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Most of the existing research on mathematical morphology is restricted to the deterministic case. This thesis addresses the void in the results on the stochastic properties of morphological filters.

The primary results include analysis of the stochastic properties of morphological operations, such as dilation, erosion, closing and opening. Two unbiased morphological filters are introduced and a quantitative description of the probability distribution function of morphological operations on independent, identically distributed random signals is obtained. Design of an optimal morphological filter in the sense of a criterion proposed here is also discussed.

A brief, but systematic description of the definitions and properties of deterministic morphological operations on sets is presented to establish the necessary background for the analysis of the filter stochastic properties.

STOCHASTIC PROPERTIES OF MORPHOLOGICAL FILTERS

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STOCHASTIC PROPERTIES OF MORPHOLOGICAL FILTERS

CHAPTER 1. INTRODUCTION

During the past decade, the research on digital image processing attracted a lot of attention among engineers and scientists. One of the most interesting topics in this field is mathematical morphology. The underlying concepts of mathematical morphology were first proposed by G. Matheron and F. Serra in 1964 [1] and their results gained considerable popularity recently. Morphological operations are highly nonlinear signal smoothing techniques where each signal is viewed as a set in Euclidean space. Since the morphological operation modifies signals, it is often referred to as a morphological filter. The filters are the set operations which transform the graph of the signal and provide a quantitative description of its geometrical structure. Two properties of morphological filters are of particular interest [10]: 1) The filters smooth transient change in signal intensity (e.g. noise) and 2) The filters are invariant to sustained change in signal intensity (e.g. monotonic trends), most notably very sharp change or edges. That is the filters can suppress noise without destroying the important signal (e.g. image) details, such as edges and lines. These properties have made the technique very attractive for enhancing and restoring images, and for noise filtering [2]-[9],[12]-[23].

Morphological transforms locally modify geometric features of signal by exploring the concept of structuring element (or function). Maragos and Schafer [16] [19] [20] introduced a unified theory of translation invariant systems with applications to morphological analysis and coding of images. They examined the set theoretic interpretation of morphological filters in the framework of mathematical morphology and introduced the representation of classical linear filters in terms of morphological operations. They also extended the theory of median, order statistic and stack filters by using mathematical morphology to analyze them and by relating them to those morphological operations that commute with thresholding.

As the demand for high precision signal processing or high resolution image processing increases, optimal filter design is necessary. In particular the noise contaminating signal should be removed as efficiently as possible. The following issues pertain to optimal morphological filter design. 1) A criterion to evaluate the smoothness degree of a filtered signal versus the filter computational complexity. 2) A method to analyze the filtered signal statistics quantitatively. The main difficulty in the latter arises from the nonlinearity of morphological filters [19]. A successful investigation of the stochastic properties of the filters can help to simplify their complexity.

The analysis of morphological filters, referred in this thesis as M-Filter, has largely been restricted to the deterministic case. This thesis attempts to address the stochastic performance of M-Filters in a systematic manner. In Chapter II, the necessary mathematical background, such as Minkowski addition and subtraction is presented and definitions of basic M-Filters (dilation, erosion, closing and opening) are given. These definitions are based on set operations. In Chapter III, the concept of umbra of a function is briefly reviewed. This establishes a connection between set operation and functional operation. Thus, the operational concepts and properties of mathematical morphology applied on sets are also suitable for functions. In Chapter IV, the basic properties of M-Filters applied to a deterministic signal sequence are discussed. These properties prepare the necessary background for the later discussion. In Chapter V, the general stochastic properties of M-Filters are examined. We present a lemma which explores the duality of M-Filters in the stochastic sense. This lemma simplifies the discussion of the M-Filter stochastic properties. Several lemmas concerning the scaling of the means and variances of M-Filters, the separation of a constant signal in additive noise, and the variance bounds are introduced. Two lemmas describing an invariance property of filtering an i.i.d. stochastic field are also presented. All of these concepts build up a base to analyze the M-Filters and to describe the filter stochastic characteristics. An intuitive notion that M-Filters are biased is formalized in Chapter VI. Two unbiased morphological filters called "average dilation erosion" filter (ADE) and "average closing opening" filter (ACO) are defined and their insensitivity to the change of the structuring functions is analyzed. In Chapter VII, the probability distribution functions (P.D.F.s) of the M-Filters on any i.i.d. stochastic field with a simple structuring function sequence are characterized[1]. We give the analytical solutions

1. The phrase "M-Filter on a process" abbreviates "the output of a M-Filter operating on an input stochastic process".

of the probability density functions (p.d.f.s), provided that the P.D.F. of the i.i.d. input stochastic field is continuous. The means and variances of the M-Filters on an i.i.,d. stochastic field with symmetric uniform distribution are obtained. A simple computer simulation illustrates the obtained results. As a special case, the analytical solutions of the P.D.F.s, means and variances of M-Filters on the binary stochastic process are presented. All these results provide the necessary background for selecting a structuring function sequence to design optimal M-Filters. In Chapter VIII, a criterion and a design method for an optimal morphological filter design are proposed. The final chapter contains the conclusions and suggestions for further research.

CHAPTER 2. BASIC MINKOWSKI OPERATIONS AND MATHEMATICAL MORPHOLOGY

This chapter is concerned with basic concepts of Minkowski algebra which form the basis for the analysis and design of morphological filters. The original reference to this material is Serra [1][2].

2.1 Basic concepts.

Let A, B, S denote the sets such that $S \supseteq A$ and $S \supseteq B$, we then can define their union, intersection and difference as

Union $A \cup B = \{x \in S: x \in A \text{ or } x \in B\}$. (2.1)

Intersection $A \cap B = \{x \in S: x \in A \text{ and } x \in B\}$. (2.2)

Difference $A \setminus B = \{x \in S: x \in A \text{ and } x \notin B\}$. (2.3)

We also define the complement A^c as

$$A^c = \{x \in S: x \notin A\} = S \setminus A. \quad (2.4)$$

Some basic identities are now presented.

Distribution of union over intersection:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \quad (2.5)$$

Distribution of intersection over union:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \quad (2.6)$$

DeMorgan's law:

$$(A \cup B)^c = A^c \cap B^c, \quad (2.7)$$

$$(A \cap B)^c = A^c \cup B^c. \quad (2.8)$$

2. All proofs in this chapter are modified versions of those found in the Bibliography. However, an original contribution provides a systematic introduction of set operations using a minimum of mathematical notions.

2.2 Operations on subsets of Euclidean space.

Euclidean space R^d is a d-dimensional (d-D) linear vector space, on which the following operation are defined.

Addition

$$x+y=(x_1+y_1,\cdots\cdots, x_d+y_d). \tag{2.9}$$

Scalar multiplication

$$c\cdot x=cx=(cx_1,\cdots\cdots,cx_d), \tag{2.10}$$

where $x=(x_1,\cdots\cdots, x_d)$, $y=(y_1,\cdots\cdots, y_d)\in R^d$ and $c\in R$. This vector structure allows for the definition of new set operations special to Euclidean space.

Multiplication by real numbers

$$cA=\{c\cdot a:\ a\in A\} \tag{2.11}$$

for $c\in R$ and $R^d\supseteq A$. The case $c=-1$ leads to the particular case of reflection

$$A^s=-A=\{-a:\ a\in A\}\quad \text{for } R^d\supseteq A. \tag{2.12}$$

If $A=A^s$ then set A is said to be symmetric.

Translation by x

$$A_x=\{z:\ z=a+x \text{ and } a\in A\}\quad \text{for a given x in } R^d. \tag{2.13}$$

Lemma 2.1: Let $R^d\supset A$, then

$$(A^c)_x=(A_x)^c\ . \tag{2.14}$$

□

Proof: $(A^c)_x=\{z:\ z=a+x,\ a\in R^d\backslash A\}=\{z:\ z\in (R^d)_x\backslash A_x\}=\{z:\ z\in R^d\backslash A_x\}=(A_x)^c$

Q.E.D.

Eq.(2.14) states that the complement of a set A translated by a vector x is equal to the translation of the complement of set A. The importance of Eq.(2.14) arises from providing a mean to prove the following properties of Minkowski addition and subtraction.

Minkowski addition

$$A \oplus B = \bigcup_{y \in B} A_y \tag{2.15}$$

or $$A \oplus B = \{z: z = x + y,\ x \in A \text{ and } y \in B\} \text{ for } R^d \supseteq A,\ B. \tag{2.16}$$

The operation of Minkowski addition by a set B enlarges, translates, and deforms a set A.

Minkowski subtraction

$$A \ominus B = \bigcap_{y \in B} A_y \tag{2.17}$$

or $$A \ominus B = (A^c \oplus B)^c. \tag{2.18}$$

Note that Eqs.(2.15)(2.16) are equivalent. Eq.(2.17) and Eq.(2.18) are equivalent also. This can be shown as follows

$$(A \ominus B)^c = (\bigcap_{y \in B} A_y)^c = \bigcup_{y \in B} (A_y)^c = \bigcup_{y \in B} (A^c)_y = (A^c \oplus B)$$

where the DeMorgan's law and Lemma 2.1 (see Eqs.(2.8)(2.14)) were applied. Thus

$$A \ominus B = (A^c \oplus B)^c.$$

2.3 Four basic operations of mathematical morphology

The four operations mentioned below can be thought as special cases of Minkowski addition and subtraction. However these special cases constitute the core of morphological filters.

2.3.1 Dilation $A \oplus B^s$

The set A is enlarged and, at least in the case where the structuring element B is a ball, smoothed. In particular, the action of dilation fills in cavities, repairs fissures, and joins together a fragmented image.

2.3.2 Erosion $A \ominus B^s$

This operation shrinks the set A, tending to produce smaller fragments, even separating the connected sets into several subsets. This can be helpful in the estimation of the number of parts composing an image.

2.3.3 Opening $A_B=(A \ominus B^s) \oplus B$

The operation of opening A by B is the result of attempting to reverse an erosion by a dilation. The opening A_B of a set A by a set B has an appearance similar to that of the original set A, but is built only on the portion of the image that survives the initial erosion. Therefore small disconnected fragments of the image disappear under opening. This is useful in eliminating possible image defects or noise.

2.3.4 Closing $A^B=(A\oplus B^s)\ominus B$

This operation attempts to reverse a dilation by an erosion. Again, A^B bears an approximate resemblance to A. Together with the opening, the closing operation is useful in cleaning up an image. The action of closing tends to close up small holes, to join up close but separated subsets, and to smooth out the boundaries of an image.

The geometric explanation of the four basic Minkowski operations, dilation, erosion, opening and closing is given in references [1][19]. The applications of these operations in quantitative image analysis are further described in the following chapters.

2.4 Equalities and inclusions of the geometric operations

Because of the importance of Minkowski addition and subtraction, we present some identities for these operations. Let A, B, C, are the subspaces of d-dimensional (d-D) real space R^d, then

$$A\oplus B=B\oplus A, \tag{2.19}$$

$$(A\oplus B)\oplus C=(A\oplus C)\oplus B=A\oplus(B\oplus C), \tag{2.20}$$

$$A\oplus(B\cup C)=(A\oplus B)\cup(A\oplus C), \tag{2.21}$$

$$(A\cup B)\oplus C=(A\oplus C)\cup(B\oplus C). \tag{2.22}$$

Eqs.(2.19)(2.20)(2.21)(2.22) can be proved from the definitions of Minkowski addition (see Eqs.(2.15)(2.16)). Next, two properties which are often used together with the DeMorgan's law in the proofs of other properties are given:

$$(A\ominus B)^c=A^c\oplus B, \tag{2.23}$$

and $$(A\oplus B)^c=A^c\ominus B. \tag{2.24}$$

Eq.(2.23) has been proved to be equivalent to the definition of Minkowski subtraction (see Eqs.(2.17)(2.18)). Eq.(2.24) can be proved in a similar way.

Proof of Eq.(2.24): By the DeMorgan's law and Lemma 2.1

$$(A\oplus B)^c=(\bigcup_{y\in B}A_y)^c=\bigcap_{y\in B}(A_y)^c=\bigcap_{y\in B}(A^c)_y=A^c\ominus B.$$ **Q.E.D.**

Other properties are listed below:

$$(A\ominus B)\ominus C=(A\ominus C)\ominus B, \tag{2.25}$$

$$A\ominus(B\oplus C)=(A\ominus B)\ominus C, \tag{2.26}$$

$$A\ominus(B\cup C)=(A\ominus B)\cap(A\ominus C), \tag{2.27}$$

$$(B\cap C)\ominus A=(B\ominus A)\cap(C\ominus A). \tag{2.28}$$

The four properties, in fact, are the application of Eqs.(2.23)(2.24), which can be proved in an identical manner. For example:

Proof of Eq.(2.26)

$$A\ominus(B\oplus C)=((A\ominus(B\oplus C))^c)^c=(A^c\oplus(B\oplus C))^c=((A^c\oplus B)\oplus C)^c$$

$$=(A^c\oplus B)^c\ominus C=(A\ominus B)\ominus C.$$ **Q.E.D.**

For Minkowski addition and subtraction, the following inclusions are often used.

Let X, Y, B and C are subsets of real space R^d, we have

$$\text{if } Y\supseteq X,\ \text{then } Y\oplus B\supseteq X\oplus B \ \text{ and } Y\ominus B\supseteq X\ominus B, \tag{2.29}$$

$$\text{if } C\supseteq B,\ \text{then } X\oplus C\supseteq X\oplus B \ \text{ and } X\ominus C\supseteq X\ominus B. \tag{2.30}$$

Eqs.(2.29)(2.30) follow from the definitions of Minkowski addition and subtraction directly. In general, Minkowski subtraction is not an inverse of Minkowski addition, but instead it satisfies:

Lemma 2.2: $(A\oplus B^s)\ominus B \supseteq A \supseteq (A\ominus B^s)\oplus B$. (2.31)

□

Eq.(2.31), in fact, is an anti-extensive property for opening and extensive property for closing. The operations of opening and closing in the image algebra satisfy not only anti-extensive and extensive property but also increasing and idempotence properties. i.e.

anti-extensive and extensive: $A^B \supseteq A \supseteq A_B$, (2.32)

increasing: if $C\supset A \Rightarrow C_B \supset A_B$ and $C^B \supset A^B$, (2.33)

idempotence: $(A_B)_B = A_B$ and $(A^B)^B = A^B$. (2.34)

Proof: the increasing property can be derived from the definitions of closing and opening directly. For idempotence, we have

$$A_B \supseteq (A_B)_B$$

since the opening is anti-extensive . On the other hand,

$$(A_B)_B = \{[(A\ominus B^s)\oplus B]\ominus B^s\}\oplus B.$$

The expression between braces { } is just $(A\ominus B^s)^{B^s}$ which includes $(A\ominus B^s)$, therefore,

$$(A_B)_B \supseteq (A\ominus B^s)\oplus B = A_B.$$

Thus the idempotence for opening is established. In the same way, the idempotence for closing can be proved. **Q.E.D.**

Also, other properties related to opening and closing operations are presented here.

$$(A\oplus B)_B = A\oplus B, \tag{2.35}$$

$$(A\ominus B)^B = A\ominus B, \tag{2.36}$$

$$A_B\ominus B^s = A\ominus B^s, \tag{2.37}$$

$$A^B\oplus B^s = A\oplus B^s . \tag{2.38}$$

Again, these properties are the applications of Lemma 2.2. We analyze Eq.(2.37) as an example.

Proof: By Lemma 2.2

$$A_B \ominus B^s = [(A \ominus B^s) \oplus B] \ominus B^s \supseteq (A \ominus B^s)$$

and $$A \supseteq A_B.$$

Therefore, from Eq.(2.29)

$$(A \ominus B^s) \supseteq A_B \ominus B^s,$$

which leads to Eq.(2.37). **Q.E.D.**

This chapter summarized some of the formulas for Minkowski addition, subtraction and the basic mathematical morphological operations. Most of above results can be found in bibliography [1][2][4][25][26]. All the equalities and inclusions constitute the foundation for the research of morphological filters. They can be directly applied to show the properties of morphological filters on a function.

CHAPTER 3. THE RELATION BETWEEN UMBRA OF FUNCTION AND SET

Traditional signal processing deals with the operations on functions. In order to apply the concepts of mathematical morphology, it is necessary to transform a function to a set. This is achieved here by introducing the concept of the umbra of a function. The morphological operations (such as dilation, erosion, closing and opening) can be applied to the umbra, and the processed umbra is used to reconstruct a new function called filtered signal.

3.1 Definition of umbra and example

The concept of the umbra of a function was first proposed by Sternberg [2] [5]. The umbra U(f) of a function f is a subset of $R^d \times R$ which consists of all points that occupy the space below the function to minus infinity (see Figs.(3.1)(3.3)). The formal definition for the umbra of a function f is given as follows:

Definition 3.1: [19] The umbra U(f) of a function f(x) is a set :

$$U(f(x))=\{(x,t)\in R^d \times R:\ t\leq f(x)\}. \tag{3.1}$$

□

Definition 3.1 establishes the link between functions and sets and allows all morphological operations on a set to be applied to the umbra of a function. The correspondence between the set and the umbra of a function is one-to-one [19]. Similarly, a filtered function can be obtained from its umbra by only keeping the outline of the umbra. i.e.

$$f(x)=\operatorname*{Sup}_{t\in R} U\{f(x)\}. \tag{3.2}$$

The four basic morphological operations can also be applied to process the umbra of an input function as a special case of set operations. Most general case of morphological transform of a function is the transform of U(f) by a structuring function g(x) that is a subset of $R^d \times R$. Thus, Minkowski addition $f\oplus g$ and subtraction $f\ominus g$ can be defined as follows [19].

$$U(f)\oplus g=U(f\oplus g), \tag{3.3}$$

$$U(f)\ominus g=U(f\ominus g). \tag{3.4}$$

Also, we define

$$U(f)^g=(U(f)\oplus g^s)\ominus g)=U((f\oplus g^s)\ominus g)=U(f^g), \tag{3.5}$$

$$U(f)_g=(U(f)\ominus g^s)\oplus g)=U((f\ominus g^s)\oplus g)=U(f_g). \tag{3.6}$$

The following example illustrates $U(f\ominus g)$ in 1-D case. From Eq.(3.4), $U(f\ominus g)=U(f)\ominus g$, then

Example:

1. Given a discrete function f(x) and a structuring function g(x) as in Figs.(3.1)(3.2), where

$$f=\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}4\\2\end{bmatrix}, \begin{bmatrix}5\\4\end{bmatrix}, \begin{bmatrix}6\\0\end{bmatrix}\},$$

$$g=\{\begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\}=\{g(-1),\ g(0),\ g(1)\}.$$

2. The shadowed part shown in Fig.(3.3) is the umbra U(f) of the function f, which can be expressed by a group vectors as follows:

$$U(f)=\{ \begin{bmatrix}0 & \\ 1 & -\infty\end{bmatrix}, \begin{bmatrix}1 & \\ 2 & -\infty\end{bmatrix}, \begin{bmatrix}2 & \\ 3 & -\infty\end{bmatrix}, \begin{bmatrix}3 & \\ 4 & -\infty\end{bmatrix}, \begin{bmatrix}4 & \\ 2 & -\infty\end{bmatrix}, \begin{bmatrix}5 & \\ 4 & -\infty\end{bmatrix}, \begin{bmatrix}6 & \\ 0 & -\infty\end{bmatrix}\}$$

$$=\{ U_0, U_1, U_2, U_3, U_4, U_5, U_6 \},$$

where

$$U_0=\begin{bmatrix}0 & \\ 1 & -\infty\end{bmatrix} = \{ \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}, \dots\dots, \begin{bmatrix}0\\-\infty\end{bmatrix} \}$$

and other elements in U(f) have the same form.

3. According to the definition of $U(f)\ominus g$ (see Eq.(2.17)), we have

$$U(f)\ominus g=U(f)_{g(-1)}\cap U(f)_{g(0)}\cap U(f)_{g(1)}.$$

$U(f)_{g(-1)}$, $U(f)_{g(0)}$ and $U(f)_{g(1)}$ are equivalent to moving the origin of U(f) to g(-1), g(0) and g(1) respectively (see Figs.(3.4)(3.5)(3.6)). Thus, their intersection constructs $U(f)\ominus g$ (see Fig.(3.7)). From the definition of a filtered function (see Eq.(3.2)), a new function $f\ominus g$ is obtained. The filtered function, according to Eq.(3.2), is

$$f\ominus g=\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}4\\1\end{bmatrix}, \begin{bmatrix}5\\0\end{bmatrix} \},$$

which is shown in Fig.(3.8).

3.2 Morphological operations on function

In the same way, $f\oplus g$ can also be obtained by replacing the intersection operation by the union operation. In fact, $f\oplus g$ (or $f\ominus g$) can be expressed using Sup (or Inf) operation which is presented in the following conclusion.

Conclusion 3.1[3]: $$(f\oplus g)(x)= \sup_{y\in (B^S)_x} \{f(y)+g(x-y)\}, \qquad (3.7)$$

$$(f\ominus g)(x)=\inf_{y\in(B^S)_x}\{f(y)+g(x-y)\}, \tag{3.8}$$

where B is the domain of structuring function g(z). □

Proof: Given a x, it can be shown by the definitions of umbra and Minkowski addition that

$$(U(f\oplus g)(x))=(U(f)\oplus g)(x)=\{\begin{bmatrix} y \\ f(y) & -\infty \end{bmatrix}+\begin{bmatrix} z \\ g(z) \end{bmatrix}: y+z=x,\ z\in B\}$$

$$=\{\begin{bmatrix} y+z \\ f(y)+g(z) & -\infty \end{bmatrix}: y+z=x,\ z\in B\}=\{\begin{bmatrix} y+z=x \\ U(\mathrm{Sup}(f(y)+g(z))) \end{bmatrix}: z\in B\}$$

$$=U(\sup_{y=x-z\ \&\ z\in B}\{f(y)+g(x-y)\})=U(\sup_{y\in(B^S)_x}\{f(y)+g(x-y)\}).$$

By Eq.(3.3), $U(f)\oplus g=U(f\oplus g)$. Thus,

$$(f\oplus g)(x)=\sup_{y\in(B^S)_x}\{f(y)+g(x-y)\}.$$

In the same way, Minkowski subtraction can be proved by replacing Sup by Inf.

Q.E.D.

Eqs.(3.7)(3.8) establish the relationship between set operations (Minkowski addition and subtraction) and functional operations (Sup and Inf). This is important and is often used in the sequel. The function $(f\ominus g)$ in the example (pp.14,15), which was obtained from the set operation, can also be derived from by Eq.(3.8). Conclusion 3.1 can be applied to Minkowski four operations, i.e. dilation, erosion, opening and closing. In order to be consistent with the concept of symmetrical set (see Chapter II), let $g^s(x)=-g(-x)$, $x\in -B$ denote the symmetrical function of g with respect to the origin and $g_r(x)=-g(x)$, $x\in B$ denote the reflected function of g. We have:

3. Authors of [19] have shown an incomplete form of the conclusion without proof. Author of [1] showed a special case of the conclusion without proof.

Conclusion 3.2:

Erosion of f by g: $$(f \ominus g^s)(x) = \operatorname*{Inf}_{y \in B_x} \{f(y) - g(y-x)\}. \tag{3.9}$$

Dilation of f by g_r: $$(f \oplus (g_r)^s)(x) = \operatorname*{Sup}_{y \in B_x} \{f(y) + g(y-x)\}. \tag{3.10}$$

Opening of f by g: $$(f_g)(x) = [(f \ominus g^s) \oplus g](x)$$

$$= \operatorname*{Sup}_{z \in (B^S)_x} \{ \operatorname*{Inf}_{y \in B_z} \{f(y) - g(y-z)\} + g(x-z)\}. \tag{3.11}$$

Closing of f by g_r: $$(f^{g_r})(x) = [(f \oplus (g_r)^s) \ominus g_r](x)$$

$$= \operatorname*{Inf}_{z \in (B^S)_x} \{ \operatorname*{Sup}_{y \in B_z} \{f(y) + g(y-z)\} - g(x-z)\}. \tag{3.12}$$

□

All the above equalities can be derived from the definitions of dilation, erosion, opening and closing, using Eqs.(3.5)(3.6)(3.7)(3.8). A special case results from g(x)=0, thus yielding g_r=g. The analysis follows Conclusion 3.2 directly. One only needs to replace g(x) by B in Eqs.(3.9)(3.10)(3.11)(3.12). This can be shown as follows:

Erosion of f by B: $$(f \ominus B^s)(x) = \operatorname*{Inf}_{y \in B_x} \{f(y)\}. \tag{3.13}$$

Dilation of f by B: $$(f \oplus B^s)(x) = \operatorname*{Sup}_{y \in B_z} \{f(y)\}. \tag{3.14}$$

Opening of f by B: $$(f_B)(x) = \operatorname*{Sup}_{z \in (B^S)_x} \{ \operatorname*{Inf}_{y \in B_z} \{f(y)\}\}. \tag{3.15}$$

Closing of f by B: $$(f^B)(x) = \operatorname*{Inf}_{z \in (B^S)_x} \{ \operatorname*{Sup}_{y \in B_z} \{f(y)\}\}. \tag{3.16}$$

If a function f(x) and a structuring function g(x) are digitized, Sup and Inf operations can be replaced by Max and Min operations, respectively. Since the functional operations are equivalent to the set operations, all the properties for set operations discussed in Chapter II can be directly applied to the functional operations. An example in terms of set operation is given in the proof of one of the properties (see Property 4.2 in the next chapter).

3.3 Other morphological filters

In general, various filters can be constructed by Minkowski operations, similarly to the erosion, dilation, opening and closing. Combinations of such filters can be employed to build up new filters. Let nB denote n times dilations by B (i.e. $nB=B\oplus B\oplus B\oplus\cdots\oplus B$, n times), then [3],

$$\text{"Low Pass Filter"}=f\oplus nB, \tag{3.17}$$

$$\text{"High Pass Filter"}=f-(f\oplus nB), \tag{3.18}$$

$$\text{"Band Pass Filter"}=(f\oplus nB)-(f\oplus mB) \quad (n<m). \tag{3.19}$$

Furthermore, Maragos [19] proposed two new filters called open-closing and close-opening. The open-closing filter is defined by $(f_g)^g(x)$, i.e., the opening followed by the closing by the same structuring function (or set). The close-opening filter is defined by $(f^g)_g(x)$, i.e., the closing followed by the opening by the same structuring function (or set). The two filters are used for image noise suppression and for providing fixed points of median filter [20].

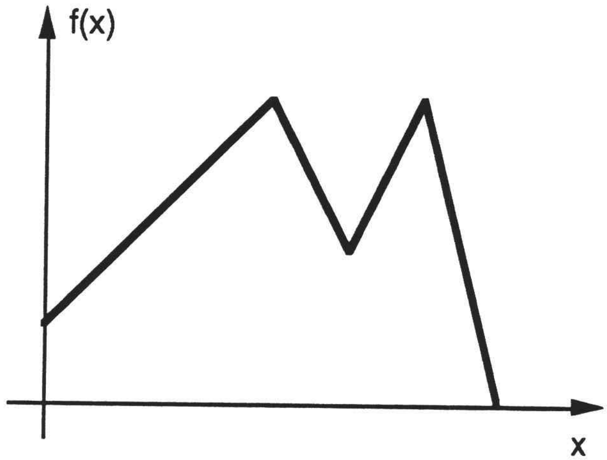

Fig.(3.1)

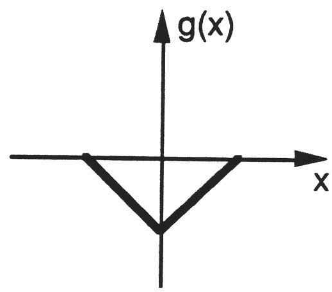

Fig.(3.2)

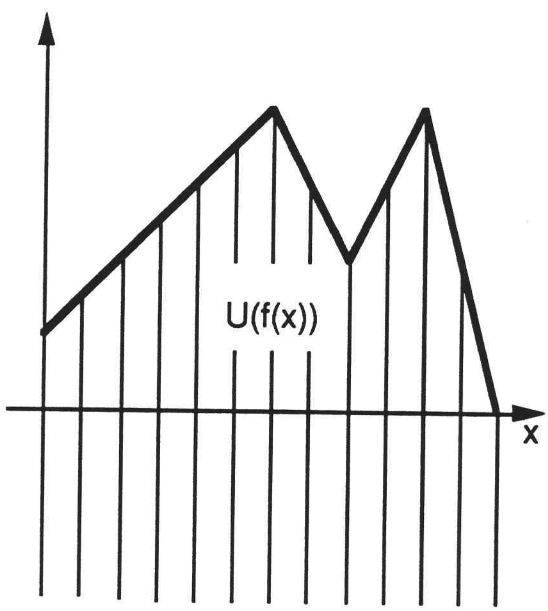

Fig.(3.3)

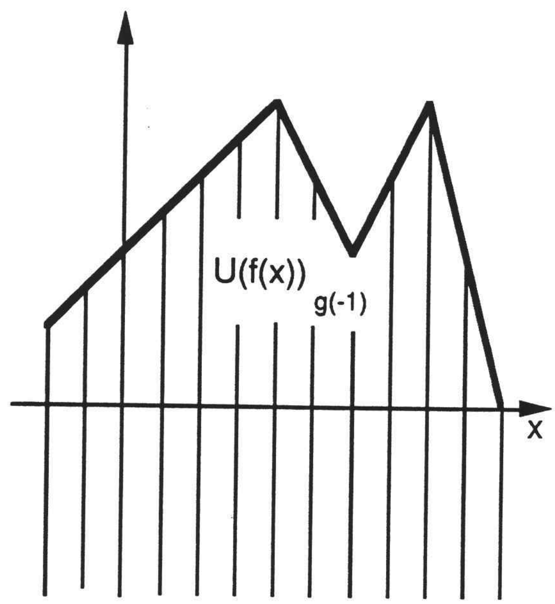

Fig.(3.4)

Fig.(3.1): Function f(x).
Fig.(3.2): Structuring function g(x).
Fig.(3.3): Umbra of function f(x)
Fig.(3.4): Umbra of function f(x) translated by g(-1).

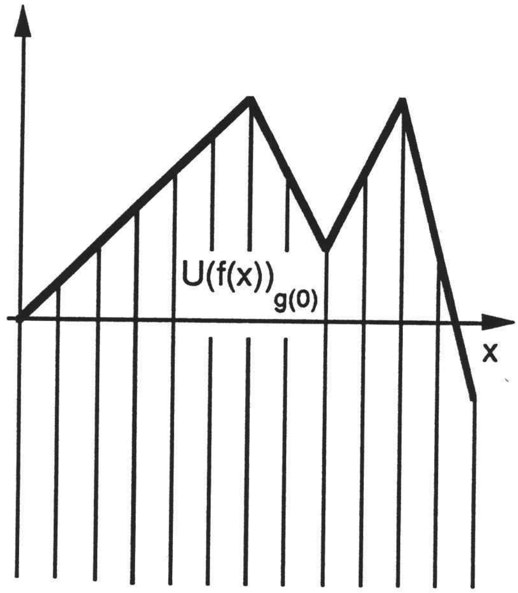

Fig.(3.5)

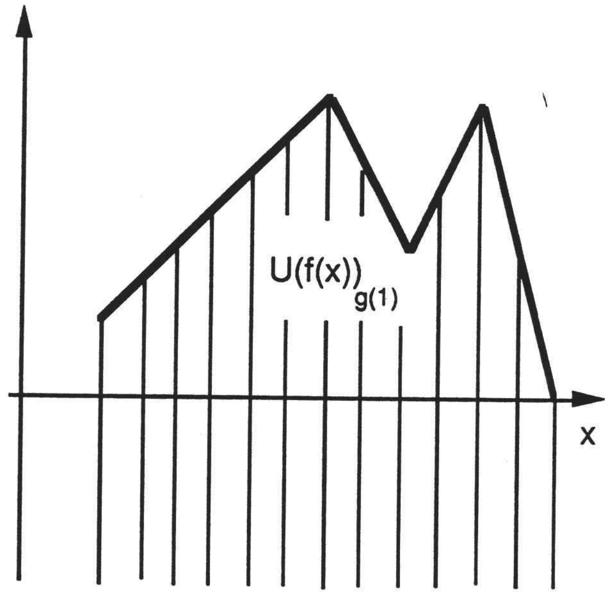

Fig.(3.6)

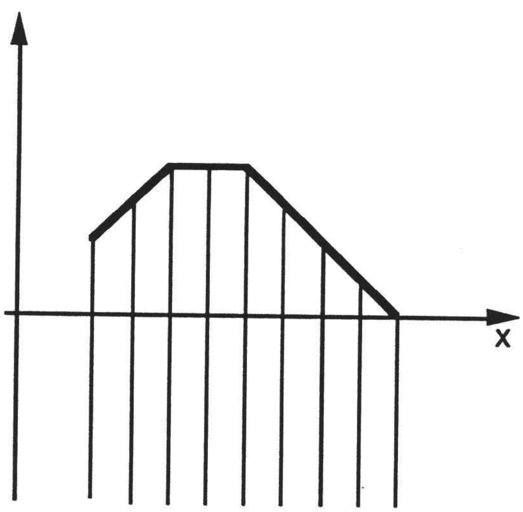

Fig.(3.7)

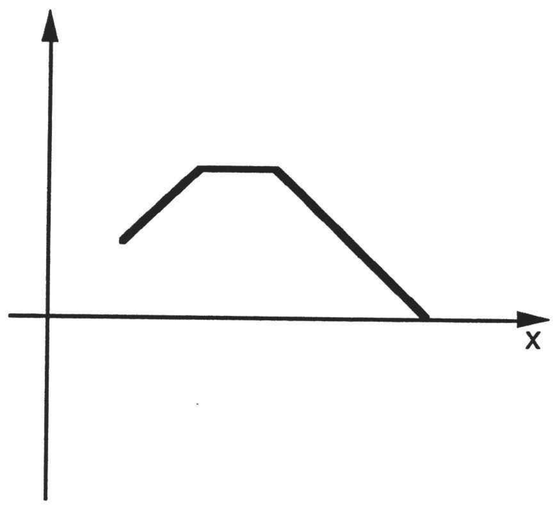

Fig.(3.8)

Fig.(3.5): Umbra of function f(x) translated by g(0).
Fig.(3.6): Umbra of function f(x) translated by g(-1).
Fig.(3.7): Intersection of $U(f)_{g(-1)}$, $U(f)_{g(0)}$ and $U(f)_{g(1)}$.
Fig.(3.8): New function (or filtered signal) $f\ominus g$.

CHAPTER 4. BASIC PROPERTIES OF DETERMINISTIC MORPHOLOGICAL FILTERS

Let D denote a multi-dimensional (m-D) index set, and let R denote a one dimensional real space.Let B (called support region) be a finite m-D subindex set of D, and $\{f(i)\}$ be an input signal sequence such that f: $D \rightarrow R$, $i \in D$. Also let $\{g(j)\}$, g: $B \rightarrow R$, be a structuring function sequence which is used to modify the geometrical characteristics of the input signal sequence $\{f(i)\}$.

4.1 Basic morphological filters on deterministic function sequence

The basic morphological filters (M-Filters), erosion, dilation, opening and closing, on the input signal sequence $\{f(i)\}$ by the structuring function sequences $\{g(j)\}$ and $\{g_r(j)\}$ can be derived as follows from Conclusion 3.2:

Erosion of f by g:

$$(f \ominus g^s)(i) = \underset{j \in B_i}{\mathrm{Min}}\{f(j) - g(j-i)\}. \tag{4.1}$$

Dilation of f by g_r:

$$(f \oplus (g_r)^s)(i) = \underset{j \in B_i}{\mathrm{Max}}\{f(j) + g(j-i)\}. \tag{4.2}$$

Opening of f by g:

$$(f_g)(i) = ((f \ominus g^s) \oplus g)(i) = \underset{k \in (B^S)_i}{\mathrm{Max}}\{\underset{j \in B_k}{\mathrm{Min}}\{f(j) - g(j-k)\} + g(i-k\}. \tag{4.3}$$

Closing of f by g_r:

$$(f^{g_r})(i) = ((f \oplus (g_r)^s \ominus g_r)(i) = \underset{k \in (B^S)_i}{\mathrm{Min}}\{\underset{j \in B_k}{\mathrm{Max}}\{f(j) + g(j-k)\} - g(i-k)\}. \tag{4.4}$$

In the above B^s denotes the symmetrical set of the support region B (B^s=-B), B_i denotes the support region of the structuring function sequence {g(j)} with the origin moved to the point i (B_i={b+i, b∈B}). {g^s} denotes the symmetrical sequence of the structuring function sequence {g} ($g^s(j')=-g(j)$, $j'=-j \in B^s$) and {g_r} denotes the reflected sequence of {g} ($g_r(j)=-g(j)$, $j \in B$). Eqs.(4.1)-(4.4) follow from Conclusion 3.2, by replacing Inf and Sup operations by Min and Max operations, respectively. Dilation, erosion, closing and opening operations on the deterministic sequence {f(i)} by the structuring function sequence {g(j)} are called deterministic M-Filters.

4.2 Properties of basic morphological filters

Comparing Eqs.(4.1)(4.3) with Eqs.(4.2)(4.4), we notice a dual property of the four filters. This means that the results for erosion and opening filters can be obtained by studying dilation and closing filters. This observation is formalized in Property 4.1.

Property 4.1: Dilation and erosion, opening and closing are dual filters in the following sense:

$$(f_r \oplus (g_r)^s)(i)=-(f \ominus g^s)(i), \tag{4.5}$$

$$((f_r)_{g_r})(i)=-(f_g)(i), \tag{4.6}$$

where $\{f_r(i)\}$, $f_r(i)=-f(i)$, denotes the reflected sequence of the input function sequence {f(i)}. □

<u>Proof</u>: From Eqs.(4.1)-(4.4), it can be shown that

$$(f_r \oplus (g_r)^s)(i) = \underset{j \in B_i}{\mathrm{Max}}\{-f(j)+g(j-i)\} = -\underset{j \in B_i}{\mathrm{Min}}\{f(j)-g(j-i)\} = -(f \ominus g^s)(i),$$

which gives Eq.(4.5). Validity of Eq.(4.6) can be shown as follows

$$((f_r)^{g_r})(i) = \underset{k \in (B^S)_i}{\mathrm{Min}} \{\underset{j \in B_k}{\mathrm{Max}}\{-f(j)+g(j-k)\}-g(i-k)\}$$

$$= -\underset{k \in (B^S)_i}{\mathrm{Max}} \{\underset{j \in B_k}{\mathrm{Min}}\{f(j)-g(j-k)\}+g(i-k)\} = -(f_g)(i). \qquad \textbf{Q.E.D.}$$

In order to discuss the bias property of M-Filters, an ordering property is presented below.

Property 4.2 (Ordering): [19] Opening is an antiextensive filter ($f_g \le f$), whereas closing is extensive ($f^{g_r} \ge f$). If $g(0)>0$, then erosion by $\{g\}$ is strictly antiextensive, whereas dilation by $\{g\}$ is strictly extensive. Thus, if $g(0)>0$

$$f \ominus g^s < f_g \le f \le f^{g_r} < f \oplus (g_r)^s. \qquad (4.7)$$

$\square$

Proof: Note that if $U(f_1) \supseteq U(f_2)$, then, $f_1 \ge f_2$.

From Eq.(2.31) in Chapter II and Eqs.(3.5)(3.6), we have

$$U(f^{g_r}) = U(f)^{g_r} \supseteq U(f) \supseteq U(f)_g = U(f_g).$$

Therefore, $\quad f_g \le f \le f^{g_r}$. (4.8)

On the other hand, due to the assumption $g(0)>0$, we have

$$(f_g)(i) = \underset{k \in (B^S)_i}{\mathrm{Max}} \{\underset{j \in B_k}{\mathrm{Min}}\{f(j)-g(j-k)\}+g(i-k)\} \ge \underset{j \in B_i}{\mathrm{Min}}\{f(j)-g(j-i)\}+g(0)$$

$$> \underset{j \in B_i}{\mathrm{Min}}\{f(j)-g(j-i)\} = (f \ominus g^s)(i), \qquad (4.9)$$

and $$(f^{g_r})(i)=\underset{k\in(B^S)_i}{\text{Min}}\{\underset{j\in B_k}{\text{Max}}\{f(j)+g(j-k)\}-g(i-k)\}\leq\underset{j\in B_i}{\text{Max}}\{f(j)+g(j-i)\}-g(0)$$

$$<\underset{j\in B_i}{\text{Max}}\{f(j)+g(j-i)\}=f\oplus(g_r)^s. \tag{4.10}$$

Eqs.(4.8)(4.9)(4.10) lead to Eq.(4.7). **Q.E.D.**

The difference in geometrical effects between opening (or closing) on a input signal sequence $\{f(i)\}$ by a structuring function sequence $\{g(j)\}$, $j\in B$, and by its support region B is quantified analytically as follows:

Property 4.3: [19] If $\{g(j)\}$ is a bounded, real-valued, structuring function sequence with a support region B, then for any input function sequence $\{f(i)\}$ we have:

$$|f_g(i)-f_B(i)|\leq\underset{k\in B}{\text{Max}}\{g(k)\}-\underset{k\in B}{\text{Min}}\{g(k)\}:=d. \tag{4.11}$$

$\square$

Note: Eq.(4.11) is satisfied for any signal sequence $\{f(i)\}$ and the difference "d" is independent of $\{f(i)\}$. Eq.(4.11) can be also modified to the closing operation. From Property 4.1, we have

$$((f_r)^{B_r})(i)=-(f_B)(i),$$

$$((f_r)^{g_r})(i)=-(f_g)(i).$$

Using Eq.(4.11), we have

$$|((f_r)^{g_r})(i)-((f_r)^{B_r})(i)|=|(f_g)(i)-(f_B)(i)|\leq\underset{k\in B}{\text{Max}}\{g(k)\}-\underset{k\in B}{\text{Min}}\{g(k)\}.$$

Since the property is valid for any function and B_r=B, the above inequality can be rewritten as

$$\left|(f^{g}{}_{r})(i)-f^{B}(i)\right| \leq \underset{k\in B}{\mathrm{Max}}\{g(k)\}-\underset{k\in B}{\mathrm{Min}}\{g(k)\}:=d. \tag{4.12}$$

Based on these properties, some stochastic properties for M-Filters are derived in the following chapters.

CHAPTER 5. GENERAL DISCUSSION OF STOCHASTIC PROPERTIES OF M-FILTERS

The dilation, erosion, closing and opening operations on a stochastic process $\{F(i)\}$ by a structuring function sequence $\{g(j)\}$ are called here stochastic M-Filters. This chapter discusses the properties of such stochastic M-Filters. Since the M-Filters are obtained by the mixed operations of minimum and maximum, the probability distribution functions (P.D.F.s) of M-Filters is related to the P.D.F.s of mixed Min and Max operations on random variables. Some mathematical background for this study in this field can be found in Appendices A and in Bibliography [28][30][48][49][50].

5.1 Symmetricity of M-Filters

Let $\{F(i)\}$, $i\in D$ be an m-D stochastic process with a P.D.F.

$$P[\cap(F(i)\leq f(i)):\ i\in D].$$

Let $\{F_r(i)\}$, $\{F_r(i)\}=\{-F(i)\}$, be the reflected process of $\{F(i)\}$. We observe that the joint P.D.F. of the reflected process $\{F_r(i)\}$ is symmetric to that of $\{F(i)\}$, i.e., for all sequences $\{f(i)\}$,

$$P[\cap(F_r(i)\leq f(i)):\ i\in D]=P[\cap(F(i)\geq -f(i)):\ i\in D]. \tag{5.1}$$

Eq.(5.1) implies a dual relationship in the stochastic sense between dilation and erosion or closing and opening filters. The dual relationship is stated in the following lemma.

Lemma 5.1: For any bounded, real-valued, structuring function sequence $\{g(j)\}$, the P.D.F.s of dilation (closing) of $\{F_r(i)\}$ by $\{g_r(j)\}$ are symmetric to the P.D.F.s of erosion (opening) of $\{F(i)\}$ by $\{g(j)\}$. This can be formalized as follows:

$$P[\cap(F_r \oplus (g_r)^s)(i) \leq m(i):\ i \in D] = P[\cap(F \ominus g^s)(i) \geq -m(i):\ i \in D], \qquad (5.2)$$

$$P[\cap((F_r)_{g_r})(i) \leq m(i):\ i \in D] = P[\cap F_g(i) \geq -m(i):\ i \in D]. \qquad (5.3)$$

□

Proof: Applying Property 4.1 to almost every realization of the stochastic processes $\{F(i)\}$ and $\{F_r(i)\}$, we have, with probability one (w.p.1), that

$$(F_r \oplus (g_r)^s)(i) = -(F \ominus g^s)(i).$$

Therefore,

$$P[\cap(F_r \oplus (g_r)^s)(i) \leq m(i):\ i \in D] = P[\cap(-(F \ominus g^s)(i)) \leq m(i):\ i \in D]$$

$$= P[\cap(F \ominus g^s)(i) \geq -m(i):\ i \in D],$$

which gives Eq.(5.2). Likewise, Eq.(5.3) can be proved. **Q.E.D.**

Note: $\{(F_r \oplus (g_r)^s)(i)\}$, $\{(F \ominus g^s)(i)\}$, $\{((F_r)^{g_r})(i)\}$ and $\{F_g(i)\}$ are stochastic fields of the dilation, erosion, closing and opening operations on $\{F(i)\}$ and $\{F_r(i)\}$.

According to Lemma 5.1, the investigation of the stochastic behavior of the four morphological filters can be reduced to that of two filters: the dilation and closing filters or the erosion and opening filters. We concentrate here on the dilation and closing operations, but all the results can be simply extended to the erosion and opening operations. Furthermore, if the stochastic field $\{F(i)\}$ has a symmetrical P.D.F., i.e., if, for all sequences $\{f(i)\}$,

$$P[\cap(F(i)\leq f(i)):\ i\in D]=P[\cap(F(i)\geq -f(i)):\ i\in D], \tag{5.4}$$

then, by Eqs.(5.2)(5.3), we have, for all sequences $\{m(i)\}$

$$P[\cap(F\oplus(g_r)^s)(i)\leq m(i):\ i\in D]=P[\cap(F\ominus g^s)(i)\geq -m(i):\ i\in D], \tag{5.5}$$

$$P[\cap((F)^{g_r})(i)\leq m(i):\ i\in D]=P[\cap F_g(i)\geq -m(i):\ i\in D]. \tag{5.6}$$

In the simple case of $g(j)=0$, $j\in B$, $g_r(j)=-g(j)=0$ and the structuring function sequence is the set B. Therefore,

$$P[\cap(F\oplus B^s)(i)\leq m(i):\ i\in D]=P[\cap(F\ominus B^s)(i)\geq -m(i):\ i\in D], \tag{5.7}$$

$$P[\cap(F^B)(i)\leq m(i):\ i\in D]=P[\cap F_B(i)\geq -m(i):\ i\in D]. \tag{5.8}$$

As an extension of Lemma 5.1, we conclude that the means of the dilation and closing filters are the same as those of the erosion and opening filters in amplitude but opposite in the signs and that the variances are the same for the dilation and erosion filters or the closing and opening filters if the stochastic process $\{F(i)\}$ has a symmetrical P.D.F.. Formally

Conclusion 5.1: If the P.D.F. of stochastic field $\{F(i)\}$ satisfies Eq.(5.4), then, w.p.1, we have

$$(F\oplus(g_r)^s)(i)=-(F\ominus g^s)(i). \tag{5.9}$$

$$F^{g_r}(i)=-F_g(i) \tag{5.10}$$

If the means and variances of dilation, erosion, closing and opening exist, then,

$$E[(F\oplus(g_r)^s)(i)]=-E[(F\ominus g^s)(i)], \tag{5.11}$$

$$E[(F^{g_r})(i)]=-E[F_g(i)], \tag{5.12}$$

$$Var[(F\oplus(g_r)^s)(i)]=Var[(F\ominus g^s)(i)], \tag{5.13}$$

$$Var[(F^{g_r})(i)]=Var[F_g(i)]. \tag{5.14}$$

□

5.2 Scaling

In particular, when the structuring function sequence $\{g(j)\}=0$, $j\in B$, the following lemma can be applied to the "scaled stochastic field" $\{K\cdot F(i)\}$.

Lemma 5.2: Let $M[F(i)]$ be a M-Filter on the stochastic field $\{F(i)\}$, with the structuring function sequence $\{g\}$. Denote

$$E[M[F(i)]]=E(i), \quad Var[M[F(i)]]=\sigma^2(i),$$

and assume that the structuring function sequence satisfies $g(j)=0$ for all $j\in B$. For a "scaled stochastic field" $\{K\cdot F(i)\}$,

$$E[M[K\cdot F(i)]]=K\cdot E(i) \tag{5.15}$$

and $$Var[M[K\cdot F(i)]]=K^2\sigma^2(i), \tag{5.16}$$

where K is a constant and $K\geq 0$. □

Proof: Since $g(j)=0$ for all $j\in B$ and $K\geq 0$, we have

$$(KF\oplus B^s)(i)=\max_{j\in B_i}\{KF(j)\} = K\max_{j\in B_i}\{F(j)\} = K(F\oplus B^s)(i),$$

$$(KF)^B(i)= \underset{k\in(B^S)_i}{\mathrm{Min}} \{\underset{j\in B_k}{\mathrm{Max}}\{KF(j)\}\} = K \underset{k\in(B^S)_i}{\mathrm{Min}} \{\underset{j\in B_k}{\mathrm{Max}}\{F(j)\}\} = K(F^B)(i)$$

w.p.1. Similarly, it follows that

$$(KF\ominus B^s)(i) = K(F\ominus B^s)(i), \qquad (KF)_B(i) = K(F_B)(i)$$

w.p.1.

The above results and the definitions of the mean and variance lead to

$$E[M[K\cdot F(i)]]=E[K[M(F(i))]]=KE[M[F(i)]]=K\cdot E(i),$$

$$Var[M[K\cdot F(i)]]=Var[K[M(F(i))]]=K^2Var[M[F(i)]]=K^2\sigma^2(i). \qquad \textbf{Q.E.D.}$$

This lemma shows that the study of the means and variances of M-Filters on a family of "scaled stochastic fileds" $\{KF(i)\}$ can be reduced to that of the original stochastic field $\{F(i)\}$.

5.3 Separation

The following lemma shows that the analysis of a constant signal c plus a stochastic field $\{N(i)\}$ is equivalent to the analysis of the stochastic field only.

Lemma 5.3: If a stochastic field $\{F(i)\}$, $i\in D$, is a constant c plus a stochastic field $\{N(i)\}$, i.e. $F(i)=c+N(i)$, the M-Filters on such process are equal to the sum of the M-Filters on the constant signal, c, and on the stochastic field, $\{N(i)\}$, separately. i.e.

$$(F\ominus g^s)(i) = c+(N\ominus g^s)(i), \tag{5.17}$$

$$(F\oplus (g_r)^s)(i) = c+(N\oplus (g_r)^s)(i), \tag{5.18}$$

$$(F_g)(i) = c+(N_g)(i), \tag{5.19}$$

$$(F^{g_r})(i) = c+(N^{g_r})(i). \tag{5.20}$$

□

<u>Proof</u>: According to Eqs.(4.1)-(4.4), it can be shown, w.p.1, that

$$(F\oplus(g_r)^s)(i)=\underset{j\in B_i}{Max}\{c+N(j)+g(j-i)\}=c+(N\oplus(g_r)^s)(i)$$

$$(F^{g_r})(i)= \underset{k\in(B^S)_i}{Min}\ \{\underset{j\in B_k}{Max}\{c+N(j)+g(j-k)\}-g(i-k)\}$$

$$= \underset{k\in(B^S)_i}{Min}\ \{c+\underset{j\in B_k}{Max}\{N(j)+g(j-k)\}-g(i-k)\}= c + (N^{g_r})(i).$$

Thus, Eqs.(5.18) and (5.20) are proved. Similarly, Eqs.(5.17) and (5.19) can be proved. **Q.E.D.**

5.4 Upper bounds of variances of closing and opening filters

For a general structuring function sequence, the calculation of P.D.F. of closing and opening is a quite complicated task. Therefore it may be necessary to restrict the analysis to the estimation of the upper bounds of moments. The following lemma gives an upper bound of the variance for the closing operation.

Lemma 5.4: If $\{g(j)\}$ is a bounded, real-valued, structuring function sequence with a support region B, then, for any stochastic field $\{F(i)\}$,

$$Var[F^{g_r}(i)]\leq Var[F^B(i)]+4d^2+4dE|F^B(i)-\bar{F}^B(i)|, \tag{5.21}$$

where $d:=\underset{j\in B}{\mathrm{Max}}\{g(j)\}-\underset{j\in B}{\mathrm{Min}}\{g(j)\}$, and $\bar{F}^B(i)=E[F^B(i)]$. $\square$

<u>Proof</u>: For almost every realization of the stochastic field $\{F(i)\}$, it can be shown by Eq.(4.12) that

$$-d\le f^{g_r}(i)-f^B(i)\le d.$$

Thus $$P(-d\le F^{g_r}(i)-F^B(i)\le d)=1$$

and $-d+F^B(i)-\bar{F}^B(i)+\bar{F}^B(i)-\bar{F}^{g_r}(i)\le F^{g_r}(i)-\bar{F}^{g_r}(i)\le d+F^B(i)-\bar{F}^B(i)+\bar{F}^B(i)-\bar{F}^{g_r}(i)$.

Because $|\bar{F}^B(i)-\bar{F}^{g_r}(i)|\le d$, we have

$$-2d+F^B(i)-\bar{F}^B(i)\le F^{g_r}(i)-\bar{F}^{g_r}(i)\le 2d+F^B(i)-\bar{F}^B(i).$$

The above equation can be rewritten in a simple form by squaring each side

$$[F^{g_r}(i)-\bar{F}^{g_r}(i)]^2\le \mathrm{Max}\{(F^B(i)-\bar{F}^B(i)-2d)^2,\ (F^B(i)-\bar{F}^B(i)+2d)^2\}$$

$$=(F^B(i)-\bar{F}^B(i))^2+4d^2+4d|F^B(i)-\bar{F}^B(i)|.$$

That is $E[F^{g_r}(i)-\bar{F}^{g_r}(i)]^2\le E[(F^B(i)-\bar{F}^B(i))^2]+4d^2+4dE|F^B(i)-\bar{F}^B(i)|$,

which yields Eq.(5.21). **Q.E.D.**

Lemma 5.4 shows that the variance of the closing filter by $\{g(j)\}$ depends on $Var[F_B(i)]$ and the maximum "span" d of the structuring function sequence $\{g(j)\}$. In order to reduce the variance of the output process of a closing filter, a "flat" (i.e. small d) structuring function sequence $\{g(j)\}$ should be considered. On the other hand, if the variance of $F^B(i)$ is known, the bound of the variance for $F^{g_r}(i)$ is given by Eq.(5.21). This discussion also applies to the opening filter because of the duality between closing and opening filters (see Eqs.(5.2)(5.3)).

5.5 Invariances

In general, even if the P.D.F. of $\{F(i)\}$ is i.i.d., the P.D.F.s of stochastic field of dilation, erosion, closing and opening operations on $\{F(i)\}$ are not an i.i.d.. However the resulting 1-D P.D.F.s are independent of i. This statement can be formalized in the following lemma.

Lemma 5.5: The 1-D P.D.F.s of stochastic field of dilation, erosion, closing and opening operations on an i.i.d. stochastic field $\{F(i)\}$ satisfy

$$P[(F\oplus(g_r)^s)(i)\leq m]=P[(F\oplus(g_r)^s)(h)\leq m], \tag{5.22}$$

$$P[(F\ominus g^s)(i)\leq m]=P[(F\ominus g^s)(h)\leq m], \tag{5.23}$$

$$P[(F^{g_r})(i)\leq m]=P[(F^{g_r})(h)\leq m], \tag{5.24}$$

$$P[F_g(i)\leq m]= P[F_g(h)\leq m] \tag{5.25}$$

for all i, $h\in D$. □

<u>Proof</u>: Let $B=\{b_1, b_2,\ldots,b_N\}$. For dilation filter, we have

$$(F\oplus(g_r)^s)(i)=\underset{j\in B_i}{\mathrm{Max}}\{F(j)+g(j-i)\}$$

$$=\mathrm{Max}\{F(b_1+i)+g(b_1),\ldots\ldots,F(b_N+i)+g(b_N)\}, \tag{5.26}$$

Similarly, $(F\oplus(g_r)^s)(h)=\mathrm{Max}\{F(b_1+h)+g(b_1),\ldots\ldots,F(b_N+h)+g(b_N)\}$.

The i.i.d. assumption about $\{F(i)\}$ implies Eq.(5.22). Likewise, for closing filter, we have

$$(F^{g_r})(i)=\underset{k\in(B^S)_i}{\mathrm{Min}}\ \{\underset{j\in B_k}{\mathrm{Max}}\{F(j)+g(j-k)\}-g(i-k)\}$$

$$=\mathrm{Min}\{\underset{j\in B_{i-b_1}}{\mathrm{Max}}\{F(j)+g(j-i+b_1)\}-g(b_1),\ldots\ldots,\underset{j\in B_{i-b_N}}{\mathrm{Max}}\{F(j)+g(j-i+b_N)\}-g(b_N)\}$$

$$=\mathrm{Min}\{\mathrm{Max}\{F(i)+g(b_1),\ldots,F(i-b_1+b_N)+g(b_N)\}-g(b_1),\ldots\ldots,$$

$$\mathrm{Max}\{F(i-b_N+b_1)+g(b_1),\ldots,F(i)+g(b_N)\}-g(b_N)\}, \tag{5.27}$$

$$(F^{g_r})(h)=\mathrm{Min}\{\mathrm{Max}\{F(h)+g(b_1),\ldots,F(h-b_1+b_N)+g(b_N)\}-g(b_1),\ldots\ldots,$$

$$\mathrm{Max}\{F(h-b_N+b_1)+g(b_1),\ldots,F(h)+g(b_N)\}-g(b_N)\}.$$

Therefore, $P[(F^{g_r})(i)\leq m]=P[(F^{g_r})(h)\leq m]$ using again the i.i.d. assumption. The invariance of P.D.F.s for erosion and opening filters can be proved in the same way. **Q.E.D.**

In the case of two structuring function sequences, $g_1(j_1)=0$, $j_1\in B_1$ and $g_2(j_2)=0$, $j_2\in B_2$, the following lemma presents invariance of 1-D P.D.F.s for M-Filters on two i.i.d. stochastic fields $\{F_1(i_1)\}$ and $\{F_2(i_2)\}$.

Lemma 5.6: The 1-D P.D.F.s of stochastic field of dilation, erosion, closing and opening operations on an i.i.d. stochastic field $\{F_1(i_1)\}$ by B_1 are identical with those of dilation, erosion, closing and opening operations on $\{F_2(i_2)\}$ by B_2, provided that the set B_1 is one to one correspondence to the set B_2, and 1-D P.D.F. of $\{F_1(i_1)\}$ is identical to that of $\{F_2(i_2)\}$, i.e., for all m, we have

$$P[F_1(i_1)\leq m]=P[F_2(i_2)\leq m]. \tag{5.28}$$

Formally,

$$P[(F_1\oplus(B_1)^s)\leq m]=P[(F_2\oplus(B_2)^s)\leq m], \tag{5.29}$$

$$P[(F_1\ominus(B_1)^s)\leq m]=P[(F_2\ominus(B_2)^s)\leq m], \tag{5.30}$$

$$P[(F_1)^{B_1}\leq m]=P[(F_2)^{B_2}\leq m], \tag{5.31}$$

$$P[(F_1)_{B_1}\leq m]= P[(F_2)_{B_2}\leq m]. \tag{5.32}$$

□

<u>Proof</u>: Due to the one to one correspondence between the sets B_1 and B_2, we can write $B_1=\{b_{11}, b_{12},\ldots,b_{1N}\}$ and $B_2=\{b_{21}, b_{22},\ldots,b_{2N}\}$. From Lemma 5.5, we have

$$P[(F_1\oplus(B_1)^s)\leq m]=P[(F_1\oplus(B_1)^s)(i_1)\leq m],$$

and

$$P[(F_1)^{B_1}\leq m]=P[(F_1)^{B_1}(i_1)\leq m].$$

Considering the assumptions $g_1(j_1)=0$ and $g_2(j_2)=0$, we have by Eqs.(5.26)(5.27),

$$(F_1\oplus(B_1)^s)(i_1)=\mathrm{Max}\{F_1(b_{11}+i_1),\ldots\ldots,F_1(b_{1N}+i_1),\}$$

$$(F_2\oplus(B_2)^s)(i_2)=\mathrm{Max}\{F_2(b_{21}+i_2),\ldots\ldots,F_2(b_{2N}+i_2)$$

$(F_1)^{B_1}(i_1)=$

$$\mathrm{Min}\{\mathrm{Max}\{F_1(i_1),\ldots,F_1(i_1-b_{11}+b_{1N})\},\ldots\ldots,\mathrm{Max}\{F_1(i_1-b_{1N}+b_{11}),\ldots,F_1(i_1).\}$$

$(F_2)^{B_2}(i_2)=$

$$\mathrm{Min}\{\mathrm{Max}\{F_2(i_2),\ldots,F_2(i_2-b_{21}+b_{2N})\},\ldots\ldots,\mathrm{Max}\{F_2(i_2-b_{2N}+b_{21}),\ldots,F_2(i_2).\}$$

The i.i.d. assumptions of $\{F_1(i_1)\}$ and $\{F_2(i_2)\}$ and Eq.(5.27) lead to Eqs.(5.29)(5.31). Similarly, Eqs.(5.30)(5.32) can be proved. **Q.E.D.**

Note: There is no constraint that the dimensions of i_1 and i_2 are the same. This implies the discussion for P.D.F. of a high dimensional stochastic field can be simplified by studying the P.D.F. of a stochastic process.

5.6 Summary and example

All of the results in this chapter represent tools to analyze the characteristics of a stochastic field filtered by M-Filters. As an example of applying the above tools, we consider the following example. Assume that the domain D of an input field {F(i)} can be partitioned into several parts such that the input signal in each local domain is considered as a constant plus a stochastic field (e.g. noise). Assume also that the size of each local domain compared with domain B of structuring function sequence {g(j)} is large, using Lemma 5.3, we can discuss the stochastic properties of each local domain. If the noise is an i.i.d. process, Lemma 5.5 shows the invariance of the P.D.F. of the filtered signal at every pixel in each local domain. Lemma 5.6 simplifies the discussion in m-D case to that of 1-D case. Considering only the M-Filters with the domain B of the structuring function sequence {g(j)}, we can apply Lemmas 5.2 and 5.4. If the stochastic field is scaled by a constant, Lemma 5.2 relates the mean and variance of the original stochastic field to those of the "scaled" one. Finally Lemma 5.4 gives the bounds for variance of the M-Filter output. Such analysis can be applied "locally" to an input signal and can be used in the M-Filter design.

CHAPTER 6. BIASED AND UNBIASED FILTERS

Biased and unbiased filters are often of interest in the discussion of filter characteristics. Here we extend this notion to the M-Filters.

6.1 Biased filters

For an unbiased filter, the mean of filtered signal is the same as that of the original input signal at each moment. Filters which do not satisfy this property are called biased filters. The bias or unbias, which is an intrinsic property for scaled linear filters, depends on the class of the input signal sequences for nonlinear filters. Thus, a nonlinear filter may be unbiased for some input signal but biased for others. The following lemma provides some useful inequalities.

Lemma 6.1: For any stochastic field $\{F(i)\}$ and any structuring function sequence $\{g(j)\}$, $j \in B$, the closing and opening filters satisfy, w.p.1,

$$F_g(i) \le F(i) \le (F^{g_r})(i). \tag{6.1}$$

If $g(0)>0$, then the dilation and erosion satisfy, w.p.1,

$$(F \ominus g^s)(i) < F_g(i) \le F(i) \le F^{g_r}(i) < F \oplus (g_r)^s(i). \tag{6.2}$$

In particular, we have

$$E[(F \ominus g^s)(i)] < E[F_g(i)] \le [F(i)] \le E[F^{g_r}(i)] < E[F \oplus (g_r)^s(i)]. \tag{6.3}$$

□

<u>Proof</u>: By the definitions of dilation, erosion, opening and closing (see Eqs.(4.1)-(4.4)) and Property 4.2, any sample path sequence $\{f(i)\}$ of $\{F(i)\}$ at each i satisfies

$$f_g(i) \leq f(i) \leq f^{g_r}(i),$$

yielding Eq.(6.1). This leads to

$$E[F_g(i)] \leq E[F(i)] \leq E[F^{g_r}(i)].$$

If $g(0)>0$, then any sample path sequence $\{f(i)\}$ at each i satisfies

$$(f \ominus g^s)(i) < f_g(i) \leq f(i) \leq f^{g_r}(i) < (f \oplus (g_r)^s)(i),$$

which proves Eqs.(6.2)(6.3). **Q.E.D.**

Lemma 6.1 implies that dilation and erosion filters are in general biased filters. For closing and opening filters, we have the following conjecture:

Conjecture 6.1: If structuring function sequence $\{g(j)\} \neq \{g^s(j)\}$, $j \in B$, contains at least two distinct elements, the closing and opening filters are biased filters, i.e.,

$$E[F_g(i)] < E[F(i)] < E[F^{g_r}(i)]. \tag{6.4}$$

□

Lemma 6.1 and Conjecture 6.1 indicate the output sequences of the filters deviate from the input sequences in their expectations. Because the stochastic field $\{F(i)\}$ may not be i.i.d., the deviation is different at different points even if $\{F(i)\}$ has a symmetrical P.D.F. with respect to its expectation $E[F(i)]=c$, i.e.,

$$P[\cap((F(i)-c)\leq f(i)):\ i\in D]=P[\cap((F(i)-c)\geq -f(i)):\ i\in D]. \tag{6.5}$$

This means that it is difficult to cancel the deviation by translating the signals by a constant. To solve this problem, two new filters are proposed in the following section.

6.2 ADE and ACO filters and unbias

ADE (average dilation erosion) and ACO (Average closing opening) filters are defined by

$$ADE[F(i)]=\frac{(F\oplus(g_r)^s)(i)+(F\ominus g^s)(i)}{2}, \tag{6.6}$$

$$ACO[F(i)]=\frac{(F^{g_r})(i)+F_g(i)}{2}. \tag{6.7}$$

The sample paths of ADE and ACO filters are shown in Figs.(6.1)(6.2) in 1-D case. The input signal is an i.i.d. stochastic field with symmetric, uniform distribution. All filters, dilation, ADE, erosion, closing, ACO and opening, operate on the stochastic signal by the structuring function sequence $\{g(j)\}$, $j\in B=\{-2,-1,0,1,2\}$. We notice that the results of ADE and ACO filters are close to the mean of the signal. The following lemma shows that with some constraints on the input stochastic process, these two filters are unbiased.

Lemma 6.2: The ADE and ACO filters are unbiased if the input stochastic field $\{F(i)\}$ has symmetrical P.D.F. with respect to its constant expectation c (see Eq.(6.5)). In this case

$$E[ADE(F(i))]=E[\frac{(F\oplus(g_r)^s)(i)+(F\ominus g^s)(i)}{2}]=E[F(i)]=c, \tag{6.8}$$

and $$E[ACO(F(i))]=E[\frac{(F^{g_r})(i)+F_g(i)}{2}]=E[F(i)]=c. \tag{6.9}$$

$\square$

<u>Proof</u>: Let N(i)=F(i)-c, thus N(i) has a symmetrical P.D.F. by Eq.(6.5). This implies, by Conclusion 5.1, that

$$E[(N\oplus(g_r)^s)(i)]=-E[(N\ominus g^s)(i)].$$

In addition, from Lemma 5.3, we have

$$E[(F\oplus(g_r)^s)(i)]=c+E[(N\oplus(g_r)^s)(i)]=c-E[(N\ominus g^s)(i)],$$

$$E[(F\ominus g^s)(i)]=c+E[(N\ominus g^s)(i)],$$

which shows that $E[ADE(F(i))]=E[c]=c$ for all i.

Similarly, Eq.(6.8) can be proved. **Q.E.D.**

Remark:

Since ADE and ACO filters do not place any constraints on the structuring function sequence {g(j)}, the variation of the structuring function sequence {g(j)} can not change the unbias property of the filters. In addition, the median filter is unbiased under assumption(6.5) and for an i.i.d. input stochastic field [38][42]. The ADE and ACO filters are unbiased under assumption of Eq.(6.5) only.

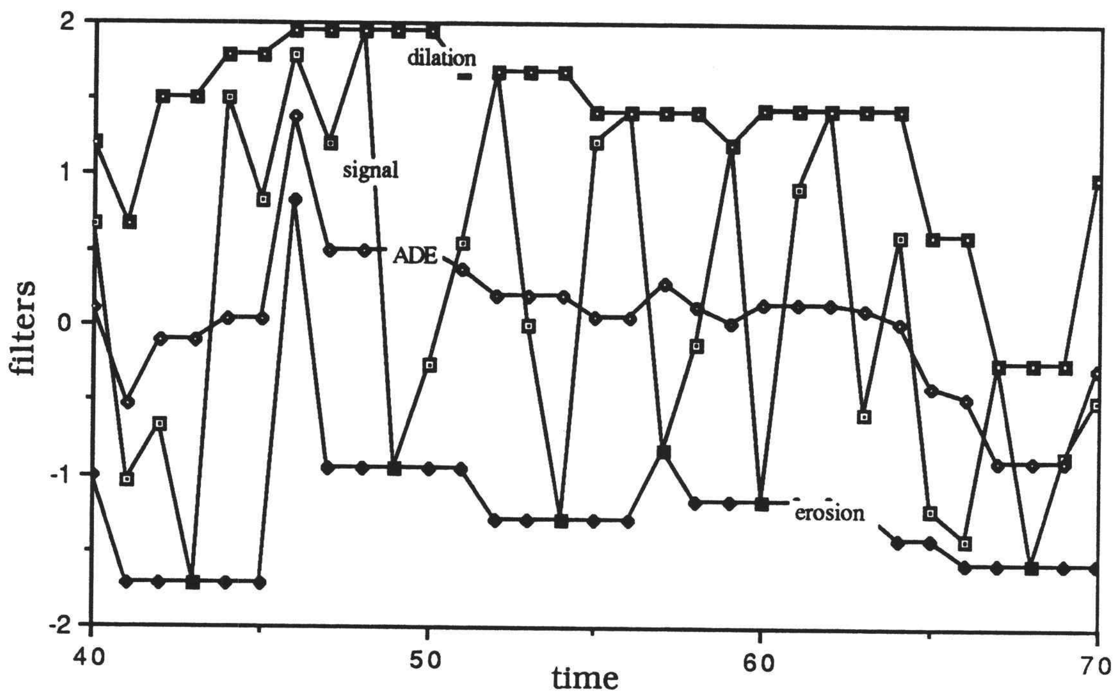

Fig.(6.1): Comparison between dilation, erosion and ADE filters on an i.i.d. stochastic process (signal) with symmetric uniform distribution.

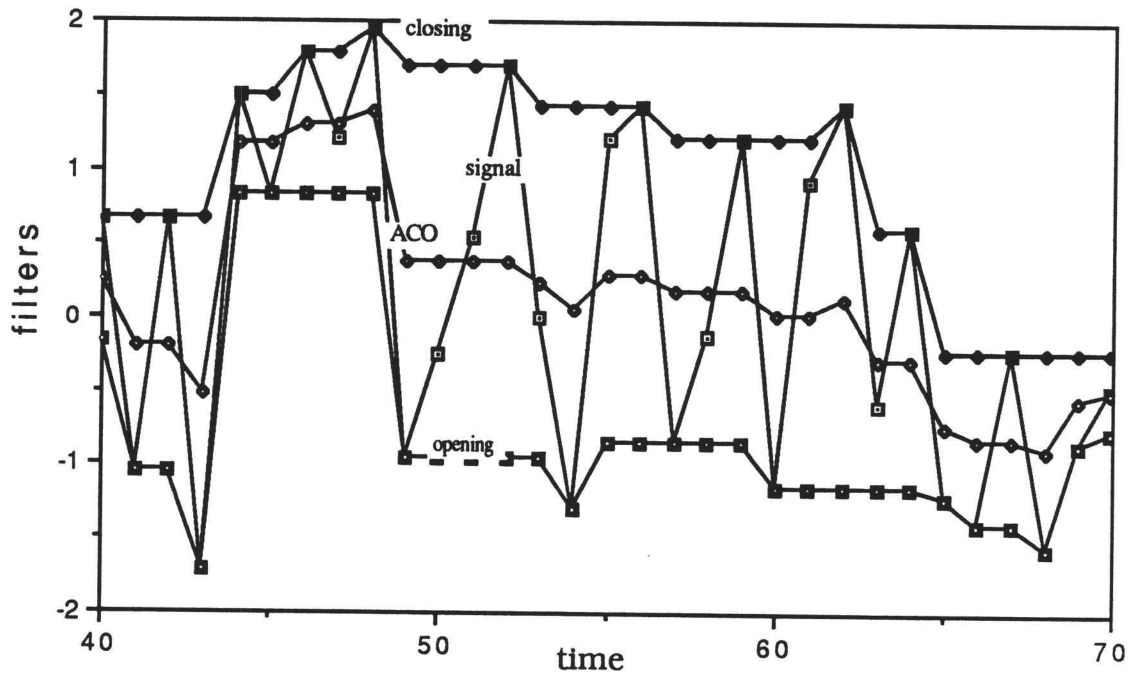

Fig.(6.2): Comparison between closing, opening and ADE filters on an i.i.d. stochastic process (signal) with symmetric uniform distribution.

CHAPTER 7. THE PROBABILITY DISTRIBUTION FUNCTIONS, MEANS AND VARIANCES OF M-FILTERS

Consider a simple structuring function process $\{g(j)\}$, $g(j)=0$, where $j\in B$, and $N\geq 1$ is the number of elements of the index set B.

7.1 P.D.F.s of dilation and erosion filters

Theorem 7.1: If a stochastic field $\{F(i)\}$ is i.i.d., the P.D.F.'s for dilation and erosion filters do not depend on i and

$$P[(F\oplus B^s)\leq m]=P^N[F\leq m], \tag{7.1}$$

$$P[(F\ominus B^s)\leq m]=1-P^N[F>m]. \tag{7.2}$$

If additionally the P.D.F. of F(i) is continuous, then the probability density functions (p.d.f.) of dilation $p_d(m)$ and erosion $p_e(m)$ are given by

$$p_d(m)=Np[m]P^{N-1}[F\leq m], \tag{7.3}$$

$$p_e(m)=Np[m]P^{N-1}[F>m], \tag{7.4}$$

where, p[m] is the p.d.f. of random variable F(i). □

Proof: In 1-D case, since $g(j)=0$, $j\in B$, then, for dilation filter, we have

$$P[(F\oplus B^s)\leq m]=P[\underset{j\in B_i}{\mathrm{Max}}\{F(j)\}\leq m]=P^N[F\leq m].$$

The last equality of the above equation is due to the i.i.d. property of the stochastic field $\{F(i)\}$. From Lemma 5.1 we have for erosion filter that

$$P[(F\ominus B^s)\leq m]=P[(-F\oplus B^s)\geq -m]=1-P[(-F\oplus B^s)<-m]$$
$$=1-P^N[-F<-m]=1-P^N[F>m].$$

By taking derivative of Eqs.(7.1)(7.2) with respect to m, we obtain

Eqs.(7.3)(7.4), respectively. These results can be extended to m-D case by applying Lemma 5.6. **Q.E.D.**

As a consequence of Theorem 7.1, we have the following corollary.

Corollary 7.1: If an i.i.d. stochastic field {F(i)} has a symmetric uniform distribution between [-a a], the p.d.f., mean and variance of the dilation filter are given by

$$p_d[m]=\frac{N}{2a}\left(\frac{m+a}{2a}\right)^{N-1}, \tag{7.5}$$

$$E[(F\oplus B^s)]=a\cdot\frac{N-1}{N+1}, \tag{7.6}$$

$$Var[(F\oplus B^s)]=\frac{(2a)^2N}{(N+1)^2(N+2)}. \tag{7.7}$$

$\square$

Proof: For the i.i.d. stochastic field with uniform distribution between [-a a], the p.d.f. of dilation on the stochastic field can be obtained by substituting p[m]=1/2a into Eq.(7.3), i.e.

$$p_d[m]=\frac{N}{2a}\left(\frac{m+a}{2a}\right)^{N-1},$$

which is Eq.(7.5).

By the definitions of the expectation and variance, we have

$$E[(F\oplus(B)^s)]=\int_{-a}^{+a}mp_d[m]dm=a\cdot\frac{N-1}{N+1},$$

and $Var[(F\oplus B^s)]=E[(F\oplus B^s)^2-E^2[(F\oplus B^s)]$

$$=\int_{-a}^{+a}m^2p_d[m]dm-E^2[(F\oplus B^s)]=\frac{(2a)^2N}{(N+1)^2(N+2)}.$$ **Q.E.D.**

The p.d.f., mean and variance of erosion filter can be obtained by applying Conclusion 5.1 to Corollary 7.1, and using the symmetricity of uniform distribution.

7.2 P.D.F.s of closing and opening filters

Before discussing the p.d.f. of closing filter, a lemma concerning the 1-D stochastic process of closing is presented.

Lemma 7.1: Let $B=\{i,i+1,i+2,\ldots,i+N-1\}$, $B'=B_{-i}=\{0,1,2,\ldots,N-1\}$, and let $\{Y(k)\}$, $\{Z(i)\}$, $\{Y'(k)\}$ and $\{Z'(i)\}$ be stochastic processes defined as follows

$$Y(k)=\underset{j\in B_k}{Max}\{F(j)\}, \qquad Z(i)=\underset{k\in(B^S)_i}{Min}\{Y(k)\}, \tag{7.8}$$

$$Y'(k)=\underset{j\in B'_k}{Max}\{F(i)\}, \qquad Z'(i)=\underset{k\in B'_i}{Min}\{Y'(k)\}, \tag{7.9}$$

then the relationship between P.D.F.'s of the two stochastic processes $\{Z(i)\}$ and $\{Z'(i)\}$ is given by

$$P[F^B(i+N-1)\le m]=P[Z(i+N-1)\le m]=P[Z'(i)\le m]. \tag{7.10}$$

□

Proof: From Eqs.(7.8)(7.9), the P.D.F. of Z'(i) is

$$P[Z'(i)\le m]=P[\underset{k\in B'_i}{Min}\{Y'(k)\}\le m]=1-P[Y'(i)>m,\cdots,Y'(i+N-1)>m]$$

$$=1-P[Max\{F(i),\cdots,F(i+N-1)\}>m,\cdots,Max\{F(i+N-1),\cdots,F(i+2N-2)\}>m]. \tag{7.11}$$

Also, by the definition of closing, we have $F^B(i+N-1)=Z(i+N-1)$ and

$$P[Z(i+N-1)\le m]=P[\underset{k\in(B^S)_{i+N-1}}{Min}\{Y(k)\}\le m]=1-P[Y(0)>m,\cdots,Y(N-1)>m]$$

$$=1-P[\mathrm{Max}\{F(i),\cdots,F(i+N-1)\}>m,\cdots,\mathrm{Max}\{F(i+N-1),\cdots,F(i+2N-2)\}>m]. \tag{7.12}$$

Comparing Eq.(7.11) with Eq.(7.12) proves Lemma 7.1. **Q.E.D.**

Furthermore if a stochastic process is i.i.d., using Lemma 5.5, Lemma 7.1 can be extended to

$$P[F^B(i)\leq m]=P[Z(i)\leq m]=P[Z'(i)\leq m]. \tag{7.13}$$

With this lemma, the p.d.f. of closing on an i.i.d. stochastic process $\{F(i)\}$ can be described by the following theorem.

Theorem 7.2: If a stochastic process $\{F(i)\}$ is i.i.d., the P.D.F.'s $P[F^B(i)\leq m]$ and $P[F_B(i)\leq m]$ of the stochastic processes $\{F^B(i)\}$ and $\{F_B(i)\}$ do not depend on i and

$$P[F^B\leq m]=NP^N[F\leq m]-(N-1)P^{N+1}[F\leq m], \tag{7.14}$$

$$P[F_B\leq m]=1-NP^N[F>m]+(N-1)P^{N+1}[F>m]. \tag{7.15}$$

If additionally the P.D.F. of F(i) is continuous, then the p.d.f.s of closing $p_c(m)$ and opening $p_o(m)$ are given by

$$p_c(m)=(N^2P^{N-1}[F\leq m]-(N^2-1)P^N[F\leq m])p(m), \tag{7.16}$$

$$p_o(m)=(N^2P^{N-1}[F>m]-(N^2-1)P^N[F>m])p(m) \tag{7.17}$$

for $N\geq 1$. □

<u>Proof</u>: We use the notation in Lemma 7.1 (see Eqs.(7.8)(7.9)).

$P[Z'(1)\leq m]=P[Y'(1)\leq m,Y'(2)>m,Y'(3)>m,\cdots,Y'(N)>m]$

$+P[Y'(2)\leq m,Y'(3)>m,\cdots,Y'(N)>m]+\cdots+P[Y'(j)\leq m,Y'(j+1)>m,\cdots,Y'(N)>m]$

$$+\cdots+P[Y'(N)\le m]. \tag{7.18}$$

According to the definition of process $\{Y'(k)\}$ in Lemma 7.1 and i.i.d. assumption, the first term of the right hand side in Eq.(7.18) can be rewritten as follows

$$P[Y'(1)\le m, Y'(2)>m, Y'(3)>m, \cdots, Y'(N)>m]$$

$$=P[F(1)\le m, F(2)\le m, F(3)\le m, \cdots, F(N)\le m, F(N+1)>m]=P^N[F\le m]P[F>m]. \tag{7.19}$$

The second term of the right hand side in Eq.(7.18) is

$$P[Y'(2)\le m, Y'(3)>m, \cdots, Y'(N)>m]$$

$$=P[F(2)\le m, F(3)\le m, F(4)\le m, \cdots, F(N)\le m, F(N+1)\le m, F(N+2)>m]$$

$$=P^N[F\le m]P[F>m]. \tag{7.20}$$

Therefore, the jth term of the right hand side in Eq.(7.18) can be expressed by

$$P[Y'(j)\le m, Y'(j+1)>m, \cdots, Y'(N)>m]$$

$$=P[F(j)\le m, F(j+1)\le m, F(j+2)\le m, \cdots, F(N+j-1)\le m, F(N+j)>m]$$

$$=P^N[F\le m]P[F>m] \tag{7.21}$$

for $1\le j\le N-1$.

The last term of Eq.(7.18) is given by

$$P[Y'(N)\le m]=P[F(N)\le m, F(N+1)\le m, F(N+2)\le m, \cdots, F(2N-1)\le m]$$

$$=P^N[F\le m]. \tag{7.22}$$

Because $P[Z'(1)\le m]$ is the sum of Eqs.(7.19)-(7.22), we have

$$P[Z'(1)\le m]=(N-1)P^N[F\le m]P[F>m]+P^N[F\le m]$$

$$=(N-1)P^N[F\le m](1-P[F\le m])+P^N[F\le m]=NP^N[F\le m]-(N-1)P^{N+1}[F\le m].$$

According to Lemma 7.1 and Lemma 5.5, we have

$P[(F^B)\le m]=P[Z'(i)\le m]=P[Z'(1)\le m]=NP^N[F\le m]-(N-1)P^{N+1}[F\le m]$

for all i. Eq.(7.15) can be proved by Lemma 5.1:

$P[F_B\le m]=P[(-F^B)\ge -m]=1-P[(-F^B)<-m]$

$=1-NP^N[-F<-m]+(N-1)P^{N+1}[-F<-m]=1-NP^N[F>m]+(N-1)P^{N+1}[F>m]$.

Eqs.(7.16) and (7.17) follow from differentiating with respect to m, Eqs.(7.14) and (7.15), respectively. In addition, the P.D.F.'s and p.d.f.'s in 1-D case can be extended to m-D case by considering Lemma 5.6. **Q.E.D.**

Eqs.(7.14)(7.15) are illustrated in Figs.(7.1)(7.2), respectively. We notice the P.D.F.s of closing and opening are point-wise monotonic in N. This observation is stated in the following.

Corollary 7.2: The P.D.F. of closing (opening) filter on any i.i.d. stochastic process {F(i)} by B with N+1 elements is, for each m, less (greater) than or equal to that of closing (opening) filter by B' with N elements, i.e.

$$P_{N+1}[F^B\le m]\le P_N[F^B\le m], \tag{7.23}$$

$$P_{N+1}[F_B\le m]\ge P_N[F_B\le m], \tag{7.24}$$

where, $P_N[F^B\le m]=NP^N[F\le m]-(N-1)P^{N+1}[F\le m]$,

$P_N[F_B\le m]=1-NP^N[F>m]+(N-1)P^{N+1}[F>m]$. □

<u>Proof:</u> Since $NP^N[F\le m](1-P[F\le m])^2\ge 0$,

we have $NP^N[F\le m](1-2P[F\le m]+P^2[F\le m])\ge 0$,

$$NP^N[F\leq m]-P^{N+1}[F\leq m](N-1+N+1)+NP^{N+2}[F\leq m]\geq 0,$$

$$NP^N[F\leq m]-(N-1)P^{N+1}[F\leq m]-(N+1)P^{N+1}[F\leq m]+NP^{N+2}[F\leq m]\geq 0,$$

$$NP^N[F\leq m]-(N-1)P^{N+1}[F\leq m]\geq(N+1)P^{N+1}[F\leq m]+NP^{N+2}[F\leq m].$$

This leads to Eq.(7.23). Similarly Eq.(7.24) can be proved. **Q.E.D.**

For the stochastic field, {F(i)}, with an i.i.d. uniform distribution between [-a a], the p.d.f., mean and variance after the closing operation on the process by B can be characterized as follows:

Corollary 7.3: If a stochastic field {F(i)} is i.i.d. with uniform distribution between [-a a], the p.d.f. $p_c(m)$, mean $E[(F^B)]$ and variance $Var[(F^B)]$ of the closing on the process {F(i)} is as follows:

$$p_c[m]=\frac{1}{2a}\left[N^2\left(\frac{m+a}{2a}\right)^{N-1}-(N^2-1)\left(\frac{m+a}{2a}\right)^N\right], \tag{7.25}$$

$$E[(F^B)]=a\left[\frac{2}{N+1}-\frac{6}{N+2}+1\right] \tag{7.26}$$

and

$$Var[(F^B)]=a^2\left(\frac{8}{N+1}+\frac{24}{N+2}-\frac{32}{N+3}+\frac{8}{(N+1)(N+2)}-\frac{4}{(N+1)^2}-\frac{36}{(N+2)^2}\right) \tag{7.27}$$

for $N\geq 1$. □

The <u>proof</u> of Corollary 7.3 is straightforward. Eq.(7.25) can be derived by substituting the p.d.f. of the uniform distribution into Eq.(7.16), then Eqs.(7.26)(7.27) can be derived according to the definitions of the mean and variance (see Appendix B). Again, due to the symmetricity of the uniform distribution, we can obtain the p.d.f., mean and variance of opening filter on the stochastic process by using Conclusion 5.1.

7.3 Binary M-Filters

As a special case, consider i.i.d. binary stochastic process with only two values, 0 or 1. Such process has many applications in signal processing [11][18], (e.g. black and white images). Using Theorems 7.1 and 7.2, we derive, here, the P.D.F.s and moments of M-Filters for such a binary stochastic process.

Corollary 7.4: Let $\{F(i)\}$ be an i.i.d. binary stochastic field with the following P.D.F.:

$$P[F=1]=p \text{ and } P[F=0]=1-p:=q, \tag{7.28}$$

the P.D.F.s of M-Filters on the binary stochastic process by B are described by

$$P[(F\oplus B^s)=1]=1-q^N \quad \text{and} \quad P[(F\oplus B^s)=0]=q^N, \tag{7.29}$$

$$P[(F\ominus B^s)=1]=p^N \quad \text{and} \quad P[(F\ominus B^s)=0]=1-p^N, \tag{7.30}$$

$$P[(F^B)=1]=1-Nq^N+(N-1)q^{N+1} \text{ and } P[(F^B)=0]=Nq^N-(N-1)q^{N+1}, \tag{7.31}$$

$$P[(F_B)=1]=Np^N-(N-1)p^{N+1} \text{ and } P[(F_B)=0]=1-Np^N+(N-1)p^{N+1}. \tag{7.32}$$

The means are given by

$$E[F\oplus B^s]=1-q^N, \tag{7.33}$$

$$E[F\ominus B^s]=p^N, \tag{7.34}$$

$$E[F^B]=1-Nq^N+(N-1)q^{N+1}, \tag{7.35}$$

$$E[F_B]=Np^N-(N-1)p^{N+1}. \tag{7.36}$$

The variances are given by

$$Var[F\oplus B^s]=(1-q^N)q^N, \tag{7.37}$$

$$Var[F\ominus B^s]=(1-p^N)p^N, \tag{7.38}$$

$$Var[F^B]=(1-Nq^N+(N-1)q^{N+1})(Nq^N-(N-1)q^{N+1}), \tag{7.39}$$

$$Var[F_B]=(1-Np^N+(N-1)p^{N+1})(Np^N-(N-1)p^{N+1}). \tag{7.40}$$

Corollary 7.4 can be proved by substituting Eq.(7.28) into Theorems 7.1 and 7.2, and by considering the definitions of mean and variance.

7.4 Discussion

The theoretical means and variances as well as experimental means and variances for closing filter on an i.i.d. stochastic process with symmetric uniform distribution are given in Figs.(7.3)(7.4). The theoretical results were calculated according to Eqs.(7.26)(7.27) and the experimental results were obtained by computer simulation. The computer ran a "closing program" on an i.i.d. stochastic process with a uniform distribution ranging from -5 to 5 (denoted by U(-5 5)) and 2000 sample paths were generated by the computer. It can be seen in the experimental results both the means and variances are very close to the theoretical results, which illustrates the validity of the theoretical analysis. Table (7.1) gives a quantitative comparison. The small discrepancies between the theoretical results and experimental results are probably caused by the finite number of sample paths produced by the Pseudo-random number generator and the numerical limitation of the Pseudo-random number generator simulated by a digital computer.

Fig.(7.5) shows the experimental results of the mean and variance of the closing operation on a stochastic process with an i.i.d. uniform distribution ranging from -5 to 5. Fig.(7.6) shows results for an i.i.d. Gaussian stochastic process, G(0, 100) in 2-D case. The set B is a symmetrical n×n square window ($N=n^2$). The experimental means and variances of closing on the uniform and Gaussian stochastic processes are also given in Tables (7.2) and (7.3) quantitatively. Comparing Table (7.1) with Table (7.2), we notice that if the element number N of the index set B in 1-D case is the same as that of the square window in 2-D case (e.g. when N=9 in 1-D case and n=3 in 2-D case), the means and variances of the closing filters are close to each other. This illustrates the validity of Theorem 7.2 both in 1-D and 2-D cases.

Theoretically, if the set B contains only the zero element, the result of the closing operation on any stochastic field by the set B is the same as the stochastic field itself. As the number N of elements of set B increases, the filtered signal will be smoother (see Figs.(7.7)-(7.14)). If the number of elements of set B is large enough, the result of closing operation on the stochastic field by the set B would be the maximum value of the process at each point. This statement is apparent if we notice that the closing operation is nothing but the first maximum operation followed by the minimum operation in a given neighborhood defined by B.

Although increasing the element number N of the set B can reduce the variance of the closing operation, it will take a longer computational time. By considering both the variance and computational complexity, it is necessary to have a comprehensive criterion to evaluate the efficiency of morphological filter. This issue will be discussed later.

Theorem 7.2 can be applied to various i.i.d. fields, and simple structuring function sequences. In general it is difficult to give the analytical formulas of means and variances. However, the numerical solutions of the means and variances for the M-Filters can be readily obtained.

The mean and variance of closing by B give a bound of variance when the structuring function sequence $\{g_r(j)\}$, supported by the set B, is not equal to zero (i.e., $g_r(j) \neq 0 \; j \in B$). The variance bound based on Lemma 5.4 describes the relationship between the variance of closing by $\{g_r(j)\}$ and the variance of closing by its support region B. For example, if the field has a symmetrical uniform distribution and the mean and variance of closing by B are given for the i.i.d. stochastic field (see Eqs.(7.26)(7.27)), the variance bound of closing can be derived according to Eq.(5.21). Also, it can be seen that a small "span" of a structuring function sequence, $\{g(j)\}$, on a given support region, B, implies a small variance bound. That means that the smoother the structuring function sequence $\{g(j)\}$, the smoother the filtered signal. This agrees with the concept of Maragos and Schafer [19 p.1162] "any structuring function g should be seen first as a geometrical pattern" before designing a filter from a stochastic viewpoint.

window	means of		variances of	
lengths (N)	theory	experiment	theory	experiment
1	0.000000	0.000782	8.333333	8.426160
3	1.500000	1.507348	4.416667	4.357319
5	2.380952	2.383434	2.664399	2.555929
7	2.916667	2.941170	1.770833	1.797748
9	**3.272727**	**3.271369**	**1.258953**	**1.226142**
11	3.525641	3.536760	0.939819	0.947938
13	3.714286	3.734053	0.727891	0.745433
15	3.860294	3.861979	0.580155	0.550225
17	3.976608	3.973113	0.473137	0.491404
19	4.071426	4.082875	0.393166	0.394334
21	4.150198	4.166646	0.331854	0.313533
23	4.216667	4.224529	0.283825	0.267615
25	4.273504	4.285433	0.245505	0.230063
27	4.322660	4.328930	0.214445	0.204037
29	4.365592	4.380407	0.188924	0.172878
31	4.403409	4.415167	0.167698	0.155241
33	4.439675	4.439432	0.149856	0.138197
35	4.466967	4.469017	0.134716	0.128985
37	4.493927	4.487499	0.121758	0.121912
39	4.518293	4.524189	0.110583	0.100349

Table (7.1): Comparison between theoretical and experimental means and variances of closing filter in 1-D case.

window lengths (n)	means	variances
1	-0.083480	8.396547
3	**3.282714**	**1.289326**
5	4.254735	0.234895
7	4.407100	0.098471
9	4.621335	0.038939
11	4.767135	0.018249
13	4.811302	0.009527
15	4.868992	0.005023
17	4.887948	0.003509
19	4.908752	0.002339
21	4.926284	0.001470
23	4.934706	0.001064
25	4.949734	0.000773

Table (7.2): Experimental results of means and variances of closing on 2-D stochastic process with i.i.d. uniform distribution U(-5 5).

window lengths (n)	means	variances
1	0.058106	97.846240
3	9.023750	21.197334
5	13.604583	11.974811
7	16.697535	9.910424
9	18.525057	7.008729
11	20.070976	5.209347
13	21.154730	4.832991
15	22.393742	4.731099
17	23.081654	3.974100
19	23.914974	3.756544
21	24.598885	3.601431
23	25.388340	3.801970
25	26.043087	2.949000

Table (7.3): Experimental results of means and variances of closing on 2-D stochastic process with i.i.d. Gaussian distribution G(0, 100).

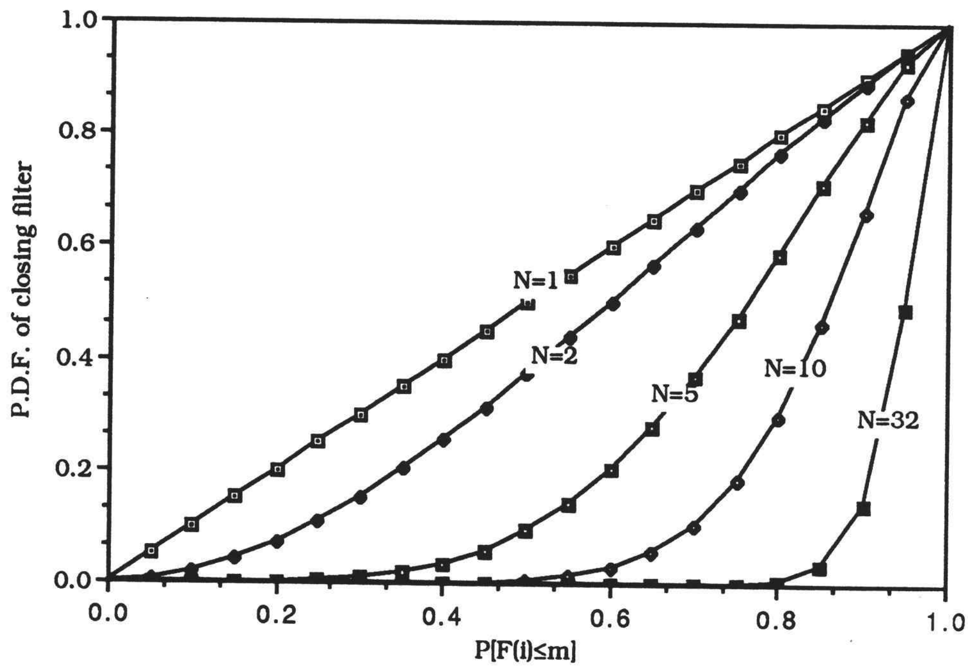

Fig.(7.1): Relationship of P.D.F.s between stochastic process {F(i)} and the closing process {$F^B(i)$}

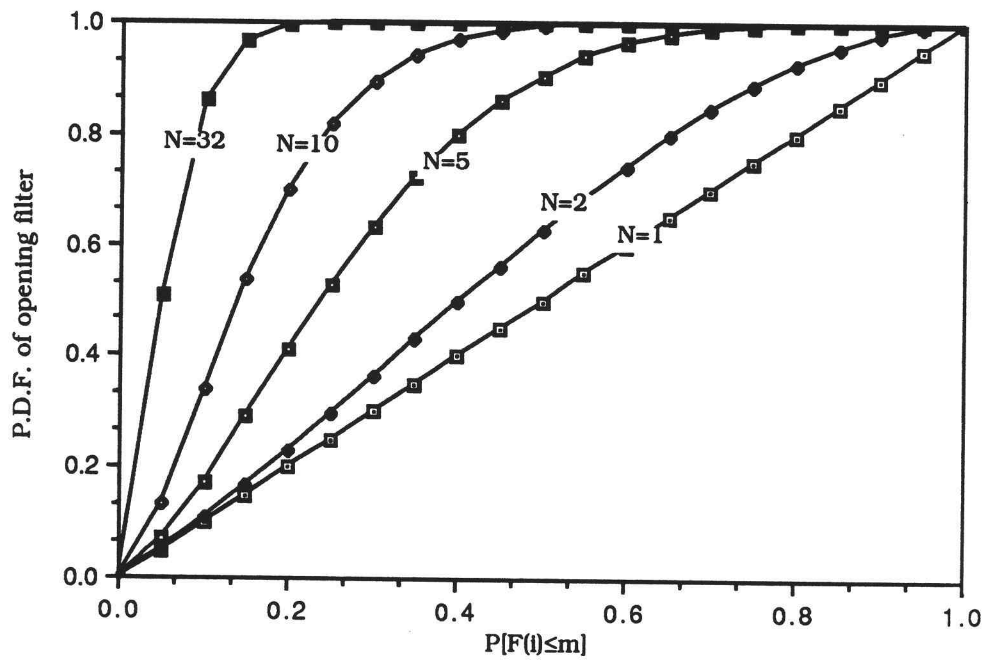

Fig.(7.2): Relationship of P.D.F.s between stochastic process {F(i)} and the opening process {F_B(i)}

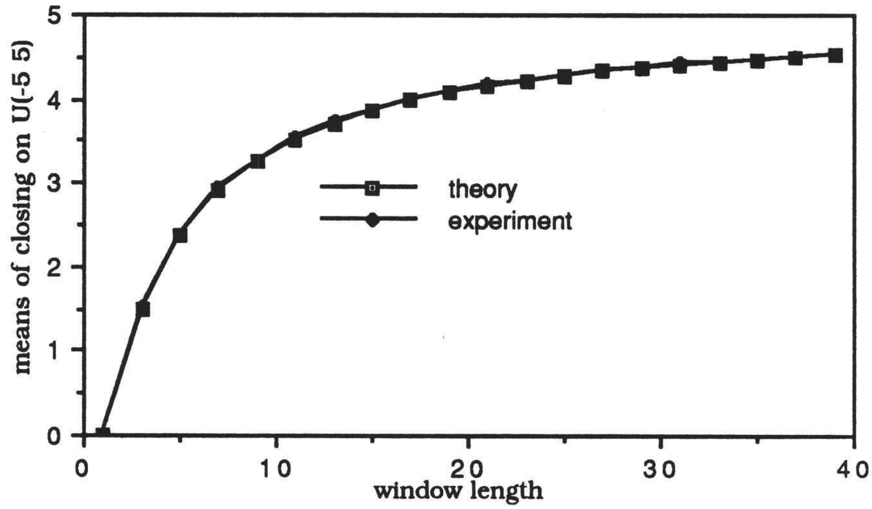

Fig.(7.3): Comparison between experimental and theoretical means of the closing operation by a symmetric interval B on 1-D i.i.d. uniform stochastic process U(-5 5).

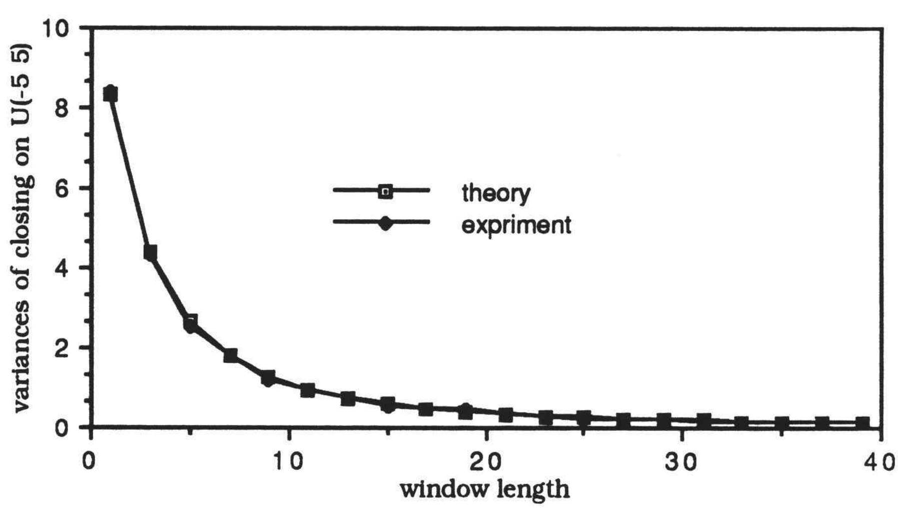

Fig.(7.4): Comparison between experimental and theoretical variances of the closing operation by a symmetric interval B on 1-D i.i.d. uniform stochastic process U(-5 5).

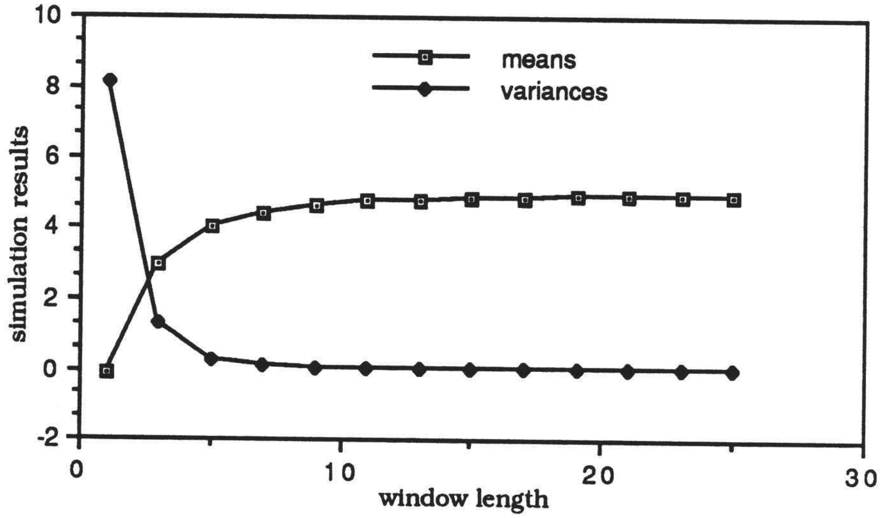

Fig.(7.5): Experimental results of means and variances of the closing operation by window B on 2-D stochastic process with i.i.d. uniform distribution U(-5 5).

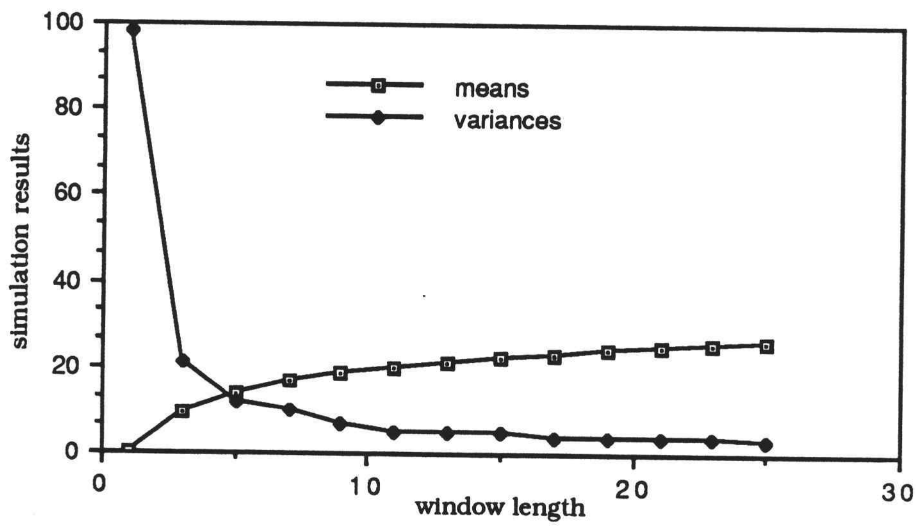

Fig.(7.6): Experimental results of means and variances of the closing operation by window B on 2-D stochastic process with i.i.d. Gaussian distribution G(0, 100).

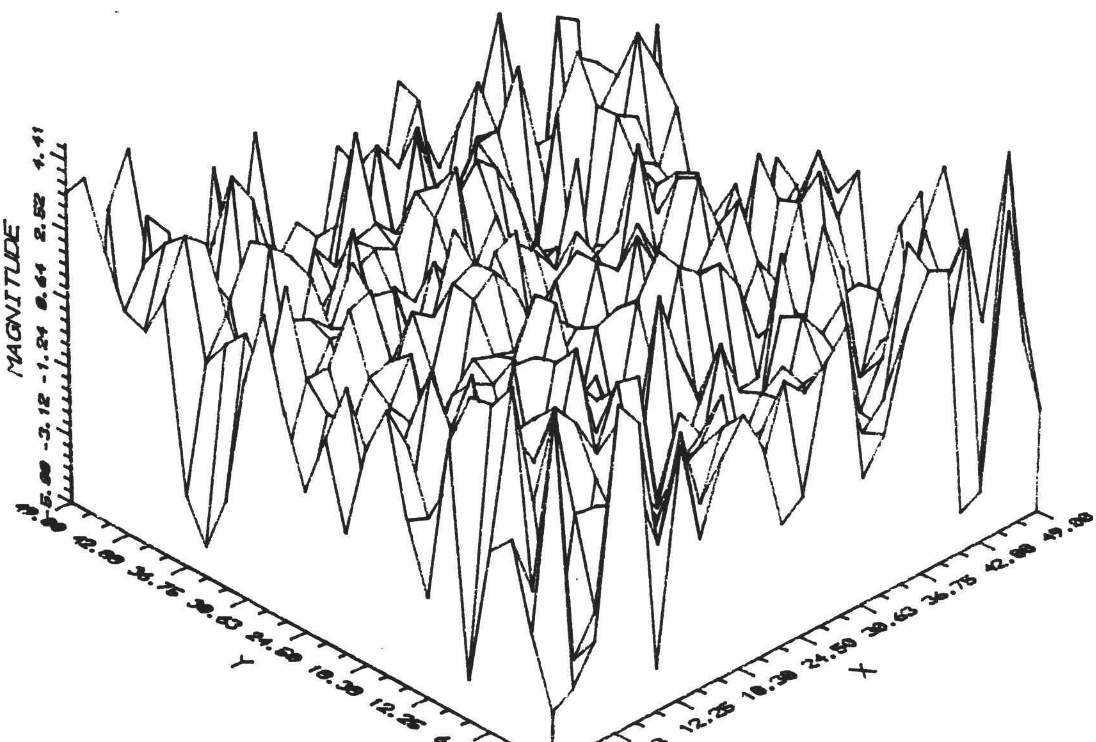

Fig.(7.7) 2-D stochastic process with i.i.d. uniform distribution U(-5, 5).

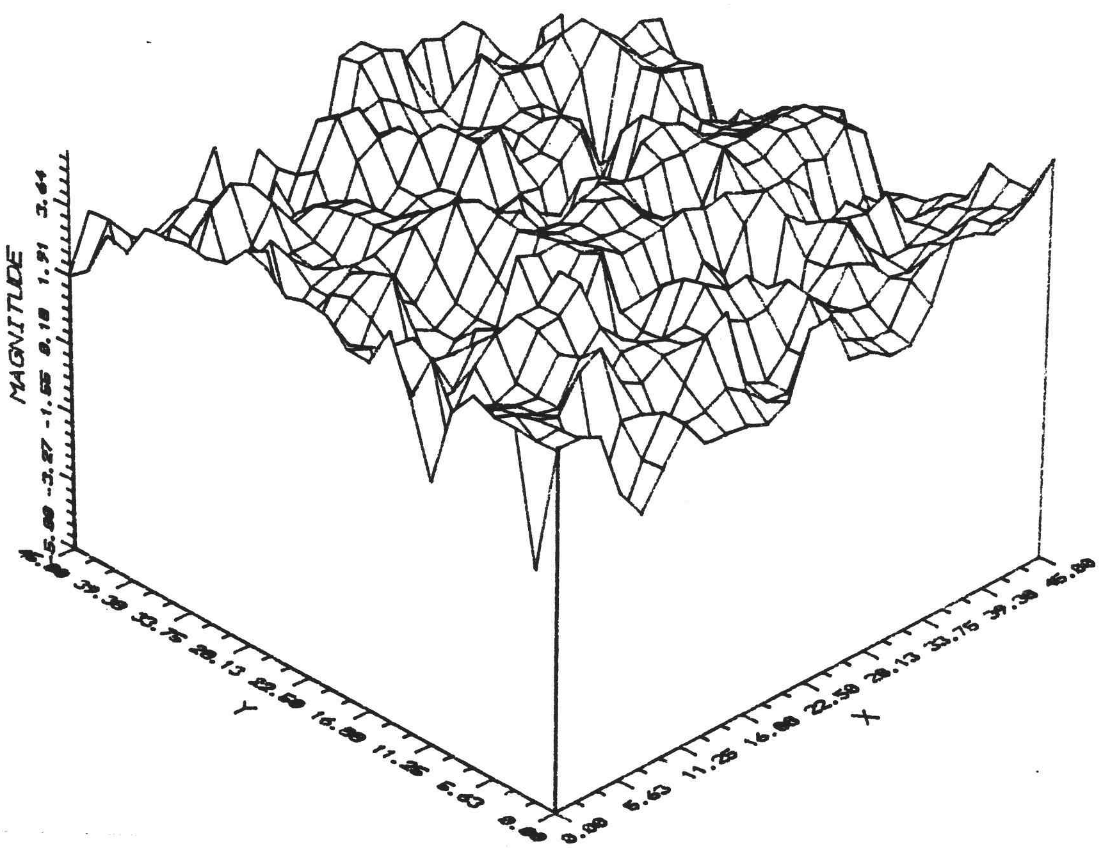

Fig.(7.8) Closing on a 2-D i.i.d uniform stochastic process U(-5, 5) by a square window with window length n=3.

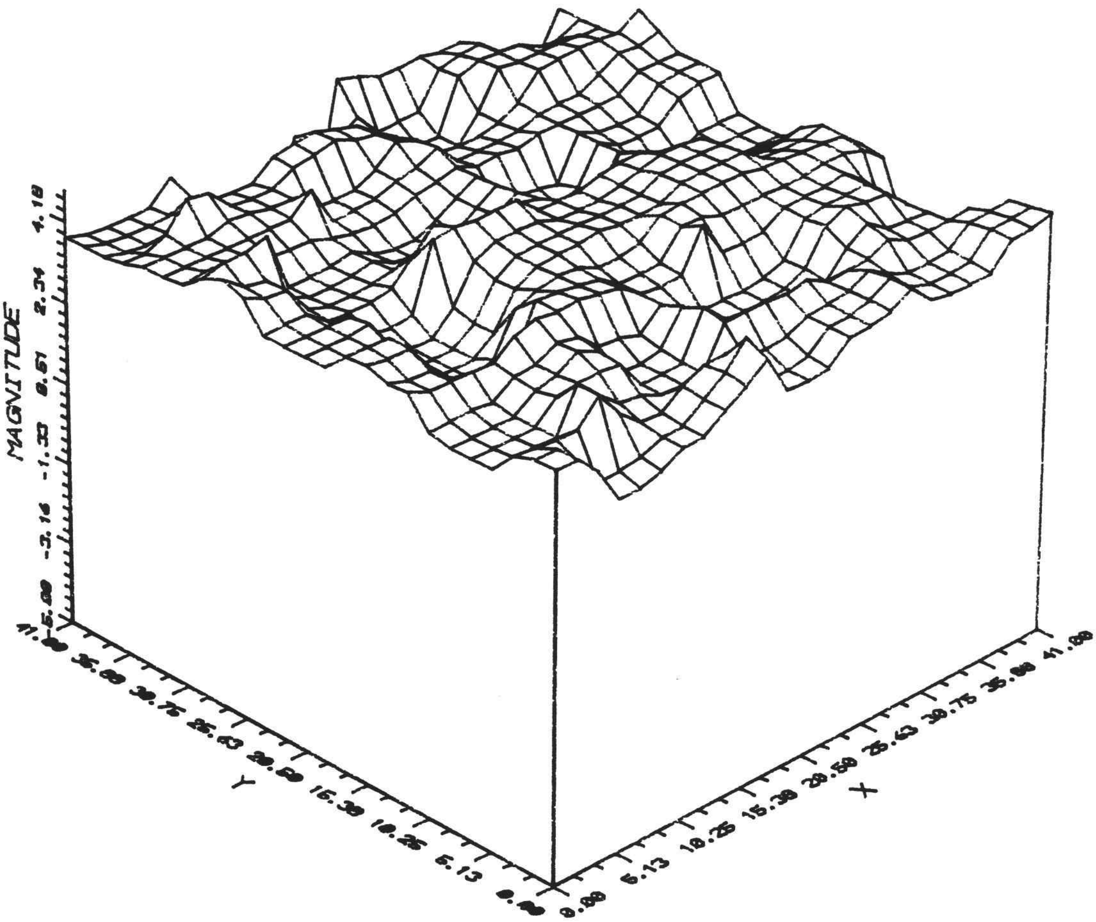

Fig.(7.9) Closing on a 2-D i.i.d uniform stochastic process U(-5, 5) by a square window with window length n=5.

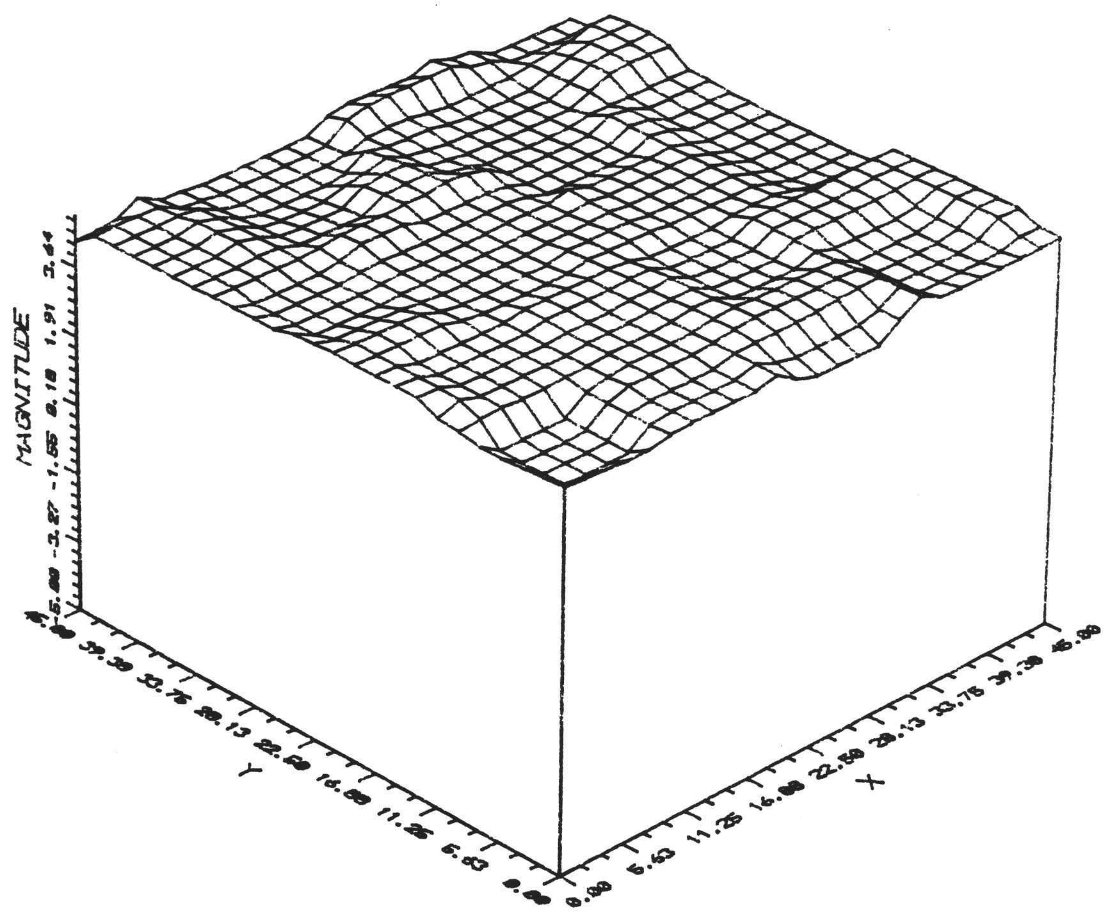

Fig.(7.10) Closing on a 2-D i.i.d uniform stochastic process U(-5, 5) by a square window with window length n=9.

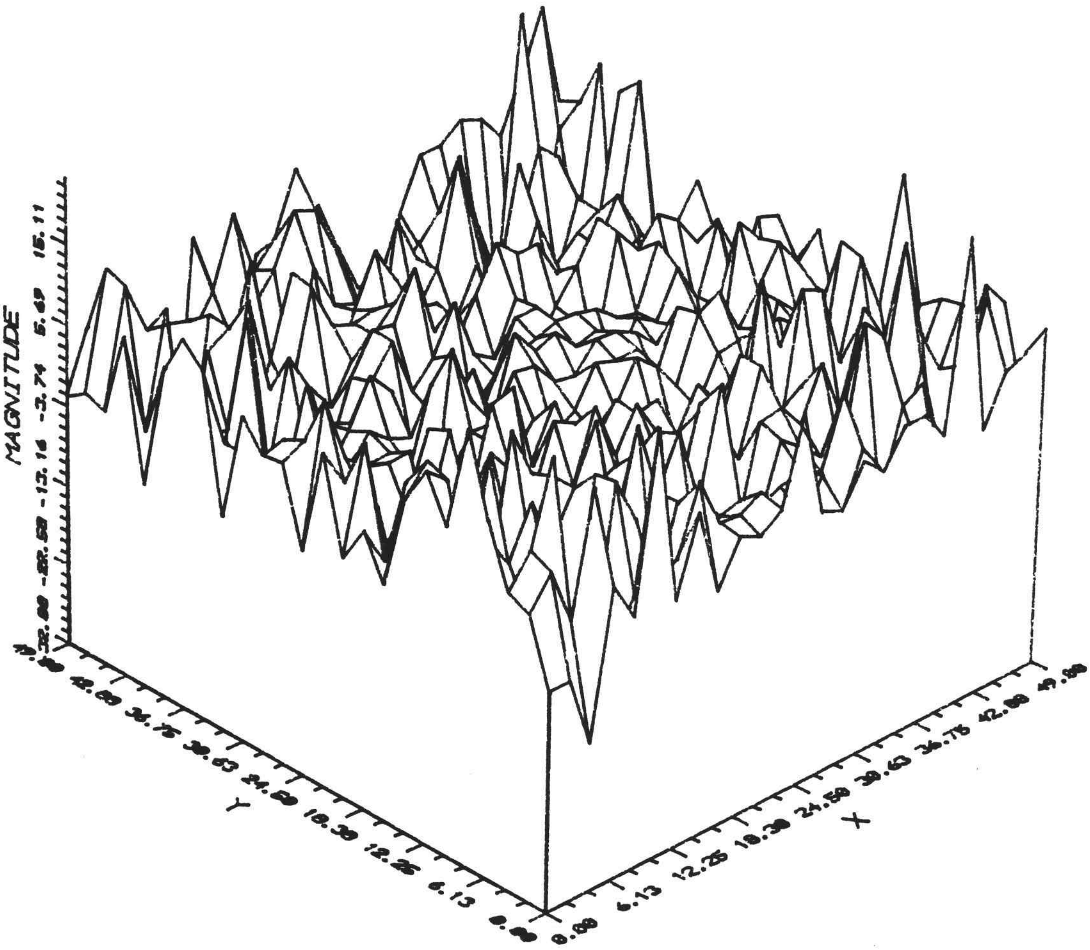

Fig.(7.11) 2-D stochastic process with i.i.d. Gaussian distribution G(0, 100).

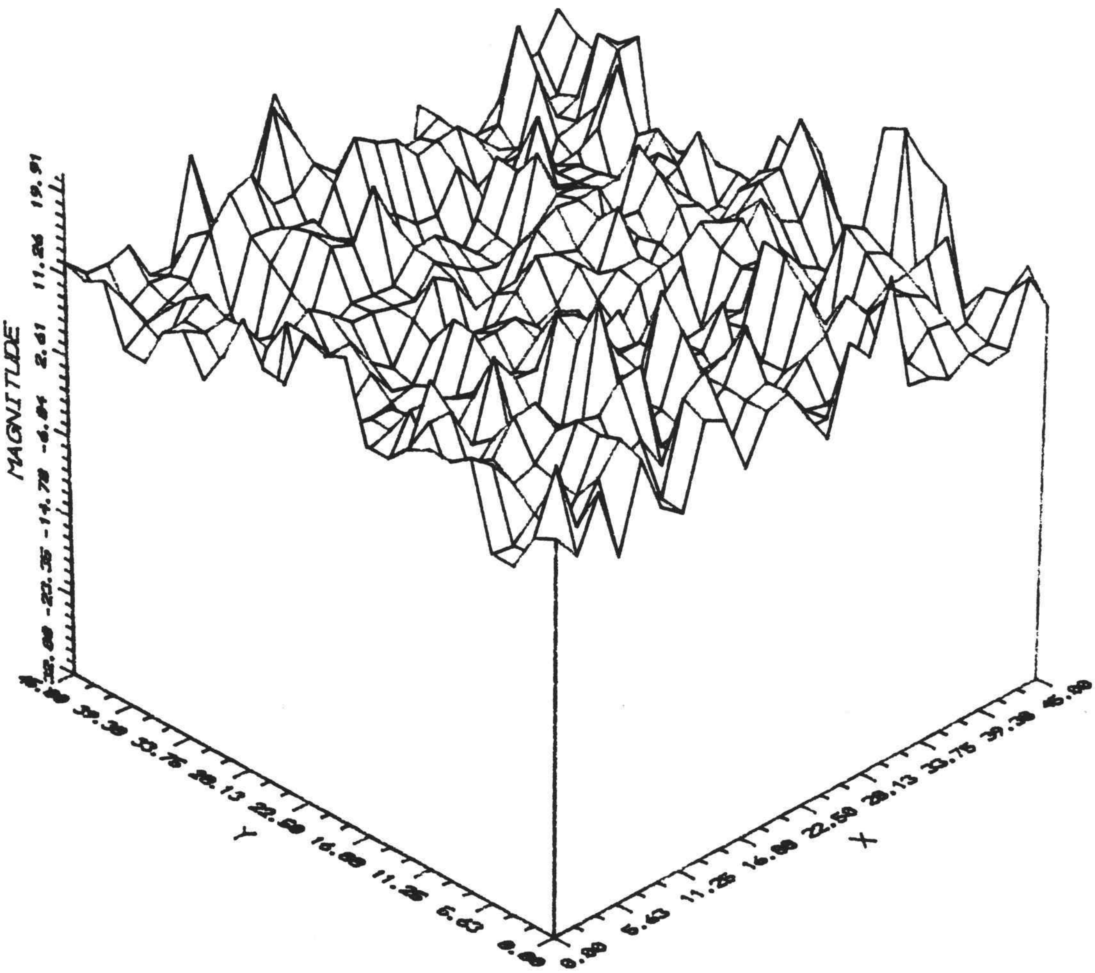

Fig.(7.12) Closing on a 2-D i.i.d Gaussian stochastic process G(0, 100) by a square window with window length n=3.

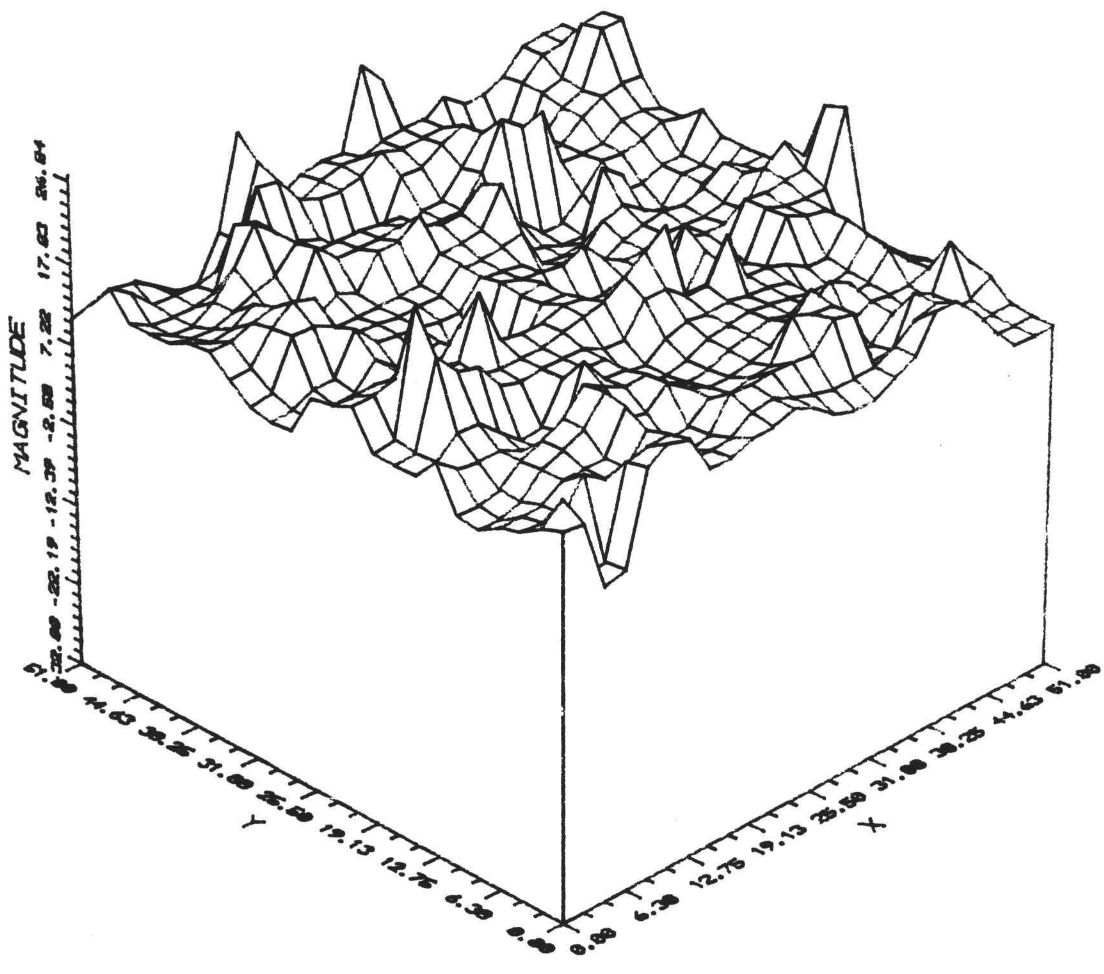

Fig.(7.13) Closing on a 2-D i.i.d Gaussian stochastic process G(0, 100) by a square window with window length n=5.

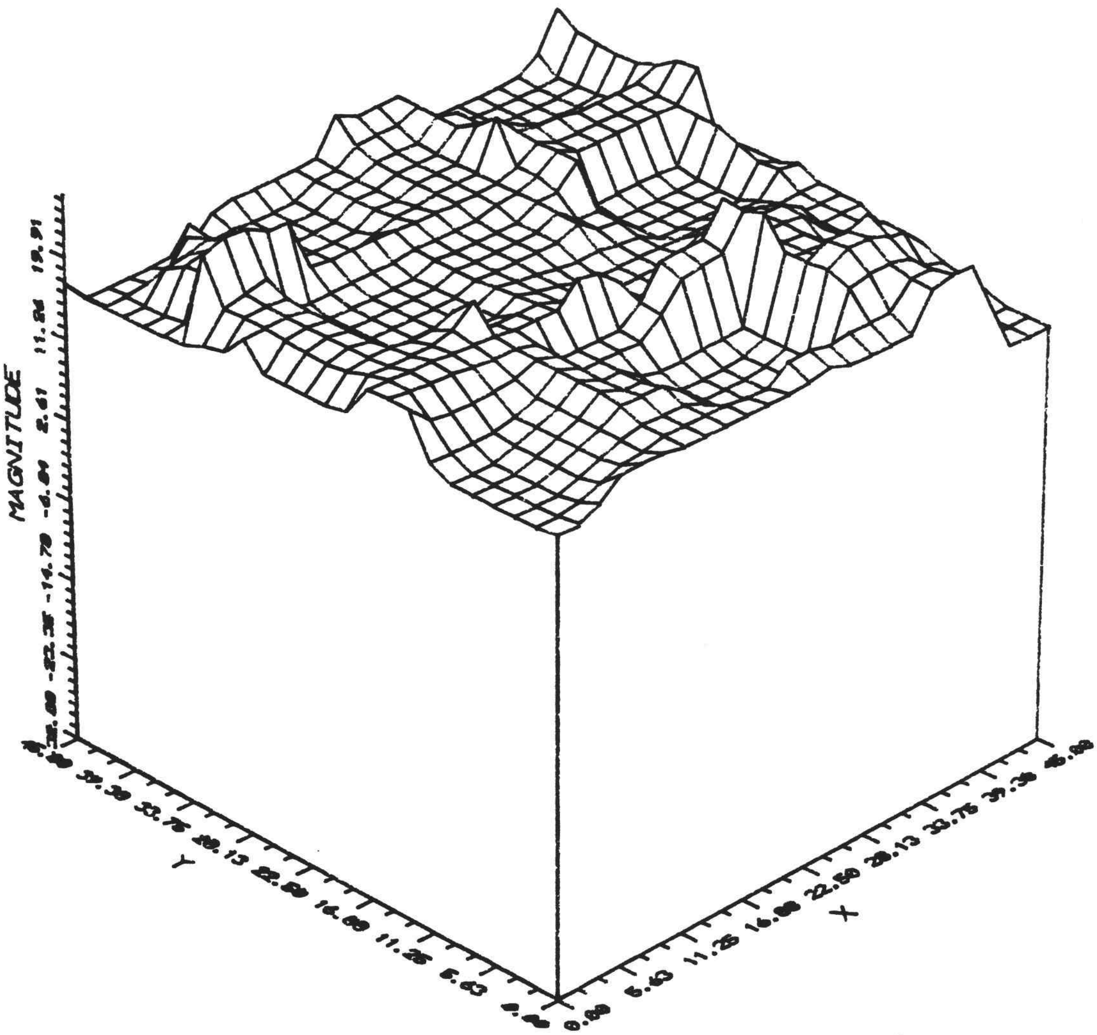

Fig.(7.14) Closing on a 2-D i.i.d Gaussian stochastic process G(0, 100) by a square window with window length n=9.

CHAPTER 8. CRITERIA AND OPTIMAL M-FILTER DESIGN

The performance quality criterion of M-Filters is an important and difficult issue. As mentioned by Maragos and Schafer in [19], the main difficulties in morphological filter analysis and design arise from the nonlinearity and the lack of analytical criteria to choose the structuring sets or functions. In this chapter, we present a comprehensive criterion which includes the measure of smoothness of filtered signal as well as the element number N of support region B of structuring function sequence $\{g(j)\}$, $j \in B$. The criterion is defined as

$$J(N) = \sum_{i \in D} (V[M(i)] + C(N)) \qquad N \geq 1, \; C(N) \geq 0. \tag{8.1}$$

In Eq.(8.1), M(i) denotes one of the M-Filters with an arbitrary structuring function sequence $\{g(j)\}$ $j \in B$ (B is a finite index set). N is the number of elements of set B. C(N) reflects the calculational complexity for the filter. D is the domain of the filter M(i). V[M(i)] is the measure of smoothness of the filtered signal at i, such as variance of M(i),

We say the design of a morphological filter is optimal in the sense of Eq.(8.1), if there exists a N, such that the criterion J(N) is minimized. An optimal filter is expected to compromise between the smoothness of filtered signal and the number of elements of set B.

For example, if the structuring function sequence is a square window with $N=n^2$, then, an optimal filter may be obtained, for a given 2-D signal, by adjusting the parameter N (or n), such that Eq.(8.1) reaches a minimum. Of course, the parameter N can be replaced by the area of the support region. In fact, the minimization of J(N) belongs to the class of integer programming problems [51][52].

To illustrate the above concept, we take C(N)=kN, k>0 and use the variances (Var[M(i)]) discussed in Chapter VII for i.i.d. uniform stochastic process (see Eq.(7.27)) as the measure of smoothness (V[M(i)]). Using Lemma 5.5, the criterion J(N) (Eq.(8.1)) can be simplified as follows.

$$J(N)=Var[M]+kN=Var[(F^B)]+kN$$

$$=a^2\left(\frac{8}{N+1}+\frac{24}{N+2}-\frac{32}{N+3}+\frac{8}{(N+1)(N+2)}-\frac{4}{(N+1)^2}-\frac{36}{(N+2)^2}\right)+kN. \qquad (8.2)$$

In accordance with the above criterion, different optimal window lengths are shown in Figs.(8.1)(8.2) for different value of k. The optimal window length is 11 with k=0.1 in Fig.(8.1) and 5 with k=0.5 in Fig.(8.2), where a=5. Different optimal window size corresponds to different value of k, which depends on the priorities of designers.

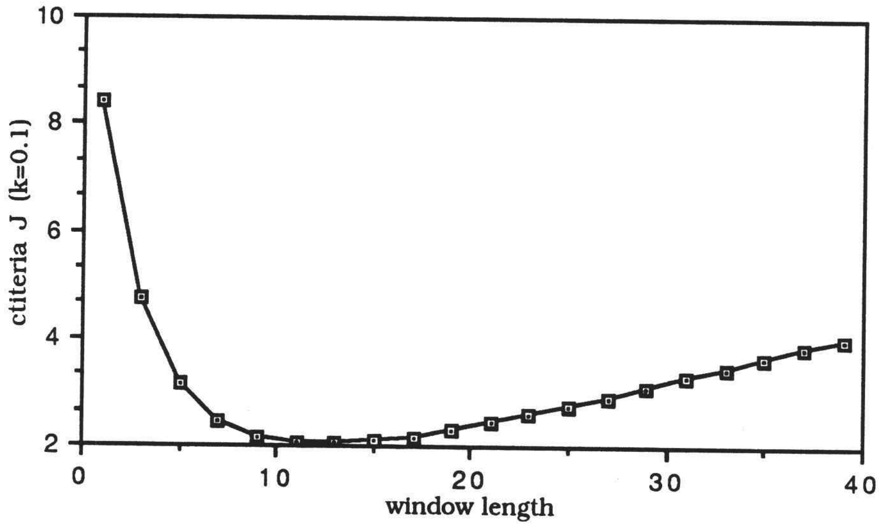

Fig.(8.1): Optimal closing filter design on i.i.d. uniform stochastic process U(-5 5) by symmetric interval in the sense of criterion Eq.(8.2) with weight coefficient k=0.1.

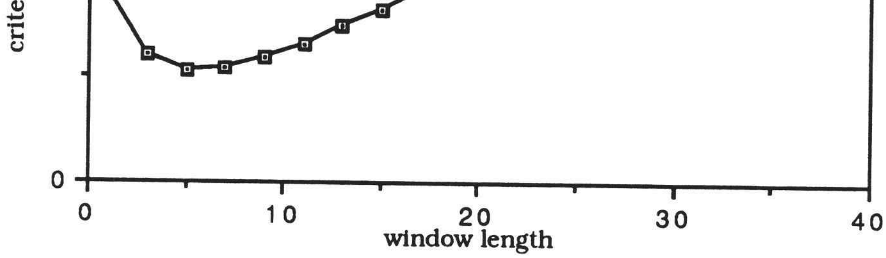

Fig.(8.2): Optimal closing filter design on i.i.d. uniform stochastic process U(-5 5) by symmetrical interval in the sense of criterion Eq.(8.2) with weight coefficient k=0.5.

CHAPTER 9. CONCLUSION AND FURTHER RESEARCH

The analysis of stochastic properties of M-Filters is presented in the form of lemmas and theorems. Characterization of the core of morphological operations (dilation, erosion, closing and opening filters) is emphasized in this thesis. The lemmas simplify the study of the stochastic properties of M-Filters, by exploring the concept of duality between dilation and erosion filters, closing and opening filters. As the result, two "unbiased filters", ADE and ACO, are proposed. In particular, the P.D.F.s and p.d.f.s of M-Filters on any i.i.d. stochastic field by the simple structuring function sequence are derived. As a special case, the P.D.F.s of M-Filters on an i.i.d. binary stochastic field, are presented. The analytical solutions of the means and variances are given for M-Filters on the stochastic field with i.i.d. uniform P.D.F. and binary P.D.F., respectively.

In order to evaluate the filter designs, a comprehensive criterion is proposed, which combines the measure of smoothness of a filtered signal and the computational complexity. This makes it possible for a designer to propose a "best" filter.

The presented results were obtained under several assumptions, necessary to simplify the mathematical theory of nonlinear morphological operations. Some of these assumptions restrict the real applications and further study should be considered. The following problems may be considered for further research:

The P.D.F.s of M-Filters derived in this thesis consider only the simple structuring function sequence $g(j)=0$, $j \in B$, and an i.i.d.

stochastic field. Therefore, in the increasing order of complexity, the P.D.F.s of M-Filters on i.i.d. stochastic field with $g(j)\neq0$, $j\in B$, the P.D.F.s of M-Filters on non i.i.d. stochastic field with $g(j)=0$, $j\in B$, and the P.D.F.s of M-Filters on non i.i.d. stochastic field with $g(j)\neq0$, $j\in B$, may be researched. The P.D.F.s of M-Filters on i.i.d. stochastic field with $g(j)\neq0$, $j\in B$ can be analyzed in a similar way to the procedure of Theorems 7.1 and 7.2, but the simple analytical solution may be difficult to obtain.

Although the ADE and ACO filters have the same means as that of median filter for an i.i.d. stochastic field with symmetric P.D.F., the comparison of variances between median, ADE and ACO filters has not been completed. Probably, the variances of ADE and ACO filters by the simple structuring function sequence $\{g(j)=0\}$, $j\in B$ are close to those of median filters. This conjecture needs further investigation.

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APPENDICES

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APPENDIX 1. P.D.F.s OF MAX-MIN OPERATIONS

This appendix briefly describes some mathematical results describing maximum and minimum operations on random variables. All of these results can be found in [48][49][50].

1. Probability distribution functions (P.D.F.) of M=max(X, Y) and N=min(X, Y) [49]

Let X, Y be two random variables mutually independent, their P.D.F.s are defined as $F_x(x)$ and $F_y(y)$ respectively. The question is how to obtain the P.D.F.s of M=max(X, Y) and N=min(X, Y).

Since M=max(X, Y) not larger than z is equivalent to that all X and Y is not larger than z, we have

$$P\{M \le z\}=P\{X \le z,\ \ Y \le z\}.$$

The independence of X and Y results in the P.D.F. of M=max(X, Y) by

$$F_{max}(z)=P\{M \le z,\ \ Y \le z\}=P\{X \le z\}P\{Y \le z\}$$

i.e., $F_{max}(z)=F_x(z)\ F_y(z)$.

Similarly, the P.D.F. of N=min(X, Y) can be obtained by

$$F_{min}(z)=P\{N \le z\}=1-P\{N>z\}=1-P\{X>z,\ \ Y>z\}=1-P\{X>z\}P\{Y>z\}$$

i.e., $F_{min}(z)=1-[1-F_x(z)][1-\ F_y(z)]$.

All results above can be extended to the case of n random variables. Let X_1, X_2,....,X_n be n random variables mutually independent and their P.D.F.s be $F_{x1}(x_1)$,$F_{x2}(x_2)$,.....,$F_{xn}(x_n)$. Then the

P.D.F.s of $M_n=\max(X_1, X_2,\ldots,X_n)$ and $N_n=\min(X_1, X_2,\ldots,X_n)$ are described by

$$F_{max}(z)=F_{x1}(x_1)\,F_{x2}(x_2),\ldots,F_{xn}(x_n),$$

$$F_{min}(z)=1-[1-F_{x1}(x_1)][1-F_{x2}(x_2)]\ldots[1-F_{xn}(x_n)].$$

Particularly, when $X_1, X_2,\ldots,X_n$ are independent, identically distributed (i.i.d.), random variables with the distribution function $F(z)$

$$F_{max}(z)=[F(z)]^n$$

$$F_{min}(z)=1-[1-F(z)]^n$$

2. The relationships between the random variable with uniform P.D.F. and the random variable with other P.D.F.s [50]

Let $X_1, X_2,\ldots,X_n$ be n i.i.d. random variables with continuous, strictly increasing, cumulative distribution function $F(x)$. Without loss of generality, let us set

$$Y_i=F(X_i) \qquad (i=1,\ldots\ldots n)$$

and compute the P.D.F. of Y_i :

$$P\{Y_i\leq y\}=P\{F(X_i)\leq y\}=P\{X_i\leq F^{-1}(y)\}=F[F^{-1}(y)]=y \quad \text{for } 0\leq y\leq 1,\ i=1,\ldots n,$$

where F^{-1}, the inverse function of F, is uniquely defined by our assumptions about F. Further, since $0\leq F(x)\leq 1$,

$$P\{Y_i\leq y\}=\begin{cases} 0 & \text{if} \quad y<0 \\ 1 & \text{if} \quad y>1 \end{cases} \qquad \text{for} \quad i=1\ldots n.$$

Thus, Y_i is uniformly distributed over [0, 1] for every i=1,.....n regardless of the form the continuous, strictly increasing function F takes. Notice that the maximal and minimal relationships among the $\{X_i\}$, $\max(X_1, X_2,.....,X_n)$ and $\min(X_1, X_2,.....,X_n)$, are preserved by the transformation $Y_i=F(X_i)$. This means that instead of investigating the general sample $(X_1, X_2,.....,X_n)$ we may study the particular sample $(Y_1, Y_2,.....,Y_n)$ taken from the uniform distribution on [0, 1].

3. Fluctuations of sums of random variables [48]

Let $X_1, X_2,....,X_n$ be i.i.d. random variables with an arbitrary P.D.F. and G(x) be a monotone nondecreasing function of bounded variation in $(-\infty, \infty)$. Then

$$\gamma(s)=\int_{-\infty}^{\infty} e^{-sx}dG(x)$$

is convergent for Re(s)=0. Knowing $\gamma(s)$, the function

$$\gamma^{+}(s)=\int_{-0}^{\infty} e^{-sx}dG(x)$$

is determined uniquely for Re(s)≥0. Now define an operator **A** such that

$$\gamma^{+}(s)=\mathbf{A}\gamma(s).$$

The operator **A** is linear and $\mathbf{A}^2=\mathbf{A}$. Now suppose that

$$G(x)=\sum_{n=0}^{\infty} \frac{a^n}{n!} F_n(x),$$

where $F_n(x)$ is the nth iterated convolution of a distribution function F(x) with itself; $F_0(x)=1$ if $x\geq 0$ and $F_0(x)=0$ if $x<0$. If

$$\phi(s)=\int_{-\infty}^{\infty} e^{-sx}dF(x)$$

for Re(s)=0, then the Laplace-Stieltjes transform of G(x) is given by

$$\gamma(s)=e^{a\phi(s)}$$

for Re(s)=0. In what follows, the following two statements will be used and the truth of them can be seen immediately.

(i) If $\mathbf{A}\phi(s)=\phi(s)$, then $\mathbf{A}\gamma(s)=e^{a\phi(s)}$,

(ii) If $\mathbf{A}\phi(s)=0$, then $\mathbf{A}\gamma(s)=1$.

Theorem: Let $X_1, X_2,....,X_n$ be i.i.d. random variables for which

$$E\{e^{-sX_r}\}=\phi(s)$$

when Re(s)=0. Let $Y_n=\text{Max}(0, X_1, X_1+X_2,....,X_1+....+X_n)$ and

$$E\{e^{-sY_n}\}=\Phi_n(s)$$

for Re(s)≥0. If $0\leq w<1$ and Re(s)≥0, then we have

$$\sum_{n=0}^{\infty} \Phi_n(s)\, w^n=\exp\Big(\sum_{n=1}^{\infty} \frac{w^n}{n}\mathbf{A}\{[\phi(s)]^n\}\Big)$$

or equivalently,

$$\sum_{n=0}^{\infty} \Phi_n(s)\, w^n=\exp(-\mathbf{A}\{\log[1- w\phi(s)]\}). \qquad \square$$

Therefore, it follows that

$$\Phi_n(s)=\sum_{k_1+2k_2+...+nk_n=n} \frac{[\mathbf{A}(\phi(s))]^{k_1}[\mathbf{A}(\phi(s))^2]^{k_2}\,....[\mathbf{A}(\phi(s))^n]^{k_n}}{k_1!k_2!....k_n!1^{k_1}2^{k_2}....n^{k_n}}$$

and the distribution of Y_n can be obtained by inversion.

APPENDIX 2. MEAN AND VARIANCE OF CLOSING OPERATION

Proof: Eq.(7.25) can be proved by substituting the P.D.F. and p.d.f. of the stochastic process with i.i.d. uniform distribution function into Eq.(7.16). From the definition of mean, we have

$$E[(F^B)(i)]=\int_{-a}^{+a} mp_c(m)\, dm =\int_{-a}^{+a} \frac{m}{2a}\left[N^2\left(\frac{m+a}{2a}\right)^{N-1}-(N^2-1)\left(\frac{m+a}{2a}\right)^{N}\right] dm$$

$$=\int_{-a}^{+a} \frac{m}{(2a)^{N+1}}\left[2aN^2(m+a)^{N-1}-(N^2-1)(m+a)^{N}\right] dm$$

$$=\int_{-a}^{+a} \frac{m+a-a}{(2a)^{N+1}}\left[2aN^2(m+a)^{N-1}-(N^2-1)(m+a)^{N}\right] dm$$

$$=\frac{1}{(2a)^{N+1}}\int_{-a}^{+a} [(2aN^2(m+a)^{N}-(N^2-1)(m+a)^{N+1}$$

$$-2a^2N^2(m+a)^{N-1}+a(N^2-1)(m+a)^{N}]dm$$

$$=\frac{1}{(2a)^{N+1}}\left[\frac{2aN^2+aN^2-a}{N+1}(m+a)^{N+1}-\frac{N^2-1}{N+2}(m+a)^{N+2}-2a^2N(m+a)^{N}\right]_{-a}^{a}$$

$$=\frac{3aN^2-a}{N+1}-\frac{2a(N^2-1)}{N+2}-aN=a\left[\frac{3N^2-1}{N+1}-\frac{2(N^2-1)}{N+2}-N\right]$$

$$=a\left[\frac{3(N^2+1+2N-1-2N)-1}{N+1}-\frac{2(N^2+4+4N-4-4N)-2}{N+2}-N\right]$$

$$=a\left[\frac{3(N+1)^2}{N+1}-\frac{6N+4}{N+1}-\frac{2(N+2)^2-10-8N}{N+2}-N\right]$$

$$=a\left[3(N+1)-6+\frac{2}{N+1}-2(N+2)+8-\frac{6}{N+2}-N\right]=a\left[1+\frac{2}{N+1}-\frac{6}{N+2}\right]$$

which is Eq.(7.26).

From the definition of variance, we have

$Var[(F^B)(i)]=E[(F^B)^2(i)]-E^2[(F^B)(i)]$,

and

$E[(F^B)^2(i)]=i)]=\int_{-a}^{+a} m^2p_c(m)dm$

$$=\int_{-a}^{+a} \frac{m^2}{2a}\left[N^2(\frac{m+a}{2a})^{N-1}-(N^2-1)(\frac{m+a}{2a})^N\right]dm$$

$$=\frac{1}{(2a)^{N+1}}\int_{-a}^{+a} m^2[2aN^2(m+a)^{N-1}-(N^2-1)(m+a)^N]dm$$

$$=\frac{1}{(2a)^{N+1}}\int_{-a}^{+a} (m^2+a^2+2am-a^2-2am+2a^2-2a^2)[2aN^2(m+a)^{N-1}$$

$$-(N^2-1)(m+a)^N]dm$$

$$=\frac{1}{(2a)^{N+1}}\int_{-a}^{+a} ((m+a)^2-2a(m+a)+a^2)[2aN^2(m+a)^{N-1}-(N^2-1)(m+a)^N]dm$$

$$=\frac{1}{(2a)^{N+1}}\int_{-a}^{+a} [2aN^2(m+a)^{N+1}-(N^2-1)(m+a)^{N+2}-4a^2N^2(m+a)^N$$

$$+2a(N^2-1)(m+a)^{N+1}+2a^3N^2(m+a)^{N-1}-a^2(N^2-1)(m+a)^N]dm$$

$$=\frac{1}{(2a)^{N+1}}\int_{-a}^{+a} [(4aN^2-2a)(m+a)^{N+1}-(N^2-1)(m+a)^{N+2}+2a^3N^2(m+a)^{N-1}$$

$$+(-4a^2N^2-a^2N^2+a^2)(m+a)^N]dm$$

$$=\frac{1}{(2a)^{N+1}}[(2a)^{N+2}\frac{4aN^2-2a}{N+2}-(2a)^{N+3}\frac{N^2-1}{N+3}+(2a)^N\frac{2a^3N^2}{N}+(2a)^{N+1}\frac{-5a^2N^2+a^2}{N+1}]$$

$$=(2a)\frac{4aN^2-2a}{N+2}-(2a)^2\frac{N^2-1}{N+3}+\frac{2a^3N^2}{2aN}+(2a)^{N+1}\frac{-5a^2N^2+a^2}{N+1}$$

$$=\frac{-4a^2}{N+2}+8a^2\frac{N^2+4+4N-4-4N-8+8}{N+2}$$

$$-2a^2\frac{N^2+9+6N-9-6N-18+18-1}{N+3}+a^2N-5a^2\frac{N^2+1+2N-2N-1-2+2}{N+1}+\frac{a^2}{N+1}$$

$$=\frac{-4a^2}{N+2}+8a^2[\frac{(N+2)^2}{N+2}+\frac{4}{N+2}-4]$$

$$-2a^2\frac{(N+3)^2-6(N+3)+8}{N+3}+a^2N-5a^2\frac{(N+1)^2-2(N+1)+1}{N+1}+\frac{a^2}{N+1}$$

$$=\frac{-4a^2}{N+2}+8a^2(N+2)+\frac{32a^2}{N+2}-32a^2-4a^2(N+3)+24a^2$$

$$-\frac{32a^2}{N+3}+a^2N-5a^2(N+1)+10a^2-\frac{5a^2}{N+1}+\frac{a^2}{N+1}$$

$$=\frac{28a^2}{N+2}-\frac{32a^2}{N+3}-\frac{4a^2}{N+1}+8a^2+16a^2-32a^2-4a^2N-12a^2+24a^2+a^2N$$

$$-5a^2N-5a^2+10a^2$$

$$=\frac{28a^2}{N+2}-\frac{32a^2}{N+3}-\frac{4a^2}{N+1}+a^2,$$

In addition,

$$E^2[(F^B)(i)]=a^2\left[1+\frac{2}{N+1}-\frac{6}{N+2}\right]^2$$

$$=a^2\left[\frac{4}{(N+1)^2}+\frac{36}{(N+2)^2}+1-\frac{24}{(N+1)(N+2)}-\frac{12}{N+2}+\frac{4}{N+1}\right].$$

Therefore, we have

$$Var[(F^B)(i)]=a^2[\frac{28}{N+2}-\frac{32}{N+3}-\frac{4}{N+1}+1$$

$$-\frac{4}{(N+1)^2}-\frac{36}{(N+2)^2}-1+\frac{24}{(N+1)(N+2)}+\frac{12}{N+2}-\frac{4}{N+1}]$$

$$=a^2[\frac{40}{N+2}-\frac{32}{N+3}-\frac{8}{N+1}-\frac{4}{(N+1)^2}-\frac{36}{(N+2)^2}+\frac{24}{N+1}-\frac{24}{N+2}]$$

$$=a^2[\frac{16}{N+1}+\frac{16}{N+2}-\frac{32}{N+3}-\frac{4}{(N+1)^2}-\frac{36}{(N+2)^2}]$$

which is Eq.(7.27). **Q.E.D.**