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The goal of this research is to develop an equation describing the two, dimensional motion of an inviscid incompressible fluid in the rectangular wavemaker of constant depth. The boundary value problem of the rectangle is transformed to the upper half plane with the use of Jacobian elliptical functions. The boundary value problem is then transformed to the unit disc. The solution to the mixed value problem of the disc is found using a general solution satisfying the Laplace equation in polar coordinates. In order to solve the coefficients of the general solution, a system of equations is developed using a method similar to the one applied for the coefficients of a Fourier series. The system is converted to matrix form and the coefficients are calculated using Mathematica. Four approximate solutions are calculated for depths of 3.96 m and 4.42 m with N equal to 2 and 10.

THE BOUNDARY VALUE PROBLEM
FOR
THE RECTANGULAR WAVEMAKER

by

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a	intersection point of the surface and upper wall
a_o	intersection point of the upper wall and wavemaker
α	parameter for adjusting the unit disc
b	intersection point of the lower wall and bottom
b_o	hinge point
c	intersection point of the bottom and far wall
cn $(=\sqrt{1-\mathrm{sn}^2})$	a Jacobian Elliptical Function
d	intersection point of the far wall and surface
dn $(=\sqrt{1-k^2\mathrm{sn}^2})$	a Jacobian Elliptical Function
ϵ_o	phase angle of the instantaneous wavemaker displacement
$\eta(x,t)$	free surface elevation equation
f(y)	shape function of the wavemaker displacement
g	gravitational constant
h	stillwater depth (depth of rectangle)
i $(=\sqrt{-1})$	imaginary unit
Im(•)	imaginary component of a complex variable
k	propagating wave number
k_o $(=\omega^2/g)$	wave number
K	$\mathrm{sn}^{-1}[1,k]$
K′	$\mathrm{sn}^{-1}[1,k']$
L	wave length
m	parameter used in Mathematica for the Jacobian Elliptical Functions
m′	complementary parameter used in Mathematica for the Jacobian Elliptical Functions
ω $(=2\pi/T)$	wave frequency
$\bar{\phi}$	scalar velocity potential
q $(=re^{i\theta})$	complex variable for the unit disc
Q $(=Re^{i\Theta})$	complex variable for the adjusted unit disc
Re(•)	real component of a complex variable
sn	a Jacobian Elliptical Function

T		wave period
w	$(=u+iv)$	complex variable for the initialized rectangle
z	$(=x+iy)$	complex variable for the physical domain
Z	$(=X+iY)$	complex variable for the upper half plane

THE BOUNDARY VALUE PROBLEM FOR THE RECTANGULAR WAVEMAKER

1. INTRODUCTION

At Oregon State University there are two types of wave makers; one is rectangular and the other is circular. Ocean engineers study the dynamics of the waves through the use of these machines. One aspect of the study is developing an equation that represents the motion of the wave. This paper explores the treatment of one such equation for the rectangular wave tank and is an extension of Yoshihiro Tanaka's "Irregular Points in Wavemaker Boundary Value Problems."

The equation to be developed is restricted by the boundaries of the wave tank. Each part of the boundary (Figure 1.1) affects the wave differently. As the wave moves away from the wavemaker, it radiates its energy, and all of its energy is radiated when the wave has traveled three depth lengths. The mathematical representation of these effects are called boundary conditions. In order to satisfy conditions throughout the inside of the tank, a specific equation, called a Laplace equation, must be used.

Due to the shape of the tank, difficulties arise in developing the equation. For each of the six points in Figure 1.1, the boundary conditions vary depending upon the path approaching any one of these points. An example of the varying boundary conditions would be at the hinge point, b_o, where the wavemaker meets the lower wall. If the approach to b_o is made along the wavemaker, the boundary condition is different from when the approach is made along the lower wall. The corners of the wavemaker pose another problem, similar to that of trying to 'roll' a box. The corners of a box make it difficult to roll. A ball with its smooth surface is much easier to roll. Due to the problems mentioned above, a rectangle is not an ideal boundary from which to develop an equation.

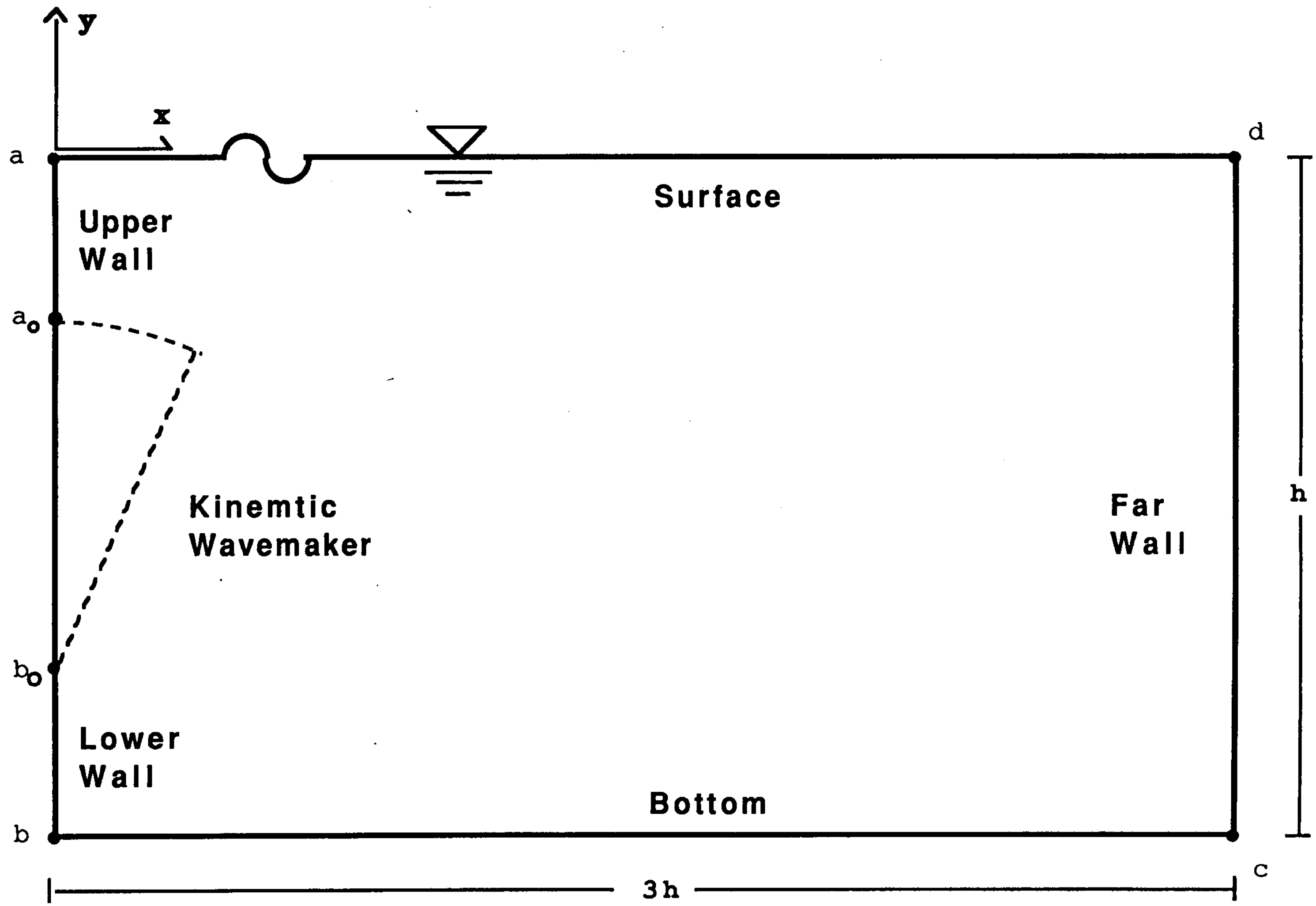


Figure 1.1 The sketch for the generic wavemaker in a two dimensional channel

In order to overcome these difficulties all the points in the rectangle, including the boundaries, are mapped into a circle. A way to think of this mapping is to imagine the rectangle molded into a circle. The perimeter is now the perimeter of the circle, and the inside of the rectangle is now arranged inside the circle. This molding, called a transformation, is performed in two steps so pre-existing equations can be used. The first step transforms the rectangle to the upper half plane. This step can be imagined as a person opening a Japanese fan. When the fan is closed, it is rectangular in shape and all the points are within the braces of the fan. When opened, the fan spreads out and the braces are straight resembling the boundary that is the horizontal axis in the plane. The next step transforms the upper half plane to the circle with a radius of one. This step can be imagined as a playful uncle greeting his nieces and nephews. At first the arms are extended wide and are straight like the horizontal axis. As he gathers his nieces and nephews, the arms encircling them form the boundary where they're all snuggled inside.

Once the transformations are completed, the next task is to transfer the boundary conditions of the rectangle onto the boundary of the circle. This task is also performed in two steps. The first task transfers the boundary conditions of the rectangle to boundary conditions of the upper half plane along the horizontal axis. The horizontal axis will also be partitioned in six ways that correspond to the six parts of the boundary of the rectangle. Then, these boundary conditions of the upper half plane are transferred to the boundary of the circle, also partitioned in six parts. With each of these steps, the boundary conditions are rewritten using the coordinates that are appropriate for that shape; the rectangle and the upper half plane use x and y, but the circle uses r and θ.

Now that the shape has been transformed into the circle and the boundary conditions have been transferred, the equation is developed. Even though the new shape eliminates some of the difficulties, the development of the equation remains difficult.

Due to the different boundary conditions associated with each of the six parts of the wave tank, the precise equation that meets all of the requirements will not be developed. Instead, an approximation is made using a general equation. Once the approximation equation is developed in the circle, it is transferred back into the rectangle by changing the variables of r and θ back to those of x and y. Thus, an equation is developed that satisfies the restrictions owing to the shape and to the boundary conditions.

2. THE BOUNDARY VALUE PROBLEM IN THE PHYSICAL DOMAIN

The two dimensional, irrotational motion of an inviscid, incompressible fluid in this rectangle basin of constant water depth, h, may be computed from $\vec{u}(x,y,t) = -\vec{\nabla}\bar{\phi}(x,y,t)$ where the scalar velocity potential $\bar{\phi}$ is defined by

$$\bar{\phi}(x,y,t) = \mathrm{Re}\{\phi(x,y)\ \exp(-i\omega t)\}. \tag{2.1}$$

The potential, $\bar{\phi}(x,y,t)$, for waves generated by the simple-harmonic motion of the wavemaker apparatus about its mean position $x = 0$ with the frequency $\omega = 2\pi/T$, is a solution to the Laplace equation $\phi_{xx} + \phi_{yy} = 0$ for $0 < x < 3h$ and $-h < y < 0$. T is the wave period. Also, the potential must satisfy the following boundary conditions (Figure 2.1).

At $0 < x < 3h$ and $y = -h$, the Bottom Boundary Condition is

$$\phi_y = 0. \tag{2.2}$$

At $0 < x < 3h$ and $y = 0$, the Free Surface Boundary Condition is

$$\phi_y - k_o\phi = 0 \tag{2.3}$$

with $k_o = \omega^2/g$. g is the gravitational acceleration.

At $x = 0$ and $b_o < y < a_o$, the Wavemaker Boundary Condition is

$$\phi_x = -\omega\ f(y)\ \exp[-i\epsilon_o]. \tag{2.4}$$

ϵ_o is an arbitrary phase angle of the instantaneous wavemaker displacement and f(y) is the shape function of the wavemaker displacement.

At $x = 3h$ and $-h < y < 0$, the Far Wall Boundary Condition is

$$\phi_x - ik\phi = 0. \tag{2.5}$$

k is the propagating wave number. When the tank has a length such that the wave radiates all of its energy before reaching the far

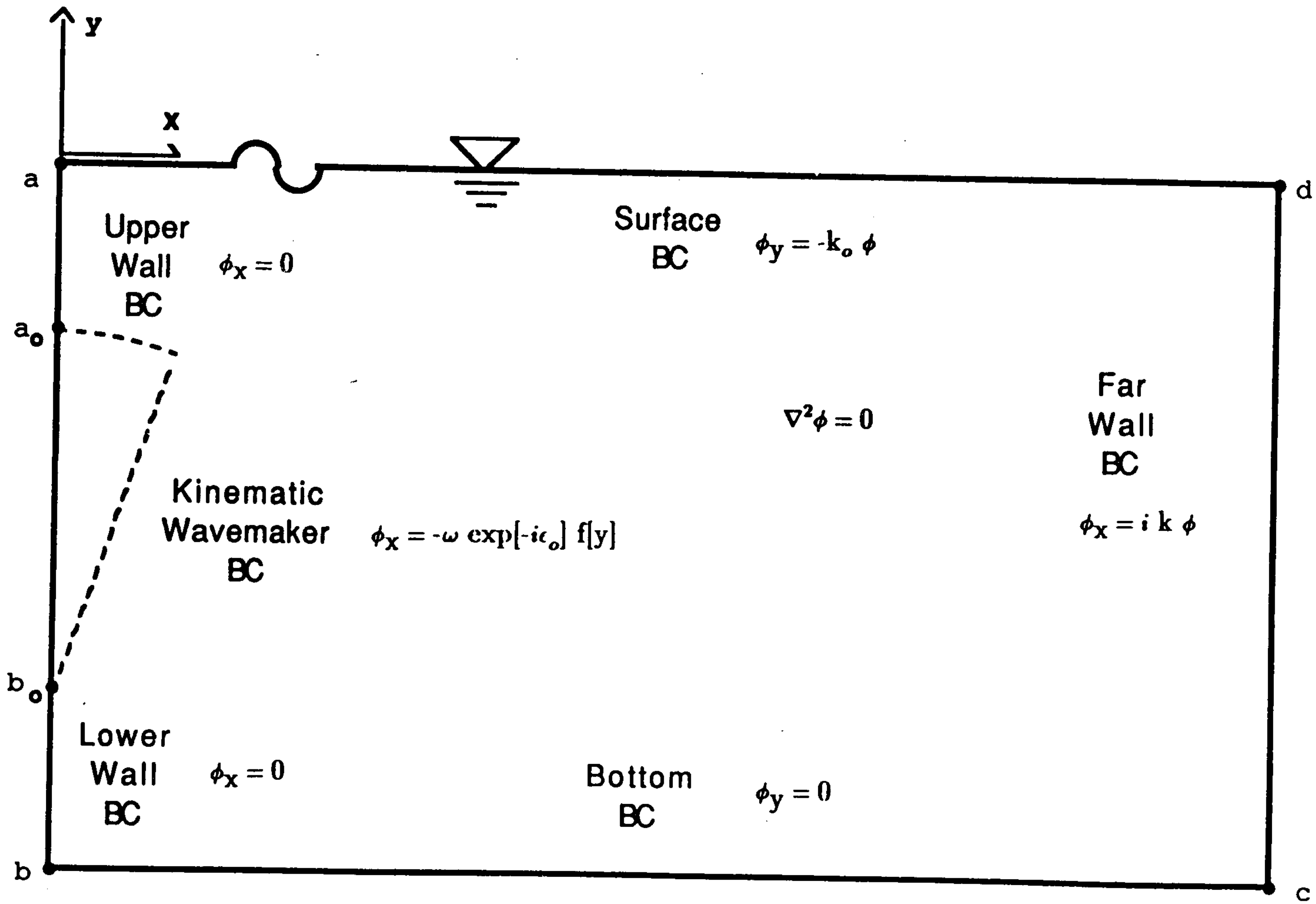


Figure 2.1 The boundary value problem in the physical domain

wall, k is $2\pi/L$; else, k is 0. L is the wave length rather than the length of the tank.

At $x=0$ and $a_o<y<0$ and at $x=0$ and $-h<y<b_o$, both the Upper Wall Boundary Condition and the Lower Wall Boundary Condition, respectively, are

$$\phi_X = 0. \tag{2.6}$$

The linearized wavemaker boundary value problem in the rectangle has been expressed in terms of a unknown complex-valued velocity potential, $\phi(x,y)$; thus, the geometry of the rectangle can be considered in the complex plane, $z=x+iy$. Since it is in the complex plane, the rectangle can be transformed to the unit disc.

3. THE MAPPING OF THE RECTANGLE TO THE UNIT DISC

The mapping of the rectangle to the unit disc is done in two stages: first, the rectangle is mapped to the upper half plane and then the upper half plane is mapped to the unit disc. The first stage of the mapping (Figure 3.1) is accomplished using the Schwarz-Christoffel Transformation,

$$\mathrm{dw} = (a-t)^{-\alpha}(b-t)^{-\beta}(c-t)^{-\gamma}(d-t)^{-\lambda}\mathrm{dt}.$$

The points $a=-1$, $b=1$, $c=\frac{1}{\sqrt{\mathrm{m}}}$, and $d=-\frac{1}{\sqrt{\mathrm{m}}}$ are on the real axis and are the mapping of the vertices of the rectangle (the value of m will be determined shortly). The exponents α, β, γ, and λ are the measure of the angles divided by π. For the rectangle the exponents have the value $\frac{1}{2}$. By integrating both sides of the transformation, the mapping of the plane to the rectangle is

$$\mathrm{w} = \int_{Z_0}^{Z} \frac{\mathrm{dt}}{(a-\mathrm{t})^{\frac{1}{2}}(b-\mathrm{t})^{\frac{1}{2}}(c-\mathrm{t})^{\frac{1}{2}}(d-\mathrm{t})^{\frac{1}{2}}}.$$

Since $a = -b$ and $c = -d$, the integral becomes

$$\mathrm{w} = \int_{Z_0}^{Z} \frac{\mathrm{dt}}{\sqrt{(a^2-\mathrm{t}^2)\ (c^2-\mathrm{t}^2)}}.$$

With the change of variable $\mathrm{t} = at$ and $\mathrm{dt} = a\mathrm{d}t$, the integral becomes

$$\mathrm{w} = \int_{Z_0}^{Z} \frac{\mathrm{d}t}{\sqrt{(1-t^2)\ (1-k^2t^2)}}. \tag{3.1}$$

$k^2 = \frac{a^2}{c^2} = \mathrm{m}$ and is evaluated in Appendix I. This integral is the

Jacobian Elliptical Integral. The right hand side of Eq.(3.1) is $\mathrm{sn}^{-1}[Z,\mathrm{k}]$; this relationship gives $\mathrm{sn}[\mathrm{w},\mathrm{k}] = Z$. $\mathrm{sn}[\mathrm{w},\mathrm{k}]$ is the Jacobian elliptical function and is discussed in Appendix I.

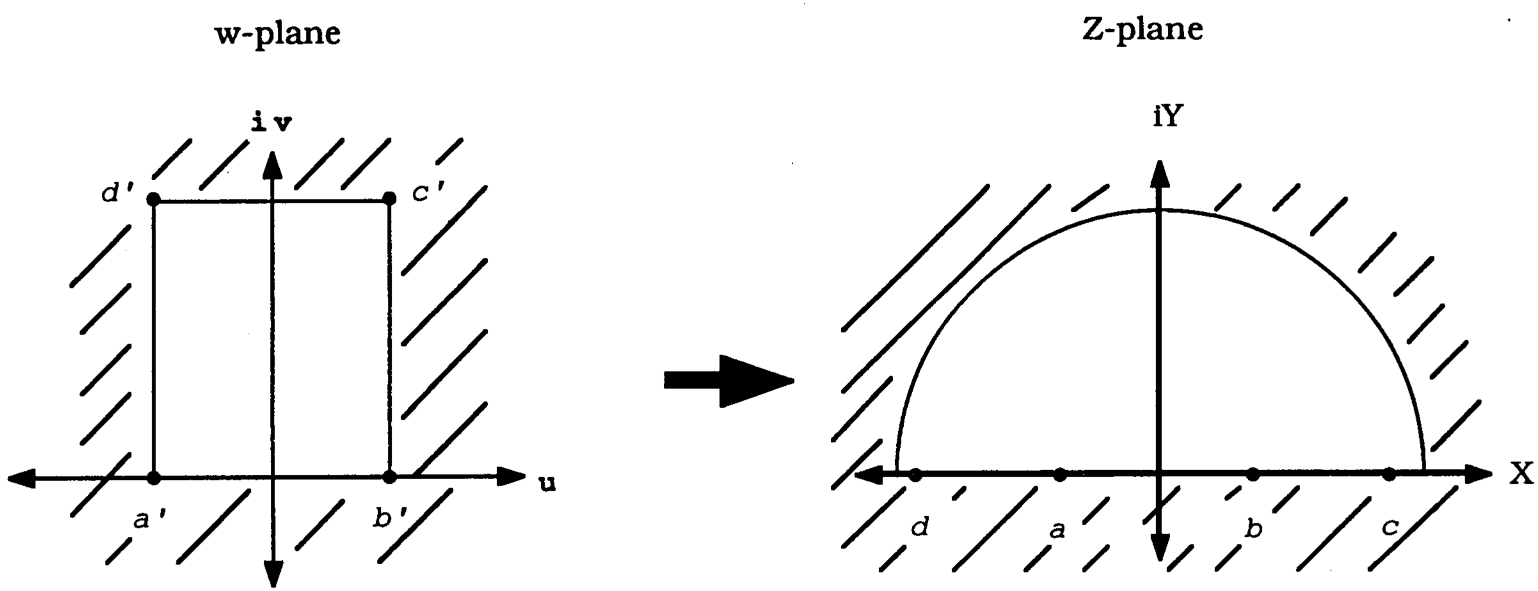


Figure 3.1 The mapping of a rectangle to the upper half plane

Since w is complex, the complex form of the Jacobian elliptical function must be introduced. In Appendix II, the complex form is developed for Eq.(3.1) by using the method presented by Bowman. With this complex form of the Jacobian elliptical function, the mapping of the rectangle to the upper half plane is performed by

$$\mathrm{sn}[u+iv,k]=\frac{\mathrm{sn}[u,k]\ \mathrm{dn}[v,k']+i\ \mathrm{cn}[u,k]\ \mathrm{dn}[u,k]\ \mathrm{sn}[v,k']\ \mathrm{cn}[v,k']}{1-\mathrm{dn}^2[u,k]\ \mathrm{sn}^2[v,k']}, \tag{3.2}$$

where cn[u,k] and dn[u,k] are related to sn[u,k] by the identities $\sqrt{1-\mathrm{sn}^2[u,k]}$ and $\sqrt{1-m\ \mathrm{sn}^2[u,k]}$, respectively. (For the relationship of dn[u,k] and sn[u,k], I use m rather than k^2. This use is done to keep in form with Mathematica which uses m as the parameter.)

In order to use sn[u+iv,k], the rectangle of the wave maker (Figure 3.2i) must be initialized. This initialization is done in three steps. In order to keep the orientation of the domain positive, the first step rotates the rectangle by $\frac{\pi}{2}$ radians. This rotation is accomplished by

$$w'=i*z.$$

The next step centers the rectangle and makes the rectangle symmetric to the imaginary axis (Figure 3.2iii). This step is required so that the rectangle maps to the whole upper half plane rather than just the positive side of the imaginary axis. The centering of the rectangle is accomplished by

$$w''=w'-\frac{h}{2}.$$

The third step for initializing the rectangle restricts the dimensions of the rectangle. This step is required to have an inverse function. In order to have an inverse, the rectangle must have the dimensions $-K<u<K$ and $0<v<K'$, where $K=\mathrm{sn}^{-1}[1,k]$ and $K'=\mathrm{sn}^{-1}[1,k']$ Thus, the third step required for initialization of the rectangle is

$$w=\frac{2K}{h}\ w''.$$

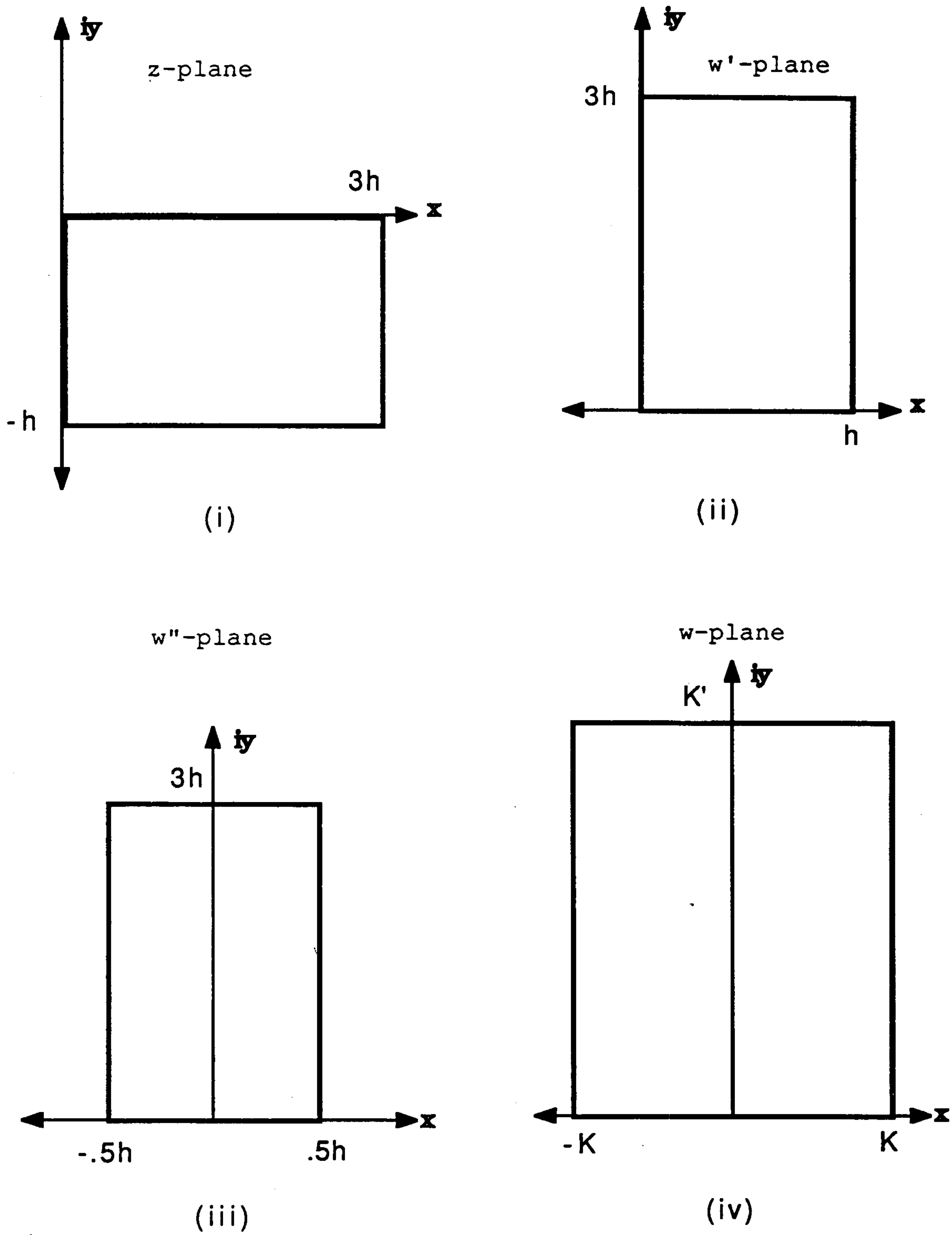


Figure 3.2 Initialization of the rectangle

Therefore, the function that initializes the rectangle for use of $sn[u+iv,k]$ is

$$w = \frac{2K}{h}(i*z - \frac{h}{2}).$$

In order to use $sn[u+iv,k]$, u and v, the real and imaginary components of the initialization equation, need to be in terms of x and y. When $x+iy$ is substituted for z into the initialization equation,

u is $-\frac{2K}{h}(\frac{h}{2}+y)$ and v is $\frac{2K}{h}$ x.

Hence, the equation for the mapping of the rectangle with dimensions $0<x<3h$ and $-h<y<0$ onto the upper half plane (Figure 3.3) is

$$Z = \frac{sn[-\frac{2K}{h}(\frac{h}{2}+y),k]\ dn[\frac{2K}{h}x,k']}{1-dn^2[-\frac{2K}{h}(\frac{h}{2}+y),k]\ sn^2[\frac{2K}{h}x,k']} + i\,\frac{cn[-\frac{2K}{h}(\frac{h}{2}+y),k]\ dn[-\frac{2K}{h}(\frac{h}{2}+y),k]\ sn[\frac{2K}{h}x,k']\ cn[\frac{2K}{h}x,k']}{1-dn^2[-\frac{2K}{h}(\frac{h}{2}+y),k]\ sn^2[\frac{2K}{h}x,k']}. \tag{3.3}$$

One point to be noted is that the calculation of the modulus (Appendix I) depends upon the ratio of the sides. Before the initialization steps are applied, the ratio of the sides of the physical tank is 3. Since the real value of Eq.(3.3) is an odd function, only the positive half of the rectangle needs to be considered. In Figure (3.2 iii) the sides of the positive half of the rectangle have a ratio of 6.

The next stage is the mapping from the upper half plane to the unit disc (Figure 3.4); this stage is done in two steps. Both functions are standard and can be found in most complex analysis book. The first is the function

$$q = \frac{(i - Z)}{(i + Z)}, \tag{3.4}$$

which is analytic and maps the upper half plane to the unit disc with the real axis mapped to the boundary of the disc. This mapping

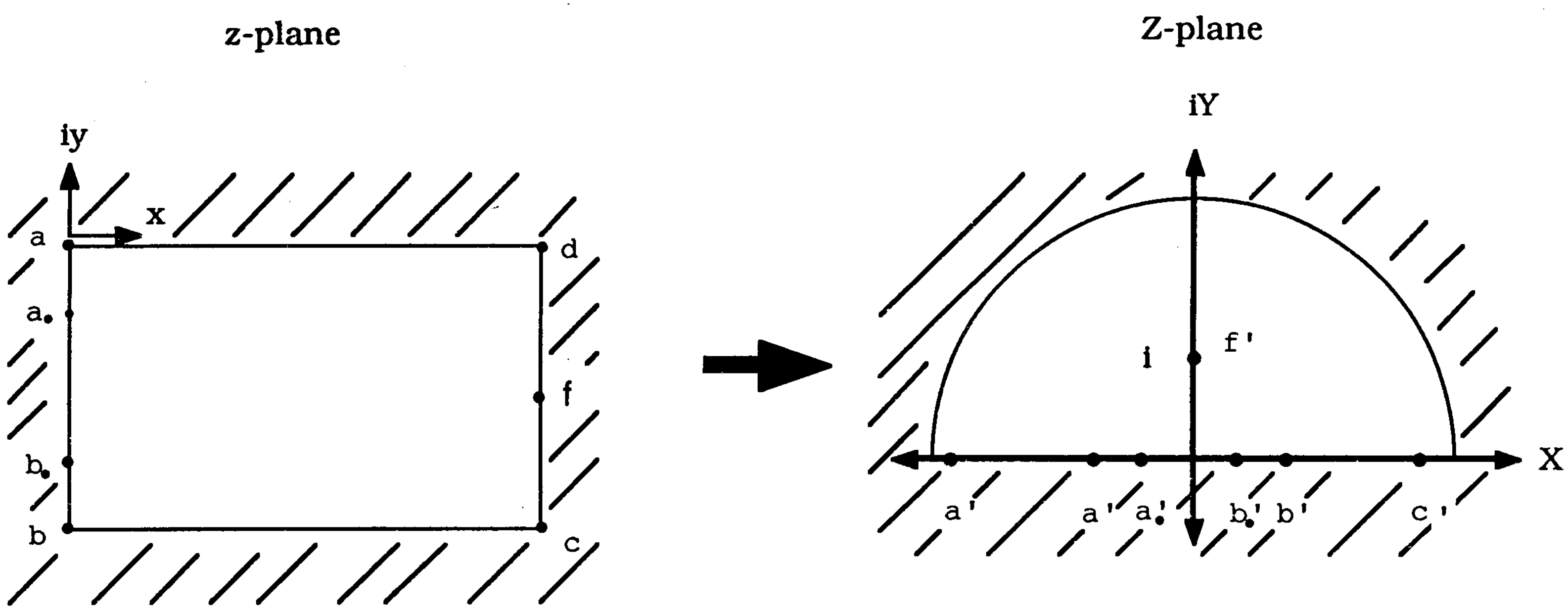


Figure 3.3 The mapping of the rectangle to the upper half plane

is not quite sufficient; the problem is that the far wall of the rectangle is mapped to a very small portion of the disc near r = 1 and $\theta = \frac{\pi}{2}$. This area of the disc can be enlarged using

$$Q = \frac{(q - \alpha)}{(1 - \alpha * q)}, \tag{3.5}$$

where α is real and $-1 < \alpha < 1$. By adjusting the value of α, the mapping of the area near the far wall of the rectangle onto the disc can be enlarged (appendix III). Thus, the mapping from the upper half plane to the adjusted unit disc (Figure 3.4) is the composition of Eqs.(3.4) and (3.5),

$$Q = \frac{\frac{i-Z}{i+Z} - \alpha}{1 - \alpha \; \frac{i-Z}{i+Z}}.$$

This mapping simplifies to

$$Q = \frac{(1-\alpha)^2 - Z^2(1+\alpha)^2 + i \; 2Z(1-\alpha^2)}{(1-\alpha)^2 + Z^2(1+\alpha)^2}. \tag{3.6}$$

When $X + iY$ is substituted for Z in this equation, the mapping of the upper half plane to the adjusted unit disc becomes

$$Q = \frac{(1-\alpha)^2 - (X^2+Y^2)(1+\alpha)^2 + i \; 2X(1-\alpha)}{(1-\alpha)^2 + 2Y - 2\alpha^2 Y + (X^2+Y^2)(1+\alpha)^2}. \tag{3.7}$$

The polar coordinates of the adjusted unit disc are

$$R = \sqrt{\frac{(1-\alpha)^2 - 2Y(1-\alpha^2) + (X^2+Y^2)(1+\alpha)^2}{(1-\alpha)^2 + 2Y(1-\alpha^2) + (X^2+Y^2)(1+\alpha)^2}} \tag{3.8}$$

and

$$\Theta = \text{Arctan}\left[\frac{2X(1-\alpha^2)}{(1-\alpha)^2 - (X^2+Y^2)(1+\alpha)^2}\right]. \tag{3.9}$$

When the real and imaginary components of Eq.(3.3) are substituted for X and Y in the polar coordinates of Eq.(3.8) and (3.9), the polar coordinates of the mapping from the rectangle to the adjusted unit disc are

$$R=\frac{(1-\alpha)^2(1-{d_1}^2{s_2}^2)^2-(1+\alpha)^2({s_1}^2{d_2}^2+{c_1}^2{d_1}^2{s_2}^2{c_2}^2)+2(1-\alpha^2)(c_1d_1s_2c_2)({d_1}^2{s_2}^2-1)}{(1-\alpha)^2(1-2{d_1}^2{s_2}^2)^2+(1+\alpha)^2({s_1}^2{d_2}^2+{c_1}^2{d_1}^2{s_2}^2{c_2}^2)-2(1-\alpha^2)(c_1d_1s_2c_2)({d_1}^2{s_2}^2-1)} \tag{3.10}$$

and

$$\Theta=\mathrm{Arctan}\left[\frac{2\ (1-\alpha^2)\ s_1\ d_2\ ({d_1}^2\ {s_2}^2-1)}{\left((1-\alpha)^2\ \left(1-{d_1}^2{s_2}^2\right)^2-(1+\alpha)^2\left({s_1}^2{d_2}^2+{c_1}^2{d_1}^2{s_2}^2{c_2}^2\right)\right)}\right]. \tag{3.11}$$

(Note: A shorthand notation is used by substituting s_1 for sn[u,k], c_1 for cn[u,k], d_1 for dn[u,k], s_2 for sn[v,k'], c_2 for cn[v,k'], and d_2 for dn[v,k'].)

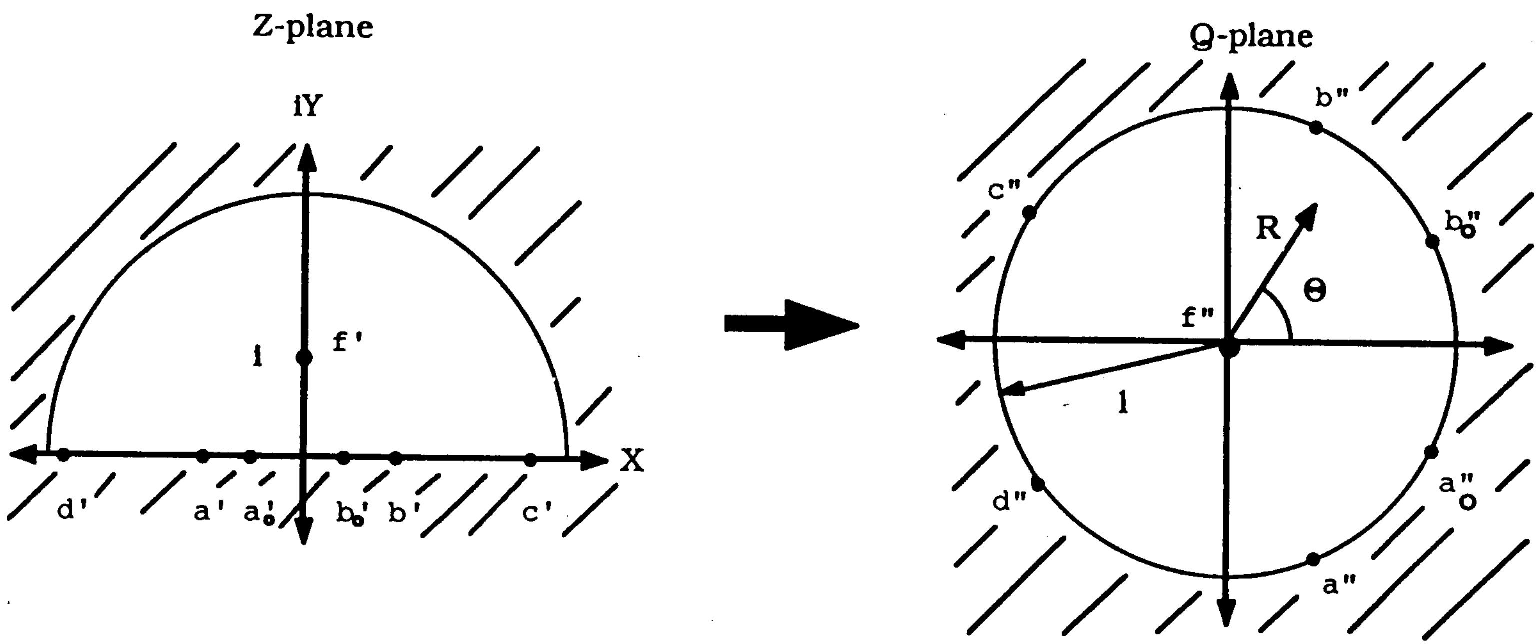


Figure 3.4 The mapping of the upper half plane to the unit disc

4. THE BOUNDARY VALUE PROBLEM IN THE UPPER HALF PLANE

After the transformation of the rectangle to the unit disc is completed, the boundary conditions on the rectangle can be transformed onto the upper half plane where the boundary is the real axis. Additionally, the transformation of $\phi(x,y)$ must be shown to still satisfy the Laplace equation. In order to do this transformation, let

$$\phi(x,y) = U(X,Y) = U(X(x,y),Y(x,y)),$$

where from Eq. (3.3)

$$X = \mathrm{Re}[\mathrm{sn}(x,y)] = \frac{\mathrm{sn}[u,\ k]\ \mathrm{dn}[v,\ k']}{1-\mathrm{dn}^2[u,\ k]\ \mathrm{sn}^2[v,\ k']}$$

$$Y = \mathrm{Im}[\mathrm{sn}(x,y)] = \frac{\mathrm{cn}[u,\ k]\ \mathrm{dn}[u,\ k]\ \mathrm{sn}[v,\ k']\ \mathrm{cn}[v,\ k']}{1-\mathrm{dn}^2[u,\ k]\ \mathrm{sn}^2[v,\ k']}$$

with $u = -\frac{2K}{h}\left(\frac{h}{2}+y\right)$ and $v = \frac{2K}{h}x$.

The partial derivatives of $\phi(x,y)$ are transformed by

$$\phi_X = U_X X_X + U_Y Y_X \tag{4.1}$$

and

$$\phi_Y = U_X X_Y + U_Y Y_Y. \tag{4.2}$$

The partials of X and Y are

$$X_X = \frac{2K\ s_1\ s_2\ c_2\ (-m'+2\ d_1^2\ d_2^2+m'\ d_1^2\ s_2^2)}{h(1-d_1^2\ s_2^2)^2},$$

$$X_Y = \frac{2K\ c_1\ d_1\ d_2\ (-1+d_1^2\ s_2^2+2m\ s_1^2\ s_2^2)}{h(1-d_1^2\ s_2^2)^2},$$

$$Y_X = \frac{2K\ c_1\ d_1\ d_2\ (1-d_1^2\ s_2^2-2m\ s_1^2\ s_2^2)}{h(1-d_1^2\ s_2^2)^2},$$

and

$$Y_Y = \frac{2K\ s_1\ s_2\ c_2\ (-m'+2\ d_1^2\ d_2^2+m'\ d_1^2\ s_2^2)}{h(1-d_1^2\ s_2^2)^2}.$$

The first boundary condition to be transformed is the Bottom Boundary Condition, $\phi_y = 0$. Since $y = -ih$ at the bottom of the rectangle, $\mathrm{cn}[u,k] = 0$, $\mathrm{sn}[u,k] = 1$, and $\mathrm{dn}[u,k] = \sqrt{m'}$. These values are substituted into the partials of X and Y and the partials with respect to y become

$$X_y = 0$$

and

$$Y_y = \frac{2K\ m'\ \mathrm{sn}[v,k']\ \mathrm{cn}[v,k']}{h(1 - m'\ \mathrm{sn}^2[v,k'])}.$$

The substitution of these two partials into Eq.(4.2) transforms the boundary condition to

$$0 = U_X * 0 \ + \ U_Y \frac{2K\ m'\ \mathrm{sn}[v,k']\ \mathrm{cn}[v,k']}{h(1 - m'\ \mathrm{sn}^2[v,k'])}.$$

Since $\mathrm{sn}[v,k']$ and $\mathrm{cn}[v,k']$ are not identically zero for $0 < x < 3h$, the Bottom Boundary Condition transformed onto the plane is

$$U_Y = 0 \tag{4.3}$$

for $1 < X < \frac{1}{\sqrt{m}}$ and $Y = 0$.

The second boundary condition to be transformed is the Surface Boundary Condition, $\phi_y = k_o\ \phi$. Since $y = 0$ at the top of the rectangle, $\mathrm{sn}[u,k] = -1$, $\mathrm{cn}[u,k] = 0$, and $\mathrm{dn}[u,k] = -\sqrt{m'}$. These values are substituted into the partials of X and Y and the partials with respect to y become

$$X_y = 0$$

and

$$Y_y = \frac{-2K\ m'\ \mathrm{sn}[v,k']\ \mathrm{cn}[v,k']}{h(1 - m'\ \mathrm{sn}^2[v,k'])}.$$

The substitution of these partials into Eq.(4.2) changes the boundary condition to

$$k_o\ U \ = \ U_Y \frac{-2K\ m'\ \mathrm{sn}[v,k']\ \mathrm{cn}[v,k']}{h(1 - m'\ \mathrm{sn}^2[v,k'])}.$$

Since $sn[v,k']$ and $cn[v,k']$ are not zero for $0 < x < 3h$, the Surface Boundary Condition transformed onto the plane becomes

$$U_Y = U \frac{-k_o\ h(1 - m'\ sn^2[v,k'])}{2\ K\ m'\ sn[v,k']\ cn[v,k']}.$$

Since the boundary condition in the plane needs to be in terms of

X and Y, $\sqrt{1 - sn^2[v,k']}$ is substituted for $cn[v,k']$ and $\frac{\sqrt{X^2-1}}{\sqrt{m'}\ X}$ is

substituted for $sn[v,k']$. Thus, the Surface Boundary Condition in the plane is

$$U_Y = U \frac{-k_o\ h\ m'}{2\ K\ \sqrt{X^2-1}\ \sqrt{m'\ X^2\ - X^2\ + 1}} \tag{4.4}$$

for $-\frac{1}{\sqrt{m}} < X < -1$ and $Y = 0$.

The third boundary condition to be transformed to the plane is the Wavemaker Boundary Condition, $\phi_X = -\omega\ \exp[-i\epsilon_o]\ f(y)$. Since $x = 0$ on the wavemaker part of the rectangle, $sn[v,k'] = 0$, $cn[v,k'] = 1$, and $dn[v,k'] = 1$. These values are substituted into the partials of X and Y and the partials with respect to x become

$$X_X = 0$$

and

$$Y_X = \frac{2\ K}{h}\ cn[u,k]\ dn[u,k].$$

The substitution of these partials into Eq.(4.2) changes the boundary condition to

$$-\omega\ \exp[-i\epsilon_o]\ f(y) = U_Y \frac{2\ K}{h}\ cn[u,k]\ dn[u,k].$$

Since $cn[u,k]$ and $dn[u,k]$ are not zero for $b_o < y < a_o$, the boundary condition can be written as

$$U_Y = \frac{-h\ \omega\ \exp[-i\epsilon_o]\ f(y)}{2\ K\ cn[u,k]\ dn[u,k]}.$$

In order to put the boundary condition in terms of X and Y, $\sqrt{1-\mathrm{sn}^2[u,k]}$ is substituted for $\mathrm{cn}[u,k]$, $\sqrt{1-m\ \mathrm{sn}^2[u,k]}$ is substituted for $\mathrm{dn}[u,k]$, and X is substituted for $\mathrm{sn}[u,k]$. Thus, the transformed Wavemaker Boundary Condition is

$$U_Y = \frac{-h\ \omega\ \exp[-i\epsilon_o]\ f\left(y=[-\frac{h}{2}-\frac{h}{2\ K}\ \mathrm{sn}^{-1}X]\right)}{2\ K\ \sqrt{1-X^2}\ \ \sqrt{1-m\ X^2}} \tag{4.5}$$

for $a'_o < X < b'_o$ and $Y = 0$.

The next boundary condition to be transformed is the Far Wall Boundary Condition, $\phi_X = ic\phi$. Since $x = 3h$ for the far wall part of the rectangle, $\mathrm{sn}[v,k'] = 1$, $\mathrm{cn}[v,k'] = 0$, and $\mathrm{dn}[v,k'] = \sqrt{m}$. These values are substituted into the partials of X and Y and the partials with respect to x become

$$X_X = 0$$

and

$$Y_X = -\frac{2\ K}{h}\ \frac{\sqrt{m}\ \mathrm{cn}[u,k]\ \mathrm{dn}[u,k]}{1-\mathrm{dn}^2[u,k]}.$$

The substitution of these partials into Eq. (4.1) changes the boundary condition to

$$i\ k\ U = -\ U_Y\ \frac{2\ K}{h}\ \frac{\sqrt{m}\ \mathrm{cn}[u,k]\ \mathrm{dn}[u,k]}{1-\mathrm{dn}^2[u,k]}.$$

Since $\mathrm{cn}[u,k]$ and $\mathrm{dn}[u,k]$ are not zero for $-h < y < 0$, the boundary condition becomes

$$U_Y = -\frac{i\ k\ h}{2\ K}\ \frac{1-\mathrm{dn}^2[u,k]}{\sqrt{m}\ \mathrm{cn}[u,k]\ \mathrm{dn}[u,k]}\ U.$$

In order to put the boundary condition in terms of X and Y, $\sqrt{1-\mathrm{sn}^2[u,k]}$ is substituted for $\mathrm{cn}[u,k]$, $\sqrt{1-m\ \mathrm{sn}^2[u,k]}$ is substituted for $\mathrm{dn}[u,k]$, and $\frac{1}{\sqrt{m}\ X}$ is substituted for $\mathrm{sn}[u,k]$. The boundary condition becomes

$$U_Y = -\frac{i\ k\ h}{2\ K}\ \frac{\sqrt{m}\ (\frac{1}{\sqrt{m}\ X})^2}{\sqrt{1-(\frac{1}{\sqrt{m}\ X})^2}\ \sqrt{1-m\ (\frac{1}{\sqrt{m}\ X})^2}}\ U,$$

which simplifies to

$$U_Y = -\frac{i\ k\ h}{2\ K} \frac{1}{\sqrt{m\ X^2 - 1}\ \ \sqrt{X^2 - 1}}\ U \tag{4.6}$$

for $\frac{1}{\sqrt{m}} < X < \infty$ and $Y = 0$ and $-\infty < X < -\frac{1}{\sqrt{m}}$ and $Y = 0$.

The final transformations are the Upper and Lower Wall Boundary Conditions, $\phi_x = 0$. Since $x = 0$ for both the upper and lower wall portion of the rectangle, $sn[v, k'] = 0$, $cn[v, k'] = 1$, and $dn[v, k'] = 1$. These values are substituted into the partials of X and Y and the partials with respect to x become

$$X_x = 0$$

and

$$Y_x = \frac{2\ K}{h}\ cn[u, k]\ dn[u, k].$$

The substitution of these partials into Eq.(4.1) changes the boundary condition to

$$0 = U_y\ \frac{2\ K}{h}\ cn[u, k]\ dn[u, k].$$

Since $cn[u, k]$ and $dn[u, k]$ are not identically zero for $-h \leq x \leq 0$, the Upper and Lower Boundary Conditions become

$$U_y = 0 \tag{4.7}$$

for $-1 < X < a'_o$ and $b'_o < X < 1$, respectively.

For the boundary value problem in the upper half plane, the last requirement is to show that U(X,Y) satisfies the Laplace equation. The partials of X and Y satisfy the Cauchy-Riemann equations $X_x = Y_y$ and $X_y = -Y_x$. With $\phi(x,y)$ being harmonic and X and Y satisfying the Cauchy-Riemann equations, U is harmonic and satisfies the Laplace equation (Jeffrey, 1992).

The transformed boundary conditions in the plane for the Laplace equation U are shown in Figure 4.1.

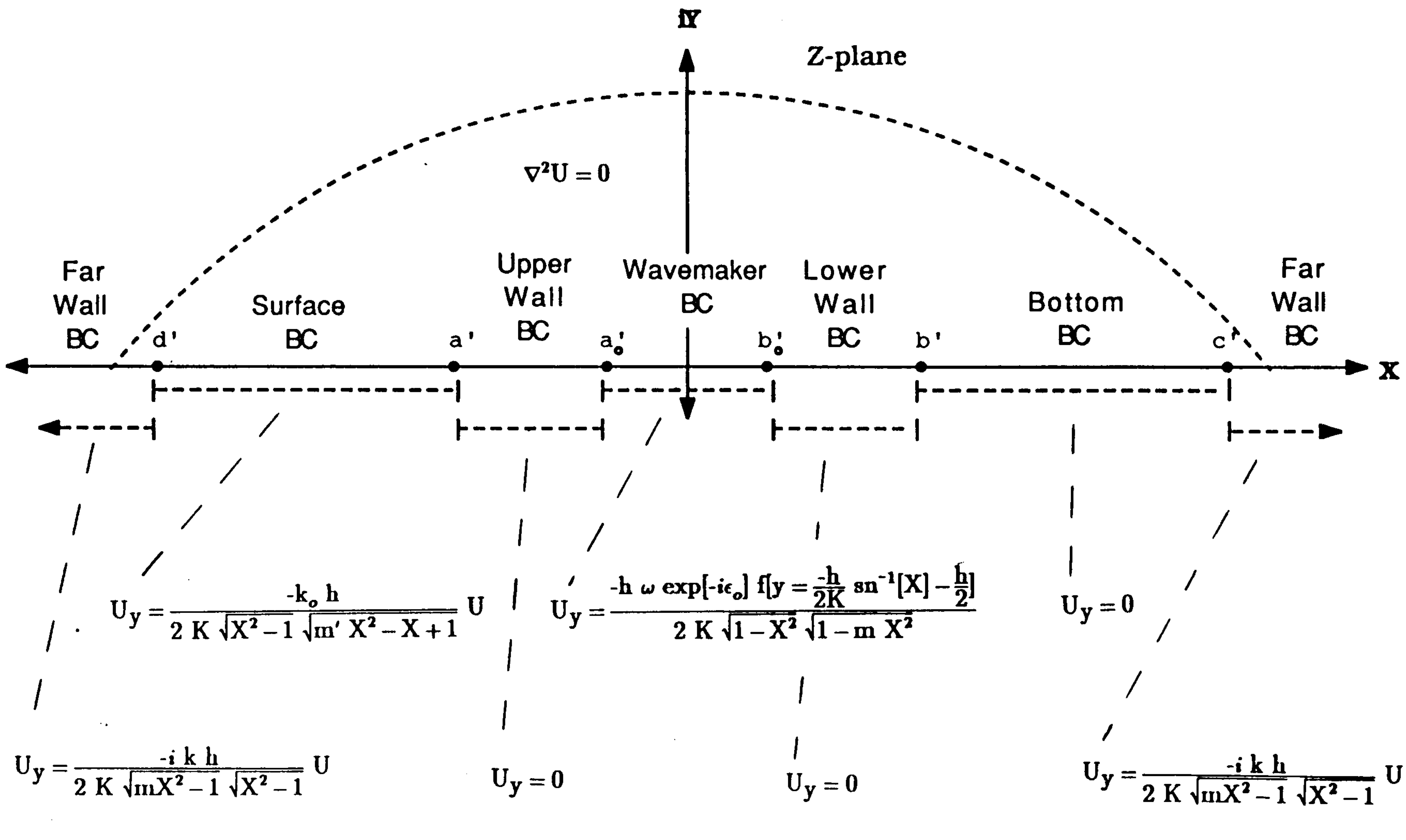


Figure 4.1 The bondary value problem in the upper half plane

5. THE BOUNDARY VALUE PROBLEM IN THE UNIT DISC

The final steps of setting up the boundary value problem on the disc are the transformation of the boundary conditions to the unit disc and the verification that the transformed potential satisfies the Laplace equation. To do this transformation let

$$U(X,Y)=\Phi(R,\Theta)=\Phi(R(X,Y),\Theta(X,Y))$$

where from Eqs.(3.8) and (3.9)

$$R = \sqrt{\frac{(1-\alpha)^2-2Y(1+\alpha^2)+(X^2+Y^2)(1+\alpha)^2}{(1-\alpha)^2+2Y(1+\alpha^2)+(X^2+Y^2)(1+\alpha)^2}}$$

and

$$\Theta = \mathrm{Arctan}\left[\frac{2X(1-\alpha^2)}{(1-\alpha)^2-(X^2+Y^2)(1+\alpha)^2}\right].$$

The partial derivatives are transformed by

$$U_X = \Phi_R R_X + \Phi_\Theta \Theta_X$$

and

$$U_Y = \Phi_R R_Y + \Phi_\Theta \Theta_Y. \tag{5.1}$$

The partials of R and Θ are

$$R_X = \frac{4(1-\alpha)(1+\alpha)^3\ X\ Y}{\mu^{\frac{3}{2}}\ \nu^{\frac{1}{2}}},$$

$$R_Y = \frac{2(-1+\alpha^2)\big((1-\alpha)^2+\ (X^2-Y^2)(1+\alpha)^2\big)}{\mu^{\frac{3}{2}}\ \nu^{\frac{1}{2}}},$$

$$\Theta_X = \frac{-2(-1+\alpha^2)\big((1-\alpha)^2+\ (X^2-Y^2)(1+\alpha)^2\big)}{\mu\ \nu},$$

and

$$\Theta_Y = \frac{4(1-\alpha)(1+\alpha)^3\ X\ Y}{\mu\ \nu},$$

where $\mu = (1-\alpha)^2 + 2Y(1+\alpha^2) + (X^2+Y^2)(1+\alpha)^2$
$\nu = (1-\alpha)^2 - 2Y(1+\alpha^2) + (X^2+Y^2)(1+\alpha)^2$.

Since the derivatives in the boundary conditions in the upper half plane are with respect to Y, R_X and Θ_X are not required. For the boundary conditions in the plane, $Y=0$; thus, the partials with respect to Y are

$$R_Y = \frac{2(-1+\alpha^2)}{\left((1-\alpha)^2+X^2(1+\alpha)^2\right)} \quad \text{and} \quad \Theta_Y = 0.$$

Since α does not equal 1 or -1, R_Y is not zero.

The transformation of the Bottom Boundary Condition in the upper half plane to the unit disc is accomplished by substituting R_Y, Θ_Y, and Eq.(4.3) into Eq.(5.1) to get

$$0 = \Phi_R \frac{2(-1+\alpha^2)}{\left((1-\alpha)^2+X^2(1+\alpha)^2\right)}.$$

Since the fraction on the right side is not identically zero, the Bottom Boundary Condition on the disc is

$$\Phi_R = 0 \tag{5.2}$$

for $R = 1$ and $\Theta_b < \Theta < \Theta_c$. ($\Theta_b$ is the argument for point b″ on the boundary of the disc and the arguments for the other five points use the same form.)

The transformation of the Surface Boundary Condition is accomplished by substituting the partials and Eq.(4.4) into Eq.(5.1) to get

$$\frac{-k_o\ h}{2\ K\ \sqrt{X^2-1}\ \sqrt{m'X^2-X^2+1}}\ \Phi = \Phi_R \frac{2(-1+\alpha^2)}{\left((1-\alpha)^2+X^2(1+\alpha)^2\right)}.$$

Since the right side does not equal zero, the boundary condition becomes

$$\Phi_R + \frac{k_o\ h\left((1-\alpha)^2+X^2(1+\alpha)^2\right)}{4\ K\ (-1+\alpha^2)\ \sqrt{X^2-1}\ \sqrt{m'X^2-X^2+1}}\ \Phi = 0.$$

The substitution of $\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}$ for X changes the Surface Boundary Condition on the disc to

$$\Phi_R + \frac{k_o\, h\left((1-\alpha)^2 + \frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}(1+\alpha)^2\right)}{4\ K\ (-1+\alpha^2)\sqrt{\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}-1}\sqrt{m'\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}-\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}+1}}\ \Phi\ = 0$$

which simplifies to

$$\Phi_R + \frac{k_o\, h\,\sqrt{2}\,(-1+\alpha^2)}{4\ K\sqrt{\left(2\alpha+(1+\alpha^2)\mathrm{Cos}[\Theta]\right)\left(4\alpha+m'(1-\alpha)^2((3-2\alpha^2)\ \mathrm{Cos}[\Theta]-1)\right)}}\ \Phi = 0$$

for R = 1 and $\Theta_d < \Theta < \Theta_a$. (5.3)

The transformation of Wavemaker Boundary Condition is accomplished by substituting the partials and Eq.(4.5) into Eq.(5.1) to get

$$\frac{-h\ \omega\ \exp[-i\epsilon_o]\ f\left(y=[-\frac{h}{2}-\frac{h}{2\ K}\ \mathrm{sn}^{-1}X]\right)}{2\ K\sqrt{1-X^2}\sqrt{1-m\ X^2}} = \Phi_R\,\frac{2(-1+\alpha^2)}{\left((1-\alpha)^2+X^2(1+\alpha)^2\right)}.$$

The right side is not equal to zero; the boundary condition becomes

$$\Phi_R = \frac{-h\ \omega\ \exp[-i\epsilon_o]\ f\left(y=[-\frac{h}{2}-\frac{h}{2\ K}\ \mathrm{sn}^{-1}X]\right)\left((1-\alpha)^2+X^2(1+\alpha)^2\right)}{4\ K\ \sqrt{1-X^2}\ \sqrt{1-m\ X^2}\ (-1+\alpha^2)}.$$

The substitution of $\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}$ for X changes the Wavemaker Boundary Condition on the disc to

$$\Phi_R = \frac{-h\omega\ \exp[-i\epsilon_o]\ f\left(y=[-\frac{h}{2}-\frac{h}{2\ K}\ \mathrm{sn}^{-1}\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}]\right)\left((1-\alpha)^2+\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}(1+\alpha)^2\right)}{4\ K\ (-1+\alpha^2)\sqrt{1-\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}}\sqrt{1-m\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}}}$$

which can be rewritten as

$$\Phi_R = \frac{-h\omega\ \exp[-i\epsilon_o] f\left(y=[-\frac{h}{2}-\frac{h}{2\ K}\ \mathrm{sn}^{-1}\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}]\right)(-1+\alpha^2)\ \mathrm{Sec}[\frac{\Theta}{2}]\left(\alpha+\alpha^2-(-1+\alpha)\mathrm{Tan}[\frac{\Theta}{2}]\right)}{4\ K\left(1+\alpha+(\alpha^2-\alpha)\mathrm{Tan}[\frac{\Theta}{2}]\right)\sqrt{\left(2\alpha+(1+\alpha^2)\mathrm{Cos}[\Theta]\right)\left((1+\alpha)^2-m(1-\alpha)^2\mathrm{Tan}^2[\frac{\Theta}{2}]\right)}}$$

for R = 1 and $\Theta_{a_o} < \Theta < \Theta_{b_o}$. (5.4)

The transformation of the Far Wall Boundary Condition is accomplished by substituting the partials and Eq.(4.6) into Eq.(5.1) to get

$$-\frac{i\ k\ h}{2\ K}\ \frac{1}{\sqrt{m\ X^2-1}\ \sqrt{X^2-1}}\ \Phi\ =\ \Phi_R\ \frac{2(-1+\alpha^2)}{\left((1-\alpha)^2+X^2(1+\alpha)^2\right)}.$$

Since the right hand side is not zero, the equation can be rewritten as

$$\Phi_R + \frac{i\ k\ h}{4\ K}\ \frac{\left((1-\alpha)^2+X^2(1+\alpha)^2\right)}{(-1+\alpha^2)\sqrt{m\ X^2-1}\ \sqrt{X^2-1}}\ \Phi\ =\ 0.$$

The substitution of $\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}$ for X changes the Far Wall Boundary Condition to

$$\Phi_R + \frac{i\ k\ h}{4\ K}\ \frac{\left((1-\alpha)^2+(1-\alpha^2)\ \mathrm{Tan}[\frac{\Theta}{2}]\right)}{(-1+\alpha^2)\sqrt{m\ \frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}-1}\ \sqrt{\frac{(1-\alpha)\ \mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}-1}}\ \Phi\ =\ 0$$

which can be rewritten as

$$\Phi_R + \frac{i\ k\ h}{4\ K}\ \frac{(-1+\alpha^2)\ \mathrm{Sec}[\frac{\Theta}{2}]}{\sqrt{\left(2\alpha+(1+\alpha^2)\mathrm{Cos}[\Theta]\right)\left((1+\alpha)^2-m(1-\alpha)^2\mathrm{Tan}^2[\frac{\Theta}{2}]\right)}}\ \Phi=0$$

for R = 1 and $\Theta_c<\Theta<\pi$ and $-\pi<\Theta<\Theta_d$. (5.5)

The transformation of the Upper and Lower Boundary Conditions is accomplished by substituting the partials and Eq(4.7) into Eq.(5.1) to get

$$0\ =\ \Phi_R\ \frac{2(-1+\alpha^2)}{\left((1-\alpha)^2+X^2(1+\alpha)^2\right)}.$$

Since the right hand side is not zero, both the Upper and Lower Boundary Conditions are

$$\Phi_R=0 \tag{5.6}$$

for R = 1 and $\Theta_a<\Theta<\Theta_{a_o}$ and $\Theta_b<\Theta<\Theta_{b_o}$, respectively.

What remains is to verify that $\Phi(X,Y)$ satisfies the Laplace equation. Since the partials of R and Θ satisfy the Cauchy-Riemann equations in the polar form, $\frac{\partial R}{\partial X} = R\frac{\partial \Theta}{\partial Y}$ and $\frac{\partial R}{\partial Y} = -R\frac{\partial \Theta}{\partial X}$, and since U is harmonic, $\Phi(X,Y)$ is also harmonic and satisfies the Laplace equation.

Figure (5.1) shows the transformed Boundary Value Problem on the unit disc.

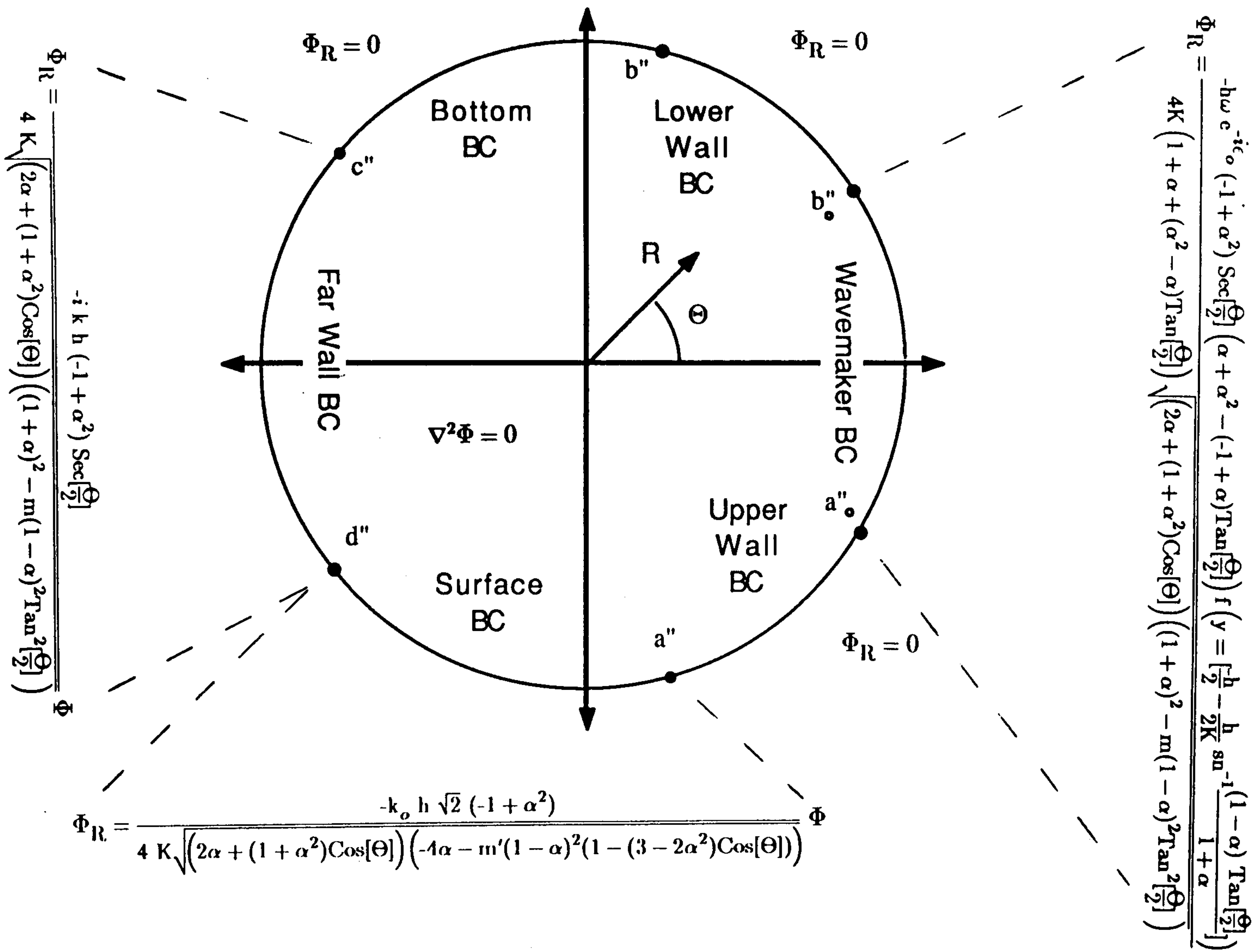


Figure 5.1 The Boundary Value Problem on the unit (

6. THE SOLUTION TO THE MIXED BOUNDARY VALUE PROBLEM IN THE UNIT DISC

Figure (5.1) shows that the boundary value problem is a mixed value problem because the transformed boundary conditions on the disc are of two types. The boundary conditions of the Surface and the Far Wall are Robin problems and the boundary conditions of the Bottom, the Wavemaker, and the Upper and Lower Wall are Neumann problems. Since all the boundary conditions are functions of Θ, the mixed value problem can be generalized by

$$\left.\frac{\partial\Phi}{\partial R}\right|_{R=1} + a(\Theta)\ \Phi(1,\Theta) = h(\Theta)\ . \tag{6.1}$$

A solution to Eq.(6.1) is

$\Phi(R,\Theta) = \frac{\gamma_o}{2} + \sum_{j=1}^{\infty} R^j\ [\gamma_j \cos j\Theta + \beta_j \sin j\Theta]$, (Guenther and Lee, 1988) because $\Phi(R,\Theta)$ satisfies the Laplace equation in polar coordinates,

$$\Delta\Phi = \frac{1}{R}\ \frac{\partial}{\partial R}\left(R\ \frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2}\ \frac{\partial^2\Phi}{\Theta^2}.$$

The task now is to solve the coefficients γ_j and β_j that satisfy Eq(6.1). In order to accomplish this task, $\left.\frac{\partial\Phi}{\partial R}\right|_{R=1}$ is needed;

$$\left.\frac{\partial\Phi}{\partial R}\right|_{R=1} = \sum_{j=1}^{\infty} j\ [\gamma_j \cos j\Theta + \beta_j \sin j\Theta].$$

Substitute $\left.\frac{\partial\Phi}{\partial R}\right|_{R=1}$ and $\Phi(1,\Theta)$ into Eq.(6.1) to get

$$\frac{\gamma_o}{2}\ a(\Theta) + \sum_{j=1}^{\infty}[(a(\Theta)+j)\ \gamma_j \cos j\Theta + (a(\Theta)+j)\ \beta_j \sin j\Theta].$$

$$= h(\Theta).$$

In place of $\sum_{j=1}^{\infty}$, $\lim_{N\to\infty} \sum_{j=1}^{N}$ can be used. This summation can be used to make approximations. In order to solve the coefficients the following equation can be used;

$$\lim_{N\to\infty} a(\Theta)\frac{\gamma_o^{(N)}}{2}+\sum_{j=1}^{N}[(a(\Theta)+j)\gamma_j^{(N)}\cos j\Theta+(a(\Theta)+j)\beta_j^{(N)}\sin j\Theta]$$

$$=h(\Theta). \tag{6.2}$$

In order to solve the values of γ_j and β_j, a 2N+1 system of equations needs to be developed. The method used for this development is similar to the method of finding Fourier coefficients. Multiplying each term of Eq.(6.2) by $\cos i\Theta$, where $i=0,1,\ldots,N$, and integrating from $-\pi$ to π produces N+1 equations. Then multiplying each term of Eq.(6.2) by $\sin l\Theta$, where $l=1,\ldots,N$ and integrating from $-\pi$ to π produces N equations. These N equations combined with the first N+1 equations produce a system of 2N+1 equations from which the coefficients γ_j and β_j can be calculated.

Multiplying both sides of Eq.(6.2) by $\cos i\Theta$ gives

$$\frac{\gamma_o}{2}\, a(\Theta)\cos i\Theta+\sum_{j=1}^{N}\left\{(a(\Theta)+j)\gamma_j\cos j\Theta\cos i\Theta+(a(\Theta)+j)\beta_j\sin j\Theta\cos i\Theta\right\}$$
$$=h(\Theta)\cos i\Theta\,.$$

Integrating from $-\pi$ to π produces

$$\int_{-\pi}^{\pi}\frac{\gamma_o}{2}\, a(\Theta)\cos i\Theta+\sum_{j=1}^{N}\left\{\int_{-\pi}^{\pi}(a(\Theta)+j)\gamma_j\cos j\Theta\cos i\Theta+\int_{-\pi}^{\pi}(a(\Theta)+j)\beta_j\sin j\Theta\cos i\Theta\right\}$$

$$=\int_{-\pi}^{\pi}h(\Theta)\cos i\Theta.$$

This last equation can be rewritten as

$$\frac{\gamma_o}{2}A_i+\sum_{j=1}^{N}\mathcal{A}_{ij}\gamma_j+\sum_{j=1}^{N}\mathfrak{B}_{ij}\beta_j+0 = \mathcal{H}_i \tag{6.3}$$

for $i = 0,\ldots,N$, where

$$A_i=\int_{-\pi}^{\pi}a(\Theta)\cos i\Theta\, d\Theta,$$

$$\mathcal{A}_{ij}=\int_{-\pi}^{\pi}(a(\Theta)+j)\cos i\Theta\cos j\Theta\, d\Theta,$$

$$\mathfrak{B}_{ij} = \int_{-\pi}^{\pi} (a(\Theta) + j) \cos i\Theta \sin j\Theta \, d\Theta,$$

and

$$\mathcal{H}_i = \int_{-\pi}^{\pi} h(\Theta) \cos i\Theta \, d\Theta.$$

Figure (5.1) shows $a(\Theta)$ to be the piecewise function

$$a(\Theta) = \begin{cases} a_1(\Theta) & , \ \Theta_a \le \Theta \le_c \\ a_2(\Theta) & , \ \Theta_d \le \Theta \le \Theta_a \\ a_3(\Theta) & , \ -\pi \le \Theta \le \Theta_d \text{ and } \Theta_c \le \Theta \le \pi, \end{cases} \tag{6.4}$$

where $a_1(\Theta) = 0$,

$$a_2(\Theta) = \frac{h\ k_o\ \sqrt{2}\ (-1+\alpha^2)}{4K\ \sqrt{\left(2\alpha + (1+\alpha^2)\mathrm{Cos}[\Theta]\right)\left(4\alpha + m'(1-\alpha)^2((3-2\alpha^2)\mathrm{Cos}[\Theta]-1)\right)}},$$

and

$$a_3(\Theta) = \frac{-i\ k\ h\ (-1+\alpha^2)\ \mathrm{Sec}[\frac{\Theta}{2}]}{4K\ \sqrt{\left(2\alpha + (1+\alpha^2)\mathrm{Cos}[\Theta]\right)\left((1+\alpha)^2 + m(1-\alpha)^2\ \mathrm{Tan}^2[\frac{\Theta}{2}]\right)}}.$$

Furthermore, $h(\Theta)$ is the piecewise function

$$h(\Theta) = \begin{cases} h_1(\Theta) & , \ -\pi \le \Theta \le \Theta_{a_o} \text{ and } \Theta_{b_o} \le \Theta \le \pi \\ h_2(\Theta) & , \ \Theta_{a_o} \le \Theta \le \Theta_{b_o}, \end{cases} \tag{6.5}$$

where $h_1(\Theta) = 0$ and

$$h_2(\Theta) = \frac{-h\omega e^{-i\epsilon_o} f\left[y = -\frac{h}{2} - \frac{h}{2K}\mathrm{sn}^{-1}\left[\frac{(1-\alpha)\mathrm{Tan}[\frac{\Theta}{2}]}{1+\alpha}\right]\right] (-1+\alpha^2)\ \mathrm{Sec}[\frac{\Theta}{2}] \left(\alpha + \alpha^2 - (-1+\alpha)\mathrm{Tan}[\frac{\Theta}{2}]\right)}{4K\left(1+\alpha+(\alpha^2-\alpha)\mathrm{Tan}[\frac{\Theta}{2}]\right)\sqrt{\left(2\alpha+(1+\alpha^2)\mathrm{Cos}[\Theta]\right)\left((1+\alpha)^2 - m(1-\alpha)^2\mathrm{Tan}^2[\frac{\Theta}{2}]\right)}}.$$

The first matrix to be calculated is A_i. $\int_{-\pi}^{\pi} a(\Theta) \cos i\Theta \, d\Theta$ can be calculated by the sum of the integrals of the four partitions defined in Eq.(6.4). A_i is

$$\int_{-\pi}^{\Theta_d} a_3(\Theta) \cos i\Theta \, d\Theta + \int_{\Theta_d}^{\Theta_a} a_2(\Theta) \cos i\Theta \, d\Theta + 0 + \int_{\Theta_c}^{\pi} a_3(\Theta) \cos i\Theta \, d\Theta \tag{6.6}$$

for $i = 0, 1, \ldots, N$.

In order to calculate $\mathcal{A}_{ij}$, $\int_{-\pi}^{\pi} (a(\Theta) + j) \cos j\Theta \cos i\Theta \, d\Theta$, is separated into sum of the two parts $\int_{-\pi}^{\pi} a(\Theta) \cos j\Theta \cos i\Theta \, d\Theta$ and $\int_{-\pi}^{\pi} j \cos j\Theta \cos i\Theta \, d\Theta$. The first of these two integrals is calculated

from the sum of the partitions,

$$\int_{-\pi}^{\Theta_d} a_3(\Theta) \cos j\Theta \cos i\Theta \, d\Theta + \int_{\Theta_d}^{\Theta_a} a_2(\Theta) \cos j\Theta \cos i\Theta \, d\Theta$$

$$+ 0 + \int_{\Theta_c}^{\pi} a_3(\Theta) \cos j\Theta \cos i\Theta \, d\Theta.$$

The second integral is $j\pi$, because $\int_{-\pi}^{\pi} \cos j\Theta \cos i\Theta \, d\Theta = \begin{cases} \pi & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$.

Thus, $\mathcal{A}_{ij}$ is

$$\int_{-\pi}^{\Theta_d} a_3(\Theta) \cos j\Theta \cos i\Theta \, d\Theta + \int_{\Theta_d}^{\Theta_a} a_2(\Theta) \cos j\Theta \cos i\Theta \, d\Theta$$

$$+ 0 + \int_{\Theta_c}^{\pi} a_3(\Theta) \cos j\Theta \cos i\Theta \, d\Theta + j\,\pi \tag{6.7}$$

for $i = 0, 1, \ldots, N$ and $j = 1, \ldots, N$.

The calculation of $\mathcal{B}_{ij}$, $\int_{-\pi}^{\pi}(a(\Theta)+j)\sin j\Theta\cos i\Theta\, d\Theta$, is performed in the same manner as $\mathcal{A}_{ij}$ with the exception of the final term $j\pi$; the final term for $\mathcal{B}_{ij}$ is zero because $\int_{-\pi}^{\pi}\sin j\Theta\cos i\Theta\, d\Theta = 0$ for all i and j. Thus, $\mathcal{B}_{ij}$ is

$$\int_{-\pi}^{\Theta_d} a_3(\Theta)\ \sin j\Theta\ \cos i\Theta\ d\Theta + \int_{\Theta_d}^{\Theta_a} a_2(\Theta)\ \sin j\Theta\ \cos i\Theta\ d\Theta + 0 + \int_{\Theta_c}^{\pi} a_3(\Theta)\ \sin j\Theta\ \cos i\Theta\ d\Theta$$

for $i=0,1,\ldots,N$ and $j=1,\ldots,N$. (6.8)

$\mathcal{H}_i$ is calculated by partitioning $\int_{-\pi}^{\pi} h(\Theta)\ \cos i\Theta\, d\Theta$ into the sum of three integrals,

$$\int_{-\pi}^{\Theta_{a_o}} h_1(\Theta)\ \ \cos i\Theta\ d\Theta + \int_{\Theta_{a_o}}^{\Theta_{b_o}} h_2(\Theta)\ \ \cos i\Theta\ d\Theta + \int_{\Theta_{b_o}}^{\pi} h_1(\Theta)\ \ \cos i\Theta\ d\Theta$$

Since $h_1(\Theta)=0$, $\mathcal{H}_i$ reduces to

$$\int_{\Theta_{a_o}}^{\Theta_{b_o}} h_2(\Theta)\ \ \cos i\Theta\ d\Theta \qquad (6.9)$$

for $i=0,1,\ldots,N$. Hence, the combination of Eqs.(6.7), (6.8), and (6.9) is the N+1 part of the system of equations.

For the second part of the system, multiplying each term of Eq.(6.2) by sin lΘ gives

$$\frac{\gamma_o}{2}\, a(\Theta)\sin l\Theta + \sum_{j=1}^{N}\left\{(a(\Theta)+j)\gamma_j\cos j\Theta\sin l\Theta + (a(\Theta)+j)\beta_j\sin j\Theta\sin l\Theta\right\}$$

$$= h(\Theta)\sin l\Theta.$$

Then, integrating from $-\pi$ to π produces

$$\int_{-\pi}^{\pi}\frac{\gamma_o}{2}\, a(\Theta)\sin l\Theta + \sum_{j=1}^{N}\left\{\int_{-\pi}^{\pi}(a(\Theta)+j)\gamma_j\cos j\Theta\sin l\Theta + \int_{-\pi}^{\pi}(a(\Theta)+j)\beta_j\sin j\Theta\sin l\Theta\right\}$$

$$= \int_{-\pi}^{\pi} h(\Theta)\sin l\Theta.$$

This last equation can be rewritten as

$$\frac{\gamma_o}{2}\, C_l + \sum_{j=1}^{N} \mathcal{C}_{lj}\gamma_j + 0 + \sum_{j=1}^{N} \mathcal{D}_{lj}\beta_j \;=\; \mathcal{K}_l \quad \text{for } l = 1,\ldots,N, \tag{6.10}$$

where

$$C_l = \int_{-\pi}^{\pi} a(\Theta)\ \sin l\Theta\ d\Theta,$$

$$\mathcal{C}_{lj} = \int_{-\pi}^{\pi} (a(\Theta) + j)\ \cos j\Theta\ \sin l\Theta\ d\Theta,$$

$$\mathcal{D}_{lj} = \int_{-\pi}^{\pi} (a(\Theta) + j)\ \sin j\Theta\ \sin l\Theta\ d\Theta,$$

and

$$\mathcal{K}_l = \int_{-\pi}^{\pi} h(\Theta)\ \sin l\Theta\ d\Theta.$$

Notice that C_l is the same as A_i with the substitution of sin $l\Theta$ for cos $i\Theta$; C_l is the sin $l\Theta$ counterpart of A_i. Thus, C_l can be calculated in a manner similar to the calculation of A_i; C_l is

$$\int_{-\pi}^{\Theta_d} a_3(\Theta)\ \sin l\Theta\ d\Theta + \int_{\Theta_d}^{\Theta_a} a_2(\Theta)\ \sin l\Theta\ d\Theta + 0 + \int_{\Theta_c}^{\pi} a_3(\Theta)\ \sin l\Theta\ d\Theta \tag{6.11}$$

for $l = 1,\ldots,N$.

Since $\mathcal{C}_{lj}$ is the sin $l\Theta$ counterpart of $\mathcal{B}_{ij}$, $\mathcal{C}_{lj}$ can be calculated in a manner similar to the calculation of $\mathcal{B}_{ij}$; $\mathcal{C}_{lj}$ is

$$\int_{-\pi}^{\Theta_d} a_3(\Theta)\ \cos j\Theta\ \sin l\Theta\ d\Theta + \int_{\Theta_d}^{\Theta_a} a_2(\Theta)\ \cos j\Theta\ \sin l\Theta\ d\Theta + 0 + \int_{\Theta_c}^{\pi} a_3(\Theta)\ \cos j\Theta\ \sin l\Theta\ d\Theta$$

for $l, j = 1,\ldots,N$. (6.12)

Also, since $\mathcal{D}_{lj}$ is the sin $l\Theta$ counterpart of $\mathcal{A}_{ij}$, $\mathcal{D}_{lj}$ can be calculated in a manner similar to the method applied to $\mathcal{A}_{ij}$; $\mathcal{D}_{lj}$ is

$$\int_{-\pi}^{\Theta_d} a_3(\Theta) \sin j\Theta \sin l\Theta \, d\Theta + \int_{\Theta_d}^{\Theta_a} a_2(\Theta) \sin j\Theta \sin l\Theta \, d\Theta + 0 + \int_{\Theta_c}^{\pi} a_3(\Theta) \sin j\Theta \sin l\Theta \, d\Theta + j\,\pi \qquad (6.13)$$

for $l, j = 1,\ldots,N$. The last term in Eq.(6.13) is $j\pi$ because

$$\int_{-\pi}^{\pi} \sin l\Theta \sin j\Theta \, d\Theta = \begin{cases} 0, & l \neq j \\ \pi, & l = j \end{cases} .$$

Finally, since $\mathcal{K}_l$ is the sin $l\Theta$ counterpart of $\mathcal{H}_i$, $\mathcal{K}_l$ can be calculated in a manner similar to the method applied to $\mathcal{H}_i$; $\mathcal{K}_l$ is

$$\int_{\Theta_{a_o}}^{\Theta_{b_o}} h_2(\Theta) \quad \sin l\Theta \, d\Theta \quad \text{for } l = 1,\ldots,N. \qquad (6.14)$$

The combination of Eqs.(6.11), (6.12), (6.13), and (6.14) makes a N numbered system of equations. Combining these N equations with the N+1 equations produces the 2N+1 linear system of equations with the 2N+1 unknowns, γ_o, $\gamma_1,\ldots\gamma_N$, $\beta_1,\ldots,\beta_N$. The 2N+1 system of equations is

$$\frac{\gamma_o}{2} A_0 + \mathcal{A}_{01}\gamma_1 + \mathcal{A}_{02}\gamma_2 + \ldots + \mathcal{A}_{0N}\gamma_N + \mathcal{B}_{01}\beta_1 + \mathcal{B}_{02}\beta_2 + \ldots + \mathcal{B}_{0N}\beta_N = \mathcal{H}_0$$

$$\frac{\gamma_o}{2} A_1 + \mathcal{A}_{11}\gamma_1 + \pi\gamma_1 + \mathcal{A}_{12}\gamma_2 + \ldots + \mathcal{A}_{1N}\gamma_N + \mathcal{B}_{11}\beta_1 + \mathcal{B}_{12}\beta_2 + \ldots + \mathcal{B}_{1N}\beta_N = \mathcal{H}_1$$

$$\frac{\gamma_o}{2} A_2 + \mathcal{A}_{21}\gamma_1 + \mathcal{A}_{22}\gamma_2 + 2\pi\gamma_2 + \ldots + \mathcal{A}_{2N}\gamma_N + \mathcal{B}_{21}\beta_1 + \mathcal{B}_{22}\beta_2 + \ldots + \mathcal{B}_{2N}\beta_N = \mathcal{H}_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{\gamma_o}{2} A_N + \mathcal{A}_{N1}\gamma_1 + \mathcal{A}_{N2}\gamma_2 + \ldots + N\pi\gamma_N + \mathcal{A}_{NN}\gamma_N + \mathcal{B}_{N1}\beta_1 + \mathcal{B}_{N2}\beta_2 + \ldots + \mathcal{B}_{NN}\beta_N = \mathcal{H}_N$$

$$\frac{\gamma_o}{2} C_1 + \mathcal{C}_{11}\gamma_1 + \mathcal{C}_{12}\gamma_2 + \ldots + \mathcal{C}_{1N}\gamma_N + \mathcal{D}_{11}\beta_1 + \pi\beta_1 + \mathcal{D}_{12}\beta_2 + \ldots + \mathcal{D}_{1N}\beta_N = \mathcal{K}_1$$

$$\frac{\gamma_o}{2} C_2 + \mathcal{C}_{21}\gamma_1 + \mathcal{C}_{22}\gamma_2 + \ldots + \mathcal{C}_{2N}\gamma_N + \mathcal{D}_{21}\beta_1 + \mathcal{D}_{22}\beta_2 + 2\pi\beta_2 + \ldots + \mathcal{D}_{2N}\beta_N = \mathcal{K}_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{\gamma_o}{2} C_N + \mathcal{C}_{N1}\gamma_1 + \mathcal{C}_{N2}\gamma_2 + \ldots + \mathcal{C}_{NN}\gamma_N + \mathcal{D}_{N1}\beta_1 + \mathcal{D}_{N2}\beta_2 + \ldots + N\pi\beta_N + \mathcal{D}_{NN}\beta_N = \mathcal{K}_N .$$

Solving the coefficients of the system can be done either by Gaussian elimination or by the use of matrices. In Appendix V, the coefficients are calculated with the use of matrices. Once the coefficients are calculated, the solution to the mixed value boundary problem becomes

$$\Phi(R,\Theta) = \lim_{N\to\infty} \frac{\gamma_o^{(N)}}{2} + \sum_{j=1}^{N} R^j \left\{\gamma_j^{(N)} \cos j\Theta + \beta_j^{(N)} \sin j\Theta\right\}.$$

The substitution of Eq.(3.11) for R and Eq.(3.12) for Θ transforms $\Phi(R,\Theta)$, the solution for the disc, back into $\phi(x,y)$, the solution for the rectangle.

The following are two examples of approximate solutions to the boundary value problem for a variable-draft hinged wavemaker with depth of 3.96 m and 4.42 m. This type of wavemaker has no upper wall because the wavemaker apparatus extends to the surface. Figure (6.1) is a variable-draft hinged wavemaker with a still water depth of 3.96 m. and with a wavemaker apparatus length of .55 m. Figure (6.2) is the same type of wavemaker with the exception of the depth of 4.42 m. Since the apparatus reaches the surface and has a length of .55 m, $a_o = 0$ and $b_o = -.55i$. The gravitation constant is 9.8. Let the wave period, T, be 3.45 s. Then, $\omega = \frac{2\pi}{3.45}$ and $k_o = \frac{4\pi^2}{3.45^2 * 9.8}$. For a simple

case, let the wavemaker displacement be a function of depth, $f(y) = y$, and let the phase angle, ϵ_o, be 0. Approximate the wave length, L, by

using the deep water wave length which is $L_o = \frac{gT^2}{2\pi}$. Then, the

propagating wave number, c, is $\frac{4\pi^2}{3.45^2 * 9.8}$.

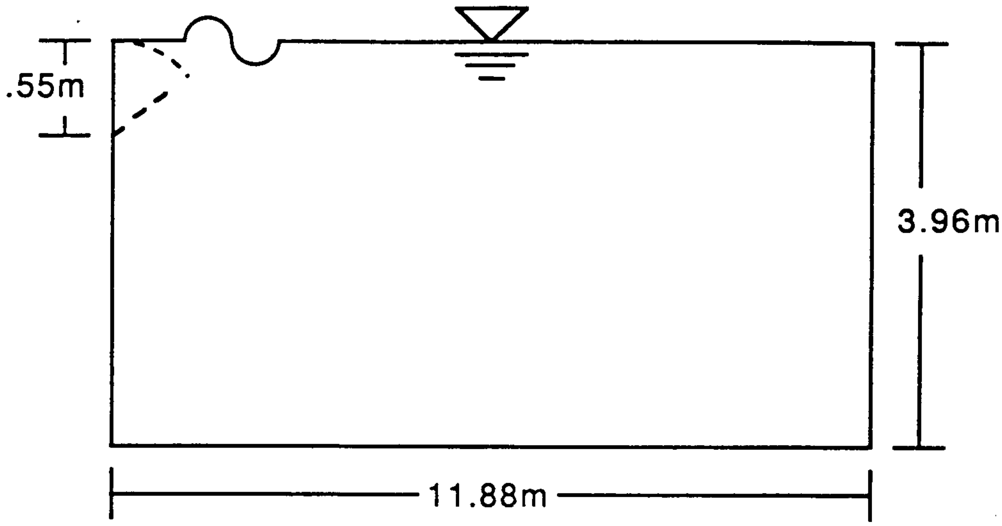


Figure 6.1 Variable-draft hinged wavemaker with depth 3.96 m

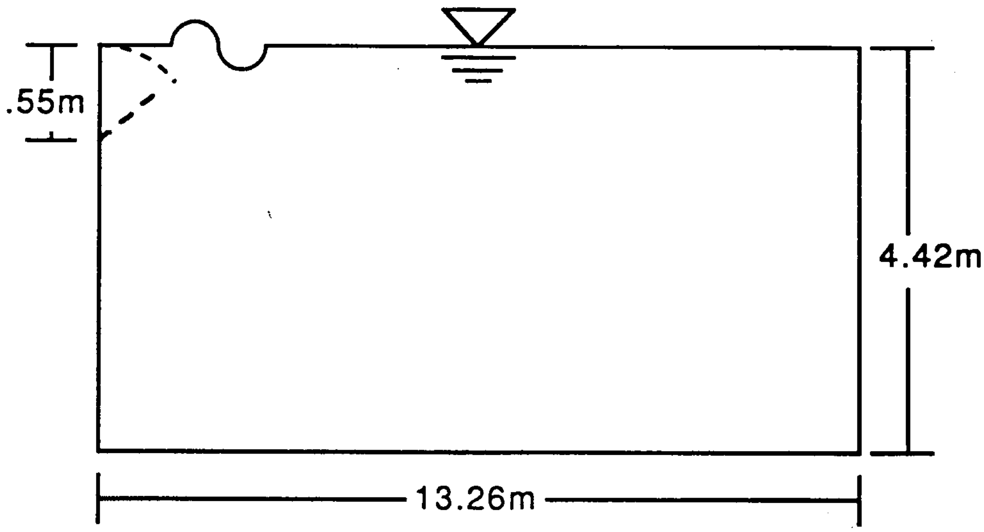


Figure 6.2 Variable-draft hinged wavemaker with depth 4.42 m

For the still water depth of 3.96 m, the solution for N = 2 is

$$
\begin{aligned}
\phi(x,y) = {} & 13.2396 - 154.450 \\
& + R[(16.8104 - i\ 0.807126)\ \cos\Theta \\
& \quad + (-0.007040 + i\ 0.007276)\ \sin\Theta] \\
& + R^2[(1.740632 + i\ 0.262197)\ \cos 2\Theta \\
& \quad + (-0.006793 + i\ 0.003826)\ \sin 2\Theta)]
\end{aligned}
$$

and for N = 10 the solution is

$$
\begin{aligned}
\phi(x,y) = {} & 14.3141 - i\ 154.62\text{-}33248.1 \\
& + R[(16.811 - i\ 0.870828)\ \cos\Theta \\
& \quad + (-0.00705027 + i\ 0.00739797)\ \sin\Theta] \\
& + R^2[(1.74586 + i\ 0.294695)\ \cos 2\Theta \\
& \quad + (-0.00686734 + i\ 0.00373608)\ \sin 2\Theta)] \\
& + R^3[(3.72785 - i\ 0.125151)\ \cos 3\Theta \\
& \quad + (-0.00647711 + i\ 0.00240748)\ \sin 3\Theta)] \\
& + R^4[(2.34358 + i\ 0.0134998)\ \cos 4\Theta \\
& \quad + (-0.00638975 + i\ 0.00197604)\ \sin 4\Theta)] \\
& + R^5[(1.3774 + i\ 0.0132964)\ \cos 5\Theta \\
& \quad + (-0.00625563 + i\ 0.00149049)\ \sin 5\Theta)] \\
& + R^6[(1.84152 - i\ 0.0315419)\ \cos 6\Theta \\
& \quad + (-0.00617576 + i\ 0.00125997)\ \sin 6\Theta)] \\
& + R^7[(1.10302 + i\ 0.0151858)\ \cos 7\Theta \\
& \quad + (-0.0061302 + i\ 0.00110551)\ \sin 7\Theta)] \\
& + R^8[(1.09558 - i\ 0.00997334)\ \cos 8\Theta \\
& \quad + (-0.006067 + i\ 0.000937063)\ \sin 8\Theta)] \\
& + R^9[(1.11311 - i\ 0.00537119)\ \cos 9\Theta \\
& \quad + (-0.00603971 + i\ 0.000841688)\ \sin 9\Theta)] \\
& + R^{10}[(0.758557 + i\ 0.00354203)\ \cos 10\Theta \\
& \quad + (-0.00600342 + i\ 0.000778069)\ \sin 10\Theta)].
\end{aligned}
$$

R is

$$
\sqrt{\frac{3.99742 - .00257905\ \mathrm{Im}[\mathrm{sn}[u,v]] + 4.15988*10^{-7}\ (\mathrm{Im}[\mathrm{sn}[u,v]]^2 + \mathrm{Re}[\mathrm{sn}[u,v]]^2)}{3.99742 + .00257905\ \mathrm{Im}[\mathrm{sn}[u,v]] + 4.15988*10^{-7}\ (\mathrm{Im}[\mathrm{sn}[u,v]]^2 + \mathrm{Re}[\mathrm{sn}[u,v]]^2)}}
$$

and Θ is

$$\text{Arctan}\left[\frac{1.99871(1-\text{Im}[\text{sn}[u,v]]^2-\text{Re}[\text{sn}[u,v]]^2)+1.99871(1+\text{Im}[\text{sn}[u,v]]^2+\text{Re}[\text{sn}[u,v]]^2)}{.00257905\ \text{Re}[\text{sn}[u,v]]}\right],$$

where

$$u = -.793331(1.98+y)$$

and

$$v = .793331\ x.$$

For the still water depth of 4.42 m, the solution for $N=2$ is

$$\begin{aligned}\phi(x,y) = {} & 15.843828 - i\ 165.059626 \\ & + R[(20.012421 - i\ 1.062219)\ \cos\Theta \\ & \quad + (-0.008606 + i\ 0.008675)\ \sin\Theta] \\ & + R^2[(2.078897 + i\ 0.348708)\ \cos 2\Theta \\ & \quad + (-0.008274 + i\ 0.004584)\ \sin 2\Theta]\end{aligned}$$

and for N = 10 the solution is

$$\begin{aligned}\phi(x,y) = {} & 17.1214 - i\ 165.277 \\ & + R[(20.012 - i\ 1.14678)\ \cos\Theta \\ & \quad + (-0.0086174 + i\ 0.00884136)\ \sin\Theta] \\ & + R^2[(2.08636 + i\ 0.391816)\ \cos 2\Theta \\ & \quad + (-0.00837387 + i\ 0.00446366)\ \sin 2\Theta] \\ & + R^3[(4.43832 - i\ 0.164294)\ \cos 3\Theta \\ & \quad + (-0.00785352 + i\ 0.00288105)\ \sin 3\Theta] \\ & + R^4[(2.79303 + i\ 0.0188989)\ \cos 4\Theta \\ & \quad + (-0.00773794 + i\ 0.00235953)\ \sin 4\Theta] \\ & + R^5[(1.64081 + i\ 0.0182178)\ \cos 5\Theta \\ & \quad + (-0.00755902 + i\ 0.00178366)\ \sin 5\Theta] \\ & + R^6[(2.1936 - i\ 0.0410971)\ \cos 6\Theta \\ & \quad + (-0.00745287 + i\ 0.00150566)\ \sin 6\Theta] \\ & + R^7[(1.31435078897 + i\ 0.020578)\ \cos 7\Theta \\ & \quad + (-0.0073923 + i\ 0.00132095)\ \sin 7\Theta]\end{aligned}$$

$$+R^8[(1.30496 - i\ 0.0128194)\ \cos 8\Theta$$
$$+(-0.00730803 + i\ 0.00132095)\ \sin 8\Theta]$$
$$+R^9[(1.32613 + i\ 0.00669471)\ \cos 9\Theta$$
$$+(-0.0072719 + i\ 0.00100383)\ \sin 9\Theta]$$
$$+R^{10}[(0.903689 + i\ 0.0049632)\ \cos 10\Theta$$
$$+(-0.00722348 + i\ 0.000932709)\ \sin 10\Theta].$$

R is

$$\sqrt{\frac{3.99742 - .00257905\ \mathrm{Im}[\mathrm{sn}[u,v]] + 4.15988*10^{-7}\ (\mathrm{Im}[\mathrm{sn}[u,v]]^2 + \mathrm{Re}[\mathrm{sn}[u,v]]^2)}{3.99742 + .00257905\ \mathrm{Im}[\mathrm{sn}[u,v]] + 4.15988*10^{-7}\ (\mathrm{Im}[\mathrm{sn}[u,v]]^2 + \mathrm{Re}[\mathrm{sn}[u,v]]^2)}}$$

and Θ is

$$\mathrm{Arctan}\left[\frac{1.99871(1 - \mathrm{Im}[\mathrm{sn}[u,v]]^2 - \mathrm{Re}[\mathrm{sn}[u,v]]^2) + 1.99871(1 + \mathrm{Im}[\mathrm{sn}[u,v]]^2 + \mathrm{Re}[\mathrm{sn}[u,v]]^2)}{.00257905\ \mathrm{Re}[\mathrm{sn}[u,v]]}\right],$$

where

$$u = -.710768\ (2.21 + y)$$

and

$$v = .710768\ x.$$

7. CONCLUSION AND RECOMMENDATIONS

An approximate solution has been developed that describes the behavior of the motion of an inviscid fluid in a rectangular wave tank. The solution was developed using Jacobian elliptical functions in a conformal mapping that transforms the boundary of the rectangle into the boundary of the unit disc. With the use of this mapping, the critical boundary points of the corners become smooth points on the circumference of the disc.

The mapping function consisting of the Jacobian elliptical functions varies from Tanaka's mapping by using the rectangle as the physical domain rather than the channel with infinite length. The use of the channel was based upon the radiation condition such that the wave can be considered at the point of infinity when its remaining energy is negligible. With the use of the channel for the physical domain rather than the rectangle, the mapping to the upper half plane does not utilize the Jacobian elliptical functions.

The mapping of the rectangle with use of the Jacobian elliptical functions gives the researcher flexibility with regard to the length at which the wave's energy is negligible. If a different length is to be studied, the boundary value problem remains essentially the same. The difference is the modulus, which would change due to the new ratio of lengths. The boundary conditions keep the same form or type of equation; the Jacobian elliptical functions are the part of the boundary conditions that change with the new modulus. This change is evident in Figures (I.1), (I.2), and (I.3), where the graphs of the functions change with each modulus. Also to be noted, the value of alpha would change due to the different position of point c on the rectangle.

The difficulty arises from the method of solving the mixed boundary value problem on the disc. With the use of the general equation that satisfies the Laplace equation in polar coordinates, only an approximate solution is developed; the explicit solution is not found. Due to the impracticality of computing a sum with an upper limit of infinity, an approximate solution is calculated using a large N. Another reason the solution is only an approximation is the singularity at $\Theta = \Theta_a$ on the boundary of the disc. This singularity comes from the Surface Boundary Condition. An approximation is used for the integration that is required in solving for the coefficients, β_j and γ_j. An explicit solution would eliminate the first reason and incorporate the second reason required.

Further study is required for developing an explicit solution for the mixed value problem on the disc. Research in this area has already been done by Ian N. Sneddon and can be found in his book, <u>Mixed Boundary Value Problems in Potential Theory</u>.

BIBLIOGRAPHY

Ahlfors, Lars V. 1979. Complex Analysis. 3rd edition. McGraw-Hill Publishing, New York.

Bowman, F. 1953. Introduction to Elliptic Functions with applications. English Universities Press LTD, London.

Guenther, R. B. and J. W. Lee. 1988. Partial Differential Equations of Mathematical Physics and Integral Equations. Prentice Hall, Englewood Cliffs, New Jersey.

Haberman, Richard. Elementary Applied Partial Differential Equations. Prentice Hall, Englewood Cliffs, New Jersey.

Hudspeth, R.T., J.W. Leonard, and Min-Chu Chen. May 1981. "Design Curves for Hinged Wavemakers: Experiments." Reprinted from Journal of the Hydraulics Division. Oregon State University Sea Grant College Program. Corvallis, Oregon.

Jeffrey, Alan. 1992. Complex Analysis and Applications. CRC Press, Boca Raton, Florida.

Milne-Thomson, L. M. 1968. Theoretical Hydrodynamics. 5th edition. MacMillan, New York.

Tanaka, Yoshihiro. 1988. "Irregular Points in Wavemaker Boundary Value Problem." Ph.d Thesis, Oregon State University, Corvallis, Oregon.

Wolfram, Stephen. 1991. Mathematica: A System for Doing Mathematics by Computer. 2nd edition. Addison-Wesley, New York.

APPENDICES

APPENDIX I
JACOBIAN ELLIPTICAL FUNCTIONS

The method for developing the Jacobian elliptical functions parallels the method used for trigonometric sine and cosine. Like sine and cosine, the Jacobian elliptical functions are developed from two integrals;

$$u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{I.1}$$

and

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{I.2}$$

where $0 < k < 1$ and $-1 < x < 1$ and where the square roots in the denominator are positive. Just as $\sin^{-1} x$ is defined as $\int_0^x \frac{dt}{\sqrt{1-t^2}}$, the right hand of Eq.(I.1) is defined as $\mathrm{sn}^{-1}\, x$ and by taking the inverse $\mathrm{sn}\, u = x$. The companion functions are defined as

$$\mathrm{cn}\, u = \sqrt{1 - \mathrm{sn}^2\, u} \tag{I.3}$$

and

$$\mathrm{dn}\, u = \sqrt{1 - k^2 \mathrm{sn}^2\, u}\ . \tag{I.4}$$

Since sn u has a range of {-1,1} for $-K < u < K$, cn u and dn u are positive and are even functions.

The Jacobian elliptical functions are not only dependent on u but also on k, called the modulus. When $k = 0$, sn u is just sin u. And when $k = 1$, sn u is tanh u. As k varies from 0 to 1, K ranges from $\frac{\pi}{2}$ to ∞. Due to the effect of the modulus, it is included in the notation of the Jacobian elliptical functions; the functions are denoted by $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, and $\mathrm{dn}(u,k)$.

From the Eqs.(I.1) and (I.2), the values of the Jacobian elliptical functions for $x=0$ and $x=K$ and for all k are

$$\begin{aligned} &\mathrm{sn}\,0=0 && \mathrm{cn}\,0=1 && \mathrm{dn}\,0=1 \\ &\mathrm{sn}\,K=1 && \mathrm{cn}\,K=1 && \mathrm{dn}\,K=k', \end{aligned} \tag{I.5}$$

where $k'^2+k^2=1$ and k' is called the complementary modulus. Associated with the complementary modulus is the relationship $\mathrm{sn}^{-1}[1,k']=K'$.

For practical purposes, the modulus can be approximated by

$$4\ \exp\left[\frac{-\pi K'}{2K}\right]. \tag{I.6}$$

The next properties to be developed are the derivatives of sn u, cn u, and dn u. (For the remainder of Appendix I, the modulus for the Jacobian elliptical functions will be understood to be k unless specified differently.) The derivative of Eq.(I.1) with respect to x is

$$\frac{du}{dx}=\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Taking the reciprocal of this equation gives

$$\frac{dx}{du}=\sqrt{(1-x^2)(1-k^2x^2)}.$$

By substituting sn u for x, the derivative becomes

$$\frac{d(\mathrm{sn}\,u)}{du}=\sqrt{(1-\mathrm{sn}^2\,u)(1-k^2\mathrm{sn}^2\,u)},$$

which can be written as

$$\frac{d(\mathrm{sn}\,u)}{du}=\mathrm{cn}\,u\ \ \mathrm{dn}\,u. \tag{I.7}$$

In order to find the derivative of cn u, substitute Eq.(I.7) into the derivative of Eq.(I.3) to get

$$\frac{d(\mathrm{cn}\,u)}{du}=-\mathrm{sn}\,u\ \ \mathrm{dn}\,u.$$

Similarly, for the derivative of dn u, substitute Eq.(I.7) into the derivative of Eq.(I.4) to get

$$\frac{d(\mathrm{dn}\ u)}{du} = -\mathrm{sn}\ u\ \mathrm{cn}\ u.$$

Another property needed to be developed is the addition property given by

$$\mathrm{sn}[u+v] = \frac{\mathrm{sn}\ u\ \mathrm{cn}\ v\ \mathrm{dn}\ v + \mathrm{sn}\ v\ \mathrm{cn}\ u\ \mathrm{dn}\ u}{1-k^2\mathrm{sn}^2\ u\ \mathrm{sn}^2\ v}\ . \tag{I.8}$$

Eq.(I.8) is developed by letting

$$z = \frac{\mathrm{sn}\ u\ \mathrm{cn}\ v\ \mathrm{dn}\ v + \mathrm{sn}\ v\ \mathrm{cn}\ u\ \mathrm{dn}\ u}{1-k^2\mathrm{sn}^2\ u\ \mathrm{sn}^2\ v}.$$

$\frac{dz}{du}$ is equal to $\frac{dz}{dv}$. Therefore, z can be written

as a function of $u+v$; $z=f(u+v)$. By letting $v=0$, $z=\mathrm{sn}\ u$ and $f(u)=\mathrm{sn}\ u$. Therefore, $f(u+v)=\mathrm{sn}[u+v]$ and the addition property is satisfied.

The addition properties of $\mathrm{cn}[u+v]$ and $\mathrm{dn}[u+v]$ are developed by substituting Eq.(I.8) into Eqs.(I.3) and (I.4) to get

$$\mathrm{cn}[u+v] = \frac{\mathrm{cn}\ u\ \mathrm{cn}\ v - \mathrm{sn}\ u\ \mathrm{sn}\ v\ \mathrm{dn}\ u\ \mathrm{dn}\ v}{1-k^2\ \mathrm{sn}^2\ u\ \mathrm{sn}^2\ v}$$

and

$$\mathrm{dn}[u+v] = \frac{\mathrm{dn}\ u\ \mathrm{dn}\ v - k^2\ \mathrm{sn}\ u\ \mathrm{sn}\ v\ \mathrm{cn}\ u\ \mathrm{cn}\ v}{1-k^2\ \mathrm{sn}^2\ u\ \mathrm{sn}^2\ v}.$$

The last property to develop is that the Jacobian elliptical functions are doubly periodic. By substituting the values of Eq.(I.5) and by letting $v=K$, the addition properties become

$$\mathrm{sn}[u+K] = \frac{\mathrm{cn}\ u}{\mathrm{dn}\ u} \qquad \mathrm{cn}[u+K] = \frac{-k'\ \mathrm{sn}\ u}{\mathrm{dn}\ u} \qquad \mathrm{dn}[u+K] = \frac{k'}{\mathrm{dn}\ u}.$$

By substituting K for u, the previous equations become

$$\mathrm{sn}[2K]=0 \qquad \mathrm{cn}[2K]=-1 \qquad \mathrm{dn}[2K]=1.$$

Then, by letting $v=2K$, the addition properties give

$$\mathrm{sn}[u+2K] = -\mathrm{sn}\ u \qquad \mathrm{cn}[u+2K] = -\mathrm{cn}\ u \qquad \mathrm{dn}[u+2K] = \mathrm{dn}\ u.$$

dn u is periodic with a period of 2K. Next, by letting $u=u+2K$, the previous three properties give

$$\mathrm{sn}[u+4K]=\mathrm{sn}\,u \qquad \mathrm{cn}[u+4K]=\mathrm{cn}\,u \qquad \mathrm{dn}[u+4K]=\mathrm{dn}\,u.$$

sn u and cn u are periodic with a period of 4K.

The periods that were just developed were done using real arguments. In order to show that the functions are doubly periodic, the use of complex arguments is required. Developed in Appendix II,

$$\mathrm{sn}[u+iv,k]=\frac{\mathrm{sn}[u,k]\ \mathrm{dn}[v,k']+i\ \mathrm{cn}[u,k]\ \mathrm{dn}[u,k]\ \mathrm{sn}[v,k']\ \mathrm{cn}[v,k']}{1-\mathrm{dn}^2[u,k]\ \mathrm{sn}^2[v,k']}.$$

By letting $v=K'$,

$$\mathrm{sn}[u+iK']=\frac{1}{k\mathrm{sn}\,u}.$$

Then, by substituting $u+iK'$ for u in the last equation,

$$\mathrm{sn}[u+i2K']=\frac{1}{k\ \mathrm{sn}[u+iK']}=\frac{1}{k\ \frac{1}{k\ \mathrm{sn}\,u}}=\mathrm{sn}\,u.$$

Thus, sn u is periodic with a $i2K'$ period using complex arguments. Hence, the term doubly periodic refers to a function that has the ratio of its periods to be not real.

Figures (I.1), (I.2), and (I.3) are the graphs of the Jacobian elliptical functions with various parameters. The graphs in Figure (I.1) use the parameter $k^2=.5$. This parameter is used to highlight the 'sine-like' nature of the Jacobian elliptical functions. K'' is $\mathrm{sn}^{-1}[1,.5]$. The graphs in Figure (I.2) are computed using the parameter that is used for the dimensions of the wavemaker tank; $k^2=1.041985942*10^{-7}$. With such a small parameter, dn u has only the range, {.99999994701, 1}. The graphs in Figure (I.3) are computed using the complementary parameter, $k^2=.9999998958$. With this parameter, dn u has the range, $\{3.228002378*10^{-4}, 1\}$. Also, sn u and cn u lose some of their 'sine-like' qualities.

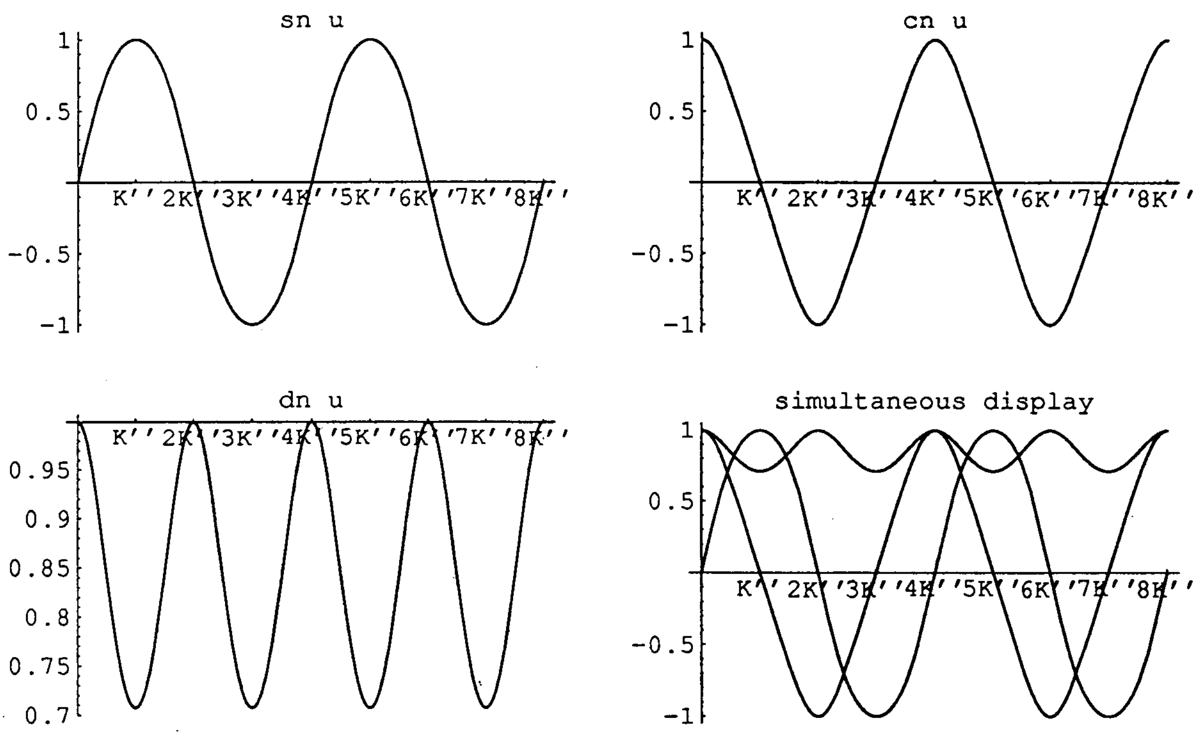


Figure I.1 The Jacobian elliptical functions with k^2=.5

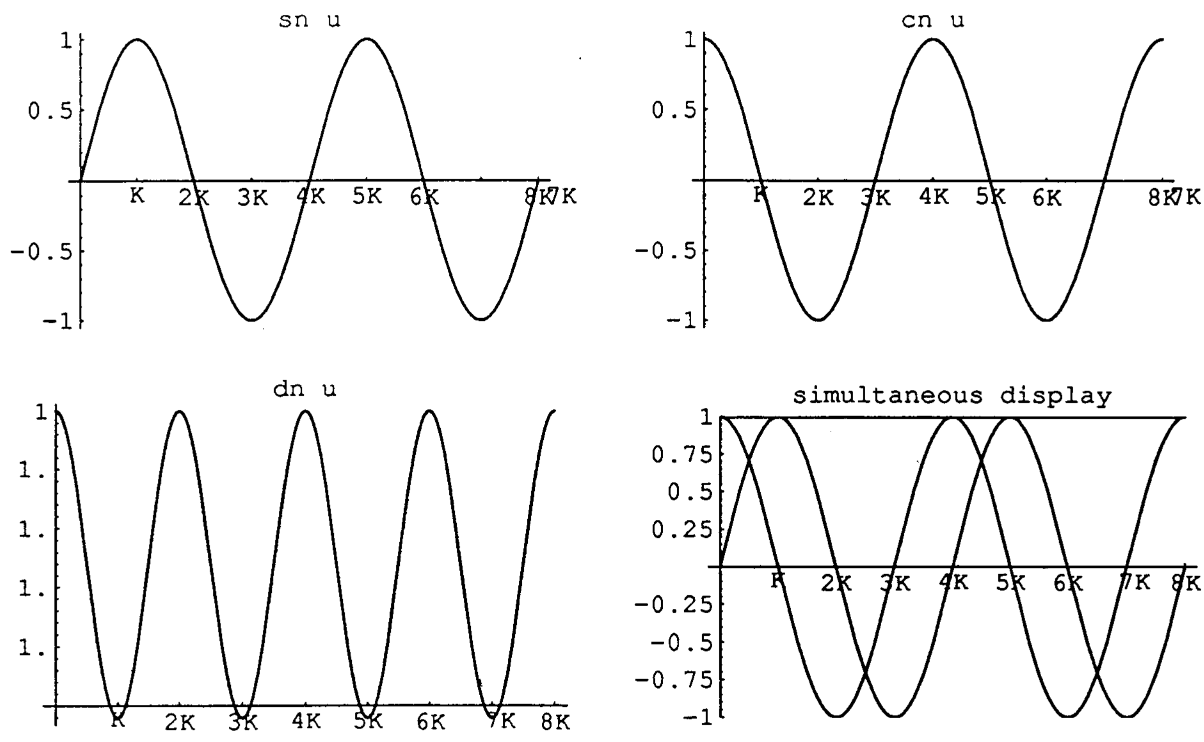


Figure I.2 The Jacobian elliptical functions with $k^2=1.041985942*10^{-7}$

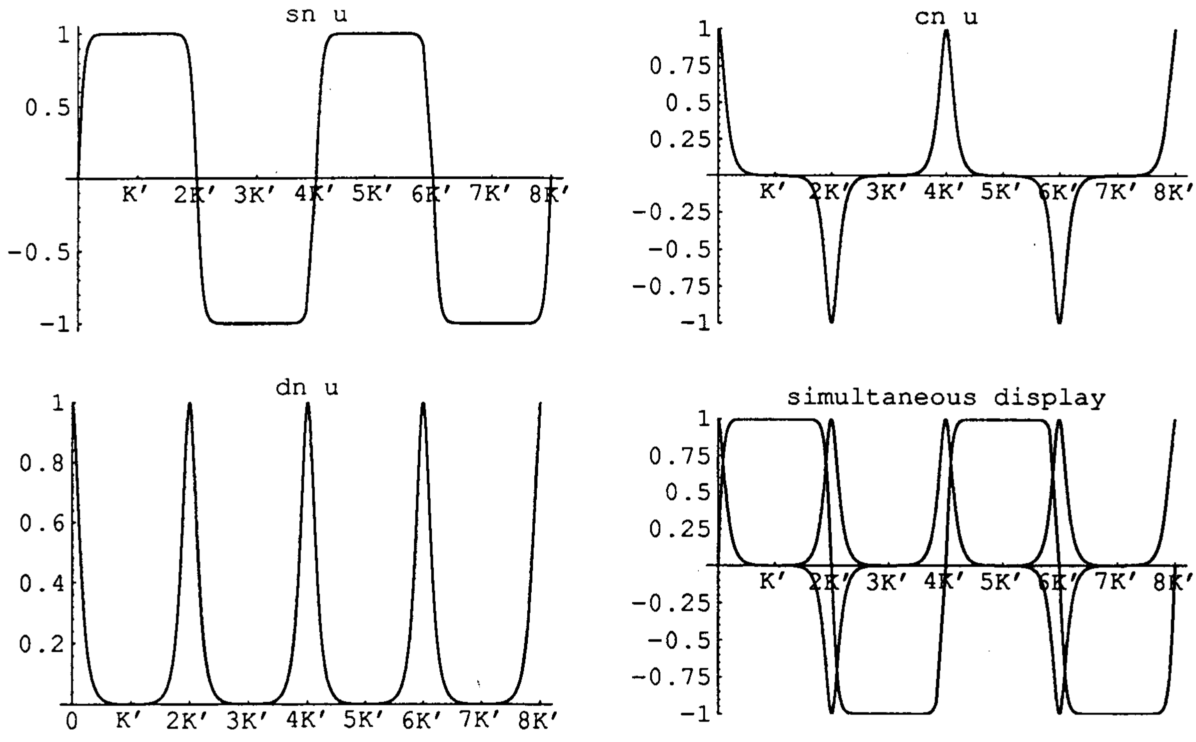


Figure I.3 The Jacobian elliptical functions with k^2=.9999998958

APPENDIX II
THE DEVELOPMENT OF THE COMPLEX FORM OF THE JACOBIAN ELLIPTICAL FUNCTIONS

Bowman develops the complex form of the Jacobian elliptical functions from

$$z = \int_0^y \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}. \tag{II.1}$$

Consider integration along the imaginary axis to get

$$z = \int_0^{iy} \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}. \tag{II.2}$$

Then, substitute $i\eta$ for w and $id\eta$ for dw to get

$$z = \int_0^y \frac{id\eta}{\sqrt{(1+\eta^2)(1+k^2\eta^2)}}.$$

Since the integration is real, z is strictly imaginary and iv can be substituted for z to get

$$iv = \int_0^y \frac{id\eta}{\sqrt{(1+\eta^2)(1+k^2\eta^2)}},$$

which reduces to

$$v = \int_0^y \frac{d\eta}{\sqrt{(1+\eta^2)(1+k^2\eta^2)}}.$$

Also, iv can be subsituted for z in Eq.(II.2) to get

$$\mathrm{sn}[iv,k] = iy.$$

In order to find the value of y, substitute tan ψ for η, tanϕ for y and $\sec^2\psi$ dψ for dη in the last integral to get

$$v = \int_0^\phi \frac{\sec^2\psi \; d\psi}{\sqrt{(1+\tan^2\psi)(1+k^2\tan^2\psi)}}.$$

The upper limit is ϕ because $\psi = \tan^{-1} y = \tan^{-1} \tan \phi = \phi$. When the modulus relationship $k^2 + k'^2 = 1$ is used, the integral becomes

$$v = \int_0^{\phi} \frac{\sec^2\psi \; d\psi}{\sqrt{(1+\tan^2\psi)(1+(1-k'^2)\tan^2\psi)}}.$$

The use of some standard trigonometric identities simplifies the denominator in the integral to

$$\begin{aligned}\sqrt{(1+\tan^2\psi)(1+(1-k'^2)\tan^2\psi)} &= \sqrt{\sec^2\psi \; (1+\tan^2\psi \; - k'^2 \tan^2\psi)} \\ &= \sqrt{\sec^2\psi \; (\sec^2\psi \; - \; k'^2 \tan^2\psi)} \\ &= \sqrt{\sec^4\psi \; (1-k'^2 \sin^2\psi)} \\ &= \sec^2\psi \; \sqrt{1-k'^2 \sin^2\psi}.\end{aligned}$$

Thus, the integral reduces to

$$v = \int_0^{\phi} \frac{d\psi}{\sqrt{1 \; - \; k'^2 \sin^2\psi}}.$$

The substitutions of x for $\sin \psi$ and $\frac{dx}{\sqrt{1-x^2}}$ for $d\psi$ changes the integral to

$$v = \int_0^{\sin \phi} \frac{dx}{\sqrt{1-x^2}\sqrt{1-k'^2 \; x^2}}.$$

(The latter substitution is done because $dx = \cos \psi \; d\psi$ implies $d\psi = \frac{dx}{\cos \psi}$ which can be rewritten $\frac{dx}{\sqrt{1 \; - \; \sin^2\psi}}$ and is $\frac{dx}{\sqrt{1 \; - \; x^2}}$ by the substitution of x for $\sin \psi$.)

From Appendix I, $v = \mathrm{sn}^{-1} \sin \phi$ and $\mathrm{sn}[v,k'] = \sin \phi$. Since both $\mathrm{sn}[u,k]$ and $\sin \beta$ have the similar identities, $\mathrm{sn}^2[u,k] + \mathrm{cn}^2[u,k] = 1$ and $\sin^2\beta + \cos^2\beta = 1$, $\sqrt{1-\mathrm{cn}^2[v,k]} = \sqrt{1-\cos^2\phi}$, which can be simplified to $\mathrm{cn}[v,k'] = \cos \phi$.

Recall that the goal is to find the value of y. So far, $y = \tan \phi$, $\mathrm{sn}[v,k'] = \sin \phi$, and $\mathrm{cn}[iv,k'] = \cos \phi$. Combining these three relationships gives

$$y = \tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{\mathrm{sn}[v,k']}{\mathrm{cn}[v,k']}.$$

Therefore, $\mathrm{sn}[iv,k] = iy$ can be restated as

$$\mathrm{sn}[iv,k] = i\,\frac{\mathrm{sn}[v,k']}{\mathrm{cn}[v,k']}. \tag{II.3}$$

To get the complex form in terms of functions with real arguments, substitute Eq.(II.3) into the identities

$$\mathrm{cn}[u,k] = \sqrt{1 - \mathrm{sn}^2[u,k]} \quad \text{and} \quad \mathrm{dn}[u,k] = \sqrt{1 - k^2\,\mathrm{sn}^2[u,k]}$$

to get

$$\mathrm{cn}[iv,k] = \sqrt{1 + \frac{\mathrm{sn}^2[v,k']}{\mathrm{cn}^2[v,k']}} = \frac{1}{\mathrm{cn}[v,k']} \tag{II.4}$$

and

$$\mathrm{dn}[iv,k] = \sqrt{1 + k^2\,\frac{\mathrm{sn}^2[v,k']}{\mathrm{cn}^2[v,k']}} = \frac{\mathrm{dn}[v,k']}{\mathrm{cn}[v,k']}. \tag{II.5}$$

Combine Eqs.(II.3), (II.4), and (II.5) to make the complex forms

$$\mathrm{sn}[iv,k'] = i\,\frac{\mathrm{sn}[v,k']}{\mathrm{cn}[v,k']}, \tag{II.6}$$

$$\mathrm{cn}[iv,k] = \frac{1}{\mathrm{cn}[v,k']}, \tag{II.7}$$

$$\mathrm{dn}[iv,k] = \frac{\mathrm{dn}[v,k']}{\mathrm{cn}[v,k']}. \tag{II.8}$$

The last three equations are the Jacobian elliptical functions for imaginary values in terms of a real argument. Since w is complex, the calculation of $\mathrm{sn}[u+iv,k]$ is required. Subsituting $u+iv$ into the addition property given by Eq.(I.8) gives

$$\mathrm{sn}[u+iv,k] = \frac{\mathrm{sn}[u,k]\ \mathrm{cn}[iv,k]\ \mathrm{dn}[iv,k] + \mathrm{cn}[u,k]\ \mathrm{dn}[u,k]\ \mathrm{sn}[iv,k]}{1 - k^2\ \mathrm{sn}^2[u,k]\ \mathrm{sn}^2[iv,k]}.$$

Substituting Eqs.(I.6), (I.7), and (I.8) into the previous equation gives

$$\mathrm{sn}[\mathrm{u}+i\mathrm{v},k] = \frac{\mathrm{sn}[\mathrm{u},k]\ \dfrac{1}{\mathrm{cn}[\mathrm{v},k']}\ \dfrac{\mathrm{dn}[\mathrm{v},k']}{\mathrm{cn}[\mathrm{v},k']} + i\ \mathrm{cn}[\mathrm{u},k]\ \ \mathrm{dn}[\mathrm{u},k]\ \ \dfrac{\mathrm{sn}[\mathrm{v},k']}{\mathrm{cn}[\mathrm{v},k']}}{1-\ k^2\ \mathrm{sn}^2[\mathrm{u},k]\ \dfrac{\mathrm{sn}^2[\mathrm{v},k']}{\mathrm{cn}^2[\mathrm{v},k']}},$$

which simplifies to

$$\mathrm{sn}[\mathrm{u}+i\mathrm{v},k] = \frac{\mathrm{sn}[\mathrm{u},k]\ \mathrm{dn}[\mathrm{v},k']\ +\ i\ \mathrm{cn}[\mathrm{u},k]\ \mathrm{dn}[\mathrm{u},k]\ \mathrm{sn}[\mathrm{v},k']\ \mathrm{cn}[\mathrm{v},k']}{\mathrm{cn}^2[\mathrm{v},k']\ +\ k^2\ \mathrm{sn}^2[\mathrm{u},k]\ \mathrm{sn}^2[\mathrm{v},k']}.$$

The substitution and manipulation of the identities reduces the denominator and the complex form of one of the Jacobian elliptical functions is

$$\mathrm{sn}[\mathrm{u}+i\mathrm{v},k] = \frac{\mathrm{sn}[\mathrm{u},k]\ \mathrm{dn}[\mathrm{v},k']\ +\ i\ \mathrm{cn}[\mathrm{u},k]\ \mathrm{dn}[\mathrm{u},k]\ \mathrm{sn}[\mathrm{v},k']\ \mathrm{cn}[\mathrm{v},k']}{1-\mathrm{dn}^2[\mathrm{u},k]\ \mathrm{sn}^2[\mathrm{v},k']}.$$

APPENDIX III
THE CALCULATION OF ALPHA

The following is the development of the equation that calculates alpha. Alpha is the parameter for the mapping function, Eq.(3.5), that is used to make an adjusted disc. The alpha equation uses both a chosen point on the boundary of the disc in the $\bar{q}$-plane and the argument of the point in the $\bar{Q}$-plane that is mapped from the chosen point (Figure III.1). Let, $\bar{q} = \bar{a} + \bar{b}i$ and be a point on the boundary with a positive argument. Let $\bar{Q} = \bar{A} + \bar{B}i$ and be the point to which $\bar{q}$ is mapped. From Eq.(3.5) $\bar{Q}$ is

$$\frac{\bar{q}-\alpha}{1-\alpha\bar{q}}.$$

Substitute $\bar{a}+\bar{b}i$ for $\bar{q}$ to get

$$\bar{Q} = \frac{\bar{a}+\bar{b}i-\alpha}{1-\alpha(\bar{a}+\bar{b}i)}.$$

This function in standard form is

$$\bar{Q} = \frac{((\bar{a}-\alpha)(1-\alpha\bar{a})-\alpha\bar{b}^2)+i(\alpha\bar{b}(\bar{a}-\alpha)+(1-\alpha\bar{a})\bar{b})}{(1-\alpha\bar{a})^2+\alpha^2\bar{b}^2)}.$$

Thus, $\bar{A}$ is $\dfrac{((\bar{a}-\alpha)(1-\alpha\bar{a})-\alpha\bar{b}^2)}{(1-\alpha\bar{a})^2+\alpha^2\bar{b}^2)}$ and $\bar{B}$ is $\dfrac{(\alpha\bar{b}(\bar{a}-\alpha)+(1-\alpha\bar{a})\bar{b})}{(1-\alpha\bar{a})^2+\alpha^2\bar{b}^2)}$.

To introduce θ, use the relationship

$$\tan\theta = \frac{\bar{B}}{\bar{A}},$$

which gives

$$\tan\theta = \frac{(\alpha\bar{b}(\bar{a}-\alpha)+(1-\alpha\bar{a})\bar{b})}{((\bar{a}-\alpha)(1-\alpha\bar{a})-\alpha\bar{b}^2)}.$$

To solve for α, first multiply the denominator to both sides to get

$$\bar{a}\tan\theta - \alpha\,\bar{a}^2\tan\theta - \alpha\tan\theta + \alpha^2\,\bar{a}\tan\theta - \alpha\,\bar{b}^2\tan\theta = \alpha\bar{a}\bar{b} - \alpha^2\bar{b} + \bar{b} - \alpha\,\bar{a}\,\bar{b}.$$

This equation simplifies to

$$\alpha^2(\bar{a}\tan\theta + \bar{b}) - \alpha(\bar{a}^2+\bar{b}^2+1)\tan\theta + \bar{a}\tan\theta - \bar{b} = 0.$$

The quadratic formula gives

$$\alpha = \frac{(\bar{a}^2+\bar{b}^2+1)\tan\theta + \sqrt{(\bar{a}^2+\bar{b}^2+1)^2\tan^2\theta - 4\bar{a}^2\tan^2\theta - \bar{b}^2}}{2\,(\bar{a}\tan\theta+\bar{b})}.$$

Since $\bar{q}$ is on the boundary of the unit disc, $\bar{a}^2+\bar{b}^2=1$; thus, the equation to calculate alpha is

$$\alpha = \frac{\tan\theta+\bar{b}\ \sqrt{\tan^2\theta-1}}{\bar{a}\tan\theta+\bar{b}}.$$

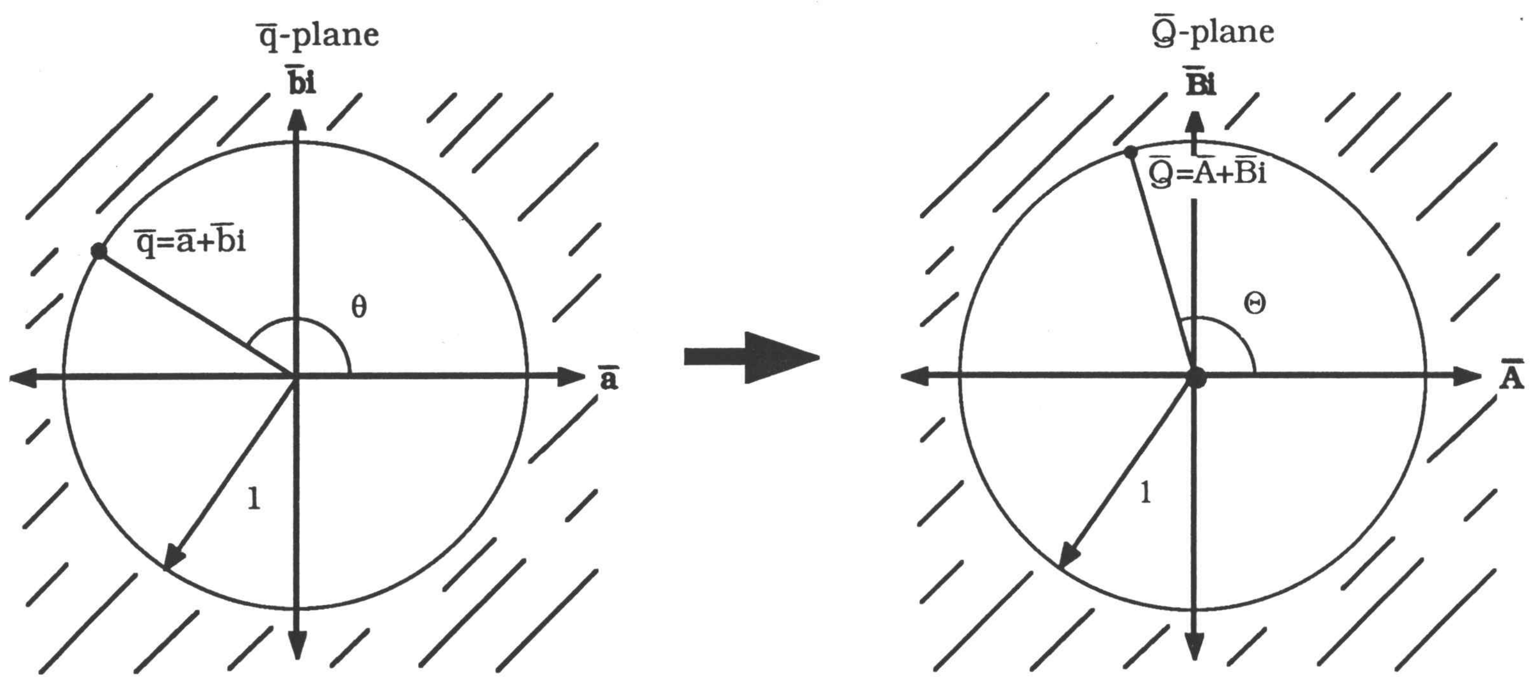


Figure III.1 The adjusted disc mapping of alpha

APPENDIX IV

THE DEVELOPMENT OF THE MATRIX FOR THE SYSTEM OF EQUATIONS

For the system of equations, Eq.(6.15), a method for solving the coefficients by using Mathematica needs to be developed. By converting the system of equations into the matrix form, $WX = V$, the linear algebra commands of Mathematica can be used. The linear solve command is used to calculate the coefficients.

For simplicity the examples in this appendix use $N = 2$ to demonstrate the process. For $N = 2$, the system of equations is

$$\frac{\gamma_o}{2}A_o + \mathcal{A}_{01}\gamma_1 + \mathcal{A}_{02}\gamma_2 + \mathcal{B}_{01}\beta_1 + \mathcal{B}_{02}\beta_2 = \mathcal{H}_o$$

$$\frac{\gamma_o}{2}\ A_1 + \ \mathcal{A}_{11}\gamma_1 + \pi\gamma_1 + \mathcal{A}_{12}\gamma_2 + \mathcal{B}_{11}\beta_1 + \mathcal{B}_{12}\beta_2 = \mathcal{H}_1$$

$$\frac{\gamma_o}{2}A_2 + \mathcal{A}_{21}\gamma_1 + \mathcal{A}_{22}\gamma_2 + 2\pi\gamma_2 + \mathcal{B}_{21}\beta_1 + \mathcal{B}_{22}\beta_2 = \mathcal{H}_2$$

$$\frac{\gamma_o}{2}C_1 + \mathcal{C}_{11}\gamma_1 + \mathcal{C}_{12}\gamma_2 + \mathcal{D}_{11}\gamma_1 + \pi\gamma_1 + \mathcal{D}_{12}\gamma_2 = \mathcal{K}_1$$

$$\frac{\gamma_o}{2}C_2 + \mathcal{C}_{21}\gamma_1 + \mathcal{C}_{22}\gamma_2 + \mathcal{D}_{21}\gamma_1 + \mathcal{D}_{22}\gamma_2 + 2\pi\gamma_2 = \mathcal{K}_2 .$$

The method that is developed converts this system into the matrix multiplication,

$$\begin{bmatrix} A_o & \mathcal{A}_{01} & \mathcal{A}_{02} & \mathcal{B}_{01} & \mathcal{B}_{02} \\ A_1 & \mathcal{A}_{11}+\pi & \mathcal{A}_{12} & \mathcal{B}_{11} & \mathcal{B}_{12} \\ A_2 & \mathcal{A}_{21} & \mathcal{A}_{22}+2\pi & \mathcal{B}_{21} & \mathcal{B}_{22} \\ C_1 & \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{D}_{11}+\pi & \mathcal{D}_{12} \\ C_2 & \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{D}_{21} & \mathcal{D}_{22}+2\pi \end{bmatrix} \begin{bmatrix} \gamma_o \\ \gamma_1 \\ \gamma_2 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \mathcal{H}_o \\ \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{K}_1 \\ \mathcal{K}_2 \end{bmatrix}. \qquad \text{(IV.1)}$$

There are two reasons that restrain the conversion of the left hand side of the system into the matrix W. The first is that Mathematica does not use 0 as an index number. The second reason is incorporation of the π-terms that are in the second through fifth equations of the system. W is not just

$$\begin{bmatrix} A_i & \mathcal{A}_{ij} & \mathfrak{B}_{ij} \\ C_l & \mathfrak{C}_{lj} & \mathfrak{D}_{lj} \end{bmatrix};$$

it is a combination of seven matrices shown by

$$\begin{bmatrix} A_o & \mathcal{A}_{o1} & \mathcal{A}_{o2} & \mathfrak{B}_{o1} & \mathfrak{B}_{o2} \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} \mathcal{A}_{11}+\pi & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22}+2\pi \end{bmatrix} \cdot \begin{bmatrix} \mathfrak{B}_{11} & \mathfrak{B}_{12} \\ \mathfrak{B}_{21} & \mathfrak{B}_{22} \end{bmatrix}$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \begin{bmatrix} \mathfrak{C}_{11} & \mathfrak{C}_{12} \\ \mathfrak{C}_{21} & \mathfrak{C}_{22} \end{bmatrix} \begin{bmatrix} \mathfrak{D}_{11}+\pi & \mathfrak{D}_{12} \\ \mathfrak{D}_{21} & \mathfrak{D}_{22}+2\pi \end{bmatrix}.$$

The first step is to introduce the π-terms. This introduction is done by making the π-matrix, which is a square, diagonal matrix whose terms are $i\pi$;

$$\begin{bmatrix} \pi & 0 \\ 0 & 2\pi \end{bmatrix}.$$

The π-matrix is made by the addition of three matrices, such as

$$\begin{bmatrix} \pi & \pi \\ 2\pi & 2\pi \end{bmatrix} - \begin{bmatrix} 0 & \pi \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 2\pi & 0 \end{bmatrix}.$$

This calculation is done by

```
Table[i*Pi,{i,n},{j,n}]-Table[If[i<j,i*Pi,0],{i,n},{j,n}]
  -Table[If i>j, i*Pi, 0],{i,n},{j,n}].
```

To introduce the π-terms into W, add the π-matrix to $\mathcal{A}_{ij}$ and $\mathfrak{D}_{lj}$ to get

$$\begin{bmatrix} \mathcal{A}_{11}+\pi & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22}+2\pi \end{bmatrix} \text{ and } \begin{bmatrix} \mathfrak{D}_{11}+\pi & \mathfrak{D}_{12} \\ \mathfrak{D}_{21} & \mathfrak{D}_{22}+2\pi \end{bmatrix}.$$

Label these matrices as $\mathcal{A}_{ij}/\pi$ and $\mathfrak{D}_{ij}/\pi$, respectively.

Now, that all the pieces of W are developed, the next step is to combine the matrices to make W. The first task to make W is to combine the transposes of $\mathcal{A}_{ij}/\pi$ and $\mathfrak{B}_{ij}$ to get

$$\begin{bmatrix} \mathcal{A}_{11}+\pi & \mathcal{A}_{21} \\ \mathcal{A}_{12} & \mathcal{A}_{22}+2\pi \\ \mathfrak{B}_{11} & \mathfrak{B}_{21} \\ \mathfrak{B}_{12} & \mathfrak{B}_{22} \end{bmatrix},$$

called matrix1. This task is performed by

```
Table[If i<N+1, Transpose[piAijmatrix][[i]],
  Transpose[Bijmatrix][[i-N]], {i,2*N}].
```

Next, combine the 1,...,N terms of A_i to the previous matrix to get

$$\begin{bmatrix} A_1 & A_2 \\ \mathcal{A}_{11}+\pi & \mathcal{A}_{21} \\ \mathcal{A}_{12} & \mathcal{A}_{22}+2\pi \\ \mathfrak{B}_{11} & \mathfrak{B}_{21} \\ \mathfrak{B}_{12} & \mathfrak{B}_{22} \end{bmatrix},$$

called matrix2. This task is performed by

```
matrix2=Prepend[matrix1, Aimatrix].
```

Transpose matrix2 to get

$$\begin{bmatrix} A_1 & \mathcal{A}_{11}+\pi & \mathcal{A}_{12} & \mathfrak{B}_{11} & \mathfrak{B}_{12} \\ A_2 & \mathcal{A}_{21} & \mathcal{A}_{22}+2\pi & \mathfrak{B}_{21} & \mathfrak{B}_{22} \end{bmatrix}.$$

This task is performed by

```
matrix3=Transpose[matrix2].
```

Note that matrix3 is the matrix form of the left hand side of the second and third equations of the system.

Similarly, C_i, $\mathcal{C}_{ij}$ and $\mathcal{D}_{ij}/\pi$ are combined into the matrix form of the right hand side of the fourth and fifth equations,

$$\begin{bmatrix} C_1 & \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{D}_{11}+\pi & \mathcal{D}_{12} \\ C_2 & \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{D}_{21} & \mathcal{D}_{22}+2\pi \end{bmatrix}$$

This task is performed using the set of commands:

```
matrix4 = Table[If i < N+1, Transpose[Cijmatrix][[i]],
  Transpose[piDijmatrix][[i-N]], {i,2*N}];
```

```
matrix5 = Prepend[matrix4,Cimatrix];
```

and

```
matrix6 = Transpose[matrix5].
```

Combining the two matrices forms

$$\begin{bmatrix} A_1 & \mathcal{A}_{11}+\pi & \mathcal{A}_{12} & \mathcal{B}_{11} & \mathcal{B}_{12} \\ A_2 & \mathcal{A}_{21} & \mathcal{A}_{22}+2\pi & \mathcal{B}_{21} & \mathcal{B}_{22} \\ C_1 & \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{D}_{11}+\pi & \mathcal{D}_{12} \\ C_2 & \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{D}_{21} & \mathcal{D}_{22}+2\pi \end{bmatrix},$$

which is made by

```
matrix7 = Table[If[i < N+1, matrix3[[i]], matrix6[[i-N]], {i,2*N}],
```

This matrix is the conversion of all but the first equation in the system.

The next step is to develop the first equation and combine it to matrix7. In order to do this step, combine the $i = 0$ terms of $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$ to get

$$\begin{bmatrix} \mathcal{A}_{01} & \mathcal{A}_{02} & \mathcal{B}_{01} & \mathcal{B}_{02} \end{bmatrix}.$$

This task is performed by

```
Not1 = Table[If[i < N+1, Aimatrix[[i]], Bimatrix[[i-N]], {i, 2*N}].
```

Then, combine A_0 to Not1 to get

$$\begin{bmatrix} A_0 & \mathcal{A}_{01} & \mathcal{A}_{02} & \mathcal{B}_{01} & \mathcal{B}_{02} \end{bmatrix}.$$

This task is performed by

```
Not2 = Prepend[Not1, Ainot].
```

Combine this last matrix with matrix7 to get the matrix form of the left hand side of the equations of the system,

$$\begin{bmatrix} A_0 & \mathcal{A}_{01} & \mathcal{A}_{02} & \mathfrak{B}_{01} & \mathfrak{B}_{02} \\ A_1 & \mathcal{A}_{11}+\pi & \mathcal{A}_{12} & \mathfrak{B}_{11} & \mathfrak{B}_{12} \\ A_2 & \mathcal{A}_{21} & \mathcal{A}_{22}+2\pi & \mathfrak{B}_{21} & \mathfrak{B}_{22} \\ C_1 & \mathfrak{C}_{11} & \mathfrak{C}_{12} & \mathfrak{D}_{11}+\pi & \mathfrak{D}_{12} \\ C_2 & \mathfrak{C}_{21} & \mathfrak{C}_{22} & \mathfrak{D}_{21} & \mathfrak{D}_{22}+2\pi \end{bmatrix}.$$

This last task is accomplished by

```
Thematrix = Prepend[matrix7, Not2].
```

For the right hand side of the equations, combine $\mathcal{H}_i$ and $\mathcal{K}_l$ for $i, l = 1, \ldots, N$ to get

$$\begin{bmatrix} \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{K}_1 & \mathcal{K}_2 \end{bmatrix}.$$

This task is performed by

```
otherside1 = Table[If[i < N+1, Himatrix[[i]], Kimatrix[[i-N]],
  {i,2*N}].
```

And, to attach the $\mathcal{H}_0$ to get

$$\begin{bmatrix} \mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{K}_1 & \mathcal{K}_2 \end{bmatrix}$$

use

```
otherside2 = Prepend[otherside2, Hinot].
```

Thus, the two matrices of the system have been developed and the coefficients can be solved by

```
LinearSolve[Thematrix, otherside2].
```

The remaining step is to label the coefficients as γ_i or β_j. The labeling is done by

gammacoefficientnot = coefficients[[1]],

gammacoefficients = Table[coefficients[[i]], {i,2,N+1}],

and

betacoefficients = Table[coefficients[[i]], {i,N+2,2N+1}].

APPENDIX V
THE SURFACE ELEVATION EQUATION

The equation that describes the surface elevation of the wavemaker is

$$\eta(x,t) = -\frac{\omega}{g}\,\mathrm{Re}[\phi(x)\ i\ e^{-i\omega t}]$$

which can be rewritten as

$$\eta(x,t) = \frac{\omega}{g}\ \{\mathrm{Im}[\phi(x)]\ \cos\omega t - \mathrm{Re}[\phi(x)]\ \sin\omega t\}.$$

Figure V.1i is the three dimensional graph of the surface elevation equation that is calculated with the solution found by using $h=3.96$ and $N=2$. Figures V.1ii and V.1iii are the two dimensional graphs for $t=1$ and $t=2$, respectively, of the surface elevation equation.

Figure V.2i is the three dimensional graph of the surface elevation equation that is calculated with the solution found by using $h=3.96$ and $N=10$. Figures V.2ii and V.2iii are the two dimensional graphs for $t=1$ and $t=2$, respectively, of the surface elevation equation.

Figure V.3i is the three dimensional graph of the surface elevation equation that is calculated with the solution found by using $h=4.42$ and $N=2$. Figures V.3ii and V.3iii are the two dimensional graphs for $t=1$ and $t=2$, respectively, of the surface elevation equation.

Figure V.4i is the three dimensional graph of the surface elevation equation that is calculated with the solution found by using $h=4.42$ and $N=10$. Figures V.4ii and V.4iii are the two dimensional graphs for $t=1$ and $t=2$, respectively, of the surface elevation equation.

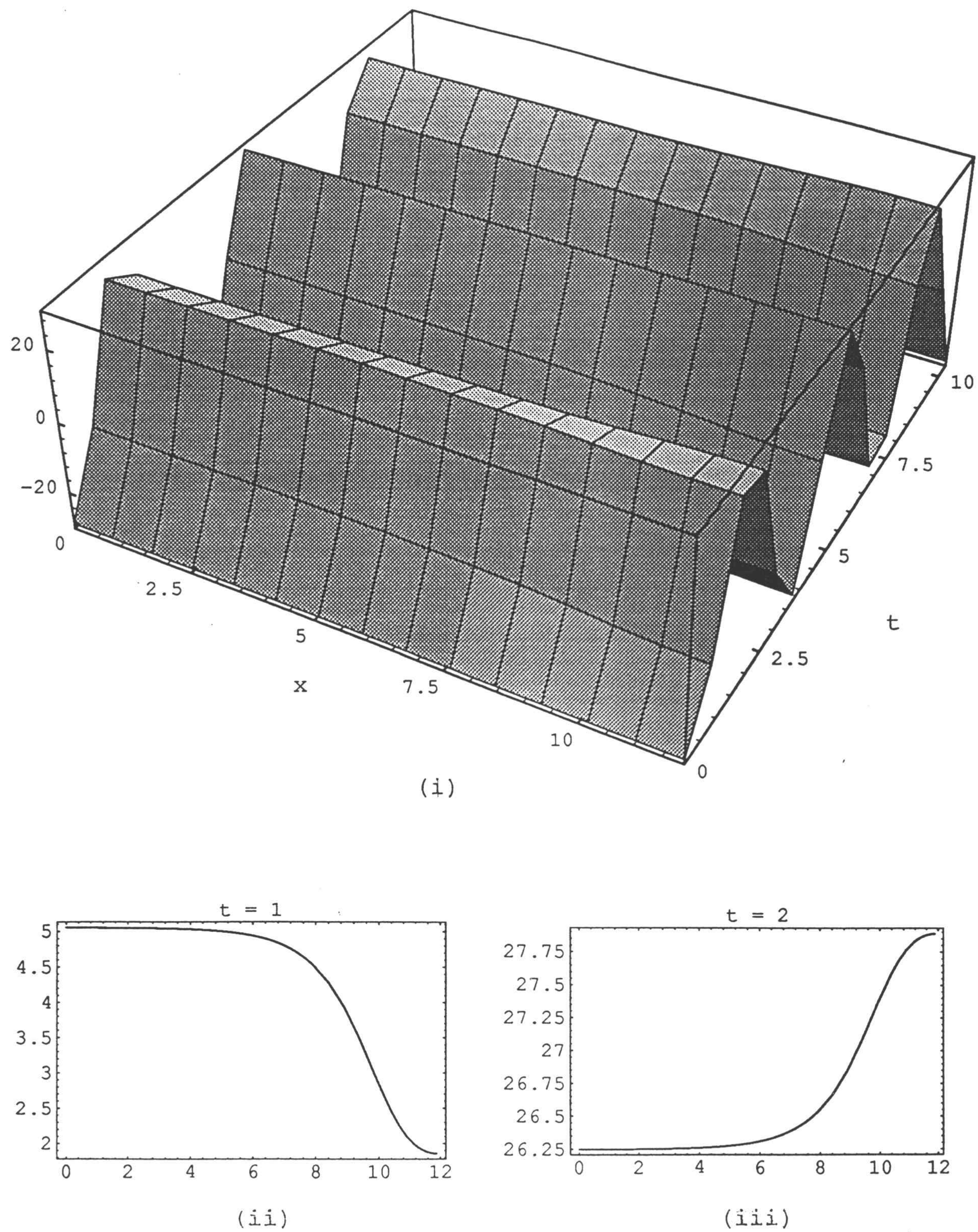


Figure V.1 The surface elevation for h=3.96 and N=2

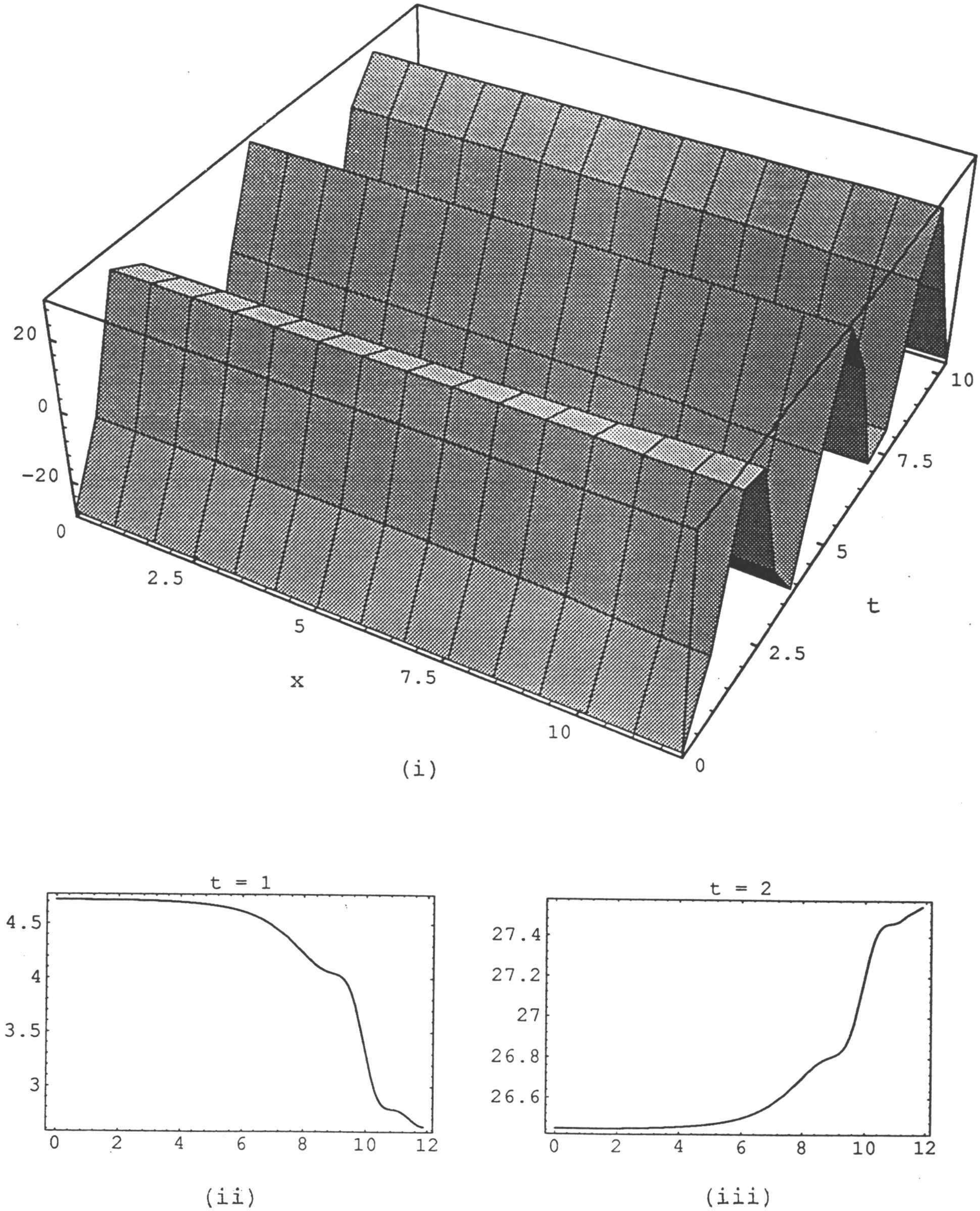
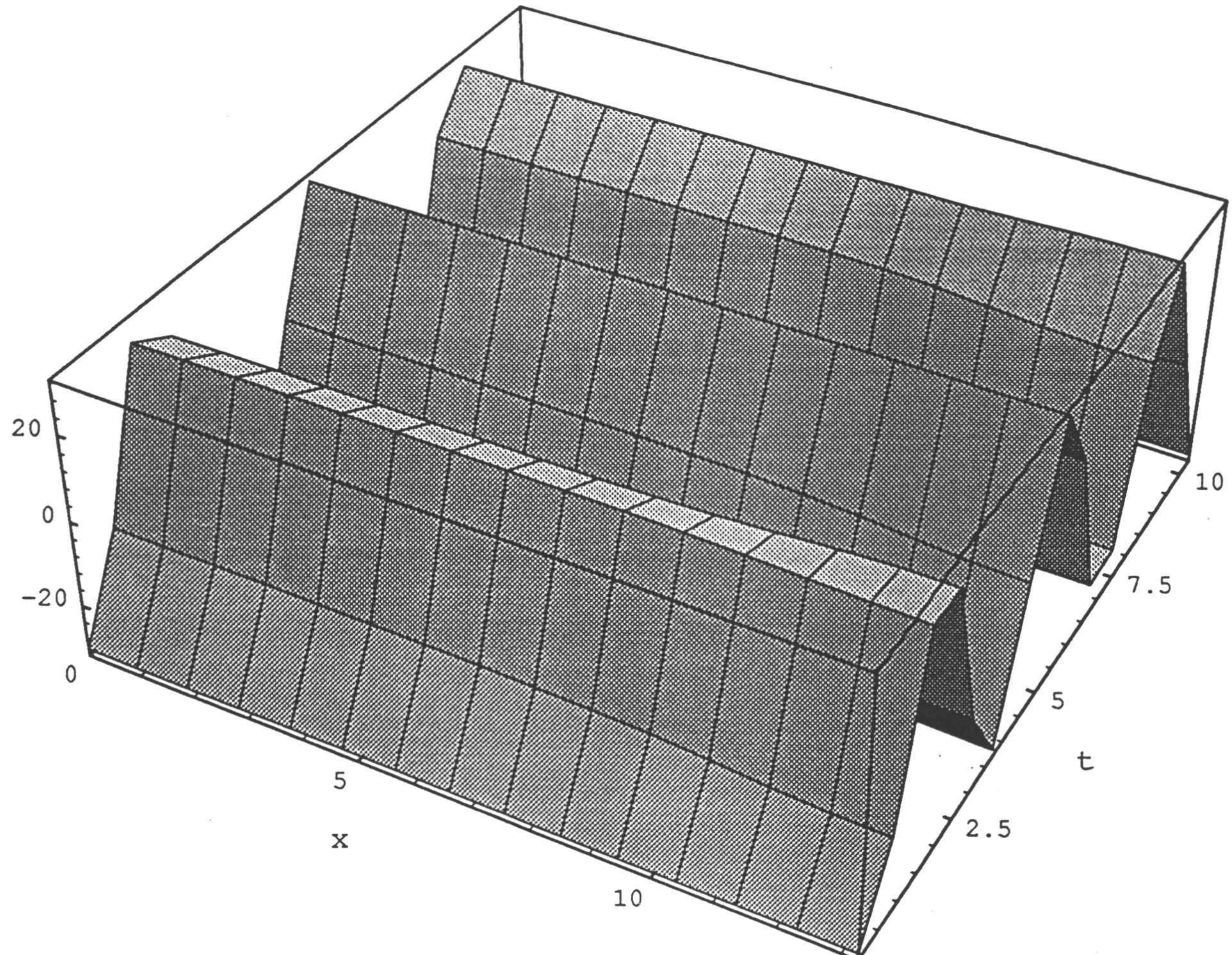
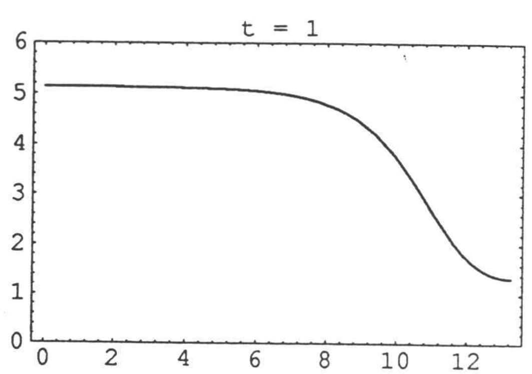


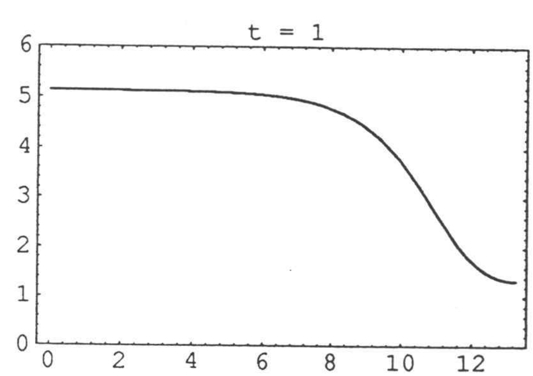
Figure V.2 The surface elevation for h=3.96 and N=10



(i)



(ii)



(iii)

Figure V.3 The surface elevation for h=4.42 and N=2

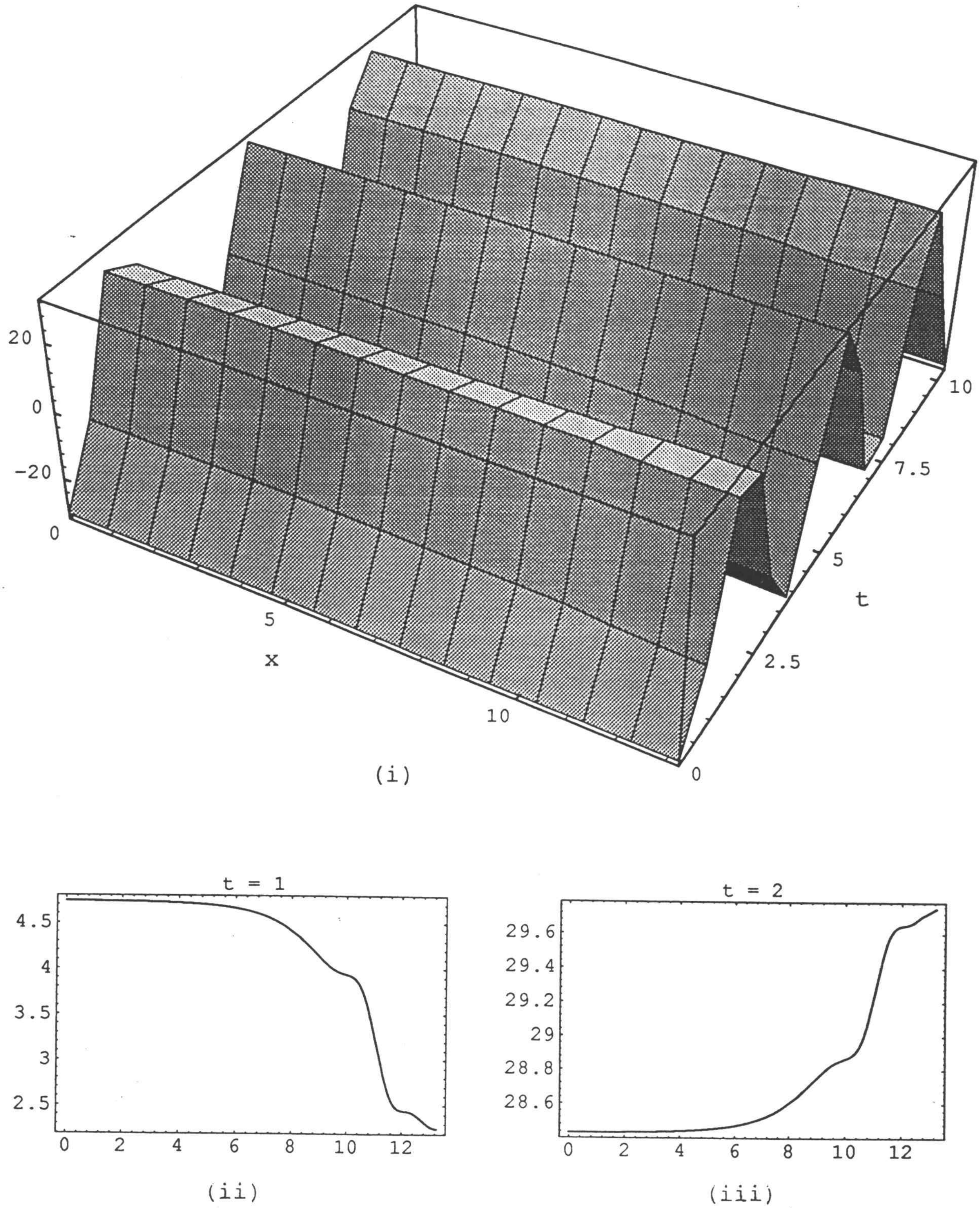


Figure V.4 The surface elevation for h=4.42 and N=10

```
                    (*  solution.m  *)

(* solution.m is a series of equations formatted on Mathematica
that gives a numerical approximation of the mixed boundary
differential equation problem on the rectangle.  This is done
in several steps.

      Step 1:  Creates the mapping from the rectangle to the
               unit disc.

      Step 2: Calculates the alpha variable allowing for
              adjustments to the unit disc.

      Step 3: Calculates the angles fo specific points on the
              disc to be used as limits of integration

      Step 4: Calculates the matrices used to solve the Fourier
              coefficients.

      Step 5: Produces the 2n+1 by 2n+1 matrix and the 2n+1
              by 1 matrix that represent the system of
              equations for the coefficients.  Alos, produces
              the Phi equation which satisfies the boundary
              conditions.

      Step 6: Replaces the variables (r,theta) with (x,y) in
              equation Phi, ie. maps back to the rectangle. *)

 (*-------------------------------------------------------*)

                    (* CONSTANTS

 anot             top meeting point of the wavemaker and the wall

 bnot             hinge point

 c                propagating wave number

 epsilonnot       phase angle of wavemaker displacement

 h                still water depth

 Knot             wave constant

 lengthfactor     length of of depth for radiation condition

 n                number of iterations

 omega            wave frequency

 ScriptF[theta_]  shape equation of the wavemaker displacement

 T                wave period

 Thetac           the desired angle of point c on the disc
*)
```

```
(*CALCULATION OF THE PARAMETERS FOR JACOBIAN ELLIPTICAL EQNS.*)

ratio = 2*lengthfactor  (*ratio of depth to width used in
                  calculation of parameters for the Jacobian
                  Elliptical Equations.  2 is the introduced
                  from the initialization of the rectangle in
                  order to map to the upper plane. *)

m = N[16 Exp[-ratio Pi], 16]     (* parameter for J.E.I.*)
mi = N[1 - m, 16]                (* complementary parameter *)


(*---------------------------------------------------------*)


                    (* MAPPING FUNCTIONS *)


s[u_] = JacobiSN[u, m]
si[u_] = JacobiSN[u, mi]

cn[u_] = JacobiCN[u, m]
cni[u_] = JacobiCN[u, mi]

d[u_] = JacobiDN[u, m]
di[u_] = JacobiDN[u,mi]

K := N[InverseJacobiSN[1, m], 14]
Ki := N[InverseJacobiSN[1, mi], 14]     (* Ki/K = ratio *)

di[Ki] = Sqrt[m]
di[-Ki] = Sqrt[m]
di[0] = 1

d[K] = Sqrt[mi]
d[-K] = Sqrt[mi]
d[0] = 1


s[0] = 0
s[K] = 1
s[-K] = 1               (* Initilized conditions for proper *)
                        (* calculations, due to 0/0 *)
                        (* example:  cn[K] = # 10^-16, *)
                        (* which cannot be used, else affect *)
si[0]= 0                (* the mapping of the rectangle *)
si[Ki] = 1              (* to the upper half plane *)
si[-Ki] = 1


cn[0] = 1
cn[K] = 0
cn[-K] = 0

cni[0] = 1
cni[Ki] = 0
cni[-Ki] = 0
```

```
sn[u_, v_] = resn[u,v] + I imsn[u,v]     (* Jacobian E.F.
                                           for complex numbers *)
   resn[u_, v_] = (s[u] di[v])/(1 - (d[u] si[v])^2)
                      (* real component of the J.E.F *)
   imsn[u_, v_] = (cn[u] d[u] si[v] cni[v])/(1 - (d[u] si[v])^2)
                      (* imaginary component of the J.E.F.*)

imsn[K,v_] = 0              (*Initilized conditions for proper*)
imsn[-K,v_] = 0             (*calculations, due to 0/0*)
imsn[u_,0] = 0
imsn[u_,Ki] = 0

resn[0, v_] = 0


w1[z_] = I*z                   (*rotation of rectangle.*)

w2[z_] = z - (h/2)             (*centering.*)

w3[z_] = 2*z                   (*unitizing.*)

w4[z_] = z/prok                (*proportion wrt K.*)
prok = h/K                     (*adjusts the rectangle to*)
                               (*correct dimensions.*)

      map4[z_] = (2/prok) (-(h/2) + I z)
                               (*composition of 1-4.*)

w5[z_] = sn[w5u[z], w5v[z]]    (*J.E.F. *)
      w5u[z_] = Re[map4[z]]    (*for complex numbers.*)
      w5v[z_] = Im[map4[z]]    (*rectangle to half plane.*)

w6[z_] = (I - z)/(I + z)       (*plane to unit disc.*)

cir[z_] = w6[w5[z]]            (*eqn. rectangle to disc.*)

w7[z_] := (z - alpha)/(1 - alpha*z)
                               (*adjusted disc; use of the*)
                               (*alpha eqn. in conjunction.*)

adjcir[z_] := w7[cir[z]]
                      (*eqn. rectangle to adjusted disc*)



      (*---------------------------------------------*)
```

```
        (* CALCULATION OF POINT c ONTO THE UNIT DISC *)

pointC = cir[3 h - I h]
                 (* pointC is the mapping of c onto the unit
                    disc and is the reference point for
                    calculating alpha for the adjusted disc
                    equation in w7[z] *)

      (*---------------------------------------------*)

                     (* CALCULATION OF ALPHA *)

      (* alpha is the adjustment factor in w7[z].  It is
         calculated using the real and imaginary values of
         pointC and the desired angle Thetac.  *)

alpha = N[(Tan[Thetac]
      + Im[pointC] Sqrt[Tan[Thetac]^2-1])/
      (Tan[Thetac] Re[pointC] + Im[pointC])]

      (*example: move the point -0.999999584 + 0.0009121 I
      to a point on the boundary of the disc with an angle
      of 3 Pi/4.
      alphaeqn[-1,-0.999999584,0.0009121] = -0.9978*)

      (*-----------------------------------------------*)

              (* CALCULATIONS OF THE ANGLES *)

    (* calculates the angles of points A,C,D,Anot,Bnot on the
       disc, which are used as the limits of integration *)

Thetaa = N[((10001/10000) Arg[adjcir[0]])]
              (* 10001/10000 is an adjustment due to the
                 singularity in the function a2[theta] *)

Thetad = -Thetac        (* Thetad and Thetac are symmetric on
                           the boundary of the disc *)

ThetaAnot = N[(10000/10001) Arg[adjcir[anot]]]
                        (* angle of Anot, the mapping of anot
                           on the disc *)

ThetaBnot = N[Arg[adjcir[bnot]]]
                        (* angle of Bnot, the mapping of bnot
                           on the disc *)

      (*---------------------------------------------*)
```

```
(*  CALCULATION OF THE MATRICES USED FOR EVALUATION OF
                  THE FOURIER COEFFICIENTS  *)


                  (*  Functions  *)

      (* dPhi/dr + a[theta] Phi = h[theta] *)

      (* a[theta] = a1[theta] + a2[theta] + a3[theta]
                  a1[theta] = 0 *)


a2[theta_] := (h Knot Sqrt[2] (-1+alpha^2))/(4 K*
              Sqrt[(2*alpha+ Cos[theta] (1 + alpha^2))*
              (4 alpha + mi (1 - alpha)^2 ((3
              - 2 alpha^2) Cos[theta] - 1))])

                  (* for Thetad..Thetaa, else = 0 *)

a3[theta_] := (I (-1 + alpha^2)*c*h*Sec[theta/2])/
      (4 K*((2*alpha + (1 + alpha^2)*Cos[theta])*
      ((1 + alpha)^2 - m (1-alpha)^2 Tan[theta/2]^2))^(1/2))

               (* for Thetac ..Pi, -Pi..Thetad, else = 0 *)

      (* h[theta] = h1[theta] + h2[theta]
h1[theta] = 0 *)

h2[theta_] := (-(-1 + alpha^2)*E^(-I*epsilonnot)*h*omega*
      Sec[theta/2]*ScriptF[theta]*(alpha + alpha^2
      - (-1 + alpha) Tan[theta/2]))/
      (4*K*(1 + alpha + (alpha^2 - alpha)*Tan[theta/2])*
      ((2*alpha + (1+ alpha^2)*Cos[theta])*
      ((1 + alpha)^2 - m*(1 - alpha)^2 Tan[theta/2]^2))^(1/2))

                  (* for ThetaAnot..ThetaBnot, else = 0 *)
                  (* with zero at 0 *)


                  (*  Matrices [i,j] *)

      (* calculates a[0] *)
Ainot = (NIntegrate[a2[theta],{theta,Thetad,Thetaa}]
              + NIntegrate[a3[theta],{theta, Thetac, Pi}]
              + NIntegrate[a3[theta],{theta, -Pi, Thetad}])/2


              (* Matrix A *)

      (*calculates a[1],a[2],..,a[n] *)
Aimatrix = Table[((NIntegrate[a2[theta] Cos[i theta],
                                {theta,Thetad,Thetaa}]
              + NIntegrate[a3[theta] Cos[i theta],
                                {theta, Thetac, Pi}]
```

```
            (* Matrix Script A  *)

      (* calculates   a[1,1]..a[1,n]
                      a[n,1]..a[n,n] *)

Aijmatrix = Table[(NIntegrate
   [a2[theta] Cos[i theta] Cos[j theta],{theta,Thetad,Thetaa}]
              + NIntegrate
   [a3[theta] Cos[i theta]*Cos[j theta], {theta, Thetac, Pi}]
              + NIntegrate
   [a3[theta] Cos[i theta] Cos[j theta], {theta, -Pi, Thetad}]),
      {i,n},{j,n}]


            (* Matrix Script B *)

      (* calculates   b[1,1]..b[1,n]
                      b[n,1]..b[n,n] *)

Bijmatrix = Table[(NIntegrate
  [a2[theta] Cos[i theta] Sin[j theta],{theta,Thetad,Thetaa}]
              + NIntegrate
  [a3[theta] Cos[i theta]  Sin[j theta],{theta, Thetac, Pi}]
              + NIntegrate
  [a3[theta] Cos[i theta] Sin[j theta],{theta, -Pi, Thetad}]),
    {i,n},{j,n}]


            (* Matrix C *)

      (* calculates   c[1],..,c[n] *)

Cimatrix = Table[((NIntegrate
  [a2[theta] Sin[i theta],{theta,Thetad,Thetaa}]
              + NIntegrate
  [a3[theta] Sin[i theta],{theta, Thetac, Pi}]
              + NIntegrate
  [a3[theta] Sin[i theta],{theta, -Pi, Thetad}])/2),
              {i,n}]


            (* Matrix Script C *)

      (* calculates   c[1,1]..c[1,n]
                      c[n,1]..c[n,n] *)

Cijmatrix = Table[(NIntegrate
  [a2[theta] Sin[i theta] Cos[j theta],{theta,Thetad,Thetaa}]
              + NIntegrate
  [a3[theta] Sin[i theta]  Cos[j theta],{theta, Thetac, Pi}]
              + NIntegrate
  [a3[theta] Sin[i theta] Cos[j theta],{theta, -Pi, Thetad}]),
      {i,n},{j,n}]
```

```
                    (* Matrix Script D *)

      (* calculates   d[1,1]..d[1,n]
                      d[n,1]..d[n,n] *)

Dijmatrix = Table[(NIntegrate
  [a2[theta] Sin[i theta] Sin[j theta],{theta,Thetad,Thetaa}]
             + NIntegrate
  [a3[theta] Sin[i theta]  Sin[j theta],{theta, Thetac, Pi}]
             + NIntegrate
  [a3[theta] Sin[i theta] Sin[j theta],{theta, -Pi, Thetad}]),
      {i,n},{j,n}]

                    (* calculates h[0] *)

Hinot = NIntegrate[h2[theta], {theta,ThetaAnot,ThetaBnot}]


                    (* Matrix Script H *)

      (* calculates   h[1],..,h[n] *)

Himatrix = Table[NIntegrate
 [h2[theta] Cos[i theta], {theta,ThetaAnot,ThetaBnot}], {i,n}]


                    (* Matrix Script K *)

      (*calculates    k[1],..,k[n] *)

Kimatrix = Table[NIntegrate
 [h2[theta] Sin[i theta],{theta,ThetaAnot,ThetaBnot}] , {i,n}]


      (* calculates   Pi  0       for n = 2
                      0  2Pi                  *)

pimatrix = Table[i*Pi, {i, n}, {j, n}]
             - Table[If[i < j, i*Pi, 0], {i, n}, {j, n}]
             - Table[If[i > j, i*Pi, 0], {i, n}, {j, n}]


      (* calculates   a[1,1]+Pi   a[1,2]        for n = 2
                      a[2,1]    a[2,2]+2Pi             *)

piAijmatrix := Aijmatrix + pimatrix


              (* similarly for Dijmatrix *)

piDijmatrix := Dijmatrix + pimatrix
```

```
(* Calculation of Phi equation *)

      (* calculates   a[1,1]+Pi  a[2,1]   for n = 2
                      a[1,2]     a[2,2]+2Pi
                      b[1,1]     b[2,1]
                      b[1,2]     b[2,2]   *)
 matrix1 = Table[If[i < n + 1, Transpose[piAijmatrix][[i]],
      Transpose[Bijmatrix][[i - n]]], {i, 2*n}]


      (* calculates
         a[1]       a[2]  for n = 2
         a[1,1]+Pi  a[2,1]
         a[1,2]     a[2,2]+2Pi
         b[1,1]     b[2,1]
         b[1,2]     b[2,2]  *)
 matrix2 = Prepend[matrix1, Aimatrix]


      (* calculates
         a[1]  a[1,1]+Pi  a[1,2]      b[1,1]  b[1,2]
         a[2]  a[2,1]     a[2,2]+2Pi  b[2,1]  b[2,2] *)
 matrix3 = Transpose[matrix2]


      (* calculates   c[1,1]     c[2,1]
                      c[1,2]     c[2,2]
                      d[1,1]+Pi  d[2,1]
                      d[1,2]     d[2,2]+2Pi  *)
 matrix4 = Table[If[i < n + 1, Transpose[Cijmatrix][[i]],
      Transpose[piDijmatrix][[i - n]]], {i, 2*n}]


      (* calculates   c[1]        c[2]
                      c[1,1]      c[2,1]
                      c[1,2]      c[2,2]
                      d[1,1]+Pi   d[2,1]
                      d[1,2]      d[2,2]+2Pi  *)
 matrix5 = Prepend[matrix4, Cimatrix]


      (* calculates
         c[1]  c[1,1]  c[1,2]  d[1,1]+Pi  d[1,2]
         c[2]  c[2,1]  c[2,2]  d[2,1]     d[2,2]+2Pi*)
 matrix6 = Transpose[matrix5]


(* calculates
   a[1]  a[1,1]+Pi  a[1,2]      b[1,1]     b[1,2]
   a[2]  a[2,1]     a[2,2]+2Pi  b[2,1]     b[2,2]
   c[1]  c[1,1]     c[1,2]      d[1,1]+Pi  d[1,2]
   c[2]  c[2,1]     c[2,2]      d[2,1]     d[2,2]+2Pi*)
matrix7 = Table[If[i < n + 1, matrix3[[i]], matrix6[[i - n]]],
               {i, 2*n}]
```

```
          (* calculates   a[1]  a[2]  b[1]  b[2]  *)
Not1 = Table[If[i < n + 1, Aimatrix[[i]], Bimatrix[[i - n]]],
              {i, 2*n}]


      (* calculates   a[0]  a[1]  a[2]  b[1]  b[2]  *)
Not2 = Prepend[Not1, Ainot]  (* the part where index k = 0  *)


(* calculates
      a[0]   a[1]        a[2]        b[1]        b[2]
      a[1]  a[1,1]+Pi  a[1,2]      b[1,1]      b[1,2]
      a[2]  a[2,1]     a[2,2]+2Pi  b[2,1]      b[2,2]
      c[1]  c[1,1]     c[1,2]      d[1,1]+Pi  d[1,2]
      c[2]  c[2,1]     c[2,2]      d[2,1]      d[2,2]+2Pi*)
Thematrix = Prepend[matrix7, Not2]  (* the complete matrix;rhs*)


      (* calculates   h[1]  h[2]  k[1]  k[2]  *)
otherside1 = Table[If[i < n + 1, Himatrix[[i]],
              Kimatrix[[i - n]]], {i, 2*n}]


      (* calculates   h[0]  h[1]  h[2]  k[1]  k[2]  *)
otherside2 = Prepend[otherside1, Hinot]


(* solves
      a[0]   a[1]        a[2]        b[1]        b[2]          = h[0]
      a[1]  a[1,1]+Pi  a[1,2]      b[1,1]      b[1,2]        = h[1]
      a[2]  a[2,1]     a[2,2]+2Pi  b[2,1]      b[2,2]        = h[2]
      c[1]  c[1,1]     c[1,2]      d[1,1]+Pi  d[1,2]        = k[1]
      c[2]  c[2,1]     c[2,2]      d[2,1]      d[2,2]+2Pi  = k[2]
      *)
fouriercoefficients = LinearSolve[Thematrix,otherside2]


      (* the solution for gammanot *)
gammacoefficientnot = fouriercoefficients[[1]]


      (* the solutions for gamma1..gamman *)
gammacoefficients = Table[fouriercoefficients[[i]],{i,2,n+1}]


      (* the solutions for beta1..betan *)
betacoefficients = Table[fouriercoefficients[[i]],{i,n+2,2*n+1}]


      (* the expanded Phi equation with the coefficients *)
Phi[r_,theta_] := (gammacoefficientnot
              + Sum[r^i (gammacoefficients[[i]] Cos[i theta]
              + betacoefficients[[i]] Sin[i theta]),{i,n}])
```

```
R = ((1 - 2*alpha + alpha^2 + X^2 + 2*alpha*X^2 + alpha^2*X^2
     - 2*Y + 2*alpha^2*Y + Y^2 + 2*alpha*Y^2 + alpha^2*Y^2)/
     (1 - 2*alpha + alpha^2 + X^2 + 2*alpha*X^2
     + alpha^2*X^2 + 2*Y - 2*alpha^2*Y + Y^2 + 2*alpha*Y^2
     + alpha^2*Y^2))^(1/2)


Theta = ArcTan[(1 - X^2 - Y^2 + alpha^2*(1 - X^2 - Y^2)
     - 2*alpha*(1 + X^2 + Y^2))/
     (X^2 + alpha^2*(X^2 + (-1 + Y)^2) + (1 + Y)^2
     + 2*alpha*(-1 + X^2 + Y^2)),
     (2*(1 - alpha^2)*X)/(X^2 + alpha^2*(X^2 + (-1 + Y)^2)
     + (1 + Y)^2 + 2*alpha*(-1 + X^2 + Y^2))]


     X := (s[u] di[v])/(1 - (d[u] si[v])^2)

     Y := (cn[u] d[u] si[v] cni[v])/(1 - (d[u] si[v])^2)

             u := -2 (k/h) (h/2 + y)
             v := 2 (k/h) x

phi[R,Theta] = Phi[R,Theta]
```