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Title TWO-PARAMETER CHART FOR SELECTING EMPIRICAL DENSITY FUNCTIONS

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This thesis treats the problem of estimating the frequency distribution from which a random sample of a continuous random variable has been drawn.

Two parameters, θ and α_3^2, based on the normalized samples are computed. These parameters are rational functions of the third and fourth central moments. A point on a (θ, α_3^2) chart is constituted by them. The chart is partitioned into regions such that if a (θ, α_3^2) point is located in a given region of the chart the frequency distribution which best fits these data is determined. There are 13 frequency distributions considered in partitioning the chart; they are known as the Pearson's system of frequency curves.

TWO-PARAMETER CHART FOR SELECTING EMPIRICAL DENSITY FUNCTIONS

by

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TWO-PARAMETER CHART FOR SELECTING EMPIRICAL DENSITY FUNCTIONS

CHAPTER I

INTRODUCTION

Suppose one wishes to study some statistical characteristics of a particular population. In order to obtain a reasonable (statistical) interpretation, he may need to know the distribution of that random variable. This thesis presents a method for determining the density function (distribution) which corresponds to each population.

The steps are as follows:

1. Collect a set of data X_i, $(i = 1, 2, \cdots n)$, as a sample of size n from the population. (Techniques of sampling may be used.) The data may be obtained from some sort of experiments.

2. Compute $\overline{X}$ (sample mean) and s^2 (sample variance) from the observed values of X.

3. Transform the old observed values of X_i to new ones t_i, where

$$t_i = (X_i - \overline{X})/s, \quad i = 1, 2, \cdots n. \tag{1.1}$$

By the above transformation, the variable t would have zero sample mean and unit sample variance.

From now on, we will use the values of t's to compute the two moments a_3 and a_4, where

$$(1.2) \qquad a_k = \frac{1}{n} \sum_{i=1}^{n} t_i^k, \quad k = 1, 2, 3, 4.$$

The two parameters which will be used in the later discussion are θ and a_3^2. These are rational functions of a_3 and a_4, which we will discuss in the following chapters.

4. Locate the (θ, a_3^2) point on a 'selection chart'. The different regions of the chart correspond to different types of density functions.

5. Determine the parameters of that type of density function which has been selected by reference to the 'selection chart'.

CHAPTER II

DETERMINATION OF PARAMETERS

The Pearson Frequency Functions are to be found among the solutions of the differential equation,

$$\frac{1}{y}\frac{dy}{dt} = \frac{a - t}{b_0 + b_1 t + b_2 t^2} \tag{2.1}$$

under the restriction that [1/]

$$(b_0 + b_1 t + b_2 t^2)t^n \cdot f(t) \Big|_{t=r}^{t=s} = 0 \tag{2.2}$$

where $y = f(t)$ is a solution; r and s are the extreme values of the variable t, provided that the first $(n + 1)$ moments over this range exist.

By integrating both sides of (2.1) after multiplying by $(b_0 + b_1 t + b_2 t^2)t^n \cdot f(t)$, we have

$$\int_r^s (b_0 + b_1 t + b_2 t^2)t^n \frac{dy}{dt} \cdot dt = \int_r^s (a-t)t^n \cdot f(t)dt$$

where $y = f(t)$ is a probability density function.

[1/] See Appendix 6.2 .

Integrating by parts on the left side,

$$(b_0+b_1t+b_2t^2)t^nf(t)\Big|_{t=r}^{t=s} - \int_r^s [nb_0t^{n-1}+(n+1)b_1t^n+(n+2)b_2t^{n+1}]\, f(t)\,dt$$

$$= \int_r^s (at^n-t^{n+1})f(t)dt\ .$$

The first expression vanishes by (2.2). Then we obtain

$$nb_0\alpha_{n-1}+(n+1)b_1\alpha_n+(n+2)b_2\alpha_{n+1}+a\alpha_n-\alpha_{n+1} = 0 \tag{2.3}$$

where

$$\alpha_k = \int_r^s t^kf(t)dt \qquad (k = 0, 1, 2, \cdots)$$

We call (2.3) the recursion formula for the moments.

Since we had stated in Chapter I that the variable had been normalized (i.e. for the variable t, $\alpha_1 = 0$ and $\alpha_2 = 1$), we obtain simultaneous equations in a, b_0, b_1 and b_2 by setting $n = 0, 1, 2$ and 3.

$$b_1 + a = 0 \tag{2.4}$$

$$b_0 + 3b_2 = 1 \tag{2.5}$$

$$3b_1 + 4b_2\alpha_3 + a = \alpha_3 \tag{2.6}$$

$$3b_0+4b_1\alpha_3 + 5b_2\alpha_4 + a\alpha_3 = \alpha_4 \tag{2.7}$$

For these four equations, the solution is

$$
\begin{aligned}
a &= \frac{-\alpha_3}{2(\sqrt{\theta}-1)} \\
b_0 &= \frac{\sqrt{\theta}+2}{4(\sqrt{\theta}-1)} \\
b_1 &= \frac{\alpha_3}{2(\sqrt{\theta}-1)} \\
b_2 &= \frac{\sqrt{\theta}-2}{4(\sqrt{\theta}-1)}
\end{aligned}
\tag{2.8}
$$

where θ is a paremeter which depends on the two moments, α_3 and α_4.

$$\theta = 36 \left(\frac{\alpha_4 - \alpha_3^2 - 1}{\alpha_4 + 3}\right)^2 . \tag{2.9}$$

We see that in order for (2. 8) to be valid, the value of θ must be different from one. The case $\theta = 1$, will be included in the later discussion of the differential equation (2. 1).

Note that [2]

$$0 \leq \theta < 36. \tag{2.10}$$

For the whole discussion in the following chapters, the two

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[2] See Section 6. 1, Appendix I.

parameters θ and α_3 will be considered. Corresponding to all possibilities and relations of these parameters, there will be various types of density functions.

It is useful to make the preliminary statements, for the integration of (2. 1) and the development of the various forms of $f(t)$ that arise:

1. We must have $f(t) \geq 0$, over the range of variable t.

2. The area under the curve $y = f(t)$ over the range of variation must be finite. This being true then we always determine the constant of integration so that the area is unity.

3. The range of t in each case is taken as the maximum one for which (1) and (2) may be secured which contains the point $t = 0$.

4. It is sufficient throughout to take $\alpha_3 \geq 0$ since the curve for $\alpha_3 = -k$ is the reflection of that for $\alpha_3 = k$ through the line $t = 0$.

One consequence which we can obtain from (2. 8) is that b_0 cannot vanish for any kind of Pearson Frequency Function whose moment of fourth order does exist.

CHAPTER III

CHART AND SUMMARIZED TABLE OF DENSITY FUNCTIONS

The chart that had been set up in this chapter depends on the two parameters θ and α_3^2 which are rational functions of α_3 and α_4. The horizontal axis is the θ-axis which has the range of values from zero to 7.84. The region for which $7.84 \leq \theta < 36$ is called the "Heterotypic Area". The nature of solutions to (1.1) in this region will be discussed in Section 6.8, Appendix I. The vertical axis is the α_3^2-axis. Since α_3 is real, then α_3^2 is a non-negative quantity. And the range of α_3^2 is $[0, \infty)$. In the chart, the maximum value of α_3^2 is about twelve units.

On the 'selection chart', there are six curves. The equations for these are

$$\theta = 0$$

$$\theta = 1$$

$$\theta = 4$$

$$\alpha_3^2 = 0$$

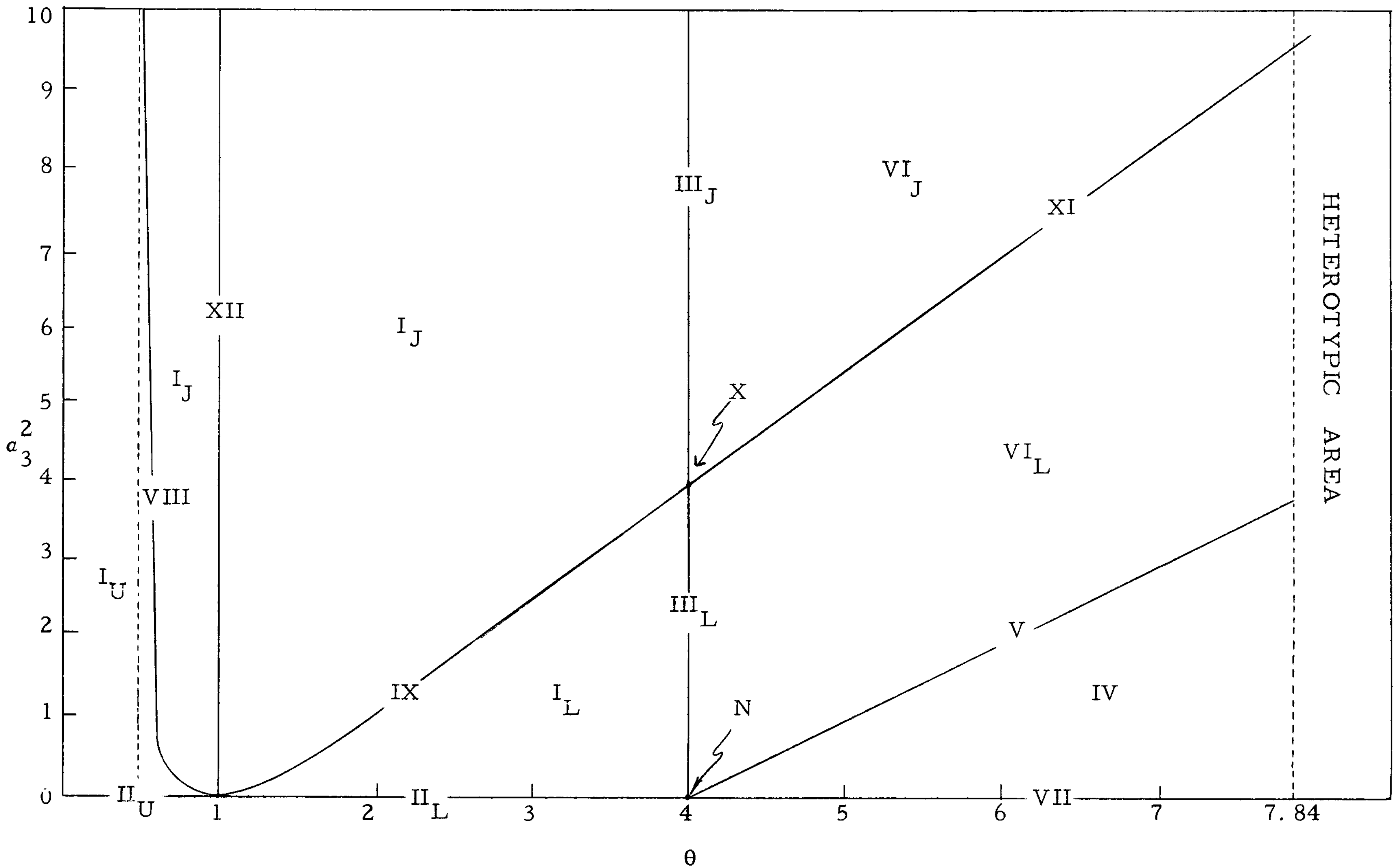
$$\alpha_3^2 = \theta - 4$$

and

$$\alpha_3^2(3\sqrt{\theta}-2) = 4(\sqrt{\theta}+2)(\sqrt{\theta}-1)^2 .$$

The last curve is tangent to the line $\alpha_3^2 = 0$ at the point $\theta = 1$, and has the line $\theta = 4/9$ as its asymptote. These six curves divide the region $0 \leq \theta < 7.84$, and $\alpha_3^2 \geq 0$, into small areas. Each area contains the points which correspond to a specific type of probability density function. Other types of density function correspond to the points on the curves or to their points of intersection. Each region (area, curve or point) of the chart is labelled by the corresponding type number of the probability density function. The subscript L, j, U indicate bell-shpaed, J-shaped and U-shaped curves respectively. The "N" stands for the normal curve. The formulas for these probability density functions are listed in the table which follows the chart together with other pertinent facts.

In the application of this to the selection of the density for an observed statistical table, θ and α_3^2 would be calculated numerically, the (θ, α_3^2) point located on the chart, the type determined from the labels, and the formula for that density would be found in the table. The constants which are found in the formula of the density are functions of θ and α_3. Finally, the density $y = f(t)$ is transformed to $y = g(X)$.



The (θ, a_3^2)-chart for selecting density functions.

Table 1. Summarized Table of Density Functions.

Type No.	Conditions on Parameters: θ	α_3^2	Additional Conditions	Density Functions	Range of Variable t	Remarks
Main Types						
I	$\theta \neq 1$	>0	$0<\theta<4$; $\alpha_3^2(3\sqrt{\theta}-2) \neq 4(2+\sqrt{\theta})(\sqrt{\theta}-1)^2$	$y_0(t-r_1)^{m_1}(r_2-t)^{m_2}$	(r_1, r_2)	U-, J-, bell-shaped
IV	$\theta>4$	>0	$\alpha_3^2<\theta-4$	$y_0 e^{\pi n}[(t+r)^2+s^2]^{-m} e^{-2n\tan^{-1}\left[\frac{t+r}{s}\right]}$	$(-\infty, \infty)$	Bell-shaped
VI	$\theta>4$	>0	$\alpha_3^2>\theta-4$	$y_0(t-r_2)^{m_2}(t-r_1)^{m_1}$	(r_1, ∞)	Bell- and J-shaped
Transitional Types						
"Normal Curve"	$\theta=4$	$=0$	-----	$y_0 e^{-t^2/2}$	$(-\infty, \infty)$	Bell-shaped
III	$\theta=4$	>0	-----	$y_0(A+t)^{A^2-1}e^{-At}$	$(-A, \infty)$	Bell- and J-shaped
II	$\theta \neq 1$	$=0$	$0<\theta<4$	$y_0(S^2-t^2)^M$	$(-S, S)$	U- or Bell-shaped
V	$\theta>4$	>0	$\alpha_3^2=\theta-4$	$y_0(t+r)^{-2(m+1)}e^{-\frac{2rm}{t+r}}$	$(-r, \pm\infty)$	Bell-shaped
VII	$\theta>4$	$=0$	-----	$y_0(t^2+S^2)^{-m}$	$(-\infty, \infty)$	Bell-shaped
VIII	$\theta<1$	>0	$\alpha_3^2(3\sqrt{\theta}-2)=4(2+\sqrt{\theta})(\sqrt{\theta}-1)^2$	$y_0(t-r_1)^{-2m}$	(r_1, r_2)	Bell-shaped
IX	$1<\theta<4$	>0	$\alpha_3^2(3\sqrt{\theta}-2)=4(2+\sqrt{\theta})(\sqrt{\theta}-1)^2$	$y_0(r_2-t)^{-2m}$	(r_1, r_2)	J-shaped
X	$\theta=4$	$=4$	-----	$y_0 e^{-t}$	$(-1, \infty)$	J-shaped
XI	$4<\theta<7.84$	>0	$\alpha_3^2(3\sqrt{\theta}-2)=4(2+\sqrt{\theta})(\sqrt{\theta}-1)^2$	$y_0(t-r_2)^{-2m}$	(r_1, ∞)	J-shaped
XII	$\theta=1$	--	-----	$y_0\left[\frac{r_2-t}{t-r_1}\right]^{m_2}$	(r_1, r_2)	J-shaped

CHAPTER IV

IDENTIFICATION OF DENSITY FUNCTIONS

The complete discussion for each frequency type is given in this chapter. The order in which the types are considered is the same as the order in which they are listed in the table of Chapter II.

MAIN TYPES: I, IV, VI.

For $\theta \neq 4$ and $b_2 \neq 0$, the denominator on the right of (2.1) is always a quadratic function in t, which we can write in the form

$$(4.1) \qquad (b_0 + b_1 t + b_2 t^2) = b_2(t - r_1)(t - r_2)$$

in which neither r_1 nor r_2 can be zero (since $b_0 > 0$), and

$$r_1 = \frac{-b_1 + \sqrt{b_1^2 + 4b_0 b_2}}{2b_2}$$

$$r_2 = \frac{-b_1 - \sqrt{b_1^2 + 4b_0 b_2}}{2b_2}$$

By substituting the values of the b's from (2.8), we get

$$(4.2) \qquad r_1 = \frac{-a_3 + \sqrt{D}}{\sqrt{\theta} - 2}$$

(4.3) $$r_2 = \frac{-\alpha_3 - \sqrt{D}}{\sqrt{\theta - 2}}$$

where $D = \alpha_3^2 - \theta + 4$. When $D \neq 0$, $r_1 \neq r_2$ and the differential equation (2.1) can be written in the form

(4.4) $$\frac{1}{y} \cdot \frac{dy}{dt} = \frac{m_1}{t-r_1} + \frac{m_2}{t-r_2}$$

By integrating, we get

(4.5) $$y = y_0 (t - r_1)^{m_1} \cdot (t - r_2)^{m_2}$$

where m_1 and m_2 are constants which depend on θ and α_3 only.

(4.6) 3/ $$m_1 = -2\left[1+ \frac{1}{\sqrt{\theta-2}} \right] + \frac{\sqrt{\theta}\,\alpha_3}{\sqrt{\theta-2}} \cdot \frac{1}{\sqrt{D}}$$

(4.7) 3/ $$m_2 = -2 \left[1+ \frac{1}{\sqrt{\theta-2}} \right] - \frac{\sqrt{\theta}\,\alpha_3}{\sqrt{\theta-2}} \frac{1}{\sqrt{D}}$$

From (4.2) and (4.3), we can have the following:

3/ See Section 6.3, Appendix I.

1. r_1 and r_2 are real and opposite in sign, for $0 \leq \theta < 4$.

2. r_1 and r_2 are complex, for $4 < \theta$ and $\alpha_3^2 < \theta - 4$.

3. r_1 and r_2 real and have the same sign, for $4 < \theta$ and $\alpha_3^2 > \theta - 4$.

These conditions with another additional condition that $\alpha_3 \neq 0$, gives us respectively the " Main Types" of frequency functions designated I, IV and VI. The points which correspond to these types fall in simply determined areas on the (θ, α_3^2)-chart. And the boundaries of these areas are the curve

(4.8)[4/]
$$\alpha_3^2 (3\sqrt{\theta} - 2) = 4(2+\sqrt{\theta})(\sqrt{\theta} - 1)^2$$

MAIN TYPE I. For the regions labelled I_U, I_j and I_L,

$$0 < \theta < 4$$

$$\theta \neq 1$$

$$\alpha_3 \neq 0$$

and
$$\alpha_3^2(3\sqrt{\theta}-2) \neq 4(\sqrt{\theta} + 2)(\sqrt{\theta} - 1)^2.$$

Since we consider $\alpha_3 > 0$ only, we see from (4.2) and (4.3) for

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4/ See Section 6.5, Appendix I.

regions I_U and I_j that

$$r_1 < 0 < r_2$$

and
$$|r_1| < |r_2| .$$

The range is taken to be (r_1, r_2) and then (4.5) is written

(4.9)
$$y = y_0(t - r_1)^{m_1} \cdot (r_2 - t)^{m_2}$$

It is easily seen that [5/] the area under the curve is finite only when $m_1 + 1 > 0$ and $m_2 + 1 > 0$ and if these inequalities hold, moments of all orders exist.

From (4.6) and (4.7) we have

$$m_1 + 1 = \frac{\sqrt{\theta}}{2-\sqrt{\theta}} \left(1 - \frac{\alpha_3}{\sqrt{D}}\right)$$

$$m_2 + 1 = \frac{\sqrt{\theta}}{2-\sqrt{\theta}} \left(1 + \frac{\alpha_3}{\sqrt{D}}\right)$$

In the case $\theta < 4$, we have

$$\frac{\alpha_3^2}{\alpha_3^2 - \theta + 4} = \frac{\alpha_3^2}{D} < 1$$

or

5/ See Section 6.4, Appendix I.

$$1 \pm \frac{\alpha_3}{\sqrt{D}} > 0 \tag{4.10}$$

Thus $m_1 + 1 > 0$ and $m_2 + 1 > 0$ only if $\theta > 0$, since (2.9) shows that $\theta > 0$.

Further the Type I curve will be U-shaped, J-shaped, or bell-shaped if m_1 and m_2 are both negative, opposite in sign, or both positive, respectively.

For $0 < \theta < 1$, $0 < \frac{\sqrt{\theta}}{2-\sqrt{\theta}} < 1$; and from (4.10)

$$0 < (1 - \frac{\alpha_3}{\sqrt{D}}) < 1, \tag{4.11}$$

hence $m_1 = (\frac{\sqrt{\theta}}{2-\sqrt{\theta}})(1 - \frac{\alpha_3}{\sqrt{D}}) - 1 < 0$.

For $1 < \theta < 4$, $m_1 > 0$ only if

$$\frac{\sqrt{\theta}}{2-\sqrt{\theta}}(1 - \frac{\alpha_3}{\sqrt{D}}) > 1$$

or

$$\alpha_3^2 (3\sqrt{\theta} - 2) < 4(2+\sqrt{\theta})(\sqrt{\theta}-1)^2 . \tag{4.12}$$

Also, from

$$m_2 = (\frac{\sqrt{\theta}}{2-\sqrt{\theta}})(1+\frac{\alpha_3}{\sqrt{D}}) - 1$$

when $1 < \theta < 4$, it is similarly seen that $m_2 > 0$ only when the inequality (4.12) holds. Thus the curve

$$\alpha_3^2 (3\sqrt{\theta}-2) = 4(2+\sqrt{\theta})(\sqrt{\theta}-1)^2 \tag{4.13}$$

is tangent to the line $\alpha_3^2 = 0$ at $\theta = 1$ and divides the Type I area on the chart into three parts. On the left lie the points corresponding to U-shaped curves, above it the points correspond to J-shaped curves, and on the right they correspond to bell-shaped curves.

MAIN TYPE IV. For

$$\theta > 4,$$

$$\alpha_3^2 \neq 0,$$

and

$$\alpha_3^2 < \theta - 4, \quad \text{we have}$$

$$D = (\alpha_3^2 - \theta + 4) < 0 ;$$

and the m's are complex conjugates. From (4.6) and (4.7), we obtain

$$m_1 = -2\left[1 + \frac{1}{\sqrt{\theta}-2}\right] + \frac{i\sqrt{\theta}\,\alpha_3}{D(\sqrt{\theta}-2)} \cdot \sqrt{-D}$$

$$m_2 = -2\left[1 + \frac{1}{\sqrt{\theta}-2}\right] - \frac{i\sqrt{\theta}\,\alpha_3}{D(\sqrt{\theta}-2)} \cdot \sqrt{-D}$$

or

$$m_1 = -m + ni \quad \text{and} \quad m_2 = -m - ni$$

where $m = 2[1 + \frac{1}{\sqrt{\theta} - 2}]$, $n = \frac{\sqrt{\theta}\, a_3 \sqrt{-D}}{D(\sqrt{\theta}-2)}$ and $i = \sqrt{-1}$.

From (4.2) and (4.3), we have

$$r_1 = \frac{-a_3 + \sqrt{D}}{\sqrt{\theta}-2} \quad \text{and} \quad r_2 = \frac{-a_3 - \sqrt{D}}{\sqrt{\theta}-2}$$

or

$$r_1 = -r + si \quad \text{and} \quad r_2 = -r - si$$

where $r = \frac{a_3}{\sqrt{\theta}-2}$ and $s = \frac{\sqrt{-D}}{\sqrt{\theta}-2}$.

Using these parameters, (4.5) becomes

$$\begin{aligned} y &= y_0 (t+r-si)^{-m+ni} \cdot (t+r+is)^{-m-ni} \\ &= y_0 [(t+r)^2 + s^2]^{-m} \cdot [\frac{t+r-si}{t+r+si}]^{ni} . \end{aligned}$$

Using the complex identity[6/]

$$[\frac{a-bi}{a+bi}]^{ci} = e^{2c.\tan^{-1}(b/a)} \tag{4.13 A}$$

for all real values a, b and c, the frequency function can be written

$$\begin{aligned} y &= y_0 [(t+r)^2 + s^2]^{-m} \cdot e^{2n[\pi/2 - \tan^{-1}(\frac{t+r}{s})]} \\ &= y_0 \cdot e^{\pi n} [(t+r)^2 + s^2]^{-m} \cdot e^{-2n\tan^{-1}(\frac{t+r}{s})} . \end{aligned} \tag{4.14}$$

6/ See Section 6.6, Appendix I.

It is readily seen that $m > 0$, $n < 0$ and

$$e^{\pi n} < e^{-2n\tan^{-1}\left(\frac{t+r}{s}\right)} < e^{-\pi n};$$

hence the range $(-\infty, \infty)$ can be used.

This type corresponds to the region which falls below the line $\alpha_3^2 = \theta - 4$ on the chart.

MAIN TYPE VI. Consider the conditions

$$\theta > 4$$

$$\alpha_3^2 > \theta - 4$$

and
$$\alpha_3^2(3\sqrt{\theta}-2) \neq 4(\sqrt{\theta}+2)(\sqrt{\theta}-1)^2.$$

The above conditions define the remaining area on the chart. The frequency function is still in the form,

$$y = y_0\,(t-r_1)^{m_1} \cdot (t-r_2)^{m_2} \tag{4.15}$$

which is exactly the same as (4.5).

From (4.2) and (4.3), we see that $r_2 < r_1 < 0$. And from (4.6), (4.7), we see that $m_2 < 0$ and $m_1 \gtrless 0$ accordingly as

$$\alpha_3^2(3\sqrt{\theta}-2) \gtrless 4(2+\sqrt{\theta})(\sqrt{\theta}-1)^2.$$

We note that 7/

(4.16) $$a - r_2 = b_2(r_2 - r_1) \cdot m_2 > 0$$

since now $b_2 > 0$, and that

(4.17) $$a - r_1 = b_2(r_1 - r_2) \cdot m_1$$

has the same sign as m_1 because $r_2 < r_1 < 0$ and $r_1 - r_2 > 0$. Finally $a < 0$, from (2.8).

Thus for $\alpha_3 > 0$ and $m_1 > 0$, the point $t = a$ on the aixs of t lies to the right of both $t = r_1$ and $t = r_2$.

The range is taken (r_1, ∞). The curve is bell-shaped when $m_1 > 0$. If $m_1 < 0$, the curve is J-shaped, and the point $t = a$ is now lying to the left of $t = r_1$.

In order to have finite area and the first four moments exist, we must have

$$m_1 + m_2 < -5 \quad \text{and} \quad m_1 + 1 > 0.$$

And similarly for the n^{th} moment to be finite we must have

$$-(m_1 + m_2) > n + 1,$$

7/ See Section 6.7, Appendix I.

which is the same condition as in the case of Type IV, giving the same deadline $\theta = 7.84$. [8/]

If the origin is shifted to the point $t = r_2$, then we can let

$$t - r_2 = z \qquad \text{and} \qquad r_1 - r_2 = r.$$

For Type VI function, the expression can be obtained by changing variable from t to z as above,

$$y = y_0 \cdot z^{m_2} \cdot (z - r)^{m_1} \tag{4.18}$$

the range of variable t is (r, ∞).

THE NORMAL DENSITY FUNCTION.

$$\theta = 4 \qquad \text{and} \qquad \alpha_3^2 = 0.$$

Under the above conditions, and from (2.8), we have $a = 0$, $b_0 = 1$, $b_1 = 0$ and $b_2 = 0$; then (2.1) becomes

$$\frac{1}{y} \cdot \frac{dy}{dt} = -t.$$

By integrating both sides we obtain

8/ See Section 6.8, Appendix I.

(4.19) $$y = y_0 \cdot e^{-t^2/2}$$

which is the normal density function. The range is taken to be $(-\infty, \infty)$.

TRANSITIONAL TYPE III.

$$\theta = 4 \quad \text{and} \quad \alpha_3^2 \neq 0$$

These conditions lead us to obtain the values of a and b's in (2.8) as follows:

$$a = -\alpha_3/2$$

$$b_0 = 1$$

$$b_1 = \alpha_3/2$$

and $$b_2 = 0 .$$

Then, by substituting these specific values in (2.1), it becomes

(4.20) $$\frac{1}{y} \cdot \frac{dy}{dt} = -\frac{\alpha_3 + 2t}{2 + \alpha_3 t}$$

Let $A = 2/\alpha_3$. Then we can write (4.20) in the form

(4.21) $$\frac{1}{y} \cdot \frac{dy}{dt} = -\frac{1 + At}{A + t} = -A + \frac{A^2 - 1}{A + t}$$

By integrating both sides and taking anti-logarithms of the two members,

(4.22) $$y = y_0 (t + A)^{A^2 - 1} \cdot e^{-At}$$

Since we are considering only the case $\alpha_3 > 0$, then the range of the variable t is taken to be $(-A, \infty)$.

It is readily verified that, since $A^2 - 1 < -1$, then the conditions at the end of Chapter II are satisfied. For $A^2 > 1$ (i.e., for $\alpha_3^2 < 4$), the curve is bell-shaped; and for $A^2 < 1$ or $\alpha_3^2 > 4$, it is J-shaped with an infinite ordinate at $t = -A$.

TRANSITIONAL TYPE II.

$$\alpha_3^2 = 0$$

$$0 < \theta < 4$$

and $$\theta \neq 1 .$$

In this case, (4.2) and (4.3) can be reduced to

(4.23) $$r_1 = -\frac{\sqrt{D}}{2-\sqrt{\theta}} \quad \text{and} \quad r_2 = \frac{\sqrt{D}}{2-\sqrt{\theta}}$$

where $D = (-\theta + 4) > 0$. Also (4.6) and (4.7) can be reduced to

(4.24) $$m_1 = m_2 = -2\left[1 - \frac{1}{2-\sqrt{\theta}} \right]$$

which is greater or less than zero according as $\theta \gtrless 1$.

Then the frequency function is a special case of Type I. By setting $-r_1 = r_2 = S$ which is always greater than zero, and $m_1 = m_2 = M$ which can be either positive or negative, we can write (4. 5) in the form

$$y = y_0(S^2 - t^2)^M \tag{4. 25}$$

which is the probability density function of the transitional Type II. Note that the frequency curve of this type is symmetrical about the mean because $\alpha_3 = 0$.

When $0 < \theta < 1$, $-1 < M < 0$ and the curve is U-shaped with domain $(-S,S)$. When $1 < \theta < 4$, $0 < M$ and the curve is bell-shaped with the same domain.

Since $M > -1$ for $0 < \theta < 4$, moments of all order exist for the density function.

TRANSITIONAL TYPE V.

When $\alpha_3 \neq 0$ and $D = 0$ or $\alpha_3^2 = \theta - 4$, $\theta > 4$ and $r_1 = r_2 = -|r|$. Since the mεs in (4. 6) and (4. 7) are not finite, we have to return to the direct integration of (2. 0). Since the denominator in the right member has equal zeros, (2. 1) may be written

$$\frac{1}{y} \cdot \frac{dy}{dt} = \frac{a - t}{b_2(t + r)^2} = \frac{a + r}{b_2(t + r)^2} - \frac{t + r}{b_2(t + r)^2} \tag{4. 26}$$

where $r = \dfrac{\alpha_3}{\sqrt{\theta-2}}$ which is always positive, $a = -\dfrac{\alpha_3}{2(\sqrt{\theta-1})}$ and $b_2 = \dfrac{\sqrt{\theta-2}}{4(\sqrt{\theta-1})}$.

Let $m = -1 + 1/2b_2$ or $m = \frac{\sqrt{\theta}}{\sqrt{\theta-2}}$, then

$$\frac{a+r}{b_2} = 2mr$$

Substituting the values of b_2 and $\frac{a+r}{b_2}$ in terms of m and r in (4.26), then it becomes

$$\frac{1}{y} \cdot \frac{dy}{dt} = \frac{2rm}{(t+r)^2} - \frac{2(m+1)}{(t+r)}$$

Integrating both sides, we get

$$(4.27) \qquad y = y_0(t+r)^{-2(m+1)} \cdot e^{-2mr/(t+r)}$$

which is the probability density function of this type.

Note that r has the same sign as α_3. The range is taken to be $(-\infty, r)$ or $(-r, \infty)$ which depends on whether α_3 is negative or positive. The curve is always bell-shaped.

In order for the n^{th} moment to exist, we must have as usual $4 < \theta < 7.84$, which is leading to the same conclusion as in Type IV or Type VI case. (See Section 6.8, Appendix I.)

TRANSITIONAL TYPE VII.

$$\alpha_3 = 0 \quad \text{and} \quad \theta > 4 .$$

This may be regarded as a special case of Type IV. Under the above conditions, we obtain

$$r = 0, \qquad s = \frac{\sqrt{\theta - 4}}{\sqrt{\theta} - 2} > 0$$

and

$$n = 0, \qquad m = 2\left[\, 1 + \frac{1}{\sqrt{\theta} - 2} \,\right] > 0$$

Then the frequency function (4. 14) can be written in the form

$$y = y_0(t^2 + s^2)^{-m} \tag{4.28}$$

The range of variable t is $(-\infty, \infty)$.

TRANSITIONAL TYPE VIII.

$$\alpha_3 \neq 0$$

$$\theta < 1$$

and

$$\alpha_3^2\,(3\sqrt{\theta} - 2) = 4(2+\sqrt{\theta})(\sqrt{\theta} - 1)^2$$

The function is a special case of Type I in which $m_1 < 0$ and $m_2 = 0$. Let $m_1 = 2m$, $m > 0$. Then the frequency function becomes

$$y = y_0(t - r_1)^{-2m} \tag{4.29}$$

where $m = 1/2 - \dfrac{\sqrt{\theta}}{2(2-\sqrt{\theta})}\,(1 - \dfrac{\alpha_3}{\sqrt{D}})$.

The range is (r_1, r_2). The curve is J-shaped with an infinite ordinate at $t = r_1$ and a finite one at $t = r_2$.

TRANSITIONAL TYPE IX.

$$\alpha_3 \neq 0$$

$$1 < \theta < 4$$

and
$$\alpha_3^2 (3\sqrt{\theta}-2) = 4(2+\sqrt{\theta})(\sqrt{\theta}-1)^2$$

We have another special case of Type I function in which $m_1 = 0$ and $m_2 = -2m > 0$. The density function is in the form

$$y = y_0(r_2 - t)^{-2m} \tag{4.30}$$

where $m = 1/2 - \frac{\sqrt{\theta}}{2(2-\sqrt{\theta})} \left(1 + \frac{\alpha_3}{\sqrt{D}}\right)$

The range is still taken to be (r_1, r_2). The curve is J-shaped with finite ordinates at both $t = r_1$ and $t = r_2$.

TRANSITIONAL TYPE X.

The frequency function is a special case of Type III, which has the density function as in (4.22). For $A^2 = 1$ or $\alpha_3^2 = 4$ and $\theta = 4$, (4.22) can be reduced to [9/]

[9/] See Section 6.9, Appendix I.

(4.31) $$y = y_0 \cdot e^{-t}$$

which represents a J-shaped curve with a finite ordinate at $t = -1$. The range is $(-1, \infty)$.

On the chart, the points corresponding to Type III function fall on the line $\theta = 4$, the Type X function being represented by a single point $(4,4)$ on this line.

TRANSITIONAL TYPE XI.

$$\alpha_3 \neq 0$$

$$4 < \theta < 7.84, \quad \text{(See Section 6.8)}$$

and $$\alpha_3^2 (3\sqrt{\theta} - 2) = 4(2 + \sqrt{\theta})(\sqrt{\theta} - 1)^2$$

The function is a special Type IV in which $m_1 = 0$ and $m_2 = -2m < 0$, and we may write (4.15) in the form

(4.32) $$y = y_0(t - r_2)^{-2m}$$

with the range still being (r_1, ∞). The curve is J-shaped with finite ordinate at $t = r_1$.

TRANSITIONAL TYPE XII.

$$\alpha_3 \neq 0$$

and $$\theta = 1 .$$

If $\theta = 1$, the four linear equations (2.4), (2.5), (2.6) and (2.7) from which the values of a, b_0, b_1 and b_2 in (2.8) are derived, are inconsistent. We can however set the values of (2.8) in the differential equation (2.1) and from its limiting form as $\theta \rightarrow 1$ derive the function appropriate to this case.

We obtain

$$(4.33) \qquad \frac{1}{y} \cdot \frac{dy}{dt} = \frac{-2a_3 - 4(\sqrt{\theta} - 1)t}{(\sqrt{\theta} + 2) + 2a_3 t + (\sqrt{\theta} - 2)t^2}$$

and if $\theta = 1$, (4.33) becomes

$$(4.34) \qquad \frac{1}{y} \cdot \frac{dy}{dt} = \frac{-2a_3}{3 + 2a_3 t - t^2} = \frac{2a_3}{(t - r_1)(t - r_2)}$$

where $r_1 = a_3 - \sqrt{a_3^2 + 3}$ and $r_2 = a_3 + \sqrt{a_3^2 + 3}$.

Then,

$$\frac{1}{y} \cdot \frac{dy}{dt} = \frac{-2a_3}{r_2 - r_1}\left[\frac{1}{t - r_1} - \frac{1}{t - r_2}\right] = \frac{-a_3}{\sqrt{a_3^2 + 3}}\left[\frac{1}{t - r_1} - \frac{1}{t - r_2}\right]$$

On integration,

$$(4.35) \qquad y = y_0 (t - r_1)^{m_1} (t - r_2)^{m_2}$$

where $-m_1 = m_2 = \dfrac{a_3}{\sqrt{a_3^2 + 3}}$.

We observe that $(\alpha_3 > 0)$

$$r_2 > 0 > r_1$$

(4.36) $$|r_2| > |r_1|$$

$$m_2 = -m_1 > 0 .$$

Taking the range to be (r_1, r_2), we write (4.35) as

(4.37) $$y = y_0 \left[\frac{r_2 - t}{t - r_1}\right]^{m_2}$$

The curve is J-shaped with an infinite ordinate at $t = r_1$. Finally, we note that for $\alpha_3 = 0$, the special case of Type XII arises. We have $m_1 = m_2 = 0$. And the reduced form of (4.37) is

(4.38) $$y = y_0 \quad \text{(a constant)}$$

thus including the rectangular (or uniform) distribution function among the Pearson System of Frequency Functions.

Among the discussions, a system of criteria for the various types of functions has been set up in terms of θ and α_3, in which these parameters may be readily calculated.

The (θ, α_3^2)-chart which makes these criteria visual is comparatively simple to construct and is striking in appearance. One can conclude that, besides the lines

$$\theta = 0$$

$$\theta = 1$$

$$\theta = 4$$

$$\theta = 7.84$$

$$\alpha_3 = 0$$

and $$\alpha_3^2 = \theta - 4 ,$$

it contains only the curve

$$\alpha_3^2 (3\sqrt{\theta} - 2) = 4(\sqrt{\theta} + 2)(\sqrt{\theta} - 1)^2$$

on which the points corresponding to the functions of Types VIII, IX, X and XI are found.

CHAPTER V

NUMERICAL EXAMPLE

Let $\{X_i\}$ be a set of data for which we want to determine the density function of the variable X. The values of X_i's are:

9.1	5.2	4.5	6.4	4.5	5.1	6.0
5.5	5.6	6.9	4.6	4.9	6.5	4.8
5.6	8.0	5.4	7.4	4.0	5.3	4.6
5.1						

At first we need to know a_3 and a_4 which are defined in Chapter I. Define

$$\beta_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k \tag{5.1}$$

and we have

$$t_i = \frac{X_i - \beta_1}{s}$$

where $s^2 = \beta_2 - \beta_1^2$. Then $a_1 = 0$, $a_2 = 1$ and

$$a_3 = \frac{1}{s^3}\left[\beta_3 - 3\beta_1\beta_2 + 2\beta_1^3\right] \tag{5.2}$$

$$a_4 = \frac{1}{s^4}\left[\beta_4 - 4\beta_3\beta_1 + 6\beta_2\beta_1^2 - 3\beta_1^4\right] \tag{5.3}$$

From the use of (5. 1), (5. 2) and (5. 3) with the above data, we have the following table:

k	1	2	3	4
β_k	5. 6818	33.7973	211. 4134	1393.7320
α_k	0. 0000	1. 0000	1. 1666	3. 8342

From the value of θ in (2. 9)

$$\theta = 36 \left(\frac{\alpha_4 - \alpha_3^2 - 1}{\alpha_4 + 3} \right)^2$$

we can have $\theta = 1.6693$, and $\alpha_3^2 = (1.1666)^2 = 1.3609$.

We find that the point (1. 6693, 1. 3609) falls on the region of the chart in Chapter III which corresponds to the Main Type I density function.

$$y = y_0 (t - r_1)^{m_1} (r_2 - t)^{m_2}$$

the constants r_1, r_2, m_1 and m_2 can be computed from (4. 2), (4. 3), (4. 6) and (4. 7), respectively. Then

$$r_1 = -1.0661$$

$$r_2 = 4.3616$$

$$m_1 = -0.2831$$

$$m_2 = 1.9328$$

And the density constant y_0 can be calculated by the table in Appendix II together with the table of Beta-function,

$$y_0 = 0.03631$$

Finally, the density function which fits these data is in the form

$$f(t) = 0.03631(t + 1.0661)^{-0.2831}(4.3616-t)^{1.9328}$$

which can be transformed to the density function $g(X)$ of the original variable X.

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APPENDICES

APPENDIX I

PROOFS AND VERIFICATIONS OF SOME RELATIONS

Contents of Appendix I

6.1 TO SHOW THAT $0 \leq \theta < 36$.

By using the fact that

$$\int_r^s (t^2 + \lambda t)^2 \cdot f(t)dt = a_4 + 2\lambda a_3 + \lambda^2$$

which is always a non-negative quantity since $f(t) \geq 0$ and $r \leq t \leq s$ for any real number λ, then

$$a_4 + 2a_3 + \lambda^2 \geq 0 .$$

This requires that,

$$4a_3^2 - 4a_4 \leq 0,$$

(6.1.1) $$a_4 \geq a_3^2$$

Consider the expression

$$\phi = \frac{a_4 - a_3^2 - 1}{a_4 + 3}$$

which can be written in the form,

(6.1.2) $$-1 + \frac{2a_4 - a_3^2 + 2}{a_4 + 3} = \phi = 1 - \frac{a_3^2 + 4}{a_4 + 3}$$

From the property of (6.1.1), we see that both of the fractions in

(6. 1. 2) are always positive.

Hence,

(6. 1. 3) $$-1 < \phi = \frac{\alpha_4 - \alpha_3^2 - 1}{\alpha_4 + 3} < 1$$

But from (2. 9)

$$\theta = 36 \left(\frac{\alpha_4 - \alpha_3^2 - 1}{\alpha_4 + 3}\right)^2 = 36\phi^2$$

Then, it follows that

$$0 < \theta < 36. \qquad \text{by (6. 1. 3).}$$

6. 2 THE DISCUSSION OF THE RESTRICTION (2. 2)

$$(b_0 + b_1 t + b_2 t^2) t^n \cdot f(t) \Big]_{t=r}^{t=s} = 0$$

where b's are real constants, f(t) is a density function, and r, s are the extreme values of the variable t. Note that r and s can be the numbers which belong to the set of extended real number and n is a finite positive integer. Let

$$R(t) = (b_0 + b_1 t + b_1 t^2) t^n \cdot f(t) .$$

The problem is to show that $R(r) = R(s) = 0$.

The expression $R(t)$ can be considered at the extreme values r, s by the following three cases:

Case I. We had established that we can put the quadratic expression in t, $b_0 + b_1 t + b_2 t^2$, in the form of $b_2(t-r_1)(t-r_2)$. If r_1 and r_2 are both finite, then $t^n \cdot f(t)$ is bounded. And it is readily seen that $R(t)$ vanishes at both extreme limits of the variable t(i. e., at $t = r_1$ and $t = r_2$), or $R(r_1) = R(r_2) = 0$.

Case II. In the case that the extreme limits of variable t are both unbounded (i. e., the range of t is $(-\infty, +\infty)$), $f(t)$ has to be in the form $A(t) \cdot e^{-p(t)}$, and

$$R(t) = (b_0 + b_1 t + b_2 t^2) t^n A(t) e^{-p(t)} = P_k(t) e^{-p(t)}$$

where $p(t)$ is a quadratic polynomial in t and $P_k(t)$ is a polynomial in t of degree k. The exponential is the predominant factor in the product when t is sufficiently large. Since $\lim_{|t| \to \infty} e^{-p(t)} = 0$, $\lim_{|t| \to \infty} R(t) = 0$.

Case III. This is the case which is the combination of the first two. In some types of density functions, the extreme values of t may not be both finite or infinite (i. e., t may have (A, ∞) as its range). We observe that $(b_0 + b_1 t + b_2 t^2)$ vahishes at $t = A$, where $t^n f(t)$ is finite at the same point. Note that A is a finite value. At the same time $\lim_{t \to \infty} R(t) = 0$ by generosity of case two.

From these above three possible cases of the extreme values of t, (r, s), we can conclude that

$$R(s) = R(r) = 0$$

or

$$(b_0 + b_1 t + b_2 t^2) t^n \cdot f(t) \Big]_{t=r}^{t=s} = 0 ,$$

where r and s are the elements of the set of extended real numbers.

6.3 TO OBTAIN THE RESULTS OF (4.6) and (4.7)

We can write (2.1) in the form of (4.2)

$$\frac{1}{y} \cdot \frac{dy}{dt} = \frac{a - t}{b_2 (t - r_1)(t - r_2)} \tag{6.3.1}$$

By equating the right member to the right of (4.4), we have

$$\frac{a - t}{b_2 (t - r_1)(t - r_2)} = \frac{m_1}{t - r_1} + \frac{m_2}{t - r_2}$$

or

$$\frac{a - t}{b_2} = m_1 (t - r_2) + m_2 (t - r_1)$$

which leads us to

$$m_1 = (a - r_1)/b_2(r_1 - r_2)$$

(6.32)

$$m_2 = (a - r_2)/b_2(r_2 - r_1)$$

Substituting for a, b_2 from (2.8) and r_1, r_2 from (4.2), (4.3) we obtain

$$m_1 = -2\left[1 + \frac{1}{\sqrt{\theta - 2}}\right] + \frac{\sqrt{\theta}\,\alpha_3}{\sqrt{\theta - 2}} \cdot \frac{1}{\sqrt{D}}$$

which is the result of (4.6).

By the same kind of substitution and algebraic manipulation we can have

$$m_2 = -2\left[1 + \frac{1}{\sqrt{\theta - 2}}\right] - \frac{\sqrt{\theta}\,\alpha_3}{\sqrt{\theta - 2}} \cdot \frac{1}{\sqrt{D}}$$

which is (4.7).

6.4 THE NECESSARY CONDITIONS FOR THE FREQUENCY FUNCTION OF (4.9) TO HAVE FINITE AREA AND MOMENTS OF ALL ORDERS

(4.9) $$y = y_0(t - r_1)^{m_1} \cdot (r_2 - t)^{m_2}$$

where (r_1, r_2) is the range of the variable t.

The area under this frequency curve can be evaluated by the definite integral

(6.4.1) $$B = \int_{r_1}^{r_2} y_0(t - r_1)^{m_1} \cdot (r_2 - t)^{m_2} dt$$

which can be separated into the sum of two integrals

$$(6.4.2) \qquad B = \int_{r_1}^{c} y_0 (t-r_1)^{m_1} (r_2-t)^{m_2} dt + \int_{c}^{r_2} y_0 (t-r_1)^{m_1} (r_2-t)^{m_2} dt$$

where $r_1 < c < r_2$, (since $r_1 \neq r_2$).

Consider the first integral of (6.4.2)

$$(6.4.3) \qquad C = \int_{r_1}^{c} y_0 (t-r_1)^{m_1} (r_2-t)^{m_2} dt .$$

Since $y_0 (r_2-t)^{m_2}$ is bounded for all t in $[r_1, c]$, then C is finite only when the improper integral

$$C_1 = \int_{r_1}^{c} (t-r_1)^{m_1} dt$$

is finite. This implies that m_1 has to be greater than minus one. Let

$$D = \int_{c}^{r_2} y_0 (t-r_1)^{m_1} (r_2-t)^{m_2} dt$$

Similarly $y_0 (t-r_1)^{m_1}$ is bounded for $c \leq t \leq r_2$. In order for D to be finite, m_2 has to be greater than minus one.

We see that the value of B in (6.4.1) is finite if C and D are both finite. Finally, $m_1 > -1$ and $m_2 > -1$ are the necessary conditions for obtaining finite area under the frequency function (4.9).

Since the range of the variable t is finite, then t^n is also bounded for any finite positive number n. We can say that whenever the above conditions hold, the integral

$$\int_{r_1}^{r_2} y_0(t-r_1)^{m_1}(r_2-t)^{m_2} \cdot t^n dt$$

always exists. Then the conditions $m_1 > -1$ and $m_2 > -1$ imply moments of all orders exist.

6.5 THE EQUATION (4.8) INDICATES THE BOUNDARIES OF THE MAIN TYPES I, IV AND VI

It is already seen from (6.4) that $m_1 + 1 > 0$ and $m_2+1 > 0$ are the necessary conditions for a frequency function to have finite area and moments of all orders. Consider the special case that $m_1 = 0$ or $m_2 = 0$. From (4.6) and (4.7) the values of m_1 and m_2 are

(6.5.1) $$-2\left[1 + \frac{1}{\sqrt{\theta}-2}\right] \pm \frac{\sqrt{\theta}\,\alpha_3}{\sqrt{\theta}-2} \cdot \frac{1}{\sqrt{D}}$$

By equating (6.5.1) to zero we get,

$$-2\left[1+\frac{1}{\sqrt{\theta}-2}\right] = \mp \frac{\sqrt{\theta}\,\alpha_3}{\sqrt{\theta}-2} \cdot \frac{1}{\sqrt{D}}$$

$$4(\sqrt{\theta}-1)^2 = \frac{\theta\,\alpha_3^2}{D}$$

Since $D = \alpha_3^2 - \theta + 4$, then

$$4(\sqrt{\theta}-1)^2(\alpha_3^2-\theta+4) = \theta\,\alpha_3^2$$

which implies

$$\text{(6.5.2)} \qquad \alpha_3^2\,(3\sqrt{\theta}-2) = 4(\sqrt{\theta}-1)^2(\sqrt{\theta}-2)$$

which is the relation between θ and α_3^2 found in (4.8).

6.6 VERIFICATION OF COMPLEX IDENTITY (4.13 A)

$$\text{(4.13 A)} \qquad \left[\frac{a-bi}{a+bi}\right]^{ci} \qquad e^{2c.\tan^{-1}(b/a)}$$

where a, b, c are real and $i = \sqrt{-1}$.

Consider the expression $\dfrac{a-bi}{a+bi}$,

$$\frac{a-bi}{a+bi} = \frac{a^2-b^2-2abi}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2} + \frac{-2abi}{a^2+b^2} = e^{-pi}$$

where $p = \tan^{-1}\left[\dfrac{2ab}{a^2-b^2}\right] = 2\cdot\tan^{-1}(b/a)$.

Then,

$$\left[\frac{a-bi}{a+bi}\right]^{ci} = \left[e^{-pi}\right]^{ci} = e^{pc} = e^{2c\cdot\tan^{-1}(b/a)}$$

which is the complete verification of (4.13 A). And furthermore,

$$e^{2c\cdot\tan^{-1}(b/a)} = e^{2c[\pi/2-\tan^{-1}(a/b)]} = e^{c}\cdot e^{-2c\tan^{-1}(a/b)}.$$

6.7 VERIFICATION OF (4.16)

(4.16) $$0 < a-r = b_2(r_2 - r_1)m_2$$

This equation can be obtained immediately from (6.3.2). Since in this case, $m_2 < 0$, $r_1 < r_2 < 0$ and $0 < b_2$, then each side of (4.16) is always positive.

6.8 DISCUSSION ON "HETEROTYPIC AREA" OF DENSITY FUNCTION TYPES IV, V AND VI

Case I, Type IV.

(4.14) $$y = y_0\cdot e^{n\pi}[(t+r)^2+s^2]^{-m}\cdot e^{-2n\cdot\tan^{-1}\left(\frac{t+r}{s}\right)}$$

where

$$n = \frac{\sqrt{\theta}\,\alpha_3\sqrt{-D}}{D(\sqrt{\theta}-2)} \qquad m = 2\left[1+\frac{1}{\sqrt{\theta}-2}\right]$$

$$r = \frac{\alpha_3}{\sqrt{\theta}-2} \qquad s = \frac{\sqrt{-D}}{\sqrt{\theta}-2}$$

and $D = \alpha_3^2 - \theta + 4$.

The exponential is bounded by $e^{n\pi}$ in $[-\infty,\infty]$ and the term of highest degree in $t^k[(t+r)^2+s^2]^{-m}$ is t^{k-2m}. Hence,

$$a_k = \int_{-\infty}^{\infty} t^k \cdot f(t)dt$$

does not exist unless $k - 2m < -1$ or $2m > k + 1$. If a_9 does not exist while a_8 does exist, let $k = 8$, then

$$2m > 9 \qquad \text{or} \quad 1 + \frac{1}{\sqrt{\theta} - 2} < 9/4$$

which implies $\theta < 7.84$.

Case II, Type V.

$$y = y_0 \cdot (t + r)^{-2(m+1)} \cdot e^{-2mr/(t+r)} \tag{4.27}$$

where $r = \dfrac{a_3}{\sqrt{\theta} - 2}$ and $m = \dfrac{\sqrt{\theta}}{\sqrt{\theta} - 2}$.

The exponential term tends to one as t tends to infinity. Hence, $a_k = \int_{-\infty}^{\infty} t^k \cdot f(t)dt$, is finite only if

$$k - 2(m+1) < -1$$

or

$$2m > k - 1 .$$

By substituting the value of m in the above inequality, and let $k = 8$, we get

$$\frac{2\sqrt{\theta}}{\sqrt{\theta} - 2} > 7$$

or

$$5\sqrt{\theta} < 14$$

which implies $\theta < 7.84$.

Case III, Type VI; J-shaped.

(4.18) $$y = y_0 \cdot z^{m_2} \cdot (z-r)^{m_1}$$

where $r < z < \infty$, $z = t - r_2$

$$a_k(z) = \int_r^\infty z^k f(t)dz = y_0 \int_r^\infty (z^{k+m_1+m_2} + \text{terms of lower degree})dz.$$

Hence $a_k(z)$ does exist unless $k + m_1 + m_2 < -1$. Since

$m_1 + m_2 = -\frac{4(\sqrt{\theta}-1)}{\sqrt{\theta}-2}$, then

$$-\frac{4(\sqrt{\theta}-1)}{\sqrt{\theta}-2} < -k-1$$

or $$4(\sqrt{\theta}-1) > (k+1)(\sqrt{\theta}-2)$$

which leads to the same condition, $\theta < 7.84$.

Finally, from these three cases, we conclude that for the 8^{th} moment to exist we must have $\theta < 7.84$. And the area where $\theta \geq 7.84$, we call the "Heterotypic Area" which is next to the regions corresponding to the Main Types IV, V and VI.

6.9 REDUCTION OF (4.22) TO OBTAIN (4.31)

(4.22) $$y = y_0 \cdot (t+A)^{A^2-1} \cdot e^{-At}$$

Under the condition $A = 1$ and $(t + A)$ is finite, the frequency function (4.22) can be reduced to

$$y = y_0 \cdot e^{-t} . \tag{4.31}$$

APPENDIX II

VALUE AND EVALUATION OF DENSITY CONSTANTS

The values of the density constants (y_0) which correspond to each type of density functions in the table of Chapter III are listed consecutively by types as follows:

Type no.	Density constants (y_0)
I	$[\beta(m_1+1, m_2+1)(r_2-r_1)^{m_1+m_2+1}]^{-1}$
IV	$S^{2m-1}[G(2m-2, 2n)]^{-1}$ [10/]
VI	$[(r_1-r_2)^{m_1+m_2+1}\beta(m_1+1, -m_1-m_2-1)]^{-1}$
"Normal"	$(2\pi)^{-1/2}$
III	$A^{A^2}e^{-A^2}[\Gamma(A^2)]^{-1}$
II	$[(2S)^{2M+1}\beta(M+1, M+1)]$
V	$(2rm)^{2m+1}[\Gamma(2m+1)]^{-1}$
VII	$S^{2m-1}\Gamma(m)[\sqrt{2\pi}\,\Gamma(\frac{2m-1}{2})]^{-1}$

10/ The value of G function can be found from the Table for Statisticians and Biometricians, Cambridge Univ. Press, Part I, 2nd edition (1924), p. lxxxi.

Type no.	Density constants (y_0)
IX	$(1-2m)(r_2-r_1)^{2m-1}$
X	e^{-1}
XI	$(2m-1)(r_2-r_1)^{1-2m}$
XII	$[(r_2-r_1)\beta(1-m_2, 1+m_2)]^{-1}$

These constants are not all verified here. We will pick Type III and Type XII as the examples of evaluation of y_0.

TYPE III.

(4.22) $$y = y_0(t+A)^{A^2-1} \cdot e^{-At}$$

$$t \varepsilon (-A,\infty)$$

Then,

(7.1) $$\frac{1}{y_0} = \int_{-A}^{\infty} (t+A)^{A^2-1} \cdot e^{-At} dt$$

Let $z = A(t+A)$ or $t = (z-A^2)/A$.

By substituting into (7.1), we have

(7.2) $$\frac{1}{y_0} = \int_0^{\infty} (z/A)^{A^2-1} \cdot e^{-z+A^2} \cdot (1/A)dz = e^{A^2} A^{-A^2} \int_0^{\infty} z^{A^2-1} \cdot e^{-z} dz$$
$$= e^{A^2} \cdot A^{A^2} \cdot \Gamma(A^2)$$

where $\Gamma(n) = \int_0^\infty z^{n-1} \cdot e^{-z} dz$, the gamma function.

Finally, $$y_0 = \frac{A^{A^2} \cdot e^{-A^2}}{\Gamma(A^2)}$$

TYPE XII.

(4.37) $$y = y_0 \left[\frac{r_2 - t}{t - r_1} \right]^{m_2}$$

Then,

(7.2) $$\frac{1}{y_0} = \int_{r_1}^{r_2} (r_2 - t)^{m_2} \cdot (t - r_1)^{-m_2} dt$$

Let $t - r_1 = (r_2 - r_1) \cdot z$, therefore

$$r_2 - t = (r_2 - r_1) - (r_2 - r_1) \cdot z = (r_2 - r_1) \cdot (1 - z)$$

(7.3) $$\frac{1}{y_0} = \int_0^1 (1-z)^{m_2} \cdot z^{-m_2} (r_2 - r_1) dz = (r_2 - r_1) \cdot \beta(1 - m_2, 1 + m_2)$$

where $\beta(m, n) = \beta(n, m) = \int_0^1 z^m (1-z)^n dz$, the beta-function.