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In 1932 A. Ya. Khinchin gave the first partial solution of the celebrated 1931 $\alpha\beta$ Conjecture of L. G. Schnirelmann and E. Landau on the density of sums of sets of integers, which was completely proved in 1942 by H. B. Mann.

Khinchin's theorem is proved along with theorems of P. Scherk and E. Landau, which are respectively equivalent to Khinchin's theorem and a special case of Khinchin's theorem. These theorems are used to obtain two special cases of the $\alpha\beta$ Theorem. Some aspects of the history of Schnirelmann density and several remarks concerning the literature are also given.

KHINCHIN'S THEOREM ON THE SCHNIRELMANN DENSITY OF SUMS OF SETS OF INTEGERS

by

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CHAPTER I

INTRODUCTION

Our aim in §§ 1.1 and 1.2 is to introduce the reader to Schnirelmann density and give the purpose of this thesis. A description of the content of the thesis is postponed until § 1.3.

§1.1 Preliminary Remarks

Let A be a (possibly empty) set of nonnegative integers and let $A(n)$ be the number of positive integers of A which do not exceed n. In accordance with L. G. Schnirelmann [34, 36] we define the density of A, denoted here by $d(A)$, by

$$d(A) = g\ell b \left\{ \frac{A(n)}{n} : n \geq 1 \right\}. \tag{1.1}$$

In 1930 Schnirelmann used this definition to study Goldbach's Conjecture and other classical problems of additive number theory and it served as the basis for an entirely new branch of mathematics.

Let B be a second set of nonnegative integers. The sum of A and B is the set $A+B$ given by

$$A+B = A \cup B \cup \{a+b : a \in A, \ b \in B\}. \tag{1.2}$$

(The Schnirelmann densities $d(A)$, $d(B)$, and $d(A+B)$ will often be denoted by the Greek letters α, β, and γ respectively.) Moreover, if $A_1, A_2, \cdots, A_h$ $(h \geq 2)$ are sets of nonnegative integers then their sum S is defined recursively by

$$S = \sum_{i=1}^{h} A_i = A_h + \sum_{i=1}^{h-1} A_i .$$

Many European mathematicians, notably E. Landau, I. Schur, and A. Khinchin, turned their attention in an effort to answer a question of Schnirelmann [15, p. 21] : to what extent is the density of the sum of an arbitrary number of sets determined solely by the densities of the summands ? Schnirelmann [36] , Landau [17] , and Schur [37] obtained several inequalities relating γ to α and β, and Landau [18, p. 7] was the first to carefully state the conjecture,

$$\gamma \geq \alpha + \beta , \tag{1.3}$$

under the assumption that $0 < \alpha + \beta < 1$, a conjecture first made by him and Schnirelmann in 1931 in Göttingen [15, p. 26] .

§1.2 A Theorem of A. Khinchin

The conjecture (1.3), now often called the $\alpha\beta$ Theorem, remained unsolved for nearly a dozen years until H. B. Mann's proof in 1942 [20]. However the special cases $\alpha = \beta$ and $\beta = 1-2\alpha$ (or $\alpha = 1-2\beta$) were established by A. Ya. Khinchin less than six months after the conjecture was made. In 1932 Khinchin [16] succeeded in proving a theorem from which one easily obtains (1.3) for the cases mentioned. Khinchin's original proof, as well as a proof of a special case of this theorem due to Landau [18], and the proof of an equivalent theorem due to P. Scherk [33], is elementary but extremely complicated and has never been clearly rewritten or discussed to any extent in English. Briefly, the intent of this thesis is to do this for these three theorems.

§1.3 Proofs of Khinchin's Theorem and Related Topics

In this thesis we treat five theorems, namely Khinchin's 1932 theorem and four theorems related to this theorem. The five are called Khinchin's First Theorem [16, p. 28-29], Landau's Theorem [18, p. 71-72] (a special case of Khinchin's First Theorem proved differently by Landau), Scherk's Theorem [33, p. 263] (which is equivalent to Khinchin's First Theorem), Khinchin's Second Theorem [13, p. 161-162] (which is equivalent to the first of two parts of Khinchin's First Theorem), and a special case of Khinchin's Second Theorem [14, p. 57].

The first three theorems are similar in that from each one easily obtains the special cases $\alpha = \beta$ and $\beta = 1-2\alpha$ (or $\alpha = 1-2\beta$) of the $\alpha\beta$ Theorem (1. 3) and each is proved by successive applications of an inversion formula, defined in Chapter III. It is the main purpose of this thesis to present in detail the proofs of these three theorems. As is the case of much of the literature on Schnirelmann density, these papers of Khinchin, Landau, and Scherk are not to be found in English. Khinchin's 1932 paper [16] as well as the papers of Landau [18] and Scherk [33] are written in German, and have been translated for purposes of inclusion in this thesis. It is often necessary to include numerous intermediate steps and statements to clarify these proofs. Many of the original symbols and notations have been changed in order to accomplish these objectives and to maintain as much uniformity as possible.

The other two theorems can only be used to obtain the special case $\alpha = \beta$ of the $\alpha\beta$ Theorem. Furthermore, their proofs are quite dissimilar to the first three and do not involve an inversion formula. Hence we have treated these theorems much more briefly and their proofs are not given.

Since Scherk's Theorem is equivalent to Khinchin's First Theorem, and Landau's Theorem is equivalent to a special case of Khinchin's First Theorem, we often refer to each of these simply as Khinchin's Theorem, which explains the title of this thesis.

For one wishing to read the simplest proof of Khinchin's

Theorem, where only two summands are involved, the author suggests Chapter V together with Chapter III. If one wishes to read the simplest known proof of Khinchin's Theorem, where an arbitrary number of summands are involved, then the author suggests Chapter VI together with Chapter III.

In Chapter II Khinchin's First Theorem, Scherk's Theorem, and Khinchin's Second Theorem are stated and the equivalences previously mentioned are established. Each of these theorems is then used to obtain its respective special cases of the $\alpha\beta$ Theorem.

The inversion of a set of nonnegative integers is defined in Chapter III, followed by the proofs of several related lemmas and theorems. This definition of inversion is due to Landau [18]. The notion of inversion has proved to be important in verifying Khinchin's First Theorem, Landau's Theorem, and Scherk's Theorem.

Khinchin's First Theorem is proved in Chapter IV. However, his original proof has been slightly modified in that Landau's inversion, rather than his own inversion, is used.

Chapter V contains the proof of Landau's Theorem. It is a special case of Khinchin's First Theorem and was used in Chapter II to obtain (1.3) for $\alpha = \beta$ and $\beta = 1-2\alpha$ (or $\alpha = 1-2\beta$).

In Chapter VI we prove Scherk's Theorem after establishing three lemmas that are needed in the proof. Each of these lemmas is stated as a remark by Scherk [33, p. 264], without proof.

In Chapter VII several remarks are given which do not fit naturally in the earlier chapters. We begin by mentioning how the $\alpha\beta$ Theorem (1.3) can be generalized to the sum of more than two sets. Next we state Khinchin's definition of the inversion of a set of nonnegative integers and give several related lemmas and theorems. Then we discuss the strengthening which occurred due to the slight modifications made in the hypotheses of the original forms of Khinchin's First Theorem and Landau's Theorem. We then briefly compare the proofs of Khinchin, Landau, and Scherk, followed by an equally brief discussion of Khinchin's Second Theorem and its special case. Finally, a few remarks about the accessibility of the papers of Khinchin, Landau, and Scherk, are given.

To conclude the thesis, a detailed discussion of the history of Schnirelmann density, together with many remarks concerning the literature, is presented in Chapter VIII.

CHAPTER II

KHINCHIN'S FIRST AND SECOND THEOREMS AND SCHERK'S THEOREM

In this chapter Khinchin's First Theorem, Scherk's Theorem, and Khinchin's Second Theorem are stated and the equivalences mentioned in § 1.3 are established. Each of the theorems is then used to obtain its respective special cases of the $\alpha\beta$ Theorem.

§ 2.1 Statements of the Three Theorems

The following theorem was proved in 1932 by Khinchin [16, p. 28-29]. This theorem will be proved in Chapter IV by mathematical induction on the integer N.

Khinchin's First Theorem. For $h \geq 2$ let C be the sum of the sets $A_1, A_2, \cdots, A_{h-1}$, B of nonnegative integers. Let

$$\text{(a)} \quad 0 < \alpha < \frac{1}{h}$$

and for each i, where $i = 1, 2, \cdots, h-1$, let

$$\text{(b)} \quad A_i(n) \geq \alpha n - \frac{i-1}{h} \qquad (1 \leq n \leq N),$$

where N is an arbitrary positive integer. Then the following statements hold:

(i) If $\frac{h-1}{h} \leq \lambda \leq 1-\alpha$ and if $B(n) \geq \alpha n-\lambda$ for $1 \leq n \leq N$, then $C(N) \geq h\alpha N-h\lambda+h-1$.

(ii) If $0 \leq \mu < \alpha$ and if $B(n) \geq (1-h\alpha)n-h\mu$ for $1 \leq n \leq N$, then $C(N) \geq (1-\alpha)N-\mu$.

P. Scherk [33, p. 263] stated and proved the next theorem in 1938. Although several similarities in the theorem of Scherk and Khinchin's First Theorem will be immediately recognized, their equivalence is by no means obvious. This equivalence is established in § 2.2. Scherk's proof will be given in Chapter VI, also by induction on N.

Scherk's Theorem. For $h \geq 2$ let C be the sum of the sets $A_1, A_2, \cdots, A_{h-1}, B$ of nonnegative integers. Let the numbers α, β, and δ, and the function f_β be defined by the following conditions:

(a) $0 < \alpha < \frac{1}{h}$

(b) $\beta = \alpha$ or $\beta = 1-h\alpha$

(c) $0 \leq \delta < 1-\beta$

(d) $f_{1-h\alpha}(\delta) = \frac{\delta}{h}$

(e) $f_\alpha(\delta) = \max\{0, h\delta-h+1\}$.

Let N be an arbitrary positive integer. For each i, where $i = 1, 2, \cdots, h-1$, let

$$\text{(f)} \quad A_i(n) \geq \alpha n - \frac{i-1}{h} \qquad (1 \leq n \leq N).$$

Furthermore, let

$$\text{(g)} \quad B(n) \geq \beta n - \delta \qquad (1 \leq n \leq N).$$

Then, $\quad C(N) \geq [(h-1)\alpha + \beta] N - f_\beta(\delta)$.

In 1939 Khinchin [13, p. 161-162] was able to give a different proof of a theorem equivalent to part (i) of his first theorem. This equivalence is verified in § 2.3. The proof of this theorem will be briefly discussed in § 7.5. The theorem is stated in the following way.

Khinchin's Second Theorem. For $h \geq 2$ let C be the sum of the sets $A_1, A_2, \cdots, A_{h-1}$, B of nonnegative integers. Let N be an arbitrary positive integer and let the numbers α and σ be defined by the following conditions:

$$\text{(a)} \quad 0 < \alpha < \frac{1}{h}$$

$$\text{(b)} \quad \sigma = \min \{ B(n) - \alpha(n+1) + 1 : 1 \leq n \leq N \}.$$

Further, for each i, where $i = 1, 2, \cdots, h-1$, let

$$\text{(c)} \quad A_i(n) \geq \alpha n - \frac{i-1}{h} \qquad (1 \leq n \leq N).$$

Then, $C(N) \geq \begin{cases} h\alpha N, & \text{if} \quad \sigma \geq \frac{1}{h} - \alpha \\ h\alpha(N+1)-1+h\sigma, & \text{if} \quad 0 \leq \sigma < \frac{1}{h} - \alpha . \end{cases}$

§ 2.2 The Equivalence of Khinchin's First Theorem and Scherk's Theorem

Apparently Scherk found it unnecessary to show that his theorem is equivalent to Khinchin's First Theorem, since from both of these one obtains identical special cases of the $\alpha\beta$ Theorem. He does, however, claim this equivalence in his paper [33, p. 263]. We now verify this.

Khinchin's First Theorem implies Scherk's Theorem. We assume the validity of Khinchin's First Theorem together with the hypotheses of Scherk's Theorem. We must consider two cases due to the double role played by the number β.

Case 1 $(\beta = \alpha)$. It is necessary to further complicate matters by considering the following subcases.

Subcase a $(\frac{h-1}{h} \leq \delta < 1-\alpha)$. Thus δ satisfies the hypotheses of part (i) of Khinchin's First Theorem and so substitution of δ for λ in the conclusion yields

(2.1) $$C(N) \geq h\alpha N - h\delta + h - 1.$$

Since $\delta \geq \frac{h-1}{h}$ a simple calculation shows that $h\delta-h+1 \geq 0$ and consequently

(2.2) $$f_{\alpha}(\delta) = h\delta-h+1 .$$

Substituting (2.2) into (2.1) shows that

$$C(N) \geq h\alpha N - f_{\alpha}(\delta)$$

which is the conclusion of Scherk's Theorem for $\beta = \alpha$.

<u>Subcase b</u> $(0 < \delta < \frac{h-1}{h})$. This assumption together with condition (g) of Scherk's hypotheses yields

$$B(n) \geq \alpha n - \delta > \alpha n - \frac{h-1}{h} .$$

Letting $\lambda = \frac{h-1}{h}$ we see that B satisfies the hypotheses of part (i) of Khinchin's First Theorem. Consequently

(2.3) $$C(N) \geq h\alpha N - h\left(\frac{h-1}{h}\right) + h-1 = h\alpha N.$$

A simple calculation shows that $h\delta-h+1 < 0$ since $\delta < \frac{h-1}{h}$, and consequently $f_{\alpha}(\delta) = 0$. Substituting this into (2.3) gives

$$C(N) \geq h\alpha N - f_{\alpha}(\delta) ,$$

which is again the conclusion of Scherk's Theorem for $\beta = \alpha$.

Case 2 ($\beta = 1-h\alpha$). This assumption used in condition (c) of Scherk's hypothesis yields $0 \leq \frac{\delta}{h} < \alpha$. We also have by Scherk's hypotheses that

$$B(n) \geq \beta n-\delta = (1-h\alpha)n-h(\frac{\delta}{h}) .$$

Hence we may apply part (ii) of Khinchin's First Theorem, with μ replaced by $\frac{\delta}{h}$. Thus

$$C(N) \geq (1-\alpha)N - \frac{\delta}{h} = (1-\alpha)N-f_{1-h\alpha}(\delta),$$

which is the conclusion of Scherk's Theorem for $\beta = 1-h\alpha$. This establishes the implication.

Scherk's Theorem implies Khinchin's First Theorem. Similarly to the proof just completed, we now assume that Scherk's Theorem holds, together with the hypotheses of Khinchin's First Theorem. We must consider two cases.

Case 1 ($B(n) \geq \alpha n-\lambda$). By the hypotheses of part (i) of Khinchin's Theorem,

$$(2.4) \qquad 0 \leq \frac{h-1}{h} \leq \lambda \leq 1-\alpha .$$

If we let $\beta = \alpha$ in condition (b) of Scherk's hypotheses, (2.4) allows us to let $\delta = \lambda$ in (c). Hence by Scherk's Theorem (with $\beta = \alpha$)

(2.5) $$C(N) \geq h\alpha N - f_\alpha(\lambda) .$$

By (2.4), $h\lambda - h + 1 \geq 0$, so

(2.6) $$f_\alpha(\lambda) = h\lambda - h + 1 .$$

After substituting (2.6) into (2.5) we have

$$C(N) \geq h\alpha N - h\lambda + h - 1,$$

thus establishing part (i) of Khinchin's First Theorem.

Case 2 $(B(n) \geq (1-h\alpha)n - h\mu)$. By the hypotheses of part (ii) of Khinchin's First Theorem we see that $0 \leq h\mu < h\alpha$. If we let $\beta = 1 - h\alpha$ in Scherk's condition (b), we can therefore let $\delta = h\mu$ in condition (c), since $0 \leq h\mu < h\alpha = 1 - \beta$. Consequently by Scherk's Theorem (with $\beta = 1 - h\alpha$) we have

$$C(N) \geq (1-\alpha)N - f_{1-h\alpha}(h\mu) = (1-\alpha)N - \mu,$$

establishing part (ii) of Khinchin's First Theorem. Hence the implication is established.

§ 2.3 The Equivalence of Khinchin's Second Theorem and Part (i) of His First Theorem

Khinchin's First Theorem-part (i) implies Khinchin's Second Theorem. Suppose that the conditions of Khinchin's Second Theorem are satisfied. Since

$$\sigma = \min \{ B(n)-\alpha(n+1)+1 : 1 \leq n \leq N \},$$

we see that $\sigma \leq B(n)-\alpha(n+1)+1$ for each n, and so

(2.7) $$B(n) \geq \alpha(n+1)-1+\sigma \qquad (1 \leq n \leq N) .$$

We now consider two cases.

Case 1 $(\sigma > \frac{1}{h} - \alpha)$. This assumption used in (2.7) gives

$$B(n) \geq \alpha(n+1)-1+\frac{1}{h}-\alpha = \alpha n - \frac{h-1}{h} \qquad (1 \leq n \leq N) .$$

Hence B satisfies the hypotheses of part (i) of Khinchin's First Theorem with $\lambda = \frac{h-1}{h}$. Consequently

$$C(N) \geq h\alpha N - h\left(\frac{h-1}{h}\right)+h-1 = h\alpha N,$$

which is the conclusion of Khinchin's Second Theorem for $\sigma \geq \frac{1}{h} - \alpha$.

Case 2 $(0 \leq \sigma < \frac{1}{h} - \alpha)$. Since $\sigma < \frac{1}{h} - \alpha$ we have $\frac{h-1}{h} < 1-\alpha-\sigma$. Further, since $\sigma \geq 0$, $1-\alpha-\sigma \leq 1-\alpha$. Consequently,

(2.8) $$\frac{h-1}{h} \leq 1-\alpha-\sigma \leq 1-\alpha .$$

Thus we may let $\lambda = 1-\alpha-\sigma$ by the hypotheses of part (i) of Khinchin's First Theorem. We also have by (2.7) that

$B(n) \geq \alpha n-(1-\alpha-\sigma)$ and so B satisfies the hypotheses of part (i) of Khinchin's First Theorem. Consequently,

$$C(N) \geq h\alpha N-h(1-\alpha-\sigma)+h-1 = h\alpha(N+1)+h\sigma-1$$

which is the conclusion of Khinchin's Second Theorem for $0 \leq \sigma < \frac{1}{h}-\alpha$. This establishes the implication.

<u>Khinchin's Second Theorem implies Khinchin's First Theorem-part(i).</u>

Assume that the hypotheses of part (i) of Khinchin's First Theorem are satisfied. Since $B(n) \geq \alpha n-\lambda \quad (1 \leq n \leq N)$ then

$$B(n)-\alpha(n+1)+1 \geq \alpha n-\lambda-\alpha(n+1)+1 = 1-\alpha-\lambda \quad (1 \leq n \leq N).$$

Consequently

$$\sigma = \min\{B(n)-\alpha(n+1)+1 : 1 \leq n \leq N\} \geq 1-\alpha-\lambda .$$

Also $\lambda \leq 1-\alpha$ by the hypotheses of part (i) of Khinchin's First Theorem. Thus

$$0 \leq 1-\alpha-\lambda \leq \sigma .$$

Hence by Khinchin's Second Theorem if $\sigma < \frac{1}{h}-\alpha$, we have

$$C(N) \geq h\alpha(N+1)-1+h\sigma \geq h\alpha(N+1)-1+h(1-\alpha-\lambda) = h\alpha N-h\lambda+h-1 ,$$

while if $\sigma \geq \frac{1}{h}-\alpha$, we have

$$C(N) \geq h\alpha N \geq h\alpha N - h\lambda + h - 1,$$

since $\lambda \geq \frac{h-1}{h}$. Thus in either case we have the conclusion of part (i) of Khinchin's First Theorem, and so the implication is verified.

§ 2.4 The Special Case $\alpha = \beta$ of the $\alpha\beta$ Theorem

The $\alpha\beta$ Theorem was stated in the following way in § 1.1.

$\alpha\beta$ Theorem. If C is the sum of two sets A and B of non-negative integers, where $d(A) = \alpha$, $d(B) = \beta$, and $d(C) = \gamma$, and $0 < \alpha + \beta < 1$, then

(2.9) $$\gamma \geq \alpha + \beta .$$

Some authors (e.g. [23, p. 261]) refer to the inequality

(2.10) $$\gamma \geq \min \{1, \alpha + \beta\}$$

as the $\alpha\beta$ Theorem. This inequality follows easily from what we have called the $\alpha\beta$ Theorem. Thus if $\alpha + \beta = 0$ then inequality (2.10) is immediate. If $0 < \alpha + \beta < 1$, then by inequality (2.9) we have $\gamma \geq \alpha + \beta = \min \{1, \alpha + \beta\}$. Finally if $\alpha + \beta \geq 1$, then by Schnirelmann's Theorem (see Theorem 8.1 or [18, p. 56]) we have $\gamma = 1 \geq \min\{1, \alpha + \beta\}$.

We will formally state the case $\alpha = \beta$ of the $\alpha\beta$ Theorem and will use each of the three theorems of § 2.1 to prove it.

<u>$\alpha\beta$ Theorem $(\alpha = \beta)$.</u> If $0 < \alpha < \frac{1}{2}$, then $\gamma \geq 2\alpha$.

Proof. Let $d(A) = \alpha = d(B)$, $d(C) = \gamma$, where $C = A+B$. Then for all n, $A(n) \geq \alpha n$ and $B(n) \geq \alpha n$ by (1.1).

<u>a. (By use of Khinchin's First Theorem - part (i)).</u> Before continuing with the proof we make a remark. We will be applying the case $h = 2$ and $\lambda = \frac{1}{2}$ of part (i) of Khinchin's First Theorem. Part (i) of Landau's Theorem (see § 5.1) is just this specialization where he allows λ to be any number in the interval $[\frac{1}{2}, 1-\alpha]$. These other values of λ are unneeded to prove this case of the $\alpha\beta$ Theorem. (Consequently the proofs of the $\alpha\beta$ Theorem for $\alpha = \beta$ are identical when using either Khinchin's First Theorem or Landau's Theorem).

We first notice that

$$B(n) \geq \alpha n > \alpha n - \frac{1}{2} .$$

Letting $h = 2$ and $\lambda = \frac{1}{2}$ in part (i) of Khinchin's First Theorem we have

$$C(N) \geq 2\alpha N - 2(\frac{1}{2}) + 2 - 1 = 2\alpha N.$$

Consequently $\dfrac{C(N)}{N} \geq 2\alpha$ and hence, since N is an arbitrary positive integer, we have $\gamma = g\ell b\{\dfrac{C(n)}{n} : n \geq 1\} \geq 2\alpha$.

b. (By use of Scherk's Theorem). Letting $h = 2$, $\beta = \alpha$, and $\delta = 0$ in Scherk's Theorem we have

$$C(N) \geq 2\alpha N - f_\alpha(0) = 2\alpha N.$$

Since N is an arbitrary positive integer, $\gamma = g\ell b\{\dfrac{C(n)}{n} : n \geq 1\} \geq 2\alpha$.

c. (By use of Khinchin's Second Theorem). Letting $h = 2$ in Khinchin's Second Theorem we have the following special case proved by Khinchin in 1940 [12, p. 57]: If $C = A+B$, $0 < \alpha < \frac{1}{2}$, $\sigma = \min\{B(n) - \alpha(n+1) + 1 : 1 \leq n \leq N\}$, and $A(n) \geq \alpha n$ $(1 \leq n \leq N)$, then

$$C(N) \geq \begin{cases} 2\alpha N, & \text{if } \sigma \geq \frac{1}{2} - \alpha \\ 2\alpha(N+1) - 1 + 2\sigma, & \text{if } 0 \leq \sigma < \frac{1}{2} - \alpha. \end{cases}$$

Since $B(n) \geq \alpha n$ $(1 \leq n \leq N)$, we have

$$\sigma = \min\{B(n) - \alpha(n+1) + 1 : 1 \leq n \leq N\} \geq \min\{\alpha n - \alpha(n+1) + 1 : 1 \leq n \leq N\}$$

$$= \min\{1 - \alpha : 1 \leq n \leq N\}$$

$$= 1 - \alpha > \frac{1}{2} - \alpha .$$

Hence $C(N) \geq 2\alpha N$ and consequently, since N is an arbitrary positive integer, we have $\gamma = g\ell b\{\frac{C(n)}{n} : n \geq 1\} \geq 2\alpha$.

§ 2.5 The Special Case $\beta = 1-2\alpha$ (or $\alpha = 1-2\beta$) of the $\alpha\beta$ Theorem

Both part (ii) of Khinchin's First Theorem and Scherk's Theorem can be used to verify the $\alpha\beta$ Theorem for the case $\beta = 1-2\alpha$ (or $\alpha = 1-2\beta$). We will state the case $\beta = 1-2\alpha$ as a formal theorem. The other case follows immediately by interchanging the roles of A and B, and α and β, in the proof.

<u>$\alpha\beta$ Theorem ($\beta = 1-2\alpha$).</u> If $0 < \alpha \leq \frac{1}{2}$, then $\gamma \geq 1-\alpha$.

Before proving this statement we shall first show that this is in fact the special case indicated of the $\alpha\beta$ Theorem.

The hypothesis of the $\alpha\beta$ Theorem $0 < \alpha+\beta < 1$ becomes $0 < \alpha + (1-2\alpha) < 1$ which reduces to $0 < \alpha < 1$. The implied restriction $\beta \geq 0$ becomes $\alpha \leq \frac{1}{2}$ since $\beta = 1-2\alpha$. Consequently the hypothesis of our special case is $0 < \alpha \leq \frac{1}{2}$. Since $\beta = 1-2\alpha$, then $\alpha + \beta = 1-\alpha$ and so we have the conclusion for the special case.

Proof. We now prove the special case of the $\alpha\beta$ Theorem by treating two cases.

<u>Case 1</u> ($\alpha = \frac{1}{2}$). If $\alpha = \frac{1}{2}$, then $\gamma \geq \frac{1}{2} = 1-\alpha$, since by definition $d(C) \geq d(A)$.

Case 2 $(0 < \alpha < \frac{1}{2})$. Let $d(A) = \alpha$, $d(B) = \beta = 1-2\alpha$, and $C = A+B$. Then by (1.1)

$$A(n) \geq \alpha n, \quad B(n) \geq (1-2\alpha)n \ . \tag{2.11}$$

We complete the proof by using each of the following two theorems.

a. (By use of Khinchin's First Theorem - part (ii). We will be applying the case $h = 2$ and $\mu = 0$ of part (ii) of Khinchin's First Theorem. Part (ii) of Landau's Theorem (see § 5.1) is just this specialization except that he allows μ to be any real number in the interval $[0, \alpha)$. (Consequently the proofs of the $\alpha\beta$Theorem for the case $\beta = 1-2\alpha$ are identical when using Khinchin's First Theorem or Landau's Theorem.)

Letting $h = 2$ and $\mu = 0$ in Khinchin's First Theorem we see by (2.11) that the hypotheses of (ii) are satisfied. Consequently

$$C(N) \geq (1-\alpha)N,$$

and since N is an arbitrary positive integer, we have $\gamma = g\ell b \{\frac{C(n)}{n} : n \geq 1\} \geq 1-\alpha$.

b. (By use of Scherk's Theorem). Letting $h = 2$, $\beta = 1-2\alpha$, and $\delta = 0$ in Scherk's Theorem we see by (2.11) that the hypotheses are satisfied. Consequently,

$$C(N) \geq [\alpha + (1-2\alpha)] N - f_{1-2\alpha}(0) = (1-\alpha)N,$$

and since N is an arbitrary positive integer, we have
$\gamma \geq g\ell b \, \{\frac{C(n)}{n} ; \; n \geq 1 \} \geq 1-\alpha$.

CHAPTER III

THE INVERSION OF A SET OF NONNEGATIVE INTEGERS

In this chapter we define the inversion of a set of nonnegative integers and give several related formulas. The inversion, due to Landau [18], is used throughout the proceeding Chapters IV, V, and VI. A second kind of inversion, due to Khinchin [16], is discussed in § 7.2.

Landau makes the following definition [18, p. 72].

Definition 3.1. Let M be an arbitrary positive integer. The *inversion* of a set A of nonnegative integers with respect to the interval $[1, M]$ is the set A^* given by

$$A^* = \{ M+1-x : x \in \overline{A} \cap [1, M] \}.$$

Landau and Scherk developed several formulas related to the inversion. We first prove a lemma of Landau [18, p. 72]. This is followed by a lemma and a formula of Scherk [33, p. 265]. A second formula, due to Landau [18, p. 73], will then be shown to be a special case of Scherk's formula. Scherk's formula first appeared in the literature in 1938 and is actually a generalization of Landau's formula, which appeared in 1937.

<u>Lemma 3.1.</u> For any integer n, where $0 \leq n \leq M$,

$$A^*(M-n) = M-n-[A(M)-A(n)] \ .$$

Proof. To calculate $A^*(M-n)$ we must find the number of positive integers of the form $M+1-x$ not exceeding $M-n$, where $x \in \overline{A} \cap [1, M]$. The conditions

$$0 < M+1-x \leq M-n, \quad x \in \overline{A} \cap [1, M]$$

are equivalent to

$$n+1 \leq x \leq M, \quad x \in \overline{A} \ .$$

Thus all we need do is compute the number of integers missing from A in the interval $[n+1, M]$. This is just

$$[M-A(M)] - [n-A(n)] = M-n-[A(M)-A(n)] \ ,$$

and the lemma is proved.

The next lemma is proved by Scherk [33, p. 265]. As Scherk's proof is quite brief we have supplied the necessary details. First we make the following definition.

<u>Definition 3.2.</u> For $h \geq 2$ let C be the sum of the sets $A_1, A_2, \cdots, A_h$ of nonnegative integers. For each i, where $i = 1, 2, \cdots, h$, we define the set S_i by

$$S_i = A_1+A_2 + \cdots + A_{i-1} + C^* + A_{i+1} + \cdots + A_h .$$

Lemma 3.2. If $M+1$ is missing from C, then for each i, where $i = 1, 2, \cdots, h$, we have

$$S_i \subseteq A_i^*$$

on $[1, M]$.

Proof. Suppose $i \in \{1, 2, \cdots, h\}$ and let $a \in S_i \cap [1, M]$. We want to show that $a \in A_i^*$. By definition of S_i,

$$a = a_1 + a_2 + \cdots + a_{i-1} + c + a_{i+1} + \cdots + a_h , \tag{3.1}$$

where $a_j \in A_j$ or $a_j = 0$ $(j = 1, 2, \cdots, i-1, i+1, \cdots, h)$ and $c \in C^*$ or $c = 0$. In fact at least one a_j or c must be nonzero, for otherwise $a = 0 \notin [1, M]$. The addition of $M+1-c-a$ to the right and left-hand members of (3.1) yields

$$M+1-c = a_1 + a_2 + \cdots + a_{i-1} + (M+1-a)+a_{i+1} + \cdots + a_h . \tag{3.2}$$

Now we assume that $a \in \overline{A_i^*} \cap [1, M]$ and arrive at a contradiction. We define the integer x by $a = M+1-x$. Then $x \in [1, M]$. Further, $x \in A_i$, for if $x \in \overline{A_i}$ then $a \in A_i^*$ by definition of A_i^*. Consequently $M+1-a = x \in A_i$. Thus by (3.2) $M+1-c \in C$. Now if $c \in C^*$, then $c = M+1-x$ where $x \in \overline{C}$, and so $x=M+1-c \in \overline{C}$. Also if $c = 0$, then $M+1-c = M+1-c \in \overline{C}$ by hypothesis. But this contradicts the fact that $M+1-c \in C$. Consequently the assumption $a \in \overline{A_i^*}$ is false, and so $a \in A_i^*$, which establishes the lemma.

The following formula follows immediately from Lemmas 3.1 and 3.2.

<u>Theorem 3.1 (Scherk's inversion formula).</u> For all n, where $0 \leq n \leq M$, and for each i, where $i = 1, 2, \cdots, h$,

$$A_i(M) - A_i(n) = M-n-A_i^*(M-n) \leq M-n-S_i(M-n),$$

if $M+1 \notin C$.

The next theorem is shown to be a special case of Scherk's inversion formula.

<u>Theorem 3.2 (Landau's inversion formula).</u> Let A and B be sets of nonnegative integers. If $C = A+B$ and if $T = A+C^*$, then for all n, where $0 \leq n \leq M$,

$$B(M)-B(n) = M-n-B^*(M-n) \leq M-n-T(M-n),$$

if $M+1 \notin C$.

To show that Landau's inversion formula is a special case of Scherk's inversion formula we need only let $h = 2$, $i = 2$, $A_1 = A$, $A_2 = B$, and $S_2 = T$ in Scherk's fomrula. Consequently for all n, where $0 \leq n \leq M$, if $M+1 \nmid C$ then

$$B(M) - B(n) = M-n-B^*(M-n) \leq M-n-T(M-n),$$

which is the conclusion of Landau's formula.

CHAPTER IV

THE PROOF OF KHINCHIN'S FIRST THEOREM

In this chapter Khinchin's First Theorem is proved. The original proof of Khinchin [16] has been modified slightly in that the inversion of Landau (Definition 3.2) is used in place of Khinchin's original inversion (Definition 7.1).

§ 4.1 Khinchin's First Theorem

We stated Khinchin's First Theorem in § 2.1. So as to eliminate any unnecessary strain on the reader, we repeat it here.

Khinchin's First Theorem. For $h \geq 2$ let C be the sum of the sets $A_1, A_2, \cdots, A_{h-1}, B$ of nonnegative integers. Let

$$\text{(a)} \quad 0 < \alpha < \frac{1}{h}$$

and for each i, where $i = 1, 2, \cdots, h-1$, let

$$\text{(b)} \quad A_i(n) \geq \alpha n - \frac{i-1}{h} \qquad (1 \leq n \leq N),$$

where N is an arbitrary positive integer. Then the following statements hold:

(i) If $\frac{h-1}{h} \leq \lambda \leq 1-\alpha$ and if $B(n) \geq \alpha n - \lambda$ for $1 \leq n \leq N$, then $C(N) \geq h\alpha N - h\lambda + h - 1$.

(ii) If $0 \leq \mu < \alpha$ and if $B(n) \geq (1-h\alpha)n - h\mu$ for $1 \leq n \leq N$, then $C(N) \geq (1-\alpha)N - \mu$.

§ 4. 2 Introduction to the Proof

Khinchin's First Theorem is proved by mathematical induction on the integer N. The proof is long, quite complicated, but completely elementary. We first verify the theorem for the case $N = 1$. We then assume that the theorem is valid for all N where $1 \leq N \leq M$, M a fixed positive integer. The induction is completed by arriving at a contradiction after assuming that the theorem fails for $N = M+1$. The induction will be applied separately to (i) and (ii). We use the induction hypotheses of (i) and (ii) to prove that (i) follows for $N = M+1$, after which we use (i) for $N = M+1$ together with the induction hypotheses of (i) and (ii) to prove that (ii) follows for $N = M+1$. This completes the induction.

Throughout the second step of the induction we need the following lemma.

Lemma 4. 1. Suppose that Khinchin's First Theorem is true for $1 \leq N \leq M$. If $M+1 \in C$, then the theorem holds for $N = M+1$.

Proof. Since $M+1 \in C$ and $\alpha < \frac{1}{h}$ we have in the first case that

$$C(M+1) = 1+C(M) \geq 1+h\alpha M-h\lambda+h-1 > h\alpha+h\alpha M-h\lambda+h-1 = h\alpha(M+1)-h\lambda+h-1,$$

and so (i) is true for $N = M+1$. In the second case, since $\alpha > 0$ and $M+1 \in C$ we have that

$$C(M+1) = 1+C(M) \geq 1+(1-\alpha)M-\mu > (1-\alpha)+(1-\alpha)M-\mu = (1-\alpha)(M+1)-\mu,$$

and so (ii) is true for $N = M+1$. Thus the lemma is proved.

§ 4.3 The Proof for $N = 1$

We notice that for each i, where $i = 1, 2, \cdots, h-1$, $A_i(1) \geq \alpha - \frac{i-1}{h}$ by hypotheses. Thus in particular $A_1(1) \geq \alpha > 0$ and so $A_1(1) = 1$. Consequently $1 \in A_1$. Hence $1 \in C$, and so $C(1) = 1$. If $\lambda \geq \frac{h-1}{h}$ and $N = 1$, then

$$h\alpha N-h\lambda+h-1 \leq h\alpha < 1 = C(1),$$

which proves (i) for $N = 1$. If $0 \leq \mu < \alpha$ and $N = 1$, then

$$(1-\alpha)N-\mu = 1-\alpha-\mu < 1 = C(1),$$

which proves (ii) for $N = 1$.

§ 4.4 The Induction - Part I

We assume both parts (i) and (ii) of the theorem are valid for all N where $1 \leq N \leq M$. We further assume the hypotheses hold but the conclusion fails for part (i) of the theorem when $N = M+1$, and obtain a contradiction. In this way we show that part (i) follows for $N = M+1$. By Lemma 4.1 we may suppose that $M+1 \nmid C$. Hence

$$(4.1) \qquad h\alpha M - h\lambda + h - 1 \leq C(M) = C(M+1) < h\alpha(M+1) - h\lambda + h - 1 .$$

We define the number ϕ by

$$(4.2) \qquad \phi = \frac{C(M)}{h} + \lambda - \frac{h-1}{h} - \alpha M,$$

where it follows by using (4.1) in (4.2) that

$$(4.3) \qquad 0 \leq \phi < \alpha .$$

The induction hypothesis of (ii) states that

$$C(M-k) \geq h\alpha(M-k) - h\lambda + h - 1,$$

for all integers k where $0 \leq k < M$. This inequality is also true for $k = M$ by a simple calculation since $\lambda \geq \frac{h-1}{h}$. We solve for $h\alpha M$ in (4.2) and substitute this expression for $h\alpha M$ in the last inequality to obtain

$$C(M-k) \geq C(M) - h\phi - h\alpha k \qquad (0 \leq k \leq M),$$

which may be rewritten as

$$(4.4) \qquad C(M-k) - C(M) \geq -h\phi - h\alpha k \qquad (0 \leq k \leq M).$$

Referring to Lemma 3.1 (with $n = M-k$) we have

$$(4.5) \qquad C(M) - C(M-k) = k - C^*(k) \qquad (0 \leq k \leq M),$$

and so by (4.4)

$$C^*(k) \geq k - h\phi - h\alpha k = (1-h\alpha)k - h\phi \qquad (0 \leq k \leq M).$$

This result together with (4.3) shows that C^* satisfies the hypotheses of (ii) (with μ replaced by ϕ) and thus it enables us to apply the induction hypothesis of (ii) to the set S_h defined by

$$S_h = A_1 + A_2 + \cdots + A_{h-1} + C^* .$$

Consequently

$$(4.6) \qquad S_h(M-n) \geq (1-\alpha)(M-n) - \phi \qquad (0 \leq n \leq M),$$

where the case $n = M$ follows from (4.3). The first step below follows from Scherk's inversion formula (Theorem 3.1, with $i = h$ and $A_h = B$) and Definition 3.2. We have

$$B(M)-B(n) \leq M-n-S_h(M-n)$$

$$\leq M-n-(1-\alpha)(M-n)+\phi \qquad [(4.6)]$$

(4.7) $$= \alpha(M-n)+\phi \qquad (0 \leq n \leq N).$$

We now divide the positive integer $C(M)$ by the positive integer h. From the Division Algorithm there exist unique integers q and r such that

(4.8) $$C(M) = qh+r, \qquad 0 \leq r < h .$$

To continue the proof we first consider the case $r = h-1$. We later consider the case $0 \leq r < h-1$. (In each case a contradiction is obtained.)

<u>Case 1</u> ($r = h-1$). We first see that

$$B(M) \leq \alpha M+\phi \qquad [(4.7),\ n=0]$$

$$= \frac{C(M)}{h} + \lambda - \frac{h-1}{h} \qquad [(4.2)]$$

$$= q + \frac{r}{h} + \lambda - \frac{h-1}{h} \qquad [(4.8)]$$

$$= q + \frac{h-1}{h} + \lambda - \frac{h-1}{h} \qquad [r = h-1]$$

$$= q+\lambda$$

$$< q+1 , \qquad [\lambda < 1 \text{ by hypothesis}]$$

and since $B(M)$ is an integer,

$$B(M) \leq q . \tag{4.9}$$

We record the formula

$$\alpha M+\phi = q+\lambda , \tag{4.10}$$

obtained from the first and fifth steps above, for future reference.

We next see that

$$\begin{aligned} B(M) &= B(M+1) && [M+1 \notin B] \\ &\geq \alpha(M+1)-\lambda && [\text{hypotheses of (i)}] \\ &= q-\phi+\alpha && [(4.10)] \\ &> q-\phi+\phi && [(4.3)] \\ &= q, \end{aligned}$$

contradicting (4.9).

<u>Case 2</u> ($0 \leq r < h-1$). We see that

$$
\begin{aligned}
B(M) &= B(M+1) && [M+1 \nmid B] \\
&\geq \alpha(M+1)-\lambda && [\text{hypotheses of (i)}] \\
&= (\alpha M-\lambda)+\alpha \\
&= \frac{C(M)}{h} - \frac{h-1}{h} - \phi + \alpha && [(4.2)] \\
&= q + \frac{r}{h} - \frac{h-1}{h} - \phi + \alpha && [(4.8)] \\
&> q - \frac{h-1-r}{h} - \phi + \phi && [(4.3)] \\
&= q - \frac{h-1-r}{h} \\
&> q-1, && [r \geq 0]
\end{aligned}
$$

and since $B(M)$ is an integer,

(4.11) $$B(M) \geq q .$$

We record the formula

(4.12) $$\alpha M + \frac{h-1}{h} - \lambda + \phi = q + \frac{r}{h} ,$$

obtained from the third and fifth steps above, for future reference.

Next we have, for $0 \leq n \leq M$, that

$$\begin{aligned}
B(n) &\geq B(M) - \alpha(M-n) - \phi && [(4.7)] \\
&= \alpha n + B(M) - \alpha M - \phi \\
&\geq \alpha n + q - \alpha M - \phi && [(4.11)] \\
&= \alpha n - \lambda + \frac{h-1-r}{h} && [(4.12)] \\
&> \alpha n - \lambda + \frac{1}{h} && [r < h-1] \\
&> \alpha n - 1 + \frac{1}{h} && [\lambda < 1 \text{ by hypothesis}] \\
&= \alpha n - \frac{h-1}{h} .
\end{aligned}$$

Thus we may let $\lambda = \frac{h-1}{h}$ in the hypotheses of part (i). We apply (i) to the set $C = A_1 + \cdots + A_{h-1} + B$. Consequently

$$C(M-k) \geq h\,\alpha(M-k) - h\left(\frac{h-1}{h}\right) + h - 1 = h\alpha k \qquad (0 \leq k \leq M),$$

where the case $k = M$ follows trivially.

We solve for $h\alpha M$ in (4.2) and substitute this expression for $h\alpha M$ into the last inequality to obtain

$$C(M-k) \geq C(M) + h\lambda - h + 1 - h\phi - h\alpha k \qquad (0 \leq k \leq M).$$

Therefore

$$(4.13) \qquad C(M) - C(M-k) \leq h\alpha k + h\phi - h\lambda + h - 1 \qquad (0 \leq K \leq M).$$

Use of (4.13) and Lemma 3.1 (with $n = M-k$) yields

$$C^*(k) = k - C(m) + C(M-k) \geq k - h\alpha k - h\phi + h\lambda - h + 1$$

$$(4.14) \qquad = (1-h\alpha)k - h\left(\frac{h-1}{h} - \lambda + \phi\right) \qquad (0 \leq k \leq M) .$$

If we can show that $0 \leq \frac{h-1}{h} - \lambda + \phi < \alpha$, then C^* will satisfy the hypotheses of (ii) by (4.14). To do this let $k = 0$ in (4.14). A simple calculation shows that

$$(4.15) \qquad \frac{h-1}{h} - \lambda + \phi \geq 0 .$$

Also, by hypotheses $\frac{h-1}{h} \leq \lambda$ and so by (4.3) and (4.15)

$$(4.16) \qquad 0 \leq \frac{h-1}{h} - \lambda + \phi < \alpha .$$

Consequently the number $\frac{h-1}{h} - \lambda + \phi$ satisfies the hypotheses of (ii) and so by (4.14) we may apply the induction hypothesis of (ii) to the set S_h defined by $S_h = A_1 + A_2 + \cdots + A_{h-1} + C^*$ and obtain

$$(4.17) \qquad S_h(k) \geq (1-\alpha)k - \left(\frac{h-1}{h} - \lambda + \phi\right) \qquad (0 \leq k \leq M),$$

where the case $k = 0$ follows from (4.16). Again by Scherk's inversion formula (with $n = M-k$, $i = h$, and $A_h = B$) together with Definition 3.2, we have

$$(4.18) \qquad B(M) - B(M-k) \leq k - S_h(k) \qquad (0 \leq k \leq M) .$$

Therefore for $0 \leq k \leq M$,

$$\begin{aligned} B(M-k) &\geq B(M)-k+S_h(k) && [(4.18)] \\ &\geq q-k+S_h(k) && [(4.11)] \\ &\geq q-k+(1-\alpha)k - \left(\frac{h-1}{h}-\lambda+\phi\right) && [(4.17)] \\ &= q-\alpha k - \frac{h-1}{h} + \lambda - \phi \\ &= q-\alpha k+\alpha M-q-\frac{r}{h} && [(4.12)] \\ &= \alpha(M-k)-\frac{r}{h}, \end{aligned}$$

or equivalently,

$$B(n) \geq \alpha n - \frac{r}{h} \qquad (0 \leq n \leq M) . \tag{4.19}$$

We now make another application of the induction hypotheses of (ii). By (4.19),

$$B(n) \geq \alpha n - \frac{(r+1)-1}{h} \qquad (0 \leq n \leq M),$$

and so B satisfies the requirements placed on the $(r+1)$-st set by condition (b) of the hypotheses. Consider the set S_{r+1} defined by

$$S_{r+1} = A_1 + A_2 + \cdots + A_r + B + A_{r+2} + \cdots + A_{h-1} + C^*.$$

By (4.16) and (4.14) we may replace μ and B in the hypotheses of (ii) by $\frac{h-1}{h} - \lambda + \phi$ and C^* respectively, to obtain

(4.20) $$S_{r+1}(M) \geq (1-\alpha)M - \left(\frac{h-1}{h} - \lambda + \phi\right).$$

We may interchange the positions of B and C^* in the above sum for S_{r+1} to obtain

$$S_{r+1} = A_1 + A_2 + \cdots + A_r + C^* + A_{r+2} + \cdots + A_{h-1} + B.$$

Hence because of Definition 3.2 (with $A_h = B$) we have by Scherk's inversion formula (with $n = 0$ and $i = r+1$) that

$$A_{r+1}(M) \leq M - S_{r+1}(M).$$

Thus,

$$\begin{aligned} A_{r+1}(M) &\leq M-(1-\alpha)M + \left(\frac{h-1}{h} - \lambda + \phi\right) && [(4.20)] \\ &= \alpha M + \left(\frac{h-1}{h} - \lambda + \phi\right) \\ &= q + \frac{r}{h} && [(4.12)] \\ &< q + 1, && [r < h-1] \end{aligned}$$

and since $A_{r+1}(M)$ is an integer,

$$A_{r+1}(M) \le q\ . \tag{4.21}$$

However,

$$\begin{aligned}
A_{r+1}(M) &= A_{r+1}(M+1) && [M+1 \notin A_{r+1}] \\
&\ge \alpha(M+1) - \frac{(r+1)-1}{h} && [\text{condition (b)}] \\
&= \alpha(M+1) - \frac{r}{h} \\
&= \lambda + q - \frac{h-1}{h} - \phi + \alpha && [(4.12)] \\
&\ge \frac{h-1}{h} + q - \frac{h-1}{h} - \phi + \alpha && [\text{hypotheses of (i)}] \\
&> q - \phi + \phi && [(4.3)] \\
&= q,
\end{aligned}$$

which contradicts (4.21) and completes the induction on (i).

§ 4.5 The Induction - Part II

We now assume the validity of (i) for all N where $1 \le N \le M+1$ and assume that (ii) is valid for all N where $1 \le N \le M$. We further assume the hypotheses hold but the conclusion fails for part (ii) of the theorem when $N = M+1$, and obtain a contradiction. In this way we show that part (ii) follows for $N = M+1$. By Lemma 4.1 we may suppose that $M+1 \notin C$. Hence

$$(1-\alpha)M-\mu \leq C(M) = C(M+1) < (1-\alpha)(M+1) - \mu \ . \tag{4.22}$$

We define the number τ by

$$\tau = \alpha M - M + C(M) + \mu \ , \tag{4.23}$$

where it follows by using (4. 22) in (4. 23) that

$$0 \leq \tau < 1-\alpha < 1 \ . \tag{4.24}$$

The induction hypothesis of (ii) states that

$$C(M-k) \geq (1-\alpha)(M-k) - \mu \ ,$$

where $0 \leq k < M$. This inequality is also true for $k = M$ by a simple calculation since $\mu \geq 0$. We solve for $M-\alpha M$ in (4. 23) and substitute this expression for $M-\alpha M$ in the last inequality to obtain

$$C(M-k) \geq C(M) - (1-\alpha)k - \tau \qquad (0 \leq k \leq M),$$

which may be rewritten as

$$C(M-k) - C(M) \geq -(1-\alpha)k - \tau \qquad (0 \leq k \leq M) \ . \tag{4.25}$$

Referring to Lemma 3. 1 (with $n = M-k$) we have

$$C^*(k) = k - C(M) + C(M-k) \qquad (0 \leq k \leq M), \tag{4.26}$$

and so by (4.25),

(4.27) $$C^*(k) \geq k-(1-\alpha)k-\tau = \alpha k-\tau \qquad (0 \leq k \leq M).$$

To complete the proof, we first consider the case where $\tau \geq \frac{h-1}{h}$. We later consider the case where $\frac{r-1}{h} \leq \tau < \frac{r}{h}$ for some r where $1 \leq r \leq h-1$. (In each case we will arrive at a contradiction.)

<u>Case 1</u> $(\tau \geq \frac{h-1}{h})$. By (4.24) $\tau < 1-\alpha$ so

$$\frac{h-1}{h} \leq \tau < 1-\alpha,$$

and consequently by (4.27) C^* satisfies the hypotheses of (i). If we let S_h be defined by

$$S_h = A_1 + A_2 + \cdots + A_{h-1} + C^*,$$

then we have that

(4.28) $$S_h(M) \geq h\alpha M - h\tau + h - 1 .$$

Another application of Scherk's inversion formula (with $n = 0$, $i = h$, and $A_h = B$) together with Definition 3.2 gives us

$$B(M) \leq M - S_h(M).$$

Hence,

$$
\begin{aligned}
B(M) &\leq M-h+1-h\alpha M+h\tau && [(4.28)] \\
&= M-h+1-hM+hC(M)+h\mu && [(4.23)] \\
&= hC(M)-(h-1)M-h+1+h\mu \\
&< hC(M)-(h-1)M-h+1+1 && [\mu < \alpha < \tfrac{1}{h} \text{ by hypothesis}] \\
&= hC(M)-(h-1)(M+1)+1,
\end{aligned}
$$

and since $B(M)$ is an integer,

$$B(M) \leq hC(M) - (h-1)(M+1) \,. \tag{4.29}$$

However,

$$
\begin{aligned}
B(M) &= B(M+1) && [M+1 \notin B] \\
&\geq (1-h\alpha)(M+1)-h\mu && [\text{hypotheses of (ii)}] \\
&= (-h\alpha M-h\mu)+M+1-h\alpha \\
&= -h\tau-hM+hC(M)+M+1-h\alpha && [(4.23)] \\
&> -h(1-\alpha)-hM+hC(M)+M+1-h\alpha && [(4.24)] \\
&= hC(M)-(h-1)(M+1),
\end{aligned}
$$

which contradicts (4.29).

<u>Case 2</u> $(\frac{r-1}{h} \leq \tau < \frac{r}{h}$ for some r where $1 \leq r \leq h-1)$.

By (4.27)

$$C^*(k) \geq \alpha k - \tau \qquad (0 \leq k \leq M),$$

and by the above assumption $\tau < \frac{h-1}{h}$. Consequently

$$C^*(k) > \alpha k - \frac{h-1}{h} \qquad (0 \leq k \leq M).$$

Thus we may let $\lambda = \frac{h-1}{h}$ and $B = C^*$ in the hypotheses of (i). We apply (i) to the set S_h defined by $S_h = A_1 + A_2 + \cdots + A_{h-1} + C^*$ to obtain

$$S_h(k) \geq h\alpha k - h(\frac{h-1}{h}) + h - 1 = h\alpha k \qquad (0 \leq k \leq M), \tag{4.30}$$

where the case $k = 0$ follows trivially. Again applying Scherk's inversion formula (with $n = M-k$, $i = h$, and $A_h = B$) together with Definition 3.2 we have

$$B(M) - B(M-k) \leq k - S_h(k) \qquad (0 \leq k \leq M). \tag{4.31}$$

We also see that

$$B(M) = B(M+1) \qquad [M+1 \notin B]$$

$$\geq (1-h\alpha)(M+1)-h\mu \qquad [\text{hypotheses of (ii)}]$$

$$= M+1-h\alpha M-h\alpha-h\mu$$

$$= hC(M)-(h-1)M+1-h\tau-h\alpha \qquad [(4.23)]$$

$$> hC(M)-(h-1)M+1-r-h\alpha \qquad [\tau < \frac{r}{h}]$$

$$> hC(M)-(h-1)M-r, \qquad [\alpha < \frac{1}{h} \text{ by hypotheses}]$$

and since $B(M)$ is an integer,

(4.32) $$B(M) \geq hC(M) - (h-1)M-r+1 .$$

Therefore for $0 \leq k \leq M$,

$$B(M-k) \geq B(M)-k+S_h(k) \qquad [(4.31)]$$

$$\geq B(M)-k+h\alpha k \qquad [(4.30)]$$

$$\geq hC(M)-(h-1)M-r+1-k+h\alpha k \qquad [(4.32)]$$

$$= hC(M)-hM+M-r+1-(1-h\alpha)k$$

$$= h\tau-h\alpha M-h\mu+M+1-r-(1-h\alpha)k \qquad [(4.23)]$$

$$= (1-h\alpha)(M-k)-h(\mu-\tau+\frac{r-1}{h}) .$$

Consequently for $0 \leq n \leq M$ we have

$$(4.33) \qquad B(n) \geq (1-h\alpha)n - h(\mu - T + \frac{r-1}{h}) .$$

Since $\frac{r-1}{h} \leq T$ we have $\mu - T + \frac{r-1}{h} \leq \mu$. If we let $n = 0$ in (4.33) a simple calculation shows that $\mu - T + \frac{r-1}{h} \geq 0$. Combining these inequalities we have

$$(4.34) \qquad 0 \leq \mu - T + \frac{r-1}{h} \leq \mu < \alpha .$$

By (4.33) and (4.34) B satisfies the hypotheses of (ii), with μ replaced by $\mu - T + \frac{r-1}{h}$. Consequently

$$C(M-k) \geq (1-\alpha)(M-k) - (\mu - T + \frac{r-1}{h}) \qquad (0 \leq k \leq M),$$

where the case $k = M$ follows from (4.34). Substituting for T from (4.23) into this result we obtain

$$C(M-k) \geq C(M) - (1-\alpha)k - \frac{r-1}{h} \qquad (0 \leq k \leq M),$$

and so by Lemma 3.1 (with $n = M-k$) we have

$$C^*(k) = k - C(M) + C(M-k) \geq k - (1-\alpha)k - \frac{r-1}{h} = \alpha k - \frac{r-1}{h} \qquad (0 \leq k \leq M).$$

Thus C^* satisfies the requirements placed on the r-th set by condition (b) of the hypotheses. Again using the fact that by (4.33) and (4.34) B satisfies the hypotheses of (ii), with μ replaced by $\mu - T + \frac{r-1}{h}$, we have, letting S_r be defined by

$$S_r = A_1 + A_2 + \cdots + A_{r-1} + C^* + A_{r+1} + \cdots + A_{h-1} + B,$$

that

(4.35) $$S_r(k) \geq (1-\alpha)k - (\mu - \mathcal{T} + \frac{r-1}{h}) \qquad (0 \leq k \leq M),$$

where the case $k = 0$ follows from (4.34). Applying Scherk's inversion formula for the final time (with $n = 0$, $i = r$, and $A_h = B$) together with Definition 3.2 we have

$$\begin{aligned} A_r(M) &\leq M - S_r(M) \\ &\leq M-(1-\alpha)M-(\mu-\mathcal{T}+\frac{r-1}{h}) && [(4.35)] \\ &= \alpha M + \mu - \mathcal{T} + \frac{r-1}{h} \\ &= M - C(M) + \frac{r-1}{h} && [(4.23)] \\ &< M - C(M) + 1, && [r \leq h-1 \text{ assumption}] \end{aligned}$$

and since $A_r(M)$ is an integer,

(4.36) $$A_r(M) \leq M - C(M).$$

However,

$$
\begin{aligned}
A_r(M) &= A_r(M+1) && [\, M+1 \notin A_r] \\
&\geq \alpha(M+1) - \frac{r-1}{h} && [\text{condition (b)}] \\
&= \alpha M + \alpha - \frac{r-1}{h} \\
&= \tau + M - C(M) - \mu + \alpha - \frac{r-1}{h} && [(4.23)] \\
&= M - C(M) + \alpha - \left(\mu - \tau + \frac{r-1}{h}\right) \\
&> M - C(M) + \alpha - \alpha && [(4.34)] \\
&= M - C(M),
\end{aligned}
$$

which contradicts (4.36) and completes the induction on (ii). Thus the proof of Khinchin's First Theorem is complete.

CHAPTER V

THE PROOF OF LANDAU'S THEOREM

In this chapter we prove Landau's Theorem, which is the special case $h = 2$ of Khinchin's First Theorem. We give Landau's proof [18, p. 73-76]. To prove Landau's Theorem one could simply make the specialization $h = 2$ in the proof given in Chapter IV. However, Landau's proof involves several simplifications.

§ 5.1 Landau's Theorem

Let C be the sum of the sets A and B of nonnegative integers. Let

$$\text{(a)} \quad 0 < \alpha < \frac{1}{2},$$

and

$$\text{(b)} \quad A(n) \geq \alpha n \qquad (1 \leq n \leq N),$$

where N is an arbitrary positive integer. Then the following statements hold:

(i) If $\frac{1}{2} \leq \lambda \leq 1-\alpha$ and if $B(n) \geq \alpha n-\lambda$ for $1 \leq n \leq N$, then $C(N) \geq 2\alpha N-2\lambda+1$.

(ii) If $0 \leq \mu < \alpha$ and if $B(n) \geq (1-2\alpha)n-2\mu$ for $1 \leq n \leq N$, then $C(N) \geq (1-\alpha)N-\mu$.

§ 5.2 Introduction to the Proof

Landau's theorem will be proved by mathematical induction on the integer N. We first verify the theorem for the case $N = 1$. We then assume that the theorem is valid for all N where $1 \leq N \leq M$, M a fixed positive integer. The induction is completed by arriving at a contradiction after assuming that the theorem fails for $N = M+1$. The induction will be applied separately to (i) and (ii). We use the induction hypotheses of (i) and (ii) to prove that (i) follows for $N = M+1$, after which we use (i) for $N = M+1$ together with the induction hypotheses of (i) and (ii) to prove that (ii) follows for $N = M+1$. This completes the induction.

Throughout the second step of the induction we need the following lemma, which is merely the specialization $h = 2$ of Lemma 4.1. So as to make Chapter V self-contained we will take the time to prove this lemma here.

Lemma 5.1. Suppose that Landau's Theorem is true for $1 \leq N \leq M$. If $M+1 \in C$, then the theorem holds for $N = M+1$.

Proof. Since $M+1 \in C$ and $\alpha < \frac{1}{2}$ we have in the first case that

$$C(M+1) = 1+C(M) \geq 1+2\alpha M-2\lambda+1 > 2\alpha+2\alpha M-2\lambda+1 = 2\alpha(M+1)-2\lambda+1 ,$$

and so (i) is true for $N = M+1$. In the second case, since $\alpha > 0$ and $M+1 \in C$ we have that

$$C(M+1) = 1+C(M) \geq 1+(1-\alpha)M-\mu > 1-\alpha+(1-\alpha)M-\mu = (1-\alpha)(M+1)-\mu ,$$

and so (ii) is true for $N = M+1$. Thus the lemma is proved.

§ 5.3 The Proof for $N = 1$

We notice that $A(1) \geq \alpha > 0$ by the hypotheses, and so $A(1) = 1$. Consequently $1 \in A$. Hence $1 \in C$, and so $C(1) = 1$. If $\lambda \geq \frac{1}{2}$ and $N = 1$, then

$$2\alpha N-2\lambda+1 = 2\alpha-2\lambda+1 \leq 2\alpha < 1 = C(1) ,$$

which proves (i) for $N = 1$. If $0 \leq \mu < \alpha$ and $N = 1$, then

$$(1-\alpha)N-\mu = 1-\alpha-\mu < 1 = C(1),$$

which proves (ii) for $N = 1$.

§ 5.4 The Induction - Part I

We assume both parts (i) and (ii) of the theorem are valid for all N where $1 \leq N \leq M$. We assume the hypotheses hold but the conclusion fails for part (i) of the theorem when $N = M+1$, and obtain a contradiction. In this way we show that part (i) follows for $N = M+1$. By Lemma 5.1 we may suppose that $M+1 \notin C$. Hence

$$2\alpha M-2\lambda+1 \leq C(M) = C(M+1) < 2\alpha(M+1)-2\lambda+1 . \tag{5.1}$$

We define the number ϕ by

$$\phi = \frac{C(M)}{2} - \frac{1}{2} + \lambda - \alpha M, \tag{5.2}$$

where it follows by using (5.1) in (5.2) that

$$0 \leq \phi < \alpha . \tag{5.3}$$

The induction hypothesis of (i) states that

$$C(M-k) \geq 2\alpha(M-k) - 2\lambda + 1,$$

for all integers k where $0 \leq k < M$. This inequality is also true for $k = M$ by a simple calculation, since $\lambda \geq \frac{1}{2}$. We solve for $2\alpha M$ in (5.2) and substitute this expression for $2\alpha M$ in the last inequality to obtain

$$C(M-k) \geq C(M) - 2\phi - 2\alpha k \qquad (0 \leq k \leq M),$$

which may be rewritten as

$$C(M-k) - C(M) \geq -2\phi - 2\alpha k \qquad (0 \leq k \leq M) . \tag{5.4}$$

Referring to Lemma 3.1 (with $n = M-k$) we have

$$C^*(k) = k - C(M) + C(M-k) \qquad (0 \leq k \leq M), \tag{5.5}$$

and so by (5.4)

$$C^*(k) \geq k-2\phi-2\alpha k = (1-2\alpha)k-2\phi \qquad (0 \leq k \leq M).$$

This result together with (5.3) shows that C^* satisfies the hypotheses of (ii) (with μ replaced by ϕ) and thus it enables us to apply the induction hypothesis of (ii) to the set T defined by $T = A+C^*$. Consequently

(5.6) $$T(M-n) \geq (1-\alpha)(M-n)-\phi \qquad (0 \leq n \leq M),$$

where the case $n = M$ follows from (5.3). From (5.6) and the Landau inversion formula (Theorem 3.2)

$$B(M)-B(n) \leq M-n-T(M-n) \leq M-n-(1-\alpha)(M-n) + \phi$$

(5.7) $$= \alpha(M-n) + \phi \qquad (0 \leq n \leq M).$$

We next see that

$$B(M) = B(M+1) \qquad [M+1 \notin B]$$

$$\geq \alpha(M+1)-\lambda \qquad [\text{hypotheses of (i)}]$$

$$= \alpha M-\lambda + \alpha$$

$$= \frac{C(M)}{2} - \frac{1}{2} -\phi + \alpha \qquad [(5.2)]$$

$$> \frac{C(M)}{2} - \frac{1}{2}, \qquad [(5.3)]$$

and so $2B(M) > C(M)-1$. Hence, since $B(M)$ is an integer,

$2B(M) \geq C(M)$, and so

$$B(M) \geq \frac{C(M)}{2} = \frac{1}{2} + \phi - \lambda + \alpha M \qquad [(5.2)] \tag{5.8}$$

$$> \phi + \alpha M - \frac{1}{2}. \qquad [\lambda < 1 \text{ by hypothesis}]$$

The application of this last inequality to (5.7) yields

$$B(n) \geq B(M) - \alpha(M-n) - \phi > \phi + \alpha M - \frac{1}{2} - \alpha M + \alpha n - \phi$$

$$= \alpha n - \frac{1}{2} \qquad (0 \leq n \leq M). \tag{5.9}$$

Thus if we let $\lambda = \frac{1}{2}$ in the hypotheses of (i), we have

$$C(M-k) \geq 2\alpha(M-k) - 2(\tfrac{1}{2}) + 1 = 2\alpha(M-k) \qquad (0 \leq k \leq M),$$

where the case $k = M$ follows trivially. This result and (5.2) are then used in (5.5) to obtain

$$C^*(k) = k - C(M) + C(M-k) \geq k - (1 - 2\lambda + 2\alpha M + 2\phi) + 2\alpha(M-k)$$

$$= (1-2\alpha)k - 2(\tfrac{1}{2} - \lambda + \phi) \qquad (0 \leq k \leq M). \tag{5.10}$$

We define the number ρ by

$$\rho = \frac{1}{2} - \lambda + \phi. \tag{5.11}$$

When (5.11) is used in (5.10) we have

$$(5.12) \qquad C^*(k) \geq (1-2\alpha)k-2\rho \qquad (0 \leq k \leq M).$$

We have $\rho \geq 0$ as may be seen by letting $k = 0$ in (5.12). Furthermore, since $\lambda \geq \frac{1}{2}$ and $\phi < \alpha$, it follows that $\rho < \alpha$. Consequently

$$(5.13) \qquad 0 \leq \rho < \alpha.$$

By (5.12) and (5.13) C^* satisfies the hypotheses of (ii) (with μ replaced by ρ), and so again letting $T = A+C^*$ we have

$$(5.14) \qquad T(M-n) \geq (1-\alpha)(M-n)-\rho \qquad (0 \leq n \leq M),$$

when the case $n = M$ follows from (5.13). From the Landau inversion formula,

$$B(M)-B(n) \leq M-n-T(M-n) \qquad (0 \leq n \leq M),$$

and hence

$$
\begin{aligned}
B(n) &\geq B(M)-M+n+T(M-n) \\
&\geq B(M)-M+n+(1-\alpha)(M-n)-\rho && [(5.14)] \\
&= B(M)-\alpha(M-n)-\rho \\
&\geq \frac{1}{2}+\phi+\alpha M-\lambda-\alpha M+\alpha n-\rho && [(5.8)] \\
&= \frac{1}{2}+\phi-\lambda+\alpha n-\rho \\
&= \frac{1}{2}+\phi-\lambda+\alpha n-\frac{1}{2}+\lambda-\phi && [(5.11)] \\
&= \alpha n \qquad (0 \leq n \leq M).
\end{aligned}
$$

Thus B satisfies condition (b) of the hypotheses and since C^* satisfies the hypotheses of (ii), with μ replaced by ρ, by (5.12) and (5.13), we have, letting $S = B+C^*$,

(5.15) $$S(M) \geq (1-\alpha)M-\rho.$$

This result together with the Landau inversion formula (with $n = 0$, $T = S$, and A and B interchanged) gives

(5.16) $$A(M) \leq M-S(M) \leq M-(1-\alpha)M+\rho = \alpha M+\rho.$$

However, $M+1 \notin A$ and so by hypotheses

$$A(M) = A(M+1) \geq \alpha(M+1) = \alpha M+\alpha.$$

This inequality and (5.16) gives

$$\alpha M+\alpha \leq A(M) \leq \alpha M+\rho,$$

from which it follows that $\alpha \leq \rho$, contradicting (5.13). This completes the induction on (i).

§ 5.5 The Induction - Part II

We now use the validity of (i) for all N where $1 \leq N \leq M+1$ and assume that (ii) is valid for all N where $1 \leq N \leq M$. We assume the hypotheses hold but the conclusion fails for part (ii) of the theorem when $N = M+1$, and obtain a contradiction. In this way we show that part (ii) follows for $N = M+1$. By Lemma 5.1 we may suppose that $M+1 \notin C$. Hence

$$(1-\alpha)M-\mu \leq C(M) = C(M+1) < (1-\alpha)(M+1)-\mu . \tag{5.17}$$

We define the number τ by

$$\tau = \alpha M-M+C(M)+\mu, \tag{5.18}$$

where it follows by using (5.17) in (5.18) that

$$0 \leq \tau < 1-\alpha < 1 . \tag{5.19}$$

The induction hypothesis of (ii) states that

$$C(M-k) \geq (1-\alpha)(M-k)-\mu ,$$

where $0 \leq k < M$. This inequality is also true for $k = M$ by a simple calculation since $\mu \geq 0$. We solve for $M-\alpha M$ in (5.18) and substitute the expression for $M-\alpha M$ in the last inequality to obtain

$$C(M-k) \geq C(M) - (1-\alpha)k - \mathcal{T} \qquad (0 \leq k \leq M),$$

which may be rewritten as

$$(5.20) \qquad C(M-k) - C(M) \geq -(1-\alpha)k - \mathcal{T} \qquad (0 \leq k \leq M).$$

Referring to Lemma 3.1 (with $n = M-k$) we have

$$C^*(k) = k - C(M) + C(M-k) \qquad (0 \leq k \leq M),$$

and so by (5.20) ~~and (5.19)~~,

$$(5.21) \qquad C^*(k) \geq k - (1-\alpha)k - \mathcal{T} = \alpha k - \mathcal{T} \qquad (0 \leq k \leq M),$$

whence, since $\mathcal{T} < 1-\alpha$ by (5.19),

$$(5.22) \qquad C^*(k) > \alpha k - (1-\alpha) \qquad (0 \leq k \leq M).$$

To complete the proof, we first restrict our attention to the case $\frac{1}{2} \leq \mathcal{T} < 1-\alpha$. We later consider the case where $0 \leq \mathcal{T} < \frac{1}{2}$.

<u>Case 1</u> ($\frac{1}{2} \leq \mathcal{T} < 1-\alpha$). If we let $\lambda = 1-\alpha$, we see by (5.22) that C^* satisfies the hypotheses of (i). We may apply (i) to the

set $T = A+C^*$ to get

$$T(M) \geq 2\alpha M - 2(1-\alpha)+1 = 2\alpha M + 2\alpha - 1 .$$

Using this and the Landau inversion formula (with $n = 0$) we have

$$\text{(5.23)} \qquad B(M) \leq M - T(M) \leq M - 2\alpha M - 2\alpha + 1 .$$

Furthermore, by (5.19), an easy calculation shows that

$$\text{(5.24)} \qquad -1 < 1 - 2\alpha - 2\tau .$$

We then have that

$$\begin{aligned}
2C(M)-M-1 &= 2\tau - 2\alpha M + 2M - 2\mu - M - 1 && [(5.18)] \\
&< 2\tau - 2\alpha M + M - 2\mu + 1 - 2\alpha - 2\tau && [(5.24)] \\
&= (1-2\alpha)(M+1) - 2\mu \\
&\leq B(M+1) && [\text{hypotheses of (ii)}] \\
&= B(M) && [M+1 \notin B] \\
&\leq M - 2\alpha M - 2\alpha + 1 && [(5.23)] \\
&= M + [2C(M) - 2\tau - 2M + 2\mu] - 2\alpha + 1 && [(5.18)] \\
&= 2C(M) - M + 1 + (2\mu - 2\alpha) - 2\tau \\
&< 2C(M) - M + 1 - 1 && [\mu < \alpha ,\ \tau \geq \tfrac{1}{2}] \\
&= 2C(M) - M,
\end{aligned}$$

which implies that

$$2C(M)-M-1 < B(M) < 2C(M)-M .$$

Hence $-1 < B(M)-2C(M)+M < 0$, which is a contradiction since $B(M)-2C(M)+M$ is an integer. Thus the first contradiction is obtained.

Case 2 $(0 \leq \tau < \frac{1}{2})$. Since $\tau < \frac{1}{2}$ then by (5.21) we have

$$C^*(k) > \alpha K - \frac{1}{2} \qquad (0 \leq k \leq M).$$

Thus we may let $\lambda = \frac{1}{2}$ and $B = C^*$ in the hypotheses of (i). We apply (i) to the set $T = A+C^*$ to obtain

$$T(M-n) \geq 2\alpha(M-n)-2(\tfrac{1}{2}) + 1 = 2\alpha(M-n) \qquad (0 \leq n \leq M),$$

where the case $n = M$ follows trivially. Thus another application of the Landau inversion formula gives us

(5.25) $\quad B(M)-B(n) \leq M-n-T(M-n) \leq (1-2\alpha)(M-n) \qquad (0 \leq n \leq M).$

We also have that

$$
\begin{aligned}
B(M) &= B(M+1) && [M+1 \notin B] \\
&\geq (1-2\alpha)(M+1)-2\mu && [\text{hypotheses of (ii)}] \\
&= M+1-2\alpha-2\alpha M-2\mu \\
&= M+1-2\alpha-2\tau-2M+2C(M) && [(5.18)] \\
&= 1-2\alpha-2\tau-M+2C(M) \\
&> 1-2\alpha-2(1-\alpha)-M+2C(M) && [(5.19)] \\
&= 2C(M)-M-1,
\end{aligned}
$$

and since $B(M)$ is an integer,

$$B(M) \geq 2C(M)-M .$$

Using this result and (5.18) in (5.25) we have

$$B(n) \geq B(M)-(1-2\alpha)(M-n) \geq 2C(M)-M-(1-2\alpha)(M-n)$$

(5.26) $$= (1-2\alpha)n-2(\mu-\tau) \qquad (0 \leq n \leq M),$$

and in particular when $n = 0$ a simple calculation shows that $\mu - \tau \geq 0$. Furthermore, since $\tau \geq 0$ by (5.19) and $\mu < \alpha$ by hypotheses, we have

(5.27) $$0 \leq \mu-\tau < \alpha .$$

Thus by (5.26) and (5.27) B satisfies the hypotheses of (ii), with

μ replaced by $\mu - T$. Consequently (ii) may be applied to the set $C = A+B$ to obtain

$$C(M-k) \geq (1-\alpha)(M-k)-(\mu-T) \qquad (0 \leq k \leq M),$$

where the case $k = M$ follows from (5.27). This inequality together with (5.18) and Lemma 3.1 (with $n = M-k$) yields

$$C^*(k) = k-C(M)+C(M-k) \geq k+(\alpha M-M+\mu-T)+(1-\alpha)(M-k)-(\mu-T)$$

$$= \alpha k \qquad (0 \leq k \leq M), \tag{5.28}$$

and thus C^* satisfies condition (b) of the hypotheses. Since B satisfies the hypotheses of (ii) by (5.26) and (5.27), with $\mu - T$ in place of μ, then (ii) may be applied to the set $S = C^* + B$ to obtain

$$S(M) \geq (1-\alpha)M - (\mu-T). \tag{5.29}$$

Once more applying the Landau inversion formula (with $n = 0$, A and B interchanged, and $T = S$) we obtain

$$A(M) \leq M-S(M).$$

Consequently,

$$
\begin{aligned}
A(M) &\leq M-(1-a)M + (\mu - T) && [(5.29)] \\
&= aM + \mu - T \\
&\cancel{= M - G(M)} && \cancel{[(5.18)]} \\
&\cancel{< M - C(M) - \mu + T + a} && \cancel{[(5.27)]} \\
&< aM + a && \cancel{[(5.18)]}\ [(5.27)] \\
&= a(M+1) \\
&\leq A(M+1) && [\text{condition (b)}] \\
&= A(M), && [M+1 \notin A]
\end{aligned}
$$

which is the final contradiction needed. This completes the induction on (ii). Thus the proof of Landau's Theorem is complete.

CHAPTER VI

THE PROOF OF SCHERK'S THEOREM

In this chapter the proof of Scherk's Theorem if given. Three lemmas are established prior to the proof.

§ 6.1 Scherk's Theorem

We previously stated Scherk's Theorem in § 2.1. It is repeated here.

Scherk's Theorem. For $h \geq 2$ let C be the sum of the sets $A_1, A_2, \cdots, A_{h-1}, B$ of nonnegative integers. Let the numbers α, β, and δ, and the function f_β be defined by the following conditions:

(a) $0 < \alpha < \frac{1}{h}$ (c) $0 \leq \delta < 1-\beta$

(b) $\beta = \alpha$ or $\beta = 1-h\alpha$ (d) $f_{1-h\alpha}(\delta) = \frac{\delta}{h}$

$$\text{(e)} \quad f_\alpha(\delta) = \max\{0, h\delta-h+1\}.$$

Let N be an arbitrary positive integer. For each i, where $i = 1, 2, \cdots, h-1$, let

(f) $A_i(n) \geq \alpha n - \frac{i-1}{h}$ $(1 \leq n \leq N)$.

Furthermore, let

$$(g) \quad B(n) \geq \beta n - \delta \qquad (1 \leq n \leq N).$$

Then, $\quad C(N) \geq [(h-1)\alpha + \beta] N - f_\beta(\delta).$

§ 6.2 Three Preliminary Lemmas

Consider $f_\beta(\delta)$, where $0 \leq \delta < 1-\beta$.

Lemma 6.1. The function f_β is monotonically increasing on $[0, 1-\beta)$.

Proof. We consider two cases depending upon the value of β.

Case 1 $(\beta = 1-h\alpha)$. By condition (d), $f_{1-h\alpha}(\delta) = \frac{\delta}{h}$, and since the derivative $f'_{1-h\alpha}(\delta) = \frac{1}{h} > 0$, it follows that $f_{1-h\alpha}$ is strictly increasing for all $\delta \epsilon [0, 1-\beta)$.

Case 2 $(\beta = \alpha)$. By condition (e), $f_\alpha(\delta) = \max\{0, h\delta - h + 1\}$. For $\delta \epsilon [\frac{h-1}{h}, 1-\alpha)$ a simple calculation shows that $h\delta - h + 1 \geq 0$, and consequently $f_\alpha(\delta) = h\delta - h + 1$. Hence f_α is strictly increasing on $[\frac{h-1}{h}, 1-\alpha)$ since the derivative $f'_\alpha(\delta) = h \geq 2 > 0$. Furthermore, for $\delta \epsilon [0, \frac{h-1}{h})$ we have $f_\alpha(\delta) = 0$. Thus f_α is monotonically increasing for all $\delta \epsilon [0, 1-\alpha)$.

It will often be useful to consider the number β' defined by

$$\beta' = 1-(h-1)\alpha - \beta . \tag{6.1}$$

Lemma 6.2.

$$0 < \beta' = \begin{cases} 1-h\alpha , & \text{if} \quad \beta = \alpha \\ \alpha , & \text{if} \quad \beta = 1-h\alpha . \end{cases}$$

Proof. The equality follows immediately by substitution into (6.1). Then $\beta' > 0$ since $0 < \alpha < \frac{1}{h}$ by condition (a).

Lemma 6.3. $0 \leq f_\beta(\delta) < \beta'$ (also $0 \leq f_{\beta'}(\delta) < \beta$).

Proof. Since f_β is monotonically increasing on $[0, 1-\beta)$ and since $f_\beta(0) = 0$, we see that $f_\beta(\delta) \geq 0$. To show that $f_\beta(\delta) < \beta'$ we again consider two cases.

Case 1 ($\beta = 1-h\alpha$). On $[0, 1-\beta) = [0, h\alpha)$ we see that

$$f_{1-h\alpha}(\delta) = \frac{\delta}{h} < \frac{h\alpha}{h} = \alpha = \beta' ,$$

by Lemma 6.2.

Case 2 ($\beta = \alpha$). On $[\frac{h-1}{h}, 1-\beta) = [\frac{h-1}{h}, 1-\alpha)$ we see that

$$f_\alpha(\delta) = h\delta - h + 1 < h(1-\alpha) - h + 1 = 1 - h\alpha = \beta' .$$

Furthermore $f_\alpha(\delta) = 0 < 1-h\alpha = \beta'$ for $\delta \in [0, \frac{h-1}{h})$. Consequently $f_\alpha(\delta) < \beta'$ for $\delta \in [0, 1-\alpha)$.

The inequality $0 \leq f_{\beta'}(\delta) < \beta$ follows immediately from $0 \leq f_\beta(\delta) < \alpha'$ and Lemma 6.2.

§ 6.3 Introduction to the Proof

Scherk's Theorem is also proved by mathematical induction on the integer N. We first verify the theorem for the case $N = 1$. We then assume that the theorem holds for all N where $1 \leq N \leq M$, M a fixed positive integer, and prove the theorem for $N = M+1$, by assuming that it is false for $N = M+1$ and arriving at a contradiction. This completes the induction.

Throughout the second step of the induction we need the following lemma, analogous to Lemma 4.1.

Lemma 6.4. Suppose that Scherk's Theorem is true for $1 \leq N \leq M$. If $M+1 \in C$, then the theorem holds for $N = M+1$.

Proof. If $M+1 \in C$ we have when $\beta = \alpha$ that

$$C(M+1) = 1+C(M) \geq 1+[(h-1)\alpha+\beta]M-f_{\alpha}(\delta) = 1+h\alpha M-f_{\alpha}(\delta)$$

$$> h\alpha+h\alpha M-f_{\alpha}(\delta) = h\alpha(M+1)-f_{\alpha}(\delta),$$

since $\alpha < \frac{1}{h}$, and so Scherk's Theorem is true for $N = M+1$.

When $\beta = 1-h\alpha$,

$$C(M+1) = 1+C(M) \geq 1+[(h-1)\alpha+\beta]M-f_{1-h\alpha}(\delta) = 1+(1-\alpha)M-f_{1-h\alpha}(\delta)$$

$$> (1-\alpha)+(1-\alpha)M-f_{1-h\alpha}(\delta) = (1-\alpha)(M+1)-f_{1-h\alpha}(\delta),$$

since $\alpha > 0$, and so Scherk's Theorem again holds for $N = M+1$. Thus the lemma is proved.

§ 6.4 The Proof for $N = 1$

Using condition (a) of the hypotheses and Lemma 6.3 we have when $N = 1$ and $\beta = 1-h\alpha$ that

$$(6.2) \qquad [(h-1)\alpha+\beta] N - f_\beta(\delta) \le 1-\alpha < 1,$$

while if $N = 1$ and $\beta = \alpha$,

$$(6.3) \qquad [(h-1)\alpha+\beta] N - f_\beta(\delta) \le h\alpha < 1.$$

We notice that for each i, where $i = 1, 2, \cdots, h-1$, $A_i(1) \ge \alpha - \frac{i-1}{h}$ by hypotheses. Thus in particular $A_1(1) \ge \alpha > 0$ and so $A_1(1) = 1$. Consequently $1 \in A_1$. Hence $1 \in C$, and so $C(1) = 1$. This together with (6.2) and (6.3) gives us when $N = 1$ that

$$[(h-1)\alpha + \beta] N - f_\beta(\delta) < 1 = C(1),$$

which proves the theorem for $N = 1$.

§ 6.5 The Beginning of the Induction

We assume that the theorem is valid for all N where $1 \le N \le M$. We assume the hypotheses hold but the conclusion fails when $N = M+1$, and obtain a contradiction. In this way we show that the theorem follows for $N = M+1$. By Lemma 6.4 we may assume that $M+1 \notin C$. Hence

$$[(h-1)\alpha+\beta]\, M - f_\beta(\delta) \leq C(M) = C(M+1) < [(h-1)\alpha+\beta]\,(M+1) - f_\beta(\delta),$$

which may be rewritten as

$$(6.4) \qquad (1-\beta')M - f_\beta(\delta) \leq C(M) = C(M+1) < (1-\beta')(M+1) - f_\beta(\delta),$$

since $1-\beta' = (h-1)\alpha+\beta$ by (6.1).

Consider the number $\max\{\beta n - B(n) \colon 0 \leq n \leq M+1\}$. If $n = 0$, $\beta n - B(n) = 0$. Furthermore, by the hypotheses, $\beta n - B(n) \leq \delta < 1-\beta$ for all n such that $1 \leq n \leq M+1$. Consequently

$$(6.5) \qquad 0 \leq \max\{\beta n - B(n) : 0 \leq n \leq M+1\} < 1-\beta,$$

and hence $\max\{\beta n - B(n) : 0 \leq n \leq M+1\}$ satisfies condition (c) of the hypotheses. We therefore let

$$(6.6) \qquad \delta = \max\{\beta n - B(n) : 0 \leq n \leq M+1\}.$$

Using Lemma 3.1 (with $n = M-k$) we have

$$(6.7) \qquad C^*(k) = k - C(M) + C(M-k) \qquad (0 \leq k \leq M).$$

By the induction hypothesis and (6.1),

$$C(M-k) \geq [(h-1)\alpha+\beta]\,(M-k)-f_\beta(\delta) = (1-\beta')(M-k)-f_\beta(\delta) \quad (0 \leq k \leq M),$$

where the case $k = M$ holds since $f_\beta(\delta) \geq 0$ by Lemma 6.3. Using this in (6.7) we have

$$C*(k) \geq k-C(M)+(1-\beta')(M-k)-f_\beta(\delta)$$

(6.8) $$= (1-\beta')M-C(M)+\beta' k-f_\beta(\delta) \qquad (0 \leq k \leq M).$$

Next, consider inequality (6.4). Since

(6.9) $$(1-\beta')M-f_\beta(\delta) \leq C(M) < (1-\beta')(M+1)-f_\beta(\delta),$$

there exists some number ξ such that

(6.10) $$C(M) = (1-\beta')M-f_\beta(\delta) + \xi,$$

where ξ satisfies the inequality

(6.11) $$0 \leq \xi < 1-\beta' ,$$

since by (6.9)

$$0 \leq C(M) - (1-\beta')M+f_\beta(\delta) < 1-\beta' .$$

Substitution from (6.10) into the right-hand member of (6.8) gives

(6.12) $$C*(k) \geq \beta' k-\xi \qquad (0 \leq k \leq M).$$

By Lemma 6.2, β' is either α or $1-h\alpha$ and so has the same domain of values as β. Hence we may replace β by β' in Scherk's Theorem. Therefore by (6.11) ξ satisfies condition (c) of the hypotheses, and so by (6.12) C^* satisfies condition (g). Let S_h be the set defined by

$$S_h = A_1 + A_2 + \cdots + A_{h-1} + C^*.$$

Applying the induction hypothesis we therefore have for $0 \leq n \leq M$ that

$$S_h(M-n) \geq (1-\beta)(M-n)-f_{\beta'}(\xi), \tag{6.13}$$

where the case $n = M$ holds by Lemma 6.3. Using Scherk's inversion formula (with $i = h$ and $A_h = B$) we have

$$B(M)-B(n) \leq M-n-S_h(M-n) \qquad (0 \leq n \leq M).$$

Use of this result together with (6.13) gives

$$B(n) \geq B(M)-M+n+S_h(M-n) \geq B(M)-M+n+(1-\beta)(M-n)-f_{\beta'}(\xi)$$

$$= (B(M)-\beta M)+\beta n-f_{\beta'}(\xi) \qquad (0 \leq n \leq M). \tag{6.14}$$

Referring to (6.6), let n_1 be one of the integers $0, 1, \cdots, M+1$ such that

(6.15) $$B(n_1) = \beta n_1 - \delta .$$

If n_1 is any of the integers $0, 1, \cdots, M$, then by (6.14) and (6.15),

(6.16) $$\beta n_1 - \delta = B(n_1) \geq (B(M) - \beta M) + \beta n_1 - f_{\beta'}(\xi) .$$

Consequently,

$$\begin{aligned}
B(M) &\leq \beta M + f_{\beta'}(\xi) - \delta && [(6.16)] \\
&< \beta M + \beta - \delta && [\text{Lemma } 6.3] \\
&= \beta(M+1) - \delta \\
&\leq [B(M+1) + \delta] - \delta && [(6.6)] \\
&= B(M+1) \\
&= B(M), && [M+1 \notin B]
\end{aligned}$$

a contradiction. Hence $n_1 = M+1$, and so $B(M+1) = \beta(M+1) - \delta$, or

(6.17) $$B(M) = \beta(M+1) - \delta ,$$

since $M+1 \notin B$. Substituting (6.17) into (6.14) we obtain

(6.18) $$B(n) \geq \beta(M+1) - \delta - \beta M + \beta n - f_{\beta'}(\xi) = \beta(N+1) - \delta - f_{\beta'}(\xi) \quad (0 \leq n \leq M).$$

§ 6.6 Conclusion of the Induction - Case 1

We first consider the case $\beta = \alpha$. We have

$$C(M) = (1-\beta')M - f_{\alpha}(\delta) + \xi \qquad [(6.10)]$$

$$= h\alpha M - f_{\alpha}(\delta) + \xi \qquad [\text{Lemma } 6.2]$$

$$= h[B(M)-\alpha+\delta] - f_{\alpha}(\delta) + \xi \qquad [(6.17)]$$

$$(6.19) \qquad = hB(M) + h\delta - f_{\alpha}(\delta) - (h\alpha - \xi).$$

Furthermore, by (6.11) and Lemma 6.2 we see that $0 \leq \xi < 1-\beta' = h\alpha$, and by condition (a) of the hypotheses $0 < h\alpha < 1$. Therefore

$$(6.20) \qquad 0 < h\alpha - \xi < 1.$$

We next claim that $\delta < 1 - \frac{1}{h}$, for if not then $\delta \geq 1 - \frac{1}{h}$ and so $h\delta - h + 1 \geq 0$ which in turn implies that $f_{\alpha}(\delta) = h\delta - h + 1$. This result substituted into (6.19) yields

$$C(M) - hB(M) - h + 1 = -(h\alpha - \xi),$$

which gives a contradiction since the left-hand side is an integer while by (6.20) the right-hand side is some real number in the interval $(-1, 0)$. Consequently

$$(6.21) \qquad \delta < 1 - \frac{1}{h},$$

and another simple calculation gives us that $f_a(\delta) = 0$. Substitution of this into (6.19) gives us

$$C(M) = hB(M)+h\delta - (h\alpha-\xi),$$

which implies that $h\delta - (h\alpha-\xi)$ is an integer. Furthermore

$$\begin{aligned} 0 &< 1 - (h\alpha-\xi) && [(6.20)] \\ &\leq 1+h\delta - (h\alpha-\xi) && [\text{condition } (c)] \\ &< 1+h\delta && [(6.20)] \\ &< h, && [(6.21)] \end{aligned}$$

and consequently, since $1+h\delta-(h\alpha-\xi)$ is an integer,

(6.22) $$1 \leq 1+h\delta - (h\alpha-\xi) \leq h-1 .$$

We define the integer r by

(6.23) $$r = 1+h\delta - (h\alpha-\xi),$$

and note that by (6.22) we have

(6.24) $$1 \leq r \leq h-1.$$

By (6.23) a simple calculation shows that

(6.25) $$\frac{r-1}{h} = \delta - (\alpha - \frac{\xi}{h}) .$$

Consequently, for $0 \leq n \leq M$, we have

$$\begin{aligned} B(n) &\geq \alpha(n+1)-\delta-f_{\beta'}(\xi) && [(6.18)] \\ &= \alpha(n+1)-\delta-f_{1-h\alpha}(\xi) && [\text{Lemma } 6.2] \\ &= \alpha(n+1)-\delta-\frac{\xi}{h} && [\text{condition (d)}] \\ &= \alpha n-[\delta-(\alpha-\frac{\xi}{h})] \\ &= \alpha n-\frac{r-1}{h}, && [(6.25)] \end{aligned}$$

and so B satisfies the requirements placed on the r-th set by condition (f) of the hypotheses. We also have by (6.12) that C^* satisfies condition (g) of the hypotheses, where, as at that stage of the argument, we have replaced β and δ by β' and ξ respectively. Again these replacements are justified by Lemma 6.2 and (6.10) respectively. Thus we may apply the induction hypothesis to the set S_r defined by

$$S_r = A_1 + A_2 + \cdots + A_{r-1} + B + A_{r+1} + \cdots + A_{h-1} + C^*$$

and obtain that

$$(6.26) \qquad S_r(M-n) \geq (1-\alpha)(M-n) - \frac{\xi}{h} \qquad (0 \leq n \leq M),$$

where the case $n = M$ holds by (6.11).

We may interchange the positions of B and C^* in the above sum for S_r to obtain

$$S_r = A_1 + A_2 + \cdots + A_{r-1} + C^* + A_{r+1} + \cdots + A_{h-1} + B .$$

Hence, because of Definition 3.2 (with $A_h = B$) we have by Scherk's inversion formula (with $n = 0$ and $i = r$) that

$$A_r(M) \leq M - S_r(M) .$$

Consequently,

$$
\begin{aligned}
A_r(M) &\leq M - (1-\alpha)M + \frac{\xi}{h} && [(6.26),\ n=0] \\
&= \alpha M + \frac{\xi}{h} \\
&= B(M) - \alpha + \delta + \frac{\xi}{h} && [(6.17) \text{ with } \beta = \alpha] \\
&= B(M) + \frac{r-1}{h} && [(6.25)] \\
(6.27) \qquad &< B(M) + 1. && [(6.24)]
\end{aligned}
$$

However, we also have that

$$
\begin{aligned}
A_r(M) &= A_r(M+1) && [M+1 \notin A_r] \\
&\geq \alpha(M+1) - \frac{r-1}{h} && [\text{condition (f)}] \\
&= B(M) + \delta - \frac{r-1}{h} && [(6.17) \text{ with } \beta = \alpha] \\
&= B(M) + \delta - (\delta - \frac{h\alpha - \xi}{h}) && [(6.25)] \\
&= B(M) + \frac{h\alpha - \xi}{h} \\
&> B(M), && [(6.20)]
\end{aligned}
$$

which together with (6.27) implies that $B(M) < A_r(M) < B(M) + 1$, or

$$0 < A_r(M) - B(M) < 1,$$

which is a contradiction since $A_r(M)-B(M)$ is an integer. Thus we have obtained the desired contradiction and the theorem is proved for $\beta = \alpha$.

§ 6.7 Conclusion of the Induction - Case 2

We next consider the case $\beta = 1-h\alpha$. It follows from (6.11) and Lemma 6.2 that $0 \leq \xi < 1-\beta' = 1-\alpha$. Furthermore, since $\alpha > 0$, we have

$$(6.28) \qquad 0 < \alpha + \xi < 1.$$

Now from (6.17) we have since $\beta = 1-h\alpha$ that

$$(6.29) \qquad B(M) = (1-h\alpha)(M+1) - \delta.$$

Furthermore, from (6.10) and Lemma 6.2 we have when $\beta = 1-h\alpha$ that

$$(6.30) \qquad C(M) = (1-\alpha)M - \frac{\delta}{h} + \xi .$$

Multiplying (6.30) through by h, solving for $h\alpha M$, and substituting this expression for $h\alpha M$ into (6.29) we obtain

(6.31) $$B(M) = M(1-h) + hC(M) + 1 - h(\alpha+\xi).$$

We define the number r by

(6.32) $$r = h(\alpha+\xi),$$

where it follows from (6.31) that r is an integer, and from (6.28) that $0 < r < h$ or

(6.33) $$1 \leq r \leq h-1.$$

From (6.32), (6.33), and condition (a) of the hypotheses,

$$\xi = \frac{r}{h} - \alpha < \frac{r}{h} \leq 1 - \frac{1}{h},$$

and hence $h\xi - h+1 < 0$, which implies that $f_\alpha(\xi) = 0$ by condition (e). Substituting this into (6.18) (where $\beta' = \alpha$ since $\beta = 1-h\alpha$) we have for $0 \leq n \leq M$ that

$$B(n) \geq (1-h\alpha)(n+1)-\delta = (1-h\alpha)n - (h\alpha+\delta-1),$$

and in particular, when $n = 0$, a simple calculation shows that $h\alpha+\delta-1 \geq 0$. We also have that $h\alpha-1 < 0$ by condition (a) of the hypotheses. These last two results together with condition (c) give us

$$0 \leq h\alpha + \delta - 1 < \delta < 1 - \beta . \tag{6.34}$$

Consequently the number $h\alpha + \delta - 1$ satisfies condition (c) and so we may apply the induction hypothesis to $C = A_1 + A_2 + \cdots + A_{h-1} + B$. Consequently,

$$C(n) \geq (1-\alpha)n - f_{1-h\alpha}(h\alpha+\delta-1) = (1-\alpha)n - \frac{h\alpha+\delta-1}{h} \qquad (0 \leq n \leq M),$$

where the case $n = 0$ holds since the function $f_{1-h\alpha}$ is non-negative by Lemma 6.3. Therefore, by Lemma 3.1 (with $n = m-k$) we have for all k where $0 \leq k \leq M$ that

$$\begin{aligned}
C^*(k) &= k - C(M) + C(M-k) \\
&= k - (1-\alpha)M + \frac{\delta}{h} - \xi + C(M-k) && [(6.10)] \\
&\geq k - (1-\alpha)M + \frac{\delta}{h} - \xi + (1-\alpha)(M-k) - \frac{h\alpha+\delta-1}{h} && [(6.35)] \\
&= \alpha k - (\alpha + \xi - \frac{1}{h}) \\
&= \alpha k - \frac{r-1}{h}, && [(6.32)]
\end{aligned}$$

and so C^* satisfies the requirements placed on the r-th set by condition (f). If we let S_r be given by

$$S_r = A_1 + A_2 + \cdots + A_{r-1} + C^* + A_{r+1} + \cdots + B$$

we then have by the induction hypothesis that

$$S_r(M) \geq (1-\alpha)M - \frac{\delta}{h} \ . \tag{6.36}$$

Also, by Definition 3.2 (with $A_h = B$) and Scherk's inversion formula (with $i = r$ and $n = 0$), we have

$$A_r(M) \leq M - S_r(M) \ .$$

Consequently,

$$\begin{aligned}
A_r(M) &\leq M - (1-\alpha)M + \frac{\delta}{h} && [(6.36)] \\
&= \alpha M + \frac{\delta}{h} \\
&= M - C(M) + \xi && [(6.30)] \\
&< M - C(M) + 1 - \alpha && [(6.11),\ \text{Lemma 6.2}] \\
&< M - C(M) + 1. && [\alpha > 0 \text{ by condition (a)}] \qquad (6.37)
\end{aligned}$$

However, we also see that

$$\begin{aligned}
A_r(M) &= A_r(M+1) && [M+1 \notin A_r] \\
&\geq \alpha(M+1) - \frac{r-1}{h} && [\text{condition (f)}] \\
&= \alpha M + \alpha - \frac{r-1}{h} \\
&= \alpha M + \alpha - (\alpha + \xi - \frac{1}{h}) && [(6.32)] \\
&= \alpha M - \xi + \frac{1}{h} \\
&= M - C(M) - \frac{\delta}{h} + \frac{1}{h} && [(6.30)] \\
&= M - C(M) + \frac{1-\delta}{h} \\
&> M - C(M) . && [\delta < 1 \text{ by conditions (a), (b), (c)}]
\end{aligned}$$

This result combined with (6.37) yields $M-C(M) < A_r(M) < M-C(M)+1$, or

$$0 < A_r(M) + C(M) - M < 1,$$

which is a contradiction since $A_r(M) + C(M) - M$ is an integer. Thus we have obtained the final contradiction. Hence the proof of Scherk's Theorem is complete.

CHAPTER VII

FURTHER REMARKS

In this chapter several remarks, which do not fit naturally in the previous chapters, are discussed. Each section is independent of the others.

§ 7. 1 The Generalized αβ Theorem

In § 2. 4 and § 2. 5 we stated the αβ Theorem and used the theorems of Khinchin and Scherk to obtain several special cases of it. The αβ Theorem can be generalized to the sum of more than two sets as follows. For $h \geq 2$ let C be the sum of the sets $A_1, A_2, \cdots, A_h$ of nonnegative integers. Let $d(C) = \gamma_h$ and for each i, where $i = 1, 2, \cdots, h$, let $d(A_i) = \alpha_i$. Then the following theorem can be proved by mathematical induction on h by using the αβ Theorem.

Theorem 7. 1. If $0 < \sum_{i=1}^{h} \alpha_i < 1$, then $\gamma_h \geq \sum_{i=1}^{h} \alpha_i$.

The theorems of Khinchin and Scherk can be used to obtain several special cases of Theorem 7. 1 by a procedure very much like that employed in §§ 2. 4-2. 5. Two such special cases follow.

Theorem 7.2. Let $h \geq 2$. If $C = \sum_{i=1}^{h} A_i$ and $d(A_i) = \alpha$ $(i = 1, 2, \cdots, h)$, and if $0 < \alpha < \frac{1}{h}$, then $d(C) = \gamma_h \geq h\alpha$.

Theorem 7.3. Let $h \geq 2$. If $C = \sum_{i=1}^{h} A_i$, $d(A_i) = \alpha$ $(i = 1, 2, \cdots, h-1)$, and $d(A_h) = 1 - h\alpha$, and if $0 < \alpha \leq \frac{1}{h}$, then $d(C) = \gamma_h \geq 1 - \alpha$.

Both H. Rohrbach [30, p. 211-213] and A. Brauer [3, p. 324-328] state Khinchin's First Theorem and use it to obtain Theorems 7.2 and 7.3 as well as several other closely related density inequalities.

§ 7.2 The Inversion of A. Khinchin

In 1932 Khinchin introduced the notion of a set inversion. It was the following definition that he used in proving his first theorem [16, p. 29]. This definition is analogous to Landau's Definition 3.1 of an inversion.

Definition 7.1. Let M be an arbitrary positive integer. The inversion of a set A of nonnegative integers with respect to the interval $[1, M]$ is the set $A^{\sim}$ given by

$$A^{\sim} = \{ M+1-x : x \in A \cap [1,M] \} .$$

We now present two lemmas and a theorem of Khinchin concerning his inversion. We do not prove any of these since they are not used to obtain any later results, but simply refer the reader to Khinchin's paper [16, p. 29-30]. Lemma 7.1, Lemma 7.2, Definition 7.2, and Theorem 7.4 are analogous to Lemma 3.1, Lemma 3.2, Definition 3.2, and Theorem 3.1 respectively.

<u>Lemma 7.1.</u> For any integer n, where $0 \leq n \leq M$,

$$A^{\sim}(n) = A(M) - A(M-n).$$

Before stating the next lemma we make the following definition.

<u>Definition 7.2.</u> For $h \geq 2$ let C be the sum of the sets $A_1, A_2, \cdots, A_h$ of nonnegative integers. For each i, where $i = 1, 2, \cdots, h$, we define the set D_i by

$$D_i = A_1 + A_2 + \cdots + A_{i-1} + \overline{C}^{\sim} + A_{i+1} + \cdots + A_h .$$

<u>Lemma 7.2.</u> If $M+1$ is missing from C, then for each i, where $i = 1, 2, \cdots, h$, we have

$$D_i \subseteq \overline{A}_i^{\sim}$$

on $[1, M]$.

The following formula of Khinchin is immediate from Lemmas 7.1 and 7.2.

Theorem 7.4. (Khinchin's inversion formula) For all n, where $0 \leq n \leq M$, and for each i, where $i = 1, 2, \cdots, h$,

$$\overline{A}_i(M) - \overline{A}_i(M-n) = \overline{A}_i^{\sim}(n) \geq D_i(n),$$

if $M+1 \notin C$.

Throughout the proofs of Khinchin's First Theorem, Landau's Theorem, and Scherk's Theorem, given in Chapters IV, V, and VI, we continually applied Lemma 3.1 and Theorem 3.1 (or, as in the proof of Landau's Theorem, its special case Theorem 3.2) to obtain an estimate of an expression involving the difference of two counting functions of the form $A_i(M)-A_i(M-k)$. We will show that Lemma 7.1 and Theorem 7.4 could just as easily have been applied to obtain this estimate.

We notice that for any set A_i of nonnegative integers,

$$\overline{A}_i(k) = k-A_i(k). \tag{7.1}$$

Using (7.1) together with Lemma 7.1, with A_i replaced by $\overline{A}_i$, we obtain

$$\overline{A_i}^{\sim}(k) = \overline{A_i}(M) - \overline{A_i}(M-k) = [M-A_i(M)] - [(M-k)-A_i(M-k)]$$

(7.2) $$= k-A_i(M) + A_i(M-k) \qquad (0 \leq k \leq M).$$

Letting $n = M-k$ and $A = A_i$ in Lemma 3.1 we have

(7.3) $$A_i^*(k) = k-A_i(M)+A_i(M-k) \qquad (0 \leq k \leq M),$$

and so by (7.2) and (7.3),

(7.4) $$A_i^*(k) = \overline{A_i}^{\sim}(k) \qquad (0 \leq k \leq M).$$

By using (7.4) and Scherk's inversion formula (Theorem 3.1) for $n = M-k$ we have

(7.5) $$\overline{A_i}^{\sim}(k) = A_i^*(k) \geq S_i(k) \qquad (0 \leq k \leq M),$$

if $M+1 \notin C$, while by Khinchin's inversion formula (Theorem 7.4) and (7.4) we have

(7.6) $$A_i^*(k) = \overline{A_i}^{\sim}(k) \geq D_i(k) \qquad (0 \leq k \leq M).$$

Consequently by (7.2) and (7.6),

(7.7) $$A_i(M)-A_i(M-k) \leq k-D_i(k) \qquad (0 \leq k \leq M),$$

and also by (7.3) and (7.5),

(7.8) $$A_i(M)-A_i(M-k) \leq k-S_i(k) \qquad (0 \leq k \leq M).$$

Thus if we can find identical lower bounds on $D_i(k)$ and $S_i(k)$, then by (7.7) and (7.8) the same upper bound can be obtained for $A_i(M)-A_i(M-k)$, regardless of which inversion formula is used. Finding this upper bound is the only purpose of applying an inversion formula in the proofs of Khinchin's Theorem. For no other reason than simplicity of notation we have chosen to use the inversion of Landau (Chapter III) in the preceding chapters since complements are not involved.

§ 7.3 A Remark Concerning the Hypotheses of Khinchin's First Theorem and Landau's Theorem

In the original statement of what we have called Khinchin's First Theorem [16, p. 28-29], one slight difference appears in the hypothesis of part (i). Khinchin only considers values of λ in the half-open interval $[\frac{h-1}{h}, 1-\alpha)$, where $h \geq 2$, while we have stated the theorem with $\lambda \in [\frac{h-1}{h}, 1-\alpha]$. The only change that needs to be made in the proof of Khinchin's First Theorem to extend the range of values for λ to this closed interval is the replacement of the inequality $\lambda < 1-\alpha$ by $\lambda \leq 1-\alpha$ throughout the proof. With this change Landau's Theorem becomes a special case of Khinchin's First Theorem. Moreover, the equivalence between Khinchin's first and second theorems could not have been established without inequality (2.8), which rests on the fact that $\lambda \leq 1-\alpha$.

A change was also made in the hypothesis of part (i) of Landau's Theorem. Landau only considered λ to be either $\frac{1}{2}$ or $1-\alpha$, while we have stated the theorem with $\lambda \epsilon [\frac{1}{2}, 1-\alpha]$. As before, we were able to easily obtain this stronger result with no additional work, as no changes were necessary in the proof. Each time the number λ was considered in his proof, Landau did not deal with the cases $\lambda = \frac{1}{2}$ and $\lambda = 1-\alpha$, as such. Rather, he used the hypotheses $\lambda \epsilon \{\frac{1}{2}, 1-\alpha\}$, and $\alpha < \frac{1}{2}$ to obtain that $\frac{1}{2} \leq \lambda \leq 1-\alpha$, and used this fact continually throughout the proof.

With this change, Landau's Theorem is more readily obtained as a special case of Khinchin's First Theorem, for all that need be done to make the specialization is to set h equal to 2.

§7.4 A Brief Comparison of the Proofs of Khinchin, Landau, and Scherk

The three proofs of Khinchin, Landau, and Scherk are similar in their use of induction, contradiction, and inversions, and of course many of the arguments leading up to the use of these three are similar.

A reduction in the number of applications of Scherk's inversion formula is probably the most striking difference in Scherk's proof over Khinchin's. Scherk only needed to apply this inversion formula three times, while Khinchin's proof required six such applications, a fact which results in Scherk's induction being somewhat shorter than Khinchin's.

Although Landau's Theorem is just the special case $h = 2$ of Khinchin's First Theorem, he has done considerably more than make this specialization in Khinchin's proof. In fact, Landau was able to considerably reduce the length of his proof over Khinchin's. However, the same number of applications of the inversion formula were required.

The reader may have noticed that identical results were obtained (except with $h = 2$) through inequality (5.7) of Landau's proof as were obtained through inequality (4.7) of Khinchin's. The reader can then verify that in the remainder of Part I of the induction, Landau's discussion would yield improper results if one were dealing with the sum of more than two sets. Of course one must be careful not to merely replace $\frac{1}{2}$ by $\frac{1}{h}$, since $\frac{1}{2}$ is also equal to $\frac{h-1}{h}$ when $h = 2$. On the other hand if we set $h = 2$ in the proof of Khinchin's First Theorem we obtain a proof of Landau's Theorem which is more complicated than Landau's proof.

A similar correspondence holds between the beginning of Part II of the induction through inequality (4.27) of Khinchin's proof and the beginning of Part II of the induction through inequality (5.22) of

Landau's proof. Here again for the remainder of the proof of this part Landau was able to make several simplifications.

§ 7.5 Remarks Concerning the Proof of Khinchin's Second Theorem

In 1939 Khinchin [13] proved a theorem that is equivalent to part (i) of his 1932 theorem (see §§ 2.1-2.2). In 1940 he published the proof of still another theorem [14], which is merely a special case of the 1939 theorem. To prove these theorems Khinchin once again employs a very complicated, but nevertheless ingenious, inductive argument. He is able to proceed by a direct argument to the desired result when considering one particular case. However, for the remaining case Khinchin must assume that the conditions of the hypotheses are satisfied for $N = M+1$, but that the conclusion fails. He then proceeds, by several very involved arguments, to arrive at a contradiction, and in this way completes the proof.

The most striking difference between these proofs and his earlier proof, as well as the proofs of Landau and Scherk, is that neither of the inversions of Chapter III or of § 7.2 are used. On the other hand, Khinchin makes use of a "translation" of a set A of nonnegative integers, quite unrelated to the notion of an inversion. Very simply, to "translate" a given set A of nonnegative integers with respect to some natural number n, in the sense of Khinchin, we merely subtract $n+1$ from each $a \in A$ and delete all non-positive elements that remain. That is, the translation A_n of a set A of nonnegative integers with respect to some natural number n, is given by the set of natural numbers

$$A_n = \{a-n-1 : a \in A\} \cap \{1, 2, \cdots, n, \cdots\}.$$

Several "translation formulas" are then developed. Khinchin applies these in somewhat the same way as he applied the inversion formula in his 1932 theorem.

§7.6 Remarks on the Papers Containing the Theorems of Khinchin, Landau, and Scherk

The articles [13, 14, 16] by Khinchin were found in two different publications of the Akademiya Nauk S. S. S. R. (Soviet Academy of Science). Although his 1932 paper [16] is widely known, little reference is made in the literature to his papers [13, 14]. Landau's book [18], containing the proof of Landau's Theorem, is a very well-known tract on some advances made in additive number theory up to 1937.

Finally, the 1938 article of Scherk [33] deserves special mention. One briefly scanning the literature might never happen upon Scherk's paper. As nearly as the author can tell, it is cited only in the bibliography of Rohrbach's 1938 expository paper [30].

Scherk's article is contained in a publication of Jednota Česko-Slovenských Matematiků a Fysiků (Czechoslovakian Society of Mathematics and Physics), and a copy was obtained from Brown University.

CHAPTER VIII

SOME ASPECTS OF SCHNIRELMANN DENSITY AND ITS HISTORY

§ 8.1 The Contributions of V. Brun, L. Schnirelmann, and I. Vinogradov to Goldbach's Conjecture

For almost 225 years mathematicians have been trying to prove a famous conjecture of Christian Goldbach (1690-1764), that every positive even integer except 2 can be expressed as a sum of two primes. The main difficulty has been that primes are defined in terms of multiplication while the problem involves addition, and any connections between the multiplicative and additive properties of integers are, in general, very difficult to find.

The first attempt of any consequence was made by the Norwegian mathematician Viggo Brun (1885-) in 1920. Brun [4] extended the idea of the sieve method of Eratosthenes and succeeded in showing that all positive even integers are representable as the sum of two numbers, each having at most nine prime factors. Brun later reduced this number to four but made no other significant contribution to the solution of Goldbach's Conjecture.

Interest in the problem spread to the Soviet Union where several mathematicians went to work on the problem in the 1920's and 1930's. Their results make a solution of the conjecture no longer seem inaccessible. In 1922 the eminent Russian teacher Aleksandr Yakovlevich Khinchin (1894-1959) [12] was the first in Moscow to study the problem. In 1930 an important result, quite unexpected and surprising to many, was obtained by a young Russian scholar Lev Genrichovitch Schnirelmann (1905-1938). Schnirelmann [34] supplemented Brun's sieve method together with his own new ideas concerning the "density" of sums of sets of integers and showed that every positive integer can be expressed as the sum of at most 300,000 primes [5, p. 31] ! Although this accomplishment seems to be of very little importance in comparison with the goal of proving Goldbach's Conjecture, nevertheless it was a first step.

A conjecture somewhat weaker than that of Goldbach is that each positive odd integer larger than 7 is the sum of three odd primes. In 1937 the Russian mathematician Ivan Matveevich Vinogradov (1891-), using analytic methods due to G.H. Hardy and J.E. Littlewood [11], succeeded in showing that all positive odd integers greater than a certain effectively computable constant k are expressible as the sum of three odd primes [39]. However, Vinogradov's proof does not permit us to appraise k. In 1938 and 1940 Nils Pipping [27, 28] verified the problem for all n with $7 < n \leq 100,000$ by numerical methods and even more recently, in 1956 K.G. Borozdkin [2] proved that

$$k \leq e^{e^{16.038}} < 3^{3^{15}} .$$

Thus the problem remains to be verified for all odd integers n where $100,000 < n \leq k$, a task beyond the capability of present day computers.

The original papers of Brun [4], Schnirelmann [34], and Vinogradov [39] are hard to obtain and difficult to read. One can find a discussion of Goldbach's Conjecture together with the results obtained and methods used by these men in the books by Courant and Robbins [5, p. 30-31], Ore [24, p. 81-85], Rademacher [29, p. 137-144], Sierpiński [38, p. 118-121], and Vinogradov [40, p. 14-16; 163-176].

§ 8.2 Schnirelmann Density

Schnirelmann's "density" has not only been helpful in further investigation of the Goldbach problem but opened an entirely new branch of mathematics. We now define the kind of "density" used by Schnirelmann, as was previously done in § 1.1.

Definition 8.1. Let A be a (possibly empty) set of nonnegative integers. We denote by $A(n)$ the number of positive integers contained in the set A which do not exceed n. Then the density, $d(A)$, of A is given by

$$d(A) = \quad g\ell b \left\{\frac{A(n)}{n} : n \geq 1 \right\}.$$

We also repeat the definitions of the sum of sets of integers made in §1.1.

Definition 8.2. The sum of two sets of nonnegative integers A and B is the set $A+B$ given by

$$A+B = A \cup B \cup \{a+b : a \in A,\ b \in B\}.$$

(The Schnirelmann densities $d(A)$, $d(B)$, and $d(A+B)$ will often be denoted by α, β, and γ respectively.)

Definition 8.3. The sum of h sets $(h \geq 2)$ of nonnegative integers $A_1, A_2, \cdots, A_h$ is the set S defined recursively by

$$S = \sum_{i=1}^{h} A_i = A_h + \sum_{i=1}^{h-1} A_i \; .$$

Schnirelmann was quite interested in obtaining relationships between the density of the sum of sets and the densities of the respective summands. The following theorem is generally attributed to Schnirelmann, but is not found in this form in either of his papers [34] or [36]. Proofs can be found in [7, p. 316], [18, p. 56], and [30, p. 206-207].

Theorem 8.1. (Schnirelmann's Theorem) If $\alpha + \beta \geq 1$, then $\gamma = 1$.

Much of the credit should certainly be extended to the German mathematician Edmund Georg Hermann Landau (1877-1938) as the early popularizer of Schnirelmann's work. Shortly after Schnirelmann's first paper appeared, Landau [17] devoted an entire article to discuss Brun and Schnirelmann's findings concerning the Goldbach Conjecture. In this paper Landau was able to simplify the proofs of several of Schnirelmann's theorems. He also discussed density to a greater extent than had Schnirelmann.

The next theorem (Theorem 8.2) is generally attributed to both Schnirelmann and Landau. Certainly Schnirelmann was aware of it in 1930 when studying Goldbach's Conjecture (see footnote [17, p. 269] and also [15, p. 24-25]). However, it and a theorem (which we have called Corollary 8.1) proved by Landau in 1930 [17, p. 269] are only special cases of a more general theorem (Theorem 8.3) proved by Schnirelmann in 1933 (in a paper written in 1931) [36, p. 652-653]. The proof of Theorem 8.2 is also found in [7, p. 316-317], [15, p. 22-23], and [18, p. 57].

Theorem 8.2. (Landau-Schnirelmann inequality)

$$\gamma \geq \alpha + \beta - \alpha\beta .$$

The generalization of Theorem 8.2 given by Schnirelmann

(Theorem 8.3) can be obtained by a simple induction on h and the use of Theorem 8.2.

Theorem 8.3. (Generalized Landau-Schnirelmann inequality)

Let $d(A_i) = \alpha_i$ $(i = 1, 2, \cdots, h)$ and $d(S) = \gamma_h$, where $S = A_1 + A_2 + \cdots + A_h$ $(h \geq 2)$. Then

$$1 - \gamma_h \leq \prod_{i=1}^{h} (1-\alpha_i).$$

To obtain Landau's special case of Theorem 8.3 we need only let $A_i = A$ and $\alpha_i = \alpha (i = 1, 2, \cdots, h)$. Then we obtain

Corollary 8.1. (Landau) $1 - \gamma_h \leq (1-\alpha)^h$.

§ 8.3 The $\alpha\beta$ Conjecture

In the fall of 1931 L.G. Schnirelmann, then a professor of mathematics at the Don Polytechnic Institute, Novocherkassk, U.S.S.R., was the guest of the Deutschen Mathematiker-Vereinigung. A resumé of his lecture on additive problems of number theory is given in [35, p. 91-92]. It was at this time that Schnirelmann and Landau discussed Theorem 8.2 and found, for all examples they could create, that it could be replaced by the sharper and simpler inequality

(8.1) ($\alpha\beta$ Conjecture) $\qquad \gamma \geq \alpha + \beta \qquad$ (if $\; 0 < \alpha+\beta < 1$).

They soon realized, after futile attempts, that if (8.1) were to be proved its proof would not be easy.

§ 9.4 The Contributions of A. Khinchin to the $\alpha\beta$ Conjecture

Upon returning from Germany, Schnirelmann reported his findings to A. Ya. Khinchin at Moscow State University. Khinchin immediately set aside all other mathematical research in favor of working on the solution of (8.1) [15, p. 27]. In March of 1932, less than six months after (8.1) had been conjectured, Khinchin published the proof of a theorem from which one can easily prove the special cases $\alpha = \beta$, and $\beta = 1-2\alpha$ (or $\alpha = 1-2\beta$) of (8.1) [16]. However the complete solution of (8.1) evaded him.

Khinchin did not acutally show how one might obtain these special cases from his theorem. The clearest proof of the case $\alpha = \beta$ is given in [18, p. 76], one of Landau's last publications before his death. The case $\beta = 1-2\alpha$ (or $\alpha = 1-2\beta$) is established by Brauer [3, p. 327-328] and is given considerable attention by Rohrbach [30, p. 218]. The papers of Errera [8, p. 301-302] and Erdős and Niven [7, p. 316] treat these cases briefly.

Khinchin [15, p. 27] mentions that he was later able to simplify the proof of his 1932 theorem (treated in Chapter IV). However correspondence between this author and P. Scherk [32] leads one to believe that Khinchin never published this simplification. However in 1939 and again in 1940, Khinchin [13, 14] was able to give a somewhat different proof of the first of two parts of his theorem, but he made no further contributions in this area.

§ 8.5. A Theorem of H. Mann

The reader is perhaps aware that the $\alpha\beta$ Conjecture (8.1) has been proved. However, none of the mathematicians previously mentioned were able to construct the proof. In 1942 Henry Berthold Mann (1905-) became interested in the problem while attending the lectures of A. T. Brauer at New York University. Mann was able to prove an inequality from which one easily obtains (8.1). Mann's method is related somewhat to Khinchin's method, but is based on an entirely different idea. The following year Artin and Scherk simplified some of Mann's work. Both the papers of Mann [20] and Artin and Scherk [1] are somewhat difficult to follow. Damewood has rewritten each proof in a much clearer form [6, p. 5-39].

Mann's inequality [20, p. 524], as well as inequalities obtained by more recent mathematicians, certainly overshadows Khinchin's theorem. A recent paper by Scherk [31] systematizes the results obtained in this direction since 1942.

§ 8.6 Additional Remarks Concerning the Literature

The original paper of Schnirelmann [34] is not only difficult to read but even more difficult to obtain. The U.S. Geological Survey Library (Washington, D.C.) is the only known U.S. library having an original, from which this author obtained a copy.

Landau [18] and Rohrbach [30] have published summaries of much of the work done in this field up to 1937 and 1938 respectively. Khinchin also published a book [15] in which is discussed much of the history of the $\alpha\beta$ Theorem, as (8.1) is now most often called. Many of the details of §8.3 were obtained from this source.

The literature on Schnirelmann density has become increasingly rich in the 37 years since Schnirelmann first defined it. For exceptionally good bibliographies the reader is referred to [21, p. 103-111], [22, p. 432-434], [25, p. 202-229], and [30, p. 235-236]. In addition, the papers [9,10] published as tributes to Khinchin give a complete bibliogrpahy of his works.

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