the division. Nisa then continued the discussion. Nisa said, "We know how to change a fraction to a decimal. Do you know how to write and read it?" No one responded. Nisa then asked, "The quotient of 1 divided by 3 is 0.33. What is the repeating digit?" A few students responded, "3." Nisa further asked, "How many repeating digits are there in this decimal?" A student replied, "Several." Nisa responded, "One digit. The digit 3 repeats over and over without end. Do you know how to write it?" A student replied that he knew. He went to the chalkboard and wrote "$0.3\dot{3}\dot{3}$." Nisa asked the students whether or not it was correct. Some students thought it was correct, other did not. Nisa then asked a student why it was not correct. That student said that there was one digit repeated. Nisa added that in this decimal the digit 3 repeated over and over without end. Thus, they would write a dot over the digit 3. Nisa also told the students how to read the repeating decimal. Nisa then had the students read them chorally. Nisa went over other examples by calling on an individual student to tell how to write each repeating decimal in a short form. However, she did not emphasize another common way of writing a repeating decimal, writing 3 dots after the decimal. She did not correct the students when they wrote a fraction with a repeating decimal form as a terminating decimal, including, "$\frac{4}{33} = 0.181818$," "$\frac{2}{11} = 0.181818$," "$\frac{7}{15} = 0.4666$," and "$\frac{3}{22} = 0.1363636$." She accepted that they were correct. At the end of the class, Nisa passed out an assignment on representing fractions by decimals and reminded the students to hand in this assignment the next morning.

After the class period, Nisa mentioned that most students understood the lesson. She mentioned that some students were confused between the dividend and the divisor. They mixed up the order of the divisor and the dividend in the $a \div b$ form and the $b\overline{)a}$. Some students thought that they were not able to divide 1 by 2. They thought that the dividend had to be greater than the divisor.

On this day, Nisa's mentor observed her teaching. After the lesson, her mentor commented that she should divide the students into groups of four or eight and then have the students write the answers on the chalkboard. He also commented that she spent too much time working on the problems. Nisa's mathematics supervisor observed her class as well. Before the class he and Nisa discussed her lesson plan.

Day 5: Division of Fractions

Nisa began the lesson with a review on multiplication of fractions. She had the students carry out the multiplication of $\frac{2}{3} \times \frac{5}{6}$. Several students gave the correct answer. To introduce a reciprocal of a fraction, Nisa asked the students, "By what number must $\frac{4}{9}$ be multiplied to make a product of 1?" A few students answered, "$\frac{9}{4}$." Nisa accepted their answer as correct and wrote "$\frac{4}{9} \times \frac{9}{4} = 1$" on the chalkboard. Nisa asked further, "By what number must $\frac{12}{49}$ be multiplied to make a product of 1?" A few students responded, "$\frac{49}{12}$." Nisa accepted the response and wrote "$\frac{12}{49} \times \frac{49}{12} = 1$" on the chalkboard. Nisa further probed, "What do you notice about these two fractions?" A few students replied, "They are reciprocal." Nisa concluded that they were reciprocal. However, Nisa did not write the definition of a reciprocal number on the chalkboard.

Nisa wrote "$6 \div 4$" on the chalkboard, and asked the students to write it in a fraction form. The students gave two responses, including, "6 over 4," and "4 over 6." Nisa asked the students to identify the dividend of the problem. Again students gave two responses, including, "4," and "6." Nisa further asked the students whether 6 was the dividend or the divisor. The students gave two answers, including, "The dividend," and "The divisor." Nisa corrected the students' error by explaining that if the division problem was written in the form $6 \div 4$, the first number was the dividend. A few students added that 4 is the divisor. Nisa then asked the students to write it as a fraction. At this time, the students replied correctly that $\frac{6}{4}$ could be written in place of $6 \div 4$. During this discussion, Nisa wrote on the chalkboard.

$6 \div 4$ can be written in a fraction form as $\frac{6}{4}$. $\frac{\text{Dividend}}{\text{Divisor}}$

Nisa asked the class to identify the dividend and the divisor of another division problem, $\frac{5}{6} \div \frac{2}{3}$. A few students responded that $\frac{5}{6}$ was the dividend and $\frac{2}{3}$ was the divisor. Nisa asked the whole class to write the problem as a fraction. A few students answered, "$\frac{5}{6}$ over $\frac{2}{3}$." Nisa wrote on the chalkboard,

$$\frac{5}{6} \div \frac{2}{3} \text{ can be written in a fraction form as } \frac{\frac{5}{6}}{\frac{2}{3}}.$$

Nisa asked the students to name this kind of fraction. A student correctly named it as a complex fraction. Nisa reminded the whole class that the dividend was $\frac{5}{6}$ and the divisor was $\frac{2}{3}$.

Nisa and the students then discussed dividing fractions. Some students said that it was difficult to divide $\frac{5}{6}$ by $\frac{2}{3}$. Some said it was easy. They indicated that they only needed to change the numerator to the denominator. Nisa further asked them to identify why they had to change the numerator to the denominator. A student replied, "To do multiplication." Nisa explained that if they divided as a fraction, it would be difficult. Thus, they should make the divisor or the denominator equal to 1. Nisa then pointed at $\frac{2}{3}$ on the chalkboard and asked the students to identify by what number $\frac{2}{3}$ must be multiplied to make the product was

1. Several students gave the correct answer, $\frac{3}{2}$. Nisa summarized that they needed to multiply it by $\frac{3}{2}$ to make the divisor equal to 1. They had to multiply both the numerator and the denominator, thus they would multiply the numerator by $\frac{3}{2}$. Nisa then asked the students to find the product of $\frac{2}{3}$ and $\frac{3}{2}$. Students gave a variety of responses, including, "1," "6 over 6 is 1," and "6 over 1." Nisa responded, "6 over 6 or 1." She further explained that they multiplied by the reciprocal to make the denominator equal to 1 and the numerator was $\frac{5}{6} \times \frac{3}{2}$. When the divisor was 1, it was easy to compute the quotient. Nisa asked, "Don't we write the one?" A student said, "We do not have to write the one." Nisa concluded, "Thus, we have $\frac{5}{6}$ multiplied by $\frac{3}{2}$ left. What is their product?" No one responded. Nisa then asked again, "What is it? Can we reduce it?" A student replied, "$\frac{5}{4}$." Nisa asked him how he reduced it. The student responded, "Divide 3 by 6." Nisa asked the student, "What is the result?" A student replied, "$\frac{5}{4}$." Nisa then asked the class to write $\frac{5}{4}$ as a mixed number. A student answered, "$1\frac{1}{4}$." The researcher observed that a few students answered Nisa's questions. Nisa did not provide more explanation as to why they could omit writing 1. She did not remind the students how to cancel the common factor nor to change an improper fraction to a mixed number.

Nisa presented the students with another division problem where the dividend was a mixed number and the divisor was a natural number, $4\frac{9}{10} \div 7$. Without asking the students how to solve this problem, Nisa told the class to change $4\frac{9}{10}$ to an improper fraction. Several students said, "$\frac{49}{10}$." Nisa then said, "$\frac{49}{10}$ divided by 7. How do you write it as a fraction?" A student called out, "Multiplied by $\frac{1}{7}$. The result is $\frac{7}{10}$." Nisa did not pay attention to his answer. Instead, she asked the class to write that problem as a fraction, getting the fraction $\frac{\frac{49}{10}}{7}$. A student called out, "Reduce them first." Nisa did not respond to this student. Nisa asked the class to identify the denominator of 7. Several students replied that 1 was the denominator of 7. Nisa further asked the students, "If we want to make the denominator equal to 1, what number do we have to multiply?" A student answered, "7." Nisa asked the class again, "What number?" Several students said, "$\frac{1}{7}$." Nisa explained that 7 multiplied by $\frac{1}{7}$, then cancelled the common factor, resulting of 1. She then asked, "If we multiply the denominator, do we have to multiply the numerator" A few students accepted. Nisa asked again, "Multiply by what number?" Several students answered chorally, "$\frac{1}{7}$." Nisa replied, "Then we got 49 multiplied by $\frac{1}{7}$, can we reduce them?" A few students replied, "Yes, cancel 49 over 7." Nisa then asked, "The answer is?" A few students replied, as a group, "$\frac{7}{10}$." Nisa repeated the students' answer, "$\frac{7}{10}$."

Nisa then asked the class, "From these two examples, can you see what the final answer looked like? A student called out, "A proper fraction." Nisa then asked again, "The final answer, can you see something?" No one responded. Nisa then explained, "You can see from dividing fractions. If we make the denominator equal to 1 by multiplying it by a fraction, that fraction is the reciprocal of the first fraction. Can anyone tell me the shortcut of dividing fractions?" A student replied, "Find the least common factor." Nisa quickly asked, "Dividing fractions." Another student said, "Invert the numerator to the denominator." Nisa further asked, "Who can tell the rapid way to find the quotient?" A student responded, "Invert the numerator to the denominator and the denominator to the numerator." Nisa further probed, "Invert the numerator to the denominator and the denominator to the numerator of what number?" Two responses were given, including "The denominator," and "The divisor." Nisa responded, "The divisor. How about the symbol?" A student responded, "Change the division symbol to the multiplication symbol." Nisa reminded the students to invert the divisor. Nisa then summarized, "From this example, $\frac{5}{6}$ divided by $\frac{2}{3}$. The final step is $\frac{5}{6}$ multiplied by $\frac{3}{2}$. Can you see whether $\frac{2}{3}$ is the reciprocal of $\frac{3}{2}$? The division symbol is changed to the multiplication symbol. This is the rapid way to find the quotient." Nisa did not write any statements on the chalkboard.

Nisa wrote another division problem on the chalkboard, $\frac{8}{9} \div \frac{4}{7}$, and had the students find the quotient using the shortcut derived from the previous example. Nisa gave the students some time to work. A student said, "I got it." Nisa quickly asked, "What is the answer?" A student said, "$1\frac{5}{9}$." Nisa probed, "How did you get this answer?" The student replied, "Invert the denominator to the numerator and the numerator to the denomination, then change the division symbol to the multiplication symbol. I got $\frac{8}{9}$ multiplied by $\frac{7}{4}$. Then cancel common factors." Nisa had that student explain how he had arrived at $\frac{14}{9}$ as an answer. However, she did not remind the students that they could cancel then multiply or multiply then cancel. Nisa told the students to take notes.

Nisa wrote a division word problem on the chalkboard, "A cloth is 18 meters long. One dress uses $2\frac{1}{4}$ meters. How many dresses can be made from the total cloth?" Nisa read the problem aloud and asked, "What operation will you use? Several students answered chorally, "Division." Nisa asked the students to identify the dividend and the divisor of the problem. The student gave correct answers. A student called out, "8 dresses." Nisa did not pay attention to that student; instead, she asked the class to write $2\frac{1}{4}$ as an improper fraction. Several students answered, "$\frac{9}{4}$." Nisa responded, "Thus we got 18 divided by $\frac{9}{4}$. What should we do next? A few students replied that they would multiply 18 by $\frac{4}{9}$. Nisa and the students carried out the multiplication of 18 times $\frac{4}{9}$, getting an answer of 8. Nisa then asked the students to summarize the rules of division fractions. A few students responded to her question. Other students talked to each other. Nisa then assigned the students six problems as the homework.

Nisa followed the presentation introduced in the textbook and her lesson plan. After class she mentioned that the lesson was not as fluently as she expected. The students learned division of fractions using the "invert and then multiply" strategy from their sixth grade mathematics classes. They did not want to know where this strategy originated. They did not pay attention to the lesson. The students came to class 15 minutes late. Nisa was not able to give the students several examples. Moreover, the lesson was in the eighth period. The students were tired and wanted to go home. On this day, her mentor did not observe her teaching.

Impact of Nisa's Knowledge of Subject Matter and of Students' Conceptions on Her Instruction

Prior to instruction, Nisa incorrectly solved division of decimals problems which required her to add zeros as place holders in the dividend and in the quotient. During her lesson, there were a few examples of division of decimals problem that required the students to add zeros as place holders in the quotient. She was not able to identify it as a repeating decimal. During the lesson on representing fractions by repeating decimals, Nisa made some errors. She did not correct the students when they carried out the division incorrectly. She did not correct the students when they did not write the symbols for repeating decimals, writing three dots after the numbers. She used the phrases "divided by" and "divide into" interchangeably throughout the lessons. Also, she incorrectly indicated that a multiplication word problem could not be solved by using addition. Nisa incorrectly said that dividing decimals could not be solved by using fractions.

Prior to instruction, Nisa correctly identified a student's error pattern where that student forgot to record a zero in the quotient when a number brought down was not great enough to be divided. During the lessons on division of decimals, there were a few examples that required the students to add zeros as the place holders in the quotient. During the lesson, Nisa prevented the students' errors in a few topics, including writing the division problem in notational forms, $a \div b$, $\frac{a}{b}$, and $b\overline{)a}$, moving the decimal point in the dividend and the divisor, placing the decimal point in the quotient, and the equivalent of fractions

Nisa planned her lessons according to the prescribed curriculum which was an algorithmically-based curriculum developed by the IPST. She did not deviate from the curriculum. She made some content errors while presenting her lessons. After teaching the lessons, Nisa was able to identify more errors students might make.

Nisa's mentor directly impacted her teaching. Nisa and her mentor discussed writing her lesson plans. She handed in the lesson plans to her mentor before she taught each lesson. Her mentor made some comments about the correctness of the content, sequencing, examples, and activities and gave them back to her. Nisa corrected them and sent them back to him one more time. His suggestions had an impact on Nisa's content knowledge and her teaching style. She did what her mentor suggested.

Comparison of Preservice Teachers

The primary focus of this comparative analysis, seeking similarities and differences, was on the preservice teachers' actions in their lessons using information gleaned from the questionnaire and interview results.

Knowledge of Mathematics and Teaching Rational Number Division

Chai and Lada, the high subject matter knowledge preservice teachers, defined division in two ways: measurement and partition. Sak and Nisa, on the other hand, defined division with only one meaning. Nisa gave the partitive interpretation of division. Sak gave the measurement definition of division.

The high subject matter knowledge preservice teachers seemed to have a broad enough concept of division to make sense of the division of fractions less than one. They were able to write story problems for given expressions. The low subject matter knowledge preservice teachers, on the other hand, were not able to write any story problems, indicating a lack of conceptual understanding of division of fractions. The high subject matter knowledge preservice teachers were able to solve division of decimal problems requiring additional zeros as place holders in either the dividend or the quotient. In contrast, the low subject matter knowledge preservice teachers could not.

All the preservice teachers focused on an algorithmic presentation following an algorithmically-based curriculum developed by the IPST. Although the high subject matter knowledge preservice teachers, Chai and Lada, had strong conceptual understanding, they followed a curriculum that did not emphasize conceptual development. Neither of them taught division of rational numbers using models or manipulatives.

In the preservice teachers' lessons, division of rational numbers was introduced to students separating the operation of division from its context. The preservice teachers used a curriculum that maintained that students' meanings of division developed independently of real-life contexts. That is, division word problems were presented after the students had gained computational proficiency. The preservice teachers presented division word problems simply as a tool to provide students with additional computational experience, following the sequence presented in the textbook. The textbook developed by the IPST was the primary source of information for the preservice teachers when preparing their lessons.

The high subject matter knowledge preservice teachers taught topics beyond the textbook. Chai introduced the shortcut of making the decimal divisor a natural number, the "moving the decimal points" strategy. He also introduced the rule of dividing fractions in symbolic form. Lada taught the representation of fractions as decimals unit beyond the mathematics textbook for Grade 7. She introduced the students to changing repeating decimals to fractions. This topic was presented in the mathematics textbooks for Grades 8 and 9. Thus, her students could see the relationship between fractions and decimals, especially a repeating decimal expressed in a fraction form. Knowing this relationship provided the basic knowledge to identify repeating decimals as rational numbers.

The low subject matter knowledge preservice teachers, Nisa and Sak, relied primarily on the scope and sequence in the IPST textbook. Nisa taught one topic beyond the textbook (moving the decimal points strategy), which required a great deal of time, and she used more time on the topic than necessary. Her mentor and mathematics supervisor also noticed this problem.

All the preservice teachers created their lesson plans following the scope and sequence in the textbook. However, during the lesson, the high subject matter knowledge preservice teachers used multiple examples some of which were in their lesson plans, some of which they created on the spot but were similar to those in their lesson plans. In contrast, the low subject matter knowledge preservice teachers used examples selected from their lesson plans. They rarely created new examples while they were teaching.

Chai's, Sak's, and Nisa's mentors directly impacted their teaching. These preservice teachers discussed writing their lesson plans with their mentors. Lada' mentor did not assist her in preparing lesson plans. However, during their lessons, Chai and Lada, the high subject matter knowledge preservice teachers, did not need much help on content knowledge from their mentors, but their mentors helped them manage the classroom. In contrast, Sak and Nisa, the low subject matter knowledge preservice teachers, needed much more help from their mentors. Sak's mentor often observed his instruction and helped him to correct his teaching. However, when his mentor was absent, Sak made many errors in teaching. Nisa made a few errors in the class. She spent a lot of time in preparing her lessons. Nisa and her mentor discussed writing her lesson plans. Her mentor made some comments about the correctness of the content, sequencing, examples, and activities. Her mentor helped her to correct her teaching after class. Nisa used her mentor's advice for improving her teaching in other sections.

The high subject matter knowledge preservice teachers exhibited more confidence in teaching division of rational numbers than the low subject matter knowledge preservice teachers. During his interview, Chai (high subject matter knowledge) mentioned that he believed that he could teach division of rational numbers and that it was not difficult to teach. Lada (high subject matter knowledge) did not seem nervous when she taught. She did not hesitate to respond to her students' questions.

Sak (low subject matter knowledge) mentioned that he did not have in-depth knowledge in mathematics. He did not understand some concepts and was not sure he could teach his students. When the class performed the division of $11 \div 20$, a student asked why they had to add zeros in the dividend. Sak neither responded to the student's question nor discussed this idea with the class because of his lack of understanding. Nisa (low subject matter knowledge) introduced the students to the "moving the decimal points" steps without helping students see the relationship between multiplying by the power of 10 and the strategy. When solving word problems, a student asked why it was a division problem; Nisa was not able to give an appropriate explanation. Moreover, she was reluctant to respond to her students' suggestions and would wait for a response that matched the approach she wanted to demonstrate to the class. Lack of confidence seemed to be related to Sak's and Nisa's somewhat shaky knowledge of division of rational numbers.

Although the preservice teachers followed the textbook carefully, they demonstrated some difficulties with the subject matter. However, the nature of their errors appeared to differ depending on their knowledge. The low subject matter knowledge preservice teachers made more errors than the high subject matter knowledge preservice teachers. The high knowledge subject matter preservice teachers made careless errors; whereas, the lower subject matter knowledge preservice teachers exhibited incorrect understanding of some concepts. For example, Chai (high subject matter knowledge) did not place the decimal point after the dividend before adding a zero while converting a fraction to a decimal. Lada incorrectly computed a basic multiplication fact.

Sak, on the other hand, incorrectly described rounding decimals. His misconceptions were translated to the students during the lesson. Moreover, Sak did not correct a student when she forgot to write the denominator for an improper fraction. Nisa, a low subject matter knowledge preservice teacher, made statements indicating that she was shaky on more fundamental concepts. She used the phrases "divided by" and "divide into" interchangeably throughout the lessons. She incorrectly indicated that a multiplication word problem could not be solved by using the addition operation. She incorrectly identified that dividing decimals could not be performed by using fractions. Further, Nisa did not place the decimal point after the dividend before adding a zero while carrying out the division problem where the divisor was greater than the dividend.

However, there were differences within the high knowledge of subject matter preservice teachers. On the questionnaire and interview, Chai solved one word problem more conceptually than Lada. However, during the lesson on division word problems, Lada's diagram and picture helped the students understand the problems. Chai did not draw a diagram or a picture. During the lessons on division of fractions, Lada showed the students more conceptually why to invert and multiply. She introduced division of fractions by changing the division problem to complex fraction form and then multiplying the dividend and the divisor by the reciprocal of the divisor. Thus, the division problem was changed to the multiplication problem. The quotient was the product of the dividend and the reciprocal of the divisor. This procedure was presented in the textbook as an explanation of the reason to invert and multiply. On the other hand, Chai did not show the students where the "invert the divisor and multiply" algorithm originated. He taught the students with the rules of dividing fractions at the outset and introduced the rule in symbolic form. His mentor directed him to do it the way that he did do it.

There were similarities and differences between the preservice teachers with low knowledge of subject matter. During the lesson on division of decimals, Sak introduced the students to a making a decimal divisor a natural number by multiplying it by the power of 10. He followed a lesson plan that was based on the textbook. Nisa introduced the students to a making a decimal divisor a natural number both by multiplying it by the power of 10 and by moving the decimal points. These strategies were mentioned in her lesson plan. The "moving the decimal points" strategy was not presented in the textbook. However, Nisa did not point out the relationship between these strategies, perhaps reflecting her low knowledge of subject matter.

Although the curriculum was provided and the preservice teachers created their lesson plans based on this curriculum, all of the preservice teachers either made careless errors or demonstrated incomplete understanding of at least one of the topics that they were teaching. However, the high subject matter knowledge preservice teachers made fewer errors. They were able to provide more explanations, and they had more confidence in teaching the lessons. Thus, the depth of subject matter knowledge appeared to be an important factor for preservice middle school mathematics teachers' teaching in a controlled national curriculum.

Knowledge of Students' Conceptions and Teaching Rational Number Division

Chai and Sak were considered to have high knowledge of students' conceptions, while Lada and Nisa had low knowledge of students' conceptions. All the four preservice teachers took a mathematics methods class. However, the class did not focus on knowledge of students' conceptions or misconceptions. Thus, the preservice teachers did not learn about students' conceptions directly from their methods class. They might gain this type of knowledge from their experience as learners or some discussion during their classes.

Prior to instruction, the high knowledge of students' conceptions preservice teachers were able to identify more common errors and the possible sources of error students might make when they learned division of fractions than the low knowledge of students' conceptions preservice teachers. The high knowledge of students' conceptions preservice teachers identified the most common errors as inverting the dividend instead of the divisor or inverting both the dividend and the divisor before multiplying numerators and denominators.

In planning lessons, all the preservice teachers developed their lesson plans based on the mathematics textbook for Grade 7 developed by the IPST. This textbook did not focus on addressing the misconceptions in the lessons. Although the preservice teachers could identify the students' conceptions, addressing the misconceptions in the lesson plans was not specifically one of their objectives. Therefore, the preservice teachers, for the most part, did not select activities designed to address the misconceptions nor create opportunities to address them. In addition, the questionnaire and interviews provided evidence that the preservice teachers themselves held some of the misconceptions, at least in certain settings. Some preservice teachers included on their list of incorrect students' responses adequate procedures and responses. As a result, the preservice teachers might have been less attuned to recognizing such errors in the students' thinking when they did occur.

During the classroom observations, the preservice teachers seemed to base their division of rational numbers instruction on the assumption that the students had at least a basic understanding of division of rational numbers concepts. They did not teach division of rational number conceptually. The efforts of the preservice teachers to engage students' thinking and deal with students' errors and misconceptions during the division of rational numbers instruction also varied somewhat from preservice teacher to preservice teacher. During the lesson, the high knowledge of students' conceptions preservice teachers often gave examples or assigned the problems that would evoke the misconceptions. For example, during the lessons on division of decimals, Chai and Sak gave the students several examples that required them to add zeros as place holders in either the dividend or the quotient. They discussed this type of problem with their students. Sak also had the students solve the problems on the chalkboard. Thus, if any students had difficulty, Chai and Sak were able to correct them right away. The low knowledge of students' conceptions preservice teachers, Lada and Nisa, gave the students few examples that required them to add zeros as place holders in either the dividend or the quotient. They rarely discussed this type of problem.

During the lesson, the high knowledge of students' conceptions preservice teachers always gave an explanation to help prevent students from making a mistake. For example, Chai reminded the students that to reduce a fraction, they needed to divide the numerator and the denominator by the number that would divide both evenly. The high knowledge of students' conceptions preservice teachers did not ignore or dismiss answers that were the result of well-known misconceptions without commenting on them and coming back to work on the topic repeatedly. For example, when a student said that $\frac{2}{5} = 2.5$, Chai guided the class to see that $\frac{2}{5} = 0.4$. He also commented that $\frac{2}{5}$ was not equal to 2.5. When some students thought that the larger number might not be the dividend, Chai gave them the example, $\frac{6}{8}$, and commented that the dividend was not necessarily greater than the divisor. Chai did not simply give the correct answer. He showed them how he got the right answer.

During the lesson on representing fractions as decimals, Sak presented the simple problem, $\frac{15}{3}$ and asked the students to find the quotient. Thus the students were able to see the relationship between the fraction and decimals. They were able to conclude that a fraction could be written as a decimal by dividing the numerator by the denominator.

During the lessons, the low knowledge of students' conceptions preservice teachers, Lada and Nisa, rarely prevented students' errors and infrequently corrected the students' misconceptions. For example, when they taught division of fractions using the "invert and then multiply" strategy, they did not remind the students that they needed to invert the "divisor." They did not know whether or not the errors students made in division of fractions were created by switching the wrong number. Consequently, the students were unsure about which number to invert.

Although the preservice teachers took one methods class, the class did not focus on knowledge of students' conceptions or misconceptions. Thus, the preservice teachers did not learn about students' conceptions directly from their methods class. However, they had some knowledge of students' conceptions. The high knowledge of students' conceptions preservice teachers were able to identify more common errors and the possible sources students might make when they learned division of fractions than the low knowledge of students' conceptions preservice teachers.

All the preservice teachers taught lessons based on a curriculum that did not focus on addressing misconceptions. However, the high knowledge preservice teachers used their knowledge of students' conceptions throughout the lessons more often than the low knowledge preservice teachers. The strategies the high knowledge preservice teachers used were discussion, presentation of examples that evoke misconceptions, and reminders for the students about possible errors. Thus, the depth of the knowledge of students' conceptions appears as an important understanding for preservice middle school mathematics teachers teaching in a controlled national curriculum as that needed in a voluntary curriculum. Moreover, after the preservice teachers taught some classes on division of rational numbers, their knowledge of students' conceptions was improved. They gained more the nature and the likely sources of related common misconceptions held by students. In addition, some preservice teachers used their knowledge of students' conceptions to prevent the errors when teaching other sections. Thus, student teaching was a practical experience that helped preservice teachers gained this type of knowledge.

CHAPTER 5
DISCUSSION AND CONCLUSIONS

Introduction

The study investigated the impact of Thai preservice middle school mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers with respect to their classroom practices in a teaching environment controlled by a required national curriculum. A question was posted at the beginning of this study: Is the depth of mathematical knowledge and the knowledge of students' conceptions as essential for preservice middle school mathematics teachers teaching in a controlled national curriculum as that needed in a voluntary curriculum?

The study included two phases. Phase 1 focused on the identification of four cases. A questionnaire and an interview were used to identify subject matter knowledge and knowledge of students' conceptions of division of rational numbers of nine preservice teachers. Four preservice teachers, Chai, Sak, Lada, and Nisa, were selected demonstrating different knowledge structures as shown in Figure 11.

		Knowledge of Students' Conceptions	
		High	Low
Subject Matter Knowledge	High	Chai	Lada
	Low	Sak	Nisa

Figure 11. Knowledge structures categorizations of preservice teachers.

Phase 2 consisted of classroom observations of the four preservice teachers. Each preservice teacher was observed three weeks, three days per week (every day the class was taught) for approximately one hour each day during the teaching of units on division of decimals, representing fractions as decimals, and division of fractions. A formal interview of each of the four preservice teachers was conducted prior to their teaching of each unit. Informal interviews were also conducted prior to the teaching of each lesson and after each lesson and each unit. All materials used in the normal teaching of the class were collected. Interviews with the preservice teachers' mentors were conducted before and after each unit. The mentors were also interviewed daily before or after instruction. Interviews with supervisors were conducted every time they supervised the preservice teachers.

Conclusions in response to the research question are drawn from data collected throughout the case studies. In addition to the conclusions and the attending discussion, comments concerning the limitations of the study, recommendations for further research, and implications of this study for mathematics teacher education are addressed.

Preservice Teachers' Knowledge Structures

Division of rational numbers was the specific topic examined in depth in the current study. The preservice teachers' knowledge of rational numbers was particularly surprising to this researcher. They were not able to correctly define a rational number. One preservice teacher incorrectly said that integers and repeating decimals were not rational numbers. Another preservice teacher incorrectly claimed that every real number was a rational number. All preservice teachers were not able to define the Closure Law. They incorrectly identified a set as closed with the operation of division. One preservice teacher incorrectly identified the Commutative Law as the Closure Law. Almost all of them mentioned that they did not remember that law. Interestingly, although the preservice teachers had taken several mathematics courses and had already studied this topic during their own school years, they still lacked some knowledge of division of rational numbers. The results were consistent with several studies (Even & Tirosh, 1995; McDiarmid & Wilson, 1991; Tirosh, 2000), who found that although many of the teacher candidates could produce correct answers for division, several could not, and few were able to give mathematical explanations for the underlying principles and meanings. As mentioned by Tirosh and Even (1995), preservice teachers' subject matter knowledge cannot be assumed to be sufficiently comprehensive and articulated for teaching. Thus, teacher education needs to explicitly consider topics included in the middle or high school curriculum, such as division of rational numbers in order to prepare them for teaching.

All of the preservice teachers correctly identified a multiplication sentence which represented a given division of fractions sentence. Their knowledge of the division of fractions was sufficient to enable them respond correctly to the questions involving computation of division of fractions. They solved each computation using the "invert divisor and multiply" algorithm. A few were able to explain what they were doing. Almost all of them solved division word problems with fractions procedurally. Moreover, some of them were not able to write a story problem representing a given numerical expression involving fractions. The same understanding of division appeared to hold for the preservice teachers in the study by Ball (1990b) who found that preservice teachers understanding of fraction division is rule-based; most of the preservice teachers were able to compute the answer to a fraction division problem but could not go beyond that point to provide a conceptual explanation that would be meaningful to their students.

The high subject matter knowledge preservice teachers seemed to have a broad enough concept of division to make sense of division of fractions less than one. They were able to write a story problem to represent a given expression. The low subject matter knowledge preservice teachers, on the other hand, were not able to write any story problems. The high subject matter knowledge preservice teachers had more conceptual understanding of division of fractions than the low subject matter knowledge preservice teachers. When given division of decimal problems that required the preservice teachers to add zeros as place holders in either the dividend or the quotient, the high subject matter preservice teachers were able to solve it correctly. In contrast, the low subject matter knowledge preservice teachers were not able to correctly solve it.

Prior to instruction, all the preservice teachers were able to identify common errors and the possible sources of errors students might make when they learned division of fractions. However, the high knowledge of students' conceptions preservice teachers were able to identify more possible sources of those errors than the low knowledge of students' conceptions preservice teachers. Although the preservice teachers were able to identify the errors that students might make when they were learning division of rational numbers, some of them were not able to identify the possible sources of those errors. The results were consistent with the studies done in a voluntary curriculum setting. As noted by Ball (1990a), Even and Markovitz (1995), and Even and Tirosh (1995), preservice teachers' abilities to analyze the reasoning behind students' responses were poor.

The preservice teachers themselves held some of the misconceptions, at least in certain settings. Some preservice teachers included on their list of incorrect students' responses adequate procedures and responses. Thus, preservice teachers' knowledge of students' conceptions of division of rational numbers appeared to be related to their subject matter knowledge of division of rational numbers. This result was consistent with the study by Tirosh (2000) who found that prospective teachers' knowledge of children's conceptions of division of fractions was strongly related to prospective teachers' subject matter knowledge of division of fractions.

The teaching phase of the study was designed to provide information about the impact of preservice teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers on their classroom practices in a teaching environment controlled by a required national curriculum. Chai, Lada, Sak, and Nisa took 42 semester hours of mathematics. They had taken 10 semester hours of practicum and field experience in teaching mathematics, including three semester hours of a mathematics methods class, one semester hour of participation and observational study, one semester hour of practicum, and five semester hours of student teaching. They all had the same courses in mathematics and teaching methods.

They taught seventh grade mathematics classes in the public schools in Bangkok, Thailand and used the required curriculum developed by the Institute for the Promotion of Teaching Science and Technology (IPST) as the main textbook in this grade level. These four preservice teachers created their lesson plans, primarily based on the sequencing and the topics presented in the textbook. The strength of the relationships between content emphasis in the textbook and content emphasis in instruction was high. The preservice teachers' sequence of instruction generally paralleled that of the textbook. The preservice teachers within the nonvoluntary curriculum had less freedom in designing and arranging their curriculum and lessons than preservice teachers in a voluntary curriculum. The results of the study by Freeman and Porter (1989) with teachers in voluntary environment indicated that:

> Teachers do not always defer to the authority of their textbooks when deciding (a) what topics to teach, (b) how much time to spend on each topic, or (c) the order in which topics are presented. (Freeman & Porter, 1989, p. 419)

Stodolsky (1989) also found that most mathematics teachers were selective in their use of textbook lessons, problem sets, and topics, although topics not included in the texts, were only occasionally added to the instructional program.

Although all the preservice teachers followed the textbook during the lesson, the high subject matter knowledge preservice teachers used multiple examples some of which were in their lesson plans, some of which they created on the spot. They could make up examples when the students asked questions. In contrast, the low subject matter knowledge preservice teachers used examples selected from their lesson plans. They rarely created new examples while they were teaching. When the students asked questions, they simply ignored the questions or gave explanations that simply repeated the previous explanation. The result was consistent with the study by Borko et al. (1992) who demonstrated that when one lacks subject matter knowledge of mathematics, it can be difficult to implement one's knowledge of mathematics as a discipline. They found that when faced with a student's question about why the "invert and multiply" algorithm works, the student teacher within a voluntary environment attempted to place the problem in the real-world context. Despite having taken several mathematics courses, the student teacher lacked a conceptual understanding of division of fractions and was unable to provide a correct representation for the student.

Moreover, the high subject matter knowledge preservice teachers displayed more confidence in teaching their lessons than the low subject matter knowledge preservice teachers by adding to their explanations. This result was consistent with that obtained by Dobey and Schafer (1984) and Katzman (1996), who noted that within a voluntary curriculum, high subject matter knowledge teachers were more confident than low subject matter knowledge teachers, and the high subject matter knowledge teachers attributed the confidence to their higher knowledge. Thus, the depth of subject matter knowledge appeared to be as important a factor for these preservice middle school mathematics teachers' teaching in a controlled national curriculum as that needed in a voluntary curriculum.

Teaching mathematics was a complicated enterprise, and mathematics content knowledge was only one of many factors. There were other factors that affected the preservice teachers' teaching within a nonvoluntary environment. All the preservice teachers focused on an algorithmic presentation following an algorithmically-based curriculum developed by the IPST. Although the high subject matter knowledge preservice teachers had a strong conceptual understanding, they followed a curriculum that did not emphasize conceptual development. None of the preservice teachers taught division of rational numbers using models or manipulatives. However, within the voluntary curriculum, the study by Freeman and Porter (1989) found that the teachers who followed their textbooks most closely were the teachers who placed the most emphasis on applications and conceptual understanding. The teachers who deviated most from their textbooks did so to augment an already heavy emphasis on drill and practice of computational skills.

In the current study, the preservice teachers relied almost exclusively on teacher-directed instruction as a teaching strategy, giving step-by-step instructions to students as they completed their lessons. They did not use calculators to reduce the tedious work. Thus it is possible that the focus of the curriculum affected the preservice teachers' presentations. Within a voluntary curriculum, the National Council of Teachers of Mathematics (NCTM, 1989) suggested that calculators for middle grades, along with other technology, should be available to "free students from tedious computations and allow them to concentrate on problem solving and other important content."

As mentioned earlier, these preservice teachers took several mathematics courses. However, the teaching and learning of those mathematics courses was limited to a traditional didactic pattern; the preservice teachers were treated as passive recipients of knowledge presented primarily through lecture, textbooks, and demonstrations with a focus on procedural fluency. McDiarmid, Ball and Anderson (1989) stated that teachers tend to teach mathematics as they were taught. Their understanding of content and pedagogy is powerfully influenced by their own experiences as students. The preservice teachers were taught procedurally while they were students. Thus, it is likely that these preservice teachers taught procedurally based on their experience as learners.

Although the curriculum was provided, all of the preservice teachers in this environment either made careless errors or demonstrated incomplete understanding of at least one of the topics that they were teaching, a result also noted by Katzman (1996) in her study of mathematics inservice teachers within a voluntary curriculum. Katzman noted that all of the teachers, irregardless of high or low content knowledge, made mathematical errors in their lessons.

In the current study, the nature of the errors between high and low subject matter knowledge preservice teachers appeared to differ. The high subject matter knowledge preservice teachers made careless errors, while the low subject matter knowledge preservice teachers gave false explanations to their students. This result appeared to hold for the inservice mathematics teachers who were in a voluntary environment in the study by Katzman (1996). Katzman indicated that a lack of mathematical knowledge, and subsequent errors exhibited in teaching, may be a fairly common classroom occurrence, although the nature of the errors may well differ depending on the knowledge level of the teachers.

In the current study, the low subject matter knowledge preservice teachers' misconceptions were translated to the students during the lessons. This result was consistent with the study of Babbitt and Van Vactor (1993), who found that within a voluntary curriculum, teachers who had deficiencies in mathematics subject matter knowledge were likely to pass their misconceptions and misunderstandings on to the students they taught. The language used by the low subject matter knowledge preservice teachers confused some students. They used the phrases "divided by" and "divide into" interchangeably throughout the lessons. Bell (1982) found that specific difficulties with language occurred in division, where pupils were often unsure if "divided into" and "divided by" meant the same thing or not.

As expected with student teachers, the four preservice teachers made a few errors when they taught. They focused on an algorithmic presentation directed by an algorithmically-based curriculum. They followed the curriculum step-by-step. The preservice teachers were consistent in teaching because of the procedural nature of the curriculum. It is possible that a nonvoluntary curriculum covers up more the errors of preservice teachers than a voluntary curriculum. Thus, type of curriculum was another factor that influenced preservice teacher's teaching.

The preservice teachers followed their mentors' suggestions as well. Chai's, Sak's, and Nisa's mentors directly impacted their teaching. The preservice teachers discussed writing their lesson plans with their mentors. Their mentors often observed their instruction. However, during their lessons, the high subject matter knowledge preservice teachers, Chai and Lada, did not need much help on content knowledge from their mentors. Chai's mentor observed every class he taught. Whenever Chai had finished instruction and if his mentor felt that the students were confused, she often summarized the topics for the students. Lada's mentor rarely observed her class. She made a few careless errors, and she acknowledged her uncertainty. In contrast, the low subject matter knowledge preservice teachers, Sak and Nisa, needed much more help from their mentors. Sak's mentor often observed his instruction. When Sak taught incorrect content or skipped some steps, his mentor told Sak privately and helped him to correct the teaching. When his mentor was absent, Sak had trouble, making some errors in teaching that he did not acknowledge. Nisa discussed writing her lesson plans with her mentor. She gave her lesson plans to her mentor before she taught each lesson. Her mentor made some comments about the correctness of the content, sequencing, examples, and activities and gave them back to her before class. Nisa corrected the plans and returned them to him for additional consideration. After the lessons, he commented on her teaching one more time by discussing the lesson with her after class. Nisa's weak subject matter knowledge was overcome by the help of her mentor. She made a few errors when she taught. If the mentors did not spend a lot of time helping, the low subject matter knowledge preservice teachers tended to have more trouble in teaching than the high subject matter knowledge preservice teachers. Thus, within this nonvoluntary curriculum, the mentor was one of the factors that influenced the preservice teachers' teaching. In contrast, the findings from the Teacher Education and Learning to Teach (TELT) study indicated that while mentors may help beginners with the emotional adjustments to teaching, and may reduce attrition among first-year- teachers, the availability of mentors by itself does not guarantee that teachers will learn better teaching than they would have learned without the mentors (Kennedy, 1991). The study by Martin (1998) also suggested that subject-matter mentoring within a voluntary environment did not straightforwardly lead to better teaching mathematics.

The mathematics textbook for Grade 7 developed by IPST did not focus on addressing possible misconceptions in the lessons. All the preservice teachers developed their lesson plans based on the textbook. Although the preservice teachers could identify the students' conceptions, addressing the misconceptions in the lesson plans was not specifically one of their objectives. Therefore, the preservice teachers, for the most part, did not select activities designed to address the misconceptions nor create opportunities to address them. However, the high knowledge of students' conceptions preservice teachers used their knowledge of students' conceptions throughout the lessons more often than the low knowledge of students' conceptions preservice teachers. The high knowledge of students' conceptions preservice teachers presented the students with examples or counterexamples to evoke the misconceptions. They did not ignore or dismiss answers that were the result of well-known misconceptions without commenting on them and coming back to work on the topic repeatedly. The low knowledge of students' conceptions preservice teachers, on the other hand, used few examples that evoked misconceptions. Within a voluntary curriculum setting, Peterson, Carpenter, and Fennema (1989) also found that teachers' pedagogical content knowledge and beliefs about student knowledge influence teachers' classroom practice. Thus, the depth of the knowledge of students' conceptions appeared to have been as important an understanding for preservice middle school mathematics teachers teaching in a controlled national curriculum as that needed in a voluntary curriculum. Having knowledge of students' common conceptions and misconceptions about the subject matter is identified as an essential understanding for teaching in a voluntary curriculum (National Council of Teachers of Mathematics [NCTM], 1991). A teacher who has solid mathematical knowledge for teaching is more capable of helping the students achieve a meaningful understanding of the subject matter (Even, 1990).

In the current study, after all the preservice teachers taught some classes, their knowledge of students' conceptions was improved. Some preservice teachers used their knowledge of students' conceptions to prevent the errors when teaching other sections. They used more examples to evoke the misconceptions. The results were consistent with the studies conducted within a voluntary curriculum. The findings suggested that experiences of interacting with students in the classroom profoundly influence teachers' pedagogical content knowledge and beliefs (Cobb, Yackel, & Wood, 1991; Peterson, Carpenter, & Fennema, 1989). Thus, student teaching in a nonvoluntary environment was one of the sources enhancing preservice teachers' knowledge of students' conceptions. Fennema et al. (1996) stated that one major way to improve mathematics instruction and learning is to help teachers understand the mathematical thought processes of their students. This knowledge is not static. It can be acquired in the context of teaching mathematics.

In the current study, preservice teachers experienced growth in their subject matter knowledge of division of rational numbers as a result of teaching and preparing to teach, as they often needed to review content as they prepared to teach. The preservice teachers also began to develop knowledge of students' conceptions as they attempted to communicate their own understanding to students.

Limitations of the Study

Several aspects limit the results of the study: the representativeness of the preservice teachers, the size of the sample, the content areas selected to be investigated, the instrument, the student assessment, and the limitation of the researcher. The individual preservice teachers in this study volunteered to participate, and thus, do not represent a randomly chosen group. The preservice teachers who volunteer may have different practices from the preservice teachers who do not, and studies based on volunteers must be interpreted with that thought in mind (Sardo-Brown, 1990).

In addition, there were only four preservice teachers who participated in the teaching part of the study, which is a very small sample of preservice teachers. Because the purpose of the study was to add to the developing knowledgebase about preservice teacher knowledge and its links to practice, and not to test an already well-defined theory, the nature and size of the sample were deemed appropriate, since they were similar to samples selected for other such studies.

Moreover, the length of the observation period and the geographical situation of the schools where the observations occurred determined that only four preservice teachers could be involved in the classroom observations. The results of this study are confined to the four preservice teachers involved. Besides the size of the sample, other characteristics of the sample introduced limitations. Specifically, the subjects were sampled from a single teachers' college, rather than randomly from the 36 teachers' colleges in Thailand.

This study purposefully narrowed its focus to the study of the subject matter knowledge of division of rational numbers and knowledge of students' conceptions of division of rational numbers held by the preservice mathematics teachers. No generalizations can be made concerning the subject matter knowledge and knowledge of students' conceptions for other topics or the constraints that may exist for the implications of such structures on teaching and learning.

The results may be limited by the degree to which the instrument was able to actually separate high, medium, and low. In the current study, the determination of high, medium, and low was based on the scores of 19 preservice teachers. The instrument might have had a different determination of high, medium, and low for a different set of preservice teachers or for a different number of preservice teachers taking the test.

The effectiveness of instruction cannot be assessed without students' learning being measured. However, students' learning was not evaluated as part of the study. Although certain inferences were suggested from students' questions and comments, it is not known if the instruction in one classroom was any more effective than in others.

The researcher was also a limitation in this study. The researcher's presence in the classroom for the 12-week observation period may have had an effect on the preservice teachers' classroom practices. The preservice teachers said that they did not change their way of teaching due to the researcher's presence in their classroom. Also, the researcher's opinions and philosophies about students' conceptions had an impact on the analysis of the data and the focus of the conclusion drawn. The researcher's strong belief that it is essential to know students' conceptions before teaching a concept has affected the direction of this study. By examining and documenting all comments and conclusions, by remaining open and truthful about all biases, the effect of the researcher on this study has been minimized.

Implications for Mathematics Teacher Education

This study extends the information available on Thai preservice teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers and how these two preservice teachers' knowledge structures relate to classroom practices. The results also suggest implications for both preservice teacher and inservice teacher as well as several avenues for research.

Investigating the four preservice teachers' subject matter knowledge of division of rational numbers provided a troublesome picture of precollege, college, and preservice mathematics education in Thailand. As suggested by the examples in this study, some preservice teachers' subject matter knowledge of division of rational numbers tended to be incomplete. Their limited understanding of division of rational numbers was surprising since they had taken several mathematics courses. This result was consistent with previous research studies of preservice teacher in voluntary curriculum who were found to have incomplete subject matter knowledge of mathematics (Ball, 1990a, 1990b; Simon, 1993; Tirosh & Graeber, 1989). Majoring in an academic subject in college did not guarantee that teachers would have that kind of subject matter knowledge they needed for teaching (Kennedy, 1991).

Division of rational numbers has already been studied by the preservice teachers during their own school years. However, as evidenced in the study, preservice teachers' subject matter knowledge was not sufficiently comprehensive and articulated for teaching even in a school system where the curriculum was predesigned for the classes (Even & Tirosh, 1995). The college-level mathematics courses often did not address the most fundamental concepts in the disciplines. Instead, professors provided massive quantities of information, with little attention given to the significance of each idea. College students might never have had to actually think about the fundamental ideas in their field, and might never have been given an opportunity to see the connections among the many ideas they had learned. Moreover, much of the content taught in college courses was different from the contents of K-12 classes. But often fundamental concepts on which a subject rests—the meaning of division, for instance—were not addressed in college. College students were assumed to have already learned these basic concepts. When they began teaching them, they had to draw on what they recalled from their own elementary or secondary education, not on what they learned in college. Thai mathematics teacher education programs may consider increasing the number of mathematics courses preservice teachers are required to take. The mathematics courses should be constructed differently, more comprehensive and articulated understanding and knowledge of division of rational numbers (and mathematics) is developed. Thai preservice teachers need to have learning environments that foster powerful constructions of mathematical concepts. As mentioned by McDiarmid and Wilson (1991),

> Although teachers may learn about some concepts from their own practice, other ideas may be less easily understood from practice. Moreover, waiting for teachers to develop conceptual understandings

> of the subject matter from teaching it seems both haphazard and callous: Who decides whose children get shortchanged while waiting for teachers to develop understandings of the subject matters they teach? (McDiarmid & Wilson, 1991, p.102)

The data reported in this study suggested that the scope of their subject matter knowledge of division of rational numbers was directly related to classroom teaching. Therefore, educational programs in Thailand need to identify the mathematical understandings of preservice teachers and develop programs to more adequately prepare them for teaching. Although improving the educational program for preservice teachers may result in changing the number of courses preservice teachers are required to take, efforts must be made to provide Thai preservice teachers with opportunities to understand the concepts underlying the mathematics that they will teach and learn how these concepts are related. As mentioned by Simon (1993), a focus on preservice teachers' understandings of concepts and relationships should make the development of dense webs of understandings a higher priority than vertical content coverage.

The data reported in this study also suggested that the preservice teachers' knowledge of students' conceptions affected teaching and teaching affected their knowledge of students' conceptions. Preservice teachers do not explicitly study students' conceptions and ways of thinking in mathematics during their own studies in school. There were possible sources that the preservice teachers acquired this type of knowledge. They might acquire knowledge of students' conceptions while they were students. They made some errors during studying division of rational numbers in their school years. They might acquire this type of knowledge during tutoring their friends. The growth of their knowledge of students' conceptions might be as a result of teaching and preparing to teach from their practicum class before they did student teaching. However, the high knowledge of students' conceptions preservice teachers used their knowledge of students' conceptions throughout the lessons more often than the low knowledge of students' conceptions preservice teachers. They presented the students with more examples or counterexamples to evoke the misconceptions. They did not ignore or dismiss answers that were the result of well-known misconceptions without commenting on them and coming back to work on the topic repeatedly. The low knowledge of students' conceptions preservice teachers, on the other hand, used few examples that evoked misconceptions.

After all the preservice teachers taught some classes, they gained more knowledge of students' conceptions. Some preservice teachers used their knowledge of students' conceptions to prevent the errors when teaching other sections. They used more examples to evoke the misconceptions. Thus, if they had a class on knowledge of students' conceptions prior to their student teaching, the preservice teachers would benefit in understanding knowledge of students' conceptions. They would be in a better position to design and arrange their lessons so the students confront their misconceptions right way. Therefore, Thai mathematics teacher education programs may consider increasing some courses that focus on knowledge of students' conceptions. The preservice teachers require taking these classes before their student teaching. Thus, the preservice teachers' own experience as learners, together with their knowledge from the courses, could be used to enhance their knowledge of common ways of thinking among students. The preservice teachers will have a strong experience before their student teaching. They will be able to plan their lesson that meets students' needs. As suggested by Tirosh (2000), one of the goals in teacher education programs should be to promote the development of preservice teachers' knowledge of common ways children think about the mathematics topics the teachers will teach.

Recommendations for Future Research

Based on the results of this study, several suggestions for related research were identified. First, more studies need to be done on preservice teachers in order to completely characterize the nature of preservice teachers' subject matter knowledge and knowledge of students' conceptions and their instruction. These additional studies need to focus on preservice teachers' knowledge of subject matter and of students' conceptions and instruction in a variety of content areas, particularly those suggested for inclusion in the middle school curriculum by the Institute for the Promotion of Teaching Science and Technology (IPST). Such content areas could include knowledge of number systems and number theory, geometry, probability, and statistics. To date, there have been few studies with preservice teachers and those studies have been limited in their content focus.

Second, the size and nature of the sample was restricted by the design of the study. There were only four preservice teachers who participated in the teaching part of the study, which was a very small sample of preservice teachers. Further research is therefore needed to explore whether or not the depth of the mathematical knowledge and knowledge of students' conceptions is as essential for preservice middle school mathematics teachers teaching in a controlled national curriculum as that needed in a voluntary curriculum in order to confirm and/or extend the findings of this study.

Third, as mentioned in the limitation section, the subjects in this study came from only one teachers' college of the 36 teachers' colleges in Thailand. This research could be duplicated in other teachers' college, or any university or college where teacher education programs were offered. As a result, a more wholistic picture of the preservice teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers would be developed.

Fourth, the length of the observation period and the geographical situation of the schools where the observations occurred determined that only four preservice teachers could be involved in the classroom observations. Clearly, the small number of informants conferred a great deal of weight on their individual characteristics relative to their individual characteristics relative to their class characteristics. For instance, if these two high subject matter knowledge preservice teachers happened to be unrepresentative of the class of high subject matter knowledge preservice teachers in general, any comparisons involving the high subject matter knowledge preservice teachers would yield skewed results. The comparison portion of the study could be improved by including more preservice teachers. Yin (1994) recommended that selecting additional cases for the sample in order to provide replications. Each additional case that replicates the findings of the first case adds to the certainty of those findings.

Fifth, this study was done with the preservice teachers in a nonvolunatry curriculum. Thus, more studies need to be done with inservice teachers in order to completely characterize the nature of teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers with respect to their classroom practices in a teaching environment controlled by a required national curriculum.

Sixth, the results of this study suggested that within a nonvoluntary curriculum the mentors directly impacted the preservice teachers' teaching. The low subject matter preservice teachers who worked closely with their mentors taught division of rational numbers well. Future studies need to be done to confirm and/or extend the findings: Does subject matter mentoring lead to better teaching of division of rational numbers?

Seventh, clearly, the textbook played a central role in a nonvoluntary classroom environment. In this study, the textbook was an algorithmically-based curriculum. All the preservice teachers focused on an algorithmic presentation following an algorithmically-based curriculum. It seems that the type of curriculum affected the preservice teachers' presentations. Future studies need to be done to confirm and/or extend the findings: Will the preservice teachers teach conceptually within a nonvoluntary conceptually-based curriculum? Will the preservice teachers teach based on the type of curriculum (conceptual/procedural)?

Eighth, the preservice teachers in a nonvolunatry curriculum followed the textbook closely. While they made some errors, it is possible that a nonvoluntary curriculum might cover up errors of the preservice teachers. Further study is needed: Does a nonvoluntary curriculum prevent more errors of the preservice teachers than a voluntary curriculum?

Ninth, the textbook used in the nonvoluntary curriculum was an algorithmically-based curriculum. The preservice teachers followed the scope and sequence of the topics in this textbook. However, the students in this curriculum still had misconceptions on division of rational numbers. It was possible that the emphasis on procedural rules of this textbook tended to reinforce some of the misconceptions. Future studies need to focus on analyzing the textbook used in a nonvoluntary context.

Tenth, the effectiveness of instruction cannot be assessed without students' learning being measured. However, students' learning was not evaluated as part of the study. Future studies need to be done to explore how differences in middle school mathematics preservice teachers' subject matter knowledge and knowledge of students' conceptions impact students' learning within a nonvoluntary classroom environment. Within a voluntary curriculum, studies have shown that teacher knowledge did relate to student achievement in the area of understanding but not computations (Begle, 1972; Mullens, Murnane, & Willett, 1996). Begle (1979) mentioned that the more a teacher knows about his subject matter, the more effective he will be as a teacher. Mewborn (2000) added that teachers of varying mathematical backgrounds were equally successful in teaching students the procedures for solving computational problems and that those with stronger mathematics background were more successful in helping students understand the mathematics they were studying. Moreover, the teachers' knowledge of whether their own students could solve different problems was significantly correlated with student achievement (Carpenter, Fennema, Peterson, & Carey, 1988). Peterson, Carpenter, and Fennema (1989) also found that teachers' pedagogical content knowledge and beliefs about student knowledge influence teachers' classroom practice, which in turn influences their students' learning and achievement.

Eleventh, the results of this study suggested that preservice teachers' confidence appeared to be related to their subject matter knowledge of division of rational numbers and their teaching. The high subject matter knowledge preservice teachers displayed more confidence in teaching their lessons than the low subject matter knowledge preservice teachers by adding to their explanations. The preservice teachers who were more confident did not seem nervous when they taught. They did not hesitate to respond to their students' questions. The preservice teachers who had less confidence were reluctant to respond to their students' suggestions or questions. However, differing levels of confidence was not explored in this study, the impact of preservice teachers' confidence in their teaching and students' learning should be researched in future studies. How do differences in preservice teachers' confidence relate to classroom teaching and students' learning?

Twelfth, the nature of the instrument to assess preservice teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers was more procedural than conceptual. Future studies need to be done using an instrument with a balance between conceptual and procedural aspects.

Thirteenth, in this study there were only 19 preservice teachers who took the test. The methods of ranking the preservice teachers based on these 19 preservice teachers. This method might not actually separate the high and the low. Future studies need to be done using the questionnaire instrument to see whether or not the scoring distribution for high, medium, and low is accurate.

Fourteenth, one preservice teacher who had low subject matter knowledge and high knowledge of students' conceptions may on the surface seem impossible. However, it could be a result of his confidence and the fact that he struggled so much in learning mathematics. Thus, he had a better understanding because of his own struggles with learning. However, he still had weak subject matter knowledge because he may have known mathematics only procedurally. Future research ought to explore the relationship of the preservice teachers' subject matter knowledge to their knowledge of students' conceptions of division of rational numbers.

Finally, future studies could explore preservice teachers' knowledge of relationships and connections within and outside of the division of rational numbers. Observations of teaching in the current study indicated that some preservice teachers lack this knowledge. They did not see the relationship between addition and multiplication, and the relationship between decimals and fractions.

Although there were some limitations and questions remained to be answered, it is hoped that the results of this study, together with the findings of future research, will contribute to the expanding knowledgebase of the field, inform teacher education, guide preservice and inservice middle school mathematics teachers, and, ultimately, improve the teaching of division of rational numbers within a nonvoluntary/restrictive curriculum.

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APPENDICES

APPENDIX A

Letter to Preservice Teachers

Dear ____________________,

My name is Charuwan Singmuang and I am a doctoral student in Mathematics Education at Oregon State University. I am beginning my dissertation soon. The project title is Thai Preservice Middle School Mathematics Teachers' Subject Matter Knowledge and Knowledge of Students' Conceptions of Division of Rational Numbers with respect to Their Classroom Practices. The purpose of the study is to describe Thai preservice middle school mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers with respect to their classroom practice in a teaching environment controlled by a required national curriculum.

I would like to ask for your help in my investigation. If you volunteer to become involved in this study, you will be asked to participate in the questionnaire. Ten participants will be asked to participate in informal audiotaped interviews. Through these activities, four participants will be selected to be observed and videotaped their teaching in the classroom.

The questionnaire will last no longer than 90 minutes and the interview will last no longer than 90 minutes asking questions about teaching and learning mathematics and knowledge of subject matter and of students' conceptions of division of rational numbers: division of integers, division of fractions, and division of decimals. If you are one of the four preservice teachers selected, this investigation will also involve at least 15 videotaped observations of your teaching on division of rational numbers. Observations will last the entire class period. You will also be interviewed formally and informally and audiotaped prior to and following the teaching of each of the following units: division of integers, division of fractions, and division of decimals. You will be interviewed formally six times and informally 30 times. Each interview will last approximately half an hour for formal interview and approximately 10 minutes for informal interview.

All information gathered from the questionnaires, interviews, and classroom observations will be kept strictly confidential, and in particular, responses given in the questionnaires and the interviews will not be shared with your supervisors or mentors. The data will be coded to protect participants; pseudonyms will be used so that participants will not be identifiable in any publication of the results of the study. Your questionnaire will be destroyed once your responses have been tallied. All audiotapes and videotapes will be kept in a secure place. They will be destroyed after the research project has been completed. Permission for researcher accesses to the questionnaire, interviews and classroom observations will not affected your grades in the student teaching. There is no risk for participating in this research project. The results from the study will provide a description of the role these two domains plays in classroom practice. With your help, the information will be used to design appropriate preservice teachers programs to support the preservice teachers to be teachers within this particular context.

Your participation in this project would be greatly appreciated. If you are interested in participating in this study, please fill out the form on the last page and return it to the researcher within one week. Your prompt reply will be greatly appreciated.

Participation is voluntary. You may refuse to participate or discontinue participation at any time without any penalty. Questions about the research study or specific procedures should be directed to Dr. Margaret Niess at Oregon State University (541-737-1818). Questions about your rights as a research subject or research-related injures should be directed to the IRB Coordinator at IRB@orst.edu. Thank you very much for your time.

Sincerely,

Charuwan Singmuang

Doctoral student, Oregon State University

I understand that my participation in this study is completely voluntary and that I may either refuse to participate or withdraw from the study at anytime without penalty. I understand that any information obtained from me will be kept confidential. A code number will be used to identify any information that I provide. The only persons who will have access to this information will be the investigator and no names will be used in any data summaries or publications.

I understand that any questions I have about the research study or specific procedures should be directed to Dr. Margaret Niess at Oregon State University (541-737-1818). If I have questions about my rights as a research subject or research-related injures, I should email the IRB Coordinator at IRB@orst.edu.

My signature below indicates that I have read and that I understand the procedures described above and give my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

Name..Signature.................................

Date..E-mail Address..........................

Phone Number..............................Best time to call........................

APPENDIX B

Letter to Mentors

Dear Mentor:

My name is Charuwan Singmuang and I am a doctoral student in Mathematics Education at Oregon State University. I am beginning my dissertation soon. The project title is Thai Preservice Middle School Mathematics Teachers' Subject Matter Knowledge and Knowledge of Students' Conceptions of Division of Rational Numbers with respect to Their Classroom Practices. The purpose of the study is to describe Thai preservice middle school mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers with respect to their classroom practice in a teaching environment controlled by a required national curriculum.

I am requesting permission for you to participate in a research project. The research will be conducted with your student teacher. Your student teacher has agreed to participate in the study. Your student teacher will be asked to participate in the questionnaire. Ten participants will be asked to participate in informal audiotaped interviews. Through these activities, four participants will be selected to be observed and videotaped their teaching in the classroom. This investigation will involve at least 15 videotaped observations of class.

If you volunteer to become involved in this study, you will be interviewed formally before and after the teaching of your student teacher on each of the following units: division of integers, division of fractions, and division of decimals. Informal interviews will also be conducted with you daily before or after the teaching of your student teacher. Thus, you will be interviewed formally six times and informally 15 times. Each interview will last approximately half an hour for formal interview and approximately 10 minutes for informal interview. The interviews will be audiotaped. The focus of the interviews is to identify your advice given to your student teacher on planning and teaching division of rational numbers.

The researcher will be the only person with access to all data collected. Confidentially will be maintained through use of coding. Pseudonyms will be used for the university and all subjects when reporting the results of the research. Videotapes will be kept in a secure place. They will be destroyed after the research project has been completed.

There is no risk for participating in this research project. With your help, the information from this study will be used to design appropriate preservice teachers programs to support the preservice teachers to be teachers within a teaching environment controlled by a required national curriculum.

Participation is voluntary. You may refuse to participate or discontinue participation at any time without any penalty. Questions about the research study or specific procedures should be directed to Dr. Margaret Niess at Oregon State University (541-737-1818). Questions about your rights as a research subject or research-related injures should be directed to the IRB Coordinator at IRB@orst.edu. Thank you for your time and participation in this research project.

Sincerely,
Charuwan Singmuang
Doctoral student, Oregon State University

I understand that my participation in this study is completely voluntary and that I may either refuse to participate or withdraw from the study at anytime without penalty. I understand that any information obtained from me will be kept confidential. A code number will be used to identify any information that I provide. The only persons who will have access to this information will be the investigator and no names will be used in any data summaries or publications.

I understand that any questions I have about the research study or specific procedures should be directed to Dr. Margaret Niess at Oregon State University (541-737-1818). If I have questions about my rights as a research subject or research-related injures, I should email the IRB Coordinator at IRB@orst.edu.

My signature below indicates that I have read and that I understand the procedures described above and gives my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

Name...Signature.................................

Date...E-mail Address..........................

Phone Number...............................Best time to call........................

APPENDIX C

Letter to Supervisors

Dear Supervisor:

My name is Charuwan Singmuang and I am a doctoral student in Mathematics Education at Oregon State University. I am beginning my dissertation soon. The project title is Thai Preservice Middle School Mathematics Teachers' Subject Matter Knowledge and Knowledge of Students' Conceptions of Division of Rational Numbers with respect to Their Classroom Practices. The purpose of the study is to describe Thai preservice middle school mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers with respect to their classroom practice in a teaching environment controlled by a required national curriculum.

I am requesting permission for you to participate in a research project. The research will be conducted with your student teacher. Your student teacher has agreed to participate in the study. Your student teacher will be asked to participate in the questionnaire. Ten participants will be asked to participate in informal audiotaped interviews. Through these activities, four participants will be selected to be observed and videotaped their teaching on division of rational numbers in the classroom. This investigation will involve at least 15 videotaped observations of your student teacher's class.

If you volunteer to become involved in this study, you will be interviewed three times following observations of the preservice teacher's teaching on division of rational numbers. The focus of the interviews is to identify your advice given to your student teacher on planning and teaching division of rational numbers. Each interview will last approximately half an hour. The interviews will be audiotaped and transcribed.

The researcher will be the only person with access to all data collected. Confidentially will be maintained through use of coding. Pseudonyms will be used for the university and all subjects when reporting the results of the research. All audiotapes will be kept in a secure place. They will be destroyed after the research project has been completed.

There is no risk for participating in this research project. With your help, the information from this study will be used to design appropriate preservice teachers programs to support the preservice teachers to be teachers within a teaching environment controlled by a required national curriculum.

Participation is voluntary. You may refuse to participate or discontinue participation at any time without any penalty. Questions about the research study or specific procedures should be directed to Dr. Margaret Niess at Oregon State University (541-737-1818). Questions about your rights as a research subject or research-related injures should be directed to the IRB Coordinator at IRB@orst.edu. Thank you for your time and participation in this research project.

Sincerely,
Charuwan Singmuang
Doctoral student, Oregon State University

I understand that my participation in this study is completely voluntary and that I may either refuse to participate or withdraw from the study at anytime without penalty.

I understand that any information obtained from me will be kept confidential. A code number will be used to identify any information that I provide. The only persons who will have access to this information will be the investigator and no names will be used in any data summaries or publications.

I understand that any questions I have about the research study or specific procedures should be directed to Dr. Margaret Niess at Oregon State University (541-737-1818). If I have questions about my rights as a research subject or research-related injures, I should email the IRB Coordinator at IRB@orst.edu.

My signature below indicates that I have read and that I understand the procedures described above and gives my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

Name.......................................Signature...............................

Date....................................... E-mail Address.........................

Phone Number............................ Best time to call........................

APPENDIX D

Letter to Parent/Guardian and Students

Dear____________,

My name is Charuwan Singmuang and I am a doctoral student in Mathematics Education at Oregon State University. I am beginning my dissertation soon. The project title is Thai Preservice Middle School Mathematics Teachers' Subject Matter Knowledge and Knowledge of Students' Conceptions of Division of Rational Numbers with respect to Their Classroom Practices. The purpose of the study is to describe Thai preservice middle school mathematics teachers' subject matter knowledge and knowledge of students' conceptions of division of rational numbers with respect to their classroom practice in a teaching environment controlled by a required national curriculum.

I am requesting permission for your son/daughter to participate in this research project. The research will be conducted with their teacher and conducted in your son's/daughter's mathematics class. Your son's/daughter's teacher has agreed to participate in the study. The teacher will be observed and videotaped his/her teaching on division of rational numbers in your child's class. Observations and videotaping will begin one week before the day division of rational numbers is introduced and continue until it is no longer the focus of instruction. The class will be observed every day these units are taught. All transactions between the teacher and the students will be videotaped. Observations and videotaping will last the entire class period. Your child will be involved in 15 videotaped observations of their teacher.

The researcher will be the only person with access to all data collected. Confidentially will be maintained through use of coding. Pseudonyms will be used for the university and all subjects when reporting the results of the research.

Videotapes will be kept in a secure place. They will be destroyed after the research project has been completed.

There is no risk for participating in this research project. The information from this study will be used to design appropriate preservice teachers programs to support the preservice teachers to be teachers within a teaching environment controlled by a required national curriculum.

Participation is voluntary. Your child may refuse to participate or discontinue participation at any time without any penalty. Questions about the research study or specific procedures should be directed to Dr. Margaret Niess at Oregon State University (541-737-1818). Questions about your child's rights as a research subject or research-related injures should be directed to the IRB Coordinator at IRB@orst.edu. Thank you for your time and participation in this research project.

Sincerely,
Charuwan Singmuang
Doctoral student, Oregon State University

I understand that the participation of my child in this study is completely voluntary and that my child may either refuse to participate or withdraw from the study at anytime without penalty.

I understand that any information obtained from my child will be kept confidential. A code number will be used to identify any information that my child provides. The only persons who will have access to this information will be the investigator and no names will be used in any data summaries or publications.

I understand that any questions I have about the research study or specific procedures should be directed to Dr. Margaret Niess at Oregon State University (541-737-1818). If I have questions about my child's rights as a research subject or research-related injures, I should email the IRB Coordinator at IRB@orst.edu.

I am willing to allow my child to participate in this study. My signature below indicates that I have read and that I understand the procedures described above and give my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

________________________________ ________________

Parent/Guardian Signature Date

________________________________ ________________

Child's Signature Date

APPENDIX E

Objectives for the Questionnaire

Objectives used to construct subject matter knowledge items.

Preservice teachers will be able to:

1. state the specific facts about division of rational numbers.
2. identify the terms of division in various contexts (dividend, divisor, quotient, remainder).
3. carry out algorithms of division of rational numbers.
4. identify the multiplication sentence which is most closely related to a given division sentence with rational number and rational number.
5. write the most appropriate symbolic sentence for obtaining the solution when given a word problem involving division of natural and/or rational number.
6. write a word problem with the same meaning when given symbolic sentence involving division of rational numbers.
7. demonstrate understandings of closure law by identifying sets as closed or not closed with the operation of division.
8. select the most appropriate explanation of the quotient when the dividend and/or divisor is zero.
9. place various numerals for rational numbers in the correct increasing or decreasing order.
10. find the solution from a given word problem which is solvable by dividing with natural and/or rational numbers.
11. construct proofs of division of rational numbers.

Objectives used to construct knowledge of students' conceptions

Preservice teachers will be able to:

12. identify preconceptions, difficulties or misconceptions.
13. describe possible sources of difficulties or misconceptions.

APPENDIX F

Table of Specifications

A table of specifications for subject matter knowledge (SMK) and knowledge of students' conceptions (KSC) of Division of Rational Numbers is presented as follows:

Objective	Item #					Total
	Computation SMK	Comprehension SMK	Application SMK	Analysis SMK	KSC	
1	1					1
2	5					1
3	6, 7, 8, 9, 10, 11					6
4		2				1
5		12, 13				2
6		14, 15				2
7		3				1
8		4				1
9			16			1
10			17, 18, 19			3
11				20		1
12					1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	11
13					1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	11
Total	8	8	4	20	22	42

APPENDIX G

English Version Questionnaire

Part I: Knowledge of Division of Rational Numbers

Section 1: Choose the correct answer and justify your answer

1. Which of the following operations is <u>not defined</u> for the rational numbers?
   a. $3+0$
   b. $0-3$
   c. 3×0
   d. $\frac{0}{3}$
   e. $\frac{3}{0}$
2. Which of the following multiplication sentence is closely related to the division sentence of $\frac{2}{3}\div\frac{5}{8}=\square$?
   a. $\frac{2}{3}\times\frac{5}{8}=\square$
   b. $\frac{2}{3}\times\frac{8}{5}=\square$
   c. $\frac{3}{2}\times\frac{5}{8}=\square$
   d. $\frac{3}{2}\times\frac{8}{5}=\square$
   e. $\frac{3}{5}\times\frac{8}{2}=\square$
3. Which of the following sets is <u>closed</u> with respect to the operation of division?
   a. $\{0,1,2,3\}$
   b. $\{1,2,4,8\}$
   c. $\{1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots\}$
   d. $\{\ldots,\frac{1}{8},\frac{1}{4},\frac{1}{2},1,2,4,8,\ldots\}$
   e. None of the above.

4. Choose the most appropriate explanation of the quotient $\frac{2}{3} \div 0 = ?$

   a. $\frac{2}{3} \div 0 = 0$, because all division with 0 results in 0.
   b. $\frac{2}{3} \div 0$ is undefined because zero is nothing.
   c. $\frac{2}{3} \div 0$ is undefined because there is no number such that 0 times the number is $\frac{2}{3}$.
   d. $\frac{2}{3} \div 0 = \frac{2}{3}$, because zero is nothing.
   e. None of the above.

Section 2: Answer the following questions and justify your answers.

5. In the division sentence $1\frac{3}{4} \div (-\frac{1}{4}) = -7$, identify dividend, divisor, and quotient.
   The dividend is..................
   The divisor is....................
   The quotient is..................
6. What is the answer for $0 \div (-2497)$?
   ....
   ....
   ....
   ....
7. What is the answer to $(-\frac{1}{4}) \div (-\frac{3}{5})$?
   ....
   ....
   ....
   ....
8. What is the result of $4\frac{1}{3}$ divided by $-1\frac{1}{9}$?
   ....
   ....
   ....
   ....

9. What is the answer for $1\frac{1}{2} \div 2\frac{1}{4}$?

10. What is the answer for $0.75\overline{)3.75}$?

11. What is 1.33 divided by (-2.1)?

12. Write an expression to represent the following word problem: $6\frac{3}{4}$ kilograms of cheese were packed in boxes, each box containing $\frac{3}{4}$ kilogram. How many boxes were needed to pack all the cheese?

13. Write an expression to represent the following word problem: A rope $4\frac{1}{2}$ feet long is to be partitioned into 30 shorter pieces. How many inches long will each of the shorter pieces be?

14. Write a story problem that would be solved by dividing 51 by 4 and for which the answers would be 13.

15. Write a story problem for which $\frac{3}{4}$ divided by $\frac{1}{4}$ would represent the operation used to solve the problem.

16. Order the following numbers from smallest to largest:

$$-\frac{7}{8}, -\frac{8}{9}, \frac{11}{15}, \frac{13}{17}$$

....

....

17. Somsak must deliver 20 tons of rambutans. If his truck can carry 3 tons at a time, how many trips must he make to finish delivery?

....

....

18. Somchai drives a car 665 kilometers on $\frac{3}{4}$ of a tank of fuel. How many kilometers will the car go on a full tank?

....

....

19. A salty beef used $\frac{5}{8}$ kilograms per piece. How many pieces will it make from 30 kilograms of meat?

....

....

20. Show that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$,
where a, b, c, and d are integers and b, c, and $d \neq 0$.

....

....

....

....

Part II: Knowledge of Students' Conceptions of Division of Rational Numbers

1. If your students are asked to compute the following expression: $0 \div 7$
   (a) list two common mistakes students in seventh or eight grade may make.
   ....
   ....
   (b) describe possible sources for each of these mistakes.
   ....
   ....
2. If your students are asked to compute the following expression: $0 \div 0$
   (a) list two common mistakes students in seventh or eight grade may make.
   ....
   ....
   (b) describe possible sources for each of these mistakes.
   ....
   ....
3. If your students are asked to compute the following expression: $\frac{1}{4} \div 4$
   (a) list two common mistakes students in seventh or eight grade may make.
   ....
   ....
   (b) describe possible sources for each of these mistakes.
   ....
   ....
4. If your students are asked to compute the following expression: $\frac{1}{4} \div \frac{3}{7}$
   (a) list two common mistakes students in seventh or eight grade may make.
   ....
   ....
   (b) describe possible sources for each of these mistakes.
   ....
   ....
5. Given the following word problem: Five friends bought $\frac{1}{5}$ kilogram of chocolate and shared it equally. How much chocolate did each person get?
   (a) write two common incorrect expressions students make when solving this problem (do not calculate the expression).
   ....
   ....
   (b) describe possible sources of these incorrect responses.
   ....
   ....

6. Kerk said that "$\frac{2}{3} \div \frac{1}{2} = \frac{3}{2} \times \frac{1}{2}$." Siri mentioned that "$\frac{2}{3} \div \frac{1}{2} = \frac{3}{2} \times \frac{2}{1}$." Who is right? Please explain your reason(s). What might each of the students be thinking?

....

....

....

7. Dang argues that he prefers to divide fractions in a way similar to multiplication. For instance $\frac{2}{9} \div \frac{1}{3} = \frac{2 \div 1}{9 \div 3} = \frac{2}{3}$. Would you accept Dang's proposal? Why or why not? What make this student think?

....

....

....

8. Somjai is having difficulty with division of fractions. Determine what procedure she is using in (1) and (2), and answer her final question as she would.

(1) $\frac{4}{6} \div \frac{2}{6} = \frac{2}{6}$

(2) $\frac{6}{10} \div \frac{2}{10} = \frac{3}{10}$

....

....

....

(3) $\frac{8}{12} \div \frac{2}{12} =$ ....

9. Determine the error pattern and tell what Kesorn will get for problem (4).

(1) $\begin{array}{r} 96 \\ 6\overline{)414} \\ \underline{360} \\ 54 \\ \underline{\underline{54}} \end{array}$ (2) $\begin{array}{r} 37 \\ 5\overline{)365} \\ \underline{350} \\ 15 \\ \underline{\underline{15}} \end{array}$ (3) $\begin{array}{r} 97 \\ 3\overline{)237} \\ \underline{210} \\ 27 \\ \underline{\underline{27}} \end{array}$ (4) $4\overline{)164}$

....

....

....

10. Jit has been doing well with much of his work in division, but recently he began having difficulty. Can you find his error pattern from (1) to (4)?

(1) $\begin{array}{r} 65 \\ 7\overline{)456} \\ \underline{42} \\ 36 \\ \underline{35} \\ \underline{\underline{1}} \end{array}$ (2) $\begin{array}{r} 9\ 4 \\ 6\overline{)5426} \\ \underline{54} \\ 26 \\ \underline{24} \\ \underline{\underline{2}} \end{array}$ (3) $\begin{array}{r} 6\ 7 \\ 8\overline{)4860} \\ \underline{48} \\ 60 \\ \underline{56} \\ \underline{\underline{4}} \end{array}$ (4) $\begin{array}{r} 5\ 4 \\ 8\overline{)4035} \\ \underline{40} \\ 35 \\ \underline{32} \\ \underline{\underline{3}} \end{array}$

....

....

Try his procedure with problem (5).

(5) $9\overline{)2721}$

11. When Ted started dividing decimals, his answers were usually correct. But now he frequently gets the wrong quotient. Can you find his pattern of errors? Use Ted's procedure with problem (4) to see if you found his error pattern.

(1) $\begin{array}{r} 3.91 \\ 6\overline{)23.5} \\ \underline{18} \\ 55 \\ \underline{54} \\ \underline{\underline{1}} \end{array}$ (2) $\begin{array}{r} 9.62 \\ 4\overline{)38.6} \\ \underline{36} \\ 26 \\ \underline{24} \\ \underline{\underline{2}} \end{array}$ (3) $\begin{array}{r} 1.644 \\ 5\overline{)8.24} \\ \underline{5} \\ 32 \\ \underline{30} \\ 24 \\ \underline{20} \\ \underline{\underline{4}} \end{array}$ (4) $3\overline{)2.57}$

....

....

APPENDIX H

Thai Version Questionnaire

แบบสอบถามเรื่องการหารจำนวนตรรกยะ

ชื่อ..รหัสประจำตัวนักศึกษา...

ระดับชั้นที่สอน...

<u>คำชี้แจง</u>

1. แบบสอบถามฉบับนี้มี 2 ตอน

ตอนที่ 1: ความรู้ทางคณิตศาสตร์เรื่องการหารจำนวนตรรกยะ

ตอนที่ 2: ความรู้เกี่ยวกับความคิดรวบยอดเรื่องการหารจำนวนตรรกยะของผู้เรียน

2. แต่ละตอนประกอบด้วยจำนวนข้อดังนี้

ตอนที่ 1 มี 20 ข้อ

ตอนที่ 2 มี 11 ข้อ

3. โปรดทำด้วยความตั้งใจตามเวลาที่กำหนดให้

ขอขอบคุณที่ให้ความร่วมมือ

<u>ตอนที่ 1</u>: **ความรู้ทางคณิตศาสตร์เรื่องการหารจำนวนตรรกยะ**

จงเลือกคำตอบที่ท่านคิดว่าถูกต้องที่สุดเพียงข้อเดียว โดยกาเครื่องหมาย X ทับตัวอักษรหน้าคำตอบนั้น

1. การดำเนินการในข้อใดต่อไปนี้<u>ไม่นิยาม</u>ในระบบจำนวนตรรกยะ (โปรดแสดงวิธีคิด)

ก. $3+0$ ข. $0-3$

ค. 3×0 ง. $\frac{0}{3}$ จ. $\frac{3}{0}$

2. ถ้า $\frac{2}{3} \div \frac{5}{8} = \square$ แล้ว ผลหารที่เติมลงใน $\square$ จะเท่ากับ<u>ผลคูณ</u>ในข้อใด (โปรดแสดงวิธีคิด)

ก. $\frac{2}{3} \times \frac{5}{8}$ ข. $\frac{2}{3} \times \frac{8}{5}$

ค. $\frac{3}{2} \times \frac{5}{8}$ ง. $\frac{3}{2} \times \frac{8}{5}$ จ. $\frac{3}{5} \times \frac{8}{2}$

3. เซตใดต่อไปนี้มีสมบัติ<u>ปิด</u>ภายใต้การหาร (โปรดแสดงวิธีคิด)

ก. $\{0, 1, 2, 3\}$ ข. $\{1, 2, 4, 8\}$

ค. $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ ง. $\{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\}$

จ. ไม่มีข้อใดถูก

4. จงเลือกคำอธิบาย<u>ที่เหมาะสม</u>กับความหมายของ $\frac{2}{3} \div 0$ (โปรดแสดงวิธีคิด)

ก. $\frac{2}{3} \div 0 = 0$ เพราะ จำนวนใดๆหารด้วย 0 ย่อมได้ผลหารเป็น 0

ข. $\frac{2}{3} \div 0$ ไม่สามารถกำหนดความหมายได้ เพราะ ศูนย์ไม่มีค่า

ค. $\frac{2}{3} \div 0$ ไม่สามารถกำหนดความหมายได้ เพราะไม่มีจำนวนใดที่คูณกับ 0 แล้วเท่ากับ $\frac{2}{3}$

ง. $\frac{2}{3} \div 0 = \frac{2}{3}$ เพราะ ศูนย์ไม่มีค่า

จ. ไม่มีข้อใดถูก

จงตอบคำถามต่อไปนี้

5. จากประโยคสัญลักษณ์ต่อไปนี้: $\boxed{1\frac{3}{4} \div (-\frac{1}{4}) = -7}$

จงบอก ตัวตั้ง ตัวหาร และ ผลหาร

ตัวตั้ง คือ............................

ตัวหาร คือ..........................

ผลหาร คือ..........................

6. จงแสดงวิธีหาคำตอบของ $0 \div (-2497)$

....

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7. จงแสดงวิธีหาคำตอบของ $(-\frac{1}{4}) \div (-\frac{3}{5})$

....

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8. จงแสดงวิธีหาคำตอบของ $4\frac{1}{3}$ หารด้วย $(-1\frac{1}{9})$

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9. จงแสดงวิธีหาคำตอบของ $1\frac{1}{2} \div 2\frac{1}{4}$

....

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10. จงแสดงวิธีหาคำตอบของ $0.75\overline{)3.75}$

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11. จงแสดงวิธีหาคำตอบของ 1.33 หารด้วย (-2.1)

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12. พิจารณาโจทย์ปัญหาต่อไปนี้ "มีเนยอยู่ $6\frac{3}{4}$ กิโลกรัมต้องการบรรจุลงในกล่อง โดยที่ 1 กล่องจะบรรจุเนยได้ $\frac{3}{4}$ กิโลกรัม อยากทราบว่า จะบรรจุเนยลงกล่องได้ทั้งหมดกี่กล่อง"

จงเขียน**ประโยคสัญลักษณ์**ที่ใช้ในการแก้โจทย์ปัญหาข้างต้น

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13. พิจารณาโจทย์ปัญหาต่อไปนี้ "เชือกเส้นหนึ่งยาว $4\frac{1}{2}$ ฟุต นำมาตัดเป็นท่อนสั้น ๆ 30 ท่อน เท่า ๆ กัน จะได้เชือกยาวท่อนละกี่นิ้ว"

จงเขียน**ประโยคสัญลักษณ์**ที่ใช้ในการแก้โจทย์ปัญหาข้างต้น

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14. จงสร้าง**โจทย์ปัญหา**ที่สามารถแก้ปัญหาได้โดยการหาร 51 ด้วย 4 และมีคำตอบเป็น 13

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15. จงสร้าง**โจทย์ปัญหา**จากประโยคสัญลักษณ์ $\frac{3}{4} \div \frac{1}{4} = \square$

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16. จงเรียงลำดับจำนวนต่อไปนี้จาก**น้อยไปหามาก** (โปรดแสดงวิธีคิด)

$-\frac{7}{8}, -\frac{8}{9}, \frac{11}{15}, \frac{13}{17}$

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17. สมศักดิ์ต้องส่งเงาะให้ลูกค้าจำนวน 20 ตัน ถ้ารถยนต์ของเขาสามารถบรรทุกได้เที่ยวละ 3 ตัน เขาจะต้องบรรทุกทั้งหมดกี่เที่ยวเงาะจึงจะหมด (โปรดแสดงวิธีคิด)

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18. สมชายขับรถระยะทาง 665 กิโลเมตร ใช้น้ำมันหมดไป $\frac{3}{4}$ ถัง ถ้ามีน้ำมันเต็มถัง รถจะไปได้ไกลเท่าไร (โปรดแสดงวิธีคิด)

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19. เนื้อเค็ม 1 ชิ้นทำจากเนื้อวัวสดหนัก $\frac{5}{8}$ กิโลกรัม ถ้าใช้เนื้อวัวสดหนัก 30 กิโลกรัม จะได้เนื้อเค็มกี่ชิ้น (โปรดแสดงวิธีคิด)

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20. จงแสดงว่า $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ เมื่อ a, b, c, และ d เป็นจำนวนเต็ม และ b, c, และ d $\neq 0$.

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<u>ตอนที่ 2</u>: ความรู้เกี่ยวกับความคิดรวบยอดเรื่องการหารจำนวนตรรกยะของผู้เรียน

1. พิจารณา $0 \div 7 = \square$

จงบอกคำตอบที่นักเรียน**<u>มักตอบผิด</u>**มา 2 คำตอบ พร้อมทั้งระบุสาเหตุหรือที่มาของการได้คำตอบนั้น

คำตอบที่นักเรียนมักตอบผิด	*สาเหตุหรือที่มาของการได้คำตอบผิด*
1)...............................	1)...............................
....	....
2)...............................	2)...............................
....	....

2. พิจารณา $0 \div 0 = \square$

จงบอกคำตอบที่นักเรียน**<u>มักตอบผิด</u>**มา 2 คำตอบ พร้อมทั้งระบุสาเหตุหรือที่มาของการได้คำตอบนั้น

คำตอบที่นักเรียนมักตอบผิด	*สาเหตุหรือที่มาของการได้คำตอบผิด*
1)...............................	1)...............................
....	....
2)...............................	2)...............................
....	....

3. พิจารณา $\frac{1}{4} \div 4 = \square$

จงบอกคำตอบที่นักเรียน**<u>มักตอบผิด</u>**มา 2 คำตอบ พร้อมทั้งระบุสาเหตุหรือที่มาของการได้คำตอบนั้น

คำตอบที่นักเรียนมักตอบผิด	*สาเหตุหรือที่มาของการได้คำตอบผิด*
1)...............................	1)...............................
....	....
2)...............................	2)...............................
....	....

4. พิจารณา $\frac{1}{4} \div \frac{3}{7} = \square$

จงบอกคำตอบที่นักเรียน**<u>มักตอบผิด</u>**มา 2 คำตอบ พร้อมทั้งระบุสาเหตุหรือที่มาของการได้คำตอบนั้น

คำตอบที่นักเรียนมักตอบผิด	*สาเหตุหรือที่มาของการได้คำตอบผิด*
1)................................	1)................................
....	....
2)................................	2)................................
....	....

5. พิจารณาโจทย์ปัญหาต่อไปนี้ "เด็ก 5 คน ซื้อช็อกโกแลตมา $\frac{1}{5}$ กิโลกรัมและแบ่งคนละเท่า ๆ กัน อยากทราบว่าเด็กจะได้รับช็อกโกแลตคนละเท่าไร" จงบอก**<u>ประโยคสัญลักษณ์</u>**ที่นักเรียน**<u>มักตอบผิด</u>**มา 2 ประโยค พร้อมทั้งระบุสาเหตุหรือที่มาของการได้ประโยคที่ผิดนั้น

ประโยคที่นักเรียนมักตอบผิด	*สาเหตุหรือที่มาของการได้ประโยคผิด*
1)................................	1)................................
....	....
2)................................	2)................................
....	....

6. เกริกกล่าวว่า $\frac{2}{3} \div \frac{1}{2} = \frac{3}{2} \times \frac{1}{2}$ ส่วน สีรีกล่าวว่า $\frac{2}{3} \div \frac{1}{2} = \frac{3}{2} \times \frac{2}{1}$ จงพิจารณาว่า**<u>คำตอบของใครถูก</u>** พร้อมทั้งให้เหตุผลประกอบ นักเรียนมีความเข้าใจผิดหรือไม่

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7. แดงเสนอแนะว่า การหารเศษส่วนควรจะกระทำด้วยวิธีการที่คล้ายคลึงกับการคูณเศษส่วน ยกตัวอย่างเช่น $\frac{2}{9} \div \frac{1}{3} = \frac{2 \div 1}{9 \div 3} = \frac{2}{3}$ จงพิจารณาว่า ข้อเสนอของแดง**<u>ถูกต้องหรือไม่</u>** เพราะเหตุใด ถ้าหากไม่ถูกต้อง ท่านจะอธิบายให้แดงเข้าใจอย่างไร

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8. ข้อมูลต่อไปนี้เป็นวิธีการคิดของสมใจเรื่องการหารเศษส่วน

1) $\frac{4}{6} \div \frac{2}{6} = \frac{2}{6}$

2) $\frac{6}{10} \div \frac{2}{10} = \frac{3}{10}$

จากข้อมูลข้างต้นแสดงว่าสมใจมีวิธีหาผลหารอย่างไร

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ถ้าใช้วิธีการของสมใจแล้ว คำตอบของ $\frac{8}{12} \div \frac{2}{12}$ จะมีค่าเท่าใด

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9. ข้อมูลต่อไปนี้เป็นขั้นตอนแสดงการหารของเกสร จงพิจารณาข้อผิดพลาดของเกสร แล้วหาคำตอบของข้อ (4) โดยใช้วิธีการของเกสร

(1) $\begin{array}{r} 96 \\ 6\overline{)414} \\ 360 \\ \hline 54 \\ 54 \\ \hline\hline \end{array}$ (2) $\begin{array}{r} 37 \\ 5\overline{)365} \\ 350 \\ \hline 15 \\ 15 \\ \hline\hline \end{array}$ (3) $\begin{array}{r} 97 \\ 3\overline{)237} \\ 210 \\ \hline 27 \\ 27 \\ \hline\hline \end{array}$ (4) $4\overline{)164}$

10. จิตต์เคยทำได้ดีเกี่ยวกับเรื่องการหาร แต่เมื่อไม่นานนี้เขาเริ่มพบความยุ่งยาก จงหาข้อผิดพลาดของจิตต์ แล้วใช้วิธีการของจิตต์แสดงการหาคำตอบของข้อ (5)

(1) $\begin{array}{r} 65 \\ 7\overline{)456} \\ 42 \\ \hline 36 \\ 35 \\ \hline 1 \\ \hline\hline \end{array}$ (2) $\begin{array}{r} 94 \\ 6\overline{)5426} \\ 54 \\ \hline 26 \\ 24 \\ \hline 2 \\ \hline\hline \end{array}$ (3) $\begin{array}{r} 67 \\ 8\overline{)4860} \\ 48 \\ \hline 60 \\ 56 \\ \hline 4 \\ \hline\hline \end{array}$ (4) $\begin{array}{r} 54 \\ 8\overline{)4053} \\ 40 \\ \hline 35 \\ 32 \\ \hline 3 \\ \hline\hline \end{array}$ (5) $9\overline{)2721}$

11. เมื่อเทศ เริ่มต้นหารทศนิยม คำตอบของเขาส่วนมากจะถูกต้อง แต่ตอนนี้เขาได้ผลหารที่ผิด จงค้นหาข้อบกพร่องของเทศแล้วใช้วิธีการของเทศแสดงการหารทศนิยมในข้อ (4)

(1) $\begin{array}{r} 3.91 \\ 6\overline{)23.5} \\ 18 \\ \hline 55 \\ 54 \\ \hline 1 \\ \hline\hline \end{array}$ (2) $\begin{array}{r} 9.62 \\ 4\overline{)38.6} \\ 36 \\ \hline 26 \\ 24 \\ \hline 2 \\ \hline\hline \end{array}$ (3) $\begin{array}{r} 1.644 \\ 5\overline{)8.24} \\ 5 \\ \hline 32 \\ 30 \\ \hline 24 \\ 20 \\ \hline 4 \\ \hline\hline \end{array}$ (4) $3\overline{)2.57}$

APPENDIX I

Interview Questions with Preservice Teachers

Part I. Background questions

1. Tell me about you background.

   What mathematics courses did you take in college?

   Do you feel that your college courses were useful? Why/Why not?

2. How do you define mathematics?

   How do you think most mathematics is explained?

   Do you find mathematics easier or harder to teach than other subjects? Why?

   Do you think the majority of your students find mathematics easy to learn or hard to learn? Why?

   Is it important to you that your students enjoy mathematics?

3. What would you say is the primary focus when teaching mathematics?

   What is the role of problem solving in a middle school math class?

   How will you balance conceptual development with skill development?

   Will you do any "drill" in your class? What skills do you want your students to have by the end of the school year, and how will you go about making sure students have those skills?

4. How will you plan for instruction in mathematics?
5. Describe a typical math class.
6. What does it mean to "learn" mathematics?
7. How important is it that teachers have a deep understanding of mathematics they teach?
8. How will you assess the mathematical learning of your students?

9. Describe some of the individual differences among students in the middle level math classroom. How would you accommodate for these differences?
10. What do you find to be the primary problems or barriers to teaching mathematics the way you believe it should be taught?

Part II. Additional questions about subject matter knowledge and knowledge of students' conceptions of division of rational numbers

1. What does division mean to you? Give examples and think of alternative interpretations.
2. Identify aspects of division of rational numbers that are difficult and easy for students to understand.
3. Indicate "True" or "False" or "not sure" and justifying your response.
    1. In a division problem, the dividend must be greater than the quotient.
    2. In a division problem, the divisor must be a whole number.
    3. When you're setting up a division problem, the greater number always goes inside the bracket.
    4. To divides fractions, invert and multiply.
    5. Any number divided by itself is one.

Part III. Probing the response on the questionnaire

Present the preservice teacher the questionnaire. Then ask the following questions: "How do you feel about answering the questionnaire?"

In clarifying responses on the questionnaire:

a. Follow any trend of thought that preservice teacher takes as long as the discussion is on the questionnaire.
b. If preservice teacher does not address unclear or omitted responses, then direct preservice teacher to clarify written or omitted explanations.

In investigating changes in responses, ask the following question:

"I've noticed that your answer for question ## has changed, can you give me some information for the reason for the change?"

This line of questioning will proceed until all of the changes have been investigated.

In general, the phrases used in the interview include:

- "Please describe that further.";
- "What do you mean by....";
- "Tell me more about....(state a key phrase preservice teacher said).";
- "Is this what you mean....(give what the researcher thinks preservice teacher said).";
- "Please explain that further.";
- "Hmm that's interesting. Tell me more about that.";
- "Hmm that's interesting. Tell me..." (a key phrase or idea that the preservice teacher used);
- "I see."

APPENDIX J

Classroom Observation Form

Preservice teacher____________________ Date______________________

Observer__________________________ Class______________________

Part I. Summary/description of the lesson. Include transcribed sections as appropriate. Number the paragraphs.

Part II. Notes for reflection on lesson. Refer back to numbered paragraphs as appropriate; complete when not covered in part I; leave blank if not applicable.

1. Mathematical Content
   a. What was being taught? Was the class time partitioned into discrete parts? (If partitioned, give an approximate time line.)
   b. What seemed to be the goal? (What are students supposed to be learning, to be able to do, to understand, etc.?)
   c. Were the underlying mathematical meanings of the content being emphasized in this lesson, or were procedural steps and facts being emphasized? Give specific examples.
   d. Was the emphasis on "doing mathematics" (e.g., framing problems, making conjectures, looking for patterns, examining constraints, determining whether an answer is valid or reasonable, knowing when a problem is solved, justifying, explaining, challenging), or was the emphasis on getting right answers? Give specific examples.
   e. Was the content of this lesson connected to other ideas that the class has been dealing with? Give specific examples.
   f. How was understanding assessed?

2. Instructional Representations and Mathematical Tools
   a. What instructional representations (concrete, pictorial, real-world, or symbolic) did the teacher or the students use in this lesson and what mathematical ideas were they targeting? Describe each instructional representation, noting whether it was introduced by the teacher or by a student. If given by a student, describe how the teacher responded to it. Describe the strengths and weaknesses (in your opinion) of each representation. (Provide evidence for your assessment, considering how the representation fits with the mathematics, how helpful it seems to be as a learning tool—that is, does it focus attention on the central idea? Does it model significant components or aspects of the idea or the thinking that underlines it?)
   b. Itemize the mathematical tools that the teacher or students used in this lesson.

      Concrete pedagogical materials
      Pictorial tools
      "Real-world" situations or stories
      Measurement tools and other mathematical objects
      Mathematical language
      Mathematical symbols and notation
      Language and skills of mathematical discourse
   c. What was the role of the textbook?
3. Classroom Discourse
   a. Did the teacher frequently verbalize reasons, understandings, and solution strategies him- or herself? Did the students do this frequently in response to prompting/encouragement from the teacher or spontaneously? How did the teacher respond to students when they did this? Give examples.
   b. Did the teacher frequently make conjectures, challenge ideas, validate and justify solutions him- or herself? Did the students do this frequently in

response to prompting/encouragement from the teacher or spontaneously? How did the teacher respond to students when they did this? Give examples.

c. What were the students doing and what was the teacher's role during discussions? How much freedom did the teacher allow in student answers? In what ways was the discourse convergent? In what ways was the press toward consensus? In what ways was the discourse divergent? Were these space and time to disagree, to remain unconvinced?

d. Were there times during the lesson when a significant number of students seemed exceptionally engaged in the mathematics at hand? What were they engaged in?

   Were there times during the lesson when a significant number of students seemed to be engaged in something other than the mathematics at hand? What were they engaged in?

4. Intellectual Space

   a. Were the problems or tasks problematic to students (i.e., was there no immediately obvious solution path)?

   b. Did students seem to be confused about anything? If so, about what? How did the teacher treat the confusion?

   c. What routines seemed to be in place in the class, and how well did students seem to know these?

   d. What role did classroom management problems seem to play?

APPENDIX K

Classroom Observation Interview Questions

Formal Classroom Observation Interview Questions (Before Instruction)

1. How many days will the unit on division of (integers, fractions, or decimals) take?
2. How long will each lesson take?
3. Did your mentor/supervisor give any advice?
4. Did your mentor/supervisor help in planning the unit?
5. What textbooks and/or supplementary materials will you use?
6. What ideas do you have for the instruction in this unit?
7. What difficulties do you think students will have in this unit?

Formal Classroom Observation Interview Questions (Following Instruction)

1. Why was this unit taught?
2. Where did you get the ideas for this unit?
3. What are common student misconceptions about this unit content?
4. How comfortable do you feel about teaching this unit?
5. What comments did your mentor/supervisor indicate on your teaching?
6. What advice did you get from the mentor/supervisor about your planning?

Informal Classroom Observation Interview Questions (Before Instruction)

1. What will you be doing in today's lesson?
2. What will be expected of the students in today's lesson?
3. How are you planning to organize the students for the lesson?
4. What are your objectives in teaching this lesson?
5. What previous lessons relate to this lesson?

6. What do the students already know about this topic?
7. Why is it important for students to learn this content?
8. How is it related to other content they have learned?
9. Did your mentor/supervisor give any advice in planning the lesson?
10. What advice did you get from the mentor/supervisor in teaching the lesson?

Informal Classroom Observation Interview Questions (Following Instruction)

1. How do you think the lesson went?
2. Why was this lesson taught?
3. Where did you get the ideas for this lesson?
4. Did your mentor give any advice in planning the lesson?
5. What advice did you get from the mentor in planning the lesson?
6. How is it connected to what you have done and to what you plan to do?
7. Are you left with any questions about the teaching/learning that went on today?
8. How is this lesson similar to or different from your usual mathematics lesson?
9. What suggestions did you get from the mentor/supervisor in improving your teaching?

Other Possible Observation Questions

1. Why did you
    a. Select the activities/behaviors for the students?
    b. Choose the classroom organization?
    c. Select the examples, definitions, and analogies used?
    d. Select the modes of representation used?
    e. Select the techniques used?
2. What are common student misconceptions about this content?

3. What are/were your expectations about
    a. Students' abilities to understand the content?
    b. Especially difficult aspects of the lesson?
    c. How students would feel/felt about the lesson?
4. How comfortable do you feel about teaching this topic?
   About questions students ask?
   About how this topic is related to other topics?
5. Why were particular decisions made during the class (researcher will identify particular situations)?

APPENDIX L

Interview Questions with Mentors and Supervisors

Interview Questions with Mentors

Formal Interview Questions (Before Instruction of each Unit)

1. Did you give your preservice teacher any suggestions in planning the unit/lesson?
2. What advice did you give to the preservice teacher for teaching this unit/lesson?

Informal Interview Questions (Before/Following Instruction)

1. Did you give your preservice teacher any suggestions for improving his/her teaching?
2. What advice did you give preservice teacher for improving his/her teaching?

Informal Interview Questions (Daily)

1. What suggestions did you give the preservice teacher?
2. Did the preservice teacher implement your suggestions?

Interview Question with Supervisors

Formal Interview Questions (Following Instruction)

1. Did you give preservice teacher any suggestions in planning the unit/lesson?
2. What advice did you give to the preservice teacher for teaching this unit/lesson?
3. Did you give preservice teacher any suggestions for improving his/her teaching?
4. What advice did you give preservice teacher for improving his/her teaching?
5. Did the preservice teacher implement your suggestions?

APPENDIX M

Division of Rational Number Units

Chapter 3: Decimals

Division of Decimals

1. When the Divisor is a Natural Number.

Consider the following problems: division of a natural by a natural and division of a decimal by a natural number.

(1) a. $147 \div 3$

$$\begin{array}{rll} & 49 & \\ 3 & \overline{)147} & \\ & \underline{12} & \longleftarrow\!\!\!-\!\!\!-\!\!\!-\; 4 \times 3 \\ & 27 & \\ & \underline{\underline{27}} & \longleftarrow\!\!\!-\!\!\!-\!\!\!-\; 9 \times 3 \end{array}$$

Check: $49 \times 3 = 147$

b. $14.7 \div 3$

$$\begin{array}{rll} & 4.9 & \\ 3 & \overline{)14.7} & \\ & \underline{12} & \longleftarrow\!\!\!-\!\!\!-\!\!\!-\; 4 \times 3 \\ & 2.7 & \\ & \underline{\underline{2.7}} & \longleftarrow\!\!\!-\!\!\!-\!\!\!-\; 0.9 \times 3 \end{array}$$

Check: $4.9 \times 3 = 147$

(2) a. $3618 \div 67$

$$\begin{array}{rll} & 54 & \\ 67 & \overline{)3618} & \\ & \underline{335} & \longleftarrow\!\!\!-\!\!\!-\!\!\!-\; 5 \times 67 \\ & 268 & \\ & \underline{\underline{268}} & \longleftarrow\!\!\!-\!\!\!-\!\!\!-\; 4 \times 67 \end{array}$$

Check: $54 \times 67 = 3618$

b. $36.18 \div 67$

$$
\begin{array}{rl}
0.54 & \\
67\overline{)36.18} & \\
\underline{33.5}\; & \longleftarrow \text{Is the product of what numbers?} \\
2.68 & \\
\underline{\underline{2.68}} & \longleftarrow \text{Is the product of what numbers?}
\end{array}
$$

Check: $0.54 \times 67 = 36.18$

Thus, the algorithms of division of a natural number by a natural number and division of a decimal by a natural number are the same.

Example 1 Find the quotient of $369.45 \div 15$

Solution

$$
\begin{array}{r}
24.63 \\
15\overline{)369.45} \\
\underline{30} \\
69 \\
\underline{60} \\
9.4 \\
\underline{9.0} \\
0.45 \\
\underline{\underline{0.45}}
\end{array}
$$

The quotient is 24.63.

In general, division of a decimal by a natural number usually places only the decimal points of the dividend and the divisor. The decimal point in the quotient must be placed directly above the decimal point in the dividend. The decimal point in each step can be omitted. For example,

$$
\begin{array}{r}
24.63 \\
15\overline{)369.45} \\
\underline{30} \\
69 \\
\underline{60} \\
9\ 4 \\
\underline{9\ 0} \\
45 \\
\underline{\underline{45}}
\end{array}
$$

When the division has a remainder, add zero to the dividend and then perform the division until it has no remainder or getting the quotient with required decimal places. For example,

Example 2 Find the quotient of $31.8 \div 25$

Solution

$$
\begin{array}{rr}
 & 1.272 \\
25 & \overline{)31.800} \\
 & \underline{25} \\
 & 6\,8 \\
 & \underline{5\,0} \\
 & 1\,80 \\
 & \underline{1\,75} \\
 & 50 \\
 & \underline{\underline{50}}
\end{array}
$$

The quotient is 1.272.

Solve the following problems.

1. Find the quotients.
   (1) $0.755 \div 5$ (3) $35.01 \div 18$
   (2) $11.98 \div 4$ (4) $0.108 \div 9$
2. Check the accuracy of the quotients.
   (1) $2.674 \div 5 = 0.5348$ (3) $23.419 \div 4 = 5.58$
   (2) $0.573 \div 12 = 0.4775$ (4) $1.845 \div 25 = 0.0738$

2. When the Divisor is a Decimal.

Division of a decimal by a decimal can be performed in the same manner as division of a decimal by a natural number by making the divisor a natural number.

Do you know how to make the divisor as a natural number?

Example 3 Find the quotient of $0.299 \div 1.3$

Solution

$$
\begin{aligned}
0.299 \div 1.3 &= \frac{0.299}{1.3} \\
&= \frac{0.299\times 10}{1.3\times 10} \\
&= \frac{2.99}{13} \\
&= 2.99 \div 13
\end{aligned}
$$

When the divisor is a natural number, find the quotient by:

$$
\begin{array}{r}
0.23 \\
13\overline{)2.99} \\
\underline{2\,6} \\
39 \\
\underline{\underline{39}}
\end{array}
$$

Thus, $0.299 \div 1.3 = 0.23$

Example 4 Find the quotient of $44.044 \div 1.21$

Solution
$$
\begin{aligned}
44.044 \div 1.21 &= \frac{44.044}{1.21} \\
&= \frac{44.044 \times 100}{1.21 \times 100} \\
&= \frac{4404.4}{121} \\
&= 4404.4 \div 121
\end{aligned}
$$

$$
\begin{array}{r}
36.4 \\
121\overline{)4404.4} \\
\underline{363} \\
774 \\
\underline{726} \\
48\,4 \\
\underline{\underline{48\,4}}
\end{array}
$$

Thus, $44.044 \div 1.21 = 36.4$

Example 5 Find the quotient of $2 \div 0.625$

Solution
$$
\begin{aligned}
2 \div 0.625 &= \frac{2}{0.625} \\
&= \frac{2 \times 1000}{0.625 \times 1000} \\
&= \frac{2000}{625}
\end{aligned}
$$

$$
\begin{array}{r}
3.2 \\
625\overline{)2000.0} \\
\underline{1875} \\
125\,0 \\
\underline{\underline{125\,0}}
\end{array}
$$

Thus, $2 \div 0.625 = 3.2$

<u>Example 6</u> Find the quotient of $2 \div 0.625$ (Answer to the nearest hundredth)

<u>Solution</u>

$$3.5 \div 0.023 = \frac{3.5}{0.023} = \frac{3.5 \times 1000}{0.023 \times 1000} = \frac{3500}{23}$$

$$\begin{array}{r}
152.173 \\
23\overline{)3500.000} \\
\underline{23} \\
120 \\
\underline{115} \\
50 \\
\underline{46} \\
40 \\
\underline{23} \\
170 \\
\underline{161} \\
90 \\
\underline{69} \\
\underline{\underline{21}}
\end{array}$$

Thus, $3.5 \div 0.023 \approx 152.17$

The rules for division of decimals:

1. To divide a decimal by a natural number, place the decimal point in the quotient directly above the decimal point in the dividend. Then divide as you would with natural numbers.
2. To divide a decimal by a decimal, multiply both the dividend and the divisor by 10, or by 100, or by 1,000, etc to make the divisor a natural number, then divide as the first rule.

Example 7 There is 18.75 meters of dress fabric. One dress requires 1.25 meters. How many dresses can be made from the entire piece of fabric?"

Solution $18.75 \div 1.25$ dresses can be made from 18.75 meters of dress fabric.

$$= \frac{18.75}{1.25}$$

$$= \frac{1875}{125}$$

$$= 15$$

Thus, 15 dresses can be made from 18.75 meters long dress fabric.

Example 8 A car travels 5.5 kilometers in 6.2 minutes. How long does it take the car to travel 42.5 kilometers. (Round the answer to the nearest tenth)

Solution A car travels 5.5 kilometers in 6.2 minutes.

Thus, a car travels 42.5 kilometers in $\frac{6.2}{5.5} \times 42.5$ minutes

$$= \frac{263.50}{5.5}$$

$$\approx 47.9 \text{ minutes}$$

Exercise 3.4

1. Find the quotients.

(1)	0.01×0.1	(9)	$1.01 \div 0.1$
(2)	2.5×2.5	(10)	$367.65 \div 5.7$
(3)	23.54×0.05	(11)	$53.32 \div 0.62$
(4)	1.72×3.42	(12)	$0.053 \div 0.25$
(5)	33.18×1.4	(13)	$0.017 \div 3.4$
(6)	14.4×12	(14)	$40.5 \div 0.125$
(7)	0.006×15	(15)	$2.093 \div 0.028$
(8)	20.942×74	(16)	$19.05 \div 0.74$ (Answer to the nearest hundredth)

2. A man bought 200 eggs, 1.60 baht each. He sold 2 baht each. What is the profit?
3. A one meter and 50 centimeters long rope is divided into 45 centimeters each. How many pieces of 45-centimeter-long rope can be made and how many centimeters are left?
4. A dozen of pencils costs 24.50 baht. We have 60.75 Baht. How many dozen pencils can we buy? How much money do we have left?
5. A human inhales and exhales 12 kiloliters of air in a day. The oxygen received in the blood is 0.05 of the volume of air. How many liters of oxygen will the blood receive from the air per day?
6. A human bone weights 0.18 of total body weight. A man weighs 76.9 kilograms. Find the weight of other tissues which are not bone.
7. A man's weight on the moon is 0.16 of his weight on the earth. If a man weighs 11.6 kilometers on the moon, what is his weight on the earth?

Representing Fractions as Decimals

A fraction can be written as a decimal by dividing the numerator by the denominator. For example, $\frac{4}{5} = 4 \div 5 = 0.8$

1. Write fractions as decimals by division.

(1) $\frac{1}{2}$ (4) $\frac{3}{8}$

(2) $\frac{2}{5}$ (5) $\frac{7}{40}$

(3) $\frac{1}{4}$ (6) $\frac{11}{20}$

2. Write fractions as decimals by division.

(1) $\frac{2}{9}$ (3) $\frac{1}{6}$

(2) $\frac{1}{3}$ (4) $\frac{16}{45}$

You can see that in Part 2, the division is never ending and has a repeating digit(s). Why do the digits repeat?

The decimals Part 2 are called repeating decimals.

$\frac{2}{9} = 0.222...$ The digit 2 repeats indefinitely.

It can be written in a short form as $0.\dot{2}$.

$\frac{1}{6} = 0.166...$ The digit 6 repeats indefinitely.

It can be written in a short form as $0.1\dot{6}$.

$\frac{1}{11} = 0.09090...$ The digits 0 and 9 repeat indefinitely.

It can be written in a short form as $0.\dot{0}\dot{9}$.

Complete the following.

Fraction	Decimal	Short Form	Reading
$\frac{4}{9}$	0.44...	$0.\dot{4}$	zero point four the digit 4 repeats.
$\frac{7}{9}$			
$\frac{4}{15}$			
$\frac{8}{15}$			

A fraction can be written as a decimal by dividing the numerator by the denominator. In Part 1, the division ends because the remainder is 0. If you continue dividing, the digit 0 will repeat in the quotient. Thus, the quotient is also a repeating decimal.

Thus, fraction $\frac{1}{2}$ which is equal to 0.5 can be written as a repeating decimal as

$$\frac{1}{2} = 0.5\dot{0}.$$

Therefore, every fraction can be written as a repeating decimal.

<u>Exercise 3.5</u>

Write each of the following fractions as a repeating decimal.

1. $\frac{2}{3}$
2. $\frac{5}{6}$
3. $\frac{8}{9}$
4. $\frac{5}{11}$
5. $\frac{6}{11}$
6. $\frac{2}{15}$
7. $\frac{7}{15}$
8. $\frac{31}{90}$
9. $\frac{11}{20}$
10. $\frac{4}{33}$
11. $\frac{19}{30}$
12. $\frac{18}{30}$
13. $\frac{8}{33}$
14. $\frac{8}{11}$
15. $\frac{26}{45}$
16. $\frac{3}{22}$

Chapter 6: Fractions

6.4 Multiplication and Division of Fractions

Division of Fractions

Since $6 \div 2$ can be written in a fraction form as $\frac{6}{2}$,

Thus, $\frac{3}{5} \div \frac{2}{3}$ can also be written in a fraction form as $\dfrac{\frac{3}{5}}{\frac{2}{3}}$.

Write $\frac{3}{5} \div 2$ and $2 \div \frac{3}{5}$ in fraction forms.

Consider the method of finding the quotient of division of fractions.

(1) $\frac{3}{5} \div \frac{2}{3}$

$$\begin{aligned}
\frac{3}{5} \div \frac{2}{3} &= \dfrac{\frac{3}{5}}{\frac{2}{3}} && \text{(Step 1)} \\
&= \dfrac{\frac{3}{5} \times \frac{3}{2}}{\frac{2}{3} \times \frac{3}{2}} && \text{(Step 2)} \\
&= \dfrac{\frac{3}{5} \times \frac{3}{2}}{1} \\
&= \frac{3}{5} \times \frac{3}{2} && \text{(Step 3)} \\
&= \frac{9}{10}
\end{aligned}$$

(2) $\frac{3}{5} \div 7$

$$\frac{3}{5} \div 7 = \frac{\frac{3}{5}}{7} \qquad \text{(Step 1)}$$

$$= \frac{\frac{3}{5} \times \frac{1}{7}}{7 \times \frac{1}{7}} \qquad \text{(Step 2)}$$

$$= \frac{\frac{3}{5} \times \frac{1}{7}}{1}$$

$$= \frac{3}{5} \times \frac{1}{7} \qquad \text{(Step 3)}$$

$$= \frac{3}{35}$$

(3) $7 \div \frac{3}{5}$

$$7 \div \frac{3}{5} = \frac{7}{\frac{3}{5}} \qquad \text{(Step 1)}$$

$$= \frac{7 \times \frac{5}{3}}{\frac{3}{5} \times \frac{5}{3}} \qquad \text{(Step 2)}$$

$$= \frac{7 \times \frac{5}{3}}{1}$$

$$= 7 \times \frac{5}{3} \qquad \text{(Step 3)}$$

$$= \frac{35}{3}$$

$$= 11\frac{2}{3}$$

Compare the fraction in Step 3 with the original fraction.

1. Find the quotients using the previous method.
   (1) $\frac{7}{8} \div \frac{4}{7}$ (3) $7 \div \frac{1}{5}$
   (2) $\frac{10}{3} \div \frac{5}{6}$ (4) $\frac{1}{5} \div 7$
2. If you want to find the quotient in Part 1 rapidly, then which step can you skip?
3. From the answer in Part 2, what is the shortcut of division of fractions?
4. Find the quotients using the shortcut.
   (1) $10 \div \frac{2}{3}$ (3) $\frac{2}{5} \div \frac{9}{10}$
   (2) $\frac{11}{3} \div 3$ (4) $\frac{21}{5} \div \frac{7}{8}$

Convert a mixed number, if any, to an improper fraction before performing the division. For example,

$$\begin{aligned} 12\frac{3}{8} \div 2\frac{3}{4} &= \frac{99}{8} \div \frac{11}{4} \\ &= \frac{99}{8} \times \frac{4}{11} \\ &= \frac{9}{2} \\ &= 4\frac{1}{2} \end{aligned}$$

Example 1 A wire of $87\frac{1}{2}$ meters long is to be partitioned into shorter pieces of 7 meters each. How many shorter pieces can be made? How many meters of wire are left?

Solution The length of the wire is $87\frac{1}{2}$ meters.

You want to partition it to smaller pieces, 7 meters each.

The number of small pieces is $87\frac{1}{2} \div 7 = \frac{175}{2} \div 7$

$$= \frac{175}{2} \times \frac{1}{7}$$

$$= \frac{25}{2} = 12\frac{1}{2}$$

12 smaller pieces can be made and the remainder is $\frac{1}{2}$, $\frac{1}{2} \times 7 = 3\frac{1}{2}$ meters.

<u>Example 2</u> Five times a fraction is $\frac{5}{33}$. Find a fraction.

<u>Solution</u> 5 times a fraction is $\frac{5}{33}$.

That fraction is $\frac{5}{33} \div 5 = \frac{5}{33} \times \frac{1}{5}$

$$= \frac{1}{33}$$

That fraction is $\frac{1}{33}$.

<u>Example 3</u> Jib's height is 135 centimeters. Aod's height is 90 centimeters. How many times is Jib taller than Aod? How many times is Aod taller than Jib?

<u>Solution</u> Jib's height is $\frac{135}{90}$ of Aod's height.

$$\frac{135}{90} = \frac{3}{2} = 1\frac{1}{2}$$

Thus, Jib's height is $1\frac{1}{2}$ of Aod's height.

Aod's height is $\frac{90}{135}$ of Jib's height.

$$\frac{90}{135} = \frac{2}{3}$$

Thus, Aod's height is $\frac{2}{3}$ of Jib's height.

<u>Exercise 6.4 b</u>

1. Find the quotients.

(1) $\frac{25}{4} \div 2$	(6) $\frac{72}{45} \div \frac{100}{7}$
(2) $4\frac{2}{3} \div 6$	(7) $2\frac{1}{4} \div \frac{24}{35}$
(3) $42 \div \frac{15}{17}$	(8) $\frac{24}{35} \div 2\frac{1}{4}$
(4) $100 \div 2\frac{2}{3}$	(9) $3\frac{1}{11} \div 2\frac{19}{33}$
(5) $\frac{15}{17} \div \frac{10}{51}$	(10) $37\frac{1}{5} \div 8\frac{2}{11}$

2. Simplify.

(1) $1\frac{2}{3}+(\frac{3}{4}+2\frac{1}{5})$

(2) $(1\frac{2}{3}+\frac{3}{4})+2\frac{1}{5}$

(3) $10\frac{1}{2}-(3\frac{1}{3}-2\frac{1}{6})$

(4) $(10\frac{1}{2}-3\frac{1}{3})-2\frac{1}{6}$

(5) $2\frac{2}{3}\times(3\frac{1}{8}\times5\frac{2}{5})$

(6) $(2\frac{2}{3}\times3\frac{1}{8})\times5\frac{2}{5}$

(7) $4\frac{2}{3}\div(1\frac{1}{6}\div7\frac{1}{2})$

(8) $(4\frac{2}{3}\div1\frac{1}{6})\div7\frac{1}{2}$

(9) $5\frac{7}{8}-(3\frac{3}{4}+1\frac{1}{3})$

(10) $(5\frac{7}{8}-3\frac{3}{4})+1\frac{1}{3}$

(11) $5\frac{7}{8}+1\frac{1}{3})-3\frac{3}{4})$

(12) $(2\frac{1}{7}\div\frac{3}{4})\times\frac{9}{16}$

(13) $2\frac{1}{7}\div(\frac{3}{4}\times\frac{9}{16})$

(14) $(1\frac{3}{14}\times2)-1\frac{2}{7}$

(15) $1\frac{3}{14}\times(2-1\frac{2}{7})$

(16) $2\frac{29}{36}+(4\frac{1}{8}\times1\frac{7}{11})$

(17) $(2\frac{29}{36}+4\frac{1}{8})\times1\frac{7}{11}$

(18) $(2\frac{1}{6}+\frac{5}{9})\div(2\frac{5}{7}-1\frac{1}{14})$

(19) $(4\frac{1}{6}-3\frac{1}{6})+(2\frac{1}{13}\div12)$

(20) $(25\times1\frac{3}{4})-(4\frac{3}{7}+7\frac{1}{2})$

(21) $(10\frac{1}{3}-3)\times(5\frac{2}{11}\times5)$

(22) $4\frac{1}{12}\div\left\{1\frac{5}{6}+(\frac{5}{26}\times1\frac{1}{12}\right\}$

3. A car travels $156\frac{1}{5}$ kilometers in $2\frac{3}{4}$ hours. What is the car's average speed?

4. A bicycle wheel has a circumference of $2\frac{2}{5}$ meters. How many times does the wheel of the bike turn if a bicycle rider travels 60 meters?

5. Mercury weighs $13\frac{3}{5}$ of water's weight. Gold weighs $19\frac{3}{5}$ of water's weight. How many times does gold weigh more than mercury?

6. A man must deliver 36 tons of rock. A truck can carry $2\frac{2}{7}$ tons at a time, how many trips must he make to finish delivery? How many tons of rock must he deliver for the last trip?

APPENDIX N

Preservice Teachers' Class Weekly Schedules

Chai's Class Weekly Schedule

Time / Day	8:00 – 8:50	8:50 – 9:40	9:40 – 10:30	10:30 – 11:20	11:20 – 12:10	12:10 – 13:00	13:00 – 13:50	13:50 – 14:40
Monday							1/4	
Tuesday						1/4		1/9
Wednesday		1/2		1/9				
Thursday		1/4	1/2					
Friday	1/9			1/2				

Lada's Class Weekly Schedule

Time / Day	8:30 – 9:20	9:20 – 10:10	10:10 – 11:00	11:00 – 11:50	11:50 – 12:40	12:40 – 13:30	13:30 – 14:20	14:20 – 15:10
Monday					1/3	1/4		
Tuesday		1/2						
Wednesday		1/2			1/4			
Thursday		1/3	1/4					
Friday						1/2		1/3

Sak's Class Weekly Schedule

Time / Day	8:00 - 8:50	8:50 - 9:40	9:40 - 10:30	10:30 - 11:20	11:20 - 12:10	12:10 - 13:00	13:00 - 13:50	13:50 - 14:40
Monday						1/3		
Tuesday								
Wednesday				1/1			1/3	1/6
Thursday		1/6						1/1
Friday	1/6			1/1		1/3		

Nisa's Class Weekly Schedule

Time / Day	8:30 - 9:20	9:20 - 10:10	10:10 - 11:00	11:00 - 11:50	11:50 - 12:40	12:40 - 13:30	13:30 - 14:20	14:20 - 15:10
Monday					1/1	1/3		
Tuesday	1/2		1/3					1/1
Wednesday		1/3				1/2		
Thursday								1/2
Friday						1/1		

APPENDIX O

Chai's Lesson Plans

Lesson Plan 2: Representing Fractions as Decimals

1. Main Ideas

   Every fraction can be written as a decimal by dividing the numerator by the denominator. The quotient sometimes is a repeating decimal.

2. Goals

   The student will be able to convert fractions to repeating decimals correctly.

3. Instructional Objectives
   1. The student will be able to define a fraction.
   2. The student will be able to write fractions as decimals.

4. Content

   A way to change a fraction to a decimal is to divide the numerator by the denominator. For example,

   $\frac{1}{2} = 0.5$

   $\frac{3}{5} = 0.6$

   In these examples, the division has a remainder of zero and the division ends.

   $\frac{1}{3} = 0.333...$ The digit 3 repeats indefinitely.

   $\frac{7}{15} = 0.466...$ The digit 6 repeats indefinitely.

   In these examples, the division does not give a remainder of zero. The digits in the quotient begin to repeat. This type of decimal number is called a repeating decimal.

   We can write a repeating decimal in a short form by:

   1. If the quotient has one repeated digit, write a dot over the repeated digit

      $0.333... = 0.\dot{3}$

      $0.466... = 0.4\dot{6}$

   2. If the quotient has more than one repeated digit, write one dot over the first repeated digit and another dot over the final repeated digit.

      $0.181818... = 0.\dot{1}\dot{8}$

      $0.123123... = 0.\dot{1}2\dot{3}$

      $0.25000... = 0.25\dot{0}$

5. Activities

Introduction

Review division of natural numbers.

Teaching

1. Present the students with the following fractions.

   1) $\frac{1}{2} = 0.5$ 2) $\frac{3}{5} = 0.6$

2. Have the students change fractions to decimals.

   1) $\frac{1}{3} = 0.333...$ 2) $\frac{7}{15} = 0.466...$

3. Write the repeating decimals in short forms.

   1) $0.333... = 0.\dot{3}$ 2) $0.466... = 0.4\dot{6}$

   3) $0.181818... = 0.\dot{1}\dot{8}$ 4) $0.123123... = 0.\dot{1}2\dot{3}$

   5) $0.25000... = 0.25\dot{0}$

Practice

- Have the students do the worksheet:
  - Write the following repeating decimals in their short forms, and indicate how to read it.

    1) 2.444... 2) 0.0666...
    3) 1.4747 4) 0.14285285...
    5) 0.813813813...

Closing the Lesson

- Have the students summarize the rule for converting fractions into decimals.
- Assign the students homework selected from the textbook.

6. Materials
   - Worksheet
   - Textbook
7. Assessments
   - Observe the students' participation.
   - Have the student do the problems.

Worksheet

Complete.

A repeating decimal	Short form	Reading
1) 2.444...		
2) 0.0666...		
3) 1.4747...		
4) 0.14285285		
5) 0.813813813...		

Lesson Plan 3: Division of Fractions

1. Main Ideas

   To divide fractions, multiply the dividend by the reciprocal of the divisor.
2. Learning Objectives

   2.1 Goals

   The student will be able to solve division word problems.

   2.2 Instructional Objectives

   1. The student will be able to compute division of fractions correctly
   2. The student will be able to solve division word problems correctly.
3. Content

To divide fractions, convert mixed numbers, if any, to improper fractions. Then change the division sign into a multiplication sign and then switch the numerator and the denominator of the divisor.

Example Evaluate $1\frac{3}{7} \div \frac{1}{2}$

Solution

$$\begin{aligned} 1\frac{3}{7} \div \frac{1}{2} &= \frac{7}{10} \div \frac{1}{2} \\ &= \frac{10}{7} \times \frac{2}{1} \\ &= \frac{20}{7} \end{aligned}$$

Example A rope, 10 feet long, is to be partitioned into shorter pieces, $1\frac{1}{4}$ feet each. How many shorter pieces can be made from the entire rope?

Solution The length of the rope is 10 feet.

The length of the shorter piece is $1\frac{1}{4} = \frac{5}{4}$ feet.

The number of the shorter piece is equal to $10 \div \frac{5}{4} = 10 \times \frac{4}{5}$

$$= \frac{40}{5} = 8$$

Thus, 8 shorter pieces can be made from the entire rope.

4. Activities

   Introduction

   Review multiplication of fractions.

   Teaching

   - Have the students do the guided worksheet. Then the teacher and the students solve the problems on the chalkboard.
   - Have the students work on the following problems:

1) $\frac{3}{5} \div \frac{4}{9}$ 2) $4 \div \frac{8}{9}$

3) $1\frac{1}{6} \div \frac{14}{15}$ 4) $2\frac{1}{3} \div 1\frac{5}{6}$ 5) $4\frac{4}{5} \div 2\frac{2}{3}$

- Have volunteer students work on the chalkboard.
- Present the students with division word problems.

Example A rope, 10 feet long, is to be partitioned into shorter pieces, $1\frac{1}{4}$ feet each. How many shorter pieces can be made from the entire rope?

Closing the lesson

1. Have the students summarize the rules for dividing fractions. Division of any number by a fraction can be performed by multiplying that number by the reciprocal of the divisor.

2. Assign the students homework selected from the textbook (exercise 6.4 b).

5. Materials
   - Worksheet
   - Textbook
6. Assessment
   - Observe the students' participation
   - Have the students do the problems in the worksheet
   - Have the students do the homework.

Mentor's Comments

Chai's mentor did not write any comments on the lesson plans.

APPENDIX P

Lada's Lesson Plans

Lesson Plan 3: Representing Fractions as Decimals

1. Main Idea

   A fraction can be written as a decimal by dividing the numerator by the denominator.
2. Learning Objectives

   2.1 Goal

   The student will be able to represent fractions as decimals and vice versa.

   2.2 Instructional Objectives

   1) The student will be able to represent fractions as decimals.
   2) The student will be able to represent decimal as fractions
   3) The student will be able to represent fractions as repeating decimals.
   4) The student will be able to represent repeating decimals as fractions.
3. Content
   - Representing fractions as terminating decimals
   - Representing fractions as repeating decimals
   - Representing repeating decimal as fractions
4. Activities
   - Arrange the students into pairs.
   - Hand out the sheets, one set for each pair.
   - Explain the instructions for doing the sheets to the class
   - Assign the students homework selected from the textbook.
5. Materials
   - Instructional Sheet
6. Assessment

   Assessment Methods
   - Answer the questions in the sheet
   - Do the exercises in the textbook

   Criteria
   - Students answer the questions on the sheet correctly
   - Students do the exercises 80% correctly

Instructional Sheet

Representing Fractions as Decimals

A fraction can be written as a decimal by dividing the numerator by the denominator. For example, $\frac{1}{2} = 1 \div 2 = 0.5$.

Write the fraction $\frac{2}{3}$ as a decimal.

You can see that $\frac{2}{3} = 2 \div 3 = 0.6666.......$ The digit 6 in the quotient repeats indefinitely.

The division does not give a remainder of zero; instead, the digits in the quotient begin to repeat. For example, $\frac{2}{3} = 0.6666.......$ is called a repeating decimal. It can be written in a short form as $0.\dot{6}$.

Example $\frac{7}{12} = 0.583... = 0.58\dot{3}$ is a repeating decimal.

$\frac{3}{22} = 0.13636... = 0.1\dot{3}\dot{6}$ is a repeating decimal.

Complete.

Fraction	Decimal	Short Form	Reading
$\frac{4}{9}$	0.444...	$0.\dot{4}$	
$\frac{5}{6}$			
$\frac{8}{9}$			
$\frac{5}{11}$			
$\frac{7}{15}$			

How can we write repeating decimals as fractions?

Repeating digits that have one repeating digit, two repeating digits, or three repeating digits can be written as fraction by using 9, 99, or 999 as the divisors, respectively. For example,

$$0.\dot{3} = \frac{3}{9} = \frac{1}{3}$$

$$0.\dot{1}\dot{5} = \frac{15}{99} = \frac{5}{33}$$

$$0.\dot{2}1\dot{3} = \frac{213}{999} = \frac{7}{333}$$

We can write a decimal with both nonrepeating digits and repeating digits as a fraction as follows:

$$\begin{aligned} 12.4\dot{1}\dot{9} &= \frac{12419-124}{990} \\ &= \frac{12295}{990} \\ &= \frac{2459}{198} \\ &= 12\frac{83}{198} \end{aligned}$$

Write $0.3\dot{6}$ as a fraction.

Exercise

1. Write the following fractions as repeating decimals

1)	$\frac{1}{4}=$	2)	$\frac{5}{3}=$ ....
3)	$\frac{7}{11}=$	4)	$\frac{8}{15}=$ ....
5)	$\frac{25}{99}=$	6)	$\frac{13}{22}=$ ....
7)	$\frac{29}{30}=$	8)	$\frac{80}{33}=$ ....

2. Write the following repeating decimals in short forms.
   1) 0.333... = ...
   2) 0.151515... = ...
   3) 0.777... = ...
   4) 0.324324... = ...
   5) 5.37153715... = ...

Lesson Plan 4: Division of Fractions

1. Main Idea

Division of fractions can be performed by changing the division sign to the multiplication sign and then change the numerator to the denominator and the denominator to the numerator.

2. Learning Objectives
   1.1 Goal
      1) The student will be able to divide fractions correctly and rapidly.
      2) The student will be able to solve division word problems for fractions correctly.

   2.2 Instructional Objectives
      1) The student will be able to write division problems in fraction forms.
      2) The student will be able to multiply fractions.
      3) The student will be able to divide fractions.
      4) The student will be able to solve division word problems for fractions.

3. Content

$6 \div 2$ can be written in a fraction form as $\frac{6}{4}$.

$\frac{2}{3} \div \frac{3}{4}$ can be written in a fraction form as $\frac{\frac{2}{3}}{\frac{3}{4}}$.

$10 \div \frac{3}{7}$ can be written in a fraction form as $\frac{10}{\frac{3}{7}}$.

Consider the following fractions:

1) $\frac{2}{5} \div \frac{4}{5} = \frac{\frac{2}{5}}{\frac{4}{5}} = \frac{\frac{2}{5} \times \frac{5}{4}}{\frac{4}{5} \times \frac{5}{4}} = \frac{\frac{2}{5} \times \frac{5}{4}}{1} = \frac{2}{5} \times \frac{5}{4} = \frac{1}{2}$

   Thus, $\frac{2}{5} \div \frac{4}{5} = \frac{2}{5} \times \frac{5}{4}$

2) $100 \div 8\frac{1}{3} = 100 \div \frac{25}{3} = \frac{100}{\frac{25}{3}} = \frac{100 \times \frac{3}{25}}{\frac{25}{3} \times \frac{3}{25}} = \frac{100 \times \frac{3}{25}}{1} = 100 \times \frac{3}{25} = 12$

   Thus, $100 \div 8\frac{1}{3} = 100 \times \frac{3}{25}$

Thus, division of fractions can be performed by converting a mixed number, if any, to an improper fraction and then changing the division sign into the multiplication sign and then inverting the divisor.

Example 1: Evaluate $1\frac{3}{7} \div 2\frac{1}{2}$

Solution $1\frac{3}{7} \div 2\frac{1}{2} = \frac{10}{7} \div \frac{5}{2} = \frac{10}{7} \times \frac{5}{2} = \frac{4}{7}$

Thus $1\frac{3}{7} \div 2\frac{1}{2} = \frac{4}{7}$

Example 2 : Evaluate $\frac{10}{11} \times \frac{2}{5} \div \frac{13}{22}$

Solution $\frac{10}{11} \times \frac{2}{5} \div \frac{13}{22} = \frac{10}{11} \times \frac{2}{5} \times \frac{22}{13} = \frac{8}{13}$

Thus $\frac{10}{11} \times \frac{2}{5} \div \frac{13}{22} = \frac{8}{13}$

Example 3: A bicycle wheel has a circumference of $2\frac{1}{5}$ meters long. How many times does the wheel of the bike turn as a bicycle rider travels 220 meters?

Solution A rider travels 220 meters.

As the wheel turns once, the rider travels $2\frac{1}{5} = \frac{11}{5}$ meters.

The rider travels 220 meters when the wheel of the bike turns

$$220 \div \frac{11}{5} = 220 \times \frac{5}{11}$$
$$= 20 \times 5 = 100 \text{ times}$$

Thus, the bike wheel turns 100 times.

4. Work
   - Exercise 6.4 b.
5. Activities

Introduction

- Review on multiplication of fractions using a question-answer technique.

For example, $\frac{2}{3} \times \frac{4}{5}$ (Multiply the numerators and the denominators)

Teaching

1) Have the students write division of natural numbers problems in fraction forms. For example, $6 \div 2$ can be written in a fraction form as $\frac{6}{4}$. Then

have the students write division of fractions problems in fraction forms.

For example, $\frac{2}{3} \div \frac{3}{4}$ can be written in fraction form as $\frac{\frac{2}{3}}{\frac{3}{4}}$.

2) Have the students consider division of fractions (From contents 1-2).
3) Have the students summarize the rules of division of fractions. (Division of fractions can be performed by converting a mixed number, if any, to an improper fraction, changing the division sign into the multiplication sign, and inverting the divisor).
4) Present the students with the examples 1-3 and then present the students several problems and have them practice.
5) Have the students do exercise 6.4 b.

Closing the lesson

- Have the students summarize the rules of division of fractions.

6. Materials
   - Textbook
7. Assessment

   Assessment Methods
   - Answer the questions
   - Do the exercises in the textbook

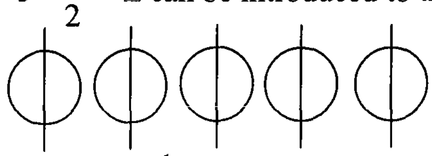
   Criteria
   - Students answer the questions correctly
   - Students do the exercises 80% correctly

Mathematics supervisor's comment

1. The supervisor commented on writing the objectives. Lada wrote that, "The student will be able to divide fractions correctly and rapidly. Lada's supervisor asked her how to measure "rapidly." She suggested Nisa leave it out.

2. In activity section, Lada's supervisor suggested she introduce the students division of fractions using the following steps:

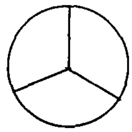
Example 1: $5 \div \frac{1}{2} = \square$ can be introduced to the students by:



How many $\frac{1}{2}$ s are there in 5?

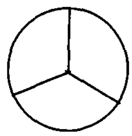
Thus, $5 \div \frac{1}{2} = 10$

Example 2: $\frac{2}{3} \div 2 = \square$ can be introduced to the students by:



Thus, $\frac{2}{3} \div 2 = \frac{1}{3}$

Example 3: $\frac{2}{3} \div \frac{1}{3} = \square$ can be introduced to the students by:



How many $\frac{1}{3}$ s are in $\frac{2}{3}$?

Thus, $\frac{2}{3} \div \frac{1}{3} = 2$

Example 4: $5 \div \frac{1}{2} = \frac{5}{\frac{1}{2}} = \frac{5 \times \frac{2}{1}}{\frac{1}{2} \times \frac{2}{1}} = \frac{5 \times \frac{2}{1}}{1} = 5 \times \frac{2}{1} = \frac{10}{1} = 10$

Example 5: $\frac{2}{3} \div 2 = \frac{\frac{2}{3}}{2} = \frac{\frac{2}{3} \times \frac{1}{2}}{2 \times \frac{1}{2}} = \frac{\frac{2}{3} \times \frac{1}{2}}{1} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$

Example 5: $\frac{2}{3} \div \frac{1}{3} = \frac{\frac{2}{3}}{\frac{1}{3}} = \frac{\frac{2}{3} \times \frac{3}{1}}{\frac{1}{3} \times \frac{3}{1}} = \frac{\frac{2}{3} \times \frac{3}{1}}{1} = \frac{2}{3} \times \frac{3}{1} = 2$

APPENDIX Q

Sak's Lesson Plans

Lesson Plan 4: Representing Fractions as Decimals

1. Main Idea

Every fraction can be written as a decimal by dividing the numerator by the denominator. Every fraction can be written as a repeating decimal.

2. Goal

The student will be able to write fractions as repeating decimals.

3. Instructional Objectives
   1. The student will be able to write fractions as decimals.
   2. The student will be able to write a repeating decimal in different forms.
   3. The student will be able to read a repeating decimal.
4. Content

Representing Fractions as Decimals

A fraction can be written in a decimal form by dividing the numerator by the denominator. For example, $\frac{1}{2} = 0.5$; $\frac{3}{5} = 0.6$.

When we divide the numerator by the denominator in some fractions, the results do not give a remainder of zero. Instead, the digit (s) in the quotient begins to repeat. For example,

$\frac{1}{3} = 0.333...$ The digit "3" in the quotient repeats indefinitely

$\frac{7}{15} = 0.4666...$ The digit "6" in the quotient repeats indefinitely.

$\frac{2}{11} = 0.1818...$ The digits "18" in the quotient repeat indefinitely.

These decimals are called repeating decimals. We show a repeating decimal by writing a dot(s) over the digit or digits that repeat.

1. Write the digit and write the dot over that digit. For example,

   $0.333... = 0.\dot{3}$

   $0.4666... = 0.4\dot{6}$

2. Write two dots over the first digit and the second digit of series of digits repeat in the quotient. For example,

   $0.1818... = 0.\dot{1}\dot{8}$

   When the quotient has no remainder, the repeating digit in a decimal is zero.

   $\frac{1}{4} = 0.25000... = 0.25\dot{0}$

Thus, every fraction can be written as a repeating decimal.

5. Activities

Introduction

Review on fractions and division.

Teaching

1. When dividing the numerator by the denominator in some fractions, the results do not give a remainder of zero. Instead, the digit (s) in the quotient begins to repeat. For example,

$\frac{1}{3} = 0.333...$ The digit "3" in the quotient repeats indefinitely

$0.333... = 0.\dot{3}$

$\frac{7}{15} = 0.4666...$ The digit "6" in the quotient repeats indefinitely.

$0.4666... = 0.4\dot{6}$

2. Write two dots over the first digit and the second digit of series of digits repeat in the quotient. For example,

$\frac{2}{11} = 0.1818...$ The digits "18" in the quotient repeat indefinitely.

$0.1818... = 0.\dot{1}\dot{8}$

3. When the quotient has no remainder, the repeating digit in the quotient is zero.

$\frac{1}{4} = 0.25000... = 0.25\dot{0}$

Thus, every fraction can be written as a repeating decimal.

4. The teacher distributes the worksheet to the students and then calls on some students put the work on the chalkboard.

Closing the Lesson

1. Have the students summarize how to change a fraction to a decimal.
2. Assign the students the homework: Exercise 3.5 from the textbook on page 61.

6. Material

Worksheet

7. Assessment Instrument
   1. Participation and answering questions
   2. Worksheet

8. Criteria

1. The students participate in the activities and are able to answer the questions.

2. The students are able to solve 80% of the worksheet correctly.

Worksheet

Complete.

Item	Fraction	Decimal	Short Form	Read
1	$\frac{4}{9}$	0.44...	$0.\dot{4}$	
2	$\frac{8}{11}$	0.72727...	$0.\dot{7}\dot{2}$	
3	$\frac{7}{9}$			
4	$\frac{4}{15}$			
5	$\frac{15}{66}$			
6	$\frac{11}{45}$			
7	$\frac{47}{99}$			
8	$\frac{40}{9}$			
9	$\frac{8}{15}$			
10	$\frac{129}{333}$			

Lesson Plan 5: Division of Fractions

1. Main Idea

   Division of any number by a fraction can be performed by multiplying that number by the reciprocal of the divisor.
2. Goal

   The student will be able to divide fractions.
3. Instructional Objectives
   1. The student will be able to divide fractions correctly and rapidly.
   2. The student will be able to solve division word problems for fractions.
4. Content

$6 \div 3$ can be written in a fraction form as $\frac{6}{3}$.

$\frac{3}{5} \div \frac{2}{3}$ can be written in a fraction form as $\dfrac{\frac{3}{5}}{\frac{2}{3}}$.

$$\frac{3}{5} \div \frac{2}{3} = \dfrac{\frac{3}{5}}{\frac{2}{3}} \qquad \text{(Step 1)}$$

$$= \dfrac{\frac{3}{5} \times \frac{3}{2}}{\frac{2}{3} \times \frac{3}{2}} \qquad \text{(Step 2)}$$

$$= \dfrac{\frac{3}{5} \times \frac{3}{2}}{1}$$

$$= \frac{3}{5} \times \frac{3}{2} \qquad \text{(Step 3)}$$

$$= \frac{9}{10}$$

Thus, $\frac{3}{5} \div \frac{2}{3} = \frac{3}{5} \times \frac{3}{2}$. We can skip steps 1 and 2. This is the shortcut for division of fractions. That is, in dividing fractions, convert a mixed number, if any, to an improper fraction, change the division sign into the multiplication sign, and invert the divisor.

Example 1: Evaluate $1\frac{3}{7} \div 2\frac{1}{2}$

$$1\frac{3}{7} \div 2\frac{1}{2} = \frac{10}{7} \div \frac{5}{2} = \frac{10}{7} \times \frac{2}{5} = \frac{4}{7}$$

Example 2: Evaluate $\left(\frac{10}{11} \times \frac{2}{5}\right) \div \frac{13}{22}$

$$\left(\frac{10}{11} \times \frac{2}{5}\right) \div \frac{13}{22} = \frac{4}{11} \div \frac{13}{22} = \frac{4}{11} \times \frac{22}{13} = \frac{8}{13}$$

Example 3: A bicycle wheel has a circumference of $2\frac{1}{5}$ meters long. How many times does the wheel of the bike turn as a bicycle rider travels 220 meters?

Solution The rider travels 220 meters.

As the wheel turns once, the rider travels $2\frac{1}{5} = \frac{11}{5}$ meters.

The rider travels 220 meters when the wheel of the bike turns

$$220 \div \frac{11}{5} = 220 \times \frac{5}{11}$$
$$= 20 \times 5 = 100 \text{ times}$$

Thus, the bike wheel turns 100 times.

5. Activities

Introduction

Review on the multiplication of fractions which the product is equal to 1.

For example, $\frac{3}{2} \times \frac{2}{3} = 1$, $\frac{7}{9} \times \frac{9}{7} = 1$

Teaching

1. Have the students write division problems in fraction forms. For example,

$$10 \div 5 = \frac{10}{5} \quad , \quad \frac{3}{4} \div \frac{2}{5} = \frac{\frac{3}{4}}{\frac{2}{5}} \quad , \quad \frac{3}{8} \div 2 = \frac{3}{8} \div \frac{2}{1} = \frac{\frac{3}{8}}{\frac{2}{1}}$$

2. Have the students consider the quotient of division of fraction:

(Step 1) $\frac{3}{5} \div \frac{2}{3} = \frac{\frac{3}{5}}{\frac{2}{3}}$ (Write a division problem as a fraction)

(Step 2) $= \dfrac{\frac{3}{5} \times \frac{3}{2}}{\frac{2}{3} \times \frac{3}{2}}$ (Make the divisor equal to 1)

$= \dfrac{\frac{3}{5} \times \frac{3}{2}}{1}$

(Step 3) $= \frac{3}{5} \times \frac{3}{2}$

$= \frac{9}{10}$

3. Present the students with another division problem and have the students solve it using the method in 2.
4. Present the Examples 1 and 2 on the chalkboard. Have the students discuss how to find the quotient when there is a mixed number. Then the teacher and the students summarize that they have to change a mixed number, if any, to an improper fraction first.
5. Write the Example 3 on the chalkboard and have the students find the quotient.
6. Have the students do the practice problems.

Closing the Lesson

1. Have the students summarize how to divide fractions.
2. Assign the students homework: Exercise 6.4 b.

6. Material

Practice problem sheet.

7. Assessment Instrument
   1. Participation and answering the questions
   2. Practice problem sheet.
   3. Exercises from the textbook.
8. Criteria

1. The students participate in the activities and are able to answer the questions.

2. The students are able to solve 80% of the problems in the sheet correctly.
3. The students are able to solve 80% of the exercises in the textbook correctly.

APPENDIX R

Nisa's Lesson Plans

Lesson Plan 4 : Representing Fractions as Decimals

1. Main Idea
   Every fraction can be written as a decimal by dividing the numerator by the denominator.
   Every fraction can be written as a repeating decimal.
2. Learning Objective After this lesson, the student will be able to:
   - write a fraction as a repeating decimal.
3. Content
   Representing Fractions as Decimals
   A fraction can be written as a decimal by dividing the numerator by the denominator. For example,

$$\frac{1}{2} = 0.5 \quad , \quad \frac{1}{4} = 0.25 \quad , \quad \frac{1}{8} = 0.125$$

The division of the numerator by the denominator of a fraction sometimes does not come out even. The division is never ending and has a repeating digit(s). For example,

$\frac{1}{3} = 0.333...$ The digit 3 repeats indefinitely.

$\frac{2}{3} = 0.666...$ The digit 6 repeats indefinitely.

$\frac{4}{33} = 0.1212...$ The digits 12 repeat indefinitely.

$\frac{7}{15} = 0.4666...$ The digit 6 repeats indefinitely.

These decimals are called repeating decimals. We can write a repeating decimal in a short form as follows:

1. If one digit repeats, write that digit and write a dot over that digit. For example,

   $0.333... = 0.\dot{3}$

   $0.666... = 0.\dot{6}$

2. If more than one digit repeats, write two dots over the first digit and the second digit repeating in the quotient. For example,

   $0.1212... = 0.\dot{1}\dot{2}$

   $0.1818... = 0.\dot{1}\dot{8}$

3. If there is a digit that does not repeat, write a dot over the repeated digit. For example,

$$0.4666... = 0.4\dot{6}$$

4. When the quotient has no remainder, the repeating digit in a decimal is zero.

$$\frac{1}{2} = 0.5000... = 0.5\dot{0}$$

4. Activities

Introduction

Review on writing a fraction as decimal by division. For example,

$\frac{1}{2} = 0.5$, $\frac{1}{4} = 0.25$, $\frac{1}{8} = 0.125$

Teaching

1. Have the students convert fractions to decimals by division. For example,

$\frac{1}{3} = 0.333...$ $\frac{2}{11} = 0.1818...$ $\frac{2}{3} = 0.666...$

$\frac{7}{15} = 0.4666...$ $\frac{4}{33} = 0.1212...$ $\frac{3}{22} = 0.13636...$

Then have the students compare with the decimals in the introduction (They are different because the digits in the quotient repeat indefinitely.

2. The teacher and the students summarize:

When dividing the numerator by the denominator of some fraction, the division is never ending and the digit in the quotient repeats indefinitely. These decimals are called repeating decimals.

3. The teacher discusses how to write and read a repeating decimal.

We can write a repeating decimal in a short form as follows:

1. If one digit repeats, write that digit and write a dot over that digit. For example,

$\frac{1}{3} = 0.333... = 0.\dot{3}$ The digit 3 repeats indefinitely.

$\frac{2}{3} = 0.666... = 0.\dot{6}$ The digit 6 repeats indefinitely.

2. If more than one digit repeats, write two dots over the first digit and the second digit repeating in the quotient. For example,

$\frac{4}{33} = 0.1212... = 0.\dot{1}\dot{2}$ The digits 1 and 2 repeat indefinitely.

3. If there is a digit that does not repeat, write a dot over the repeated digit. For example,

$\frac{7}{15} = 0.4666... = 0.4\dot{6}$ The digit 6 repeats indefinitely.

$\frac{3}{22} = 0.13636... = 0.1\dot{3}\dot{6}$ The digits 3 and 6 repeat indefinitely.

4. When the quotient has no remainder, the repeating digit is zero.

$\frac{1}{2} = 0.5000... = 0.5\dot{0}$ The digit 0 repeats indefinitely.

Thus, every fraction can be written as a repeating decimal.

4. Assign the students homework (even problems from the exercise 3.5 in the textbook, page 61)

Closing the Lesson

The teacher and the students summarize the rules for converting fractions as repeating decimals: "A fraction can be written as a repeating decimal by dividing the numerator by the denominator.

5. Instructional Materials
   - Practice problem sheet.
6. Assessment

Method	Criteria
1. Observe the students' answers	1. 80% of the students are able to answer the questions correctly.
2. Have the students do the practice sheet	2. The students are able to correctly solve 8 problems from 10 problems in the practice sheet.
3. Have the students do the exercise 3.5, even problems from page 134 in the textbook	3. The students are able to solve 7 problems from 8 problems correctly.

Mentor's Comments

After Nisa handed in her lesson plan, her mentor commented as follows:

1. The examples used in the introduction step should be terminating decimals with one decimal place, two decimal places, three decimals places, and so on. For examples, $\frac{1}{2} = 0.5$, $\frac{1}{4} = 0.25$, and $\frac{1}{8} = 0.125$.
2. The example used in the teaching step should be the following fractions: $\frac{1}{3}, \frac{1}{6}, \frac{2}{3}, \frac{4}{33}$. Thus the students would see the difference between the given examples.
3. She should summarize that "Every fraction can be represented by repeating decimals."
4. She should keep the students on task.

After observed Nisa's class, the mentor commented as follows:

1. She should make the denominator of some fractions 9, or 99, or 999 and have the students notice that the quotient would be a repeating decimal. For example,
   $\frac{1}{3} \times \frac{3}{3} = \frac{3}{9} = 0.333...$, $\frac{2}{3} \times \frac{3}{3} = \frac{6}{9} = 0.666...$, $\frac{2}{11} \times \frac{9}{9} = \frac{18}{99} = 0.181818...$
2. She should assign the students into groups of four or eight and then have them write the answers on the chalkboard. Do not spend too much time working on the problems.

Practice Sheet

Fraction	Decimal	Short Form	Reading
$\frac{1}{6}$			
$\frac{5}{9}$			
$\frac{3}{11}$			
$\frac{8}{15}$			
$\frac{47}{99}$			
$\frac{5}{33}$			
$\frac{11}{15}$			
$\frac{5}{22}$			
$\frac{7}{30}$			
$\frac{129}{333}$			

Lesson Plan 5: Division of Fractions

1. Main Idea

Division of any number by a fraction can be performed by multiplying the dividend by the reciprocal of the divisor.

2. Learning Objective After this lesson, the student will be able to:
   - divide fractions.

3. Content

Division of fractions can be performed by multiplying both the numerator and the denominator by the reciprocal of the divisor. For example,

$6 \div 4$ can be written as a fraction form as $\frac{6}{4}$.

$\frac{5}{6} \div \frac{2}{3}$ can be written as a fraction form as $\frac{\frac{5}{6}}{\frac{2}{3}}$.

Example 1 Find the quotient of $\frac{5}{6} \div \frac{2}{3}$

Solution

$$\frac{5}{6} \div \frac{2}{3} = \frac{\frac{5}{6}}{\frac{2}{3}} \qquad \text{(Step 1)}$$

$$= \frac{\frac{5}{6} \times \frac{3}{2}}{\frac{2}{3} \times \frac{3}{2}} \qquad \text{(Step 2)}$$

$$= \frac{\frac{5}{6} \times \frac{3}{2}}{1} \qquad \text{(Step 3)}$$

$$= \frac{15}{12} = \frac{5}{4} = 1\frac{1}{4}$$

Example 2 Find the quotient of $4\frac{9}{10} \div 7$

Solution

$$4\frac{9}{10} \div 7 = \frac{49}{10} \div 7$$

$$= \frac{\frac{49}{10}}{7} \qquad \text{(Step 1)}$$

$$= \frac{\frac{49}{10} \times \frac{1}{7}}{7 \times \frac{1}{7}} \qquad \text{(Step 2)}$$

$$= \frac{\frac{49}{10} \times \frac{1}{7}}{1} \qquad \text{(Step 3)}$$

$$= \frac{7}{10}$$

From those examples, we can perform the division of fractions by changing a mixed number, if any, to an improper fraction, changing the division sign into the multiplication sign, and then inverting the divisor.

Example 3 Evaluate $\frac{3}{14} \div [1\frac{7}{8} \div 1\frac{5}{7}]$

Solution

$$\begin{aligned}
\frac{3}{14} \div [1\frac{7}{8} \div 1\frac{5}{7}] &= \frac{3}{14} \div [\frac{15}{8} \div \frac{12}{7}] \\
&= \frac{3}{14} \div [\frac{15}{8} \times \frac{7}{12}] \\
&= \frac{3}{14} \div [\frac{5}{8} \times \frac{7}{4}] \\
&= \frac{3}{14} \div \frac{35}{32} \\
&= \frac{3}{14} \times \frac{32}{35} \\
&= \frac{3}{7} \times \frac{16}{35} = \frac{48}{245}
\end{aligned}$$

4. Activities

Introduction

Review on multiplication of fractions where the product is equal to 1. For example,

$$\frac{2}{3} \times \frac{3}{2} = 1 \quad , \quad \frac{7}{9} \times \frac{9}{7} = 1 \quad , \quad \frac{12}{49} \times \frac{49}{12} = 1$$

Teaching

1. Have the students write a division problem in a fraction form. For example,

$6 \div 4$ can be written as a fraction form as $\frac{6}{4}$.

$\frac{5}{6} \div \frac{2}{3}$ can be written as a fraction form as $\dfrac{\frac{5}{6}}{\frac{2}{3}}$.

2. From $\frac{5}{6} \div \frac{2}{3}$, the teacher discusses how to find the quotient.

Example 1 Find the quotient of $\frac{5}{6} \div \frac{2}{3}$

Solution

$$\frac{5}{6} \div \frac{2}{3} = \dfrac{\frac{5}{6}}{\frac{2}{3}} \qquad \text{(Step 1)}$$

$$= \dfrac{\frac{5}{6} \times \frac{3}{2}}{\frac{2}{3} \times \frac{3}{2}} \qquad \text{(Step 2)}$$

$$= \dfrac{\frac{5}{6} \times \frac{3}{2}}{1} \qquad \text{(Step 3)}$$

$$= \frac{15}{12} = \frac{5}{4} = 1\frac{1}{4}$$

Example 2 Find the quotient of $4\frac{9}{10} \div 7$

Solution

$$4\frac{9}{10} \div 7 = \frac{49}{10} \div 7$$

$$= \dfrac{\frac{49}{10}}{7} \qquad \text{(Step 1)}$$

$$= \dfrac{\frac{49}{10} \times \frac{1}{7}}{7 \times \frac{1}{7}} \qquad \text{(Step 2)}$$

$$= \dfrac{\frac{49}{10} \times \frac{1}{7}}{1} \qquad \text{(Step 3)}$$

$$= \frac{7}{10}$$

3. The teacher discusses the shortcut of dividing fractions with the students. Division of fractions can be performed by multiplying the dividend by the reciprocal of the divisor.

Example 3 Evaluate $\frac{3}{14} \div [1\frac{7}{8} \div 1\frac{5}{7}]$

Solution
$$\begin{aligned}
\frac{3}{14} \div [1\frac{7}{8} \div 1\frac{5}{7}] &= \frac{3}{14} \div [\frac{15}{8} \div \frac{12}{7}] \\
&= \frac{3}{14} \div [\frac{15}{8} \times \frac{7}{12}] \\
&= \frac{3}{14} \div [\frac{5}{8} \times \frac{7}{4}] \\
&= \frac{3}{14} \div \frac{35}{32} \\
&= \frac{3}{14} \times \frac{32}{35} \\
&= \frac{3}{7} \times \frac{16}{35} = \frac{48}{245}
\end{aligned}$$

4. Have the students do the problems in the textbook, page 131 using the shortcut method.

1) $10 \div \frac{2}{3}$ 2) $\frac{11}{3} \div 3$ 3) $\frac{2}{5} \div \frac{9}{10}$ 4) $\frac{21}{5} \div \frac{7}{8}$

5. Have the students do the practice problems in the worksheet and do the exercise 6.4, problems 1.1, 1.3, 1.5, 1.7, 1.9 from the textbook on page 134.

Closing the Lesson

The teacher and the students summarize the rules for division of fraction by:

1. Convert a mixed number, if any, to an improper fraction.
2. Division of fractions can be performed by inverting the divisor, changing the division sign into the multiplication sign, multiplying with the dividend.
3. Compute the product using the rules of multiplication of fractions.

5. Instructional Materials
   - Practice problem sheet.

6. Assessment

Assessment	Criteria
1. Observe the students' answers	1. 80% of the students are able to answer the questions correctly.
2. Have the student do the examples	2. All of the students are able to correctly solve 3 given examples out of 4 problems
3. Have the students do the practice sheet	3. The students are able to correctly solve 9 problems out of 10 problems in the practice sheet.
3. Have the students do the exercise 6.4 b, problems 1.1, 1.3, 1.5, 1.7, 1.9 from page 134 in the textbook	3. The students are able to correctly solve 4 problems out of 5 problems in the exercise.

Practice Sheet

1. Find the quotients of the following problems using the first method.

1.1) $\frac{3}{4} \div \frac{5}{8} =$

1.2) $\frac{9}{24} \div \frac{11}{25} =$

1.3) $3\frac{1}{2} \div 2\frac{2}{3} =$

2. Find the quotients of the following problems using the shortcut

2.1) $\frac{11}{24} \div \frac{5}{8} =$

2.2) $\frac{15}{16} \div \frac{5}{24} =$

2.3) $1\frac{1}{2} \div 3 =$

2.4) $17\frac{3}{5} \div 3\frac{1}{7} =$

2.5) $\frac{10}{11} \div \frac{5}{2} =$

2.6) $\frac{99}{100} \div \frac{11}{25} =$

2.7) $2\frac{5}{8} \div 1\frac{3}{4} =$