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Green and Ampt infiltration into soils of variable pore size with depth

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Abstract. Most expressions for infiltration rely upon the assumption of vertical uniformity in soil texture and hydraulic properties. We present an extension of the results of *Beven* [1982, 1984] for infiltration and lateral flow in soils with decreasing permeability with depth. Unlike Beven, we base the derivations upon joint changes in soil properties based on an overall change in characteristic pore size via Miller scaling. A set of very simple expressions for the time rate of infiltration are obtained using a Green and Ampt approach for soils with permeability that decreases with depth following linear, power law, and exponential relationships.

1. Introduction

It is widely recognized that soil hydraulic conductivity typically decreases from the surface, yet most infiltration models ignore this fact. *Beven* [1982, 1984] introduced the possibility that analytical expressions could be derived for small catchments based on soil descriptions which vary with depth. In this paper we build on these results to obtain equations which may be useful in a variety of hydrologic settings.

The basic notion that we would like to pursue is the derivation of equations for infiltration that explicitly include issues of changing soil properties with depth. *Beven* [1982] presents

$$K_s(z) = K_*(D - z)^n \tag{1}$$

$$\theta_s(z) = \theta_*(D - z)^m \tag{2}$$

to describe the vertical profiles in saturated conductivity K_s and saturated moisture content θ_s with increasing depth z (positive downward) for a soil with total depth D above an impermeable layer. Here K_*, θ_*, n, and m were taken to be parameters to be fitted to the site data, where Beven supposed that $n \approx 2m$ on the basis of scaling relations for permeability versus porosity.

Later, when developing a Green and Ampt infiltration model, *Beven* [1984] employed an exponential relationship given as

$$K_s = K_o \exp{(fz)} \tag{3}$$

$$\theta_s = \theta_o \exp{(gz)} \tag{4}$$

where f and g are fitting parameters which were seen to be unrelated when fit to real soils. Within this framework, Beven went on to construct a very interesting conceptual model for hillslope hydrology and generated a series of hydrographs. In the present paper we refine the application of the above stated relations in the context of infiltration.

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2. Analysis

First we would like to revisit the data of *Childs and Bybordi* [1969] as presented by *Beven* [1984] to provide a physical framework for our analysis (Table 1). We start by asking the question: Why do soils have lower permeability with depth? Is it due to decrease in porosity due to greater packing density of particles of essentially homogeneous particle size? This is not supported by the data in Table 1, where we observe monotonically increasing porosity with depth, while conductivity drops by a factor of 26. In fact, Figure 2 of *Beven* [1984] dispels the notion of a strong correlation between the change in conductivity and the change in porosity with depth. To obtain a more physically reasonable connection, we recall that much of spatial variability in soils can be explained by appealing to Miller similarity [*Miller and Miller*, 1956]. Miller scaling provides a quantitative formulation to relate the hydraulic properties of soils that have particle and pore size distributions which are geometrically similar but that have dissimilar mean size. Recent examples that demonstrate the utility of this approach include the works of *Rockhold et al.* [1996] and *Warrick* [1990].

Let us then suppose that the characteristic pore size varies with depth and that this is the primary factor affecting permeability. We would then expect that as the pore size decreased, conductivity would drop with the square of pore size and that the Green and Ampt wetting front pressure would increase linearly. Table 1 presents the data of *Childs and Bybordi* [1969] with an additional computed column listing the product $K_s^{1/2}\psi_{wf}$, a parameter that would be predicted to be constant if Miller similarity was valid. This notion is well supported by the observation that this product changes by <4% in the first 1.5 m of the soil profile, where both K and the Green and Ampt wetting front potential ψ_{wf} vary by 50 times this amount.

Table 1. Soil Properties With Depth

Layer, m	K_s, m h^{-1}	ψ_{wf}, m	$\Delta\theta$	$K_s^{1/2}\psi_{wf}$
0–0.3	13.2	−0.06	0.35	−0.218
0.3–0.6	7.5	−0.08	0.3355	−0.219
0.6–0.9	4.2	−0.10	0.36	−0.205
0.9–1.2	2.9	−0.125	0.36	−0.213
1.2–1.5	1.7	−0.159	0.365	−0.207
1.5–1.8	0.5	−0.178	0.37	−0.126

Data are from *Childs and Bybordi* [1969] as presented by *Beven* [1984]. Each soil layer was characterized by a saturated conductivity K_s, a Green and Ampt wetting front potential ψ_{wf}, and the available pore space $\Delta\theta$. The final column lists a product which would be constant if the depth-varying soils were similar in the sense of *Miller and Miller* [1956].

There is great advantage in employing Miller scaling, in that it provides a physical link between the vertical variation in conductivity and more widely reported values of variability in particle size. Further, it allows a reduction in the number of parameters, as we may link n and m as well as f and g used by Beven, as shown below.

We desire simple expressions in soils with vertically decreasing conductivity for infiltration rates as a function of time. To achieve these ends, we solve for Green and Ampt infiltration for soils which obey either (1) or (3), with changes in permeability explained by changes in pore size which obey Miller scaling [*Miller and Miller*, 1956].

First we will consider a vertically fining soil such that the characteristic microscopic length scale λ follows the relationship

$$\lambda(z) = \lambda_o \exp(-\beta z) \tag{5}$$

where β is a scale parameter with unit of inverse length. From Miller scaling we then know that the saturated conductivity will vary with depth following the relationship

$$K(z) = K_o \exp(-2\beta z) \tag{6}$$

where K_o is the saturated conductivity of the uppermost soil. Similarly, the Green and Ampt wetting front potential, which in general will be negative, will follow

$$\psi_{wf}(z) = \psi_{wf_o} \exp(\beta z) \tag{7}$$

We note that *Beven* [1984] held the product $\Delta\theta\psi_{wf_o}$ constant to ease computation, which, as noted by Beven and illustrated in Table 1, is not in keeping with observations.

For vertical Green and Ampt infiltration q, we know from Darcy's law that

$$\begin{aligned} q &= -K(z)\frac{dH}{dz} \\ &= -K_o \exp(-2\beta z)\frac{dH}{dz} \end{aligned} \tag{8}$$

where H is the total potential (pressure plus elevation) measured in units of hydraulic head. The wetting front is assumed to be sharp; therefore the flux is constant with depth behind the wetting front but will vary in time. Solving for total potential, we find

$$dH = -\frac{q}{K_o}\exp(2\beta z)\, dz \tag{9}$$

which may be integrated from the surface, where $H = z = 0$ to any depth z, to obtain

$$H = \frac{q}{2\beta K_o}[1 - \exp(2\beta z)] \tag{10}$$

Writing this in terms of the pressure potential h, we recall that for position measured positive in the downward direction we have $H = h - z$:

$$h = \frac{q}{2\beta K_o}[1 - \exp(2\beta z)] + z \tag{11}$$

If we take the depth of the wetting front to be z^*, we may use (7) to obtain the potential at the wetting front $h(z^*)$:

$$h(z^*) = \psi_{wf_o}\exp(\beta z^*) \tag{12}$$

Combining (11) and (12) and solving for flux, we find

$$q = 2\beta K_o\left(\frac{\psi_{wf_o}\exp(\beta z^*) - z^*}{1 - \exp(2\beta z^*)}\right) \tag{13}$$

A simple check on (13) is to let β go to 0, which yields the expected rate of infiltration for uniform media $q = -K_o(\psi_{wf_o} - z^*)/z^*$.

Recognizing that for Green and Ampt infiltration,

$$q = \Delta\theta \frac{dz^*}{dt} \tag{14}$$

we may calculate the depth of wetting as a function of time as

$$t(z^*) = \frac{\Delta\theta}{2\beta K_o}\int_0^{z^*}\left(\frac{1 - \exp(2\beta z)}{\psi_{wf_o}\exp(\beta z) - z}\right) dz \tag{15}$$

which may be evaluated numerically as needed.

If the soil fines follow a power law, such as suggested in (1), the same calculations may be carried forward. The simplest case is a linear decline in particle size, in which case we have

$$\psi_{wf}(z) = \psi_{wf_o}\beta z \tag{16}$$

$$K_s(z) = K_o\beta^{-2}z^{-2} \tag{17}$$

where β is a scale parameter with unit of inverse length. Computing as before, we find that for Green and Ampt infiltration,

$$q = -\frac{3K_o(\beta\psi_{wf_o} - 1)}{z^{*2}\beta^2} \tag{18}$$

and that the wetting front moves in with time following

$$z^* = -\left(\frac{9K_o(\beta\psi_{wf_o} - 1)t}{\beta^2\Delta\theta}\right)^{1/3} \tag{19}$$

which is, surprisingly, simpler than the implicit logarithmic form obtained for vertical infiltration into a homogeneous profile.

This computation can be carried out for other powers of depth as well. Of particular interest is the very general power law relationship introduced simultaneously by *Duan and Miller* [1997] and *Iorgulescu and Musy* [1997]. Both investigations show that the power law model may be employed in the hillslope models of *Ambroise et al.* [1996], and here we demonstrate that the power law model is easily adapted to the Miller-similar Green and Ampt infiltration approach.

In the notation of the present discussion, the general model may be obtained from the soil scale parameter

$$\lambda = \lambda_o(1 - \beta z/n)^n \tag{20}$$

where $n > 0$. It is interesting to note that (20) reduces to the exponential model given in (3) for $n = \infty$. Appealing to Miller scaling, we may deduce the expected conductivity and wetting front potentials for a soil which obeys (20) to be

$$K_s(z) = K_o(1 - \beta z/n)^{2n} \tag{21}$$

$$\psi_{wf}(z) = \psi_{wf_o}(1 - \beta z/n)^{-n} \tag{22}$$

Equations (21) and (22) are physically meaningful at depths where $z < n/\beta$, which might be interpreted as the depth to an impermeable layer. Computing as before, the flux as a function of wetting front depth is given by the simple algebraic expression

$$q = K_o\beta\left(\frac{2n-1}{n}\right)\left(\frac{\psi_{wf_o}(1 - \beta z^*/n)^{-n} - z^*}{1 - (1 - \beta z^*/n)^{1-2n}}\right) \tag{23}$$

As before, we can write the relationship between elapsed time and depth of infiltration using (14). Proceeding formally, in this case this yields

$$t(z^*) = \frac{\Delta\theta n}{K_o\beta(2n-1)}\int_0^{z^*}\frac{1 - (1 - \beta z/n)^{1-2n}}{\psi_{wf_o}(1 - \beta z/n)^{-n} - z}\,dz \tag{24}$$

3. Example of Application

To illustrate the results, we can fit the three models to the data given in Table 1 and predict the cumulative infiltration in time. To fit the models, we took 0.36 to be the average value of $\Delta\theta$, fit the conductivity function to the data of *Childs and Bybordi* [1969] to obtain values of K_o, β, and n, and then, finally, fit ψ_{wf_o} to the *Childs and Bybordi* [1969] data set using the other parameters found in fitting the conductivity data.

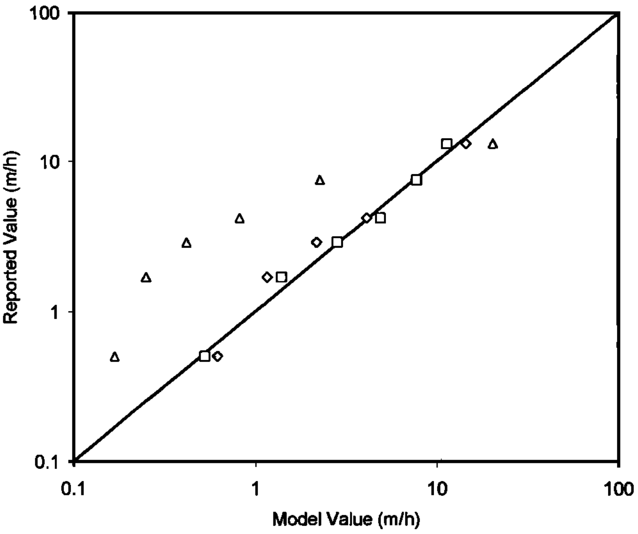

Figure 1. Model versus measured permeability based on the three models' fit to the data of *Childs and Bybordi* [1969]. The solid line is the line of perfect fit, the triangles are the linear model, the squares are the power law model, and the diamonds are the exponential model.

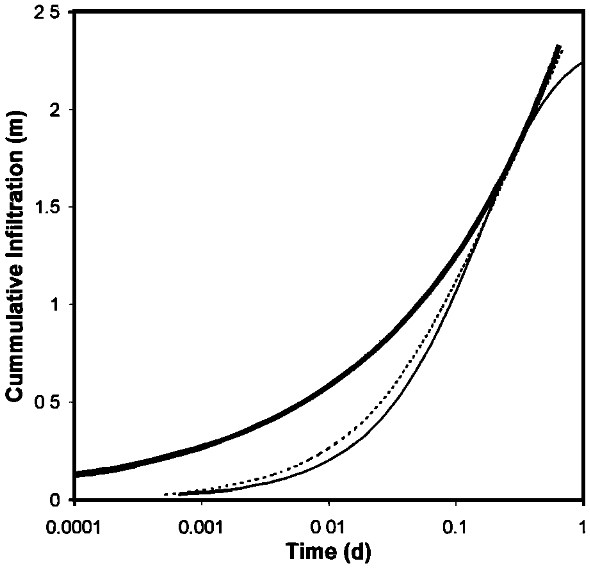

Figure 2. Predicted cumulative infiltration based on the three models' fit to the data of *Childs and Bybordi* [1969]. The heavy solid line is the linear model, the light solid line is the power law model, and the dashed line is the exponential model.

Optimal fit was taken as the parameter set that provided minimum sum of square error divided by the square of the measured values. This approach avoided giving undue weight to the large permeability values. Clearly, alternate fitting procedures could be selected as appropriate. The cumulative infiltration was then predicted using (15), (19), and (24).

The fit conductivities show that the more flexible exponential and power law models fit the data best, while the linear model was unable to fit the permeability data well (Figure 1). The predicted infiltrations of the three models (Figure 2) have several notable characteristics. The early time infiltration predicted by the linear model significantly exceeds that of the other models, predicting a factor of 2 more infiltration in the first 0.5 hours. This would give rise to significant discrepancies in short-time runoff predictions. The models are within 30% of each other at intermediate times (0.1–0.50 days). The power law model diverges from the other models at longer times as the depth of the presumed impermeable layer is approached ($t > 0.6$ days).

4. Summary

We have found that the use of the Miller similarity allows for exact solutions to the Green and Ampt vertical infiltration problem for a variety of continuously varying soil texture profiles. The expressions obtained are physically reasonable, simple to apply, and sufficiently flexible to be fit to a wide range of soil profiles. They can be applied in the full range of settings where the Green and Ampt approach has proven to be a useful quantitative model for infiltration.

References

Ambroise, B., K. Beven, and J. Freer, Toward a generalization of the TOPMODEL concepts: Topographic indices of hydrological similarity, *Water Resour. Res.*, *32*, 2135–2145, 1996.

Beven, K., On subsurface stormflow: An analysis of response times, *Hydrol. Sci. J.*, *27*, 505–521, 1982.
Beven, K., Infiltration into a class of vertically non-uniform soils, *Hydrol. Sci. J.*, *29*, 425–434, 1984.
Childs, E. C., and M. Bybordi, The vertical movement of water in stratified porous material, 1, Infiltration, *Water Resour. Res.*, *5*, 446–459, 1969.
Duan, J., and N. Miller, A generalized power function for the subsurface transmissivity profile in TOPMODEL, *Water Resour. Res.*, *33*, 2559–2562, 1997.
Iorgulescu, I., and A. Musy, Generalization of TOPMODEL for a power law transmissivity profile, *Hydrol. Processes*, *11*, 1353–1355, 1997.
Miller, E. E., and R. D. Miller, Physical theory for capillary flow phenomena, *J. Appl. Phys.*, *27*, 324–332, 1956.
Rockhold, M. L., R. E. Rossi, and R. G. Hills, Application of similar media scaling and conditional simulation for modeling water flow and tritium transport at the Las Cruses trench site, *Water Resour. Res.*, *32*, 595–609, 1996.
Warrick, A., Application of scaling to the characterization of spatial variability of soils, *SSSA Spec. Publ.*, *25*, 39–52, 1990.

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