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Using the linear model $\underline{y} = X\underline{\beta} + \underline{\varepsilon}$, where $\underline{\varepsilon} \sim N(0,\sigma^2 I)$, one and two-sided confidence bands are developed for percentiles, $y(p,\underline{x}) = \underline{x}'\underline{\beta} + z(p)\sigma$, where $z(p)$ is the 100pth percentile of the standard normal distribution. The Scheffé simultaneous confidence principle is used to form intervals that are simultaneous in $\underline{x}$ and p and intervals simultaneous in $\underline{x}$ with p fixed. The bands are similar to those considered by Steinhorst and Bowden (1971) and Turner and Bowden (1977, 1979). The new bands have the property that the band widths are proportional to a consistent estimator for the standard error of the estimator for percentiles $\hat{y}(p,\underline{x}) = \underline{x}'\hat{\underline{\beta}} + z(p)\hat{\sigma}$. The Steinhorst and Bowden (1971) and Turner and Bowden (1977, 1979) bands do not possess this property. The new procedures are found to perform well with respect to band width and asymptotic power. Following the approach of Lieberman and Miller (1963), simultaneous tolerance intervals are constructed from confidence bands for percentiles. The confidence band procedures for percentiles are extended to censored samples using an effective finite degrees of freedom. Simulation results indicate these approximate procedures have accurate confidence levels.

Confidence bands for percentiles of a single normal distribution over restricted intervals of p are developed using asymptotic results. These bands are adapted from the Wynn and Bloomfield (1971) bands for the regression line over a finite x-interval. Accuracy of the confidence levels for bands over restricted intervals of p is investigated by simulation of complete and censored samples.

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CONFIDENCE BANDS FOR PERCENTILES IN THE LINEAR REGRESSION MODEL

I. INTRODUCTION

The general linear regression model with normally distributed residuals is assumed for many statistical analyses. In some applications it is desirable to construct confidence bands for the population regression function. Such bands can be constructed using the Scheffé simultaneous confidence principal, see for example Miller (1981, p 110-114). Confidence bands for a regression function can be generalized by constructing bands for percentiles of the regression model. Consider the linear model $\underline{y} = \underline{X}\underline{\beta} + \underline{\varepsilon}$, where $\underline{y}$ is an nx1 column vector, $\underline{\beta} = (\beta_0, \beta_1, \ldots, \beta_{q-1})'$ a vector of unknown parameters, $\underline{X}$ a nxq matrix of rank q, and $\underline{\varepsilon}$ a nx1 vector of mutually independent normal random variables with mean zero and variance σ^2. Define $z(p)$ to be the 100p*th* percentile of the standard normal distribution $\Phi(\cdot)$, that is $p = \Phi(z(p))$. Then the 100p*th* percentile of a linear regression is $y(p,\underline{x}) = \underline{x}'\underline{\beta} + z(p)\sigma$. The estimators of $y(p,\underline{x})$ considered here are of the form

$$\hat{y}(p,\underline{x}) = \underline{x}'\hat{\underline{\beta}} + az(p)S , \tag{1.1}$$

where $\hat{\underline{\beta}}$ is the least squares estimator for $\underline{\beta}$, S^2 is the residual mean square estimator for σ^2, and the constant a depends on the particular band procedure. Two distinct types of confidence bands for percentiles are considered: bands simultaneous in $\underline{x}$ and p, and bands simultaneous in $\underline{x}$ with p fixed. In each case the bands have bounds

$$\hat{y}(p,\underline{x}) \pm cS(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x} + b\, z^2(p))^{\frac{1}{2}} \tag{1.2}$$

where the constant b depends on the band procedure and the constant c depends on the constants a and b. For the single normal distribution model $y = \mu + \varepsilon$ $(\beta_0 = \mu)$, confidence intervals for percentiles, $y(p) = \mu + z(p)\sigma$, simultaneous in p can be used to form confidence bands for the normal distribution function.

For the case of simple linear regression, Easterling (1969) discussed the problem of inverse prediction for the value of x at which a specified value of y, $y(p,x)$, is the 100p<u>th</u> percentile of the conditional distribution of y given x. Steinhorst and Bowden (1971) considered the problem of finding simultaneous inverse prediction intervals for the x values associated with several different y-values for the 100p<u>th</u> percentile. A solution was obtained by constructing two-sided confidence bands for the 100p<u>th</u> percentile of the conditional distribution of y given x. For bands simultaneous in <u>x</u> and p Steinhorst and Bowden used $a = 1$ in (1.1) and $b = 1$ in (1.2). In the p fixed situation they used $a = 1$ and $b = 0$ in (1.1) and (1.2) respectively. Constants c were then numerically evaluated in each case for the sample sizes $n = 10$ and 15 and specified confidence levels. Turner and Bowden (1977) generalized the procedure of Steinhorst and Bowden in the p fixed case by considering several choices of the constant a used in the percentile estimator (1.1). In a closely related problem Lieberman and Miller (1963) constructed simultaneous tolerance intervals for the regression model from confidence bands for percentiles. The two values of the constant b, 0 and 1, for the p fixed and for all p cases respectively were used to form these intervals.

One-sided (upper or lower) confidence bands for regression lines are also useful. For example, consider a simple linear regression line with negative slope calculated from a bivariate sample with x = moisture content of wood used in plywood construction and y = strength of plywood sheet. A concern in quality control is that of specifying a strength level such that at least 100p % of the plywood sheets for each moisture content level x exceed this level with high probability. An upper band on the regression line then is not particularly useful. Using only the lower band of a two-sided set is necessarily conservative. A one-sided band for the fiftieth percentile can be calculated following either Bohrer and Francis (1972) or Hochberg and Quade (1975). If a band for some percentile other than the fiftieth is desired other methods are required. Turner and Bowden (1979) constructed sharp one-sided bands for percentiles similar in form to their two-sided procedures. They considered only the fixed p case. One-sided bands for percentiles correspond to one-sided tolerance bands.

The bands for percentiles formed here use $b = \frac{1}{(2\nu)}$ in (1.2), where ν = degrees of freedom associated with S. This value of b was chosen by using large sample results to approximate the standard error of $\hat{y}(p,\underline{x})$. The same standard error term is used for bands simultaneous in $\underline{x}$ and p, and bands simultaneous in $\underline{x}$ with p fixed. Turner and Bowden (1979) mention that their standard error term for $\hat{y}(p,\underline{x})$ used in the formation of confidence bands for percentiles could possibly be improved. Using the values $b = 0$ and $b = 1$ in (1.2) for the p-fixed and for all p case respectively does not account for the variability in the estimator S. A second order Taylor series expansion is used to find a constant a that makes $\hat{y}(p,\underline{x})$ in (1.1) a nearly unbiased estimator of $y(p,\underline{x})$ that is easily calculated. Coefficients c in (1.2) for one and two-sided confidence bands for percentiles are evaluated in the p fixed and for all p cases. The model for a single normal distribution is treated as a special case throughout. The use of confidence bands for percentiles of linear regression models in forming simultaneous tolerance intervals is examined and compared to the procedures given by Lieberman and Miller. The asymptotic efficiencies of the procedure described here and those given by Steinhorst and Bowden are compared.

Censored data arise in many applications. For example, consider the following situation. An environmental study is conducted to analyze the concentration of a pollutant in a lake. Water samples are taken from various locations. For a given sample unit, if the pollutant is in concentration greater than some nominal level it is easily measured with sufficient accuracy. However, if the concentration is below this nominal level, a different more costly procedure is required to determine concentration. In an effort to avoid the expensive procedure, response values which are not measured by the easy method are recorded only as having concentration below the nominal level. These censored values contain information about the population that should be included in the statistical analysis. In Chapter III, procedures for forming one and two-sided confidence bands for percentiles from censored samples are proposed and examined through simulation. Effective degrees of freedom corresponding to the population standard deviation estimator are used.

Confidence bands for straight line regressions over finite or semi-finite intervals of the independent variable were given by Wynn and Bloomfield (1971). In Chapter IV, confidence bands for percentiles of a single normal distribution over a finite range of p are developed using the asymptotic normality of maximum likelihood estimators for the mean and standard deviation. The performance of this procedure for complete and censored samples is examined using simulation. Some suggestions are made for other possible finite range bands.

II. SIMULTANEOUS PROCEDURES FOR PERCENTILES

II.1 Preliminaries

In this chapter simultaneous procedures for percentiles of the linear regression model are developed. Attention is restricted to hyberbolic Scheffé type (1.2) procedures as opposed to the linear segment methods of Gafarian (1964) or Graybill and Bowden (1967). One and two-sided simultaneous confidence intervals for percentiles are formed and their efficiencies compared to the procedures given by Steinhorst and Bowden (1971) and Turner and Bowden (1977, 1979). Following the technique of Lieberman and Miller (1963) simultaneous tolerance intervals are derived and compared to their results.

In the linear model $\underline{y} = \underline{X}\underline{\beta} + \underline{\varepsilon}$, let $\underline{\beta} = (\beta_0, \beta_1, \ldots, \beta_{q-1})'$ and $\underline{X} = [\underline{1}, \underline{X}_1]$ where the first column is unity everywhere and the remaining columns are mean adjusted, i.e., $\underline{X}_1'\underline{1} = \underline{0}$. The procedures developed in this chapter will be based on the well known unbiased estimators

$$\hat{\underline{\beta}} = (\underline{X}'\underline{X})^{-1}\underline{X}'\underline{y} = \begin{pmatrix} \bar{y} \\ (\underline{X}_1'\underline{X}_1)^{-1}\underline{X}_1'\underline{y} \end{pmatrix}$$

and

$$S^2 = (\underline{y}'\underline{y} - \hat{\underline{\beta}}'\underline{X}'\underline{y})/(n - q) .$$

The vector $\hat{\underline{\beta}}$ is distributed as a q dimensional multivariate normal random vector with expectation $\underline{\beta}$ and covariance matrix

$$\sigma^2(\underline{X}'\underline{X})^{-1} = \sigma^2 \begin{bmatrix} 1/n & 0 \\ \underline{0} & (\underline{X}_1'\underline{X}_1)^{-1} \end{bmatrix} .$$

The random variable $(n-q)S^2/\sigma^2$ is independent of $\hat{\underline{\beta}}$ and is chi-square distributed with $\nu = n-q$ degrees of freedom.

II.2 Two-sided Bands for Percentiles

Two-sided confidence bands for $y(p,\underline{x}) = \underline{x}'\underline{\beta} + z(p)\sigma$, the 100pth percentile of the linear regression model, are formed in this section. Two types of bands are developed: bands which are simultaneous in $\underline{x}$ and $\underline{p}$, and bands which are simultaneous in $\underline{x}$ with p fixed. Bands simultaneous in $\underline{x}$ and p have the property that one can make statements at various $\underline{x}$ values and not necessarily the same p but still have one overall confidence coefficient. For instance, the percentile desired may decrease as the $\underline{x}$ value gets further from the mean. When $\underline{x}$ and p are both fixed, the noncentral t-distribution has been used to construct confidence intervals for percentiles (Kabe (1976) or Owen (1968)).

The estimator $\hat{y}(p,\underline{x}) = \underline{x}'\hat{\underline{\beta}} + z(p)S$ was used by Steinhorst and Bowden (1971) to construct simultaneous bands for percentiles, $y(p,\underline{x})$. They derived constants Z_α and $\bar{Z}_\alpha$ such that with $1-\alpha$ level of confidence the following statements hold:

$$y(p,\underline{x}) \;\varepsilon [\hat{y}(p,\underline{x}) - Z_\alpha S(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x})^{\frac{1}{2}},\ \hat{y}(p,\underline{x}) + Z_\alpha S(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x})^{\frac{1}{2}}]$$

for fixed p and

$$\hat{y}(p,\underline{x}) \;\varepsilon [\hat{y}(p,\underline{x}) - \bar{Z}_\alpha S(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x} + z^2(p))^{\frac{1}{2}}, \hat{y}(p,\underline{x}) + \bar{Z}_\alpha S(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x}+z^2(p))^{\frac{1}{2}}]$$

for all p. Turner and Bowden (1977) generalized the Steinhorst and Bowden bands by considering estimators of the form $\underline{x}'\hat{\underline{\beta}} + az(p)S$. They examined four different constants a, one of which makes aS the minimum variance unbiased estimator (UMVUE) of σ, i.e.,

$$a = [(n-q)/2]^{\frac{1}{2}}\, \Gamma[(n-q)/2]/\Gamma[(n-q+1)/2] \,, \tag{2.1}$$

where $\Gamma(\cdot)$ is the gamma function. Although Turner and Bowden (1977) concentrated their attention on the fixed p case, their estimators could be applied to the for all p situation. The bands for various constants a formed by Turner and Bowden (1977,1979) crossed one another suggesting that a "best" band has not been identified (and may not exist).

For $1-\alpha$ level confidence bands for $y(p,\underline{x})$ the constants $c_1 = c_1(\alpha,\nu,q)$ and $c_2 = c_2(\alpha,\nu,q,p)$ are derived here which satisfy

$$1-\alpha = \Pr\{|y(p,\underline{x}) - \hat{y}(p,\underline{x})| \leq c_2 S\left(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x} + \frac{z^2(p)}{2\nu}\right)^{\frac{1}{2}} \tag{2.2}$$

$$\text{for all } \underline{x} \in \mathscr{L}\}$$

for the fixed p case, and for the all p case

$$1-\alpha = \Pr\{|y(p,\underline{x}) - \hat{y}(p,\underline{x})| \leq c_1 S\left(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x} + \frac{z^2(p)}{2\nu}\right)^{\frac{1}{2}} \tag{2.3}$$

$$\text{for all } \underline{x} \in \mathscr{L} \text{ and } 0 < p < 1\}$$

where $\hat{\sigma} = aS$ with $a = 4\nu/(4\nu-1)$ in (1.1), $\nu =$ degrees of freedom, and $\mathscr{L} = \{(1,\underline{x}_1')' : \underline{x}_1 \varepsilon\, \mathbb{R}^{q-1}\}$. The estimator $\hat{\sigma}$ was chosen since the second order Taylor series approximation of $\sqrt{S^2}$ around $\sqrt{\sigma^2}$ gives the expectation of S approximately equal to σ/a. Then the expectation of $\hat{\sigma}$ is approximately σ. It should be noted that the unbiased estimator aS for σ used by Turner and Bowden could be used in place of $\hat{\sigma}$ in the derivations that follow. The term $\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x} + z^2(p)/2\nu$ was chosen by considering large sample results.

The asymptotic variance of S^2 is equal to $4\sigma^2$ times the asymptotic variance of S. Recall that the variance of S^2 for finite samples is $2\sigma^4/\nu$. Then an estimator for the variance of S is $S^2/2\nu$. This implies that an estimator of the variance of $\hat{y}(p,\underline{x}) = \underline{x}'\hat{\underline{\beta}} + z(p)\hat{\sigma}$ is $S^2(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x} + z^2(p)/2\nu)$. This estimator yields consistent bands when $\hat{\underline{\beta}}$ is consistent for $\underline{\beta}$. The term $(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x} + z^2(p))$ used by Steinhorst and Bowden (1971) has the positive lower bound $z^2(p)$ when $p \neq \frac{1}{2}$ for all sample sizes and thus does not yield consistent bands. Consequences of this lack of consistency are examined in Section II.5. In the special case of the linear model $y = \mu + \varepsilon$, that is, $q = 1$, the estimators $\underline{x}'\hat{\underline{\beta}} + z(p)\hat{\sigma}$ and $S(x'(X'X)^{-1}x + z^2(p)/2\nu)^{\frac{1}{2}}$ become $\bar{y} + z(p)\hat{\sigma}$ and $S(\frac{1}{n} + z^2(p)/2\nu)^{\frac{1}{2}}$ respectively. The bands given in (2.3) are in this case two-sided confidence bands for $\mu + z(p)\sigma$ for $0 < p < 1$. These bands can be used to form confidence bands for the cumulative distribution function (c.d.f.) of y.

The derivation of coefficients c_1 for two-sided bands in $\underline{x}$ and p requires the following lemma which is a form of the Cauchy-Schwarz inequality. The lemma and its proof are taken from Miller (1981, p. 63) and are included here only for completeness.

Lemma I: Let $\underline{t}$ and $\underline{L}$ be mx1 vectors and c a positive constant. Then $|\underline{t}'\underline{L}| \leq c(\underline{t}'\underline{t})^{\frac{1}{2}}$ for all vectors $\underline{t}$ if and only if $\underline{L}'\underline{L} \leq c^2$.

Proof If: By the Cauchy-Schwarz inequality $|\underline{t}'\underline{L}| \leq (\underline{t}'\underline{t})^{\frac{1}{2}}(\underline{L}'\underline{L})^{\frac{1}{2}}$. The inequality $\underline{L}'\underline{L} \leq c^2$ implies $|\underline{t}'\underline{L}| \leq c(\underline{t}'\underline{t})^{\frac{1}{2}}$.

Only If: Choose $\underline{t} = \underline{L}$. Then $|\underline{t}'\underline{L}| \leq c(\underline{t}'\underline{t})^{\frac{1}{2}}$ becomes $|\underline{L}'\underline{L}| \leq c(\underline{L}'\underline{L})^{\frac{1}{2}}$ which implies $\underline{L}'\underline{L} \leq c^2$. □

The application of the Lemma I requires a transformation of variables. Following the notation of Turner and Bowden (1977) let

$$\underline{W}_1' = \left(\sqrt{n}\,\frac{(\beta_0-\hat{\beta}_0)}{\sigma} \quad , \quad \frac{(\underline{\beta}^*-\hat{\underline{\beta}}^*)'Q_1'}{\sigma} \quad , \sqrt{2\nu}\frac{(\sigma-\hat{\sigma})}{\sigma}\right) \tag{2.4}$$

$$\underline{r}_1' = \left(\frac{1}{\sqrt{n}} \quad , \underline{x}_1'\underline{Q}_1^{-1} \; , \; \frac{z(p)}{\sqrt{2\nu}}\right), \quad \text{and} \quad U = \frac{S^2}{\sigma^2} \qquad ,$$

where $\underline{\beta}^* = (\beta_1, \beta_2, \ldots, \beta_{q-1})'$, $\hat{\underline{\beta}}^* = (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_{q-1})'$, and $\underline{Q}_1$ is a nonsingular matrix such that $\underline{Q}_1'\underline{Q}_1 = \underline{X}_1'\underline{X}_1$. Using these transformed variables we may write (2.3) as

$$1-\alpha = \Pr\{|\underline{r}_1'\underline{W}_1| \le c_1(\underline{r}_1'\underline{r}_1 U)^{\frac{1}{2}} \text{ for all } \underline{r}_1 \in \mathscr{G}_1\} , \tag{2.5}$$

where

$$\mathscr{G}_1 = \{(n^{-\frac{1}{2}}, \underline{t}_1', t_2') : \underline{t}_1 \in \mathbb{R}^{q-1} \text{ and } t_2 \in \mathbb{R}^1\} .$$

Applying Lemma I with $\underline{L} = \underline{W}_1/U^{\frac{1}{2}}$ and $\underline{r} = \underline{t}$ gives $1-\alpha = \Pr\{\underline{W}_1'\underline{W}_1 \le c_1^2 U\}$.

The coefficients c_1 then depend on the distribution of $\underline{W}_1$ and U. Recall that $\hat{\underline{\beta}}$ is distributed as a q dimensional multivariate normal random vector with expectation $\underline{\beta}$ and covariance matrix $\sigma^2(\underline{X}'\underline{X})^{-1}$. Then it can be seen from (2.4) that $\underline{W}_1'\underline{W}_1 = g_1 + 2\nu(1-a\sqrt{U})^2$, where g_1 is a chi-square distributed random variable with q degrees of freedom. Conditioning on $U=u$, (2.5) can be written as

$$1-\alpha = \Pr\{\underline{W}_1'\underline{W}_1 \le c_1^2 U\}$$

$$= \int_0^\infty \Pr[\underline{W}_1'\underline{W}_1 \le c_1^2 u \,|\, U=u] f_U(u)du$$

$$= \int_0^\infty \Pr[g_1 \le c_1^2 u - 2\nu(1-a\sqrt{u})^2 \,|\, U=u] f_U(u)du , \tag{2.6}$$

where $f_U(u)$ is the probability density function of a chi-square random variable divided by its degrees of freedom ν. A FORTRAN IV program was written to numerically evaluate the c_1 coefficients. The integral (2.6) was approximated by dividing $f_U(u)$ into intervals and applying Simpson's one-third rule. The conditional probability in the integral (2.6) was evaluated by the IMSL subroutine MDCH for each interval. The coefficients c_1 were then calculated iteratively by a zero finding subroutine "ROOTS" given by Lindstrom and Dodge (1981). The c_1 values are presented in Table 1 for various values of ν, α, and q. Asymptotically the Scheffé-S method is applicable since the limiting distribution of $\sqrt{2\nu}(\sigma-\hat{\sigma})/\sigma$ is the standard normal. Then for $\nu = \infty$ the coefficients are simply evaluated as $c_1 = [(q+1)F^{\alpha}_{q+1,\infty}]^{\frac{1}{2}}$. More extensive tables of coefficients c_1 are given in Appendix A for $q = 1$ or 2 parameters.

Consider now bands for percentiles $y(p,\underline{x})$ in $\underline{x}$ with p fixed characterized by Equation (2.2). An approach analogous to that given for the all p case is used to derive the coefficients c_2. Let

$$\underline{W}_2 = \begin{bmatrix} \left(1 + \frac{nz^2(p)}{2\nu}\right)^{-\frac{1}{2}} \left\{\sqrt{n}\,\frac{(\beta_0-\hat{\beta}_0)}{\sigma} + \sqrt{n}z(p)\,\frac{(\sigma-\hat{\sigma})}{\sigma}\right\} \\ Q_1\,\frac{(\underline{\beta}^*-\hat{\underline{\beta}}^*)}{\sigma} \end{bmatrix} \tag{2.7}$$

and

$$\underline{r}_2' = \left(1,\ \left(\frac{1}{n} + \frac{z^2(p)}{2\nu}\right)^{-\frac{1}{2}} \underline{x}_1' \underline{Q}_1^{-1}\right)$$

Then write (2.2) as

$$1-\alpha = \Pr\{|\underline{r}_2'\underline{W}_2| \le c_2(\underline{r}_2'\underline{r}_2 U)^{\frac{1}{2}} \text{ for all } \underline{r}_2 \in \mathcal{G}_2\},$$

Table 1. Coefficients $c_1(\alpha,\nu,q)$ for two-sided bands in $\underline{x}$ and p

	q	Degrees of freedom ν 10	20	40	60	∞
α = .10	1	2.497	2.294	2.213	2.191	2.140
	2	2.950	2.702	2.592	2.560	2.498
	3	3.320	3.035	2.904	2.863	2.786
	4	3.644	3.325	3.175	3.127	3.041
α = .05	1	3.158	2.777	2.599	2.543	2.450
	2	3.620	3.189	2.982	2.916	2.793
	3	4.007	3.530	3.298	3.223	3.079
	4	4.349	3.828	3.573	3.489	3.324
α = .01	1	4.846	3.947	3.500	3.351	3.040
	2	5.326	4.351	3.875	3.711	3.368
	3	5.755	4.697	4.190	4.013	3.644
	4	6.129	5.005	4.468	4.281	3.886

where $\mathscr{G}_2 = \{(1,\underline{t}_1')': \underline{t}_1 \in \mathbb{R}^{q-1}\}$. Applying Lemma I with $\underline{L} = \underline{W}_2/U^{\frac{1}{2}}$ and $\underline{t} = \underline{r}_2$ gives

$$1-\alpha = \Pr\{\underline{W}_2'\underline{W}_2 \leq c_2^2 U\}$$

$$= \int_0^\infty \Pr\{\underline{W}_2'\underline{W}_2 \leq c_2^2 u | U=u\} f_U(u)du \tag{2.8}$$

The conditional distribution of $\underline{W}_2$, given $U=u$, is multivariate normal with mean vector having first element
$\eta_1 = (\frac{1}{n} + \frac{z(p)}{2\nu})^{-\frac{1}{2}}(1 - a\sqrt{u})z(p)$ and zeros elsewhere and covariance matrix

$$\begin{bmatrix} (1 + \frac{nz^2(p)}{2\nu})^{-1} & \underline{0} \\ & \\ \underline{0} & \underline{I}_{q-1} \end{bmatrix} .$$

and the probability density function $f_U(u)$ is that of a chi-square random variable divided by its degrees of freedom ν. The conditional probability can be written as

$$\Pr\{\underline{W}_2'\underline{W}_2 \leq c^2 u | U=u\} = \Pr\{(1 + \frac{nz^2(p)}{2\nu})^{-1} g_2 + g_3 \leq c_2^2 u | U=u\} , \tag{2.9}$$

where g_2 is a noncentral chi-square random variable with one degree of freedom and noncentrality parameter $n(1-a\sqrt{u})z(p)$, and g_3 is a central chi-square random variable with $q-1$ degrees of freedom. The conditional probability (2.9) can be calculated as an infinite sum of cumulative probabilities of central chi-square random variables, see for example Johnson and Kotz (1970, p. 170). The integral (2.8) can now be numerically evaluated similar to that described in the for all

p case. The coefficients c_2 are calculated only for the specific examples that follow because they depend on p in addition to α, ν, and q.

Steinhorst and Bowden (1971) and Turner and Bowden (1977) present an application of their band procedures to a hypothetical data set given by Lieberman and Miller (1963). Steinhorst and Bowden also construct an inverse prediction interval for an example presented by Easterling (1969). Both of these examples are considered for comparison. Kanofsky (1968) describes a confidence region for (μ,σ) that can be used to form linear segment bands for $\mu + z(p)\sigma$. This procedure is compared to the bands resulting from the special case of $q = 1$ in the for all p situation.

Lieberman and Miller consider the simple linear relationship $E(y) = \beta_0 + \beta_1 x$, where y = rocket speed and x = the orfice size of the rocket fuel valve. They give a hypothetical data set with sample values: $n = 15$, $\bar{x} = 1.3531$, and $\Sigma(x_i - \bar{x})^2 = 0.011966$. The band procedures to be compared depend on the choice of the constants a and b in (1.1) and (1.2). Such procedures are identified in the for all $\underline{x}$ and fixed p case as

AVx the band based on the asymptotic variance $(b = 1/(2\nu))$ of $\hat{y}(p,\underline{x})$ with $a = 4\nu/(4\nu-1)$,

SBx the Steinhorst and Bowden band $(a = 1, b = 0)$,

TBUx the Turner and Bowden U band using a such that $E(aS) = \sigma$ and $b = 0$,

TBEx the Turner and Bowden E band with $a = \left[\frac{n-2}{2}\right]^{1/2} \Gamma[(n-3)/2]/\Gamma[(n-2)/2]$ and $b = 0$ based on a certain expectation, (see Turner and Bowden 1977, equation 5.2), and for the for all $\underline{x}$ and all p case as

AVxp the band based on the asymptotic variance $(b = 1/(2\nu))$ of $\hat{y}(p,\underline{x})$ with $a = 4\nu/(4\nu-1)$, and

SBxp the Steinhorst and Bowden band $(a = 1, b = 1)$.

Corresponding to (1.2) let

$$D = 2c(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x} + b\, z^2(p))^{\frac{1}{2}}$$

so that DS is the width of a band. The ratio of any two band widths is also the ratio of expected band widths. This can be seen by noting that for specified x and p , the only remaining random term is the factor S which cancels in the ratio. Table 2 presents D_{SBxp} , D_{AVxp} , and their ratio for $\alpha = .05, .10$; $p = .95$, $.75$; and various values of $|x-\bar{x}|$. For SBxp the coefficients c are 2.793 and 2.387 for $\alpha = .05$ and $\alpha = .10$ respectively. The coefficients c_1 of AVxp are given in Appendix A . The bands AVxp are, for this example, uniformly narrower than those given by Steinhorst and Bowden. Table 3 gives D_{AVx} , D_{XBx} , D_{TBUx} , and D_{TBEx} for $\alpha = .10, .01$; $p = 75, .95$; and various values of $|x-\bar{x}|$. The coefficients c_2 for these cases are $c_2(.10, 13, 2, .75) = 2.3500$ and $c_2(.01, 13, 2, .95) = 4.7182$. The coefficients c of the procedures SBx, TBUx, and TBEx are 2.506, 2.496, and 2.483 for $\alpha = .10$ and $p = .75$ and are 6.164, 6.048, and 5.789 for $\alpha = .01$ and $p = .95$ respectively. For this example, the band derived here is wider than the others for x values near $\bar{x}$ and is narrower than the others for $|x-\bar{x}| \geq .036$ when $\alpha = .10$, $p = .75$ and for $|x-\bar{x}| \geq .042$ when $\alpha = .01$, $p = .95$. The values .036 and .042 are 1.23 and 1.44 standard deviation, S_x, from zero respectively. As the sample size

Table 2. Band widths D and their ratio for the SBxp and AVxp procedures

p	α	$\lvert x-\bar{x}\rvert$	D_{SBxp}	D_{AVxp}	$\frac{D_{AVxp}}{D_{SBxp}}$
.75	.10	.000	3.451	1.644	.476
		.006	3.461	1.673	.483
		.012	3.490	1.757	.503
		.018	3.539	1.890	.534
		.024	3.606	2.061	.572
		.030	3.691	2.262	.613
		.036	3.792	2.486	.656
		.042	3.908	2.727	.698
		.048	4.037	2.980	.738
		.054	4.179	3.244	.776
		.060	4.332	3.516	.812
.95	.05	.000	6.583	2.344	.356
		.006	6.591	2.374	.360
		.012	6.612	2.461	.372
		.018	6.647	2.600	.391
		.024	6.697	2.783	.416
		.030	6.759	3.002	.444
		.036	6.835	3.250	.475
		.042	6.924	3.520	.508
		.048	7.025	3.808	.542
		.054	7.138	4.110	.576
		.060	7.261	4.423	.609

Table 3. Band widths D for bands in x with p fixed

p	α	$\lvert x-\bar{x}\rvert$	D_{AVx}	D_{SBx}	D_{TBUx}	D_{TBEx}
.75	.10	.000	1.364	1.294	1.289	1.282
		.006	1.388	1.323	1.318	1.311
		.012	1.458	1.406	1.401	1.393
		.018	1.568	1.535	1.529	1.520
		.024	1.710	1.698	1.692	1.683
		.030	1.876	1.888	1.880	1.871*
		.036	2.062	2.097	2.088	2.077
		.042	2.262	2.319	2.310	2.298
		.048	2.472	2.552	2.542	2.528
		.054	2.691	2.793	2.781	2.767
		.060	2.916	3.039	3.027	3.011
		average	1.979	1.995	1.987	1.976
.95	.01	.000	3.899	3.236	3.140	3.007
		.006	3.933	3.272	3.210	3.074
		.012	4.034	3.477	3.412	3.267
		.018	4.197	3.795	3.724	3.565
		.024	4.415	4.200	4.121	3.946
		.030	4.680	4.669	4.581	4.386
		.036	4.985	5.185	5.087	4.871*
		.042	5.323	5.735	5.627	5.388
		.048	5.688	6.310	6.192	5.929
		.054	6.075	6.905	6.775	6.487
		.060	6.480	7.514	7.373	7.059
		average	4.882	4.933	4.840	4.634

*The dashed line indicates where AVx crosses the other bands.

increases the point where AVx crosses the other bands will be closer to $\bar{x}$ since $z^2(p)/2\nu$ decreases.

For a given sample size n the band width of the SBxp procedure is as small or smaller than the AVxp band width if

$$\bar{z}_\alpha \left(\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_i (x_i-\bar{x})^2} + z^2(p)\right)^{\frac{1}{2}} \leq c_1 \left(\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_i (x_i-\bar{x})^2} \quad \frac{z^2(p)}{2\nu}\right)^{\frac{1}{2}} \qquad (2.10)$$

Let $\xi = (x-\bar{x})/S_x$, where $S_x^2 = \sum_i (x_i-\bar{x})^2/(n-1)$. Then for the orifice-speed example with $n = 15$ and $\alpha = .10$, (2.10) reduces to

$$- 0.155 - 0.166\xi^2 + 7.491\, z^2(p) \leq 0 . \qquad (2.11)$$

Inequality (2.11) defines a region in two space bounded by the hyperbola under equality. The shaded area in Figure 1 shows the region for which the SBxp band width is less than or equal to the band width of AVxp for $n=15$ and $\alpha = .10$. Note that in this example the SBxp bands are narrower only for rather extreme cases where p is near 1/2 ($z(p)$ near 0) or $|\xi| \geq 3$.

Similarly, the band width of AVx is as small or smaller than the width of SBx for sample size n if

$$c_2 \cdot \left(\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_i (x_i-\bar{x})^2} + \frac{z^2(p)}{2\nu}\right)^{\frac{1}{2}} \leq Z_\alpha \left(\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_i (x_i-\bar{x})^2}\right)^{\frac{1}{2}} . \qquad (2.12)$$

For the orifice-speed example, with $n=15$, $\alpha = .10$, and $p = .75$, inequality (2.12) reduces to

$$.230 - .054\xi^2 \leq 0 ,$$

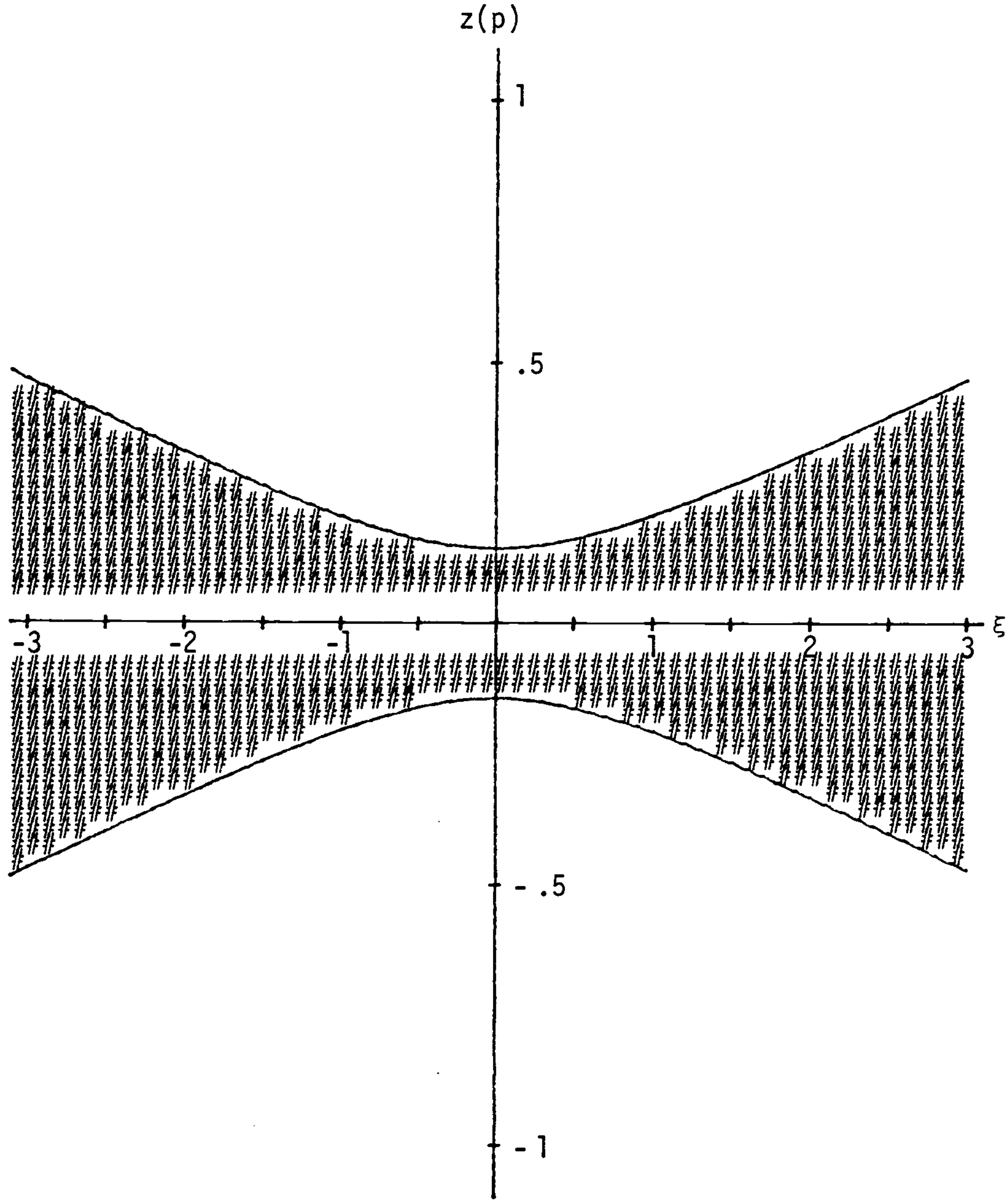

Figure 1. The region where SBxp is narrower than AVxp when $n = 15$ and $\alpha = 10$ $\left(\xi = \frac{x - \bar{x}}{S_x} \right)$.

which implies that the AVxp band width is smaller than the SBx band width when $|\xi| \geq 2.06$. For n=15, α = .01, and p = .95 the AVxp band width is smaller than the SBx band width when $|\xi| \geq 1.062$.

Easterling (1969) considers the problem of inverse prediction of the x for which a given y value, y(p,x), is the 100pth percentile of the distribution of y given x. That is, a confidence interval is constructed for the x such that $y(p,x) = \beta_0 + \beta_1 x + z(p)\sigma$. Easterling uses an example with sample quantities; n = 10, $\hat{\beta}_0 = 9$, $\hat{\beta}_1 = 2$, S = 1.225, and $\Sigma(x_i - \bar{x})^2 = 82.5$. The simultaneous inverse prediction intervals for arbitrary choices (y,p) are found using the AV_{xp} method by solving

$$|y - \hat{\beta}_0 - \hat{\beta}_1 x_0 - z(p)\hat{\sigma}| \leq c_1 S\left(\frac{1}{n} + \frac{(x_0' - \bar{x})^2}{\Sigma(x_i - \bar{x})^2} + \frac{z^2(p)}{2\nu}\right)^{\frac{1}{2}} \tag{2.13}$$

for x_0. For this example, Steinhorst and Bowden found the 90 percent inverse prediction interval [-2.21, 3.23] for $y_{.95} = 12$. The AVxp procedure yields the narrower interval [-0.52, 1.49]. For p-fixed Steinhorst and Bowden obtain the interval [-0.08, 1.10] compared to [-0.36, 1.31] found using AVx (c_2 = (.10, 8, 2, .95) = 2.57475). The latter interval is wider because x_0 is near the mean $\bar{x}$ and the sample size is small. It should be noted that a confidence interval for p, the proportion of the conditional distribution of y given x falling below a specified value, y, can be found by first solving (2.13) for z(p).

Kanofsky (1968) considers a confidence band for the single normal c.d.f. constructed from a trapezoidal confidence region R_1 for (μ,σ)

$$R_1 = \{(\mu,\sigma)\,|\,\hat{\sigma}_1 \le \sigma \le \hat{\sigma}_2\,,\ \bar{y} - Z_{\frac{1}{2}(1+\gamma_1)}\sigma/\sqrt{n} \le \mu \le \bar{y} + Z_{\frac{1}{2}(1+\gamma_1)}\sigma/\sqrt{n}\}\,,$$

where $\hat{\sigma}_1 = \left(\dfrac{\nu S^2}{\chi^2_{\nu,\lambda_1}}\right)^{\frac{1}{2}}$, $\hat{\sigma}_2 = \left(\dfrac{\nu S^2}{\chi^2_{\nu,\lambda_2}}\right)^{\frac{1}{2}}$, $Z_{\frac{1}{2}(1+\gamma_1)}$ is the $\frac{1}{2}(1+\gamma_1)$ percentile point of a standard normal distribution, and χ^2_{ν,λ_1} and χ^2_{ν,λ_2} are the $\lambda_1 = \frac{1}{2}(1+\gamma_2)$ and $\lambda_2 = \frac{1}{2}(1-\gamma_2)$ percentile points respectively of a chi-square random variable with $\nu = n-1$ degrees of freedom. The confidence band is then formed as the union of all graphs of the normal c.d.f.'s with $(\mu,\sigma) \in R_1$. Alternatively, one may find the region, say $M(R_1)$, in the plane formed by the union of all graphs of $\mu + z(p)\sigma$ for all real $z(p)$ and (μ,σ) in R_1. Then the region $M(R_1)$ is a $(1-\alpha) = \gamma_1 \cdot \gamma_2$ confidence band for $\mu + z(p)\sigma$ simultaneous in p. Given the set R_1 it can be shown that the boundaries of $M(R_1)$ are

$$\bar{y} - Z_{\frac{1}{2}(1+\gamma_1)}\,\hat{\sigma}_2/\sqrt{n} + z(p)\hat{\sigma}_2 \le y(p) \le \bar{y} + Z_{\frac{1}{2}(1+\gamma_1)}\,\hat{\sigma}_1/\sqrt{n} + Z(p)\hat{\sigma}_1$$

for $y < \bar{y}$, and

$$\bar{y} - Z_{\frac{1}{2}(1+\gamma_1)}\hat{\sigma}_1/\sqrt{n} + Z(p)\hat{\sigma}_1 \le y(p) \le \bar{y} + Z_{\frac{1}{2}(1+\gamma_1)}\hat{\sigma}_2/\sqrt{n} + Z(p)\hat{\sigma}_2$$

for $y \ge \bar{y}$.

Let $D_k = (Z_{\frac{1}{2}(1+\gamma_1)}/\sqrt{n})\,(d_1 + d_2) + Z(p)(d_2 - d_1)$,

where $d_1 = \left(\dfrac{\nu}{\chi^2_{\nu,\lambda_2}}\right)^{\frac{1}{2}}$ and $d_2 = \left(\dfrac{\nu}{\chi^2_{\nu,\lambda_2}}\right)^{\frac{1}{2}}$.

Then $S\,D_k$ is the band width constructed from the region R_1.

Let R_2 be the set of (μ,σ) such that

$$\mu + z(p)\sigma \in [\bar{y} + z(p)\hat{\sigma} - c_1 S(\tfrac{1}{n} + \tfrac{z^2(p)}{2\nu})^{\frac{1}{2}};\ \bar{y} + z(p)\hat{\sigma} + c_1 S(\tfrac{1}{n} + \tfrac{z^2(p)}{2\nu})^{\frac{1}{2}}]$$

for all p. Then R_2 is an ellipse formed by the supporting lines

$$\bar{y} + z(p)\hat{\sigma} \pm c_1 S\left(\frac{1}{n} + \frac{z^2(p)}{2\nu}\right)^{\frac{1}{2}} \text{ as } p \text{ varies over } [0,1].$$

Here $D_{AVp} = 2c_1\left(\frac{1}{n} + \frac{z^2(p)}{2\nu}\right)^{\frac{1}{2}}$. The ratio $\frac{D_{AVp}}{D_k}$ is equal to the ratio of expected band widths. The ratio of expected 95 percent confidence band widths are presented in Table 4. Since both band widths are symmetric about $p = .5$ only values of $p \geq .5$ are shown. Kanofsky takes $\gamma_1 = .9684$ and $\gamma_2 = .98$ so that $(1-\alpha) = \gamma_1 \cdot \gamma_2 = .95$. It is seen from Table 4 that the Kanofsky bands are narrower than the bands given here only for values of p near $.5$. The ratio of expected band widths is only one criterion among many measures of goodness. For example, Kanofsky uses the expected maximum width for comparisons, with width measured on the probability scale.

Table 4. Ratios, $\frac{D_{AVp}}{D_k}$, of expected 95% band widths for $\mu + z(p)\sigma$.

DF	p = .5	.55	.6	.75	.90	.95	.99
10	1.109	1.015	.944	.816	.770	.772	.792
15	1.122	1.027	.955	.824	.774	.775	.794
20	1.124	1.029	.956	.824	.773	.773	.791
30	1.124	1.030	.957	.824	.771	.771	.788
40	1.125	1.031	.958	.824	.771	.770	.786
60	1.126	1.031	.959	.825	.771	.769	.785

II.3. One-Sided Bands for Percentiles

If one needs only an upper (lower) band taking the corresponding upper (lower) boundary of a two-sided set is necessarily conservative. Bohrer and Francis (1972) constructed one-sided bands for the simple linear regression line $(p = 1/2)$ over intervals. They included as a special case the unrestricted interval $(-\infty, \infty)$. Hochberg and Quade (1975) considered the general problem of constructing one-sided confidence bands for linear regression functions $(p = 1/2)$. Turner and Bowden (1979) generalized one-sided bands to arbitrary fixed percentiles. Such bands have a boundary (+ or -) of the form (1.2) using $b = 0$. They also use the orifice-speed example to illustrate the one-sided confidence bands.

As in the two-sided bands case, $a = 4\nu/(4\nu-1)$ and $b = 1/2$ are used here for one-sided confidence bands in $\underline{x}$ and p or in $\underline{x}$ with p-fixed. The 100p<u>th</u> percentile of one normal population, $y(p) = \mu + z(p)\sigma$, is treated as a special case. Consider first one-sided (upper) bands simultaneous in $\underline{x}$ and p. Constants $c_3 = c_3(\alpha,\nu,q)$ are found such that

$$1-\alpha = \Pr\{y(p,\underline{x}) - \hat{y}(p,\underline{x}) \leq c_3 S[\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x} + \frac{z^2(p)}{2\nu}]^{\frac{1}{2}} \text{ for all } \underline{x}\varepsilon\mathscr{L} \text{ and } 0 \leq p \leq 1\} \tag{2.14}$$

is a $1-\alpha$ upper confidence band for $y(p,\underline{x})$. Apply the transformations

$$\underline{W}_3 = \begin{pmatrix} \underline{W}_{31} \\ W_{32} \end{pmatrix} = \begin{pmatrix} \underline{Q}_1(\underline{\beta}^* - \hat{\underline{\beta}}^*)/\sigma \\ \sqrt{2\nu}(\sigma-\hat{\sigma})/\sigma \end{pmatrix},$$

$$\underline{r}_3' = \left(\sqrt{n}\ \underline{x}_1 \underline{Q}_1^{-1}\ ,\ z(p)\left(\frac{n}{2\nu}\right)^{\frac{1}{2}}\right),\ \text{and}$$

$Z_3 = \sqrt{n}\,(\beta_0 - \hat{\beta}_0)/\sigma$. Then Z_3 has a standard normal distribution and $\underline{W}_{31}$ a q-1 variate normal distribution with mean vector zero and identity covariance matrix. The variables Z_3 and $\underline{W}_{31}$ are independent since $\hat{\underline{\beta}}$ and S^2 are mutually independent. Probability (2.14) can then be written as

$$1-\alpha = \Pr\left\{\frac{Z_3 + \underline{r}_3'\underline{W}_3}{(1 + \underline{r}_3'\underline{r}_3)^{\frac{1}{2}}} \leq c_3 U^{\frac{1}{2}} \text{ for all } \underline{r}_3 \varepsilon \mathbf{R}^q\right\}, \tag{2.15}$$

where $U = S^2/\sigma^2$. The following lemma and proof, taken from Hochberg and Quade, are included here for completeness.

Lemma II: Let $H_\xi = \left\{(Z,\underline{W}') : \frac{Z + \underline{r}'\underline{W}}{(1 + \underline{r}'\underline{r})^{\frac{1}{2}}} \leq \xi^{\frac{1}{2}} \text{ for all } \underline{r}\right\}$

and $J_\xi = \{(Z,\underline{W}'): Z > 0 \text{ and } Z^2 + \underline{W}'\underline{W} \leq \xi \text{ or } Z \leq 0 \text{ and } \underline{W}'\underline{W} \leq \xi\}$

then $H_\xi = J_\xi$ with probability one.

Proof: Z and $\underline{W}$ are finite and $Z \neq 0$ with probability one.

(i) If $(Z,\underline{W}') \in H_\xi$ and $Z > 0$ choose $\underline{r} = (1/Z)\underline{W}$.

Then $\frac{Z + \underline{r}'\underline{W}}{(1 + \underline{r}'\underline{r})^{\frac{1}{2}}} = (Z^2 + \underline{W}'\underline{W})^{\frac{1}{2}} \leq \xi^{\frac{1}{2}}$ which implies

$(Z,\underline{W}') \in J_\xi$. If $(Z,\underline{W}') \in H_\xi$ and $Z < 0$, choose

$r_d = d\underline{W}$ and let $d > 0$. Then $\frac{(Z + \underline{r}_d'\underline{W})}{(1 + \underline{r}_d'\underline{r}_d)^{\frac{1}{2}}} \leq \xi^{\frac{1}{2}}$

for all $d > 0$, which implies

$$(\underline{W}'\underline{W})^{\frac{1}{2}} = \lim_{d\to\infty} \frac{(Z + \underline{r}_d'\underline{W})}{(1 + \underline{r}_d'\underline{r}_d)^{\frac{1}{2}}} \leq \xi^{\frac{1}{2}},\ \text{so}\ (Z,\underline{W}') \in J_\xi.$$

(ii) If $(Z,\underline{W}') \varepsilon J_\xi$ and $Z > 0$ then by the Cauchy-Schwartz inequality

$$Z + \underline{r}'\underline{W} \leq |Z + \underline{r}'\underline{W}| \leq (Z^2 + \underline{W}'\underline{W})^{\frac{1}{2}}(1 + \underline{r}'\underline{r})^{\frac{1}{2}} \leq \xi^{\frac{1}{2}}(1+\underline{r}'\underline{r})^{\frac{1}{2}}$$

for all $\underline{r}$, which implies $(Z,\underline{W}') \varepsilon H_\xi$. Similarly, if $(Z,\underline{W}') \varepsilon J_\xi$ and $Z < 0$ we have

$$Z+\underline{r}'\underline{W} < |\underline{r}'\underline{W}| \leq (\underline{W}'\underline{W})^{\frac{1}{2}}(\underline{r}'\underline{r})^{\frac{1}{2}} \leq (\underline{W}'\underline{W})^{\frac{1}{2}}(1+\underline{r}'\underline{r})^{\frac{1}{2}} \leq \xi^{\frac{1}{2}}(1+\underline{r}'\underline{r})^{\frac{1}{2}}$$

for all $\underline{r}$, which implies $(Z,\underline{W}') \varepsilon H_\xi$.

Applying Lemma II to (2.15) gives

$$1-\alpha = \Pr\{Z_3 > 0 \text{ and } Z_3^2 + \underline{W}_3'\underline{W}_3 \leq c_3^2U \text{ or } Z_3 \leq 0 \text{ and } \underline{W}_3'\underline{W}_3 \leq c_3^2U\}. \tag{2.16}$$

Since $Z_3 > 0$ and $Z_3 \leq 0$ are mutually exclusive and Z_3 and $\underline{W}_3$ are independent (2.16) can be written as

$$1-\alpha = \Pr\{Z_3 > 0 \text{ and } Z_3^2 + \underline{W}_{31}'\underline{W}_{31} \leq \phi(c_3,U)\} + \tfrac{1}{2}\Pr\{\underline{W}_{31}'\underline{W}_{31} \leq \phi(c_3,U)\}, \tag{2.17}$$

where $\phi(c_3,U) = c_3^2U - W_{32}^2$. The sum of squares $\underline{W}_{31}'\underline{W}_{31}$ is distributed as a chi-square random variable with $q-1$ degrees of freedom. Conditioning on $U = u$, the set

$$\{(Z_3, \underline{W}_{31}'): \ Z_3^2 + \underline{W}_{31}'\underline{W}_{31} \leq \phi(c_3, u)\}$$

defines a sphere of constant density so that

$$\Pr\{Z_3 > 0 \text{ and } Z_3^2 + \underline{W}_{31}'\underline{W}_{31} \leq \phi(c_3,u)\} = \tfrac{1}{2}\Pr\{Z_3^2 + \underline{W}_{31}'\underline{W}_{31} \leq \phi(c_3,u)\}.$$

Then probability (2.17) can be written as

$$1-\alpha = \int_0^\infty \tfrac{1}{2}\{G_1[\phi(c_3,u)] + G_2[\phi(c_3,u)]\} f_U(u)du, \tag{2.18}$$

where G_1 and G_2 are chi-square c.d.f.'s with q and $q-1$ degrees of freedom respectively and $f_U(u)$ is the probability density function of a chi-square random variable divided by its degrees of freedom, ν.

In the special case of $y(p) = \mu + z(p)\sigma$, where q is equal one, take G_2 to be the indicator function equal to one when $\phi(c_3,u) \geq 0$ and equal to zero otherwise. The c_3 coefficients then are appropriate for one-sided bands for the cumulative distribution function of y.

Table 5 presents coefficients c_3 for one-sided confidence bands in $\underline{x}$ and p for various values of $\alpha,\nu,$ and q. More extensive tables for the $q=1$ and $q=2$ cases are given in Table A.2 in Appendix A. Coefficients c_3 can be calculated from (2.18) using the same method as described for the two-sided band. Asymptotically $\sqrt{2\nu}(\sigma-\hat{\sigma})/\sigma$ has the standard normal distribution so that the one-sided results of Hochberg and Quade (1975) are applicable. The infinite degrees of freedom column then is taken as $c(\alpha,K=q+1,\nu=\infty)$ from their Table 1.

Coefficients for one-sided bands in $\underline{x}$ with p fixed can be derived in a similar manner. Here constants c_4 are found such that

$$1-\alpha = \Pr\{y(p,\underline{x}) - \hat{y}(p,\underline{x}) \leq c_4 S(\tfrac{1}{n} + \underline{x}_1'(\underline{X}_1'\underline{X}_1)^{-1}\underline{x}_1 + \tfrac{z^2(p)}{2\nu})^{\frac{1}{2}} \text{ for all } \underline{x}_1\} \quad (2.19)$$

is a $1-\alpha$ upper confidence band for $y(p,\underline{x}) = \beta_0 + \underline{x}_1'\beta^* + z(p)\sigma$. Let

$$\underline{W}_4 = \underline{Q}_1(\hat{\underline{\beta}}^* - \underline{\beta}^*)/\sigma ,$$

$$\underline{r}_4 = \left(\frac{1}{n} + \frac{z^2(p)}{2\nu}\right)^{-\frac{1}{2}} \underline{x}_1'\underline{Q}_1^{-1}$$

Table 5. Coefficients $c_3(\alpha,\nu,q)$ for one-sided bands in $\underline{x}$ and p

		Degrees of freedom ν				
	q	10	20	40	60	∞
α = .10	1	2.240	2.073	2.007	1.987	1.95
	2	2.750	2.527	2.431	2.403	2.35
	3	3.152	2.887	2.767	2.730	2.66
	4	3.494	3.194	3.053	3.009	2.93
α = .05	1	2.889	2.551	2.394	2.353	2.27
	2	3.417	3.014	2.823	2.763	2.66
	3	3.831	3.379	3.161	3.091	2.96
	4	4.191	3.693	3.451	3.372	3.22
α = .01	1	4.554	3.736	3.319	3.172	2.88
	2	5.110	4.176	3.720	3.563	3.24
	3	5.556	4.543	4.054	3.885	3.54
	4	5.953	4.866	4.345	4.164	3.79

and $Z_4 = (1 + \frac{nz^2(p)}{2\nu})^{-\frac{1}{2}}(\sqrt{n}(\beta_0 - \hat{\beta}_0)/\sigma + \sqrt{n}(\sigma - \hat{\sigma})/\sigma)$.

Then (2.19) can be written as

$$1-\alpha = \Pr\left\{ \frac{Z_4 + \underline{r}_4'\underline{W}_4}{(1 + \underline{r}_4'\underline{r}_4)^{\frac{1}{2}}} \leq c_4 U^{\frac{1}{2}} \text{ for all } \underline{r}_4 \varepsilon R^{q-1} \right\} .$$

Applying Lemma II gives

$$1-\alpha = \Pr\{Z_4 > 0 \text{ and } Z_4^2 + \underline{W}_4'\underline{W}_4 \leq c_4^2 U \text{ or } Z_4 \leq 0 \text{ and } \underline{W}_4'\underline{W}_4 \leq c_4^2 U\} .$$

The derivation that follows is for the simple linear regression model. In this case the conditional distribution of W_4 , given $U=u$, is univariate standard normal. The set

$$\{(Z_4, W_4): Z > 0 \text{ and } Z^2 + W_4^2 \leq c^2 u \quad \text{or} \quad Z \leq 0 \quad \text{and} \quad W_4^2 \leq c^2 u\}$$

forms the region described by Turner and Bowden (1979) and illustrated in Figure 2. This region can be written as the disjoint union

$$\{(Z_4, W_4): Z_4^2 + W_4^2 \leq c^2 u\} \cup \{(Z_4, W_4): Z_4 \leq -(c^2 u - W_4^2)^{\frac{1}{2}} \text{ and } W_4^2 \leq c^2 u\}.$$

The set on the left describes a circle of radius $cu^{\frac{1}{2}}$. The set on the right forms an infinite strip bounded by the circle on one side and bounded above and below by $|W_4| \leq cu^{\frac{1}{2}}$. The joint conditional distribution of (Z_4, W_4) , given $U=u$, is bivariate normal with mean vector $(\eta_1, 0)$ and covariance matrix

$$\begin{bmatrix} \eta_2 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\eta_1 = (\frac{1}{n} + \frac{z^2(p)}{2\nu})^{-\frac{1}{2}}(1 - a\sqrt{u})\, z(p)$ and $\eta_2 = (1 + \frac{nz^2(p)}{2\nu})^{-1}$.

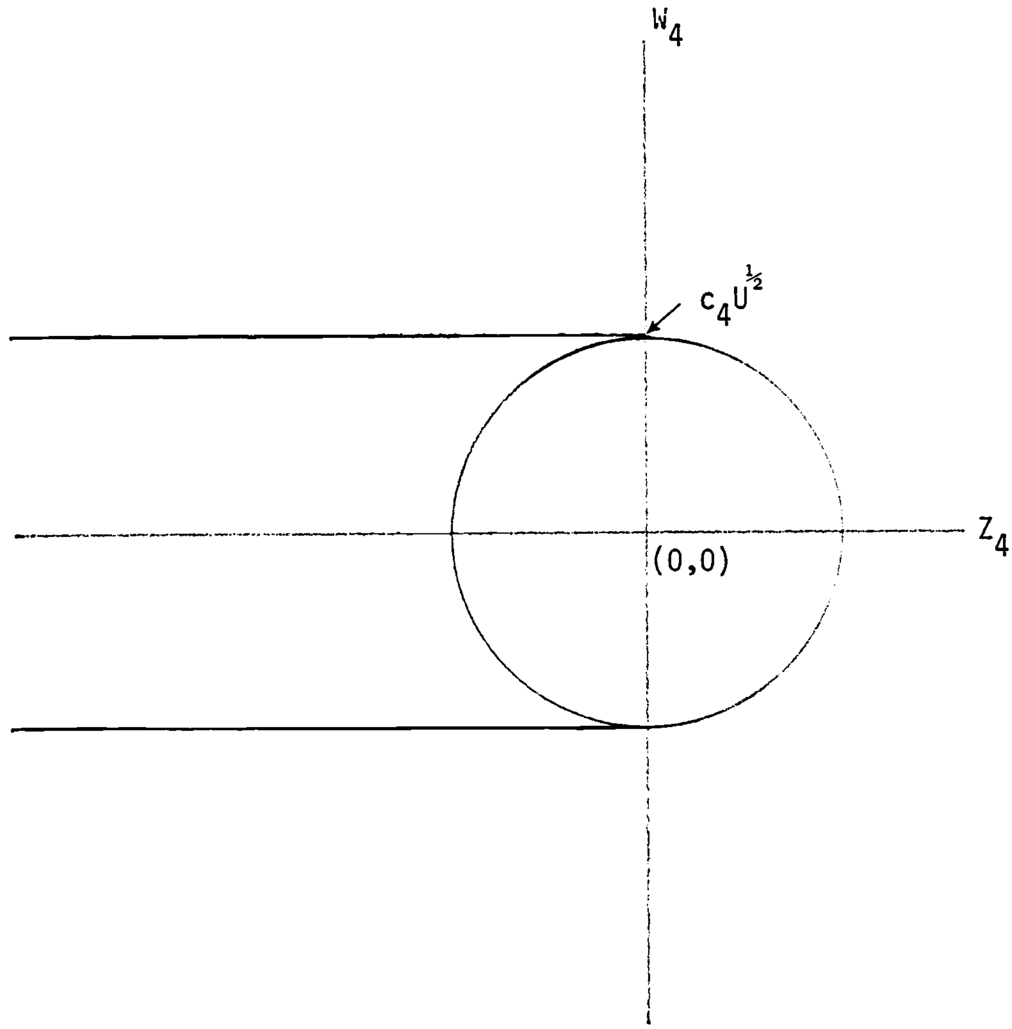

Figure 2. Region of integration in calculating coefficients c_4

The conditional distribution of $Z_4^2 + W_4^2$, given $U = u$, is therefore the same linear combination of chi-square random variables as found in the two-sided p fixed band case. The probability content of the circle can then be numerically evaluated as on page 13. Conditional on $U = u$, the probability content of the strip can be approximated by partitioning the strip into intervals on the W_4 scale and applying the Trapezoidal rule. As the number of intervals is increased the approximation approaches the true probability content of the strip. The sum of the probabilities for the circle and strip is then used in the iterative solution for c_4. The iterative process is similar to that described in the solution of c_1 values. Coefficients c_4 are calculated only for specific examples because they depend on the four quantities: α, ν, q, and p.

For a one-sided AVx upper band, the constant multiplying S is $az(p) + c_4(\underline{x}(\underline{X}'\underline{X})^{-1}\underline{x} + \frac{z^2(p)}{2\nu})^{\frac{1}{2}}$, where $a = 4\nu/(4\nu-1)$. Table 6 presents multipliers of S in the orfice speed example for the bands SBx, TBUx, TBEx, AVx, and AVxp with $p = .75, .95$, $\alpha = .10, .01$, and various values of $|x-\bar{x}|$. The coefficients c_3 used in the AVxp band are given in Appendix B. The coefficients used in the AVx band are $c_4(.10,13,2,.75) = 2.3400$ and $c_4(.01,13,2,.95) = 4.6217$. Turner and Bowden (1979) state the value of the corresponding coefficient for the TBEx band to be 2.28678, when $\alpha = .10$ and $p = .75$. The c_4 value should be smaller than this value but is not. One possible explanation for this is that Turner and Bowden may not have used as many intervals in approximating the probability content of the strip in Figure 2. This results in underestimated coefficients.

Table 6. Multipliers of S for one-sided bands in x with p fixed

p	α	$\|x-\bar{x}\|$	SBx	TBUx	TBEx	AVx	AVxp*
.75	.10	.000	1.288	1.294	1.307	1.367	1.455
		.006	1.302	1.307	1.320	1.379	1.468
		.012	1.341	1.346	1.358	1.413	1.508
		.018	1.402	1.406	1.417	1.468	1.570
		.024	1.479	1.483	1.492	1.539	1.649
		.030	1.569	1.572	1.578	1.622	1.743
		.036	1.668	1.669	1.673	1.714	1.848
		.042	1.774	1.774	1.775	1.814	1.960
		.048	1.884	1.883	1.881	1.919	2.079
		.054	1.998	1.995	1.991	2.028	2.202
		.060	2.115	2.110	2.103	2.140	2.328
		average	1.620	1.622	1.627	1.673	1.801
.95	.01	.000	3.236	3.238	3.242	3.587	3.608
		.006	3.271	3.273	3.276	3.604	3.625
		.012	3.374	3.373	3.371	3.653	3.675
		.018	3.532	3.528	3.520	3.733	3.755
		.024	3.733	3.725	3.709	3.839	3.863
		.030	3.966	3.954	3.928	3.969	3.994
		- - - - -	- - - - -	- - - - -	- - - - -	- - - - -	- - - - - **
		.036	4.223	4.206	4.169	4.118	4.145
		.042	4.496	4.474	4.426	4.284	4.313
		.048	4.782	4.755	4.694	4.463	4.493
		.054	5.078	5.045	4.972	4.652	4.685
		.060	5.381	5.342	5.256	4.851	4.886
		average	4.097	4.083	4.051	4.068	4.095

* The band in x and p , AVxp, is included for comparison.

** The dashed line indicates where AVx crosses the other bands.

The asymptotic variance bands AVx and AVxp are closer to the regression line than the other bands for $|x-\bar{x}| \geq .036$ when $p = .01$, $\alpha = .95$. The point where AVx and AVxp cross the other bands will approach zero as sample size increases.

II.4. Simultaneous Tolerance Intervals

Suppose the estimated regression of y on $\underline{x}$ is calculated for a sample of size n. Simultaneous tolerance intervals are sought for the normal populations of y given $\underline{x}$. Such intervals will contain at least 100p% of each normal distribution with confidence level at least $1-\alpha$. The proportion p refers to the sampling distribution of y values at a given $\underline{x}$. The confidence $1-\alpha$ refers to the sample from which the regression function was estimated. Two distinct types of simultaneous tolerance intervals are considered: a) those which are simultaneous in $\underline{x}$ with the proportion p fixed and b) those which are simultaneous in $\underline{x}$ and p. Lieberman and Miller (1963) gave four procedures for forming simultaneous tolerance intervals. The first method extends Wallis' (1951) technique where both $\underline{x}$ and p are fixed to intervals simultaneous in $\underline{x}$ with p fixed. This method is labeled SWx. The second and third procedures are based on the Scheffé simultaneous confidence principle. The second, simultaneous in $\underline{x}$ with p fixed, and third, simultaneous in $\underline{x}$ and p, are labeled CSx and CSxp respectively. The fourth method uses the Bonferroni inequality to combine the Scheffé simultaneous confidence region for the regression function with a one-sided confidence interval on the standard deviation. This last procedure, denoted CBSxp, is simultaneous in $\underline{x}$ and p . The Scheffé type tolerance intervals were formed by constructing confidence intervals for the interval

$$[\underline{x}'\underline{\beta} - z(\tfrac{p+1}{2})\sigma\ ,\ \underline{x}'\underline{\beta} + z(\tfrac{p+1}{2})\sigma]\ , \tag{2.20}$$

where $z(\frac{p+1}{2})$ is such that $p = \Phi(z(\frac{p+1}{2})) - \Phi(-z(\frac{p+1}{2}))$. The tolerance interval CSx, simultaneous in $\underline{x}$ with p fixed, has bounds

$$\underline{x}'\hat{\underline{\beta}} \pm c^{**}S(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x})^{\frac{1}{2}} .$$

The interval CSxp, simultaneous in $\underline{x}$ and p , has bounds

$$\underline{x}'\hat{\underline{\beta}} \pm c^{*}S(\underline{x}'(\underline{X}'\underline{X})^{-1}\underline{x}\ \ z^2(\tfrac{p+1}{2}))^{\frac{1}{2}}$$

The coefficients c^{**} and c^{*} are tabulated by Lieberman and Miller.

Using the approach of Lieberman and Miller in the formation of Scheffé type tolerance intervals, CSx and CSxp, it is shown that the coefficients c_1 and c_2 found in Section II.2 are such that the intervals

$$[\underline{x}'\hat{\underline{\beta}} - z(\tfrac{p+1}{2})\hat{\sigma} - c_i h(S,\underline{x},p),\ \underline{x}'\hat{\underline{\beta}} + z(\tfrac{p+1}{2})\hat{\sigma} + c_i h(S,\underline{x},p)]\ ,$$
$$i = 1,2 \tag{2.21}$$

are 100p% central tolerance intervals for y given $\underline{x}$, where

$$h(S,\underline{x},p) = S\left\{\underline{x}(\underline{X}'\underline{X})^{-1}\underline{x} + \frac{z^2(\frac{p+1}{2})}{2\nu}\right\}^{\frac{1}{2}}$$

The intervals in (2.21) with i=1 are simultaneous in $\underline{x}$ and p , and with i=2 are simultaneous in $\underline{x}$ with p fixed. The simultaneous tolerance interval procedures in (2.21) are referred to as

AVTxp(i=1) and AVTx (i=2) since they are based upon the asymptotic variance of the percentile estimator.

Consider first the intervals AVTxp. Recall that coefficients c_1 were found by solving

$$1-\alpha = \Pr\{\underline{W}_1'\underline{W}_1 \leq c_1^2 U\} ,$$

where W_1 and U are given in (2.3). Applying Lemma I gives

$$1-\alpha = \Pr\{|\underline{r}_1'^{*}\underline{W}_1| \leq c_1(\underline{r}_1'^{*}\underline{r}_1^{*}U)^{\frac{1}{2}} \text{ for all } \underline{r}_1^{*} \tag{2.22}$$
$$= \left(n^{-\frac{1}{2}}, \sqrt{n}\,\underline{x}_1'\,\underline{Q}_1 , \pm \frac{z^2(\frac{p+1}{2})}{2\nu}\right)$$

Rewriting (2.22) yields

$$1-\alpha = \Pr\{|(\underline{x}'\underline{\beta} \pm z(\tfrac{p+1}{2})\sigma) - (\underline{x}'\hat{\underline{\beta}} \pm z(\tfrac{p+1}{2})\hat{\sigma})| \leq c_1 h(S,\underline{x},p) \tag{2.23}$$

$$\text{for all } \underline{x} \in \mathscr{L} \text{ and } 0 \leq p \leq 1\} .$$

Let $L_{\underline{x}}(p)$ and $U_{\underline{x}}(p)$ denote respectively the lower and upper end points of the interval (2.21) with i=1 . Then (2.23) can be written as

$$1-\alpha = \Pr\{L_{\underline{x}}(p) \leq \underline{x}'\underline{\beta} - z(\tfrac{p+1}{2})\sigma,\ \underline{x}'\underline{\beta} + z(\tfrac{p+1}{2})\sigma \leq U_{\underline{x}}(p) \text{ for } \underline{x}\varepsilon\mathscr{L} \text{ and } 0 \leq p \leq 1\}. \tag{2.24}$$

Then the corresponding tolerance interval statement

$$1-\alpha \leqslant \Pr\{A_{\underline{x}}(p) \geq p \text{ for all } \underline{x} \in \mathscr{L} \text{ and } 0 \leq p \leq 1\}$$

where $A_{\underline{x}}(p) = \Phi\left(\dfrac{U_{\underline{x}}(p) - \underline{x}'\underline{\beta}}{\sigma}\right) - \Phi\left(\dfrac{L_{\underline{x}}(p) - \underline{x}'\underline{\beta}}{\sigma}\right)$, follows from (2.24).

For tolerance intervals simultaneous in $\underline{x}$ with p fixed evaluation of the coefficients c_2 in (2.21) is similar to that described for the coefficients c_1. The coefficients c_2 must satisfy

$$1-\alpha = \Pr\{\underline{W}_2'^*\underline{W}_2^* \le c_2^2 U\}\ , \tag{2.25}$$

$$\text{where}\quad \underline{W}_2^* = \begin{bmatrix} \left(1 + \dfrac{nz^2(\frac{p+1}{2})}{2\nu}\right)^{-\frac{1}{2}} \{\sqrt{n}(\frac{\beta_0 - \hat{\beta}_0}{\sigma}) \pm \sqrt{n}z(\frac{p+1}{2})(\frac{\sigma-\hat{\sigma}}{\sigma})\} \\ \underline{Q}_1\left(\dfrac{\underline{\beta}^* - \hat{\underline{\beta}}^*}{\sigma}\right) \end{bmatrix}$$

It should be noted that replacing $\underline{W}_2$ in (2.5) with $\underline{W}_2^*$ does not change the derivation of c_2 values. Applying Lemma I to (2.25) gives

$$1-\alpha = \Pr\{|\underline{r}_2'\underline{W}_2^*| \le c_2(\underline{r}_2'\underline{r}_2 U)^{\frac{1}{2}} \text{ for all } \underline{r}_2 \varepsilon\, \mathscr{G}_2\}\ , \tag{2.26}$$

where $\underline{r}_2$ and $\mathscr{G}_2$ are given in (2.5). Following the argument given for c_1 probability (2.26) leads to the tolerance interval statement

$$1-\alpha = \Pr\{A_{\underline{x}} > p \text{ for all } \underline{x}\, \varepsilon\, \mathscr{L}\}$$

$$\text{where}\quad A_{\underline{x}} = \Phi\left(\frac{U_{\underline{x}} - \underline{x}'\underline{\beta}}{\sigma}\right) - \Phi\left(\frac{L_{\underline{x}} - \underline{x}'\underline{\beta}}{\sigma}\right),$$

$L_{\underline{x}} = \underline{x}'\hat{\underline{\beta}} - z(\frac{p+1}{2})\hat{\sigma} - c_2 h(S,\underline{x},p)$, and $U_{\underline{x}} = \underline{x}'\hat{\underline{\beta}} + z(\frac{p+1}{2})\hat{\sigma} + c_2 h(S,\underline{x},p)$.

The constant which multiplies S for the interval AVTx is

$$\pm\left[az(\tfrac{p+1}{2}) + c_2\left(\underline{x}(\underline{X}'\underline{X})^{-1}\underline{x} + \frac{z^2(\frac{p+1}{2})}{2\nu}\right)^{\frac{1}{2}}\right],$$

$$\text{where}\quad a = 4\nu/(4\nu-1)\ .$$

Table 7 presents, for the orfice-speed example, the absolute value of the constant multiplying S for the intervals AVTx, AVTxp, and the

Table 7. The absolute value of constants multiplying S in simultaneous tolerance intervals

			fixed p intervals			for all p intervals		
p	α	x	SWx	CSx	AVTx	CSxp	CBSxp	AVTxp
.95	.01	1.310	6.58	6.58	4.91	7.94	5.65	4.96
		1.353	3.61	3.61	4.21	7.79	4.79	4.25
		1.400	6.99	6.99	5.02	7.97	5.77	5.07
.75	.05	1.310	3.23	3.89	2.65	3.78	3.34	2.96
		1.353	1.77	2.13	2.15	3.59	2.67	2.34
		1.400	3.43	4.14	2.73	3.82	3.43	3.05
.50	.10	1.310	1.73	2.42	1.84	2.18	2.30	2.07
		1.353	0.95	1.33	1.37	1.91	1.71	1.51
		1.400	1.84	2.58	1.90	2.22	2.39	2.15

four Lieberman and Miller procedures; CSx, CSxp, SWx, and CBSxp. The values of α, p, and x are those chosen by Lieberman and Miller. The values of c_1 used in AVTxp are taken from Appendix A, the c_2 values used in AVTx are $c_2 = 2.350$ for $p = .50$, $\alpha = .10$; $c_2 = 2.837$ for $p = .75$, $= .05$; and $c_2 = 4.782$ for $p = .95$, $\alpha = .01$. The fixed p intervals SWx and CSx are narrower than the interval AVTx for $x = \bar{x} = 1.353$ and are wider for the other two choices of x values when $p = .95$, $.75$ and $\alpha = .01$, $.05$. The interval AVTxp is uniformly narrowest among intervals simultaneous in x and p, for this example. The relative performance of the intervals AVTx and AVTxp, based on the asymptotic variance of the percentile estimator, improves as sample size increases because $z^2(\frac{p+1}{2})/2\nu$ decreases.

II.5. Relative Efficiency Comparisons of Confidence Bands

The orfice-speed example provides a rather restricted comparison of the asymptotic variance bands, AVx and AVxp, to the Steinhorst and Bowder and Turner and Bowden procedures SBx, SBxp, TBUx, TBEx. The comparison is restricted because only one specific example with $n = 15$ is treated. A more general comparison can be made by evaluating the asymptotic relative efficiencies of these procedures. Asymptotic relative efficiency can be defined as the ratio of sample sizes required so that for each point in a sequence of alternatives, converging to a hypothesized value, the two procedures have equal power, (Noether, 1955). Alternatively, the efficiency of two procedures can be defined as the limiting ratio of sample sizes required such that the asymptotic power function of two procedures have equal slope at the hypothesized value, (Blomquist, 1950). Under weak conditions these two definitions are equivalent, (Noether, 1955).

Asymptotic relative efficiencies are developed for two-sided confidence bands in x and p for the simple linear regression model with 100pth percentiles $y(p,x) = \beta_0 + \beta_1 x + z(p)\sigma$. By the Scheffé confidence principle, the $1-\alpha$ level simultaneous confidence intervals for $y(p,x)$ are constructed as

$$\{y = \beta_0{}^{o} + \beta_1{}^{o} x + z(p)\sigma^{o} : \ (\beta_0{}^{o}, \beta_1{}^{o}, \sigma^{o}) \in \underline{W}\} ,$$

where $\underline{W}$ is the set of $\underline{\theta}^{o} = (\beta_0{}^{o}, \beta_1{}^{o}, \sigma^{o})$ for which the null hypothesis

$$H_0: \underline{\theta} = \underline{\theta}^{o} \tag{2.27}$$

is not rejected at the α significance level. Let $\underline{\theta}^1 = (\beta_0{}^1, \beta_1{}^1, \sigma^1)'$ denote the true value of the parameter vector $\underline{\theta}$, then the power function at $\underline{\theta} = \underline{\theta}_1$, $Pw_n(\underline{\theta}_1)$, corresponds to the probability that the confidence bands do not contain the true regression percentile for at least some x and p. It is shown that the test statistic for (2.27) corresponding to the SBxp and AVxp procedures for bands simultaneous in x and p are asymptotically noncentral chi-square distributed with different degrees of freedom when $\theta \neq \theta^{o}$. Shirahata (1976) defined the local asymptotic relative efficiency for test statistics with these types of asymptotic distributions. He considered the general case where two test statistics of the hypothesis $H_0: \gamma = 0$, T_1 and T_2, have asymptotic distributions with degrees of freedom k_1 and k_2 and noncentrality parameters $\gamma^2\eta_1$

and $\gamma^2\eta_2$, respectively. Shirahata then showed that local asymptotic relative efficiency of T_1 to T_2 is for specified α

$$e(T_1,T_2,\alpha) = \frac{\eta_1}{\eta_2} \times \frac{1-\alpha-\Pr\{\chi^2_{k_1+2} \leq \chi^2_{k_1}(1-\alpha)\}}{1-\alpha-\Pr\{\chi^2_{k_2+2} \leq \chi^2_{k_2}(1-\alpha)\}} . \tag{2.28}$$

(2.28) is the ratio of noncentrality parameters times a term that adjusts for unequal degrees of freedom.

For a simple linear regression the confidence region for $\underline{\theta} = (\beta_0,\beta_1,\sigma)$ used to construct the asymptotic variance band, AVxp, has the form

$$\{\underline{\theta} : |\underline{r}_1{}'\underline{W}_1| \leq c_1 (\underline{r}_1'\underline{r}_1 U)^{\frac{1}{2}} \text{ for all } \underline{r}_1\} , \tag{2.29}$$

where $\underline{r}$, $\underline{W}$, and U are given in (2.3). Applying Lemma 1 of Section II.2, (2.29) can be written as

$$\{\underline{\theta} : \underline{W}_1{}' \underline{W}_1 \leq c_1{}^2 U\} . \tag{2.30}$$

The confidence bands, AVxp, formed from region (2.30) correspond to the acceptance region

$$\{\hat{\underline{\theta}} : \underline{W}_1{}' \underline{W}_1 \leq c_1{}^2 U ; \theta^o\}$$

for testing hypothesis (2.27). Let $\underline{\theta}^n = \underline{\theta}^o + \delta_n(\underline{\theta}^1 - \underline{\theta}^o)$ be a sequence of alternatives, where $\delta_n = \delta/\sqrt{n}$, $\gamma \geq 0$. Then the sequence $\underline{\theta}^n$ converges to $\underline{\theta}^o$ as n approaches infinity. Under the null hypothesis H_o: $\gamma = 0$ ($\underline{\theta} = \underline{\theta}^o$) the asymptotic distribution of $T_1 = \underline{W}_1' \underline{W}_1/U$ is central chi-square with three degrees of freedom. The coefficient $c_1{}^2$ can, in the limit, be taken as $\chi_3^2(1-\alpha)$, the $(1-\alpha)$th percentile of the central chi-square distribution with three degrees of freedom, (or equivalently $3F_{3,\infty}(1-\alpha)$). The power of the test for a point in the sequence of alternatives $\underline{\theta}^n$ is

$$Pw_n(\underline{\theta}^n) = Pr\{\underline{W}_1' \underline{W}_1/U > c_1{}^2 \; ; \underline{\theta}^n\} . \tag{2.31}$$

Rewrite (2.31) as

$$Pw_n(\underline{\theta}^n) = Pr\{(U\sigma_o^2)^{-1}(\underline{\theta}^o-\hat{\underline{\theta}})'\Sigma^{-1}(\underline{\theta}^o-\hat{\underline{\theta}}) > c_1^2 \; ; \underline{\theta}^n\} , \tag{2.32}$$

where

$$\Sigma = \begin{bmatrix} \frac{1}{n} & 0 & 0 \\ 0 & \frac{1}{\Sigma(x_i-\bar{x})^2} & 0 \\ 0 & 0 & \frac{1}{2\nu} \end{bmatrix}$$

Replacing $\underline{\theta}^o$ by $\underline{\theta}^n - \frac{\Upsilon}{\sqrt{n}} (\underline{\theta}^1 - \underline{\theta}^o)$ in (2.32) gives

$$Pw_n(\underline{\theta}^n) = Pr\{(U\sigma^2 n)^{-1}(\underline{\theta}^n-\hat{\underline{\theta}}-(\gamma/\sqrt{n})(\underline{\theta}^1-\underline{\theta}^o))'\Sigma^{-1}(\underline{\theta}^n-\hat{\underline{\theta}}-(\gamma/\sqrt{n})(\underline{\theta}^1-\underline{\theta}^o))>c_1;\underline{\theta}^n\}. \tag{2.33}$$

Assuming $\frac{\Sigma(x_i-\bar{x})^2}{n}$ has a positive limit, $\sigma_x{}^2$, as $n \to \infty$ then $(U\sigma^2 n)^{-1}(\underline{\theta}^n-\hat{\underline{\theta}})' \Sigma^{-1}(\underline{\theta}^n-\hat{\underline{\theta}})$ has a limiting central chi-square distribution with three degrees of freedom. Then

$$\lim_{n\to\infty} Pw_n\,(\underline{\theta}^n) = \Pr\{\,\chi^2_{3,\gamma^2\eta_1} > \chi^2_3\,(1-\alpha)\}\ , \qquad (2.34)$$

where $\eta_1 = (\sigma^o)^{-2}\, n^{-1}[(\underline{\theta}^1-\underline{\theta}^o)^2\, \underline{\Sigma}^{-1}(\underline{\theta}^1-\underline{\theta}^o)]$. Expanding η_1 gives

$$\eta_1 = (\sigma^o)^{-2}\ \{(\beta_0^1-\beta_0^o)^2 + \sigma_x^2\,(\beta_1^1-\beta_1^o)^2 + 2(\sigma^1-\sigma^o)^2\}\ .$$

The Steinhorst and Bowden band SBxp uses

$$\underline{W}_1^* = \frac{1}{\sigma}\begin{pmatrix} \sqrt{n}(\beta_0-\hat{\beta}_0) \\ \sqrt{n}\ \sum_i (x_i-\bar{x})^2(\beta_1-\hat{\beta}_1) \\ (\sigma-S) \end{pmatrix}\ .$$

The confidence region for (β_0,β_1,σ) used to construct the SBxp bands is

$$\{\,\underline{\theta} : \underline{W}_1^{*\prime}\,\underline{W}_1^* \leqslant \overline{Z}^2_\alpha\ U\,\}\ .$$

The acceptance region for testing (2.27) corresponding to the confidence band, SBxp, then is

$$\{\hat{\underline{\theta}} : \underline{W}_1^{*\prime}\,\underline{W}_1^* \leqslant \overline{Z}^2_\alpha\ U\ ;\ \underline{\theta}^o\}\ .$$

Since $\sqrt{2n}\ (\sigma-S)/\sigma$ is asymptotically standard normal distributed, the term $(\sigma-S)/\sigma$ is equal in distribution to $(2n)^{-1}\,\chi^2_1$. This term vanishes for large n so asymptotically the test statistic $T_2 = \underline{W}_1^{*\prime}\,\underline{W}_1^*/U$ is central chi-square distributed with two degrees of freedom when $\underline{\theta} = \underline{\theta}^o$. In the limit then the coefficient $\overline{Z}_\alpha^2$ can be taken as $\chi^2_2\ (1-\alpha)$. Taking θ^{n*} as another sequence of alternatives and following an argument similar to the derivation of (2.34) the asymptotic power of T_2 for testing (2.27) is

$$Pw_\infty(\underline{\theta}^{n*}) = \Pr\{ \chi^2_{2,\gamma^2\eta_2} > \chi^2_2(1-\alpha)\} ,$$

where $\eta_2 = (\sigma^0)^{-2}\{(\beta_0^1 - \beta_0^0)^2 + \sigma_x^2\ (\beta_1^1 - \beta_1^0)^2\}$.

The term $(\sigma^1 - \sigma^0)^2$ vanishes for the same reason that T_2 is χ^2_2 distributed rather than χ^2_3 under H_0.

Then for the Shirahata local asymptotic relative efficiency of AVxp to SBxp, bands simultaneous in x and p, substitute in (2.28) $k_1 = 3$, $k_2 = 2$ and

$$\frac{\eta_1}{\eta_2} = \frac{(\beta_0^1 - \beta_0^0)^2 + \sigma_x^2(\beta_1^1 - \beta_1^0)^2 + (\sigma^1 - \sigma^2)^2}{(\beta_0^1 - \beta_0^0)^2 + \sigma_x^2(\beta_1^1 - \beta_1^0)^2}$$

$$= 1 + \frac{2(\sigma^1 - \sigma^0)^2}{(\beta_0^1 - \beta_0^0)^2 + \sigma_x^2\ (\beta_1^1 - \beta_1^0)^2} .$$

Notice in (2.28) that the ratio of noncentrality parameters $\eta_1/\eta_2 \geqslant 1$, whereas the term that adjusts for unequal degrees of freedom is less than 1. For example, using $\alpha = .05$ with $\sigma^1 = \sigma^0$ gives $e(T_1,T_2,.05) = .780$ and with $|\sigma^1 - \sigma^0| = |\beta_0^1 - \beta_0^0| = \sigma_x\ |\beta_1^1 - \beta_1^0| = 1$ gives $e(T_1,T_2,.05) = 1.560$.

Evaluation of local asymptotic relative efficiencies among the bands AVx, SBx, TBUx, and TBEx for fixed p is complex. The complexity arises because the test statistics corresponding to the SBx, TBUx, and TBEx bands have limiting distributions which are linear combinations of two noncentral chi-square distributions under alternatives $\underline{\theta} \neq \underline{\theta}^0$.

III. CONFIDENCE BANDS FOR PERCENTILES FROM CENSORED SAMPLES

III.1. Censored Data and Estimation

The one and two-sided confidence bands in $\underline{x}$ and p, AVxp, developed in Chapter II are generalized for singly censored samples. Estimation of parameters of normal distributions from censored samples has received wide attention [see, for example, Gupta (1952), Cohen (1959), Tiku (1967), and Schmee and Hahn (1979)]. Maximum likelihood estimation is used here. The EM algorithm [Dempster, Laird, Rubin (1977)] can be used to iteratively calculate the maximum likelihood estimators $(\hat{\underline{\beta}}, \hat{\sigma})$ from censored data. A FORTRAN program given by Wolynetz (1979) is available to do these calculations. Small sample properties of the maximum likelihood estimators for the simple linear regression model have been investigated by Schmee and Hahn (1979) and Aitkin (1981). The maximum likelihood estimator $(\hat{\underline{\beta}}, \hat{\sigma})$ is asymptotically multivariate normally distributed with mean $(\underline{\beta}, \sigma)$. The asymptotic covariance matrix, denoted $\mathrm{Cov}_{\infty}(\cdot)$, is the inverse of the information matrix evaluated in the limit as n approaches infinity. For finite samples the negative of the inverse matrix of second partial derivatives of the log likelihood function evaluated at the parameter estimates is referred to as the observed covariance matrix, $\hat{\mathrm{Cov}}_n(\cdot)$. An estimator for the variance of the 100p$\underline{\text{th}}$ percentile estimator, $\underline{x}'\hat{\underline{\beta}} + z(p)\hat{\sigma}$, then is

$$(\underline{x}, z(p))\hat{\mathrm{Cov}}_n(\hat{\underline{\beta}}, \hat{\sigma}) \begin{pmatrix} \underline{x} \\ z(p) \end{pmatrix}. \tag{3.1}$$

The asymptotic normality of the maximum likelihood estimators is used to form confidence bands from censored samples. The small sample performance of these asymptotic procedures is naturally questioned. Several approximations for the small sample case are proposed and compared by simulation.

III.2. Effective Degrees of Freedom

Asymptotic theory essentially associates infinite degrees of freedom with the estimator $\hat{\sigma}$. Assignment of finite degrees of freedom to $\hat{\sigma}$ results in more conservative bands. The choice of such effective degrees of freedom is not clear. For example, one might use the number of uncensored values minus the number of estimated parameters in the model. This choice seems overly conservative, especially for heavily censored samples. The degrees of freedom suggested here are based on asymptotic results. For uncensored samples recall that an estimator of $\mathrm{Var}(S)$ is $\widehat{\mathrm{Var}}(S) = S^2/2\nu$, then $\nu = S^2/2\,\widehat{\mathrm{Var}}(S)$, where ν is the degrees of freedom $(n\text{-}q)$. Correspondingly for censored samples denote $\widehat{\mathrm{Var}}_n(\hat{\sigma}) = \frac{\hat{\sigma}^2}{n}\,d$. Then the effective degrees of freedom for $\hat{\sigma}$ from censored samples is taken as

$$\nu^* = [n/(2\,d)]\,, \tag{3.2}$$

the largest integer less than or equal $n/(2\,d)$.

III.3. Two-sided Confidence Bands for Percentiles from Censored Samples

Five methods for constructing approximate bands for the percentiles $y(p,\underline{x}) = \underline{x}'\underline{\beta} + z(p)\sigma$ are considered. The first method, FIA, applies the asymptotic normality of the maximum likelihood estimators to form approximate two-sided bands for the 100pth percentile using the Scheffé-S method. These bands have bounds

$$\underline{x}'\hat{\underline{\beta}} + z(p)\hat{\sigma} \pm \{[(q+1)F^{\alpha}_{q+1,\infty}][(\underline{x}',z(p))\mathrm{Cov}_{\infty}(\hat{\underline{\beta}},\hat{\sigma})\begin{pmatrix}\underline{x}\\ z(p)\end{pmatrix}]\}^{\frac{1}{2}}. \qquad (3.3)$$

The second method, FIO, uses the observed covariance matrix, $\hat{\mathrm{Cov}}_n(\hat{\beta},\hat{\sigma})$, in place of the asymptotic covariance matrix in (3.3). The third, FEA, and fourth, FEO, methods use the asymptotic and observed covariance matrices, respectively, with F^{α}_{q+1,ν^*} replacing $F^{\alpha}_{q+1,\infty}$ in (3.3). The last method, CEO, uses the observed covariance matrix and the coefficient $c_1(\alpha,\nu^*,q)$ of Chapter II in place of $[(q+1)F^{\alpha}_{q+1,\infty}]^{\frac{1}{2}}$ in (3.3).

A simulation study is used to evaluate and compare the confidence coefficient accuracy of the confidence band methods. First consider bands for percentiles $y(p) = \mu + z(p)\sigma$ of a single normal distribution. For a given sample $y(p)$ is contained within a band for all p if and only if

$$|(\mu + z(p)\sigma) - (\hat{\mu} + z(p)\hat{\sigma})| \leq c[(1,z(p))\mathrm{Cov}(\hat{\mu},\hat{\sigma})\binom{1}{z(p)}]^{\frac{1}{2}} \qquad (3.4)$$

for all p, where c and $\mathrm{Cov}(\hat{\mu},\hat{\sigma})$ depend on the procedure. For the purpose of simulation, without loss of generality, take $\mu = 0$

and $\sigma = 1$. Squaring both sides and simplifying (3.4) reduces to the form

$$\psi_1(z(p)) = a_1 z^2(p) + b_1\ z(p) + d_1 \leq 0 ,$$

which must hold for all p, where a_1, b_1, and d_1 depend on the parameter estimates $(\hat{\mu},\hat{\sigma})$, and the band method. Then the band does not contain the true 100pth percentile, $\mu + z(p)\sigma$, for at least some p if and only if $a_1 > 0$ or $a_1 < 0$ and $b_1 - 4a_1 d_1 > 0$. These conditions are easily evaluated to determine whether a band indeed contains all percentiles for a particular sample generated in the simulation study.

One thousand samples of size $n = 20$ were generated from a standard normal distribution on a CDC Cyber 172/720 using the polar method described by Knuth (1969, p. 104). Each observation which exceeded $\phi^{-1}(.6) = 0.253347$ was censored. This results in a censoring probability of .4 . Similarly, two sets of 1000 samples of size $n = 40$ were generated and censored with probabilities .4 and .7 . Censoring probabilities 0.7 and .4 with sample sizes $n = 20$ and 40 respectively yield the same expected number of uncensored observations, 12 . Table 8 presents, for each of the five methods, the fraction of samples out of 1000 which produced confidence bands for $\mu + z(p)\sigma$ which did not contain the true underlying percentile for some p using nominal confidence levels $1-\alpha$ with $\alpha = .01, .05,$ and .10. Examination of Table 8 indicates that the use of effective degrees of freedom greatly improves the F methods. It is also seen that there is little difference between the methods based on the

Table 8. Empirical error rates of two-sided bands for $\mu + z(p)\sigma$ from censored samples

n	Censoring Probability	α	FIA	FIO	FEA	FEO	CEO
20	.4	.01	.056	.056	.028	.027	.008
		.05	.102	.101	.067	.065	.044
		.10	.143	.145	.105	.103	.112
40	.4	.01	.027	.027	.013	.013	.006
		.05	.079	.078	.056	.054	.046
		.10	.140	.141	.118	.118	.113
40	.7	.01	.057	.058	.021	.021	.004
		.05	.112	.112	.069	.068	.044
		.10	.151	.149	.106	.107	.094

the asymptotic covariance (FIA,FEA) and the corresponding methods with observed covariance (FIO,FEO). In all cases the CEO method is most accurate. This method performed well even under heavy censoring ($n = 40$, censoring prob. = .7) with none of the differences between the empirical and nominal α values being statistically significant at the 5% level.

Consider now the simple linear regression model. For a given sample the 100p<u>th</u> percentile, $\beta_0 + \beta_1 x + z(p)\sigma$, is contained within the bounds of a given band in x and p if

$$|(\beta_0+\beta_1 x + z(p)\sigma)-(\hat{\beta}_0+\hat{\beta}_1 x + z(p)\hat{\sigma})| \leq c[\hat{\mathrm{Var}}(\hat{\beta}_0+\hat{\beta}_1 x+z(p)\hat{\sigma})]^{\frac{1}{2}} \quad (3.5)$$

for all x and p, where

$$\hat{\mathrm{Var}}(\hat{\beta}_0 + \hat{\beta}_1 x + z(p)\hat{\sigma}) = (1,x,z(p))\mathrm{Cov}(\hat{\beta}_0,\hat{\beta}_1,\hat{\sigma})\begin{pmatrix} 1 \\ x \\ z(p) \end{pmatrix}$$

and c and $\mathrm{Cov}(\hat{\beta}_0,\hat{\beta}_1,\hat{\sigma})$ depend on the band method. Without loss of generality take $\beta_0 = 0$, $\beta_1 = 1$, $\sigma = 1$. By squaring both sides and simplifying, (3.5) can be written in the form

$$\psi_2(x,z(p)) = a_2 x^2 + b_2 x + d_2 x\, z(p) + e_2 z(p) + f_2 z^2(p) \leq 0 ,$$

which must hold for all x and p, where a_2, b_2, d_2, e_2, and f_2 depend on the parameter estimates, $(\hat{\beta}_0,\hat{\beta}_1,\hat{\sigma})$, and on the band method. Then $\psi_2(x,z(p)) \leq 0$ for all x and p if and only if $d_2^2-4a_2f_2 \geq 0$ and the maximum of $\psi_2(x,z(p)) \leq 0$.

Sets of 500 random samples, $\varepsilon_1, \ldots, \varepsilon_n \sim N(0,1)$, of size n = 20, 40, and 40 were generated. For each sample y_i values were obtained as $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ using $\beta_0 = 0$, $\beta_1 = 1$, and equal numbers of x_i values equal to -2, -1, 0, 1, and 2 . The sets of random samples of size n = 20, 40, and 40 were censored with probabilities .4 , .4 , and .7 respectively. For censoring probabilities .4 and .7 the y values were censored if they exceeded .5063328 and -1.0256457, respectively. Table 9 shows the conditional probability of observing an uncensored value of y given x . Table 10 presents the fraction of samples (based on 500) which produced confidence bands for $\beta_0 + \beta_1 x + z(p)\sigma$ that did not contain the true underlying percentile for at least some x and p using nominal values $\alpha = .01, .05$, and $.10$. The CEO procedure again works well but not as well as in the one normal population model. For $n = 20$, the empirical values are all larger than the corresponding nominal values. These three differences between empirical and nominal values are all statistically significant at the 5% level. For $n = 40$ the CEO procedure is accurate in all cases.

Table 9. Conditional probability of observing an uncensored y value given x

x	Overall censoring probability .4	.7
-2	.993400	.835060
-1	.934009	.489770
0	.693688	.152529
1	.310771	.021401
2	.067631	.001241

Table 10. Empirical error rates of two-sided bands for $\beta_0 + \beta_1 x + z(p)\sigma$ from censored samples

n	Censoring probability	α	FIO	FEO	CEO
20	.4	.01	.104	.046	.022
		.05	.178	.108	.080
		.10	.218	.182	.168
40	.4	.01	.034	.020	.014
		.05	.102	.072	.056
		.10	.154	.118	.116
40	.7	.01	.068	.026	.014
		.05	.140	.066	.052
		.10	.190	.130	.118

III.4. One-sided Confidence Bands for Percentiles from Censored Samples

Since the CEO procedure performed well in the formation of two-sided bands a corresponding one-sided procedure, C3EO, is examined. The coefficients $C_3(\alpha,v,q)$ were derived in Section II.3 for complete samples. The observed covariance matrix is used to form upper one-sided bands for percentiles $y(p,\underline{x})$ in $\underline{x}$ and p. Upper one-sided bands for percentiles $y(p,\underline{x}) = \underline{x}'\underline{\beta} + z(p)\sigma$, $0 < p < 1$, are defined by the upper bound

$$\underline{x}'\hat{\underline{\beta}} + z(p)\hat{\sigma} + c_{3,\nu^*}\left[(\underline{x}',z(p))\hat{\mathrm{Cov}}_n(\hat{\underline{\beta}},\hat{\sigma})\begin{pmatrix}\underline{x}\\ z(p)\end{pmatrix}\right]^{\frac{1}{2}} .$$

For the case of a single normal distribution, the true percentile, $y(p) = \mu + z(p)\sigma$, is below this upper bound for all p if and only if

$$(\mu + z(p)\sigma)-(\hat{\mu} + z(p)\hat{\sigma}) \leqslant c_{3,\nu^*}\left[(1,z(p))\ \hat{\mathrm{Cov}}_n(\hat{\mu},\hat{\sigma})\begin{pmatrix}1\\ z(p)\end{pmatrix}\right]^{\frac{1}{2}} \quad (3.6)$$

holds for all p. Taking $\mu = 0$ and $\sigma = 1$ and denoting the elements of $\hat{\mathrm{Cov}}_n(\hat{\mu},\hat{\sigma})$ as $\frac{\hat{\sigma}^2}{n} d_{ij}$, then (3.6) can be written as

$$\frac{-\hat{\mu} + (1-\hat{\sigma})z(p)}{(d_{11} + 2z(p)d_{12} + z^2(p)d_{22})^{\frac{1}{2}}} \leqslant c_{3,\nu^*}\frac{\hat{\sigma}}{\sqrt{n}} \quad (3.7)$$

which must hold for all p. Let

$$t_1 = -\hat{\mu} - \frac{d_{12}}{d_{22}}(1 - \hat{\sigma}) ,$$

$$v_1 = (1 - \hat{\sigma})/\sqrt{d_{22}} , \text{ and } \ell_1 = \left(z(p) - \frac{d_{12}}{d_{22}}\right) d_{22}/(d_{11}d_{22} - d_{12}^2)^{\frac{1}{2}} .$$

Substituting the solutions of these equations for $\hat{\mu}$ and $1-\hat{\sigma}$ in (3.7) gives

$$\frac{t_1 + \ell_1 v_1}{(1+\ell_1^2)^{\frac{1}{2}}} \leq c_{3,\nu^*} \frac{\hat{\sigma}}{\sqrt{n}} \tag{3.8}$$

which must hold for all ℓ_1. Applying Lemma II of Chapter II, the inequality (3.8) is satisfied for all ℓ_1 if and only if

$$\begin{aligned} &t > 0 \quad \text{and} \quad t_1^2 + v_1^2 \leq c_{3,\nu^*}^2 \frac{\hat{\sigma}^2}{n} \\ \text{or}& \\ &t_1 \leq 0 \quad \text{and} \quad v_1^2 \leq c_{3,\nu^*}^2 \frac{\hat{\sigma}^2}{n} \;. \end{aligned} \tag{3.9}$$

Three sets of 1000 samples of sizes $n = 20, 40, 40$ were generated and censored with probabilities .4, .4, and .7 respectively, as in the two-sided band simulations. Table 11 presents the fraction of samples out of 1000 that gave upper confidence bands for which the true underlying percentile $y(p) = \mu + z(p)\sigma$ was above the band for some value of p using nominal confidence levels $1-\alpha$ with $\alpha = .01$, .05, and .10. By comparing entries in Tables 8 and 11, it is seen that the one-sided bands (C3EO) do not perform as well as the corresponding two-sided bands (CEO).

For the simple linear regression model the upper one-sided bands in x and p are defined by the upper bound

$$\hat{\beta}_0 + \hat{\beta}_1 x + z(p)\hat{\sigma} + c_{3,\nu^*} [\hat{\mathrm{Var}}(\hat{\beta}_0 + \hat{\beta}_1 x + z(p)\hat{\sigma})]^{\frac{1}{2}} \tag{3.10}$$

for percentiles $y(p,x) = \beta_0 + \beta_1 x + z(p)\sigma$. Taking $\beta_0 = 0$, $\beta_1 = 1$,

Table 11. Empirical error rates of one-sided bands for $\mu + z(p)\sigma$ from censored samples

n	Censoring Probability	α	C3EO
20	.4	.01	.011
		.05	.073
		.10	.141
40	.4	.01	.019
		.05	.075
		.10	.124
40	.7	.01	.010
		.05	.060
		.10	.124

and $\sigma = 1$ the percentiles $y(p,x) = x + z(p)$ will be contained in the confidence band for all x and p if and only if

$$\frac{[-\hat{\beta}_0 + (1-\hat{\beta}_1)x + (1-\hat{\sigma})z(p)]}{\left\{(1,x,z(p)) \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{bmatrix} \begin{pmatrix} 1 \\ x \\ z(p) \end{pmatrix}\right\}^{\frac{1}{2}}} \leqslant c_{3,\nu*} \frac{\hat{\sigma}}{\sqrt{n}} \tag{3.11}$$

holds for all x and p. The following transformation is used for application of Lemma II:

$$h_1 = -\hat{\beta}_0 + \xi_1(1-\hat{\beta}_1) + \xi_2(1-\hat{\sigma}), \quad h_2 = (1-\hat{\beta}_1) + \xi_3(1-\hat{\sigma})$$

$$h_3 = (1-\hat{\sigma}), \quad b_1 = x - \xi_1, \quad b_2 = z(p) - \xi_2 - \xi_3\,(b_1), \tag{3.12}$$

$$w_1 = d_{11} + 2\xi_1 d_{12} + 2\xi_2 d_{13} + \xi_1^2\, d_{22} + \xi_1\xi_2\, d_{23} + \xi_2^2\, d_{33},$$

$$w_2 = d_{22} + 2\xi_3 d_{23} + \xi_3^2\, d_{33}, \quad w_3 = d_{33},$$

$$\xi_1 = \frac{d_{13}d_{23} - d_{12}d_{33}}{d_{22}d_{33} - d_{23}}, \quad \xi_2 = \frac{d_{12}d_{23} - d_{13}d_{22}}{d_{22}d_{33} - d_{23}}, \quad \xi_3 = \frac{-d_{23}}{d_{33}},$$

$$t_2 = h_1/\sqrt{w_1}, \quad \underline{\ell}'_2 = \left(b_1\left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}, \; b_2\left(\frac{w_3}{w_1}\right)^{\frac{1}{2}}\right),$$

$$\underline{v}_2' = \left(\frac{h_2}{\sqrt{w_2}}, \; \frac{h_3}{\sqrt{w_3}}\right).$$

The first three equations in (3.12) are solved for $\hat{\beta}_0$, $\hat{\beta}_1$, and $(1-\hat{\sigma})$ with the solutions substituted in (3.11) to give

$$\frac{t_2 + \underline{\ell}_2'\underline{v}_2}{(1 + \underline{\ell}_2'\underline{\ell}_2)^{\frac{1}{2}}} \leqslant c_{3,\nu*} \frac{\hat{\sigma}}{\sqrt{n}} \text{ for all } \underline{\ell}_2 . \tag{3.13}$$

Applying Lemma II (3.13) is satisfied for all ℓ_2 if and only if

$$t_2 > 0 \quad \text{and} \quad t_2^2 + \underline{v}_2'\underline{v}_2 \leqslant c_{3,\nu*}^2 \frac{\hat{\sigma}^2}{n}$$

or (3.14)

$$t_2 \leqslant 0 \quad \text{and} \quad \underline{v}_2'\underline{v}_2 \leqslant c_{3,\nu*}^2 \frac{\hat{\sigma}^2}{n} \quad .$$

Then if a given sample does not satisfy (3.14) the true underlying percentile, $\beta_0 + \beta_1 x + z(p)\sigma$, is not bounded by the upper one-sided band. As in the two-sided band simulation, three sets of 500 samples of sizes $n = 20, 40, 40$ were generated and censored with probabilities .4 , .4 , and .7 , respectively. The fraction of samples (based on 500) which did not satisfy (3.14) are presented in Table 12 for nominal confidence levels $1-\alpha$ with $\alpha = .01, .05, .10$. The results are similar to those given for one-sided bands for $\mu + z(p)\sigma$.

III.5. Simultaneous Tolerance Intervals from Censored Samples

Two-sided tolerance bands in $\underline{x}$ and p or in $\underline{x}$ with p fixed of the form (2.21) can be used for censored data with $h(s,\underline{x},p)$ replaced by the square root of expression (3.1). The effective degrees of freedom (3.2) can be used for the coefficients c_1 and c_2 in (2.21).

Table 12. Empirical error rates of one-sided bands for $\beta_0 + \beta_1 x + z(p)\sigma$ from censored samples

n	Censoring Probability	α	C3EO
20	.4	.01	.010
		.05	.074
		.10	.158
40	.4	.01	.014
		.05	.076
		.10	.132
40	.7	.01	.014
		.05	.068
		.10	.116

IV. CONFIDENCE BANDS OVER FINITE INTERVALS

IV.1. Preview and Preliminaries

The usual Scheffé bands for the regression line give exact $1-\alpha$ confidence level only when the independent variable x is allowed to vary over the entire real line. When x is restricted to an interval (a,b) with both a and b finite or with only a or b finite the Scheffé method yields necessarily conservative bands. Similarly, the confidence bands in $\underline{x}$ and p described in Chapter II for percentiles, $y(p,\underline{x}) = \underline{x}'\underline{\beta} + z(p)\sigma$ are conservative when p is restricted to some interval contained in $[0,1]$. Wynn and Bloomfield (1971) tabulated coefficients, $c=\lambda$, for two-sided confidence bands for straight-line regressions $(p = 1/2)$ over finite intervals of the independent variable. Since the maximum likelihood estimator $(\hat{\mu},\hat{\sigma})$ of (μ,σ) is asymptotically bivariate normal, the Wynn and Bloomfield results can be used to form approximate confidence bands for $y(p) = \mu + z(p)\sigma$ over finite intervals of p. The performance of these bands is examined by simulation for complete and censored samples. The effective degrees of freedom ν^*, developed in Section III.2, are used for censored samples. Applying the Wynn and Bloomfield results to percentiles from censored samples requires the use of a more general form of the covariance matrix than for straight line regressions with complete samples. The Wynn and Bloomfield method is extended for general covariance matrices. The Bohrer and Francis (1972) one-sided bands $(p = 1/2)$ are generalized for percentiles $y(p,x) = \beta_0 + \beta_1 x + z(p)\sigma$ with fixed $p \neq 1/2$ and $y(p) = \mu + z(p)\sigma$ with $a < p < b$.

IV.2. Two-sided Bands for $\mu + z(p)\sigma$ over Finite Intervals of p.

Let $\hat{\underline{\gamma}}' = (\hat{\gamma}_0, \hat{\gamma}_1)$ have a bivariate normal distribution with mean $\underline{\gamma}' = (\gamma_0, \gamma_1)$ and covariance matrix $\sigma^2 \underline{V}$ and let $\nu\hat{\sigma}^2/\sigma^2$ be independently χ^2_ν distributed. Then coefficients λ are sought such that

$$\Pr\{-\lambda\hat{\sigma}(\underline{x}'\underline{V}\,\underline{x})^{\frac{1}{2}} \leq \underline{x}'(\hat{\underline{\gamma}}-\underline{\gamma}) \leq \lambda\hat{\sigma}(\underline{x}'\underline{V}\,\underline{x})^{\frac{1}{2}} \text{ for all } \underline{x} \text{ in } \mathscr{A}\} = 1-\alpha\,, \tag{4.1}$$

where $\mathscr{A} = \{(1,x_1)\colon a \leq x_1 \leq b\}$. Let $\underline{B}$ be a nonsingular matrix such that $\underline{V} = \underline{B}'\underline{B}$ and apply the transformation

$$y = \underline{B}'\underline{x} \quad \text{and} \quad \underline{u} = \underline{B}^{-1}(\hat{\underline{\gamma}}-\underline{\gamma})/\hat{\sigma}\,.$$

Then (4.1) reduces to

$$\Pr\{-\lambda(\underline{y}'\underline{y})^{\frac{1}{2}} \leq \underline{u}'\underline{y} \leq \lambda\,(\underline{y}'\underline{y})^{\frac{1}{2}} \text{ for all } \underline{y} = \underline{B}'\underline{x} \text{ with } \underline{x} \text{ in } U\} = 1-\alpha\,. \tag{4.2}$$

The random vector $\underline{u}$ has bivariate Students-t density

$$f_{\underline{u}}(\underline{u}) = \frac{1}{2\pi}\left(1 + \frac{\underline{u}'\underline{u}}{\nu}\right)^{-\left(\frac{\nu}{2}+1\right)}$$

The region of integration for probability (4.2) is illustrated in Figure 3. The boundaries of integration are found by maximizing $\underline{u}'\underline{y}$ subject to the restriction $\underline{u}'\underline{u} \leq \lambda^2$. The vector $\underline{u}^*$ that maximizes $\underline{u}'\underline{y}$ subject to this restriction is

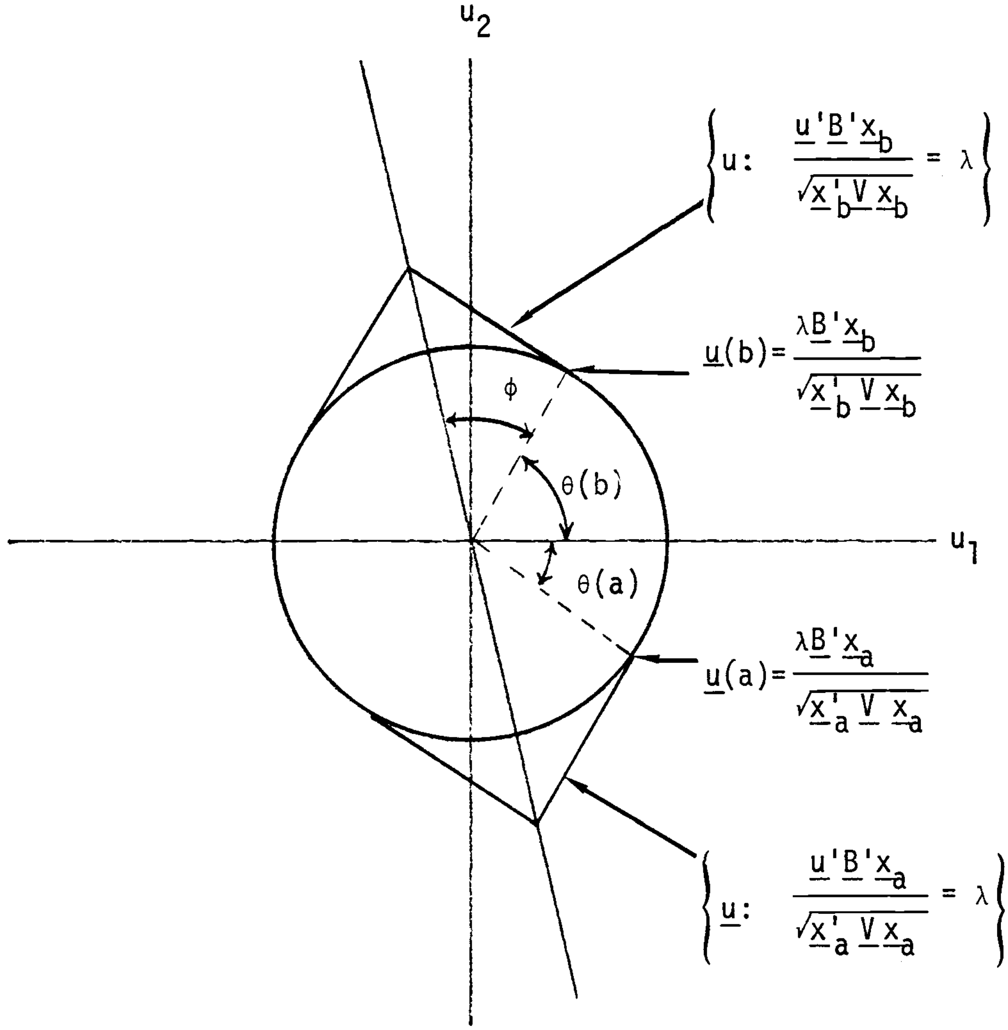

Figure 3. Region of integration for probability (4.2).

$$\underline{u}^* = \frac{\pm\lambda}{\sqrt{\underline{y}'\underline{y}}}\ \underline{y} = \frac{\pm\lambda\,\underline{B}'\underline{x}}{\sqrt{\underline{x}'\underline{V}\underline{x}}}\ , \tag{4.3}$$

As $\underline{x}$ varies over U the vector $\underline{u}^*$ in (4.3) defines two arcs of the circumference of the circle with radius λ and center the origin. The remaining boundaries are formed by the tangents at the ends of the arcs. The probability content of the region is calculated as the sum of the contents of the circle and the two regions outside the circle. Let

$$\overline{\beta} = \tan\left(\frac{\theta(b) - \theta(a)}{2}\right), \tag{4.4}$$

where $\theta(\xi)$ is the angle from zero to the line joining the origin to the point $\underline{u}(\xi)$. Let $\underline{x}_a' = (1,a)$ and $\underline{x}_b' = (1,b)$. Using the well-known half-angle formula (4.4) reduces to

$$\overline{\beta} = \frac{\sqrt{\det(\underline{V})}\ \ (b-a)}{\underline{x}_b'\underline{V}\underline{x}_a + \sqrt{(\underline{x}_b'\underline{V}\underline{x}_b)(\underline{x}_a'\underline{V}\underline{x}_a)}} \tag{4.5}$$

where $\det(\underline{V})$ is the determinant of $\underline{V}$. For simple linear regression (4.5) reduces to that given by Wynn and Bloomfield (1971,p.211). Probability (4.2) then only depends on $\overline{\beta}$ and λ. The calculations are made easier by transformation of u_1, u_2 to polar coordinates r,θ. Then (4.2) can be evaluated as

$$4\int_0^{\phi}\int_0^{-1/\cos(\theta)} \frac{1}{2\pi}\left(1+\frac{r^2}{\nu}\right)^{-(\frac{\nu}{2}+1)} r\,drd\theta + 4\int_0^{\lambda}\int_0^{\pi/2} \frac{1}{2\pi}\left(1+\frac{r^2}{\nu}\right)^{-(\frac{\nu}{2}+1)} r\,drd\theta\ , \tag{4.6}$$

where $\phi = \tan^{-1}(\bar{\beta}^{-1})$. The first term in sum (4.6) is the probability content of the region outside the circle. The second term is the content of the circle. After integration with respect to r, (4.6) equals

$$\frac{2}{\pi}\left[\phi\left(1+\frac{\lambda^2}{\nu}\right)^{-\frac{\nu}{2}} - \int_0^{\phi}\left(1+\frac{\lambda^2}{n\cos^2\theta}\right)^{-\frac{\nu}{2}} d\theta\right] + 1 - \left(1+\frac{\lambda^2}{\nu}\right)^{-\frac{\nu}{2}} . \quad (4.7)$$

(The expressions given in Wynn and Bloomfield (1971, p. 211) corresponding to (4.6) and (4.7) contain misprints.)

Taking $\underline{\gamma}' = (\mu,\sigma)$, $\hat{\underline{\gamma}}' = (\hat{\mu},\hat{\sigma})$, and $\underline{x}' = (1,z(p))$, the above results can be used to give two-sided confidence bands in p for $y(p) = \mu + z(p)\sigma$, over an interval $[p_1,p_2]$. The bands are formed by the bounds

$$\hat{\mu} + z(p)\hat{\sigma} \pm \lambda\left[(1,z(p))\ \mathrm{Cov}(\hat{\mu},\hat{\sigma})\begin{pmatrix}1\\ z(p)\end{pmatrix}\right]^{\frac{1}{2}} \quad (4.8)$$

as p varies over $[p_1,p_2]$. Let $\frac{\hat{\sigma}^2}{n}\, d_{ij}$ denote the elements of the observed covariance matrix. In the case of complete samples $d_{11} = 1$, $d_{12} = d_{21} = 0$, and $d_{22} = 1/2$. For censored samples effective degrees of freedom ν^* are used in $\lambda = \lambda(\bar{\beta}, \alpha, \nu^*)$.

The accuracy of the nominal confidence levels for bands (4.8) is examined by simulation. For a given sample these bands contain the underlying percentile, $y(p) = \mu + z(p)\sigma$, for all p in $[p_1,p_2]$ if and only if

$$|(\mu + z(p)\sigma) - (\hat{\mu} + z(p)\hat{\sigma})| \le \lambda \frac{\hat{\sigma}}{\sqrt{n}}\,[d_{11} + 2d_{12}z(p) + d_{22}z^2(p)]^{\frac{1}{2}}, \quad (4.9)$$

holds for all p in $[p_1, p_2]$. Without loss of generality take $\mu = 0$ and $\sigma = 1$. Then (4.9) holds if and only if

$$\psi_3(z(p)) = a_3 z^2(p) + b_3 z(p) + d_3 \leq 0 ,$$

for all p in $[p_1, p_2]$, where the coefficients a_3, b_3, and d_3 depend on the estimator $(\hat{\mu}, \hat{\sigma})$, λ, and the elements of the covariance matrix. Then the underlying percentile, $y(p) = \mu + z(p)\sigma$, is <u>not</u> contained within the confidence band for at least some p if and only if:

i) $a_3 > 0$ and $b_3^2 - 4a_3 d_3 < 0$ with either or both

$$a_3 z^2(p_i) + b_3 z(p_i) + d_3 \geq 0 , \quad i=1,2$$

or ii) $a_3 < 0$ and $b_3^2 - 4a_3 d_3 > 0$ with either the maximum of $\psi_3(z(p))$ in the interval $[z(p_1), z(p_2)]$ or the maximum not in the interval $[z(p_1), z(p_2)]$ with either or both

$$a_3 z^2(p_i) + b_3 z(p_i) + d_3 \geq 0 , \quad i=1,2 .$$

Five sets of 1000 samples of sizes $n = 20, 20, 40, 40, 40$ were generated from a standard normal distribution and censored with probabilities 0, .4, 0, .4, and .7, respectively. The tables of coefficients $\lambda(\overline{\beta}, \alpha, \nu)$ by Wynn and Bloomfield are extended in Appendix C to include additional degrees of freedom $\nu = 3, 7$. These additional values were included to allow for more accurate linear interpolation for small degrees of freedom. Four finite intervals were chosen for investigation: $[p_1, p_2] = [0,1/4], [0,1/2], [0,3/4]$, and $[1/4,3/4]$. Table 13 presents the fraction of samples out of 1000 that gave confidence bands which did not contain the true underlying percentile $y(p) = \mu + z(p)\sigma$ for some value of p in $[p_1, p_2]$ using nominal confidence levels $1-\alpha$ with $\alpha = .01, .05$, and $.10$. Examination of Table 13 reveals that these bands perform reasonably well. The asterisks indicate cases where the difference between the empirical and nominal α levels are statistically significant at the 5% level. The accuracy tends to improve with sample size and α.

Table 13. Empirical error rates of two-sided bands for $\mu + z(p)\sigma$ over finite intervals of p

n	Censoring probability	α	Finite interval of p [1/4, 1/2]	[0, 1/4]	[0, 1/2]	[0, 3/4]
20	0	.01	.014	.024*	.020*	.026*
		.05	.063	.065*	.055	.078*
		.10	.109	.108	.099	.121*
20	.4	.01	.009	.018*	.011	.020*
		.05	.040	.061	.054	.071*
		.10	.081	.109	.085	.108
40	0	.01	.010	.021*	.010	.015
		.05	.049	.060	.040	.052
		.10	.101	.093	.096	.091
40	.4	.01	.008	.014	.016	.020*
		.05	.040	.058	.033	.063
		.10	.081	.099	.076	.099
40	.7	.01	.007	.014	.022*	.012
		.05	.023	.051	.063	.033
		.10	.048	.085	.093	.064

IV.3. Some Miscellaneous Bands over Finite Intervals

Several of the band procedures described in Chapter II for forming one- and two-sided bands for percentiles can be extended to finite interval bands. One-sided bands for the 100pth percentile, $\beta_0 + \beta_1 x + z(p)\sigma$, simultaneous in x over $[a,b]$ with p fixed can be constructed following derivations given by Bohrer and Francis (1972) for finite bands for a regression line. One-sided bands for the 100pth percentile, $\mu + z(p)\sigma$, simultaneous in p over $[p_1,p_2]$ can be constructed in a similar manner. A brief discussion of these deviations is given in this section. A Wynn and Bloomfield type procedure for two-sided bands for regression percentiles over finite intervals of the independent variable with p fixed can be constructed. This latter construction would prove quite cumbersome because the derivation results in a nontrivial numerical integration of two dimensions. This derivation is not pursued here.

The one-sided bands for regression percentiles with p fixed constructed in Chapter II can serve as a basis from which to form bands over finite intervals. For the simple linear regression model, the coefficients c_4 can be found such that

$$\Pr\{(\beta_0 + \beta_1 x + z(p)\sigma) - (\hat{\beta}_0 + \hat{\beta}_1 x + z(p)\hat{\sigma}) \leq c_4\ h(s,x,p) \tag{4.10}$$

$$\text{for all } x \text{ in } [a,b] = 1 - \alpha ,$$

where $h(s,x,p) = s\left(\frac{1}{n} + \frac{x^2}{\Sigma x_2} + \frac{z^2(p)}{2\nu}\right)^{\frac{1}{2}}$. Recall that the x's are mean adjusted. The practical matter of finding coefficients c_4 that satisfy (4.10) is made easier by letting

$$\underline{\theta} = \left(\left(1 + \frac{nz^2(p)}{2\nu}\right)^{-\frac{1}{2}} \left\{\sqrt{n}(\beta_0 - \hat{\beta}_0) + \sqrt{n}(\sigma - \hat{\sigma})\right\},\ (\Sigma x^2)^{\frac{1}{2}}(\beta_1 - \hat{\beta}_1)\right)$$

and

$$\underline{y} = \left(1,\ \frac{x}{(\Sigma x_i^2)^{\frac{1}{2}}} \left(1 + \frac{nz^2(p)}{2\nu}\right)^{-\frac{1}{2}}\right).$$

Then the region of integration to find probability (4.10) is given by

$$R = \{(\underline{\theta}, s) : \underline{y}'\underline{\theta} \leq c_4 s \| y \| \text{ for all } y \in Y^*\}, \tag{4.11}$$

where

$$Y^* = \left\{\left(1,\ \frac{y}{(\Sigma x_i^2)^{\frac{1}{2}}} \left(1 + \frac{nz^2(p)}{2\nu}\right)^{-\frac{1}{2}}\right) : y \in [a,b]\right\}.$$

The region of integration for the case where x was allowed to vary over the entire real line was illustrated in Figure 1 of Chapter II. For comparison, Figure 4 illustrates the region (4.11) where x is restricted to the finite interval $[a,b]$. The extreme vectors in Y^* are

$$\underline{a}' = \left(1,\ \frac{a}{(\Sigma x_i^2)^{\frac{1}{2}}} \left(1 + \frac{nz^2(p)}{2\nu}\right)^{-\frac{1}{2}}\right)$$

and

$$\underline{b}' = \left(1,\ \frac{b}{(\Sigma x_i^2)^{\frac{1}{2}}} \left(1 + \frac{nz^2(p)}{2\nu}\right)^{-\frac{1}{2}}\right)$$

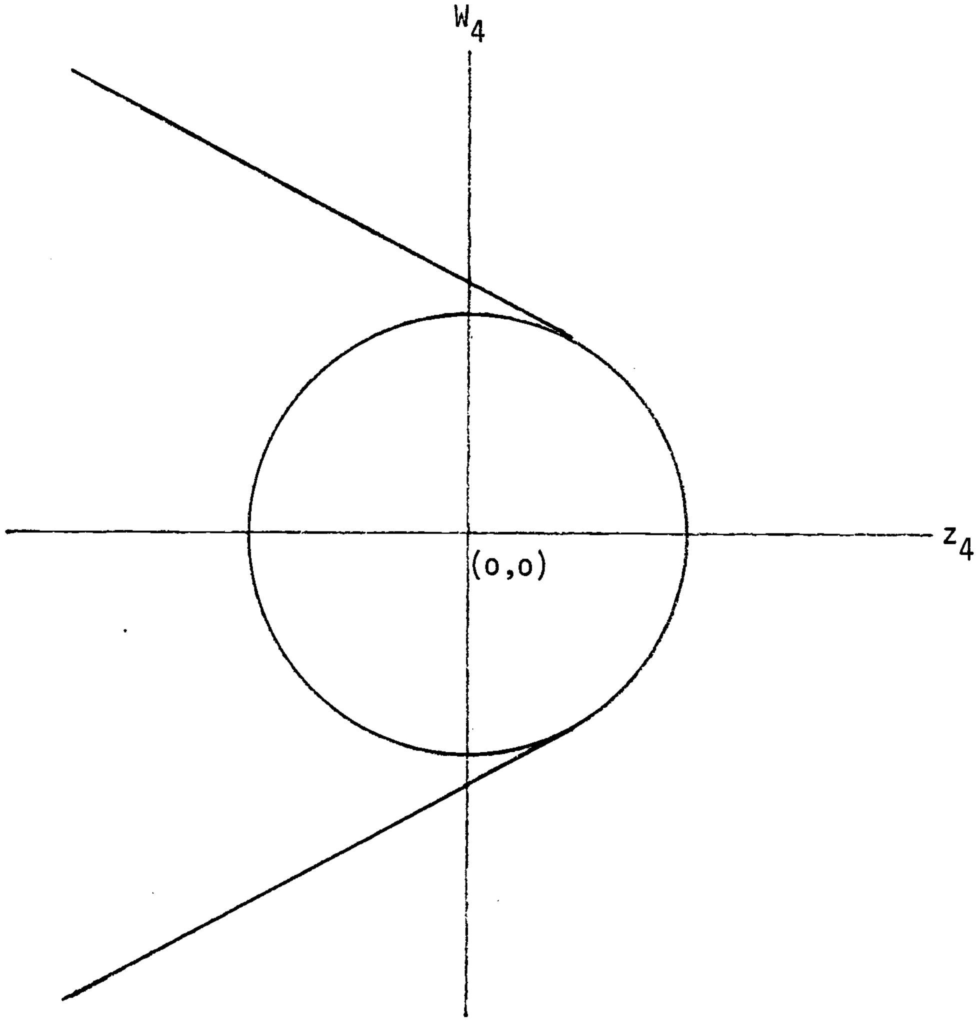

Figure 4. Region of integration (4.11) for probability (4.10).

Let $\phi(y)$ be the angle between the lines formed by joining the origin to the two vectors $\underline{a}$ and $\underline{y}$ and let $\phi^* = \phi(\underline{b})$. Then Bohrer and Francis (1972) give a lemma which shows that the region R can be partitioned into the four regions

$$R_{2_1} = \{\underline{\theta} : \|\underline{\theta}\| \leqslant c_4 S \ , \ 0 \leqslant \phi(\underline{\theta}) \leqslant \phi^* \} ,$$

$$R_{s_2} = \{\underline{\theta} : 0 \leqslant \underline{b}'\underline{\theta} \leqslant c_4 S\|b\| \ , \ \phi^* \leqslant \phi(\underline{\theta}) \leqslant \phi^* + \tfrac{\pi}{2}\} ,$$

$$R_{s_3} = \{\underline{\theta} : \phi^* + \tfrac{\pi}{2} \leqslant \phi \leqslant \tfrac{3\pi}{2} \} ,$$

and $$R_{s_4} = \{\underline{\theta} : 0 \leqslant \underline{a}'\underline{\theta} \leqslant c_4 S\|\underline{a}\| \ , \ \tfrac{3\pi}{2} \leqslant \phi(\theta) \leqslant 2\pi \} .$$

Conditioning on $U = u$, where $U = \frac{s^2}{\sigma^2}$, the probability content of the regions R_{si} can be numerically evaluated following Bohrer and Francis. The iterative procedure described in Chapter II for finding coefficients c_1, c_2, c_3, and c_4 can then be used to find coefficients c_4 of the finite interval bands.

Figure 4 can also be used to describe the region of integration used in determining coefficients of one-sided bands for $\mu + z(p)\sigma$ over finite intervals of p. In this case, the vectors $\underline{y}$ and $\underline{\theta}$ are taken as

$$\underline{y} = \left(1, \ z(p) \left(\frac{n}{2\nu}\right)^{\frac{1}{2}} \right)$$

$$\underline{\theta} = (\sqrt{n}\,(\mu-\hat{\mu}), \ \sqrt{2\nu}\,(\sigma-\hat{\sigma}))$$

The set Y^* here is $\{(1, \ z(p)(\frac{n}{2\nu})^{\frac{1}{2}}) : p \in [p_1,p_2]\}$. The derivation then follows the regression percentile procedure.

V. SUMMARY

Coefficients c for one and two-sided confidence bands for percentiles with $a = 4\nu/(4\nu-1)$ and $b = 1/(2\nu)$ in (1.2) were evaluated. Bands in $\underline{x}$ and p with $b = 1/(2\nu)$ were shown to be narrower than the Steinhorst and Bowden (1971) bands $(b=1)$ except in extreme cases. For $\underline{x}$ near its mean, the fixed p bands with $b = 1/(2\nu)$ were found to be slightly wider than those given by Steinhorst and Bowden (1971) and Turner and Bowden (1977, 1979) $(b=0)$.

It was shown that simultaneous tolerance intervals can be formed using the coefficients c for two-sided bands for percentiles. The derivation of this relationship was adapted from a Scheffé type procedure described by Lieberman and Miller (1963). Such intervals using $b = 1/(2\nu)$ are shown to be narrower than previously described methods.

The confidence bands for percentiles were extended to the censored sample situation. A method of assigning finite effective degrees of freedom to the estimator of σ was proposed. Simulation results indicate that the approximate one and two-sided bands using the coefficients c given in Chapter II and effective degrees of freedom yield accurate confidence levels.

Confidence bands for percentiles of a single normal distribution over restricted intervals of p were developed using asymptotic results. Simulation results for complete and censored samples indicate that this procedure yields accurate confidence levels even under heavy censoring.

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APPENDICES

VII. APPENDICES

Appendix A

Table A.1. Coefficients c_1 of two-sided bands for percentiles, $y(p,x) = \beta_0 + \beta_1 x + z(p)\sigma$ (q = 2).

ν	α = .10	.05	.01
3	4.394	6.195	12.326
4	3.817	5.130	9.184
5	3.505	4.580	7.688
6	3.311	4.239	6.817
7	3.177	4.011	6.248
8	3.081	3.845	5.848
9	3.007	3.717	5.544
10	2.950	3.620	5.326
11	2.903	3.540	5.139
12	2.864	3.474	5.000
13	2.833	3.419	4.867
14	2.804	3.372	4.759
15	2.781	3.330	4.668
17	2.744	3.264	4.518
20	2.702	3.189	4.351
25	2.657	3.106	4.162
30	2.628	3.050	4.036
40	2.592	2.982	3.875
60	2.560	2.916	3.711
∞	2.498	2.793	3.368

Table A.2. Coefficients c_1 of two sided bands for percentiles $y(p) = \mu + z(p)\sigma$ $(q = 1)$.

ν	α = .10	.05	.01
3	3.734	5.426	11.175
4	3.239	4.496	8.340
5	2.972	4.006	6.992
6	2.806	3.709	6.207
7	2.692	3.505	5.692
8	2.609	3.357	5.333
9	2.546	3.246	5.042
10	2.497	3.158	4.846
11	2.457	3.088	4.679
12	2.425	3.030	4.530
13	2.399	2.980	4.422
14	2.376	2.937	4.325
15	2.357	2.904	4.247
17	2.328	2.843	4.101
20	2.294	2.777	3.947
25	2.260	2.703	3.773
30	2.239	2.655	3.656
40	2.213	2.599	3.500
60	2.191	2.543	3.351
∞	2.140	2.450	3.040

Appendix B

Table B.1. Coefficients c_3 of one-sided bands for percentiles $y(p,x) = \beta_0 + \beta_1 x + z(p)\sigma$ $(q = 2)$.

ν	$\alpha =$ .10	.05	.01
3	4.086	5.833	11.808
4	3.551	4.838	8.784
5	3.263	4.317	7.362
6	3.084	4.000	6.542
7	2.961	3.783	5.998
8	2.871	3.628	5.611
9	2.804	3.510	5.332
10	2.751	3.417	5.110
11	2.708	3.342	4.935
12	2.673	3.280	4.793
13	2.644	3.228	4.673
14	2.619	3.184	4.567
15	2.598	3.145	4.479
17	2.564	3.083	4.339
20	2.527	3.014	4.176
25	2.487	2.935	3.996
30	2.462	2.885	3.874
40	2.431	2.823	3.722
60	2.403	2.763	3.563
∞	2.350	2.660	3.240

Table B.2. Coefficients c_3 of one-sided bands for percentiles $y(p) = \mu + z(p)\sigma$ $(q = 1)$.

ν	α = .10	.05	.01
3	3.345	5.024	10.801
4	2.898	4.135	7.750
5	2.668	3.700	6.704
6	2.527	3.400	5.850
7	2.415	3.213	5.350
8	2.350	3.093	5.000
9	2.285	2.980	4.750
10	2.240	2.889	4.554
11	2.202	2.850	4.450
12	2.180	2.781	4.350
13	2.157	2.750	4.290
14	2.140	2.696	4.150
15	2.119	2.670	3.945
17	2.092	2.613	3.850
20	2.073	2.551	3.736
25	2.046	2.479	3.565
30	2.027	2.442	3.450
40	2.007	2.394	3.319
60	1.987	2.353	3.172
∞	1.950	2.270	2.880

Appendix C

Table C.1. Extension of λ tables given by Wynn and Bloomfield (1971)

90% levels

Degrees of freedom

$\bar{\beta}$	3	5	7	10	15	20	30	40	60	120	∞
.00	2.35	2.01	1.89	1.81	1.75	1.72	1.68	1.68	1.67	1.66	1.65
.05	2.42	2.06	1.94	1.85	1.79	1.76	1.73	1.72	1.70	1.69	1.68
.10	2.49	2.11	1.99	1.89	1.83	1.80	1.77	1.85	1.74	1.73	1.71
.15	2.56	2.16	2.03	1.93	1.87	1.84	1.81	1.79	1.77	1.76	1.75
.20	2.62	2.21	2.08	1.97	1.90	1.87	1.84	1.82	1.81	1.79	1.78
.30	2.73	2.30	2.15	2.04	1.97	1.93	1.90	1.88	1.87	1.85	1.84
.40	2.83	2.37	2.22	2.10	2.02	1.99	1.95	1.93	1.92	1.90	1.88
.60	2.98	2.49	2.32	2.19	2.12	2.07	2.03	2.01	1.49	1.98	1.96
.80	3.08	2.57	2.39	2.26	2.17	2.13	2.09	2.07	2.05	2.03	2.01
1.00	3.15	2.62	2.44	2.31	2.22	2.17	2.13	2.11	2.09	2.07	2.05
1.5	3.24	2.69	2.50	2.37	2.27	2.23	2.18	2.16	2.15	2.12	2.10
2.0	3.27	2.72	2.53	2.39	2.29	2.25	2.20	2.18	2.16	2.14	2.12
∞	3.31	2.75	2.55	2.42	2.32	2.27	2.23	2.21	2.19	2.17	2.14

Table C.2. Extension of λ tables given by Wynn and Bloomfield (1971)

95% levels

Degrees of freedom

$\bar{\beta}$	3	5	7	10	15	20	30	40	60	120	∞
.00	3.18	2.57	2.36	2.23	2.13	2.08	2.04	2.02	2.00	1.98	1.96
.05	3.27	2.62	2.42	2.27	2.17	2.12	2.08	2.06	2.03	2.02	1.99
.10	3.36	2.68	2.47	2.31	2.21	2.16	2.12	2.10	2.07	2.05	2.03
.15	3.44	2.74	2.52	2.36	2.25	2.20	2.15	2.13	2.11	2.08	2.06
.20	3.51	2.79	2.56	2.40	2.29	2.23	2.18	2.16	2.14	2.11	2.09
.30	3.65	2.88	2.64	2.47	2.35	2.30	2.25	2.22	2.19	2.17	2.15
.40	3.77	2.97	2.71	2.53	2.41	2.35	2.29	2.27	2.24	2.21	2.19
.60	3.96	3.10	2.82	2.62	2.49	2.43	2.37	2.34	2.31	2.29	2.26
.80	4.09	3.19	2.89	2.69	2.55	2.49	2.43	2.43	2.38	2.37	2.31
1.0	4.17	3.25	2.95	2.74	2.60	2.53	2.47	2.44	2.41	2.38	2.35
1.5	4.28	3.33	3.02	2.81	2.67	2.59	2.52	2.49	2.46	2.43	2.40
2.0	4.33	3.36	3.05	2.83	2.68	2.61	2.55	2.51	2.48	2.45	2.42
∞	4.37	3.40	3.08	2.86	2.71	2.64	2.58	2.54	2.51	2.48	2.45

Table C.3. Extension of λ tables given by Wynn and Bloomfield (1971)

99% levels

Degrees of freedom

$\bar{\beta}$	3	5	7	10	15	20	30	40	60	120	∞
.00	5.84	4.03	3.50	3.16	2.95	2.84	2.75	2.70	2.66	2.62	2.58
.05	5.99	4.10	3.57	3.22	2.99	2.88	2.79	2.74	2.69	2.65	2.61
.10	6.13	4.18	3.63	3.27	3.03	2.93	2.83	2.78	2.73	2.69	2.64
.15	6.27	4.26	3.68	3.32	3.07	2.96	2.86	2.81	2.76	2.72	2.67
.20	6.39	4.33	3.74	3.36	3.11	3.00	2.89	2.84	2.80	2.75	2.70
.30	6.62	4.45	3.83	3.44	3.18	3.06	2.95	2.90	2.85	2.80	2.75
.40	6.82	4.56	3.91	3.50	3.24	3.11	3.00	2.95	2.89	2.84	2.79
.60	7.14	4.73	4.04	3.61	3.32	3.20	3.07	3.02	2.96	2.91	2.86
.80	7.36	4.85	4.13	3.68	3.39	3.25	3.13	3.07	3.01	2.95	2.90
1.0	7.50	4.94	4.20	3.74	3.43	3.30	3.17	3.11	3.05	2.99	2.94
1.5	7.69	5.05	4.29	3.81	3.50	3.36	3.22	3.16	3.10	3.04	2.98
2.0	7.77	5.10	4.33	3.85	3.53	3.38	3.25	3.19	3.15	3.06	3.01
∞	7.85	5.15	4.37	3.89	3.57	3.42	3.28	3.22	3.15	3.10	3.04