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Ergodic Theory and Dynamical Systems / Volume 33 / Issue 04 / August 2013, pp 969 - 982
DOI: 10.1017/S0143385712000107, Published online: 01 May 2012

Link to this article: http://journals.cambridge.org/abstract_S0143385712000107

How to cite this article:

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(Received 4 December 2010 and accepted in revised form 31 January 2012)

We dedicate this paper to the memory of our great friend and teacher, Dan Rudolph

Abstract. We prove that for all ergodic extensions $S_1$ of a transformation by a locally compact second countable group $G$, and for all $G$-extensions $S_2$ of an aperiodic transformation, there is a relative speedup of $S_1$ that is relatively isomorphic to $S_2$. We apply this result to give necessary and sufficient conditions for two ergodic $n$-point or countable extensions to be related in this way.

1. Introduction

Let $(X, B, \mu)$ be a Lebesgue probability space and $T : X \to X$ an ergodic $\mu$-preserving automorphism. By a speedup of $T$, we mean an automorphism of $X$ of the form $x \mapsto T^{p(x)}(x)$, where $p$ is a positive integer-valued function on $X$. We denote such an automorphism by $T^p$. It is natural to ask which automorphisms (up to isomorphism) can be obtained from $T$ in this way. If $p$ is integrable, there are significant restrictions on the possible speedups of $T$. It was proved in [N], for example, that if $S$ is isomorphic to $T^p$ and $\int pd\mu < \infty$, then the entropies of $S$ and $T$ satisfy $h(S) = (\int pd\mu)h(T)$. As another example, from [OW] we see that if $S$ is isomorphic to $T^{p(x)}$ and $\int pd\mu < \infty$, then $T$ is a factor of a finite measure preserving transformation that induces $S$. Thus, if $S$ is loosely Bernoulli, it can only be expressed as an integrable speedup of $T$ if $T$ is also loosely Bernoulli. However, if $p$ is not required to be integrable, then there are no obstructions to this relation; in [AOW] the general result was proved that for all ergodic finite measure preserving automorphisms $T$ and all aperiodic finite measure preserving $S$, there is a speedup of $T$ that is isomorphic to $S$. In this paper, we prove a conditional version of that result in the case of ergodic group extensions, and we give an application of this result to the classification of ergodic finite extensions.
Suppose that \((X, \mathcal{B}, \mu)\) and \(T\) are as above, and \(T\) is a factor of an automorphism \(S\) of the space \((Y, \mathcal{C}, \nu)\). Then by a speedup of \(S\ relative to \(T\), we mean a speedup \(S^p\), where \(p\) is measurable with respect to the factor \((X, \mathcal{B}, \mu)\). Of particular interest to us is the case where \(S\) is a group extension of \(T\) by a locally compact second countable group \(G\). We recall some basic definitions.

Let \(G\) be a locally compact second countable group with left Haar measure \(\lambda\), and let \(\sigma : X \times \mathbb{Z} \to G\) be a cocycle for \(T\). That is, \(\sigma\) is a measurable function such that for almost all \(x \in X\) and all \(n, m \in \mathbb{Z}\),

\[
\sigma(x, m + n) = \sigma(T^n x, m) \sigma(x, n).
\]

From \(T\) and \(\sigma\), we obtain an automorphism \(T_\sigma\) of \((X \times G, \mu \times \lambda)\) such that for all \(n \in \mathbb{Z}\),

\[
(T_\sigma)^n(x, g) = (T^n x, \sigma(x, n)g),
\]

which has \((T, X)\) as a factor. We refer to the map \(T_\sigma\) as a \(G\)-extension of \(T\), and to \((T, X)\) as the base factor of the extension. We write \(\sigma^{(n)}\) for the function

\[
\sigma^{(n)} : x \mapsto \sigma(x, n)
\]

and note that \(\sigma\) is determined by the function \(\sigma^{(1)}\), because of condition (1).

Given a cocycle \(\sigma\) for \(T\) and a measurable function \(\alpha : X \to G\), we obtain a new cocycle for \(T\), which we denote by \(\sigma^\alpha\), by setting

\[
\sigma^\alpha(x, n) = \alpha(T^n x) \sigma(x, n)(\alpha(x))^{-1}.
\]

Thus, given two functions \(\alpha, \beta : X \to G\), we have

\[
(\sigma^\alpha)^\beta = \sigma^{\beta\alpha}.
\]

Two cocycles \(\sigma\) and \(\sigma'\) for \(T\) are said to be cohomologous if there exists a measurable function \(\alpha : X \to G\) such that \(\sigma' = \sigma^\alpha\). In this case we say \(\sigma'\) is cohomologous to \(\sigma\) by the transfer function \(\alpha\).

We say that two \(G\)-extensions \(T_\sigma\) and \(T'_\sigma\), on spaces \(X\) and \(X'\) are \(G\)-isomorphic if there is an isomorphism \(\Phi : X \times G \to X' \times G\) between \(T_\sigma\) and \(T'_\sigma\), of the form

\[
\Phi(x, g) = (\phi(x), \alpha(x)g),
\]

where \(\phi : X \to X'\) is an isomorphism between \(T\) and \(T'\) and \(\alpha : X \to G\) is a measurable function. This is the case precisely when there is an isomorphism \(\phi\) between \(T\) and \(T'\) such that the cocycle \(\sigma'\phi\) for \(T\) is cohomologous to \(\sigma\) by the transfer function \(\alpha\), where \(\sigma'\phi\) is given by

\[
\sigma'\phi(x, n) = \sigma'\phi(x, n).
\]

Given a \(G\)-extension \(T_\sigma\), we consider speedups of \(T_\sigma\) relative to the base factor \((T, X)\). Each such relative speedup of \(T_\sigma\) determines and is determined by a speedup of the factor \(T\). Thus, if \((T_\sigma)^p\) is a relative speedup of \(T_\sigma\), we have \((T_\sigma)^p = (T^p)_{\sigma(p)}\), where \(\sigma^{(p)}\) is the cocycle for \(T^p\) determined by the values

\[
\sigma^{(p)}(x, 1) = \sigma(x, p(x)).
\]

Our first goal is to prove the following theorem.

**Theorem 1.** Let \(T_\sigma\) and \(T'_\sigma\), be \(G\)-extensions of \((T, X, \mathcal{B}, \mu)\) and \((T', X', \mathcal{B}', \mu')\), where \(G\) is a locally compact second countable group. Suppose that \(T_\sigma\) is ergodic and \(T'\) is...
aperiodic. Let \( U \) be a neighborhood of \( e_G \). Then there is a relative speedup of \( T_\sigma \) which is \( G \)-isomorphic to \( T'_\sigma \), by a \( G \)-isomorphism whose transfer function \( \alpha \) satisfies \( \alpha(x) \in U \) almost everywhere.

We remark that, as shown by Herman [H] and Zimmer [Z], a locally compact second countable group \( G \) admits ergodic \( G \)-extensions if and only if it is amenable.

Our original proof of this theorem in the case of ergodic \( G \)-extensions for compact \( G \) [BF] used techniques derived from the restricted orbit equivalence theory of Rudolph and Kammeyer [R], [KR1], [KR2], (and so ultimately from Ornstein’s isomorphism theorem [O]). However, as a referee has generously pointed out, a far simpler proof is available using the methods to be found in [AOW], which yields a stronger result, and we present that argument here.

We note that Theorem 1 may be viewed as an analogue of the orbit equivalence result for \( G \)-extensions obtained in [F], which was also obtained by other methods in [G]. The point of view which we take here has much in common with that of Golodets and Sinel’shchikov in their paper [GS], which deals with questions of orbit equivalence. In [G], a classification of finite extensions up to factor orbit equivalence was given and, after proving Theorem 1, we will adapt the methods of [G] to give an analogous classification of finite extensions with respect to the speedup relation we have introduced here.

2. Technical preliminaries

We make use of the following lemmas. Lemmas 1 and 2 are well known, but we include their proofs for the convenience of the reader.

**Lemma 1.** If \( T_\sigma \) is an ergodic \( G \)-extension of \( (T, X, \mu) \), then given sets \( A, B \subset X \) of positive measure, and a non-empty open set \( U \subset G \), there are a set \( A' \subset A \) of positive measure and \( n' \in \mathbb{N} \) such that \( T^n(A') \subset B \) and, for all \( x \in A' \), \( \sigma(x, n') \in U \).

**Proof.** Fix sets \( A, B \subset X \) of positive measure, and a non-empty open set \( U \subset G \). Choose non-empty open sets \( V_0 \) and \( V_1 \) in \( G \) so that \( e_G \in V_0 \) and \( V_1 V_0^{-1} \subset U \). Since \( T_\sigma \) is ergodic, for almost every \( (x, g) \in A \times V_0 \), there are (infinitely many) \( n \in \mathbb{N} \) such that

\[
(T_\sigma)^n(x, g) \in B \times V_1.
\]

Hence, there exists some \( g_0 \in V_0 \) such that for almost all \( x \in A \), there exists \( n \in \mathbb{N} \) such that

\[
(T_\sigma)^n(x, g_0) \in B \times V_1.
\]

For each \( n \in \mathbb{N} \), let \( A_n = \{x \in A : (T_\sigma)^n(x, g_0) \in B \times V_1\} \). Then, for some \( n' \in \mathbb{N} \), \( \mu(A_{n'}) > 0 \), and so for each \( x \in A_{n'} \) we have

\[
\sigma(x, n') g_0 \in V_1,
\]

so

\[
\sigma(x, n') \in V_1 g_0^{-1} \subset U.
\]

Setting \( A' = A_{n'} \), we have the desired result. \( \square \)

For \( A' \), \( B \) and \( n' \) satisfying the conclusions of this lemma, we say \((A', n')\) is \((B, U)\)-good, or simply \( A' \) is \((B, U)\)-good. We strengthen this lemma to obtain the following lemma.
LEMMA 2. If $T_\sigma$ is an ergodic $G$-extension of $(T, X, \mu)$, then, given $A, B \subset X$ of equal measure and a non-empty open set $U \subset G$, there is a measurable function $p : A \to \mathbb{N}$ such that $T^p | A$ is an isomorphism from $A$ to $B$ and, for almost every $x \in A$, $\sigma(x, p(x)) \in U$.

Proof. Fix $A, B \subset X$ of equal measure and a non-empty open set $U \subset G$. Fix $\varepsilon_i \downarrow 0$. Let

$$a_1 = \sup \{ \mu(A') : A' \subset A \text{ and } A' \text{ is } (B, U)\text{-good} \}.$$ 

Choose $A_1 \subset A$ and $n_1 \in \mathbb{N}$ such that $(A_1, n_1)$ is $(B, U)$-good and $\mu(A_1) > a_1 - \varepsilon_1$. If $\mu(A_1) = \mu(A)$, we are done. If $\mu(A_1) < \mu(A)$, let

$$a_2 = \sup \{ \mu(A') : A' \subset A \setminus A_1 \text{ and } A' \text{ is } (B \setminus (T^{n_1} A_1), U)\text{-good} \}$$

and choose $A_2 \subset A \setminus A_1$ and $n_2 \in \mathbb{N}$ such that $(A_2, n_2)$ is $(B \setminus (T^{n_1} A_1), U)$-good and $\mu(A_2) > a_2 - \varepsilon_2$.

Continue in this way to obtain a pairwise-disjoint sequence $\{A_i\}$ and integers $n_i \in \mathbb{N}$. If, for some $k \in \mathbb{N}$, $\mu(\bigcup_{i=1}^k A_i) = \mu(A)$, we are done. Suppose then that for all $k$, $\mu(\bigcup_{i=1}^k A_i) < \mu(A)$. If in fact $\mu(\bigcup_{i=1}^\infty A_i) < \mu(A)$, then, by Lemma 1, there are a set $A' \subset A \setminus (\bigcup_{i=1}^\infty A_i)$ of positive measure and $n' \in \mathbb{N}$ such that $(A', n')$ is $(B, U)$-good. But $\sum_{i=1}^\infty \mu(A_i) < \infty$, so $\mu(A_i) \to 0$, so $\mu(A_i) + \varepsilon_i \to 0$, so, for some $i$,

$$a_i < \mu(A_i) + \varepsilon_i < \mu(A'),$$

which contradicts the choice of $a_i$. Hence, $\mu(\bigcup_{i=1}^\infty A_i) = \mu(A)$, and we are done in this case as well. \qed

We note that this lemma can easily be strengthened to say that if a measurable function $p_1 : A \to \mathbb{N}$ is given, then the function $p$ can be chosen so that in addition to the conclusions of the lemma, $p(x) > p_1(x)$ almost everywhere.

In fact, we will use the following stronger form.

LEMMA 3. If $T_\sigma$ is an ergodic $G$-extension of $(T, X, \mu)$, then, given $A$ and $B \subset X$ of equal positive measure, $g : A \to G$ measurable, $p_1 : A \to \mathbb{N}$ measurable, and a neighborhood $U$ of $e_G$, there is a measurable $p : A \to \mathbb{N}$, with $p(x) > p_1(x)$ almost everywhere, such that $T^p : A \to B$ is an isomorphism, and $\sigma(x, p(x))g(x)^{-1} \in U$ almost everywhere.

Proof. Choose a neighborhood $V$ of $e_G$ such that $VV^{-1} \subset U$. Partition $A$ into measurable sets $\{A_i\}_{i=1}^\infty$ such that for each $i$, there is some $g_i \in G$ such that $g(A_i) \subset Vg_i$. Applying Lemma 2, for each $i$, choose a measurable function $q_i : A_i \to \mathbb{N}$ with $q_i > p_1$ on $A_i$ so that $T^{q_i} : A_i \to B$ is an isomorphism and $\sigma(x, q_i(x)) \in Vg_i$, almost everywhere in $A_i$, and so that the sets $\{T^{q_i} A_i\}$ are pairwise disjoint. Then, for each $x \in A_i$, we have $\sigma(x, p(x)) \in Vg_i$. But $g(x) \in Vg_i$, so $\sigma(x, q_i(x))g(x)^{-1} \in (Vg_i)(Vg_i)^{-1} = VV^{-1} \subset U$. Letting $p = \bigcup_{i=1}^\infty p_i$ completes the proof. \qed

We use the following terminology. A Rokhlin tower $T$ (or simply tower) for an automorphism $T$ on $(X, \mathcal{B}, \mu)$ is a pairwise-disjoint collection $\{A_i\}_{i=1}^h$ of measurable sets in $X$ such that for each $i$, $T(A_i) = A_{i+1}$. Each $A_i \in T$ is called a level of $T$, $A_1$ is the base, $h = h(T)$ is the height, and the common value $\mu(A_i)$ is the width $w_T$ of $T$. We
let $|T| = \bigcup_{i=1}^{h} T^i A_1$ and $|T| = \bigcup_{i=1}^{h-1} T^i A_1$. A column of $T$ is a tower of the form $(T^i B)_i = 0$, where $B$ is a measurable subset of the base of $T$.

A castle for $T$ is a finite collection $C = \{T_j\}_{j=1}^j$ of towers for $T$ such that $|T_j| \cap |T_{j+1}| = \emptyset$ for all $j_1 \neq j_2$. We let $|C| = \bigcup_{j=1}^{j} |T_j|$ and $|C|^o = \bigcup_{j=1}^{j} |T_j|^o$. We refer to $X \setminus |C|$ as the residual set of $C$. A level of $C$ (respectively a column of $C$) is a level (respectively column) of a tower in $C$. Thus, $\bigcup\{T : T \in C\}$ is the set of all levels of $C$, which we denote by $L(C)$.

If $T$ is a tower for $T$, then each finite measurable partition $Q = \{B_j\}_{j=1}^j$ of the base of $T$ gives rise to a castle $T_Q$ whose towers are the columns of $T$ with bases $B_j$. Given a finite partition $P$ of $|T|$, we obtain a partition $P_T$ of the base $B$ of $T$ whose atoms are maximal sets $B_j$ such that for every $i \in \{1, 2, \ldots, h_T\}$, $T^i B_j$ is contained in a single atom of $P$. That is, $P_T$ is the trace of $\bigvee_{i=1}^{h-1} T^{-i} P$ on $B$. This partition yields a castle $(T)_P$ as above.

We refer to this castle as the castle of $P$—columns of $T$. We make similar definitions for castles $C$ and partitions of $|C|$ or of the bases of the towers of $C$. We let $P(C)$ denote the partition of $X$ into the levels of $C$ and the residual set of $C$.

Given two castles $C_1$ and $C_2$ for $T$, we write $C_1 \leq C_2$ if $C_2$ can be viewed abstractly as having been obtained from $C_1$ by a cutting and stacking construction, as in [AOW]. More formally, this means the following:

(i) $|C_1| \subset |C_2|^o$;
(ii) there is a finite partition $Q$ of the bases of the towers of $C_1$ such that each level of the castle $(C_1)_Q$ is a level of $C_2$; and
(iii) for each tower of $(C_1)_Q$, there is a tower of $C_2$ that contains it.

Note that condition (i) implies that if $(A_i)_{i=1}^{h_2}$ is a tower in $C_2$ and $A_j$ is a base of a tower of $(C_1)_Q$ of height $h_1$, then we must have $j \leq h_2 - h_1$.

**Lemma 4.** Let $T_\sigma$ be a $G$-extension of the aperiodic automorphism $(T, X, B, \mu)$. Let $\{U_k\}_{k=1}^\infty$ be a neighborhood base for $G$ at $e_G$. Then there is a sequence $\{C_k\}_{k=1}^\infty$ of castles, where the towers of $C_k$ all have height $h_k$, such that:

1. for each $k$, $C_k \leq C_{k+1}$;
2. $\mu\left(\bigcup_{k=1}^\infty |C_k|\right) = 1$;
3. $\bigcup_{k=1}^\infty L(C_k)$ generates $B$;
4. for each tower $T$ in $C_k$ with base $A$, and each pair of levels $T^i A$ and $T^j A$ in $T$, where $1 \leq i < j \leq h_k$, there is some $g \in G$ so that for all $x \in A$, $\sigma(T^i x, j - i) \in U_k g$.

**Proof.** Fix a sequence of finite partitions $P_k \uparrow B$ on $X$ and a sequence $\varepsilon_k \downarrow 0$ with $\sum_k \varepsilon_k < 1$. Choose a sequence of towers $T_k$ for $T$ with residual sets of measure less than $\varepsilon_k$ such that for each $k$, $|T_k| \subset |T_{k+1}|^o$. Denote the base of $T_k$ by $B_k$ and its height by $h_k$. Choose a compact $K_1 \subset G$ so that if

$B'_1 = \{x \in B_1 \mid (\forall i, j \in \{0, \ldots, h_1 - 1\}) \sigma(T^i x, j - i) \in K_1\}$,

then $\mu(B'_1) > (1 - \varepsilon_1)\mu(B_1)$. Let $T'_1$ be the portion of $T_1$ over $B'_1$. That is, $T'_1 = \{T^i B'_1\}_{i=1}^{h_1-1}$. Partition $K_1$ into sets $\{K_{1,i}\}_{i=1}^{s_1}$ so that for each $i = 1, \ldots, s_1$, there exists $g_{1,i} \in G$ with $K_{1,i} \subset U_1 g_{1,i}$. Let $K : G \to G$ be given by

$$K_1(g) = \begin{cases} g_{1,i} & \text{if } g \in K_{1,i}, \\ e_G & \text{if } g \in G \setminus K_1. \end{cases}$$
Let $Q_1$ be the partition of $B'_1$ according to the values of $\{K_1(\sigma(T^ix, j-i))\}_{i,j=0}^{h_1-1}$ and the values of $\{P_1(T^i(x))\}_{i=0}^{h_1-1}$. $(P_1(y)$ denotes the partition element containing $y$.) We denote the resulting castle $(T'_1)_{Q_1}$ by $C'_1$.

Next consider $T_2$ with base $B_2$ and height $h_2$. Choose a compact $K_2 \subset G$ so that if

$$B'_2 = \{x \in B_2 \mid (\forall i, j \in \{0, \ldots, h_2 - 1\}) \sigma(T^ix, j-i) \in K_2\},$$

then $\mu(B'_2) > (1 - \epsilon_2)\mu(B_2)$. Let $T'_2$ be the portion of $T_2$ over $B'_2$. Partition $K_2$ into sets $\{K_{2,i}\}_{i=0}^{h_2}$ so that for each $i = 1, \ldots, h_2$, there exists $g_{2,i} \in G$ with $K_{2,i} \subset U\bar{g}_{2,i}$. Define $K_2 : G \rightarrow G$ analogously to $K_1$. Let $T'_2 = T_2 \vee \mathcal{P}(C'_1)$, and let $Q_2$ be the partition of $B'_2$ according to the values of

$$\{K_2(\sigma(T^ix, j-i))\}_{i,j=0}^{h_2-1} \vee \{P'_2(T^i(x))\}_{i=0}^{h_2-1}.$$

This gives a castle $C'_2 = (T'_2)_{Q_2}$.

Repeating this process produces a sequence of castles $C'_k$. To obtain condition (1), we restrict each $C'_k$ to the set $\bigcap_{i=k+1}^{\infty} |C'_j|$. That is, for each $k$ and each level $A'$ of $C'_k$, we replace $A'$ by the set $A = A' \cap (\bigcap_{i=k+1}^{\infty} |C'_j|)$. The resulting set of levels is a castle $C_k$, and these castles satisfy the conclusions of the lemma. \hfill \Box

3. The main result

We now give the proof of Theorem 1.

Proof. Suppose that $T_\sigma$ is an ergodic $G$-extension of $(T, X, \mu)$ and $T'_o$, is a $G$-extension of the aperiodic $(T', X', \mu')$. Fix a neighborhood $U$ of $e_G$, which we may assume to be compact. We will obtain the desired relative speedup of $T_\sigma$ and the $G$-isomorphism from it to $T'_\sigma$ as limits of a sequence of partially defined speedups and isomorphisms.

Let $\delta$ be a complete, right-invariant metric on $G$ compatible with the topology of $G$. (We note that, while such a metric must exist, there need not be a complete, two-sided invariant metric compatible with the topology. See [B].) Fix $\epsilon > 0$ so that $B(\epsilon, e_G)$, the closed $\delta$-ball of radius $\epsilon$ centered at $e_G$, is compact and contained in $U$. Fix a sequence $\epsilon_k \downarrow 0$ with $\sum_{k=1}^{\infty} \epsilon_k < \epsilon/3$. For each $k$, choose a compact neighborhood $U_k$ of $e_G$ so that $U_k U_k^{-1} \subset B(\epsilon_k, e_G)$. Choose a sequence of castles $\{C'_k\}_{k=1}^{\infty}$ for $T'_\sigma$, as in Lemma 4 with respect to these $U_k$. Denote the towers and levels of these castles by $C'_k = \{T'_k, j\}$ and $T'_{k,j} = \{A'_{k,j,i}\}$. In particular, Lemma 4 gives us, for all $i \in \{1, 2, \ldots, h_k - 1\}$ and for all levels $A'_{k,j,i}$ in $C'_k$, an element $g_{k,j,i} \in G$ so that for all $x' \in A'_{k,j,1}, \sigma'(x', i) \in U_k g_{k,j,i}$.

Make a copy $C'_1$ of $C'_1$ in $X$. That is, choose pairwise-disjoint sets $A_{1,j,i} \in B$ corresponding to the levels of $C'_1$ such that, for each $j$ and $i$, $\mu(A_{1,j,i}) = \mu(A'_{1,j,i})$. Fix $j$ and an isomorphism $\phi_{1,j} : A_{1,j,1} \rightarrow A'_{1,j,1}$.

Applying Lemma 3 repeatedly, we obtain functions $q_i : A_{1,j,1} \rightarrow \mathbb{N}$ with $q_i > q_{i-1}$ so that $T'^{q_i} : A_{1,j,1} \rightarrow A_{1,j,1}$ isomorphically and, for almost every $x \in A_{1,j,1}$ and every $i$,

$$\sigma(x, q_i(x))(\sigma'(\phi_{1,j}(x), i))^{-1} \in B(\epsilon_1, e_G). \tag{2}$$

For each $i \in [1, h_1 - 1]$, let $p_i : A_{1,j,i} \rightarrow \mathbb{N}$ be given by setting

$$p_i(x) = q_{i+1}(T^{-q_i}(x)) - q_i(T^{-q_i}(x)).$$
Then, letting \( p = \bigcup_{i=1}^{h_1-1} \phi_i \), we obtain a partially defined speedup \( T_1 := T^p \) of \( T \), defined on \( |T_{1,j}|^{p} \), for which \( T_{1,j} \) is a tower. This construction also yields a partially defined cocycle \( \sigma_1 \) for \( T_1 \), which is defined at \((x, n)\) whenever \( x \in |T_{1,j} \cap T_1^{-n}|T_{1,j} \).

Extend \( \phi_{1,j} \) to \( |T_{1,j}| \) so that on \( |T_{1,j}|^{p} \),

\[
\phi_{1,j}T_1(x) = T'\phi_{1,j}(x) \text{ a.e.}
\]

In particular, for each \( i \), \( \phi_{1,j}(A_{1,j,i}) = A'_{1,j,i} \). Define \( \alpha_{1,j} : |T_{1,j}| \to G \) by setting, for each \( x \in A_{1,j,i} \),

\[
\alpha_{1,j}(x) = \sigma'(\phi_{1,j}(x), -i)\sigma_1(x, -i).
\]

Repeating this construction on each tower of \( C_1 \), we set \( \phi_1 = \bigcup_j \phi_{1,j} \) to obtain an isomorphism from \(|C_1| \) to \(|C'_1| \) intertwining \( T_1 \) and \( T'^\prime \). Similarly, we let \( \alpha_1 = \bigcup_j \alpha_{1,j} \) and extend \( \alpha_1 \) to \( X \) by setting \( \alpha_1(x) = e_G \) for \( x \in X \setminus |C_1| \). We then see that the map \((x, g) \mapsto (\phi_1(x), \alpha_1(x)g)\) is a \( G \)-isomorphism from \((T_1)_{\sigma_1} \) to \( T'^\prime_{\sigma'} \), insofar as these maps are defined, which is to say on \(|C_1|^{p} \). In other words, for all \((x, n)\) in the domain of \( \sigma_1 \),

\[
\sigma_1^\alpha_1(x, n) := \alpha_1((T_1)^p x)\sigma_1(x, n)\alpha_1(x)^{-1} = \sigma'(\phi_1(x), n). \tag{3}
\]

We also see that, because of condition (2) and the right invariance of \( \delta \), we have for all \( x \in X \),

\[
\delta(\alpha_1(x), \varepsilon_G) < \varepsilon_1. \tag{4}
\]

(We note that in the above construction the approximate constancy of \( \sigma' \) on the levels of \( C_1^i \) was not used.)

Now we show how to iterate this construction to complete the proof of the theorem. Fix an increasing sequence of finite partitions \( \{P_k\}_{k=1}^\infty \) of \( X \) that generate \( \mathcal{B} \). Choose \( n_2 \) so that the partition \( \phi_1(P_1) \) is approximated to within \( \frac{1}{2} \) (in the partition metric) by the levels of \( C_{n_2}' \). The index \( n_2 \) must also be chosen so that \( \varepsilon_{n_2} \) is small enough to meet an additional condition, which we will describe at the end of the proof. For notational convenience, re-index \( C_{n_2}' \) and refer to it as \( C'_2 \), and do the same with \( \varepsilon_{n_2}, U_{n_2}, \) and so on. Let \( C_2 \) denote a copy of \( C'_2 \) which is the image of \( C_2' \) under \( \phi_1^{-1} \). That is, for each level \( A_{2,j,i} \) of \( C_2' \) contained in \(|C_1^i| \), the corresponding level \( A_{2,j,i} \) of \( C_2 \) is given by \( A_{2,j,i} = \phi_1^{-1}(A_{2,j,i}') \). Additional subsets of \( X \) are chosen to serve as \( A_{2,j,i} \) when \( A_{2,j,i}' \) is not contained in \(|C_1^i| \).

Our goal is to extend \( T_1 \) to a transformation \( T_2 \) on \(|C_2|^{p} \), so that \( T_2 \) is again a partially defined speedup of \( T \), with an associated cocycle \( \sigma_2 \). We will also modify \( \alpha_1 \) to a function \( \alpha_2 : X \to G \) so that on \(|C_2| \), \( \alpha_2 \) serves as a transfer function for a \( G \)-isomorphism between \((T_2)_{\sigma_2} \) and \( T'^\prime_{\sigma'} \).

Note that since \( U_2 \) is compact, and there are only finitely many towers in \( C_2' \), there is a compact set \( K \) so that for all \((x', n)\) with \( x' \in |C_2'| \) \([T'^{-n}]C_2' \), \( \sigma'(x', n) \in K \). Choose \( \xi_2 \in (0, \varepsilon_2) \) so that if \( a, b \in K \), and \( \delta(a, a') < \xi_2 \), then \( \delta(ba, ba') < \varepsilon_2 \). (This is possible by invoking the uniform continuity of the group multiplication on \( W K \), where \( W \) is a compact neighborhood of \( e_G \).)

Fix a tower \( T_{2,j} \) in \( C_2 \) and suppose that \(|T_{2,j}| \cap |C_1| \neq \emptyset \). Let \( \phi_2 : A_{2,j,1} \to A_{2,j,1}' \) be an isomorphism. For each level \( A_{2,j,m} \subset |T_{2,j}| \), define a function \( q_m : A_{2,j,1} \to \mathbb{N} \) so that \( T'^m : A_{2,j,1} \to A_{2,j,m} \) isomorphically and, for almost all \( x \in A_{2,j,1} \),

\[
\sigma'(\phi_{2,j}(x), m)\sigma(x, q_m(x))^{-1} \in B(\xi_2, \varepsilon_G).
\]
The $q_m$ are chosen so that $q_{m+1} > q_m$ and so that, as before, defining $p_2$ on each $A_{2,j,m}$ by

$$p_2(x) = q_{m+1}(T^{-q_m}(x)) - q_m(T^{-q_m}(x)),$$

the transformation $T_2(x) = T^{p_2}(x)$ is a speedup of $T$ and agrees with $T_1$ on its domain. This can be done by repeated application of Lemma 3. Explicitly, if $A_{2,j,1} \not\subseteq |C_1|$, then, by Lemma 3, there is a function $q_1 : A_{2,j,1} \rightarrow \mathbb{N}$ so that $T^{q_1} : A_{2,j,1} \rightarrow A_{2,j,2}$ isomorphically and, for almost all $x \in A_{2,j,1}$,

$$\sigma'(\phi_{2,j}(x), 1)\sigma(x, q_1(x))^{-1} \in B(\xi_2, e_G).$$

If $A_{2,j,2} \not\subseteq |C_1|$, then we choose $q_2 > q_1$ on $A_{2,j,1}$ so that $T^{q_2} : A_{2,j,1} \rightarrow A_{2,j,3}$ isomorphically and, for almost all $x \in A_{2,j,1}$,

$$\sigma'(\phi_{2,j}(x), 2)\sigma(x, q_2(x))^{-1} \in B(\xi_2, e_G).$$

We continue in this way until (unless) we first arrive at a level $A_{2,j,m} \subset |C_1|$. There $T_1$ is already defined, and we let $q_{m+1} = q_m + p_1$. We continue in this way until we reach the top level $A_{2,j,m+h_1-1}$ of this $C_1$-column. If there is another level $A_{2,j,m+h_1}$ of $|T_2,j|$, we define $q_{m+h_1}$ as before and continue until all levels of $|T_2,j|$ have been addressed.

Having defined $T_2$ on $|T_2,j|^o$, $\phi_{2,j}$ is then extendible uniquely to $|T_2,j|$ by the requirement that for almost all $x \in |T_2,j|^o$,

$$\phi_{2,j}(T_2x) = T'_{\phi_{2,j}}(x).$$

Let $\sigma_2$ denote the cocycle determined by $T_2$ and $\sigma$, which is defined for pairs $(x, n)$, where $x \in |T_2,j| \cap T_2^{-n}|T_2,j|$. We define $\alpha_2 : |T_2,j| \rightarrow G$ in two stages. First, for $x \in A_{2,j,m} \subset |T_2,j| \cap |C_1|$, if $x \in A_{1,j,l+1}$ (that is, $x$ is in the $(l+1)$st level of $C_1$), then we let

$$\tilde{\alpha}_1(x) = \sigma'(\phi_{2,j}(x), -l)^{-1}\sigma_2^{a_1}(x, -l).$$

We set $\tilde{\alpha}_1(x) = e_G$ on $|T_2,j| \setminus |C_1|$. We see that for $x \in |T_2,j| \cap |C_1|$,

$$\delta(\tilde{\alpha}_1(x), e_G) = \delta(\sigma'(\phi_{2,j}(x), -l)^{-1}, \sigma_2^{a_1}(x, -l)^{-1})$$

$$\leq \delta(\sigma'(\phi_{2,j}(x), -l)^{-1}, \sigma'(\phi_1(x), -l))$$

$$+ \delta(\sigma'(\phi_1(x), -l), \sigma_2^{a_1}(x, -l)^{-1})$$

$$= \delta(\sigma'(\phi_{2,j}(x), -l)^{-1}, \sigma'(\phi_1(x), -l))$$

$$\leq 2\varepsilon_2,$$

where the last inequality follows from the condition on the near constancy of $\sigma'$, on the columns of $C_2$. Thus, $\delta(\tilde{\alpha}_1(x)a(x), a(x)) = \delta(\tilde{\alpha}_1(x), e_G) \leq 2\varepsilon_2$. Moreover, on a $C_1$-column contained in $T_2,j$, the map $(x, g) \mapsto (\phi_{2,j}(x), \tilde{\alpha}_1(x)a_1(x)g)$ is a $G$-isomorphism from $(T_2)_2\sigma_2$ to $T_2'_{\alpha_2'}$.

Now, for any $x \in A_{2,j,m+1} \subset |T_2,j|$, let

$$\tilde{\alpha}_1(x) = \sigma'(\phi_{2,j}(x), -m)^{-1}\sigma_2^{a_1}(x, -m).$$

We see that $\delta(\tilde{\alpha}_1(x), e_G) < \varepsilon_2$. Indeed, if $x \in |T_2,j| \setminus |C_1|$ or if $x$ is in $|T_2,j|$ and in the base of $C_1$, then this is immediate from the construction. On the other hand, suppose $x \in |C_1|$. 


but not in the base of $C_1$. Say $x$ is in $A_{2, j, m+1}$ and in the $(l + 1)$st level of $C_1$. Then we have

$$\delta(\tilde{\alpha}_1(x), e_G) = \delta(\sigma'(\phi_{2,j}(x), -m)^{-1}, \sigma_2^j(x, -m)^{-1}).$$

But

$$\sigma'(\phi_{2,j}(x), -m)^{-1} = \sigma'(\phi_{2,j}(x), -l)^{-1}\sigma'(T'\sigma\phi_{2,j}(x)), -(m - l)^{-1}$$

and

$$\sigma_2^j(x, -m)^{-1} = \sigma_2^j(x, -l)^{-1}\sigma_2^j(T_2^{-l}(x), -(m - l)^{-1}).$$

Furthermore,

$$\sigma'(\phi_{2,j}(x), -l) = \sigma_2^j(x, -l)$$

and

$$\delta(\sigma'(T'\phi_{2,j}(x)), -(m - l)^{-1}, \sigma_2^j(T_2^{-l}(x), -(m - l)^{-1}) < \xi_2,$$

so, by the choice of $\xi_2$, we conclude that

$$\delta(\tilde{\alpha}_1(x), e_G) \leq \varepsilon_2.$$

Now set $\alpha_2(x) = \tilde{\alpha}_1(x)\tilde{\alpha}_1(x)\alpha_1(x)$ and observe that

$$\delta(\alpha_2(x), \alpha_1(x)) \leq 3\varepsilon_2$$

(using the right invariance of $\delta$). The map $(x, g) \mapsto (\phi_{2,j}(x), \alpha_2(x)g)$ is a $G$-isomorphism from $(T_2)_{\sigma_2}$ to $T'_{\sigma'}$ on all of $|T_2, j|$. Perform this construction on each tower of $C_2$ which meets $|C_1|$. If $T_2, j$ is a tower of $C_2$ that does not meet $|C_1|$, then employ the simpler construction that was used in the first stage of the proof to define $T_2$ and $\alpha_2$ on such a tower. Setting $\alpha_2(x) = e_G$ on $X\setminus |C_2|$ completes the second stage of the proof.

This procedure can be repeated indefinitely to produce a sequence of castles $C_k$ in $X$ for partially defined transformations $T_k$, where the levels of $C_k$ approximate the partition $P_{k-1}$ to within $1/k$, so that each $T_k$ is a speedup of $T$ defined on $|C_k|^o$, each $T_{k+1}$ extends $T_k$, and the transformation $\tilde{T} = \bigcup_k T_k$ is a speedup of $T$ defined almost everywhere. Let $\tilde{\sigma}$ denote the cocycle for $\tilde{T}$ that arises from $\sigma$.

The construction also produces a sequence of isomorphisms $\phi_k : |C_k| \rightarrow |C_k'|$ that intertwine $T_k$ and $T'$. In addition, it produces a sequence of functions $\alpha_k : X \rightarrow G$ so that for each $k$, $\delta(\alpha_{k+1}, \alpha_k) \leq 3\varepsilon_{k+1}$ and so that the map $(x, g) \mapsto (\phi_kx, \alpha_kg)$ is a $G$-isomorphism between $(T_k)_{\sigma_2}$ and $T'_{\sigma'}$ on $|C_k|$. Since $\delta$ is complete, we see that the sequence $\alpha_k$ converges uniformly to a function $\alpha$, such that for almost all $x$, $\delta(\alpha(x), e_G) \leq \varepsilon$ and hence $\alpha(x) \in U$.

In the construction of the $\phi_k$, we observe that each $\phi_{k+1}$ agrees set-wise with $\phi_k$ on the levels of $C_k$. Since the $\sigma$-algebras $B_k$ generated by the levels of $C_k$ increase to the full $\sigma$-algebra, the maps $\phi_k$ determine an isomorphism $\phi$ between $\tilde{T}$ and $T'$, which, for each $k$, agrees set-wise with $\phi_k$ on $B_k$.

Now we confirm that the map $(x, g) \mapsto (\phi x, \alpha(x)g)$ is a $G$-isomorphism from $\tilde{T}_\sigma$ to $T'_{\sigma'}$. We want to establish that for each $n$,

$$\sigma'(\phi x, n) = \alpha(\tilde{T}^n x)\tilde{\sigma}(x, n)\alpha(x)^{-1} \text{ a.e.}$$

$$= \tilde{\sigma}^n(x, n).$$
Fix \( n \in \mathbb{Z} \) and \( \eta > 0 \). For almost every \( x \), if \( k \) is sufficiently large, then
\[
\tilde{T}^n x = T_k^n x,
\]
\[
\tilde{\sigma} (x,n) = \sigma_k (x,n),
\]
and
\[
\delta (\alpha (x), \alpha_k (x)) \leq \eta.
\]
Furthermore, since the points \( \phi_k (x) \) and \( \phi_k (x) \) are in the same level of \( \mathcal{C}'_k \), the approximate constancy of \( \sigma' (\cdot, n) \) on such a level gives
\[
\delta (\sigma' (\phi x, n), \sigma' (\phi_k x, n)) \leq \eta.
\]
But we know that
\[
\sigma' (\phi_k x, n) = \alpha_k (T_k^n x) \sigma_k (x,n) \alpha_k (x)^{-1} \text{ a.e.}
\]
\[
= \alpha_k (\tilde{T}^n x) \tilde{\sigma} (x,n) \alpha_k (x)^{-1}.
\]
We only need to conclude that this value is close to \( \tilde{\sigma} \alpha (x,n) \).

Now we describe the additional condition according to which the subsequence \( \varepsilon_{n_2}, \varepsilon_{n_3}, \ldots \) (relabeled \( \varepsilon_2, \varepsilon_3, \ldots \)) must be chosen. For each \( k \geq 1 \), there is a fixed compact set \( K \) containing all the values of \( \alpha_k \) and all the values of \( \sigma_k (x, t) \), for \( x \in |\mathcal{C}_k| \cap T_k^{-t} |\mathcal{C}_k| \).

(Recall that all the values of \( \alpha_k \) lie in \( \tilde{B}(e, e_G) \), which is compact.) Regardless of how the \( \varepsilon_k \) will be chosen, we will have, for all \( k \) and \( x \),
\[
\delta (\alpha_k (x), \alpha (x)) < \sum_{m=k+1}^{\infty} \varepsilon_m.
\]
The additional condition we impose on the \( \varepsilon_k \) is that this sum is so small as to ensure that
for all \( x \in |\mathcal{C}_k| \cap T_k^{-t} |\mathcal{C}_k| \),
\[
\delta (\alpha_k (T_k^t x) \sigma_k (x, t) \alpha_k (x)^{-1}, \alpha (\tilde{T}^t x) \sigma_k (x, t) \alpha (x)^{-1}) < \frac{1}{k}.
\]
If this is done, then for the given \( n \) and \( \eta \), arguing with sufficiently large \( k \), we can conclude that
\[
\delta (\sigma' (\phi x, n), \alpha (\tilde{T}^n x) \tilde{\sigma} (x,n) \alpha (x)^{-1}) < \eta + \frac{1}{k}.
\]
Since \( \eta \) is arbitrary, we have the desired equality. \( \Box \)

We note that in the case that \( G \) is a discrete group, we immediately obtain the following stronger result.

**Corollary 1.** Let \( T_\sigma \) and \( T'_\sigma \) be \( G \)-extensions of \( T \) and \( T' \), where \( G \) is a finite or countable group. Suppose that \( T_\sigma \) is ergodic and \( T' \) is aperiodic. Then there is a relative speedup of \( T_\sigma \) which is \( G \)-isomorphic to \( T'_\sigma \), by a relative isomorphism whose transfer function \( \alpha \) satisfies \( \alpha (x) = e_G \) almost everywhere.

4. Finite and countable extensions

We now turn to the analysis of speedups of \( n \)-point extensions. First we introduce a simplification of some of our notation. Given an automorphism \( T \) and a cocycle \( \sigma \) taking
values in a group $G$, we will denote the $G$-extension $T_n$ more simply by the single letter $S$
and, in general, $G$-extensions $T'_n$ or $(T_1)_i$ will be denoted $S'$ or $S_1$, and so on.

Now fix an integer $n > 1$. Form the measure space $([n], \mathcal{P}([n]), \mu)$, where $[n] = \{1, \ldots, n\}$
and $\mu(i) = 1/n$, for each $i$. Making use of the natural action of the symmetric
group $S_n$ on $[n]$, each cocycle $\sigma$ for $T$ taking values in $S_n$ determines an automorphism $U$
of $\{X \times [n], B \times C, \mu \times \mu\}$ which has $T$ as a factor. Namely, we have the automorphism $U$
given by

$$U^n(x, i) = (T^n x, \sigma(x, n)(i)).$$

We refer to $U$ as an $n$-point extension of $T$. (Since we will only consider ergodic $n$-point
extensions, we may restrict ourselves to the uniform measure $\mu$.) We will use the
same sort of notational convention as above: the $n$-point extensions associated with pairs
$(T', \sigma')$ and $(T_1, \sigma_1)$ will be written $U'$ and $U_1$, and so on.

Given a pair of $n$-point extensions $U_1$ and $U_2$ on spaces $X_1 \times [n]$ and $X_2 \times [n]$, we say
$U_1$ is relatively isomorphic to $U_2$ if there is an isomorphism $\Phi$ from $U_1$ to $U_2$ that preserves
the fibers of these extensions. That is, there is an isomorphism of the form

$$(x, i) \mapsto (\phi(x), \alpha(x)(i)),$$

where $\alpha : X_1 \to S_n$. Equivalently, these extensions are relatively isomorphic if there are
an isomorphism $\phi$ from $T_1$ to $T_2$ and a function $\alpha : X_1 \to S_n$ such that

$$\sigma_2(\phi x, n) = \alpha(T_1^n(x))\sigma_1(x, n)\alpha(x)^{-1},$$

which is exactly the condition that the $S_n$-extensions $S_1$ and $S_2$ are $S_n$-isomorphic. We
also note that every speedup of $T_1$ (or, equivalently, every $S_n$-speedup of $S_1$) corresponds
to a speedup of $U_1$ relative to $T_1$.

Given $n$-point extensions $U_1$ and $U_2$, let us write $U_1 \sim U_2$ when there is a speedup of
$U_1$ relative to $T_1$ which is relatively isomorphic to $U_2$. This relation is evidently transitive
and apparently asymmetric; there is no reason to suppose that $U_1 \sim U_2$ implies $U_2 \sim U_1$.
In the case of ergodic finite group extensions, however (as well as for more general locally
compact second countable group extensions), we have just seen that it is symmetric, and
in fact for each locally compact second countable group $G$ there is only one equivalence
class of ergodic $G$-extensions. But, for general ergodic $n$-point extensions, we will see
that the relation is indeed asymmetric. This is due to the fact that the associated $S_n$-
 extensions, of which the ergodic $n$-point extensions are factors, need not themselves be
ergodic. By examining the ergodic components of these $S_n$-extensions, we will obtain a
characterization of this relation in other terms, and we will give an explicit example to
illustrate its asymmetry.

Fix an automorphism $T$ and an $S_n$-cocycle $\sigma$. Let $S$ be the associated $S_n$-extension
of $T$. We associate with the pair $(T, \sigma)$ a conjugacy class of subgroups of $S_n$ that will be
the basis of the characterization. We recall the discussion that can be found in [G]: let $C$
be an ergodic component of $S$. For each $x \in X$, let $C_x = \{y \in S_n : (x, y) \in C\}$. (By the
ergodicity of $T$, $|C_x| \geq 1$ is a constant.) Then, if $\beta : X \to S_n$ is any measurable function
such that $(x, \beta(x)) \in C$ almost everywhere, there is a subgroup $G$ of $S_n$ such that the
sets $\beta(x)^{-1}C_x$ are almost all equal to $G$. Moreover, letting $\alpha(x) = \beta(x)^{-1}$, and defining
a new cocycle by \( \sigma'(x, n) = \alpha(T^n x) \sigma(x, n) \alpha(x)^{-1} \), we get a new \( S_n \)-extension \( S' \) that is \( S_n \)-isomorphic to \( S \). The map

\[
(x, y) \mapsto (x, \alpha(x)y)
\]

is an \( S_n \)-isomorphism from \( S \) to \( S' \), which carries \( C \) to \( X \times G \), on which \( S' \) is ergodic. In summary: \( \sigma \) is cohomologous to a \( G \)-valued cocycle \( \sigma' \) yielding an \( S_n \)-extension \( S' \) that has \( X \times G \) as an ergodic component. (In particular, the values of \( \sigma' \) lie in \( G \).) The groups \( G \) that fit this description form a conjugacy class of subgroups of \( S_n \). We denote this conjugacy class by \( gp(T, \sigma) \).

We can easily extend the above discussion to a slightly more general context: suppose that, for some subgroup \( G \subseteq S_n \), \( X \times G \) is \( S \)-invariant, but is not an ergodic component of \( S \). Then we can apply the above arguments to an ergodic component of \( S \) that is contained in \( X \times G \), and conclude that there are a subgroup \( H \subseteq G \) and a cocycle \( \sigma' \) cohomologous to \( \sigma \) via a \( G \)-valued transfer function, so that \( X \times H \) is an ergodic component of the \( S_n \)-extension \( S' \) associated with \( \sigma' \).

For brevity, when \( X \times G \) is an ergodic component of an \( S_n \)-extension \( S \), we will say that \( \sigma \) is \( G \)-ergodic for \( T \). (This is equivalent to the ‘\( G \)-interchange property’ that Gerber introduced in [G].)

It is clear that \( gp(T, \sigma) \) is an invariant of factor isomorphism. In [G], Gerber showed that it is a complete invariant for factor orbit equivalence of ergodic \( n \)-point extensions. In connection with speedups, we now prove the following theorem.

**Theorem 2.** Let \( U_1 \) and \( U_2 \) be ergodic \( n \)-point extensions of transformations \((T_1, X_1)\) and \((T_2, X_2)\) by \( S_n \)-valued cocycles \( \sigma_1 \) and \( \sigma_2 \). Then \( U_1 \sim U_2 \) if and only if for some \( G_1 \in gp(T_1, \sigma_1) \) (and hence for every \( G_1 \in gp(T_1, \sigma_1) \)), there exists \( G_2 \in gp(T_2, \sigma_2) \) such that \( G_2 \subseteq G_1 \).

**Proof.** Suppose first that for some \( G_1 \in gp(T_1, \sigma_1) \), there exists \( G_2 \in gp(T_2, \sigma_2) \) such that \( G_2 \subseteq G_1 \). By the above discussion, for each \( i = 1, 2 \), there is a cocycle \( \sigma_i' \) cohomologous to \( \sigma_i \), so that \( X_i \times G_i \) is an ergodic component of \((T_i, \sigma_i)\). Therefore, without loss of generality, we may assume from the start that each \((T_i, \sigma_i)\) has this property.

\( S_1 \) induces an ergodic transformation \((S_1)_{X_1 \times G_2}\) on \((X_1 \times G_2)\) and, for each \((x, g_2) \in X_1 \times G_2\), we let \( j = j(x, g_2) \) denote the first return time of \((x, g_2)\) to \( X_1 \times G_1 \) under \( S_1 \). But \( j(x, g_2) \) depends only on \( x \) since, for all \((x, g_2) \in X_1 \times G_2\) and all \( n \in \mathbb{N} \),

\[
S_1^n(x, g_2) \in (X_1 \times G_2) \iff \sigma_1(x, n)g_2 \in G_2 \iff \sigma_1(x, n) \in G_2
\]

so that these conditions do not depend on \( g_2 \). We can then write \( j(x, g_2) = j(x) \), and the induced automorphism

\[
(S_1)_{X_1 \times G_2} = S_1^j \mid_{X_1 \times G_2}
\]

is an ergodic \( G_2 \)-extension of the speedup \( T_1^j \) of \( T_1 \).

Applying Theorem 1, we know that there is a \( G_2 \)-speedup \((S_1^j \mid_{X_1 \times G_2})^k \) of \( S_1^j \mid_{X_1 \times G_2} \), that is \( G_2 \)-isomorphic to \( S_2 \mid_{X_2 \times G_2} \). This gives us a speedup \((T_1^j)^k = T_1^l \) of \( T_1 \), and \((S_1^j \mid_{X_1 \times G_2})^k = S_1^l \mid_{X_1 \times G_2} \) is a \( G_2 \)-extension of \( T_1^l \).
There is an isomorphism $\phi : X_1 \rightarrow X_2$ and $\alpha : X_1 \rightarrow G_2$ so that

$$T_1^1 \approx T_2^1 \quad \text{and} \quad \sigma_2\phi = (\sigma_1^{(l)})^\alpha. \quad (5)$$

This construction gives us a speedup $U_1^1$ of $U_1$ relative to $T_1$ and conditions (5) say that $U_1^1$ is relatively isomorphic to $U_2$.

Now suppose that $U_1 \sim U_2$. Fix $G_1 \in gp(S_1)$. We want to show that there is a subgroup $G_2 \in sp(S_2)$ contained in $G_1$. As before, we may assume that $X_1 \times G_1$ is an ergodic component of $S_1$. The condition $U_1 \sim U_2$ tells us that there is an $S_n$-speedup $S_1^{k}$ of $S_1$ that is $S_n$-isomorphic to $S_2$. Since $X_1 \times G_1$ is invariant for $S_1$, it remains so for $S_1^{k}$. Arguing as before, there are a subgroup $G_2 \subset G_1$ and a cocycle $\tilde{\sigma}$ for $T_1^{k}$, cohomologous to $\sigma_1^{(k)}$, such that $X_1 \times G_2$ is an ergodic component of $\tilde{S}$, where $\tilde{S}$ is the $S_n$-extension of $T_1^{k}$ given by $\tilde{\sigma}$. But, since $\tilde{S}$ is relatively isomorphic to $S_2$, we must have $G_2 \in gp(T_2, \sigma_2)$. \hfill $\square$

As an example, we consider a pair of 3-point extensions considered by Gerber in [G]. Let $\{\gamma_i\}_{i=1}^6$ be an enumeration of the symmetric group $S_3$ so that $\{\gamma_i\}_{i=1}^3$ is the alternating group $A_3$. Let $(T_1, X_1)$ be the full 3-shift, with independent generator $P = \{P_i\}_{i=1}^3$, and $(T_2, X_2)$ the full 6-shift, with independent generator $Q = \{Q_i\}_{i=1}^6$. Define a cocycle $\sigma_1$ for $T_1$ by setting $\sigma(x, 1) = \gamma_i$ when $x \in P_i$. Define $\sigma_2$ for $T_2$ by setting $\sigma_2(x, 1) = \gamma_i$ when $x \in Q_i$. In [G], Gerber showed that $gp(S_1) = \{A_3\}$ and $gp(S_2) = \{S_3\}$ and that consequently $S_1$ and $S_2$ are not factor orbit equivalent. Using Theorem 2, we can conclude further that there is a factor speedup of $S_2$ that is isomorphically to $S_1$, but there is no factor speedup of $S_1$ that is isomorphic to $S_2$.

As a final application, we observe how the classification of extensions changes when we pass to extensions that are as close as possible to the finite case. That is, we consider extensions with countable fibers, and allow only the smallest natural group of permutations on the fibers. Let $p$ denote counting measure on $\mathbb{N}$ and $S_f$ the group of finitely supported permutations of $\mathbb{N}$. Suppose that $T$ is an automorphism of the Lebesgue probability space $(X, B, \mu)$, and $\sigma$ is an $S_f$-cocycle for $T$. Then, as in the finite case, we obtain a countable extension $U$ of $T$ on $X \times \mathbb{N}$ given by

$$U^n(x, k) = (T^n x, \sigma_1(x, n))(k).$$

Here we say that two such countable extensions $U_1$ and $U_2$ are relatively isomorphic if there are an isomorphism $\phi$ from $T_1$ to $T_2$ and a function $\alpha : X \rightarrow S_f$ such that

$$\sigma_2(\phi x, n) = \alpha(T^n_1(x))\sigma_1(x, n)\alpha(x)^{-1}.$$  

Since $S_f$ is countable, Theorem 1 applies, and the analysis of finite extensions which was given above can easily be adapted to this case. Thus, with each ergodic countable extension $U$ of $T$ by an $S_f$-valued cocycle $\sigma$, we associate a conjugacy class $gp(T, \sigma)$ of subgroups of $S_f$, and the corresponding statement of Theorem 2 holds.

**Proposition 1.** Given an ergodic $T$, there is an uncountable family of ergodic countable extensions of $T$, each of which acts on the fibers of the extension by finitely supported permutations of $\mathbb{N}$, and no one of which admits a relative speedup that is relatively isomorphic to another.
Proof. We first construct an uncountable family of subgroups of $S_f$ such that (i) each acts transitively on $\mathbb{N}$ and (ii) no conjugate of one contains another. Suppose $\mathcal{P} = \{A_k\}_{k=1}^\infty$ is a partition of $\mathbb{N}$ into two-element sets. Let

$$G_\mathcal{P} = \{\pi \in S_f : (\forall k)(\exists j) \pi(A_k) = A_j\}. $$

It is clear that $G_\mathcal{P}$ acts transitively on $\mathbb{N}$. For $\xi \in S_f$, we write $\xi \mathcal{P} = \{\xi A \mid A \in \mathcal{P}\}$ and observe that $\xi G_\mathcal{P} \xi^{-1} = G_{\xi \mathcal{P}}$. Hence, if $\mathcal{P}'$ is another partition of $\mathbb{N}$ into two-element sets and the symmetric difference of $\mathcal{P}$ and $\mathcal{P}'$ is infinite, then no conjugate of $G_\mathcal{P}$ can contain $G_{\mathcal{P}'}$. It is easy to construct an uncountable family $\{\mathcal{P}_i\}_{i \in I}$ of such partitions so that each pair has an infinite symmetric difference. The corresponding family of subgroups $\{G_{\mathcal{P}_i}\}_{i \in I}$ satisfies conditions (i) and (ii) above.

Each of the groups $G_{\mathcal{P}_i}$ is countable and amenable, and hence, by the result of Herman [H], for each $G_{\mathcal{P}_i}$, there is a cocycle $\sigma_i$ for $T$ for which the corresponding $G_{\mathcal{P}_i}$-extension $S_i$ is ergodic. But, since $G_{\mathcal{P}_i}$ acts transitively on $\mathbb{N}$, the corresponding countable extension $U_i$ is also ergodic. Indeed, for all sets $A$ and $B$ contained in $X$ of positive measure, and all $t, m \in \mathbb{N}$, if we choose $\pi \in G_{\mathcal{P}_i}$ such that $\pi(t) = m$, then Lemma 1 gives an $n^* \in \mathbb{N}$ and an $A' \subset A$ with $\mu(A') > 0$ such that $T^m(A') \subset B$ and, for all $x \in A'$, $\sigma(x, n^*) = \pi$. Condition (ii) on the groups $G_{\mathcal{P}_i}$ tells us that for all $i \neq j$ in $I$, no speedup of $U_i$ relative to $T$ is relatively isomorphic to $U_j$. \hfill $\Box$

Acknowledgement. The authors gratefully acknowledge the assistance of an anonymous referee, whose contributions enormously improved this work.

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