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The derivation of coefficients for the transfer functions of maximally flat time-delay networks generally begins from an infinite series approximation of the ideal transfer function. This thesis presents two alternative methods that are particularly suited to cases involving an unequal number of poles and zeroes in the transfer function. The specific function described in detail in this paper is the three-pole, two-zero constant input resistance case. The approximation is obtained and the network synthesized.

The Three-Pole, Two-Zero Approximation of a Maximally Flat Time-Delay Network with Constant Input Resistance

by

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TABLE OF CONTENTS

LIST OF FIGURES

THE THREE-POLE, TWO-ZERO APPROXIMATION OF A MAXIMALLY FLAT TIME-DELAY NETWORK WITH CONSTANT INPUT RESISTANCE

I. INTRODUCTION

The need for a time-delay network often arises in applications where two or more signal paths are used to transmit information to a summing point and the relative time position or difference between the two signals is of interest. If the signal is transmitted over two signal paths of unequal length, there will be a time difference indicated at the summing point. In order to correct for this time difference, a time delay may be inserted in the shorter of the two paths, with a delay equal to the difference in the propogation times of the paths.

As a particular example, consider an oscilloscope. An oscilloscope is normally used to display a voltage amplitude versus time. The vertical deflection is accomplished by an amplifier with selectable gain, and the horizontal deflection is accomplished by applying a sawtooth waveform to an amplifier which drives the horizontal deflection plates.

In order to use an oscilloscope to view the leading edge of a step input applied to the vertical amplifier, it is necessary to incorporate a delay line in the vertical signal path. A sample of the vertical signal is then obtained before the signal reaches the delay line, and this sample is used as the input signal to the horizontal circuitry. During the time the vertical signal is traversing the delay line, the sweep generator receives the sampled signal, processes it, unblanks the cathode ray tube, and starts the horizontal sweep. The display produced thus begins with a horizontal line and some time later the vertical information arrives and is displayed. The vertical delay line typically consists of a specially wound two-conductor coaxial cable with a 3dB bandwidth in excess of 100MHz and a delay time of 150 ns.

To use an oscilloscope to display two signals in an x-y mode (one signal versus the other), it is necessary to disconnect the sweep generator from the horizontal deflection amplifier, and insert instead the signal that will produce the horizontal deflection. When this is done there is usually about 100 ns more delay in the vertical channel than there is in the horizontal channel. This difference is due to the fact that the horizontal channel does not have a delay line in it, but the vertical channel does. In order to obtain accurate and convenient phase measurements using Lissajous patterns, it is desirable to equalize the delay in each channel. The insertion of an appropriate time-delay network in the horizontal channel accomplishes this.

1.1 Definition of Time Delay

The term time delay is commonly given two definitions. Consider the transfer characteristic $G_{21}(s) = e^{-sT}$, which represents the ideal time-delay transfer function. Let $s = j\omega$, so that $G_{21}(j\omega) = e^{-j\omega T}$. Here the magnitude function is equal to unity and the phase function is equal to $-\omega T$. The phase delay of the transfer function is defined as the phase function divided by the frequency, or $\frac{-\omega T}{\omega} = -T$. The group delay, or envelope delay, is defined by the derivative $\frac{-d(\omega T)}{d\omega} = -T$. This means that the group delay is equal to the slope of the phase function versus frequency. This second definition of time delay, the group delay, is the one chosen for discussion in this thesis, since the phase delay is a transcendental function, but its derivative is rational. For example, the transfer function of the maximally flat time-delay network with two poles and no zeroes is (from 10, p. 388):

$$G_{21}(s) = \frac{3}{3 + 3s + s^2} \tag{1.1}$$

$$G_{21}(j\omega) = \frac{3}{3 - \omega^2 + j3\omega} \tag{1.2}$$

The phase function, $\emptyset(\omega)$, of this expression is simply the phase function of the numerator minus the phase function of the denominator.

$$\emptyset(\omega) = \tan^{-1} 0 - \tan^{-1} \frac{3\omega}{3-\omega^2} \tag{1.3}$$

The phase delay of this transfer function is then:

$$\text{Phase delay} = \frac{\emptyset(\omega)}{\omega} = -\frac{1}{\omega} \tan^{-1} \frac{3\omega}{3-\omega^2} \tag{1.4}$$

This is a transcendental function, which means that it cannot be solved directly for ω given some phase delay except by successive trial and error substitution of ω.

However, the group delay is found to be a rational expression as follows:

$$\text{Group delay} = -\frac{d\emptyset(\omega)}{d\omega} = \frac{d}{d\omega}\left(\tan^{-1} \frac{3\omega}{3-\omega^2}\right) \tag{1.5}$$

$$= \frac{9 + 3\omega^2}{9 + 3\omega^2 + \omega^4}$$

Given some group delay, this expression may be solved directly for ω since it is a quartic equation in ω and is therefore rational.

To insure maximally flat delay versus frequency, it seems obvious that it is desirable to continue taking higher order derivatives of the phase function and set all of these derivatives equal to zero at $\omega = 0$. This ensures that the slope of time delay versus frequency will be as flat as possible over the frequency range of interest. This process becomes very laborious and awkward when the number of roots of the transfer function exceeds four or five, and, unfortunately, a computer does not relieve the drudgery since the process is algebraic rather than arithmetic.

II. APPROXIMATION

2.1 Approximation Methods

Many methods of approximating the ideal time-delay transfer characteristic appear in the literature (1,4,9,10). These approximations generally make use of some form of an infinite series expansion of e^{-Ts}, with truncation of the series determining the order of the equation. The equation is then used to synthesize a network. The equation that results from these techniques is generally an all-pole expression in s, with the understanding that if twice the delay is required, right-half plane zeroes that are images of the left-hand plane poles may be included (in which case the number of poles equals the number of zeroes).

This procedure, while fast and accurate, is not always the most practical method for obtaining the desired time delay. The networks that result may not lend themselves to ease of adjustment, or they may be restricted as to the type of source impedance required.

If the source impedance is small (voltage source) the all pole functions work well. If the source impedance is large (current source) the all pole functions still work well, but the network that is required is different than that required for the voltage source.

If the network is to provide a known delay regardless of the source impedance, some additional thought is necessary. Obviously, this requirement implies that the input impedance of the network must be constant versus frequency over the frequency range of interest if mechanical switching using relays, etc., is to be avoided when the source impedance is changed.

One method of obtaining a constant input impedance involves the use of active devices as isolating and impedance matching stages, with a fixed-precision resistor providing the input impedance. Due to considerations of reliability, cost, complexity, and stability versus time and temperature, this method is less desirable than a passive filter.

This thesis discusses a network that is desired to have the following characteristics:

1) Maximally flat time delay versus frequency.
2) Constant input impedance versus frequency.
3) A balanced input and output.
4) A 3dB bandwidth from DC to above 2MHz.
5) 50Ω impedance level per side to ground.
6) A nominal delay of 100 ns with a ± 10 ns adjustment range.

The network shown in Figure 1.1 will be considered.

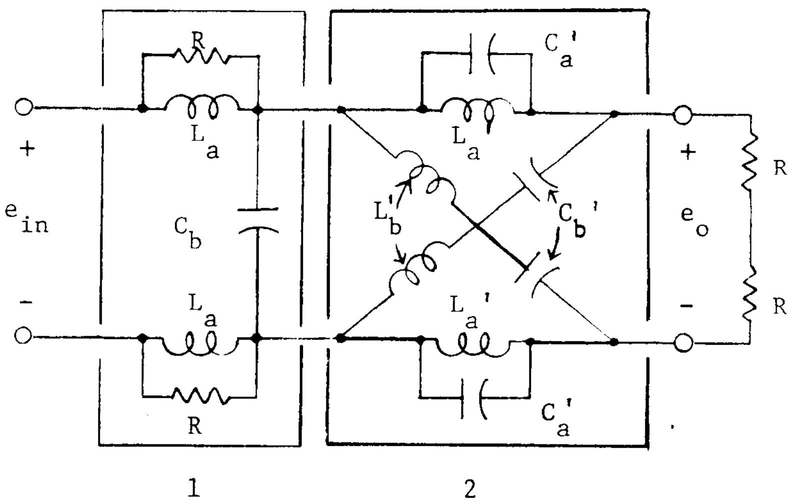


Figure 1.1. The series combination of a low-pass network and an LC lattice.

Subnetwork two of Figure 1.1 is capable of exhibiting constant input impedance and is an all-pass lattice network. Subnetwork one of Figure 1.1 is also capable of exhibiting constant input impedance if the proper relationships are chosen for the components, and is a low-pass network. Therefore the total network is capable of exhibiting constant input impedance and low-pass transmission.

The general transfer function of the network is

$$G_{21}(s) = \left(\frac{c}{s+c}\right)\left(\frac{b_0 - b_1 s + s^2}{b_0 + b_1 s + s^2}\right), \tag{2.1}$$

where the first term represents the low-pass network, and the second term represents the LC lattice. (See Appendix for derivation of $G_{21}(s)$).

Since the number of zeroes does not equal the number of poles the usual techniques involving infinite series approximations will not work here. Instead, the method of setting the first derivative of the phase angle equal to the low frequency delay and all higher order derivatives equal to zero at $\omega = 0$ is used.

2.2 Derivation of Coefficients

Multiplying and substituting $j\omega$ for s in the general transfer function (Eq. 2.1) yields

$$G_{21}(j\omega) = \frac{b_0 c - c\omega^2 - j\omega b_1 c}{b_0 c - \omega^2(b_1+c) - j(\omega^3 - \omega(b_0+b_1 c))}. \tag{2.2}$$

$$= \frac{\sqrt{(b_0 c - c\omega^2)^2 + (\omega b_1 c)^2} \;\angle \tan^{-1} \dfrac{-\omega b_1 c}{b_0 c - c\omega^2}}{\sqrt{(b_0 c - \omega^2(b_1+c))^2 + (\omega^3 - \omega(b_0+b_1 c))^2} \;\angle \tan^{-1} \dfrac{-(\omega^3 - \omega(b_0+b_1 c))}{b_0 c - \omega^2(b_1+c)}} \tag{2.3}$$

The net phase function $\emptyset(\omega)$ is found to be

$$\emptyset(\omega) = \theta_1(\omega)-\theta_2(\omega) = \tan^{-1}\frac{-\omega b_1 c}{b_0 c - c\omega^2} - \tan^{-1}\frac{-(\omega^3-\omega(b_0+b_1 c))}{b_0 c-\omega^2(b_1+c)} \quad (2.4)$$

The time delay can be determined as

$$T = \frac{d\emptyset(\omega)}{d\omega} = \frac{d\theta_1(\omega)}{d\omega} - \frac{d\theta_2(\omega)}{d\omega}. \quad (2.5)$$

$$= \frac{-\omega^2 b_1 c^2 - b_0 b_1 c^2}{\omega^4 c^2+\omega^2(b_1{}^2 c^2-2b_0 c^2)+b_0{}^2 c^2}$$

$$- \frac{\omega^4(b_1+c)+\omega^2(b_0 b_1+b_1{}^2 c-2b_0 c+b_1 c^2)+b_0 b_1 c^2+b_0{}^2 c}{\omega^6+\omega^4(c^2+b_1{}^2-2b_0)+\omega^2(b_1{}^2 c^2-2b_0 c^2+b_0{}^2)+b_0{}^2 c^2}. \quad (2.6)$$

In order to simplify the calculations and to facilitate comparison with other networks in the literature it is desirable to arbitrarily set the low frequency delay equal to one second. The final network can then be scaled for the desired delay.

$$\lim_{\omega\to 0}\frac{d\emptyset(\omega)}{d\omega} = -\frac{b_0 b_1 c^2}{b_0{}^2 c^2} - \frac{b_0 b_1 c^2+b_0{}^2 c}{b_0{}^2 c^2} = -\frac{2b_1 c+b_0}{b_0 c} = T = -1.0 \text{ sec} \quad (2.7)$$

$$\therefore \quad b_0 = \frac{2b_1 c}{c-1} \quad \text{or} \quad b_1 = \frac{b_0(c-1)}{2c} \quad (2.8)$$

To identify the other variables higher order derivatives are obtained and set equal to zero at $\omega = 0$. The second derivatives are shown below.

$$\frac{d^2\theta_1(\omega)}{d\omega^2} = \frac{4\omega^3 b_0 b_1 c^4+\omega(2b_0 b_1{}^3 c^4-6b_0{}^2 b_1 c^4)}{\omega^2(2b_0 b_1{}^2 c^4-4b_0{}^3 c^4)+b_0{}^4 c^4} \quad (2.9)$$

$$\frac{d^2\theta_2(\omega)}{d\omega^2} = \frac{\begin{array}{c}\ldots+4\omega^3(3b_0{}^2 b_1 c^2-b_0 b_1{}^3 c^2-b_0 b_1 c^4-b_0{}^2 b_1{}^2 c+2b_0{}^3 c)\\ +2\omega(3b_0{}^2 b_1 c^4-b_0 b_1{}^3 c^4-b_0{}^4 c)\end{array}}{\ldots+2b_0{}^2 c^2\omega^2(b_0{}^2+b_1{}^2 c^2-2b_0 c^2)+b_0{}^4 c^4} \quad (2.10)$$

(The reason for excluding terms of higher order than ω^3 will soon become apparent.)

These expressions are both quotients of odd to even polynomials. With expressions of this type, the higher order derivatives become increasingly complex, and it is difficult to avoid algebraic errors in the process of obtaining them. Therefore a simplification was sought.

Consider the truncated general case shown in equation 2.11 for the second derivative of a quotient of an even order numerator to an even order denominator. Only the lower order terms are shown. By continuing to take derivatives and dropping all high order terms the resulting expressions are obtained.

$$\frac{d^2\theta(\omega)}{d\omega^2} = \frac{\ldots + \omega^5F + \omega^3D + \omega E}{\ldots + \omega^4A + \omega^2C + R} \tag{2.11}$$

$$\frac{d^3\theta(\omega)}{d\omega^3} = \frac{\ldots + \omega^4(-3AE+CD+5RF) + \omega^2(-CE+3DR) + ER}{\ldots + \omega^4(2RA+C^2) + \omega^2(2CR) + R^2} \tag{2.11a}$$

$$\frac{d^4\theta(\omega)}{d\omega^4} = \frac{\ldots + \omega(6DR^3-6CER^2)}{\ldots + \omega^2(4CR^3) + R^4} \tag{2.11b}$$

$$\frac{d^5\theta(\omega)}{d\omega^5} = \frac{\ldots + \omega^2(6CER^2-6DR^3) + R^4(6DR^3-6CER^2)}{\ldots R^8} \tag{2.11c}$$

Here the even derivatives are all equal to zero if $\omega = 0$, and therefore yield no useful information. It is only the odd derivatives which are useful. The limits of the odd derivatives are:

$$\lim_{\omega\to 0} \frac{d^3\theta(\omega)}{d\omega^3} = \frac{E}{R} \tag{2.12}$$

$$\lim_{\omega\to 0} \frac{d^5\theta(\omega)}{d\omega^5} = \frac{6DR-6CE}{R^2} \tag{2.12a}$$

On examination of these limits it is seen that only the coefficients C, D, E, and R are involved. These are coefficients appearing in the second derivative (Eq. 2.11), and are the coefficients of terms of order ω^3 and lower. The coefficients of terms of higher order than ω^3 do not appear in the limits, and this is why they were not shown in equation 2.10. The second derivatives of both $\theta_1(\omega)$ (Eq. 2.9) and $\theta_2(\omega)$ (Eq. 2.10) have the same form as equation 2.11. Calculation of the third derivative limits proceeds by identifying the appropriate coefficients in equations 2.9 and 2.10, and substituting them in equation 2.12 as shown below.

$$\lim_{\omega\to 0} \frac{d^3\theta_1(\omega)}{d\omega^3} = \left.\frac{E}{R}\right|_{\theta_1} = \frac{2b_0b_1{}^3c^4-6b_0{}^2b_1c^4}{b_0{}^4c^4} \tag{2.13}$$

$$\lim_{\omega\to 0} \frac{d^3\theta_2(\omega)}{d\omega^3} = \left.\frac{E}{R}\right|_{\theta_2} = \frac{6b_0{}^2b_1c^4-2b_0b_1{}^3c^4-2b_0{}^4c}{b_0{}^4c^4} \tag{2.14}$$

Subtracting equation 2.14 from 2.13 gives the third derivative limit of the phase function (Eq. 2.4). Equating this to zero yields:

$$\lim_{\omega\to 0} \frac{d^3\emptyset(\omega)}{d\omega^3} = 2b_1{}^3c^3-6b_0b_1c^3+b_0{}^3 = 0 \tag{2.15}$$

The fifth derivative limits are found in the same manner, but equation 2.12a is used in place of equation 2.12. The result is shown below.

$$\lim_{\omega\to 0} \frac{d^5\emptyset(\omega)}{d\omega^5} = 10b_0b_1{}^3c^5-10b_0{}^2b_1c^5-2b_1{}^5c^5-b_0{}^5 = 0 \tag{2.16}$$

There are now three independent equations in three unknowns. The solution of these equations for c yields the following equation:

$$c^6 - 6c^5 + 15c^4 + 0c^3 - 45c^2 - 90c + 45 = 0 \tag{2.17}$$

Even with the above simplifications, the number of algebraic manipulations required was excessive and very time consuming. A still simpler method was therefore sought.

Since the total time delay of the network must be the sum of the time delay of the low-pass plus the time delay of the lattice, and since for maximally flat time delay this sum must be less than or equal to one second for all ω, the following relationship holds:

$$\frac{c}{c^2+\omega^2} + \frac{2b_1\omega^2+2b_0b_1}{b_0{}^2+\omega^2(b_1{}^2-2b_0)+\omega^4} \leq T \leq 1.0 \text{ for all } \omega. \tag{2.18}$$

Here the first term is the time delay of the low-pass network, and the second term is the time delay of the lattice. Rewriting this equation yields

$$cb_0{}^2+2b_0b_1c^2-c^2b_0{}^2+\omega^2(cb_1{}^2-2b_0c+2b_1c^2+2b_0b_1-b_1{}^2c^2+2b_0c^2-b_0{}^2)$$
$$+\omega^4(c+2b_1-b_1{}^2-c^2+2b_0) - \omega^6 \leq 0 \text{ for all } \omega. \tag{2.19}$$

Equating the constants to zero at $\omega = 0$ yields

$$2b_1c+b_0-b_0c = 0; \text{ or } b_0 = \frac{2b_1c}{c-1} \text{ and } b_1 = \frac{b_0(c-1)}{2c}. \tag{2.20}$$

Equating the coefficients of the ω^2 term to zero and substituting for b_1 (Eq. 2.20) yields

$$3b_0c^2-b_0{}^2c^3-3b_0c-3b_0-12c^2-12c^3 = 0; \tag{2.21}$$

or

$$b_0 = \frac{12c^2(c-1)}{(c-1)^3+4}. \tag{2.22}$$

$$\therefore \quad b_1 = \frac{b_0(c-1)}{2c} = \frac{6c(c-1)^2}{(c-1)^3+4} \tag{2.23}$$

Equating the coefficients of the ω^4 term to zero and substituting for b_0(Eq. 2.22) and b_1(Eq. 2.23) yields

$$c^6-6c^5+15c^4+0c^3-45c^2-90c+45 = 0. \tag{2.24}$$

This equation is identical to equation 2.17 which was obtained using the derivative method. The solution of equation 2.24 yielded coefficients which gave maximally flat time delay. While both methods yield the desired coefficients, the inequality method is by far the fastest and easiest to use.

Before the inequality method was used, the author used the derivative method to obtain the coefficients of the transfer function of the two-pole, two-zero lattice configuration for maximally flat time delay. The results checked with the coefficients obtained by the usual truncated infinite series approach. The three-pole, three-zero case was also calculated using derivatives and the results again checked.

III GENERAL TRANSFER FUNCTION

The final sixth order equation was solved for c using a computer program and values for b_0 and b_1 were obtained from equations 2.22 and 2.23. The resulting transfer function normalized for one second delay and one ohm input impedance is shown below.

$$G_{21}(s) = \left(\frac{3.334}{s+3.334}\right)\left(\frac{18.62 - 6.52s + s^2}{18.62 + 6.52s + s^2}\right) \cdot \qquad (3.1)$$

This equation uniquely defines the transfer function for the two-zero, three-pole maximally flat time delay network. The pole-zero plot is shown in Figure 3.1.

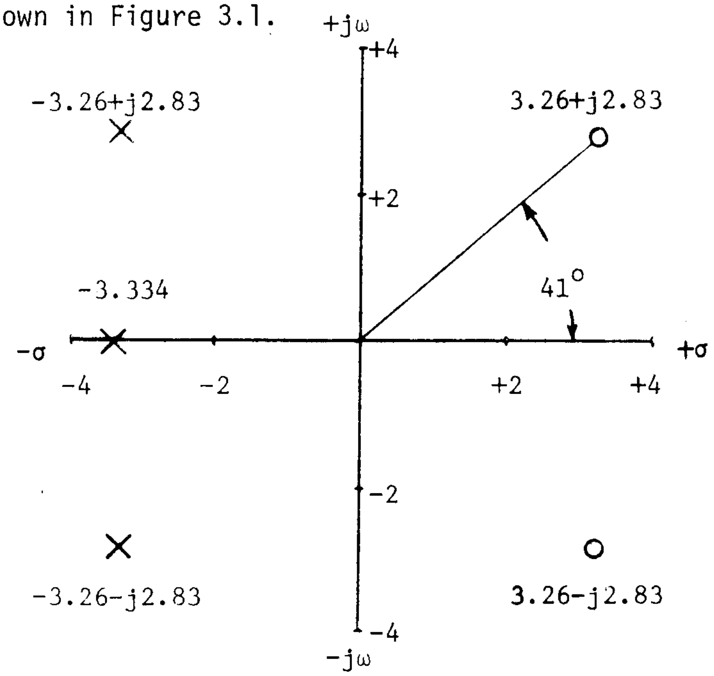


Figure 3.1 S-plane pole-zero plot of the three-pole, two-zero maximally flat time-delay network approximation normalized to one second delay.

IV SYNTHESIS

4.1 Synthesis of Network

The low-pass constant-resistance network may now be realized using standard synthesis techniques. (10, p363)

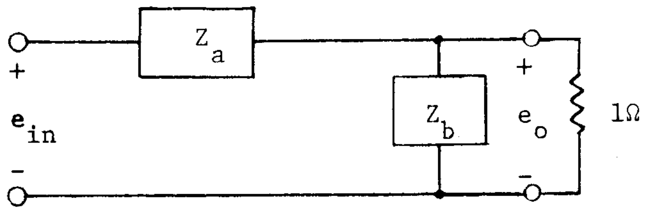


Figure 4.1 Constant-resistance network model

$$G_{21} = \frac{1}{1+Y_b} = \frac{3.334}{s+3.334}$$

$$Y_b = \frac{1}{G_{21}} - 1 = \frac{3.334}{s} = \frac{1}{\frac{3s}{10}}$$ This represents a capacitance

$$Y_a = 1 + Z_b$$

$$Z_a = \frac{1}{1+Z_b} = \frac{1}{1+\frac{10}{3s}} = \frac{\frac{3s}{10}}{\frac{3s}{10}+1}$$ This represents a parallel RL network.

By substitution of Z_a and Z_b the following network is obtained:

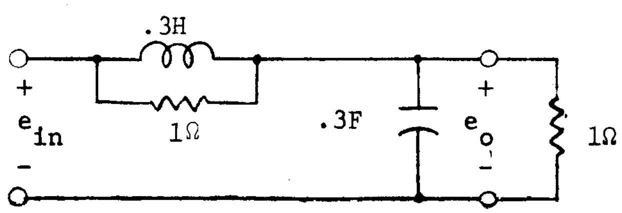


Figure 4.2 Constant-resistance low-pass network

Converting to a push-pull or balanced input with an impedance of one ohm per side results in the following network:

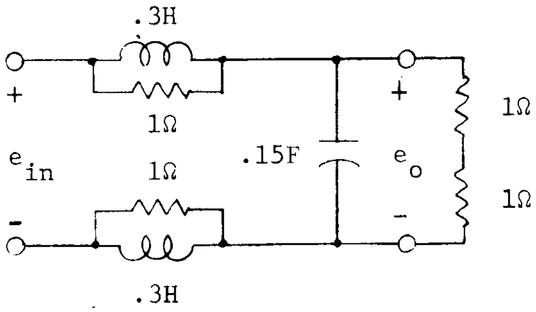
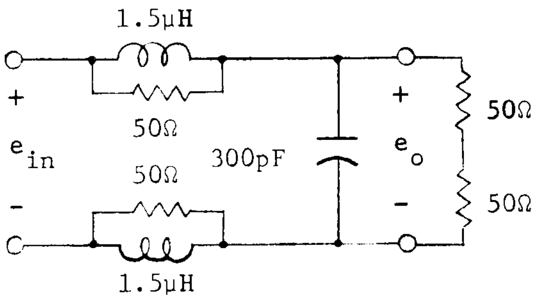


Figure 4.2a Balanced constant-resistance low-pass network

Scaling to 50Ω per side and 100 ns delay yields:



where $R_s = bR$

$L_s = \frac{b}{a} L$

$C_s = \frac{1}{ab} C$

R_s, L_s and C_s are scaled values.

where $b = \frac{R_s}{R} = \frac{50}{1} = 50$

$a = \frac{1s}{100ns} = 10^7$

Figure 4.2b Scaled constant-resistance low-pass network

The all-pass constant-resistance lattice may be realized using similar synthesis techniques. (10, p346)

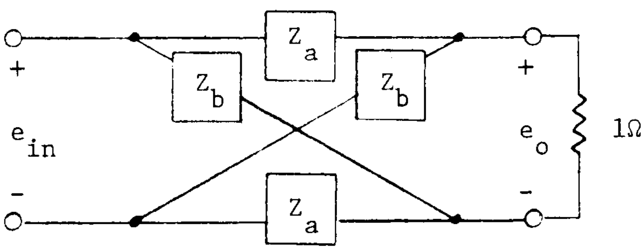


Figure 4.3 Constant-resistance lattice model

$$G_{21}(s) = \frac{18.626 - 6.52s + s^2}{18.626 + 6.52s + s^2}$$

$$Z_a = \frac{1-G_{21}}{1+G_{21}} = \frac{13.04s}{2s^2+37.252}$$

$$Z_a = \frac{1}{.1532s + \frac{1}{.35s}}$$ This represents a parallel LC network.

$$Z_b = \frac{1}{Z_a} = .1532s + \frac{1}{.35s}$$ This represents a series LC network.

By substitution of Z_a and Z_b the following network is obtained:

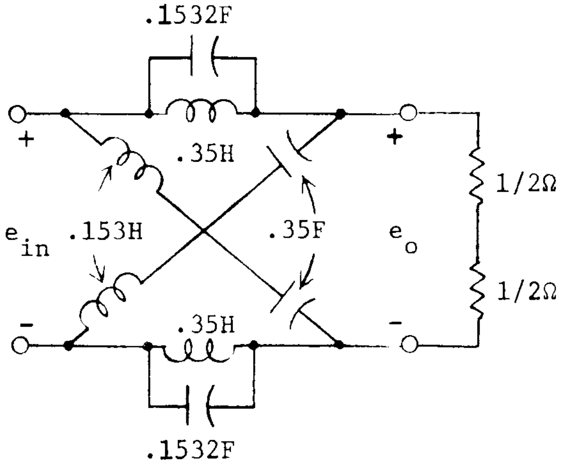


Figure 4.3a Constant-resistance lattice

Scaling the network from 1/2Ω per side to 50Ω per side and from one second to 100 ns yields:

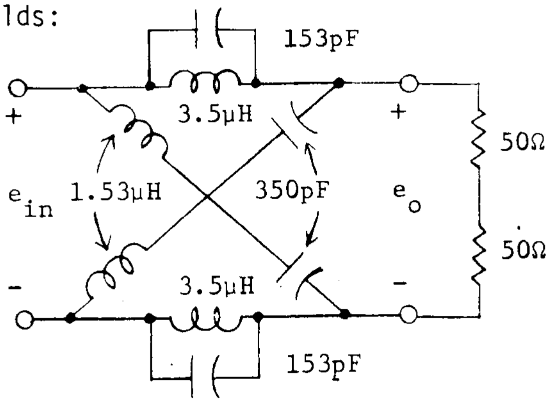


Figure 4.3b Scaled constant-resistance lattice

This lattice is of the constant-resistance type, so $Z_{in} = 100\Omega$ at all frequencies. Therefore it may be substituted for the 100Ω load on the low-pass constant-resistance network without disturbing either network.

The complete network then is as follows:

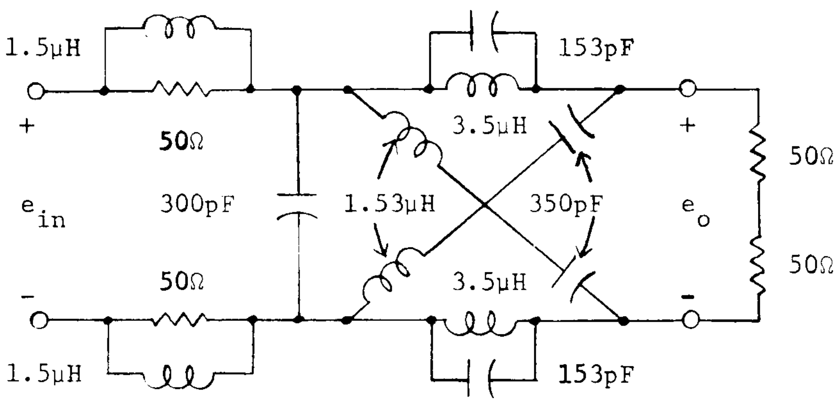


Figure 4.4 A three-pole, two-zero maximally flat time delay network with constant input resistance

This network provides the required 100 ns delay with a constant input resistance of 50Ω per side.

4.2 Adjustment of Time Delay

Since it is desired to have a ± 10% range of adjustment of the delay time, some means must be found to accomplish this. From the general transfer function the low frequency delay was found to be

$$-T = \frac{2b_1c + b_0}{b_0c} = \frac{2b_1}{b_0} + \frac{1}{c} \qquad (2.7)$$

The first term involves the coefficients associated with the lattice and the second term involves the coefficient associated with the low-pass network. Since it is desirable to maintain a constant input resistance and yet keep the number of adjustments to a minimum, and since the lattice would require at least four adjustments (two per side) to remain balanced, the author chose to make the low-pass network provide the delay variation. The 300 pF side-to-side capacitor of the low-pass network is easily made adjustable, and the 1.5μH inductors are also relatively easy to adjust. By providing a common adjustment for the inductors, the total number of adjustments required is two.

It follows that the term $\frac{1}{c}$ in the time delay expression will be variable. Relating the coefficient c to the value of the bridging capacitor and shunting inductors:

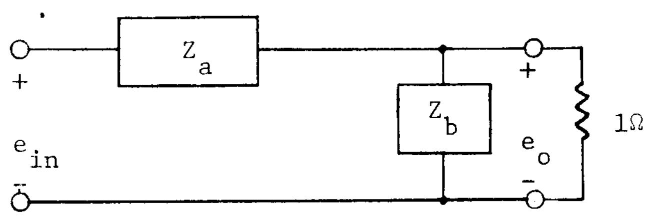


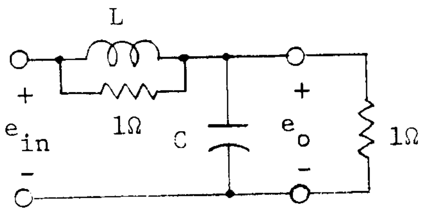
Figure 4.5 Constant-resistance network model

$$G_{21}(s) = \frac{e_o(s)}{e_{in}(s)}$$

$$Z_b = \frac{G_{21}(s)}{1-G_{21}(s)}$$

$$Z_b = \frac{\frac{c}{s+c}}{1 - \frac{c}{s+c}} = \frac{c}{s} = \frac{1}{\frac{s}{c}}$$

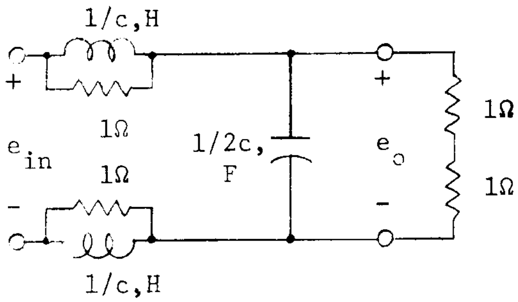
$$Z_a = \frac{1}{1+Z_b} = \frac{\frac{s}{c}}{\frac{s}{c}+1}$$



Here $\frac{1}{c}$ represents both the value of C and of L.

Figure 4.6 General form of the constant-resistance low-pass network

Converting from a single-ended to a balanced network with an impedance of one ohm per side results in the following:



Here $\frac{1}{2c}$ represents the value of the side-to-side capacitance and $\frac{1}{c}$ the value of the inductors.

Figure 4.6a General form of the balanced constant-resistance low-pass network

The desired delay is one second $\pm$ 10%. By substitution of the values of b_0 and b_1 in equation 2.7:

$$-T = 1 \pm .1 = .7 + \frac{1}{c} \quad \text{(seconds)} \tag{4.3}$$

$$\frac{1}{c} = .3 \pm .1 = .4 \text{ and } .2$$

$$c = 2.5 \text{ and } 5.0$$

$$\therefore C_{ss} = \frac{1}{2c} = .2\text{F and } .1\text{F}$$

$$L = \frac{1}{c} = .4\text{H and } .2\text{H}$$

This indicates that a $\pm$ 33% range centered around the nominal value is required of the bridging capacitor and shunting inductors to obtain a $\pm$ 10% time-delay adjustment.

The bridging capacitor should thus be 300pF $\pm$ 100pF and the inductors should be 1.5μH $\pm$.5μH. This completes the synthesis of the network.

V NETWORK CHARACTERISTICS

5.1 Experimental Network

All of the experimental data shown in this section was obtained from a network that was scaled from a 3dB bandwidth of 3.334 radians to a 3dB bandwidth of 33.34kHz. The choice of this scaling factor allowed the data points to be plotted directly against the calculated values with a minimum of data conversion. Also, the instrumentation used to obtain the data was not a limiting factor at 33.34 kHz, but at much higher frequencies it would have been. The network used to obtain the experimental data is shown in Figure 5.1.

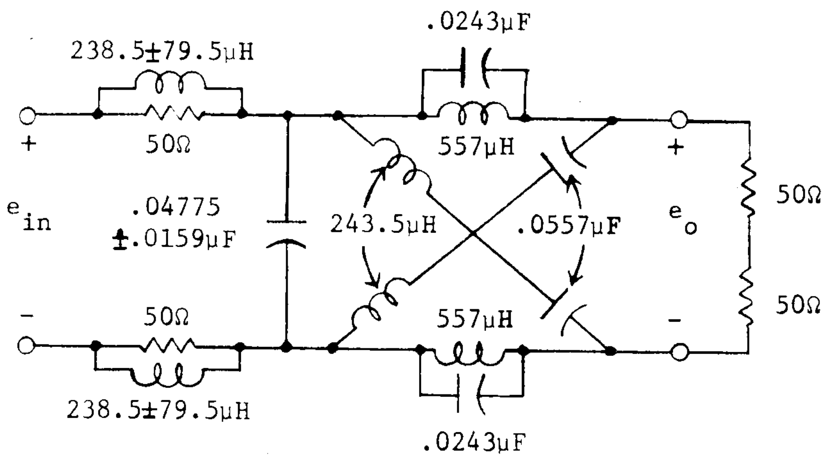


Figure 5.1 Scaled network used to obtain experimental data

5.2 Step Response

Laplace transform theory may be used to calculate the output voltage versus time due to a step input voltage. This was done using a computer program and included the response for minimum delay and maximum delay conditions as well as the nominal delay case. The results are shown in Figure 5.2. Also shown is the experimental result obtained for the nominal case.

The ringing preceding the main response is due to the LC lattice. Since the lattice is an all-pass network, all frequency components present in the step are passed. Since the time delay versus frequency is frequency limited, the higher frequency components, while phase-shifted more, are not delayed in time as much as the lower frequency components. Some phase shift between higher and lower frequencies is therefore introduced by the network which was not present in the step input. This results in cancellation or addition of various frequency components, depending on their frequency and relative phase, at the output of the network.

The resulting response may be thought of as a first output due to high frequency components resulting in an increasing output signal. (These components are phase-shifted -450°). The next part of the response is a result of cancellation between the high frequencies and middle frequencies. This results in a decreasing output signal. The middle frequencies (which are phase-shifted approximately -180°) continue to cause the output to drop resulting in an actual negative output signal. Finally, the low frequency components arrive at the output resulting in the final rising output signal.

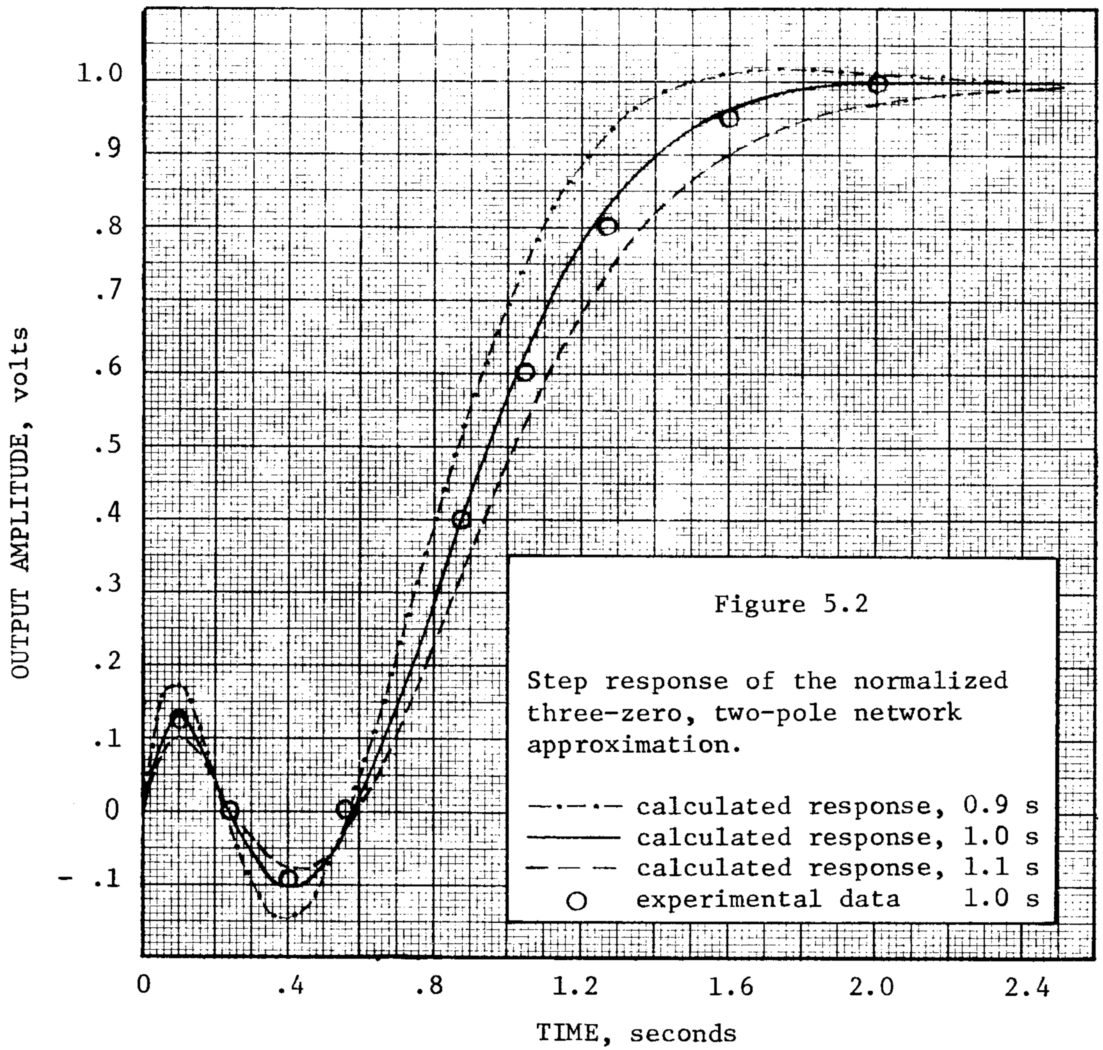


Figure 5.2

Step response of the normalized three-zero, two-pole network approximation.

5.3 Bandwidth

The bandwidth of the network is determined solely by the low-pass input network. From the lattice transfer function,

$$G_{21}(j\omega) = \frac{(b_0 - \omega^2) - jb_1\omega}{(b_0 - \omega^2) + jb_1\omega} \quad . \tag{5.1}$$

$$|G_{21}(j\omega)| = \frac{\sqrt{(b_0 - \omega^2)^2 + (b_1\omega)^2}}{\sqrt{(b_0 - \omega^2)^2 + (b_1\omega)^2}} = 1 \text{ for all } \omega \ . \tag{5.2}$$

From the low-pass network with constant input resistance,

$$G_{21}(j\omega) = \frac{c}{c + j\omega} \quad . \quad |G_{21}(j\omega)| = \frac{c}{\sqrt{c^2 + \omega^2}} \quad . \tag{5.3}$$

Thus the transfer function of the low-pass network is a declining function of frequency, with a 3dB point of $\omega = c$. The function is shown in Figure 5.3.

For the normalized constant-resistance case the 3dB frequency for 1.1 second, 1.0 second, and 0.9 second delay is 2.5 radians, 3.334 radians, and 5.0 radians (4.02MHz, 5.36MHz, and 8.03MHz for the 50Ω-100 ns case) respectively.

If the inductors are left at their nominal value (the value for one second delay) and only the capacitor is adjusted away from its nominal value (that value where $RC = \frac{L}{R}$ or $C = \frac{L}{R^2}$) the 3dB frequency will always decrease. That is, increasing the capacitance decreases the 3dB frequency, which is logical. However, decreasing the capacitance also decreases the 3dB frequency, which is not an immediately obvious conclusion (see Appendix for determination of 3dB point). For the maximum capacitance setting, C = .4F and the 3dB frequency is 3.265 radians.

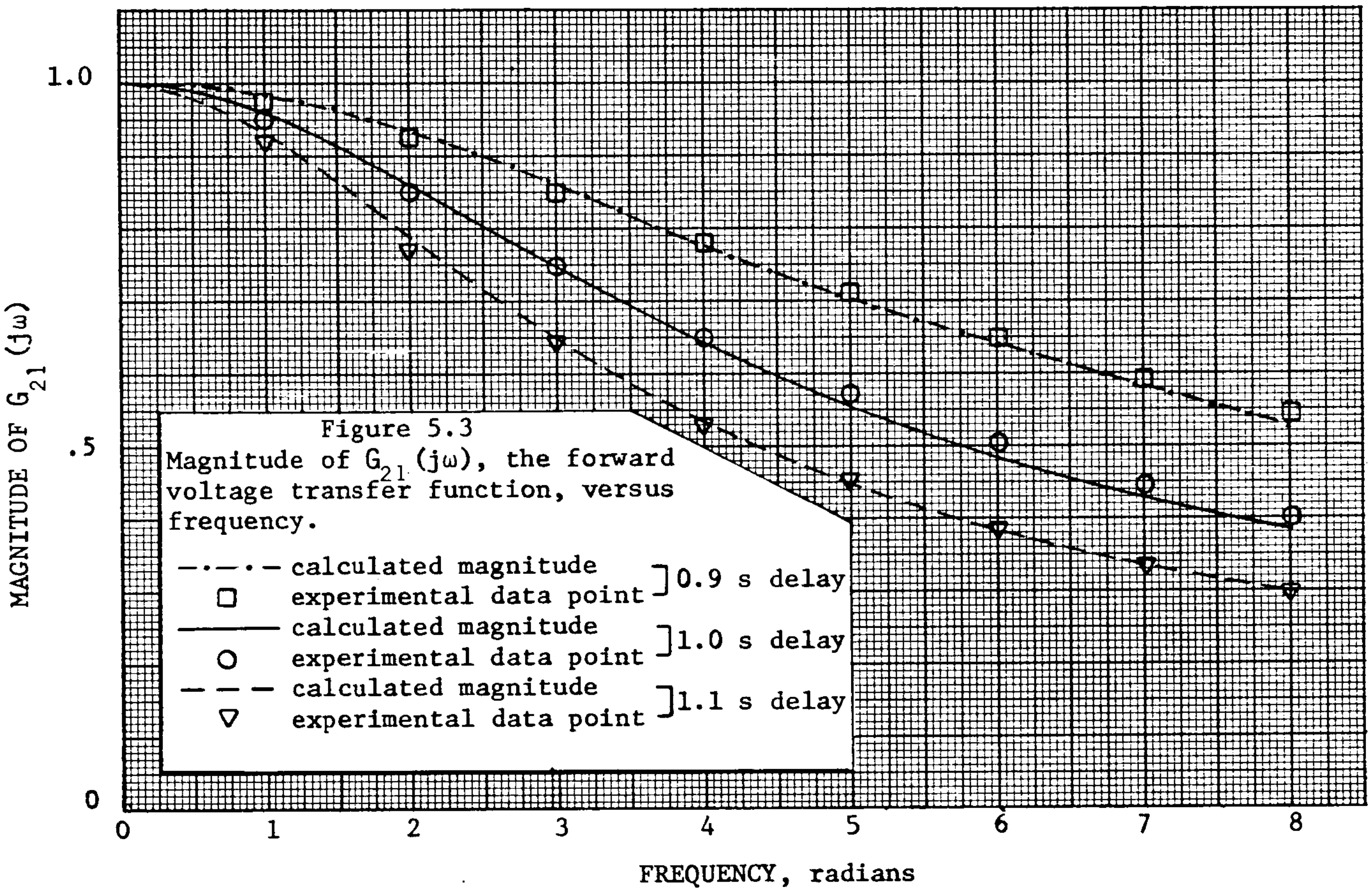


Figure 5.3
Magnitude of $G_{21}(j\omega)$, the forward voltage transfer function, versus frequency.

For the minimum capacitance setting, C = .2F and the 3dB frequency is 3.21 radians.

5.4 Phase Shift

The phase shift of the normalized network is shown in Figure 5.4. The curves were plotted from the expression for the phase function obtained on Page 8 (Eq. 2.4). A generally accepted method of evaluating the merit of a time-delay network consists of determining when the phase angle departs by 5% from the ideal case, which is a slope of -1.0 when the phase angle in radians is plotted versus ω. The 5% point for the nominal one second delay occurs at $\omega \simeq 5.0$ radians. For the minimum delay case (slope = -.9) crossover occurs at $\omega \simeq 5.25$ radians, and for the maximum delay case (slope = -1.1) at 2.6 radians. The nominal one second delay case represents the optimum linearity of phase versus frequency that is possible using two zeroes and three poles. Other techniques (**2**) that utilize ripple of the phase angle versus frequency will extend the frequency at which crossover occurs, but the ripple automatically introduces a small oscillating error versus frequency, so that any measurement made using Lissajous figures must take this error into account if maximum accuracy is desired.

A more practical evaluation of the network may be made by determining the frequency at which the phase shift exceeds the ideal case by more than a fixed number of degrees. If two degrees is used as a maximum allowable error, the nominal case exceeds this margin at 3.25 radians, the minimum delay case at 4.5 radians, and the maximum delay

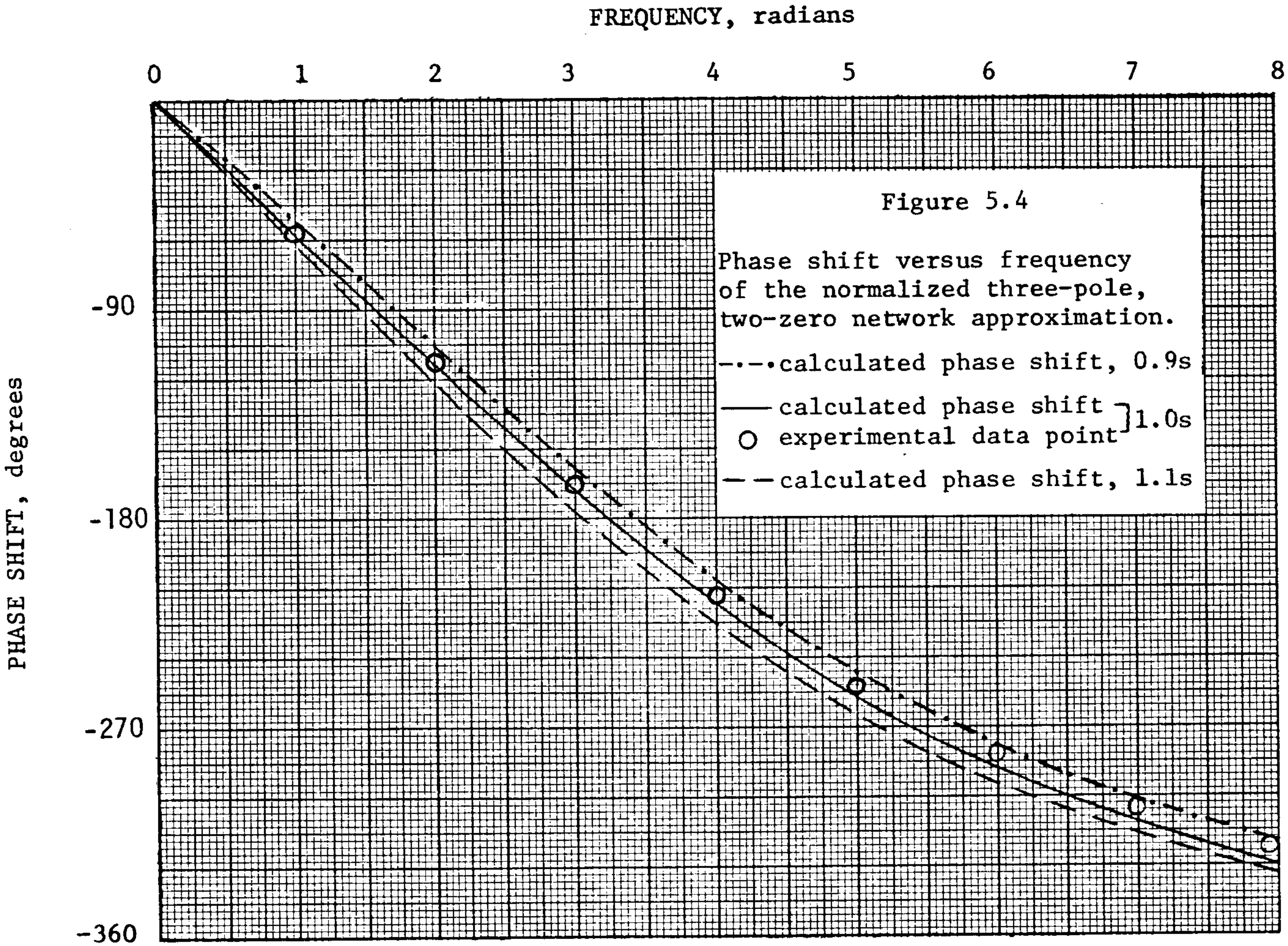


Figure 5.4

Phase shift versus frequency of the normalized three-pole, two-zero network approximation.

case at 2.0 radians. In practice, the range of required time delay adjustment will fall nearer three or four percent than ten percent, so the frequencies at which crossover occurs will be closely grouped near 3.25 radians.

5.5 Group Delay

The group time delay versus frequency of the normalized network is plotted in Figure 5.5 for nominal, maximum, and minimum time delay settings. The curves were plotted using data from a computer program and the expression for group time delay obtained on Page 8 (Eq. 2.6).

The departure from maximally flat time delay exhibited in both the minimum and maximum delay cases was expected and is normal but undesirable. Since the maximally flat case dictates fixed coefficients for the transfer function and therefore for the component values, any change of value will cause departure from the ideal. In order to correct for these variations, more adjustments are required that will alter the delay of the lattice, effectively returning the network to the maximally flat case but at a different value of delay time. Since one of the desired characteristics of the network is ease of adjustment, the author felt that the performance versus number and ease of adjustments was optimum for the two adjustment configuration used.

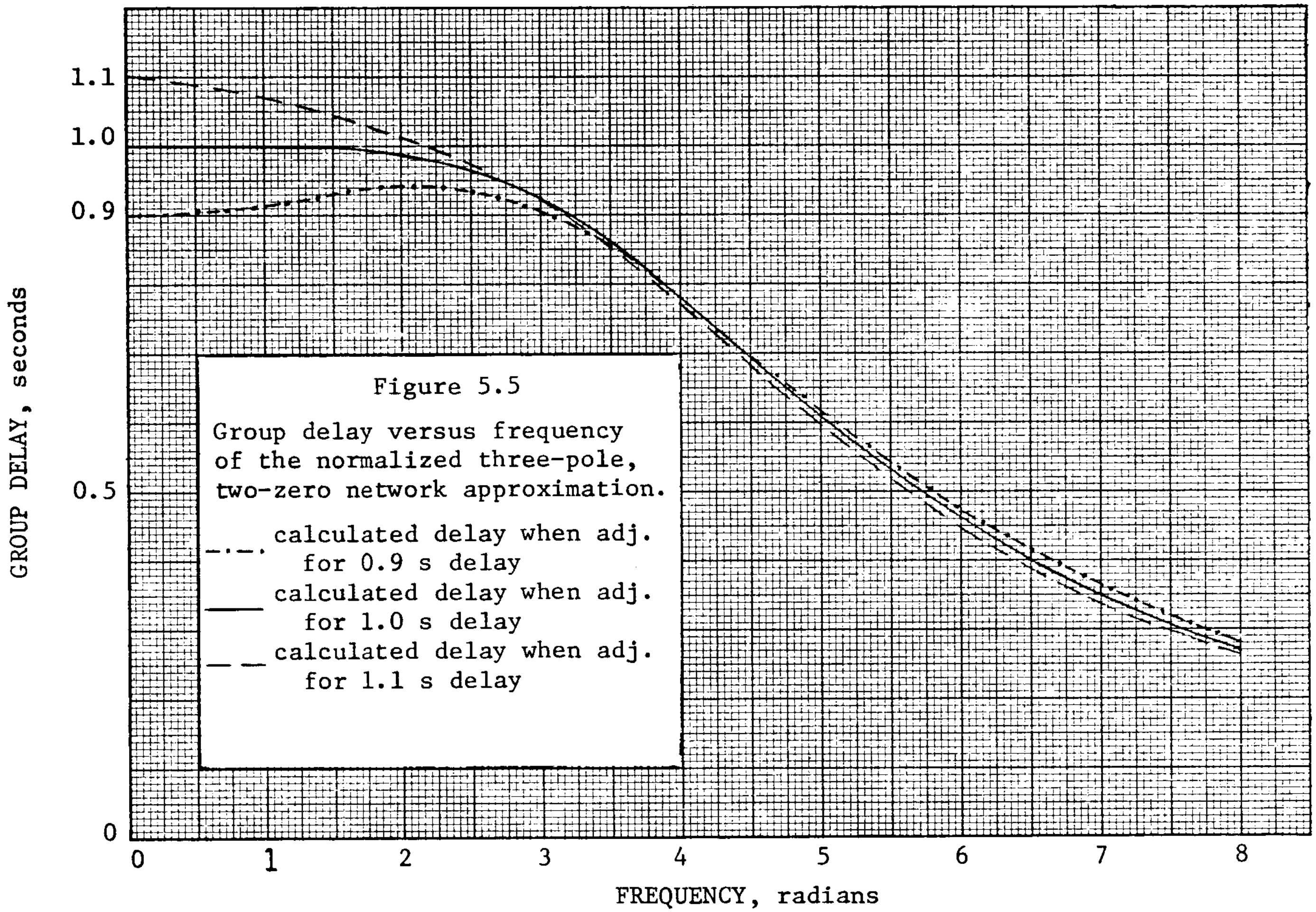


Figure 5.5

Group delay versus frequency of the normalized three-pole, two-zero network approximation.

5.6 Phase Delay

The phase delay versus frequency of the normalized network is shown in Figure 5.6. The statement was made on Page 3 that this thesis would discuss group delay rather than phase delay, but in order to obtain experimental data for delay the phase delay was measured.

The calculated data was obtained from the results of the computer program for the phase function. Since phase delay is defined as the phase angle at some frequency divided by that frequency, the computation consisted of a simple division.

From Figure 5.6 the maximum departure of the measured data from the calculated value appears to be less than three percent.

5.7 Computer Programs

Computer programs were written to calculate the phase shift, time delay, and magnitude of the transfer function versus frequency, and to print out element values for the network given two input data factors. The factors required are the impedance level per side and the time delay required of the network. The element values were determined using standard synthesis procedures but carrying out the operations with general terms rather than using specific values. The result is shown in Figure 5.7.

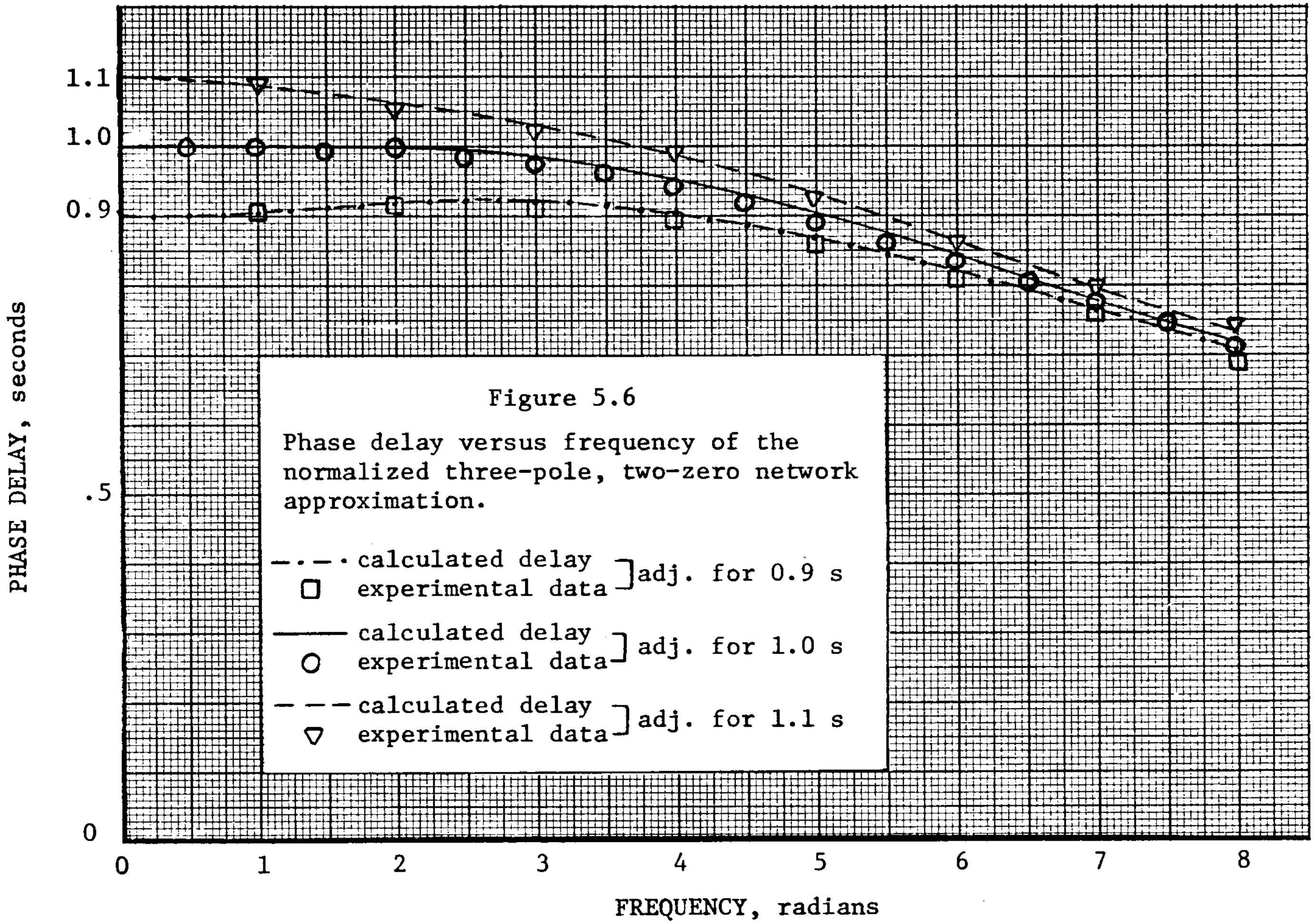
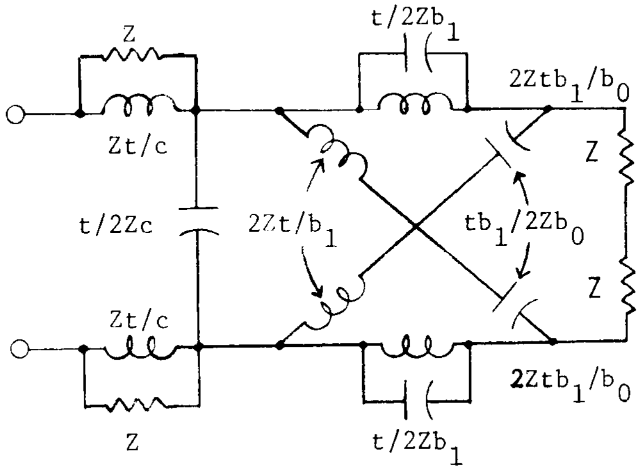


Figure 5.6

Phase delay versus frequency of the normalized three-pole, two-zero network approximation.



All element values are in ohms, farads, and henrys.

Figure 5.7 Equivalent circuit of the three-pole two-zero, constant-resistance, maximally flat time-delay network

Here Z represents the impedance per side and t represents the time delay in seconds. The coefficients c, b_0, and b_1 are 3.334, 18.626, and 6.52 respectively as determined by equations 2.21, 2.22 and 2.23.

VI SUMMARY AND CONCLUSIONS

This thesis has demonstrated two methods of obtaining coefficients of polynomials used to approximate the ideal time-delay transfer function. The particular function discussed was the three-pole, two-zero approximation with constant input resistance and adjustable delay. These methods were developed to use with functions that have an unequal number of poles and zeroes, although they are not restricted to this class.

The three-pole, two-zero approximation was obtained and synthesized. The time delay, transient response, phase shift, and voltage transfer characteristics of the network were discussed and plotted with both calculated and experimental data. The effect of the variable network elements on these characteristics was also shown.

An equivalent circuit was shown which requires only impedance and time delay scaling factors in order to obtain element values directly.

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APPENDIX

Constant-Resistance Networks

Development of the relations required to make the low-pass network exhibit constant input resistance are shown below.

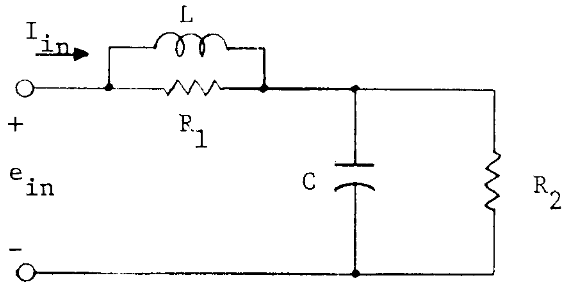


Figure A.1 The general form of a low-pass network capable of exhibiting constant input resistance

$$Z_{in}(s) = \frac{R_1 Ls}{R_1 + Ls} + \frac{\frac{R_2}{Cs}}{R_2 + \frac{1}{Cs}}$$

$$Z_{in}(j\omega) = \frac{\frac{R_1}{LC} - \omega^2 R_1 + j\omega\left(\frac{R_1}{R_2 C} + \frac{1}{C}\right)}{\frac{R_1}{R_2 LC} - \omega^2 + j\omega\left(\frac{R_1}{L} + \frac{1}{R_2 C}\right)}$$

$$Z_{in}(j\omega) = \frac{\sqrt{\left(\frac{R_1}{LC} - \omega^2 R_1\right)^2 + \left(\frac{\omega R_1}{R_2 C} + \frac{\omega}{C}\right)^2}\;\angle \tan^{-1} \frac{\left(\frac{R_1 \omega}{R_2 C} + \frac{\omega}{C}\right)}{\left(\frac{R_1}{LC} - \omega^2 R_1\right)}}{\sqrt{\left(\frac{R_1}{LR_1 C} - \omega^2\right)^2 + \left(\frac{\omega R_1}{L} + \frac{\omega}{R_2 C}\right)^2}\;\angle \tan^{-1} \frac{\left(\frac{R_1 \omega}{L} + \frac{\omega}{R_2 C}\right)}{\left(\frac{R_1}{LR_2 C} - \omega^2\right)}}$$

In order for $Z_{in}(j\omega)$ to be a real impedance at all frequencies, the imaginary part must equal zero. Therefore, the phase angle of the numerator must equal the phase angle of the denominator for all ω.

$$\therefore \quad \frac{\frac{R_1\omega}{R_2C} + \frac{\omega}{C}}{\frac{R_1}{LC} - \omega^2 R_1} = \frac{\frac{R_1\omega}{L} + \frac{\omega}{R_2C}}{\frac{R_1}{LR_2C} - \omega^2}$$

$$-\omega^2\left(\frac{R_1}{R_2C} + \frac{1}{C}\right) + \frac{R_1{}^2}{LR_2{}^2C^2} + \frac{R_1}{LR_2C^2} = -\omega^2\left(\frac{R_1{}^2}{L} + \frac{R_1}{R_2C}\right) + \frac{R_1{}^2}{LC} + \frac{R_1}{LR_2C^2}$$

$$\therefore \quad \frac{R_1}{R_2C} + \frac{1}{C} = \frac{R_1{}^2}{L} + \frac{R_1}{R\ C} \qquad \text{(from } \omega^2 \text{ terms)}$$

$$L = R_1{}^2C$$

and $$\frac{R_1{}^2}{LR_2{}^2C^2} + \frac{R_1}{LR_2C^2} = \frac{R_1{}^2}{L^2C} + \frac{R_1}{LR_2C^2} \qquad \text{(from constant terms)}$$

$$L = R_2{}^2C$$

$$\therefore \quad R_1 = R_2 \text{ and } \frac{L}{R_1} = R_1C$$

These relationships between components are necessary for the network to exhibit constant input resistance versus frequency.

The absolute values of the components have not yet been determined. This may be accomplished by consideration of the magnitude portion of the expression for $|Z_{in}(j\omega)|^2$, since it must equal $R_{in}{}^2$ for all ω.

$$R_{in}^{2} = \frac{R_1^{2}[\omega^4 R_1^{2} L^2 C^2 + \omega^2(4L^2 - 2R_1^{2}LC) + R_1^{2}]}{\omega^4 R_1^{2} L^2 C^2 + \omega^2(R_1^{4}C^2 + L^2) + R_1^{2}}$$

For this expression to be constant for all ω, the coefficients of the ω^2 terms must equal one another.

$$4L^2 - 2R_1^{2}LC = R_1^{4}C^2 + L^2$$

Substituting the previously obtained relationship $L = R_1^{2}C$:

$$4R_1^{4}C^2 - 2R_1^{4}C^2 = R_1^{4}C^2 + R_1^{4}C^2$$

$$1 \equiv 1$$

Therefore, $R_{in}^{2} = R_1^{2}$, or $R_{in} = R_1 = R_2$.

The transfer function of the low-pass network was determined as follows:

$$G_{21}(s) = \frac{e_o(s)}{e_{in}(s)} = \frac{\dfrac{\frac{R}{Cs}}{R + \frac{1}{Cs}}}{\dfrac{\frac{R}{Cs}}{R + \frac{1}{Cs}} + \dfrac{LRs}{Ls+R}}$$

but $\frac{L}{R} = RC$

$$\therefore \quad G_{21}(j\omega) = \frac{\frac{R}{L}}{\frac{R}{L} + j\omega} = \frac{\frac{1}{RC}}{\frac{1}{RC} + j\omega}$$

The phase function of $G_{21}(j\omega)$ is therefore found to be:

$$\phi(\omega) = -\tan^{-1}\frac{\omega L}{R} = \tan^{-1}\omega RC.$$

The time delay is the derivative of the phase function and is:

$$T(\omega) = \frac{d\emptyset(\omega)}{d\omega} = \frac{-1}{\frac{L}{R}(\frac{R^2}{L^2} + \omega^2)} = \frac{-1}{RC(\frac{1}{R^2C^2} + \omega^2)}$$

The above relationships are true when $C = \frac{L}{R^2}$, or when the network exhibits constant input resistance versus ω. For the case when $C \neq \frac{L}{R^2}$, the following relationships were obtained.

Transfer function:

$$G_{21}(j\omega) = \frac{\frac{R}{L} + j\omega}{\frac{R}{L} - RC\omega^2 + j2\omega} .$$

The 3dB point of $G_{21}(j\omega)$:

$$|G_{21}(j\omega)|^2 = (.707)^2 = \frac{(\frac{R}{L})^2 + \omega^2}{(\frac{R}{L} - RC\omega^2)^2 + (2\omega)^2}$$

$$\therefore \quad 3dB\ \omega^2 = \frac{\frac{R^2C}{L} - 1 + \sqrt{1 + \frac{2R^2C^2}{L^2} - \frac{2R^2C}{L}}}{R^2C^2}$$

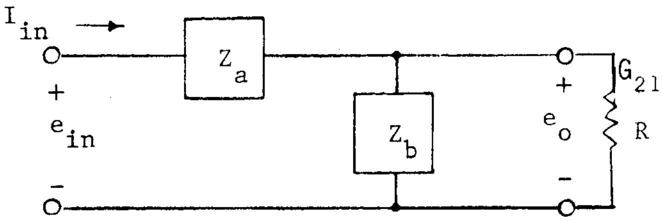
The phase function of $G_{21}(j\omega)$:

$$\emptyset(\omega) = \tan^{-1}\frac{\omega}{\frac{R}{L}} - \tan^{-1}\frac{2\omega}{\frac{R}{L} - RC\omega^2}$$

The time delay:

$$T(\omega) = \frac{1}{\frac{L}{R}(\frac{R^2}{L^2} + \omega^2)} - \frac{\frac{2R}{L} + 2RC\omega^2}{R^2C^2\omega^4 + \omega^2(4 - 2\frac{R}{L}RC) + \frac{R^2}{L^2}}$$

The equations used to synthesize the low-pass network were obtained as follows:



$$= \frac{e_0}{e_{in}} = \frac{i_{in}\ RZ_b}{e_{in}\ (R + Z_b)}$$

Figure A.2 Constant-resistance network model

For constant input impedance $i_{in} = \frac{e_{in}}{Z_{in}} = \frac{e_{in}}{R}$.

$$\therefore \quad G_{21} = \frac{e_0}{e_{in}} = \frac{Z_b}{R + Z_b} = \frac{1}{1 + RY_b}$$

For $R = 1\Omega$, $G_{21} = \frac{1}{1 + Y_b}$ or $Y_b = \frac{1}{G_{21}} - 1$

and $Z_b = \frac{G_{21}}{1 - G_{21}}$

Also $Z_{in} = R = Z_a + \frac{RZ_b}{R+Z_b}$

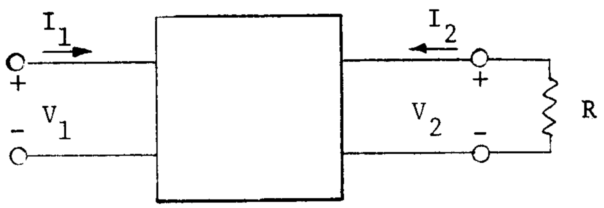
$$R^2 + RZ_b = RZ_a + Z_aZ_b + RZ_b$$

$$Z_a(R+Z_b) = R^2$$

$$Z_a = \frac{R^2}{R + Z_b} = \frac{1}{1 + Z_b} \quad \text{for } R = 1\Omega$$

$$\therefore \quad Y_a = 1 + Z_b$$

The equations used to synthesize the lattice were obtained as follows:



$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$
$$V_2 = -I_2 R$$

Figure A.3 Two-port network model

$$\therefore \frac{V_1}{I_1} = Z_{11} - \frac{Z_{12} Z_{21}}{Z_{22} + R}$$

For a symmetrical lattice, $Z_{11} = Z_{22}$

and $\frac{V_1}{I_1} = R$ (for constant input resistance)

$$\therefore \quad R^2 = Z_{22}^2 - Z_{12} Z_{11}$$

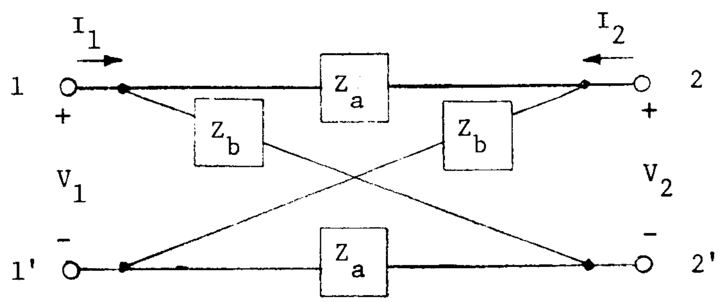


Figure A.4 Symmetrical lattice model

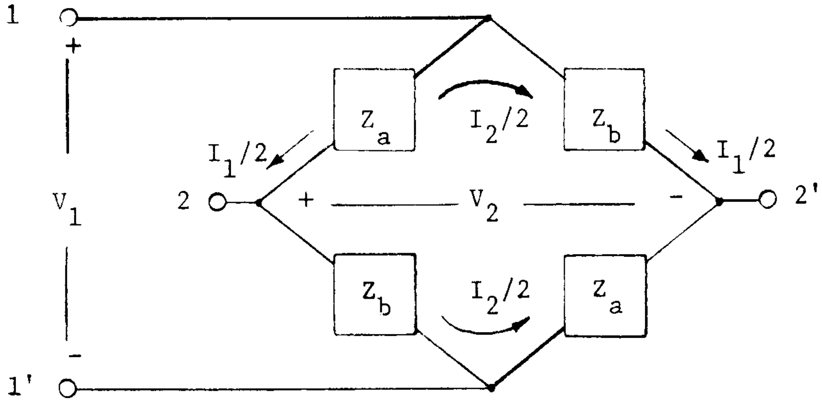


Figure A.4a Symmetrical lattice model drawn as a bridge network

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0} = \frac{Z_a + Z_b}{2} \qquad Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0} = \frac{Z_a + Z_b}{2}$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0} = \frac{1}{2}(Z_b - Z_a)$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0} = \frac{1}{2}(Z_b - Z_a)$$

$$\therefore \quad R^2 = Z_{22}{}^2 - Z_{12}Z_{11} = Z_aZ_b$$

$$G_{21} = \frac{V_2}{V_1} = \frac{Z_{21}}{R} + \frac{Z_{22}}{Z_{12}} - \frac{Z_{11}Z_{22}}{Z_{12}R}$$

$$G_{21} = \frac{1 - \frac{Z_a}{R}}{1 + \frac{Z_a}{R}}$$

$$G_{21} = \frac{1 - Z_a}{1 + Z_a} \qquad \text{for } R = 1\Omega$$

$$\therefore \quad Z_a = \frac{1 - G_{21}}{1 + G_{21}}$$

$$R^2 = 1 = Z_a Z_b \qquad Z_b = \frac{1}{Z_a}$$

If Z_a consists of a parallel LC network, then Z_b consists of a series LC network, and G_{21} can be found as follows:

$$G_{21}(s) = \frac{1 - \dfrac{\frac{Ls}{Cs}}{Ls + \frac{1}{Cs}}}{1 + \dfrac{\frac{Ls}{Cs}}{Ls + \frac{1}{Cs}}} = \frac{Ls + \frac{1}{Cs} - \frac{Ls}{Cs}}{Ls + \frac{1}{Cs} + \frac{Ls}{Cs}}$$

$$G_{21}(s) = \frac{LCs^2 - Ls + 1}{LCs^2 + Ls + 1} = \frac{s^2 - \frac{1}{C}s + \frac{1}{LC}}{s^2 + \frac{1}{C}s + \frac{1}{LC}}$$

$G_{21}(s)$ may be expressed in the general form as:

$$G_{21}(s) = \frac{b_0 - b_1 s + s^2}{b_0 + b_1 s + s^2}$$

$$\therefore \quad G_{21}(j\omega) = \frac{b_0 - \omega^2 - jb_1\omega}{b_0 - \omega^2 + jb_1\omega}$$

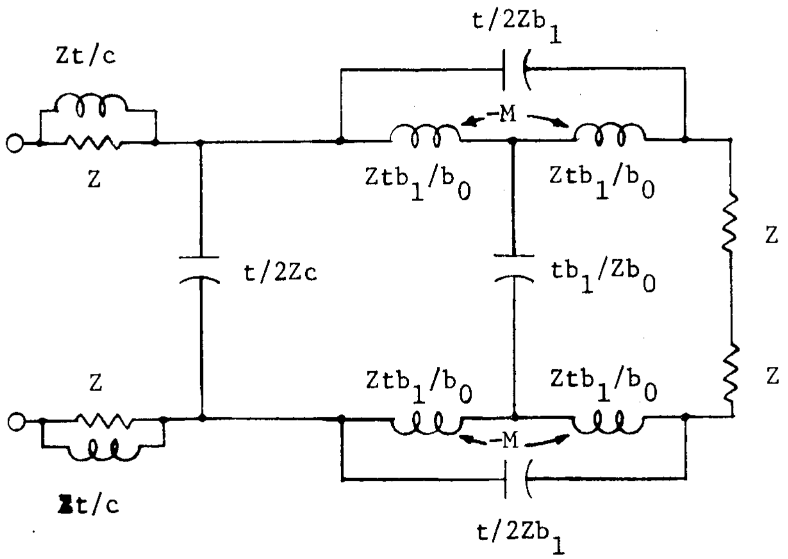
The phase function of $G_{21}(j\omega)$ is therefore found to be:

$$\emptyset(\omega) = \tan^{-1}\frac{-b_1\omega}{b_0-\omega^2} - \tan^{-1}\frac{b_1\omega}{b_0-\omega^2} = 2\tan^{-1}\frac{-b_1\omega}{b_0-\omega^2}$$

The time delay is the derivative of the phase function and is:

$$T(\omega) = \frac{d\emptyset(\omega)}{d\omega} = \frac{2b_1\omega^2-2b_0b_1}{b_0^2+\omega^2(b_1^2-2b_0)+\omega^4}$$

Since the bridged-T network is equivalent to the lattice, the following equivalent circuit is shown for information only. This circuit was obtained using Bartlett's bisection theorem.



$-M = (-b_1/2b_0 + 1/2b_1)Zt$

Figure A.5 Bridged-T equivalent circuit of Figure 5.7