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Quantum physics in the Copenhagen interpretation places an unsatisfying divide between the quantum and classical worlds. Decoherence – the destruction of superposition states of the system – helps us understand how the quantum transitions to the classical. Quantum Darwinism builds on decoherence to understand how information about the system is deposited in the environment. This gives a framework to describe the emergence of objective, classical reality from the fundamentally unpredictable quantum domain. Considering a model of a qubit interacting with a symmetric spin environment, we study how information about the system is transferred into the environment when the system has intrinsic dynamics. That is, we examine a model beyond "pure decoherence". We develop a computational technique to compute the mutual information between the system and a fragment of the environment as a function of the strength of the system's Hamiltonian. The intrinsic dynamics of the system influences the proliferation of information. However, there is always a redundant transfer of information for a weak system Hamiltonian. Only when the system evolves so rapidly that the environment can not effectively monitor its state does redundancy vanish.

Redundant Information in a Spin System Beyond Pure Decoherence

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Andrew M. Svesko, Author

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Introduction

Quantum mechanics was initially conceived as a theory of the microscopic world. So far quantum physics has been successful at doing just this, correctly predicting physical phenomena on the atomic scale where other theories have failed. Despite this success, there is an elephant in the room: How does one move from the quantum world to the classical, objective world? Put another way, classical physics primarily deals with macroscopic objects, but the macroscopic is composed of microscopic subsystems where quantum mechanics has been shown to reign surpreme. Therefore quantum mechanics should be able to describe our classical world, yet the bizarre properties of quantum states are not observed, e.g., the infamous Schrödinger's cat. This problem has plagued physicists almost since the conception of quantum mechanics, despite its experimental triumphs. Historically the strategy has been to ignore this issue, relying on the Copenhagen interpretation, yielding a fissure between the quantum and classical worlds [1–4].

More recently, this question has been addressed with the decoherence paradigm. Decoherence is the study of the loss of superpositions in a system due to the environment, and plays a role in resolving the transition from the quantum to the classical. An extension to the theory of decoherence is *Quantum Darwinism*, an information theoretic approach resolving the quantum-classical divide [5, 6]. Specifically, the objective classical world arises from the quantum substrate through the redundant propagation of information [7]. By "objective" we mean that the state of a system can be independently determined by many observers who intercept individual fragments of the environment [8]. The information an observer acquires is regarding the *pointer states* of the system, the indirect acquisition of which does not perturb the subsequent inference by other observers. It is in this sense that Quantum Darwinism takes its name: Certain states, the pointer states, survive the interaction with the environment and proliferate by replicating information theoretic copies of themselves in the environment.

Quantum Darwinism recognizes that the environment acts as a communication channel [9–11] – it is capable of transmitting information about the state of the system to observers. Recent studies

have shown that the capacity of sending such information is determined by the environment's initial state [12–14]. In particular, previous work examined how purely decohering environments store and transmit information to the observers [15]. In these types of environments, information is always redundantly proliferated except for situations of measure zero [16]. Here we examine how the inclusion of a system self-Hamiltonian alters the acquisition and transmission of quantum and classical information. Some prior results examined the quantum Brownian motion model, which has intrinsic system dynamics [17, 18]. We consider a different model – a qubit in a symmetric spin environment – and quantify the redundancy of information deposited into such an environment.

Before we get to the main results, we give a brief introduction to decoherence, information, and Quantum Darwinism. Topics include entropy, mutual information, redundancy, amplification, quantum discord, and the Holevo quantity; all meant to further clarify what is meant by decoherence and the quantum-to-classical transition. The reader who is familiar with these ideas may skip these preliminary sections.

Chapter 1

Fundamentals

1.1 The Quantum-Classical Divide

The conventional and first approach to quantum mechanics is based on the so-called Copenhagen interpretation. Let's start by observing the common postulates [19]:

I *The state of the particle is represented by a vector* $|\psi(t)\rangle$ *in a Hilbert space* $\mathcal{H}$.

II *Quantum time evolution is unitary, i.e., state vector* $|\psi(t)\rangle$ *obeys the Schrödinger equation*

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{1.1}$$

where H *is the quantum Hamiltonian operator.*

III *Immediate repetition of a measurement yields the same outcome.*

IV *Measurement outcomes are restricted to an orthonormal set* $\{|s_k\rangle\}$ *of eigenstates of the measured observable (the 'collapse postulate')*

V *The probability of finding a given outcome is*

$$p_k = |\langle s_k|\psi\rangle|^2 \tag{1.2}$$

where $|\psi\rangle$ is the preexisting state of the system.

Based on these postulates, and their common variants, we realize a fundamental difference between the procedures of classical and quantum physics: In quantum mechanics we place a distinction between a measuring apparatus and the objects being measured [4]. As Bohr noted, and is witnessed by the postulates, the Copenhagen interpretation says that a measurement changes the system, however this "collapse" cannot be described by quantum evolution itself [20]. This is obviously unsatisfactory. Afterall, if quantum mechanics is supposed to describe the physical world, then it should be able to describe a physicist's measuring device, and with it, the measurement process. The "collapse" postulate (IV) is thus a problem. Aside from the fact that the (IV) is inconsistent with (II) – the process of measurement is not described by unitary evolution – quantum mechanics allows for a particle to be described by a superposition of states; classical particles cannot be described in this way. In quantum mechanics the measurement process somehow selects a definite state from the plethora of superposition states, a state which may be analyzed in the classical sense. The Copenhagen interpretation is unable to supply an explanation for this mechanism, leading to a sharp boundary between the quantum and classical worlds. The issue arises, primarily, because the definition of "measurement" is unclear in the Copenhagen interpretation [20]: Who or what is required to induce a wavefunction collapse? If quantum mechanics, in the standard framework, does not entirely describe the transition from the quantum to the classical, then we need to know where to draw the line in its validity.

Indeed, there are some ideas that require a restructuring of the foundations of quantum mechanics, however we will not consider such alternative theories [21,22] here. Rather we assume no changes to the foundations of quantum mechanics, allowing us to focus on two broad classes of interpretations: the Many Worlds Interpretation – in which we take the view that the state vector yields a complete description of system under investigation – or the Copenhagen interpretation – in which the state vector gives a probabilistic description of the system without any explanation on how classical reality arises from the quantum world.

The crucial difference between these interpretations relies on how each views the measurement process, and this can be exemplified by a famous thought experiment [23]. Suppose a cat is placed in a closed box along with a radioactive nucleus, a Geiger counter to detect the nuclear decay, and a capsule of poison which is released when the counter detects the decay. After one half-life the state of the system (cat and poison) is in a superposition: $|\psi\rangle \propto |\psi\rangle_{alive} + |\psi\rangle_{dead}$. If we take the Copenhagen interpretation seriously, when the system is observed, either by an external observer

or the cat itself, the state of system collapses to either a dead cat or a live cat, each with its own probability given by the Born rule (postulate (V)).

Alternatively, if we take the many-worlds interpretation seriously, the state of system remains in a superposition state of a dead cat with the observer seeing a dead cat, and one with an alive cat with the observer detecting a live cat. That is, different components of the state of the system become associated with different components of the state of the measuring apparatus (observer), made rigorous through the *von Neumann measurment scheme*, leading to a branching effect; the world splits where each branch corresponds to each possible outcome of a measurement. In this way the state collapse doesn't happen at all, it only appears to do so. To summarize, while the Copenhagen intepretation lends no insight about how the collapse occurs, many worlds proposes the history of the universe continually splits into an increasingly large number of branches.

Simply put, the quantum world is full of superposition states, yet we do not experience such superpositions classically, so how does the classical world emerge from the quantum domain? In the Copenhagen intepretation the answer relies on the collapse postulate and therefore is unable to explain the emergence of classicality; classical physics does not arise from quantum mechanics [24]. The many worlds and other relative-state interpretations, on the other hand, have proposed a very different solution, one which moves beyond the wavefunction collapse.

1.2 Decoherence and the Emergence of Classicality

Broadly, decoherence starts by assuming that the Universe is composed of a system interacting with its environment [5]. Thus, the system and environment (e.g. an object in a thermal bath) become correlated – entangled – with each other, leading to non-unitary evolution for the isolated system after the interaction has taken place (the combined system and environment are still governed by unitary evolution, however). This process may be understood in a number of ways, all equivalent, and all indicating the suppression of quantum coherence (the interference fundamentally present in a quantum system). One way of viewing decoherence, and the best perspective for us, is that information about the system leaks into the environment. This leakage of information is then transferred to the surroundings, a mechanism crucial to Quantum Darwinism.

To study the basic ideas of decoherence, consider the following: Let the pure state

$$|\psi(0)\rangle_{\mathcal{S}} = \frac{1}{\sqrt{2}}(|0\rangle_{\mathcal{S}} + |1\rangle_{\mathcal{S}}) \tag{1.3}$$

denote the initial state of the system $\mathcal{S}$. Then, the system's initial density matrix is given by

$$\rho_{\mathcal{S}}(0) = |\psi(0)\rangle_{\mathcal{S}}\langle\psi(0)| \doteq \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \tag{1.4}$$

Again, in the study of decoherence, the system and its environment interact and become correlated leading to the suppression of inteference effects in the system. Crudely, we recognize decoherence has occurred when the off-diagonal terms of the density matrix of the system have vanished. After $\mathcal{S}$ and $\mathcal{E}$ have become sufficiently correlated:

$$\rho_{\mathcal{S}}(0) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \rightarrow \rho_{\mathcal{S}}(t) \approx \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{1.5}$$

To see how this occurs let us consider a simple example. Using the initial state of the system above, also let

$$|\psi(0)\rangle_{\mathcal{E}} = \frac{1}{\sqrt{2}}(|0\rangle_{\mathcal{E}} + |1\rangle_{\mathcal{E}}) \tag{1.6}$$

denote the state of the environment $\mathcal{E}$. The choice of initial states here is not an arbitrary one. In fact these states provide us with the set-up of a specific physical model: A system $\mathcal{S}$ of a single spin immersed in an environment $\mathcal{E}$ of a single spin. This model is particularly pertinent to us and we will examine a more complicated version later.

Moving on, if we assume initially that there are no correlations between the system and the environment, then the initial state of the composite system $\mathcal{SE}$ can be written as the product state:

$$\rho_{\mathcal{SE}}(0) = \rho_{\mathcal{S}}(0) \otimes \rho_r \tag{1.7}$$

where

$$\rho_r = |\psi(0)\rangle_{\mathcal{E}}\langle\psi(0)| \tag{1.8}$$

Remember, decoherence is the loss of quantum coherence. The way to see this, and how decoherence leads to classical reality, is to study the state of the system before and after $\mathcal{S}$ has interacted with the environment. Before the interaction, we have that the state of the system is, obviously,

$$\rho_{\mathcal{S}} = \mathrm{tr}_{\mathcal{E}}\, \rho_{\mathcal{SE}} = \rho_{\mathcal{S}}(0) \doteq \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \tag{1.9}$$

where $\mathrm{tr}_{\mathcal{E}}$ is the partial trace over the state of the environment. The presence of the off-diagonal terms of the density matrix is often indicative of quantum interference. Now let us turn on the interaction between the system and the environment for some period of time t. We wish to study the behavior of the state of the system at some later time, i.e, $\rho_{\mathcal{S}}(t)$. To determine this state we begin by assuming that the dynamics of the system-environment interaction is governed by a spin-coupled Hamiltonian of the form:

$$\mathbf{H}_{\mathcal{SE}} = \frac{1}{2}\sigma_{\mathcal{S}}^{z} \otimes \sigma_{\mathcal{E}}^{z} \doteq \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \tag{1.10}$$

The state of the composite system at a later time then is given by the unitary evolution

$$\rho_{\mathcal{SE}}(t) = U(t)\rho_{\mathcal{SE}}(0)U^{\dagger}(t) \tag{1.11}$$

where $U(t)$ is the expected unitary operator $U(t) = \exp(-it\mathbf{H}_{\mathcal{SE}})$. To find the state of the system $\rho_{\mathcal{S}}(t)$ we simply take a partial trace over the environment states, which, in this case, leads to

$$\rho_{\mathcal{S}}(t) \doteq \begin{pmatrix} 1/2 & 1/2\cos(t) \\ 1/2\cos(t) & 1/2 \end{pmatrix} \tag{1.12}$$

from which we notice something interesting. Consider when $t = \pi/2$. At this select time the off-diagonal elements of the density matrix vanish, indicating that there is a loss of quantum coherence in the z-basis. Therefore, the system, at time $t = \pi/2$, has become *decohered*.

There are at least two important observations to be made here, some of which won't be fully appreciated until later on. First, consider the scenario: a single two-level system, say the spin of an electron, is initially in a pure quantum state and interacts with another qubit, becoming entangled. For a specific time interval (including $t = \pi/2$) the system partially decoheres. But what caused decoherence to occur in the first place? In the process of decoherence, the environment makes a measurement on the system, indicating something fundamental: *Measurement induces decoherence*. This feature is the essential mechanism leading to the quantum-to-classical transition and will be reviewed throughout this work.

Moreover, since the environment makes a measurement on the system, $\mathcal{E}$ acquires information

about the state of the system. In other words, $\mathcal{S}$ interacts with its environment and information about the state of $\mathcal{S}$ becomes stored in the environment. Indeed, assuming the same composite system above, the process of decoherence causes the initial product state to move to an entangled state:

$$|\psi(0)\rangle_{\mathcal{SE}} = |\psi(0)\rangle_{\mathcal{S}}|\psi(0)\rangle_{\mathcal{E}} \rightarrow |\psi(t)\rangle_{\mathcal{SE}} = \frac{|0\rangle_{\mathcal{S}}|\tilde{0}(t)\rangle_{\mathcal{E}} + |1\rangle_{\mathcal{S}}|\tilde{1}(t)\rangle_{\mathcal{E}}}{\sqrt{2}} \tag{1.13}$$

where

$$|\tilde{0}(t)\rangle_{\mathcal{E}} = \frac{|0\rangle + e^{it}|1\rangle}{\sqrt{2}} \qquad |\tilde{1}(t)\rangle_{\mathcal{E}} = \frac{|0\rangle + e^{-it}|1\rangle}{\sqrt{2}} \tag{1.14}$$

from which we see that at $t = \pi/2$, $|\tilde{1}\rangle \rightarrow |1\rangle_y$ and $|\tilde{0}\rangle \rightarrow |0\rangle_y$, leading to a maximally entangled state. From here one may make a measurement on the environment and acquire information about the state of the system. This fact is crucial to Quantum Darwinism and further aids in the analysis of the quantum-to-classical transition, and will be discussed more later.

The second important observation is that the off-diagonal terms, the so-called *decoherence factors* (often written as $\Gamma(t)$ or $r(t)$), are functions periodic in time. This implies that there is some special time, the *recurrence time* τ_{rec}, which allows the state to become coherent again, after it has already decohered. But, for realistic physical systems *decoherence is an irreversible process*. Indeed, this is true and the presence of a recurrence is indicative of the limits of our model (that of a single spin in the environment). In reality the environment is exceedingly large, e.g., many spins, and such recurrences are suppressed exemplifying that decoherence is irreversible. In fact, for a model of a system qubit immersed in an environment of several qubits with different self-interaction strengths, it can be shown (as we will see) that the decoherence factor takes the form

$$r(t) = \prod_k \cos(g_k t) \rightarrow e^{-t^2/\tau^2} \tag{1.15}$$

where g_k are the system-environment coupling strengths, i.e., realistic spin environments exhibit gaussian decay, an indication of irreversibility [13]. Moreover, in many models of decoherence, the decoherence factor is found to be an exponential decay [20]

$$r(t) \propto e^{-t/\tau_d} \tag{1.16}$$

with τ_d being the characteristic time for decoherence to take place. What's more is, in the event of highly ordered initial states and symmetrical system-environment couplings (more on this later) but still for sufficiently large environment sizes, the recurrence time is longer than the age of the

Universe! Thus, even in the best possible case, once a system has been decohered, it cannot become coherent again; decoherence is still an effectively irreversible process.

1.3 Summary of Decoherence and a Preview of Quantum Darwinism

In general, decoherence describes the evolution of pure states into mixed states. Operationally, we can say that the system has become decohered when the the off-diagonal elements of the system's density matrix are suppressed. As noted above, decoherence, induced by measurement, can be viewed as the leakage of information about the system into the environment in an irreversible way. Most of all, decoherence is the first step in understanding the quantum-to-classical transition.

To have a more intuitive understanding of decoherence, let's consider a qualitative though ever-present example. Suppose we have a system of a classical macroscopic object, a chair, for instance, and the environment is a collection of photons illuminating the system, a scenario we experience every day. The incoming photons scatter off the chair, become correlated with the system, and acquire information about its state. As the system and environment interact, the superposition states (superposition of positions) do not survive the decoherence process, hinting at a mechanism which gives rise to the classical world [24].

As far as decoherence gets us, it is still not completely satisfactory. In particular, decoherence assumes the Born rule holds, it does not derive it. This fact reminds us of the issue with the Copenhagen interpretation and the collapse postulate; decoherence does not provide an explanation to why Born's rule is correct. More subtle is that decoherence doesn't clarify what we mean by the classical world. Indeed, decoherence explains why we don't observe superposition states of the system classically, but it does not clarify how we then move from the quantum to the classical. In order to solve this we must go beyond decoherence and augment it with another framework. Quantum Darwinism is such a solution.

To illustrate Quantum Darwinism, let's continue with our elementary example of a chair immersed in a photon bath. After the system and environment interact, correlations arise and, as decoherence tells us, the environment acquires information about the system. This information can then be intercepted, in particular, by us, who interact with the photon environment on a daily basis. The crucial feature of Quantum Darwinism is that the environment, subdivided into independent *fragments*, acts a witness [9, 10], and communication channel [15], in which redundant information

about the classical states of the system, the *pointer states*, are proliferated into the environment. This allows us to better describe the quantum-to-classical transition in the following sense. The classical world is described by the world we commonly experience. One way in which we characterize this world is through objective information, information which many observers can access independently without perturbing the system, e.g., we can all independently make a measurement on a chair in a room and agree on its position. For objective information to emerge from a quantum substrate we require redundancy of such information. The environment, which retains information about a particular state of the system, is then tapped into by each observer through a small fraction of the total environment. Ultimately, this allows multiple observers to determine a state of the system independently without changing it, thereby agreeing on what state it is that they see, leading to the emergence of objective classical reality - the world we commonly experience [6]. For an illustrative outline of Quantum Darwinism, see Fig. 1.1.

Simply put, Quantum Darwinism makes precise what we mean by objective, classical reality; decoherence does not. In this way, Quantum Darwinism augments the theory of decoherence by clarifying the role of pointer states and why we may interpret them as the classical states of the system, rather than just states which are robust to the interaction with the environment.

In order to better appreciate Quantum Darwinism and the role of the environment in decoherence, we require more tools. Specifically, we need to build a mathematical and physical intuition for information theory. The reader who is comfortable with topics in quantum information theory may skip this next chapter and move to the section devoted to a slightly more rigorous introduction to Quantum Darwinism.

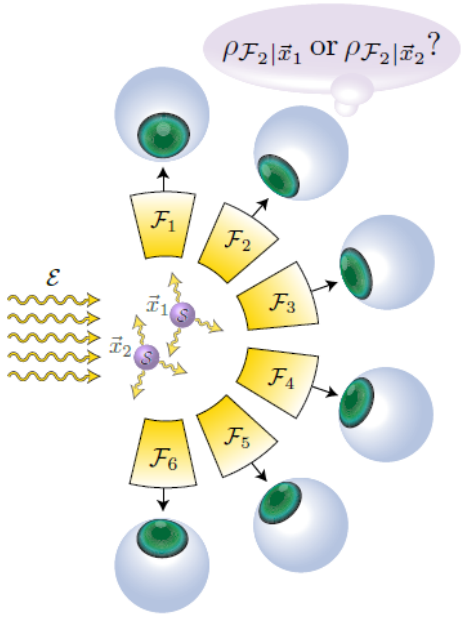

Figure 1.1: A depiction of Quantum Darwinism in a nutshell. Imagine a quantum system $\mathcal{S}$ that is illuminated by the environment $\mathcal{E}$. This environment is composed of several distinct subsystems (in this case, an environment composed of several photons), which are massed into fragments $\mathcal{F}$. One can imagine that independent observers correspond to independent fragments, each of which contains nearly sufficient information to describe the system. The environment, which acts like a quantum channel, decoheres the system, acquiring multiple copies of information about the system which are then intercepted by the many observers independently. This allows for agreement about which state the system is in amongst the observers, leading to what we infer as objective, classical reality. Briefly, often spin environments are used in place of photon environments. This figure is from Ref. [16].

Chapter 2

The Basics of Information Theory

2.1 Classical Information

We begin with *classical information theory*, whose modern history lies in coding theory[1] [25]. That is, classical information theory's primary concern is understanding how to send *classical* information – letters in an alphabet, bit strings, etc. – over communication channels. The most intuitive definition of classical information is given in terms of the *Shannon entropy*: Given a random variable X with probability distribution $p_x, x = 1, 2, ..., n$ the Shannon entropy is defined as

$$H(X) = H(p_1, p_2, ...p_n) = -\sum_x p_x \log p_x \tag{2.1}$$

where $\sum_x p_x = 1$ and $\lim_{x\to 0} x \log x = 0$. There are two central interpretations of the entropy: **(1)** $H(X)$ quantifies how much information we gain after we learn the value of X, on average; **(2)** Equivalently, $H(X)$ measures the level of uncertainty of X before we learn the value of X [26].

We can see how entropy is a reasonable description of information with a very simple example: Suppose we have some secret word that has been coded as a jumble of letters, *nligsa*, which then undergoes a decoding process (we simply unscramble it) and we discover that the word is really *signal*. Now consider another word, *puzlez*, which under the same decoding procedure we discover is *puzzle*. Between each scenario, we see that more information was gained in the first case than the information gained in the second case. Alternatively, there was a higher uncertainty of what *nligsa* was before we gained knowledge of its value, compared to the level of uncertainty of what *puzlez*

[1] For a fairly complete history of information science, consider Gleick's *The Information: a history a theory, a flood*.

was before we gained knowledge of its value.

The crucial purpose of Shannon entropy can be summarized by Shannon's *noiseless coding theorem*, which informally says:

"N random variables each with entropy $H(X)$ can be compressed into more than $NH(X)$ bits with negligible risk of information loss as N tends toward infinity; conversely, if they are compressed into fewer than $NH(X)$ bits it is virtually certain that information will be lost" [26].

In this way, entropy quantifies the optimal compression of data that may be achieved, resolving the question "what minimal physical resources are required to store information being produced by the source such that later the information can be reconstructed" [27].

In classical information theory we are not just concerned with a single random variable but rather multiple random variables. This is characterized by the *joint entropy*:

$$H(X,Y) = -\sum_{x,y} p(x,y)\log(p(x,y)) \tag{2.2}$$

where $p(x,y)$ is a joint probability distribution associated with the pair of random variables (X,Y). $H(X,Y)$, therefore, quantifies the total uncertainty about the pair (X,Y).

Related to the joint entropy is *conditional entropy* which is described as follows: Suppose we have a pair of random variables (X,Y) and we know the value of Y, i.e., we have $H(Y)$ bits of information about the pair (X,Y). There is still some remaining uncertainty about the pair, and this uncertainty is associated with the lack of knowledge about the value of X. Then, the entropy of X conditional on knowing Y is just

$$H(X|Y) = H(X,Y) - H(Y) \tag{2.3}$$

allowing us to answer how uncertain we are about X, given we know Y [27]. With these few definitions, it is straightforward to the prove several basic properties of the Shannon entropy (see Refs. [26, 27]).

A crucial quantity in classical and quantum information theory, and especially Quantum Darwinism, is the *mutual information* $I(X:Y)$, which measures how much information X and Y have in common. We can build the definition of mutual information in a rather intuitive way. We start by adding the information about X and the information about Y (quantified by $H(X)$ and $H(Y)$).

Now, information that is common to both X and Y will be counted twice. Thus, in order to get the information common to both X and Y, the mutual information, we subtract off the joint entropy, hence

$$I(X:Y) = H(X) + H(Y) - H(X,Y) \tag{2.4}$$

We can write the mutual information as the *relative entropy* between the joint distribution $p(x,y)$ and the product distribution $p(x)p(y)$ [26]

$$I(X:Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \tag{2.5}$$

where the relative entropy measures the 'distance' between two probability distributions $p(x)$ and $q(x)$

$$D(p||q) = \sum_{x} p(x) \log \frac{p(x)}{q(x)} \tag{2.6}$$

Written in the above form, we may rewrite the definition of the mutual information in the following way:

$$\begin{aligned} I(X:Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} &= \sum_{x,y} p(x,y) \log \frac{p(x|y)}{p(x)} \\ = -\sum_{x,y} p(x,y) \log p(x) - \left(-\sum_{x,y} p(x,y) \log p(x|y) \right) & \\ = H(X) - H(X|Y) & \end{aligned} \tag{2.7}$$

Therefore, an equivalent way to interpret the mutual information is that $I(X:Y)$ yields the reduction in the uncertainty of X due to the knowledge of Y. The fact that there are two equivalent definitions of $I(X:Y)$ is a feature of classical information theory [10]. As we will see, this equivalence does not hold for quantum systems, leading to an important quantity known as *quantum discord*.

For a summary of the variety of quantities describing information and various entropies, see the Venn diagram in Fig. 2.1. With these basic definitions, we now have an intuition and a quantitative measure of classical information. However, our world is fundamentally quantum, and we therefore seek to generalize the above notions to quantum systems.

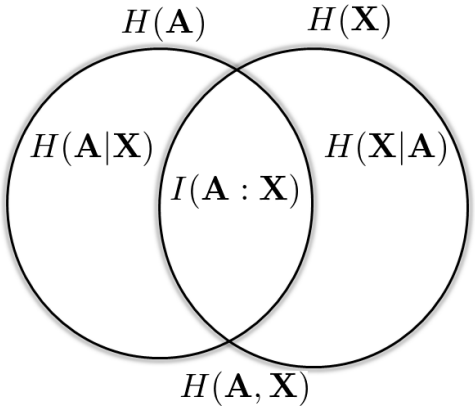

Figure 2.1: A summary of classical information. $H(\mathbf{A})$ is the entropy associated with a measurement of $\mathbf{A}$, yielding the information needed to completely determine its outcome; likewise for $H(\mathbf{X})$. $H(\mathbf{A},\mathbf{X})$ is the joint entropy of $\mathbf{A}$ and $\mathbf{X}$, while $H(\mathbf{A}|\mathbf{X})$ is the conditional entropy – the uncertainty about $\mathbf{A}$ after knowing $\mathbf{X}$, on average. This gives us the mutual information: $I(\mathbf{A}:\mathbf{X}) = H\mathbf{A} - H(\mathbf{A}|\mathbf{X})$, the information shared between $\mathbf{A}$ and $\mathbf{X}$, i.e. the information learned about $\mathbf{A}$ by measuring $\mathbf{A}$. It is also easy to realize the equivalent definition, $I(\mathbf{A}:\mathbf{X}) = H(\mathbf{A}) + H(\mathbf{X}) - H(\mathbf{A},\mathbf{X})$.

2.2 Quantum Information

The essential problem in classical information theory is how to send classical information across communication channels. By analogy, *quantum information theory* is the study of quantum communication channels [27], in which we will find that the environment plays an important role. We start by upgrading the Shannon entropy.

This is done by using the density operator formalism for quantum states. Rather than using classical probability distributions, the quantum entropy, the so-called *von Neumann* entropy, uses these density operators:

$$H(\rho) = -\operatorname{tr}(\rho \log \rho) \tag{2.8}$$

where ρ is a density operator, typically written as $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $\sum_i p_i = 1$, and satisfying the usual properties. It is straightforward to show that the von Neumann entropy can also be cast as the Shannon entropy of the eigenvalues $\{\lambda_i\}$ of the density operator

$$H(\rho) = -\sum_i \lambda_i \log \lambda_i \tag{2.9}$$

From here we quickly make two observations. First, suppose we have a pure state, $\rho = |\psi\rangle\langle\psi|$. Then, all of the eigenvalues $\lambda_i = 0$ except one, say λ_1, which has unit value, yielding $H(\rho_{pure}) = 0$.

This is one way we may characterize pure states. Also, recall that a pure state corresponds to all $p_i = 0$ except for one which takes on the value one, i.e., there is no *ignorance* about the state of the system – in correspondence with the value of entropy. Alternatively, for a maximally mixed state in a d-dimensional space, $\rho = I/d$ and so $\lambda_i = 1/d$ for each i, yielding $H(I/d) = \log d$, the maximum value of $H(\rho)$. We might have guessed this, as a maximally mixed density operator corresponds to complete ignorance about which of the mutually exclusive pure states $|\psi_i\rangle$ has been prepared – again matching our intuition about entropy in classical information theory.

As one might expect, we can move on and define other quantum entropies, starting with the quantum equivalent of the relative entropy

$$H(\rho||\sigma) \equiv \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma) \tag{2.10}$$

where ρ and σ are different density operators. Notice that when $\rho = \sigma$ the relative entropy is zero. Now suppose we have a composite system $\mathcal{SE}$. The quantum joint entropy is then

$$H_{\mathcal{SE}} = -\mathrm{tr}(\rho_{\mathcal{SE}} \log \rho_{\mathcal{SE}}) \tag{2.11}$$

where $\rho_{\mathcal{SE}}$ is the density matrix of the composite system. Note that if we may write $\rho_{\mathcal{SE}} = \rho_{\mathcal{S}} \otimes \rho_{\mathcal{E}}$, i.e., the two subsystems are uncorrelated, then the joint entropy of the composite system is the sum of the von Neumann entropies of each subsystem:

$$H_{\mathcal{SE}} = H_{\mathcal{S}} + H_{\mathcal{E}} = H(\rho_{\mathcal{S}}) + H(\rho_{\mathcal{E}}) \tag{2.12}$$

From the definition of joint entropy we can also define the quantum versions of conditional entropy and mutual information using our intuition from classical information theory:

$$H(\mathcal{S}|\mathcal{E}) = H_{\mathcal{SE}} - H_{\mathcal{E}} \tag{2.13}$$

$$I(\mathcal{S} : \mathcal{E}) = H_{\mathcal{S}} + H_{\mathcal{E}} - H_{\mathcal{SE}} \tag{2.14}$$

To see some of the differences between quantum and classical information, let's start by comparing some of the properties of quantum entropy to the basic properties of Shannon entropy. First we note that we lose the classical intuition: $H(X) \leq H(X, Y)$. That is, in quantum systems the statement "we cannot be more uncertain about the value of X than about the joint value of the pair (X, Y)"

does not hold in general. The easiest way to witness this is to consider an entangled state, say

$$|\psi_{\mathcal{SE}}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{2.15}$$

Letting the state of the composite system be the pure state $\rho_{\mathcal{SE}} = |\psi_{\mathcal{SE}}\rangle\langle\psi_{\mathcal{SE}}|$ yields a joint entropy $H_{\mathcal{SE}} = 0$. Alternatively, if we take the partial trace over one of the subsystems, say the environment $\mathcal{E}$, we find $\rho_{\mathcal{S}} = I/2$, and so $H_{\mathcal{S}} = 1$. Clearly, $H_{\mathcal{SE}} \ngeq H_{\mathcal{S}}$. This tells us that the conditional entropy $H(\mathcal{S}|\mathcal{E})$ can be negative; when the composite system is entangled.

In fact, this feature is pertinent to decoherence as well. Consider the model we introduced in the previous chapter: A system qubit interacting with an environment of a single qubit. There we found that the initial state of our system is given by

$$\rho_{\mathcal{S}}(0) \doteq \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \tag{2.16}$$

Since initially the system $\mathcal{S}$ is in a pure state $H(\rho_{\mathcal{S}}(0)) = 0$. Let's compare this to the state of the system after it has interacted with the environment at the decoherence time $t = \pi/2$. Then, $\rho_{\mathcal{S}}(\pi/2) = I/2$ and $H_{\mathcal{S}}|_{t=\pi/2} = 1$. That is, the environment has gained one bit of information about the system, quantitatively matching what were noted before: *The environment makes a measurement on the system, inducing decoherence, and acquires a single bit of information about* $\mathcal{S}$.

Another important difference between classical and quantum information is that while classical mutual information $I(X : Y)$ has other equivalent formulations, the quantum mutual information does not. Indeed, we characterize the mutual information with two expressions

$$I(\mathcal{S} : \mathcal{E}) = H_{\mathcal{S}} + H_{\mathcal{E}} - H_{\mathcal{SE}} \tag{2.17}$$

$$\mathcal{J}(\Pi_{\mathcal{S}} : \mathcal{E}) = H_{\mathcal{S}} - H(\mathcal{S}|\mathcal{E}) \tag{2.18}$$

where the second is with respect to $\Pi_{\mathcal{S}}$, a system observable. These two expressions are not equal, and the difference is measured by the *quantum discord* [28, 29]:

$$\mathcal{D}(\Pi_{\mathcal{S}} : \mathcal{E}) = I(\mathcal{S} : \mathcal{E}) - \mathcal{J}(\mathcal{S} : \mathcal{E}) = H_{\mathcal{E}} - H_{\mathcal{SE}} + H(\mathcal{S}|\mathcal{E}) \tag{2.19}$$

where we have resorted to using the familiar notation of $\mathcal{S}$ and $\mathcal{E}$. One reason why quantum discord is important is because it gives the quantum component of the correlations between the system and

environment. In a sense, it measures the "quantumness" of the composite state.

In fact, the mutual information $I(\mathcal{S}:\mathcal{E})$ naturally distributes itself between classical and quantum components, with the quantum component given by the discord, and the classical component given by the *Holevo quantity* (see below)

$$I(\mathcal{S}:\mathcal{F}) = \chi(\Pi_{\mathcal{S}}:\mathcal{F}) + \mathcal{D}(\Pi_{\mathcal{S}}:\mathcal{F}) \tag{2.20}$$

a relation we take to be a fundamental conservation law. We observe that the mutual information on the left hand side does not depend on the observable $\Pi_{\mathcal{S}}$, while the right hand side, the classical and quantum components, do. This tells us classical reality emerging from decoherence relies on the fact that measurements available to the observer are "local", i.e, measurements which do not involve global observables composed of the entangled eigenstates of the system and the whole environment, $\mathcal{SE}$.

We now have the basic elements of information theory necessary for an elementary understanding of Quantum Darwinism. Before we move on to the next section, however, let's briefly discuss some notation. As discussed before, Quantum Darwinism concerns itself with a system $\mathcal{S}$ interacting with an environment $\mathcal{E}$ that is composed of many fragments $\mathcal{F}$. A quantity of considerable interest in Quantum Darwinism is the mutual information between the system and a fragment of the environment, expressed as

$$I(\mathcal{S}:\mathcal{F}) = H_{\mathcal{S}}(t) + H_{\mathcal{F}}(t) - H_{\mathcal{SF}}(t) \tag{2.21}$$

where we have the entropies of the system $H_{\mathcal{S}}(t)$, a fragment $H_{\mathcal{F}}(t)$, and the system and fragment $H_{\mathcal{SF}}(t)$, after the system and environment have interacted for some finite time t. Using this notation, quantum discord and the Holevo quantity with respect to the pointer states are

$$\mathcal{D}(\hat{\Pi}_{\mathcal{S}}:\mathcal{F}) = I(\mathcal{S}:\mathcal{F}) - \mathcal{J}(\hat{\Pi}_{\mathcal{S}}:\mathcal{F}) = H_{\mathcal{S}}(t) - H_{\mathcal{SF}}(t) + H_{\mathcal{F}|\{\hat{\Pi}_{\mathcal{S}}\}} \tag{2.22}$$

$$\chi\left(\hat{\Pi}_{\mathcal{S}}:\mathcal{F}\right) = H\left(\sum_{\hat{s}} p_{\hat{s}}\rho_{\mathcal{F}|\hat{s}}\right) - \sum_{\hat{s}} p_{\hat{s}} H(\rho_{\mathcal{F}|\hat{s}}) \tag{2.23}$$

where now $\hat{\Pi}_{\mathcal{S}} = \sum_{\hat{s}} \pi_{\hat{s}}|\hat{s}\rangle\langle\hat{s}|$ are the so-called *pointer observables*, $\hat{s} = 1, ...D_{\mathcal{S}}$ labels the pointer states, $p_{\hat{s}}$ are the probabilities of the pointer states, and $\rho_{\mathcal{F}|\hat{s}} = \langle\hat{s}|\rho_{\mathcal{SF}}|\hat{s}\rangle/p_{\hat{s}}$ are the fragment states conditioned on the system's pointer state $\hat{s}$, i.e., information about the system transmitted by the fragment [16]. Notice that we have switched between slightly different notations for the Holevo

quantity, one in terms of a generic system observable $\Pi_{\mathcal{S}}$ and one in terms of the pointer observable $\hat{\Pi}_{\mathcal{S}}$. For further clarity, we will discuss this difference, discord, and the Holevo quantity more with a specific example in the next chapter, which we move to now.

Chapter 3

Introducing Quantum Darwinism

3.1 Decoherence, Einselection, and Pointer States

Let us reconsider a qubit system interacting with an environment of qubits with the initial state of the system as the product state

$$|\psi(0)\rangle_{\mathcal{SE}} = \frac{(|0\rangle_{\mathcal{S}} + |1\rangle_{\mathcal{S}})}{\sqrt{2}} \otimes |+\rangle_{\mathcal{E}} \tag{3.1}$$

After the system and environment have had time to exchange information, an evolutionary process governed by the Hamiltonian $\mathbf{H}_{\mathcal{SE}} = \frac{1}{2}\sigma^z_{\mathcal{S}}\sigma^z_{\mathcal{E}}$, the composite state of the system becomes

$$|\psi(t)\rangle_{\mathcal{SE}} = \frac{|0\rangle_{\mathcal{S}}|\tilde{0}(t)\rangle_{\mathcal{E}} + |1\rangle_{\mathcal{S}}|\tilde{1}(t)\rangle_{\mathcal{E}}}{\sqrt{2}} \tag{3.2}$$

where

$$|\tilde{0}(t)\rangle_{\mathcal{E}} = \frac{|0\rangle_{\mathcal{E}} + e^{it}|1\rangle_{\mathcal{E}}}{\sqrt{2}} \qquad |\tilde{1}(t)\rangle_{E} = \frac{|0\rangle_{\mathcal{E}} + e^{-it}|1\rangle_{E}}{\sqrt{2}} \tag{3.3}$$

This shows that each component of the system has become entangled with the states of the environment. As noted back in chapter one, through this entanglement process information leaks from the system into the environment. In this way, the environment plays the role as an effective measuring device continuously monitoring the system (in the context of typical inteference experiments, a "which-path detector" [20]).

Now suppose the initial state of the system starts in a specific component, say $|\psi\rangle_{\mathcal{S}} = |0\rangle_{\mathcal{S}}$. Then, after $\mathcal{S}$ and $\mathcal{E}$ have interacted, the state of the system and environment is still a product state. That

is,

$$|0\rangle_{\mathcal{S}}|+\rangle_{\mathcal{E}} \rightarrow |0\rangle_{\mathcal{S}}|\tilde{0}(t)\rangle_{\mathcal{E}} \tag{3.4}$$

Therefore the state of the system and environment can still be cast as a product state; the environment measures the system without disturbing the state of the system. On the other hand, if our system is described as the superposition above, the interaction leads to an entangled state and hence decoherence in the $\{|0\rangle_{\mathcal{S}}, |1\rangle_{\mathcal{S}}\}$ basis.

These features give us an intuitive sense of what is called *environment-induced superselection* or *einselection*. As seen above, some states of the system are more prone to decoherence than others. In our case, a superposition state is more prone to decoherence. This allows us to consider states at the other end, those least sensitive to decoherence. We call these states the *preferred states* of the system, or as we will refer to them, *pointer states* of the system. The outstanding attribute of the pointer states is that they dynamically emerge as the states of the system that are most robust to decoherence. It is in this way that the environment "superselects" the states of the system. In fact, einselection goes further than simply selecting states, it also explains why we don't observe certain states classically. We therefore often view pointer states as *quasiclassical* states.

Here we considered the concrete example of a spin interacting with a spin environment. If we were to consider the very prevalent model of a macroscopic system immersed in a photon environment, we would also find that the states most sensitive to decoherence are superposition states of positions [20]. This is why we don't readily observe objects in different locations simultaneously, even though quantum mechanically such states are completely valid.

3.2 Quantum Darwinism and the Quantum-to-Classical Transition

From decoherence we have found that the environment acts as a which-path monitor, acquiring information about the states of the system. In this way the environment acts as a resource for indirectly gaining information about the states of $\mathcal{S}$. As observers we do not typically interact with the system directly to acquire our knowledge about it; we intercept a piece of the environment which encodes information about the system. Indeed, when we look at a chair, we don't interact directly with it; photons scattering off the chair encoding information about the state (of its position say) enter our eyes.

This fact is the first step in realizing the quantum-to-classical transition. When we first attempt

to distinguish between classical and quantum we easily find that the difference is due to the notion of measurement. Or, even more elementary, *objective* measurement. That is to say, a characteristic feature of the classical world is that it is objective. When I observe the position of a chair, a friend of mine could enter the room and agree that she too finds the chair in the same position. More precisely, in classical physics the state of the system can be determined and, most importantly, agreed upon by several independent observers without perturbing the state of interest. Thus, classical states exist as objective states; leading to what we refer to as "classical reality". This is entirely different from the nature of states in closed quantum systems, where when we make a measurement on the system, the state is altered, i.e., independent observers will, in general, not agree upon the state of the system.

In light of this, a question naturally arises: How does classical reality emerge from the quantum substrate? Put another way, how do physical observables become objective physical quantities? To answer this question, we must realize yet another special role the environment plays. Indeed, from previous discussions we know that the environment acts like a which-path monitor, selecting the preferred states of the system. But this role is easily broadened. Information about the state of the system is encoded in the environment, which can then be acquired by observers without ever directly making a measurement on the system, i.e., without disturbing the system. Moreover, if we are to hope that many independent observers can acquire information about the state of the system, we expect that this information is not just present, but *redundantly* copied into the environment. This would then allow several observers, only interacting with a *fragment* of the environment, to acquire information about the system. Thus, we now ask, what kind of information is both redundantly and robustly copied and stored in a large number of distinct fragments of the environment, in such a way that many independent observers can intercept this information? That is, which kind of information leads to the observation of a (at least effectively) classical state?

Answering this question has lead to the *environment as a witness* program [20]. Since the environment acquires and encodes information about the state of the system, from which observers then incercept this information through interaction with the environment, $\mathcal{E}$ acts like a *communication channel*. This program has also lead to the study of which kind of information can be stored, in a stable way, and then proliferated by the environment, often falling under the heading, *Quantum Darwinism*.

Put succinctly, Quantum Darwinism recognizes the role of the environment by noting its ability to continually record information about a particular system. Based on our previous discussions, we

realize that the object of interest is the mutual information

$$I(\mathcal{S}:\mathcal{F}) = H_{\mathcal{S}}(t) + H_{\mathcal{F}}(t) - H_{\mathcal{SF}}(t) \tag{3.5}$$

between the system $\mathcal{S}$ and a fragment $\mathcal{F}$ of the environment $\mathcal{E}$. In a nutshell, the mutual information $I(\mathcal{S}:\mathcal{F})$ quantifies the correlations between $\mathcal{S}$ and $\mathcal{F}$. If the system and the fragment are initially uncorrelated, the mutual information gives the information $\mathcal{F}$ gained about a state of the system $\mathcal{S}$, i.e., an observer intercepts a fragment of the environment which holds information about a state of the system [15].

Not surprisingly, the states of the system that are most robust and most easily proliferate into distinct fragments of the environment are indeed the pointer states of the system selected by the system-environment interaction (the einselected states) [5, 6, 29]. Alternatively, most of the states that do not coincide with the preferred states are not redundantly stored. This fact gives us two interesting features: (1) We only need to intercept a small fraction of the environment to acquire information about the pointer observables of the system, and (2) if an observer attempted to measure other non-pointer observables on the same fragment, they would acquire close to no information, as this information is not redundantly encoded in the environment. In summary, the pointer states are the most robust and most easily determined (most redundantly recorded) states of the system, yielding a link to Darwinian evolution: pointer states are the states of the system most likely "survive" the system-environment interaction – pointer states are the "fittest" states – and proliferate copiesof themselves.

These ideas can be understood clearly by considering a simple example. Let us again use the model of a system qubit $\mathcal{S}$ interacting with an environment of qubits $\mathcal{E}$ which is to be subdivided into several fragments $\mathcal{F}$. Let's compute the mutual information between the system and a fragment of the environment $I(\mathcal{S}:\mathcal{F})$. First consider an environment of a single qubit. Then, after decoherence, the system and environment have become correlated, and thus the composite system is given by the entangled state

$$|\psi\rangle_{\mathcal{SE}} = \frac{|0\rangle_{\mathcal{S}}|0\rangle_{\mathcal{E}} + |1\rangle_{\mathcal{S}}|1\rangle_{\mathcal{E}}}{\sqrt{2}} \tag{3.6}$$

In the case of a single qubit environment, the mutual information is really

$$I(\mathcal{S}:\mathcal{E}) = H_{\mathcal{S}} + H_{\mathcal{E}} - H_{\mathcal{SE}} = 2H_{\mathcal{S}} \tag{3.7}$$

where we have invoked *Schmidt's decomposition theorem* (for completeness, see appendix A). Since

our system is a two-level spin system, $H_{\mathcal{S}} = 1$ bit, and so the mutual information $I(\mathcal{S} : \mathcal{E}) = 2$ bits. Briefly, the fact that $I(\mathcal{S} : \mathcal{E}) = 2H_{\mathcal{S}}$ is a feature of quantum mutual information, marking a crucial difference with the classical mutual information.

We proved why the quantum mutual information maximizes at twice the maximum amount of classical information about the system the environment can acquire ($2H_{\mathcal{S}}$ - 2 bits here) using Schmidt decomposition. But the quantum mutual information being larger than classical mutual information can be understood in a physically intuitive way. Consider two observers, Alice and Bob, who have a shared entangled state of two qubits given by

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \tag{3.8}$$

Now suppose Bob intends to measure what state his qubit is in and Alice intends to determine what state his qubit is in. There are a couple of ways to do this: (1) Bob makes a measurement and then transmits his findings to Alice, or (2) Alice can simply make a measurement on her qubit and determine with certainty what state Bob's qubit is in. Further suppose that Bob will make a measurement using the $\{|0\rangle, |1\rangle\}$ basis (a local measurement, of course, since he only has access to his qubit and not Alice's). Alice then decides to make a local measurement using her $0, 1$ basis, i.e., $\{|0\rangle_A, |1\rangle_A\}$, and acquire information about the state of Bob's qubit. Of course, this type of approach can be done classically (imagine coin flipping).

Let's now take this procedure further and suppose that Bob wishes to make a measurement using the $\{|0\rangle_x, |1\rangle_x\}$ basis instead. Using a simple basis transform, the entangled state can be recast as

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_{A,x}|0\rangle_{B,x} + |1\rangle_{A,x}|1\rangle_{B,x}) \tag{3.9}$$

and in this way Alice can also measure using the $\{|0\rangle_x, |1\rangle_x\}$ basis, and, importantly, can again ascertain the state of Bob's qubit. This is not the case, however, with classical measurement. In words, classically, one only measures over the observable which gives the "desired" information about the system; while quantum mechanically one can measure over many observables and acquire the desired information about the system.

It is this feature that the quantum mutual information exemplifies. We recognize that quantum mutual information can be larger than classical mutual information since quantum mechanically one can measure over all possible observables of a system and the environment – classical mutual information only gives information about a single observable.

Moving on, let's now consider what happens if we upgrade the size of the environment $\sharp\mathcal{E} = 2$, i.e., to two qubits. In this case the state of the composite system is given by

$$|\psi\rangle_{\mathcal{SE}} = \frac{|0\rangle_{\mathcal{S}}|00\rangle_{\mathcal{E}} + |1\rangle_{\mathcal{S}}|11\rangle_{\mathcal{E}}}{\sqrt{2}} \tag{3.10}$$

Now we can compute the mutual information between the system and a fragment. Assuming that we have a symmetric environment, i.e., we acquire the same information from any of one the fragments (again, assume integer size fragments), we find $I(\mathcal{S} : \mathcal{F})$ by tracing out one of the environment spins. In this case, it can be shown that $I(\mathcal{S} : \mathcal{F}) = 1$ bit. Then, as an observer intercepts the entire environment, or, as $\sharp\mathcal{F} \to \sharp\mathcal{E}$, again we find $I(\mathcal{S} : \mathcal{E}) = 2$.

Let's further increase the environment size by including another qubit, $\sharp\mathcal{E} = 3$. By following similar logic the state of the composite system is

$$|\psi\rangle_{\mathcal{SE}} = \frac{|0\rangle_{\mathcal{S}}|000\rangle_{\mathcal{E}} + |1\rangle_{\mathcal{S}}|111\rangle_{\mathcal{E}}}{\sqrt{2}} \tag{3.11}$$

Just as above, we compute the mutual information between the system and the fragment. Again the mutual information between $\mathcal{S}$ and a single fragment is found to be a single bit. Likewise, the mutual information between the system and two fragments of the environment is computed to be one bit. Lastly, as the entire environment is intercepted the mutual information increases to two bits.

The reason that the quantum mutual information stays at $H_{\mathcal{S}}$ for fragments smaller than the entire environment is because one cannot measure any possible observable and acquire the information about the state of the system. Rather, one must first have full access to the whole environment, in which case the quantum mutual information becomes larger than the classical mutual information.

As we continue this process of increasing the size of the environment, we see a trend emerge. Given some environment size, the mutual information, starting at zero (for $\sharp\mathcal{F} = 0$), settles to a single bit of information (or, for more general systems, $I(\mathcal{S} : \mathcal{F}) \to H_{\mathcal{S}}$) for a sufficient amount of fragments, before ending at twice the entropy of the system, $I(\mathcal{S} : \mathcal{F}) \to 2H_{\mathcal{S}}$, as the entire environment is intercepted (see the figure below). Plotting the mutual information versus fragment size we see the formation of the *classical plateau*, developing around $I(\mathcal{S} : \mathcal{F}) \approx H_{\mathcal{S}}$, signifying that classical objective information has proliferated into the environment [15].

Now that we understand the behavior of mutual information between the system and a fragment of the environment, let us briefly study how the classical and quantum components of the $I(\mathcal{S} : \mathcal{F})$

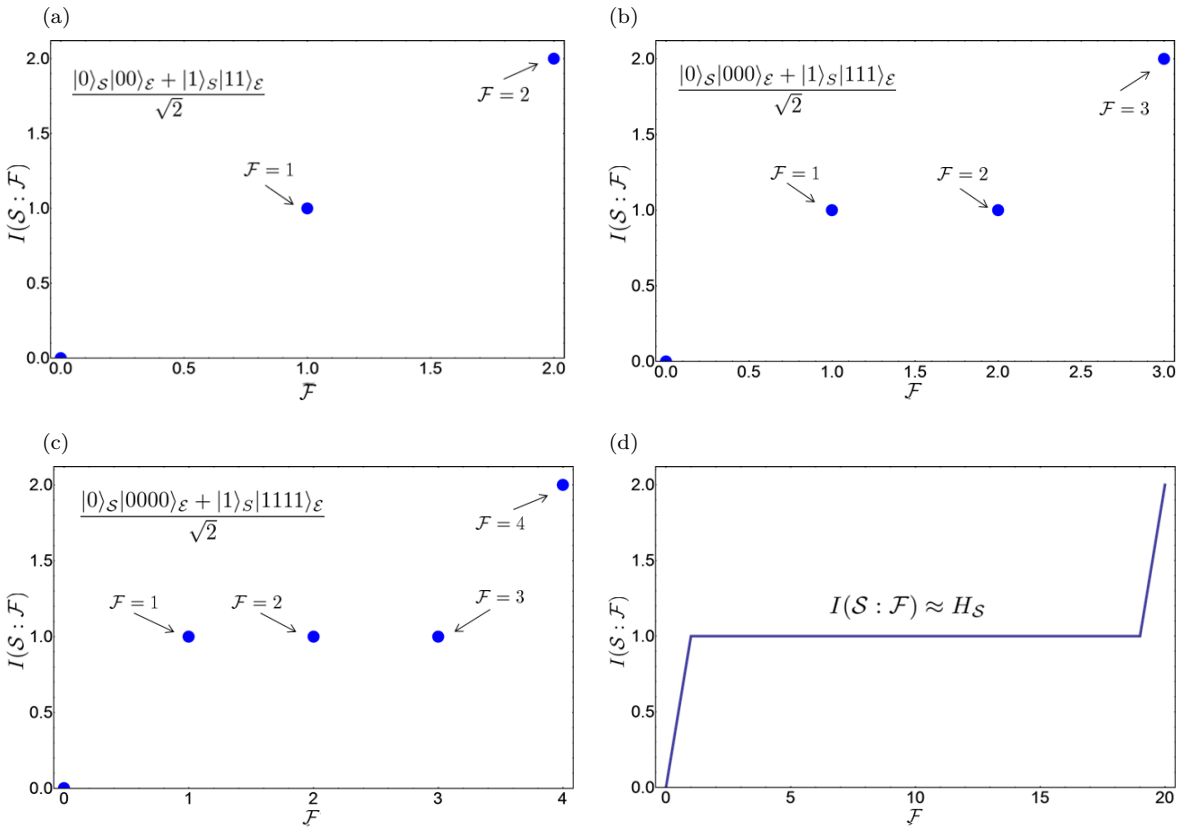

Figure 3.1: Mutual information between $\mathcal{S}$ and a fragment of the environment $\mathcal{F}$ for small environment sizes. In each case, $t = \pi/2$. (a), (b), (c) A central spin system interacting with an environment of two spins, three spins, and four spins, respectively. (d) A central spin system interacting with an environment composed of 20 qubits. Notice the development of the classical plateau forming around $I(\mathcal{S}:\mathcal{F}) \approx H_\mathcal{S}$ (here $H_\mathcal{S} = 1$ bit),indicating that classical objective information has proliferated into the environment.

depend on the number of fragments. Recall that the quantum mutual information $I(\mathcal{S}:\mathcal{F})$ is composed of classical and quantum components, the classical component being the Holevo quantity $\chi(\Pi_\mathcal{S}:\mathcal{F})$ while quantum discord $\mathcal{D}(\Pi_\mathcal{S}:\mathcal{F})$ is the quantum component. In the definition of the Holveo bound

$$\chi(\Pi_s:\mathcal{F}) = H_\mathcal{F} - \sum_s p_s H(\rho_{\mathcal{F}|_s}) \tag{3.12}$$

we will consider only projective measurements (formally called PVMs). That is, we let our set of operators $\{\Pi_s\}$ be the generic projectors

$$\{\Pi_{\hat{n}}\} = \{|n_+\rangle\langle n_+|, |n_-\rangle\langle n_-|\} \tag{3.13}$$

where

$$\hat{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{3.14}$$

$$|n_+\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle \qquad |n_-\rangle = -\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}e^{i\phi}|1\rangle \tag{3.15}$$

with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. We choose the PVM formalism as it will help us study which states emerge as the pointer basis. Thus, now χ depends on the Bloch angles θ and ϕ:

$$\chi(\Pi_{\hat{n}} : \mathcal{F}) = H_{\mathcal{F}}(t) - \left[p_+ H(\rho_{\mathcal{F}|_+}) + p_- H(\rho_{\mathcal{F}|_-})\right] \tag{3.16}$$

where

$$p_\pm = \mathrm{tr}(|n_\pm\rangle\langle n_\pm|\rho_{\mathcal{SF}}) = \mathrm{tr}(|n_\pm\rangle\langle n_\pm|\rho_{\mathcal{S}}) \tag{3.17}$$

and

$$\rho_{\mathcal{F}|_\pm} = \frac{\langle n_\pm|\rho_{\mathcal{SF}}|n_\pm\rangle}{p_\pm} \tag{3.18}$$

For clarity, let our system $\mathcal{S}$, as usual, be immersed in an environment of several qubits. Using

$$I(\mathcal{S} : \mathcal{F}) = \chi(\Pi_s : \mathcal{F}) + \mathcal{D}(\Pi_s : \mathcal{F}) \tag{3.19}$$

we plot the Holevo bound, and, subsequently, the discord as functions of the fragment size $\sharp\mathcal{F}$. Even for relatively small environment sizes ($\sharp\mathcal{E} = 20$) we can see the general behavior. When $\theta = 0$ and $\phi = 0$, i.e., for eigenstates of σ_z, the Holevo quantity χ is maximal while the quantum discord $\mathcal{D}$ is minimal. Conversely, for eigenstates of σ_x ($\theta = \pi/2$ and $\phi = 0$), χ is minimal while $\mathcal{D}$ is maximal. Since the Holevo quantity is maximized for the pointer states of the system [29], we may conclude that the preferred states of the system are the eigenstates of σ_z.

Plots (c) and (d) tell us even more, illustrating the complementarity of quantum and classical components of mutual information. That is, information that is *locally accessible*, bounded by χ, becomes maximal for the pointer observable of the system while accessible information about the other observables decreases, during which the quantum discord $\mathcal{D}$ increases. This provides us with two conclusions: (1) In the setting of Quantum Darwinism, the Holevo quantity and quantum discord are complementary functions and (2) the sum $\chi + \mathcal{D}$ remains constant, proof of the law of conservation of quantum and classical information.

Lastly, crucial to Quantum Darwinism is redundancy and amplification. In short, amplification is the replication of information about quantum states of the system while redundancy measures the number of copies disseminated into the environment. The environment, then, is not just a communication channel, rather it is an amplification channel [29]. It is these quantities, particularly

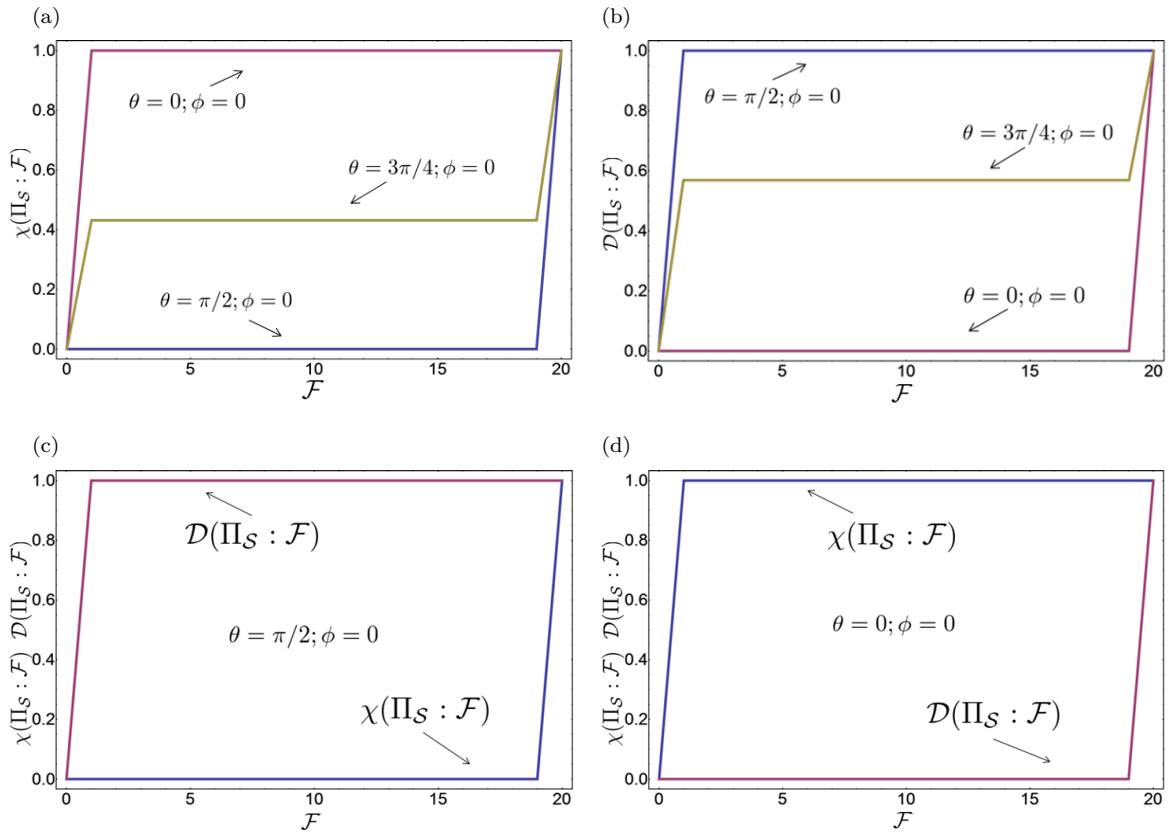

Figure 3.2: Holevo quantity $\chi(\Pi_s : \mathcal{F})$ and the quantum discord $\mathcal{D}(\Pi_s : \mathcal{F})$ for small environment sizes. In each case, $t = \pi/2$ and $\sharp\mathcal{E} = 20$. (a) A plot of χ for various Bloch angles, maximal for eigenstates of σ_z and minimal for eigenstates of σ_x. (b) A plot of $\mathcal{D}$ for various Bloch angles, maximal for eigenstates of σ_x and minimal for eigenstates of σ_z. Together, (a) and (b) indicate that the pointer states of the system are eigenstates of σ_z. (c) and (d) Symmetry of information for different sets of Bloch angles, exemplifying the conservation law $I(\mathcal{S} : \mathcal{F}) = \chi(\Pi_s : \mathcal{F}) + \mathcal{D}(\Pi_s : \mathcal{F})$.

redundancy, which allow us to make precise what we mean by the objective world. For an intuitive picture, consider the ever present scenario of a system immersed in a photon bath. Light is allowed to interact with the system, scatter off and carry away with it information about the state of the system. As mentioned before, several independent observers can come in and intercept the environment and acquire knowledge about the state of the system. Now, it is a fair assumption that an observer does not capture the entire environment – in reality we only gain our information from the portion of photons which enter our eyes, which is certainly not the entire environment – yet, the observer is still able to make objective claims about the state of the system. This fact has two consequences: (1) Since an observer does not intercept the entire environment, they must forgo some information about the state of the system, and (2) an observer need only acquire a small fraction of the environment to determine the stable states of the system. Both of these features are the first step to characterize redundancy, which we can now make more explicit.

Essentially, we wish to investigate how much information about $\mathcal{S}$ the fragment $\mathcal{F}$ acquires upon interaction and how redundant this information is. As noted, we do this by assuming that we will not gain all of the missing classical information $H_{\mathcal{S}}$ about the system. Using the information deficit δ (the fraction of $H_{\mathcal{S}}$ we do not acquire) the redundancy of information about the system is the maximum number of disjoint fragments R_δ with $I(\mathcal{S} : \mathcal{F}) > (1 - \delta)H_{\mathcal{S}}$. The redundancy can be defined in terms of the fragment size as

$$R_\delta = \frac{\sharp\mathcal{E}}{\sharp\mathcal{F}_\delta} = \frac{1}{f_\delta} \tag{3.20}$$

where $\sharp\mathcal{E}$ denotes the components of the environment, $\sharp\mathcal{F}_\delta$ is the typical size of an environment fragment needed to acquire mutual information no less than the bound above, and $f_\delta = \sharp\mathcal{E}/\sharp\mathcal{F}_\delta$ is the corresponding fraction of the environment.

More intuitively, we say that information about the system $\mathcal{S}$ is redundant if and only if the environment can be partitioned into R sub-environments, $\sharp\mathcal{F}_\delta$, each of which carry "nearly full information", i.e., hold a fraction of $(1-\delta)$ of the missing information [12]. We can in fact characterize redundancy in two ways: (1) Using classical mutual information, and hence the Holevo quantity χ, or (2) The quantum mutual information; often one will see both, and, depending on the context, one definition may seem more natural than the other.

Notice that the redundancy is maximal when $\sharp\mathcal{F}_\delta$ is minimal. That is, if a state disseminates plenty of redundant copies, we need only acquire a small fraction of the total environment to gain nearly complete information about this state of the system. Clearly, then, superpositions of positions do not proliferate redundantly while definite position states do.

Before we move on, we ask ourselves if there is a way to reduce redundancy; indeed there is. We can imagine a few scenarios how this might happen, each of which involve limiting the environment's ability to proliferate redundant information. First, often environments will be at a finite temperature, for example, photons emitted from the sun are initially partially mixed. Therefore, environments in general will have some preexisting entropy, quantified by the so-called *haziness* [15]. Second, one can imagine that a component of the environment might tilt away from the states that have the greatest potential to accept and transmit information, a concept exemplified by the *misalignment*. Thirdly, we can imagine that if the system had strong internal dynamics, (e.g., a system moving fast enough that incoming photons could not track its position well) the environment wouldn't be capable in encoding information about the state of the system. Previous work examined how haziness and misalignment reduce the redundancy [15]. In this work we examine the latter case, how the system's

intrinsic dynamics affect the redundancy.

We may encapsulate the crucial details of Quantum Darwinism with a few bullet points:

(1) *Information about the system leaks into the environment, a feature realized by decoherence theory.*

(2) *Pointer states, the so-called preferred states, of the system are dynamically selected by the environment via einselection.*

(3) *Information about the pointer states, the "fittest" information, is redundantly stored in many fragments of the environment.*

(4) *The environment acts as a communication channel: Many independent observers can intercept fragments of the environment, obtaining the redundant information about the state of the system.*

(5) *Acquiring the same information, many independent observers objectively agree on the state of the system, giving rise to the classical world.*

It is through these basic features in which we recognize that Quantum Darwinism, augmenting decoherence, describes the emergence of the classical world from the quantum domain. Now, let us move and explore these ideas further with an elementary, albeit important, model of decoherence.

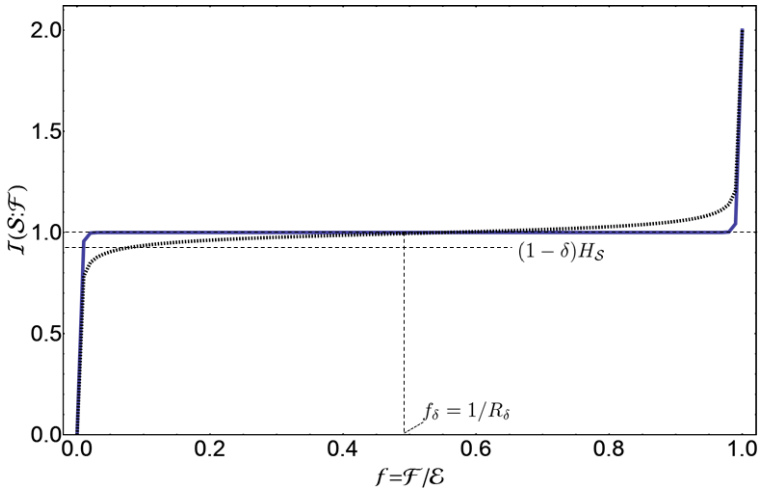

Figure 3.3: A plot of the mutual information $I(\mathcal{S} : \mathcal{F})$ of a system $\mathcal{S}$ and a fragment $\mathcal{F}$ of a symmetric environment $\mathcal{E}$ as a function of the fraction of the environment intercepted, given by $f = \sharp\mathcal{F}/\sharp\mathcal{E}$. The solid blue line indicates a system with no intrinsic dynamics, while the black dashed line is for a system when the self-evolution is weak. Notice the *classical plateau* which forms around $H_{\mathcal{S}}$, signifying that redundant classical (objective) information has propagated into the environment to be intercepted. The fragment size $\sharp\mathcal{F}_\delta$ gives the value of the mutual information within the information deficit δ of the classical plateau, $(1-\delta)H_{\mathcal{S}}$. It can be seen that the intrinsic dynamics of the system reduces the redundancy R_δ. This figure is based on an actual simulation similar to ones being performed in this paper.

Chapter 4

A Qubit Interacting with a Spin Environment

Here we put together all of the ideas previously discussed to gain a better intuition of decoherence and Quantum Darwinism by exploring an important model of decoherence: A qubit interacting with a spin environment. So far we have considered when the environment is only composed of a few spins. Here we generalize by enlarging the size of the environment. This model is exactly solvable and displays many of the important features of decoherence and Quantum Darwinism. What's more is, despite it's simplicity, the model of a qubit interacting with a spin environment has been used to explain the behavior of the Loschmidt echo appearing in NMR experiments [20], as well as models dominated by interactions with localized modes such as paramgnetic spins, electronic impurities, and defects, in the low temperature regime [30]. Moreover, as this model studies a qubit interacting with other qubits, it is naturally relevant to scenarios of quantum computation. Interestingly, an interacting spin model is often used as a substitute to the very realistic scenario of a system interacting with a photon bath.

The model consists of a central spin system $\mathcal{S}$ coupled to an environment $\mathcal{E}$ of spins. The Hamiltonian governing the dynamics of the interaction is given by

$$\mathbf{H} = \Delta\sigma_{\mathcal{S}}^{x} + \frac{1}{2}\sigma_{\mathcal{S}}^{z}\sum_{k=1}^{N} g_k\sigma_k^z \tag{4.1}$$

where $\Delta\sigma_{\mathcal{S}}^{x} = \mathbf{H}_{\mathcal{S}}$ is the system self-Hamiltonian, and $\frac{1}{2}\sigma_{\mathcal{S}}^{z}\sum_{k=1}^{N} g_k\sigma_k^z = \mathbf{H}_{\mathcal{SE}}$ is the interaction Hamiltonian. Here $\sigma_{\mathcal{S}}^{z}$ and σ_k^z are the Pauli z-spin operators of the system and the kth environment

subsystem respectively, and similarly for $\sigma_{\mathcal{S}}^x$. Moreover, g_k represents the coupling strength of the environmental spins, while Δ denotes the system-self interaction strength. For now we will consider the case when $\Delta = 0$, i.e., the system is static: $\mathcal{S}$ has no intrinsic dynamics. Later on we will consider when $\Delta \neq 0$, a more realistic model. Finally, we note that N is the number of two-level systems of the environment. In other words, N represents the size of the environment. Sometimes in the literature the size of the environment is instead notated as $\sharp\mathcal{E}$. Here we will use both freely.

4.1 A Static System and Gaussian Decoherence

In the static case, $\Delta = 0$, (or, as we will see, for a *purely decohering model*) the Hamiltonian becomes

$$\mathbf{H} = \frac{1}{2}\sigma_{\mathcal{S}}^z \sum_{k=1}^{N} g_k \sigma_k^z \tag{4.2}$$

The Hamiltonian is of the product form $\Lambda_{\mathcal{S}} \otimes \Omega_{\mathcal{E}}$ and, thus, we can easily determine the pointer states of the system. Precisely, an environment of spins monitors the z-component of the system ($\sigma_{\mathcal{S}}^z$). We therefore know what to expect for the pointer basis: the eigenbasis of $\sigma_{\mathcal{S}}^z$, i.e., $\{|0\rangle, |1\rangle\}$. That is, the eigenbasis of the z-component of the system is expected to be the preferred basis of the system (the robust states of $\mathcal{S}$), while superpositions of the eigenbasis will be prone to decoherence.

Moreover, due to the diagonal form of the Hamiltonian, the eigenbasis of the environmental spins, often denoted as $\{|\uparrow\rangle_k, |\downarrow\rangle_k\}$, are another basis that can be used to diagonalize the Hamiltonian. The energy eigenstates of the environment of this Hamiltonian are given as the products of the basis states of each environment subsystem. That is

$$|n\rangle \equiv |n_1\rangle \otimes |n_2\rangle \otimes ... \otimes |n_N\rangle = |n_1 n_2 ... n_N\rangle \tag{4.3}$$

where $0 \leq n \leq 2^N - 1$ and n_k represents the up or down component along the z-axis of the kth environment subsystem [13]. Therefore,

$$\sigma_k^z |n_k\rangle = (\pm 1)|n_k\rangle = (-1)^{n_k}|n_k\rangle \tag{4.4}$$

where $n_k = 0$ if the kth environmental spin is in the up state and $n_k = 1$ if the kth environmental spin is in the down state.

Before we move to study the dynamics of the system, let us make an assumption about our initial state. We begin by assuming that initially, the system and environment are uncorrelated. Moreover,

we make the assumption that initially the state of the composite system $\mathcal{SE}$ is pure. Then, at time $t = 0$, just before $\mathcal{S}$ and $\mathcal{E}$ begin interacting, the initial state $|\psi(0)\rangle$ can be written as the following product state:

$$|\psi(0)\rangle = (a|0\rangle + b|1\rangle) \sum_{n=0}^{2^N-1} c_n|n\rangle \tag{4.5}$$

where the '$\otimes$' is implied. The state of the composite system at some later time t is found by simple unitary evolution:

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = e^{-i\mathbf{H}t}|\psi(0)\rangle = \exp\left(-it\left[\frac{1}{2}\sigma_{\mathcal{S}}^z \sum_{k=1}^{N} g_k \sigma_k^z\right]\right)|\psi(0)\rangle \tag{4.6}$$

This can be easily solved by noting that

$$U(t)|0\rangle|n\rangle = e^{-it/2\sum_{k=1}^{N} g_k(-1)^{n_k}}|0\rangle|n\rangle = e^{-iB_n t/2}|0\rangle|n\rangle \tag{4.7}$$

where we have defined the energy associated with the eigenstate $|n\rangle$ to be

$$B_n \equiv \sum_{k=1}^{N}(-1)^{n_k} g_k \tag{4.8}$$

Putting everything together we find

$$|\psi(t)\rangle = a|0\rangle|\mathcal{E}_0(t)\rangle + b|1\rangle|\mathcal{E}_1(t)\rangle \tag{4.9}$$

where

$$|\mathcal{E}_0(t)\rangle = |\mathcal{E}_1(-t)\rangle = \sum_{n=0}^{2^N-1} c_n e^{-iB_n t/2}|n\rangle \tag{4.10}$$

The interpretation of $|\psi(t)\rangle$ is straightforward. The basis states of the system, $|0\rangle$ and $|1\rangle$, have become correlated with the relative states of the environment, $|\mathcal{E}_0(t)\rangle$ and $|\mathcal{E}_1(t)\rangle$. As these relative states become more distinguishable, i.e., become nearly orthogonal, $\langle\mathcal{E}_1(t)|\mathcal{E}_0(t)\rangle \approx 0$, information about the basis states of $\mathcal{S}$ is encoded in $\mathcal{E}$.

Due to these features, the overlap $\langle\mathcal{E}_1(t)|\mathcal{E}_0(t)\rangle$ has been termed as the *decoherence factor* $r(t)$ (sometimes denoted as $\Gamma(t)$), given by

$$r(t) \equiv \langle\mathcal{E}_1(t)|\mathcal{E}_0(t)\rangle = \sum_{n=0}^{2^N-1} |c_n|^2 e^{-iB_n t} \tag{4.11}$$

with $\sum_{n=0}^{2^N-1} |c_n|^2 = 1$.

In decoherence theory, the object of interest is the state of the system at some later time, $\rho_{\mathcal{S}}(t)$. In the present case, the time evolved state of the system is easy to solve for:

$$\rho_{\mathcal{S}}(t) = \mathrm{tr}_{\mathcal{E}} \, |\psi(t)\rangle\langle\psi(t)| = \sum_{m=0}^{2^N-1} \langle m| \left(|\psi(t)\rangle\langle\psi(t)\right) |m\rangle$$

$$= |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1| + ab^* r(t)|0\rangle\langle 1| + a^* b r^*(t)|1\rangle\langle 0| \tag{4.12}$$

Notice what happens as the relative states $|\mathcal{E}_0(t)\rangle$ and $|\mathcal{E}_1(t)\rangle$ become more distinguished, i.e., as $r(t) \to 0$. Then, the off-diagonal terms of the reduced density matrix of the state of the system vanish:

$$\rho_{\mathcal{S}}(t) \to |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1| \tag{4.13}$$

exemplifying the fact that the diagonal elements of the density matrix of the system are time independent, allowing us to identify $|0\rangle$ and $|1\rangle$ as the preferred basis of the system, as expected.

The interesting physics pertaining to this model, of course, lies with the time evolution of the decoherence factor $r(t)$, as well as how $r(t)$ depends on the size of the environment. With a set of rather generic assumptions, the form of $r(t)$ given above can be analyzed [13]. We only highlight the key results, requiring that N is sufficiently large, and the couplings g_k are concentrated near their average value such that their standard deviation exists and is finite [13]. The general behavior can be interpreted as an N-step random walk. This can be understood intuitively. Given a distribution of g_k, the contribution of the kth spin to the random energy is $\pm g_k$. Or, from our expression of the decoherence factor, $r(t)$ is the addition of 2^N vectors of length $|c_n|^2$ rotating with frequencies B_n. Then each of the 2^N steps of the random walk will have length $|c_n|^2$ following a direction weighted by the energy B_n. From here, and using the assumptions described above, $r(t)$ will exhibit Gaussian decay in time

$$r(t) \approx e^{i\bar{B}_n t} e^{-\Gamma^2 t^2} \tag{4.14}$$

where the constant of decay Γ will depend on the environment couplings g_k and, the initial state of the environment, and where $\bar{B}_n$ is related to the average value of B_n. With these fairly general assumptions, we say that the model considered displays *Gaussian decoherence*.

From the form of the decoherence factor $r(t)$ above, we notice interesting behavior. Provided that the size of the environment is finite, there exists some recurrence time τ_{rec} such that the decoherence factor will return to its initial value of one. We expect this even from the unapproximated form

of $r(t)$, as it is simply the sum of terms periodic in time. Of course, the precise value of τ_{rec} will depend on the distribution of g_k. So what does this mean for the quantum-to-classical transition? As pointed out in Ref. [31], for macroscopic but finite environment sizes the recurrence time can exceed the age of the universe! Therefore, for all intensive purposes, decoherence can be viewed practically as an irreversible process, and the quantum-to-classical transition, as described by decoherence, is safe.

4.2 Pure Decoherence and Quantum Darwinism

The model of a qubit interacting with an environment of qubits also exemplifies key features of Quantum Darwinism. We begin with a brief discussion about *pure decoherence*. Models that select a definite pointer observable which always disseminate redundant information are said to *purely decohere* $\mathcal{S}$ [15]. Purely decohering models are given by Hamiltonians of the form [16]

$$\mathbf{H} = \mathbf{H}_{\mathcal{S}} + \mathbf{H}_{\mathcal{SE}} + \mathbf{H}_{\mathcal{E}} \tag{4.15}$$

$$= \mathbf{H}_{\mathcal{S}} + \hat{\Pi}_{\mathcal{S}} \sum_{k=1}^{\sharp\mathcal{E}} \Upsilon_k + \sum_{k=1}^{\sharp\mathcal{E}} \Omega_k \tag{4.16}$$

with $\left[\hat{\Pi}_{\mathcal{S}}, \mathbf{H}_{\mathcal{S}}\right] = 0$, where $\mathbf{H}_{\mathcal{S}}$ and $\hat{\Pi}_{\mathcal{S}}$ are Hermitian operators on $\mathcal{S}$, and Υ_k and Ω_k are Hermitian operators on the kth environment subsystem. Also, in this model of several environment components interacting independently with $\mathcal{S}$, we consider the initial product state

$$\rho(0) = \rho_S(0) \otimes \left[\bigotimes_{k=1}^{\sharp\mathcal{E}} \rho_k(0) \right] \equiv \rho_{\mathcal{S}}(0) \otimes \rho_{\mathcal{E}}(0) \tag{4.17}$$

that is, as before, the system and environment are assumed to be initially uncorrelated. For clarity, let us suppose that the system begins in some arbitrary pure state

$$|\psi(0)\rangle_{\mathcal{S}} = \sum_i c_i |s_i\rangle \tag{4.18}$$

which is then coupled to the environment $\mathcal{E}$ initially as described above. We say that the system undergoes *pure decoherence* by a monolithic environment when time evolution is given by a unitary cast in the form [32]

$$U(t) = e^{-it\mathbf{H}} = \sum_i |s_i\rangle\langle s_i| \otimes U_i(t) \tag{4.19}$$

where $U_i(t)$ are unitaries acting on the environment. From here we find that the basis set $\{|s_i\rangle\}$ forms a pointer basis for the system.

Looking back at the above model of a qubit interacting with an environment of qubits (for $\Delta = 0$), we see that it falls under the category of pure decoherence. We can make a further assumption and consider the case when a qubit interacts with a *symmetric* spin environment, in which all of the environmental couplings satisfy $g_k = g$ for all k; all of the environment spins couple to the system in the same way. We also recognize that a symmetric environment also means that our environment is *permutationally invariant*, a feature we will take advantage of later.

With this additional simplification, the Hamiltonian is [15]

$$\mathbf{H} = \frac{1}{2}\sigma_{\mathcal{S}}^{z}\sum_{k=1}^{\sharp\mathcal{E}}\sigma_{k}^{z} \tag{4.20}$$

where $\frac{1}{2}\sigma_{\mathcal{S}}^{z}\sum_{k=1}^{\sharp\mathcal{E}}\sigma_{k}^{z} = \mathbf{H}_{\mathcal{SE}}$ is the interaction Hamiltonian. In this basis the system is initially described by

$$\rho_S(0) = \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} \tag{4.21}$$

We also take the initial state of the environment $\bigotimes_k \rho_k(0)$ to have $\rho_k(0) = \rho_r$ for all k. In the $\sigma_{\mathcal{S}}^{z}$ basis ρ_r takes the form

$$\rho_r = \begin{pmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{pmatrix} \tag{4.22}$$

and hence

$$\bigotimes_{k=1}^{\sharp\mathcal{E}}\rho_k(0) = \rho_r^{\otimes^{\sharp\mathcal{E}}} \tag{4.23}$$

This model also causes pure decoherence of the $\mathcal{S}$'s state into the eigenstates of $\sigma_{\mathcal{S}}^{z}$, the pointer basis of the system. One can go further still and, using this specific example, show that in pure decoherence models information disseminates into the environment regardless of whether $\mathcal{E}$ is in an initially pure or mixed state [15]. Thus, the preexisting entropy of $\mathcal{E}$ (the so-called *haziness*) does not destroy the environment's ability to acquire information. Despite the fact that nonideal initial conditions (e.g. haziness) reduce $\mathcal{E}$'s ability to store information about the system, the environment can still redundantly receive information about $\mathcal{S}$, and can transmit this information redundantly. Thus, Quantum Darwinism remains robust for (more realistic) nonideal environments.

4.3 Intrinsic Dynamics of the System

Let us return to the Hamiltonian of a central spin system interacting with a spin environment (not a *symmetric* spin environment):

$$\mathbf{H} = \Delta\sigma_{\mathcal{S}}^{x} + \frac{1}{2}\sigma_{\mathcal{S}}^{z}\sum_{k=1}^{N} g_k\sigma_k^z \tag{4.24}$$

and now ask how does the state of the system $\rho_{\mathcal{S}}$ change. That is, now we include the intrinsic dynamics of the system through the inclusion of a system self-Hamiltonian, and we ask how the state of our system might change with respect to non-zero Δ.

The first step to compute $\rho_{\mathcal{S}}(t)$ is finding the unitary which governs the time evolution of the composite system. The eigenstates of σ_k^z commute with $\mathbf{H}$, thus we can write

$$\Delta\sigma_{\mathcal{S}}^{x} + \frac{1}{2}\sigma_{\mathcal{S}}^{z}\sum_{k=1}^{N} g_k\sigma_k^z = \sum_{n=0}^{2^N-1}(\Delta\sigma_{\mathcal{S}}^{x} + B_n\sigma_{\mathcal{S}}^{z})\otimes|n\rangle\langle n| \tag{4.25}$$

The unitary then is given by

$$U(t) = \prod_{n=0}^{2^N-1} e^{-it(\Delta\sigma_{\mathcal{S}}^{x}+B_n\sigma_{\mathcal{S}}^{z})}\otimes|n\rangle\langle n| \tag{4.26}$$

Rewriting the exponential factor

$$e^{-it(\Delta\sigma_{\mathcal{S}}^{x}+B_n\sigma_{\mathcal{S}}^{z})} = \cos\left(t(\Delta\sigma_{\mathcal{S}}^{x} + B_n\sigma_{\mathcal{S}}^{z})\right) - i\sin\left(t(\Delta\sigma_{\mathcal{S}}^{x} + B_n\sigma_{\mathcal{S}}^{z})\right) \tag{4.27}$$

then using the Taylor series for $\cos x$ and $\sin x$, and keeping in mind that $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ we find that the unitary can be cast as

$$U(t) = \prod_{n=0}^{2^N-1} U_{B_n}(t)\otimes|n\rangle\langle n| \tag{4.28}$$

where

$$U_{B_n}(t) = \cos(\Omega_n t)I - i\left(\frac{\Delta\sigma_{\mathcal{S}}^{x} + B_n\sigma_{\mathcal{S}}^{z}}{\Omega_n}\right)\sin(\Omega_n t) \tag{4.29}$$

with $\Omega_n^2 = \Delta^2 + B_n^2$. Written in this form we may interpret that for every energy eigenstate $|n\rangle$ of $\mathcal{E}$ the dynamics of the system is caused by a magnetic field $\vec{\Omega}_n = (\Delta, 0, B_n)$, and thus the decoherence is a byproduct of the dispersion of the B_n "fields" [13].

Just as before we assume that initial state of the composite system can be written as a product

between the initial state of the system and the initial state of the environment:

$$|\psi(0)\rangle = |\psi(0)\rangle_{\mathcal{S}} \otimes \left(\sum_{n=0}^{2^N-1} c_n |n\rangle \right) \tag{4.30}$$

It follows that the time evolved state is given by

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = \sum_{n=0}^{2^N-1} c_n U_{B_n}(t)|\psi(0)\rangle_{\mathcal{S}} \otimes |n\rangle \tag{4.31}$$

resulting in the state of the system $\rho_{\mathcal{S}}(t)$ at some later time t

$$\rho_{\mathcal{S}}(t) = \mathrm{tr}_{\mathcal{E}} \, |\psi(t)\rangle\langle\psi(t)| = \sum_{n=0}^{2^N-1} |c_n|^2 U_{B_n}(t)\rho_{\mathcal{S}}(0)U_{B_n}^{\dagger}(t) \tag{4.32}$$

with the initial state of the system $\rho_{\mathcal{S}}(0) = |\psi(0)\rangle_{\mathcal{S}}\langle\psi(0)|$ being a pure state.

The important question for decoherence now is: What are the pointer states selected by the environment? To answer this question we follow the approach in Ref. [13]. We begin by rewriting the state $\rho_{\mathcal{S}}(t)$ in terms of the *Bloch polarization vector*. That is, recall that any density operator can be cast as

$$\rho = \frac{1}{2}(I + \vec{p} \cdot \vec{\sigma}) \tag{4.33}$$

where $\vec{p} = \mathrm{tr}(\rho\vec{\sigma})$ is the Bloch polarization vector. The reason we choose to represent $\rho_{\mathcal{S}}$ in this way is because the polarization vector tells us what eigenstates the state of system can be decomposed into. For example, consider when $p_x = p_y = 0$, then

$$\rho \to \frac{1}{2} \begin{pmatrix} 1 + p_z & 0 \\ 0 & 1 - p_z \end{pmatrix} \tag{4.34}$$

Thus, the eigenstates of ρ are simply the eigenstates of σ_z. Similar arguments hold for $p_x \neq 0$ and $p_y \neq 0$.

This method will help us identify the basis states of the system which are robust to decoherence by looking at the asymptotic behavior of the components of the polarization vector. In other words, $\vec{p}(t)$ will allow us to identify the pointer states of the system. For example, suppose that in the asymptotic limit (as t approaches the decoherence time t_{dec}) $p_x(t) \to 0$ and $p_y(t) \to 0$ while $p_z(t)$ settles at some constant value. We may then say that the eigenstates of σ_z, $\{|0\rangle, |1\rangle\}$, form the stable pointer basis for the system.

Therefore, our task now is to determine the components of the polarization vector $p_x(t), p_y(t)$ and $p_z(t)$. To solve this problem for large environment sizes, see Ref. [13]. For small environment sizes (around $N \approx 15 - 30$), the components of the polarization vector can be computed directly. We start by using

$$p_i(t) = \mathrm{tr}(\rho_{\mathcal{S}}(t)\sigma_i) \tag{4.35}$$

for $i = x, y, z$, and where

$$\rho_S(t) = \sum_{n=0}^{2^N-1} |c_n|^2 U_{B_n} \rho_{\mathcal{S}}(0) U_{B_n}^\dagger(t) \tag{4.36}$$

with

$$U_{B_n}(t) = \cos(\Omega_n t) I - i \left(\frac{\Delta \sigma_{\mathcal{S}}^x + B_n \sigma_{\mathcal{S}}^z}{\Omega_n} \right) \sin(\Omega_n t) \tag{4.37}$$

with $\Omega_n^2 = \Delta^2 + B_n^2$. To find analytic forms of $p_i(t)$, we first compute $U_{B_n}(t)\rho_{\mathcal{S}}(0)U_{B_n}^\dagger(t)$. With a little algebra we find

$$U_{B_n}(t)\rho_{\mathcal{S}}(0)U_{B_n}^\dagger(t) = \cos^2(\Omega_n t)\rho_{\mathcal{S}}(0) + \frac{i \sin(\Omega_n t)\cos(\Omega_n t)}{\Omega_n} \left[\rho_{\mathcal{S}}(0), \Delta\sigma_x^{\mathcal{S}} + B_n \sigma_z^{\mathcal{S}}\right]$$

$$+ \frac{\sin^2(\Omega_n t)}{\Omega_n^2} (\Delta\sigma_x^{\mathcal{S}} + B_n\sigma_z^{\mathcal{S}}) \rho_{\mathcal{S}}(0) (\Delta\sigma_x^{\mathcal{S}} + B_n\sigma_z^{\mathcal{S}}) \tag{4.38}$$

Now,

$$p_x(t) = \mathrm{tr}(\rho_{\mathcal{S}}(t)\sigma_x) = \sum_{n=0}^{2^N-1} |c_n|^2 \,\mathrm{tr}(U_{B_n}(t)\rho_{\mathcal{S}}(0)U_{B_n}^\dagger(t)\sigma_x) \tag{4.39}$$

Assuming that the initial state of the system is $|\psi(0)\rangle_{\mathcal{S}} = a|0\rangle + b|1\rangle$ with $a, b \in \mathbb{R}$, after a bit of algebra we find

$$\mathrm{tr}(U_{B_n}(t)\rho_{\mathcal{S}}(0)U_{B_n}^\dagger(t)\sigma_x) = 2ab\cos^2(\Omega_n t) + \frac{\sin^2(\Omega_n t)}{\Omega_n^2} \left(2ab(\Delta^2 - B_n^2) + 2(a^2 - b^2)\Delta B_n\right) \tag{4.40}$$

Similarly,

$$\mathrm{tr}(U_{B_n}(t)\rho_{\mathcal{S}}(0)U_{B_n}^\dagger(t)\sigma_y) = \frac{\cos(\Omega_n t)\sin(\Omega_n t)}{\Omega_n}(4abB_n + 2\Delta(b^2 - a^2)) \tag{4.41}$$

$$\mathrm{tr}(U_{B_n}(t)\rho_{\mathcal{S}}(0)U_{B_n}^\dagger(t)\sigma_z) = \cos^2(\Omega_n t)(a^2 - b^2) + \frac{\sin^2(\Omega_n t)}{\Omega_n^2}(4ab\Delta B_n + (b^2 - a^2)(\Delta^2 - B_n^2)) \tag{4.42}$$

To finish the calculation we simply take the sum over n, which is where we see the need for approximations when it comes to large environment sizes. Here, since everything has been computed exactly, we can study the dynamics of the components of the polarization for any size of Δ, however, let us also study the dynamics of the two limiting cases: (1) small Δ relative to N and (2) large Δ

relative to N. The figure below exemplifies the behavior of cases (1) and (2).

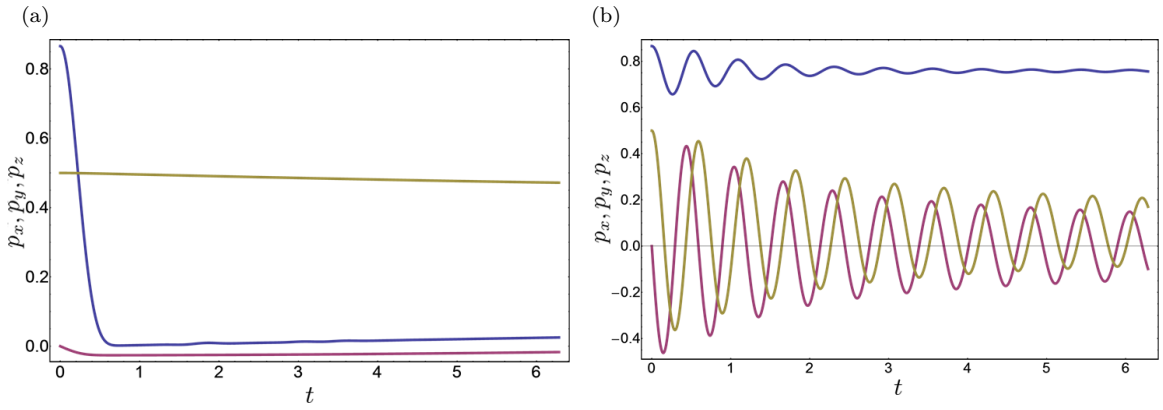

Figure 4.1: A plot of the components of the polarization vector $p_x(t)$ (blue), $p_y(t)$ (purple), and $p_z(t)$ (gold) for $\sharp\mathcal{E} = 15$, $a = \sqrt{3}/2$, $b = 1/2$, and $|\alpha_k|^2 = |\beta_k|^2 = 1/2$ for all k. (a) and (b) Components of the polarization vector with random couplings g_k for $\Delta = 0.1$, weak self-dynamics and $\Delta = 5$, strong self-dynamics, respectively. For weak self-dynamics $p_x(t)$ and $p_y(t)$ both decay rapidly from their initial values where $p_y(t)$ settles around zero and $p_x(t)$ settles at a small but finite value, while $p_z(t)$ approaches a constant value well above zero, indicating that the pointer states of the system are the eigenstates of the interaction Hamiltonian σ_z. For strong self-dynamics, after a period of fast oscillations, the eigenstates of the system self-Hamiltonian, σ_x, emerge as the pointer states.

From the figure we see that for weak self-dynamics the components $p_x(t)$ and $p_y(t)$ both decay rapidly from their initial values. Specifically, $p_y(t)$ decays toward zero while $p_x(t)$ decays to a finite value just above zero. On the other hand, $p_z(t)$ decays only slightly, settling at some constant value well above zero. This indicates that the eigenstates of σ_z, the eigenstates of the interaction Hamiltonian, (almost) emerge as the pointer states of the system, just as when $\Delta = 0$. We say almost since $p_x(t) \neq 0$ in the asymptotic regime, and thus adds small contributions to the eigenstates of σ_z. For case (2), strong self-dynamics, $p_y(t)$ and $p_z(t)$ oscillate while $p_x(t)$ settles at some finite value, indicating that the basis states of σ_x, the eigenstates of the system self-Hamiltonian, dynamically emerge as the preferred states of the system. There is a third case, the intermediary picture, where there is a balance between the strength of self-dynamics of the system and the environment, in which the pointer states are a combination of the eigenstates of σ_z and σ_x.

We now have a fairly complete picture of the dynamics of a qubit interacting with an environment of qubits. With the inclusion of a system self-Hamiltonian, the pointer states of the system depend on the strength of $\mathcal{S}$'s intrinsic dynamics Δ. For strong self-dynamics, the preferred basis states are the eigenstates of the system self-Hamiltonian while for weak self-dynamics the pointer basis states are the eigenstates of system part of the interaction Hamiltonian. So far we have only considered questions important to decoherence theory. In the next section we aim to understand how

information proliferates in the environment for this same model of a central spin system interacting with a spin environment, i.e., we study questions important to Quantum Darwinism. Studying the dissemination of information in a symmetric spin environment with the inclusion of a system self-Hamiltonian is the main premise of the next section as well as the central analysis of this work.

Chapter 5

Redundant Information Beyond Pure Decoherence

5.1 Qubit Interacting with a Symmetric Environment

Recall that we say models which select a definite pointer observable, and always disseminate redundant information, exhibit pure decoherence. Such models are given by Hamiltonians of the form [16]

$$\mathbf{H} = \mathbf{H}_{\mathcal{S}} + \mathbf{H}_{\mathcal{SE}} + \mathbf{H}_{\mathcal{E}} \tag{5.1}$$

$$= \mathbf{H}_{\mathcal{S}} + \hat{\Pi}_{\mathcal{S}} \sum_{k=1}^{\sharp\mathcal{E}} \Upsilon_k + \sum_{k=1}^{\sharp\mathcal{E}} \Omega_k \tag{5.2}$$

with $\left[\hat{\Pi}_{\mathcal{S}}, \mathbf{H}_{\mathcal{S}}\right] = 0$, where $\mathbf{H}_{\mathcal{S}}$ and $\hat{\Pi}_{\mathcal{S}}$ are Hermitian operators on $\mathcal{S}$, and Υ_k and Ω_k are Hermitian operators on the kth environment subsystem. Motivated by the previous sections, we make a generalization of the model and a couple of simplifications. First, we require that the kth self-Hamiltonians (Ω_k) on the subsystems are all zero, thereby not contributing to $\mathbf{H}$ at all.

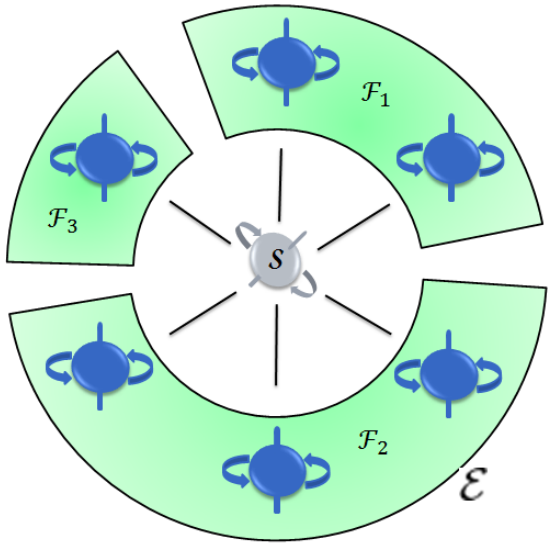

Figure 5.1: A central spin $\mathcal{S}$ interacting with a symmetric environment of spins $\mathcal{E}$. Initially $\mathcal{S}$ and $\mathcal{E}$ are uncorrelated. As the system and environment interact, information about the pointer states of the system proliferates redundantly into many fragments of $\mathcal{E}$. Independent observers intercept a fraction of these fragments and acquire information about the pointer states, independently and without disturbing the state of the system, thereby reaching consensus on the classical state of the system. Our results show that the intrinsic dynamics of the system determine how much of the environment an observer must intercept to gain information about the state of the system, and hence the fragility of redundant information.

We consider a system self-Hamiltonian ($\mathbf{H}_{\mathcal{S}}$) and an interaction operator ($\hat{\Pi}_{\mathcal{S}}$) that don't commute, $\left[\hat{\Pi}_{\mathcal{S}}, \mathbf{H}_{\mathcal{S}}\right] \neq 0$, and that the pointer observable and the interaction operators (Υ_k) do not have any degenerate eigenvalues. As before, we consider the initial product state

$$\rho(0) = \rho_S(0) \otimes \left[\bigotimes_{k=1}^{\sharp\mathcal{E}} \rho_k(0) \right] \tag{5.3}$$

Remember that we aim to study how information proliferates in the environment, and thus we must determine the mutual information between the system $\mathcal{S}$ and a fragment of the environment $\mathcal{F}$. Computing the mutual information is a straightforward task: we must determine the entropies of $\rho_{\mathcal{S}}(t), \rho_{\mathcal{SF}}(t)$, and $\rho_{\mathcal{F}}(t)$. Immediately, however, one realizes that this is not a trivial feat systems where the size of the environment $\sharp\mathcal{E}$ becomes large. To overcome this particular difficulty, here we consider a decoherence model we introduced before: A solvable qubit system interacting with a *symmetric* qubit environment

$$\mathbf{H} = \Delta\sigma_{\mathcal{S}}^x + \frac{1}{2}\sigma_{\mathcal{S}}^z \sum_{k=1}^{\sharp\mathcal{E}} \sigma_k^z \tag{5.4}$$

where $\Delta\sigma_{\mathcal{S}}^x = \mathbf{H}_{\mathcal{S}}$ is the system self-Hamiltonian, and $\frac{1}{2}\sigma_{\mathcal{S}}^z \sum_{k=1}^{\sharp\mathcal{E}} \sigma_k^z = \mathbf{H}_{\mathcal{SE}}$ is the interaction Hamiltonian. Previous work [15] has examined the case when $\Delta = 0$. In this case we have a pure decoherence model in which the interaction Hamiltonian causes pure decoherence of the state of the system into its pointer basis, or, as we have shown, the eigenstates of $\sigma_{\mathcal{S}}^z$. Here we will explore the consequences of $\Delta \neq 0$.

In either case, we still have that the system is initially described by

$$\rho_S(0) = \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} \tag{5.5}$$

and we also take the initial state of the environment $\bigotimes_k \rho_k(0)$ to have $\rho_k(0) = \rho_r$ for all k. In the $\sigma_{\mathcal{S}}^z$ basis ρ_r takes the form

$$\rho_r = \begin{pmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{pmatrix} \tag{5.6}$$

and hence

$$\bigotimes_{k=1}^{\sharp\mathcal{E}} \rho_k(0) = \rho_r^{\otimes^{\sharp\mathcal{E}}} \tag{5.7}$$

In order to calculate the entropies necessary for the mutual information, we must first find the density matrices $\rho_{\mathcal{S}}(t), \rho_{\mathcal{SF}}(t)$, and $\rho_{\mathcal{F}}(t)$. In appendix B we show that $\rho_{\mathcal{S}}(t)$ takes the form

$$\rho_{\mathcal{S}}(t) = \sum_{n}^{\sharp\mathcal{E}} \binom{\sharp\mathcal{E}}{n} r_{00}^{\sharp\mathcal{E}-n} r_{11}^n U_{0,2n}(t) \rho_{\mathcal{S}}(0) U_{0,2n}^\dagger(t) \tag{5.8}$$

with

$$U_{0,2n}(t) = \exp[-it(\Delta\sigma_{\mathcal{S}}^x + 1/2(\sharp\mathcal{E} - 2n)\sigma_S^z)] \tag{5.9}$$

We also show $\rho_{\mathcal{SF}}(t)$ is given by

$$\rho_{\mathcal{SF}}(t) = \sum_{n=0}^{\sharp\mathcal{E}/\mathcal{F}} \binom{\sharp\mathcal{E}/\mathcal{F}}{n} r_{00}^{\sharp\mathcal{E}/\mathcal{F}-n} r_{11}^n U_{\sharp\mathcal{F},2n}(t) \rho_S(0) \otimes \rho_r^{\otimes^{\sharp\mathcal{F}}} U_{\sharp\mathcal{F},2n}^\dagger(t) \tag{5.10}$$

where

$$U_{\sharp\mathcal{F},2n}(t) = \exp\left[-it\left(\Delta\sigma_{\mathcal{S}}^x + \frac{1}{2}(\sharp\mathcal{E}/\mathcal{F} - 2n)\sigma_{\mathcal{S}}^z + \frac{1}{2}\sigma_{\mathcal{S}}^z \sum_{k\in\mathcal{F}} \sigma_k^z\right)\right] \tag{5.11}$$

are unitary operators. Lastly, we show that $\rho_{\mathcal{F}}(t)$ can be written as

$$\rho_{\mathcal{F}}(t) = \sum_{n=0}^{\sharp\mathcal{E}/\mathcal{F}} \binom{\sharp\mathcal{E}/\mathcal{F}}{n} r_{00}^{\sharp\mathcal{E}/\mathcal{F}-n} r_{11}^{n}$$

$$\times \sum_{j=0}^{\sharp\mathcal{F}/2} \sum_{m,m'=-j}^{j} p_{jmm'} \operatorname{tr}\left(U_{\sharp\mathcal{F},2n-2m}(t)\rho_{\mathcal{S}}(0)U_{\sharp\mathcal{F},2n-2m'}(-t)\right) |jm\rangle\langle jm'| \otimes I_{B_j} \tag{5.12}$$

where $U_{\sharp\mathcal{F},2n-2m}(t)$ and $U_{\sharp\mathcal{F},2n-2m'}(-t)$ are unitaries derived using a specific technique involving Wigner D matrices and *Schur-Weyl (-like) duality*, as outlined in appendix C. Using this rotation technique allows us to calculate the mutual information in a much simpler way, avoiding the obvious computational cost of calculating the entropies straight out. In fact, it can be shown that this approach takes a problem that is $2^{3^{\sharp\mathcal{E}}}$ computationally complex, and reduces it to a $\sharp\mathcal{E}^4$ (approximately) complex problem, i.e., going from a problem solved in exponential time to one solved in polynomial time. The details of these calculations can be found in appendices D and E. We now highlight the main results of this work.

5.2 Symmetric vs. Non-Symmetric Environments

The model of a qubit interacting with a symmetric environment of qubits, i.e., $g_k = g$ for all k, is important because it is a completely solvable model, especially when it comes to computing the mutual information. However, the symmetric model of continuously interacting spins displays unphysical recurrences as discussed earlier. We wish to determine the regions of validity by comparing the results to a non-symmetric model.

We start by comparing the dynamics of the components of the polarization vector for both symmetric and non-symmetric environments. Here we will compare when $\Delta = 0.1$, when there is weak intrinsic system dynamics, and when $\Delta = 1$, the intermediary case system dynamics. Using our results from the previous chapter, we again only consider small environment sizes ($\sharp\mathcal{E} \approx 15-30$), which will suffice for our comparison. Below are some plots of the components of the polarization vector in these regimes.

From these plots we make two observations: (1) For long times the components of the non-symmetric environment model become more 'stable' than those computed using the symmetric environment model, and (2) for times $t \leq \pi/2$, the dynamics of $p_x(t), p_y(t)$ and $p_z(t)$ of both models is roughly the same (we can and will make this more precise), indicating our first region of validity. For longer times, i.e., $t > \pi/2$, the symmetric model has recurrences, a non-realistic feature.

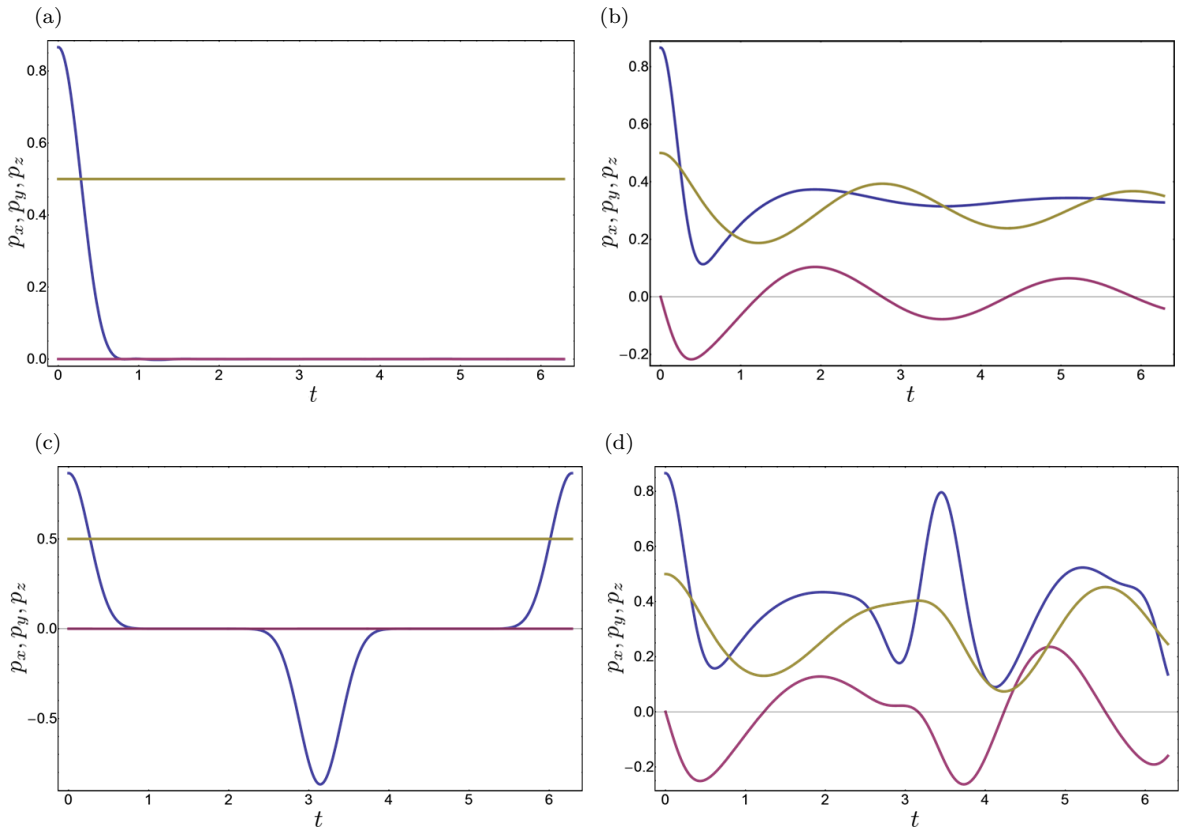

Figure 5.2: A plot of the components of the polarization vector $p_x(t)$ (blue), $p_y(t)$ (purple), and $p_z(t)$ (gold) for $\sharp\mathcal{E} = 15$, an initial state of the system with $a = \sqrt{3}/2$, $b = 1/2$, and $|\alpha_k|^2 = |\beta_k|^2 = 1/2$, using both symmetric and non-symmetric environments. (a) and (b): Using a non-symmetric environment with random couplings g_k for $\Delta = .1$ and $\Delta = 1$, repsectively. (c) and (d) Using a symmetric environment, setting all $g_k = g = 1/2$, for $\Delta = 1/10$ and $\Delta = 1$, respectively. Notice that the non-symmetric model has more 'stable' components of the polarization vector, dissimilar to that of the symmetric model, however for times shorter than $t = \pi/2$, we see that the components as calculated in each model gives fair comparison.

Although the dynamics of the components of the polarization vector computed in each model closely match in this time regime, we can do better and find a near exact match for short times. To do this, we study the decoherence factor $r(t)$ when $\Delta = 0$, in both the symmetric and non-symmetric environment models. When $\Delta = 0$, starting with with the following initial states of the system and kth environment

$$|\psi(0)\rangle_{\mathcal{S}} = a|0\rangle + b|1\rangle \qquad |\psi(0)\rangle_k = \alpha_k|0\rangle_k + \beta_k|1\rangle_k \tag{5.13}$$

it is straightforward to show that the decoherence factor takes the form [13]:

$$r(t) = \prod_{k=1}^{N} \cos(g_k t) \tag{5.14}$$

where we have made the simplifying assumption that $|\alpha_k|^2 = |\beta_k|^2 = 1/2$. In the case when $g_k = g$ for all k, i.e., for symmetric environments, this decoherence factor becomes

$$r(t) = \cos^N(gt) \tag{5.15}$$

Making a short time approximation, we find

$$\prod_{k=1}^{N} \cos(g_k t) \approx \prod_{k=1}^{N} \left(1 - \frac{(g_k t)^2}{2}\right) \approx \prod_{k=1}^{N} e^{-g_k^2 t^2/2} = \exp\left[-\sum_{k=1}^{N} g_k^2 t^2/2\right] \tag{5.16}$$

and

$$\cos^N(gt) \approx e^{-g^2 t^2 N/2} \tag{5.17}$$

If we demand that the short time decay overlap, we place a constraint on the choice of symmetric coupling g, namely, we demand that

$$g^2 = \frac{1}{N} \sum_{k=1}^{N} g_k^2 \tag{5.18}$$

When we use this constraint, we find that indeed the dynamics of the components of the polarization vector from the symmetric and non-symmetric environment models match very near exactly. It turns out that even for $\Delta \neq 0$ this constraint yields close to perfect agreement between the two models as well.

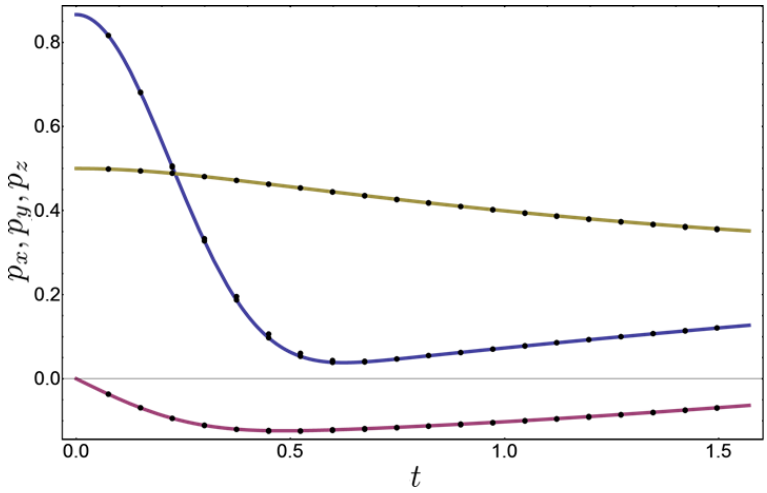

Figure 5.3: A plot of the components of the polarization vector $p_x(t)$ (blue), $p_y(t)$ (purple), and $p_z(t)$ (gold) for $\sharp\mathcal{E} = 15$, $\Delta = 1/2$, an initial state of the system with $a = \sqrt{3}/2$, $b = 1/2$, $|\alpha_k|^2 = |\beta_k|^2 = 1/2$, and $g^2 = \frac{1}{N}\sum_{k=1}^{N} g_k^2$, using both symmetric and non-symmetric environments. The two models are overlayed, with the block dots indicating the vector components computed using the symmetric environment model, indicating that indeed when $\Delta = 0$, there is a region in which the symmetric model agrees with the more realistic non-symmetric environment.

In summary, to make realistic claims about the physics of the symmetric environment model we (1) consider time regions where $t \leq \pi/2$ and (2) demand that the symmetric coupling g satisfy $g^2 = 1/N \sum_{k=1}^{N} g_k^2$. With that, let us move on to the main results of this work.

5.3 Mutual Information

Here we explore the parameter space of the mutual information for a variety of values of Δ. First consider the case when $\Delta = 0$. Initially ($t = 0$) the system $\mathcal{S}$ and the environment $\mathcal{E}$ are uncorrelated, indicating that the environment contains no information about the system. As time evolves however correlations arise between $\mathcal{S}$ and $\mathcal{E}$, encoding information about the pointer states of the system. After "good" decoherence occurs, when the environment carries sufficient information about the system to be intercepted, a plateau develops in the mutual information as a function of $\sharp\mathcal{F}$. This classical plateau exemplifies the transition from quantum-to-classical by indicating that classical information has propagated throughout the environment. That is, redundant, and therefore classical, information has disseminated throughout the environment, waiting to be intercepted by independent observers.

Now consider when $\Delta \neq 0$. For weak system coupling ($\Delta \approx 0.1$), and again for pure $\mathcal{S}$ and pure $\mathcal{E}$, the mutual information has similar behavior to the case when $\Delta = 0$; redundant information

about the pointer states of the system proliferates into the environment, and again, the classical plateau develops. For intermediate values of the Δ (here, $\Delta \approx 1-2$) the classical plateau begins to tilt, i.e., becomes less defined, indicating that the strength of the system self-interaction determines how much of the environment must be intercepted in order to acquire information about the pointer states of $\mathcal{S}$. For strong system coupling (here, $\Delta \approx 5$), the plateau becomes even less defined, exemplifying that for too large of Δ, an observer must intercept almost the entire environment to gain information about the state of $\mathcal{S}$.

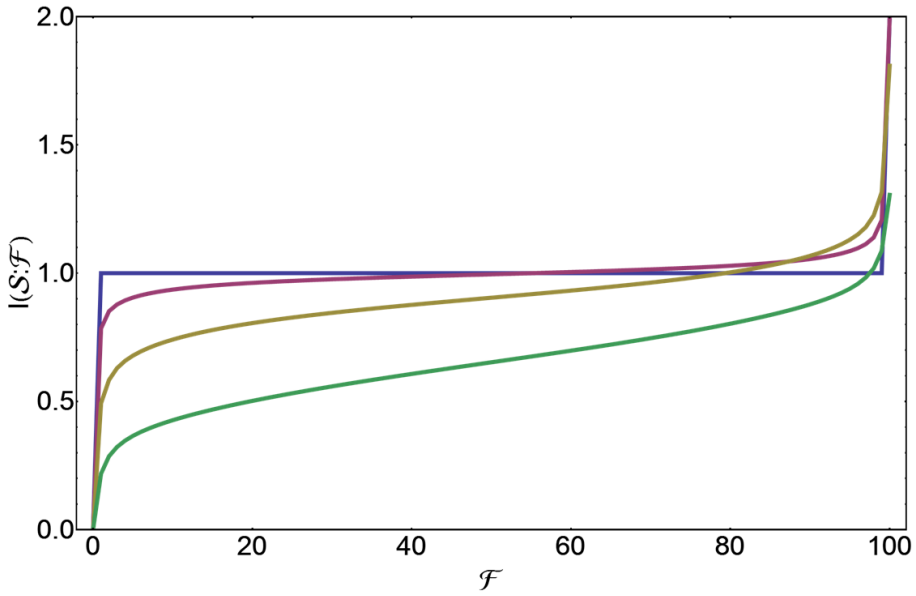

Figure 5.4: A plot of the mutual information $I(\mathcal{S} : \mathcal{F})$ of a system $\mathcal{S}$ vs. fragment size $\sharp\mathcal{F}$, for $\Delta = 0$ (blue),1/2 (purple), 2 (gold), 5 (green), and $\sharp\mathcal{E} = 100$ (color online). In each case $r_{00} = s_{00} = 1/2$, with an initially pure $\mathcal{S}$, and $H_S(0) = 0$. A plateau emerges more readily for smaller values of Δ, almost becoming non-existent for larger values of Δ, indicating strong system-self coupling reduces the redundancy. Note that for an environment size large enough, Δ will be overcome by the environment interaction such that Δ relative to $\sharp\mathcal{E}$ allows for no serious change to the classical plateau.

The reason for this behavior is as follows: Our current model can be interpreted as a universe of spins. Decoherence allows for this universe to divide into a system spin interacting with an environment of spins. For weak system coupling the system self-Hamiltonian is not strong compared to the system-environment Hamiltonian, i.e., the system is spinning "slow" enough that the environment is able to gain information about the state of the system. This leads to the presence of a classical plateau. As Δ increases, the system spins too fast for the environment to effectively monitor and acquire information about the system, as signified by the lack of a clearly defined plateau. In short, a dominant system self-Hamiltonian reduces the redundancy and forces observers to have access to

more and more of the environment in order to gain information about $\mathcal{S}$.

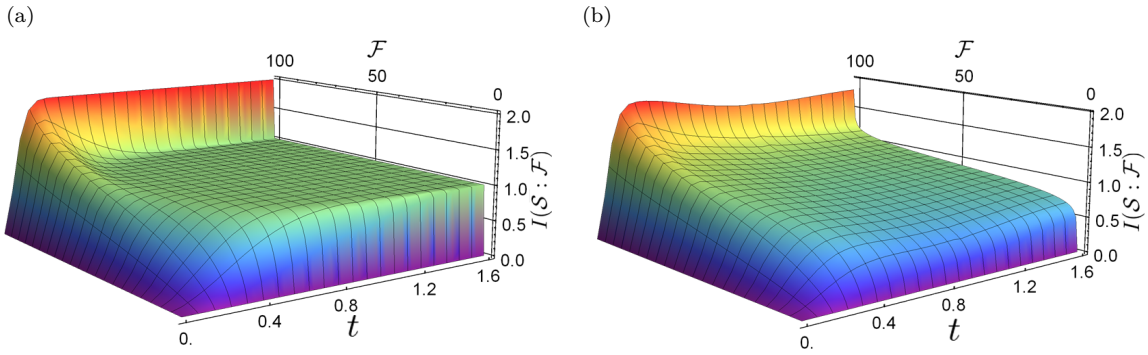

Figure 5.5: A plot of mutual information versus fragment size $\sharp\mathcal{F}$ and time t for $\sharp\mathcal{E} = 100$, and $s_{00} = r_{00} = 1/2$. (a) No system self-Hamiltonian ($\Delta = 0$). (b) Including a system self-Hamiltonian with moderate strength ($\Delta = 2$). For both cases, initially ($t = 0$) the system $\mathcal{S}$ and the environment $\mathcal{E}$ are uncorrelated, indicating that the environment contains no information about the system. As time evolves correlations arise between $\mathcal{S}$ and $\mathcal{E}$, encoding information about the pointer states of the system. When the environment carries sufficient information about the system to be intercepted, a plateau develops in the mutual information as a function $\sharp\mathcal{F}$. In (a) the classical plateau is well defined, while in (b) the plateau is not as definitive, and begins to lose its form at a later time, indicating recurrences.

5.4 Redundancy

Now we ask how redundant is the information that is stored in the environment. That is, we aim to find how the system's intrinsic dynamics, determined by the strength of Δ, affects the redundancy R_δ. Below (Fig.5.6) is a plot of the redundancy for the information deficit $\delta = 0.1$. As hinted by the mutual information, the redundancy is reduced by the system's intrinsic dynamics, but, not by much. Notice that the redundancy is constant from $\Delta = 0 \to \Delta \approx 1/10$. Thus, for small enough Δ, i.e., if the system's intrinsic dynamics is weak enough, R_δ is finite, and hence information about the pointer states of the system is redundantly encoded and then proliferated into the environment. We recognize then that redundant information is robust in the presence of a system self-Hamiltonian characterizing the intrinsic dynamics of the system.

Indeed, as the stength Δ increases beyond $\Delta \approx 1/10$, the redundancy quickly drops off due to the symmetry of the environment, however once we reach $\Delta \approx 1/5$, R_δ continues to decreases toward its minimum value (to see the full spectrum of R_δ we should plot for larger Δ). This shows that as the strength of the intrinsic dynamics increases, the capacity for the environment to encode and proliferate redundant information is reduced, as expected. However, again we emphasize, for small enough Δ, the redundancy is a constant, finite value indicating that the presence of a

system self-Hamiltonian does not entirely destroy the environment's ability to disseminate redundant information.

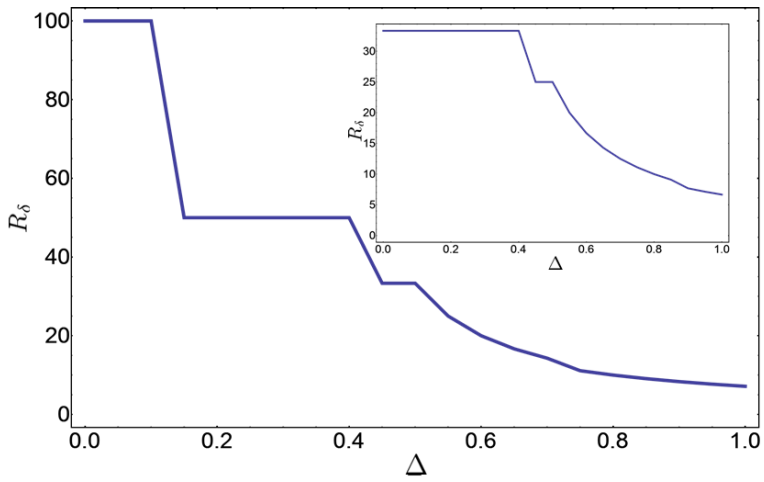

Figure 5.6: A plot of the redundancy R_δ (calculated with the mutual information) as a function of the coupling strength of the system self-Hamiltonian Δ, i.e., the system's intrinsic dynamics at $t = \pi/3$ (inset $t = \pi/4$), $s_{00} = r_{00} = 1/2$, $\sharp\mathcal{E} = 100$, and an information deficit of $\delta = 1/10$. Notice that for small enough Δ, the redundancy R_δ is non-zero and constant, indicating that the environment has the abilitiy to encode and proliferate redundant information about the pointer states of the system even when the system has intrinsic dynamics. Note that the stepping behavior occurs because R_δ is defined as a rational number for spin environments.

5.5 Conclusions and Future Work

We now have a more complete picture of how information proliferates through an environment when the system has some intrinsic dynamics. Using a specific model of a central spin system interacting with a symmetric qubit environment, we were able to show for relatively weak system dynamics the redundancy R_δ is at a constant and finite value well above its minimum. This indicates that, although the presence of a system self-Hamiltonian reduces the redundancy, the environment still maintains the capacity to encode redundant information about the pointer states of the system. Alternatively, for stronger system self-dynamics, starting around $\Delta = 1/2$, the redundancy is further reduced toward its minimum value for the associated Δ, indicating that the environment has lost its ability to transmit redundant information about the pointer states of the system.

Indeed, the fact that redundancy is insensitive to small system self-Hamiltonians is crucial for realistic models, as physical scenarios have systems with intrinsic dynamics. More foundational, we have shown that Quantum Darwinism itself is robust to the system's self-dynamics, indicating that it can be applied to a wider class of physical models.

In light of this work, there are still some interesting features of this model of a spin interacting with a symmetric spin environment to explore. First and foremost, we have found that redundant information about the system can be encoded and proliferated through the environment. The question is, what states are associated with this redundant information? As seen in previous sections, for $\Delta = 0$, the pointer states are identified as the eigenstates of the interaction Hamiltonian σ_z, and, for non-symmetric environments, for small Δ the pointer states are found to be the eigenstates of σ_z plus some small correction from the eigenstates of the system self-Hamiltonian.

We therefore want to study the emergence and stability of the pointer states. What are these states for the symmetric and non-symmetric models? How fast does the system's state approach the pointer state when embedded in the symmetric environment? These are the pertinent questions we seek to answer. We can use the Holevo quantity and the components of the Bloch polarization vector to make qualitative arguments about what we expect the pointer basis to be, for example, consider the figure below. We must be careful when we do this, however, as notice what happens when we compare the two plots of the Holevo quantity: They appear to almost be equivalent, however one occurs when $\Delta = 1/2$, i.e., when the redundancy R_δ is at a minimum. In this sense, it appears that a "pseudo"-pointer basis develops. Determining exactly what the pointer states are will give us further understanding of what states the redundant information belongs to, and is therefore important in the context of decoherence and Quantum Darwinism.

Lastly, as noted, we have moved beyond pure decoherence models by including a system self-Hamiltonian that does not commute with the interaction Hamiltonian, and shown that indeed the environment is still capable of acting as a communication channel. One way we can further generalize this model, making it more realistic, is to include an environment self-Hamiltonian in the total Hamiltonian governing the dynamics of the composite system. That is, it would be interesting to study a model described by a Hamiltonian of the form:

$$\mathbf{H} = \Delta\sigma_{\mathcal{S}}^{x} + \frac{1}{2}\sigma_{\mathcal{S}}^{z}\sum_{k=1}^{\sharp\mathcal{E}}\sigma_{k}^{z} + \sum_{k=1}^{\sharp\mathcal{E}}\Upsilon_{k} \tag{5.19}$$

where, for instance, $\Upsilon = \eta\sigma_k^x$, and where η quantifies the strength of the environment's intrinsic dynamics. Using this Hamiltonian we would ask the same questions, specifically how information proliferates in the environment, what kind of information it is, while at the same time, provide us with a more realistic model to test Quantum Darwinism.

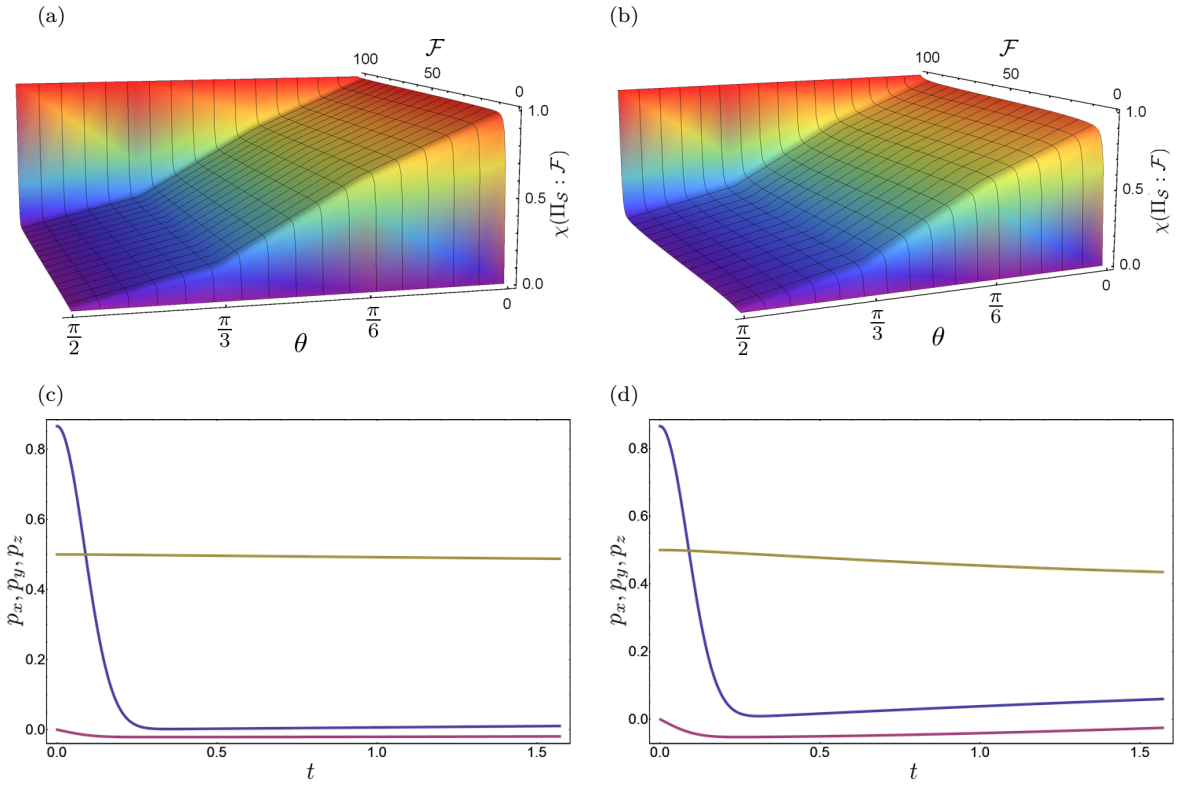

Figure 5.7: Finding a stable pointer basis. The Holevo quantity χ is plotted for $\sharp\mathcal{E} = 100$, $\phi = 0$, and $t = \pi/4$ in (a) for $\Delta = 1/5$, and (b) for $\Delta = 1/2$. (c) and (d) are plots of the components of the Bloch polarization vector $p_x(t)$ (blue), $p_y(t)$ (purple), and $p_z(t)$ (gold). In each case $r_{00} = 1/2, s_{00} = 3/4$, and $\sharp\mathcal{E} = 100$. (c) When $\Delta = 1/5$, the coupling at the edge of the redundancy plateau and (b) when $\Delta = 1/2$, the coupling when R_δ is almost at a minimum value, when the environment no longer can transmit redundant information. Comparing (b) and (d) we recognize that for larger values Δ a "pseudo"-pointer basis may form.

Chapter 6

Conclusions

Quantum physics describes well the microscopic world. Classical mechanics concerns itself with macroscopic objects. Classical objects, though, are made of quantum components and therefore quantum physics should be able to describe them. In its current framework (based on the Copenhagen interpretation and the collapse postulate), there is no mechanism that describes this quantum-to-classical transition. Quantum Darwinism provides a concrete framework for understanding how the classical world arises from the quantum world. It does so by recognizing the role of the environment as a communication channel, from which independent observers may acquire information and reach consensus about the state of the system, i.e., objectivity.

To gain intuition for decoherence we mostly considered a specific model, that of a central spin system interacting with an environment of spins. Using this example we were able to explore the elementary details of decoherence, including environment-induced superselection, and pointer states. Based on this basic principles were demonstrated that decoherence can also be viewed as the leakage of information about the system into the environment, leading to the emergence of classicality.

We went further by analyzing how information proliferates into the environment using the framework of Quantum Darwinism. Again, using the model of a system qubit interacting with an environment of qubits, we studied how Quantum Darwinsim augments decoherence and helps realize the quantum-to-classical transition. Specifically, we considered the spin-spin model in a few different cases: (1) Non-symmetric versus symmetric environments, (2) a system with and without intrinsic dynamics. For a non-symmetric environment without intrinsic dynamics, we conveyed that such a system exhibits Gaussian decoherence – the decoherence factor displays Gaussian behavior [13]. We studied the non-symmetric environment model further by including the intrinsic dynamics of the

system. There we showed that the pointer states of the system depend on the strength of the system self-Hamiltonian, Δ. For weak intrinsic dynamics, the eigenstates of the interaction Hamiltonian are approximately the pointer states, while for strong intrinsic dynamics, the eigenstates of the system's Hamiltonian are approximately the preferred states.

Moreover, we demonstrated that although the presence of a system self-Hamiltonian reduces the redundancy, the environment is still capable of encoding redundant information about the pointer states of the system. That is, we showed that the redundancy remains at a constant, finite value, indicating that the environment is able to act as a communication channel, transmitting redundant information. Therefore, even in the presence of a system self-Hamiltonian, redundant information is resilient.

Our work gives us a more complete picture of the propagation of information in an environment interacting with a quantum system. We formulated a technique that allows for the exploration of composite spin systems beyond pure decoherence, and allows for the comparison to other physically relevant models of decoherence.

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Appendix A

Evolution, Correlations, and Entropy

A.1 Operator-Sum Formalism

There is another way to mathematically represent decoherence, based on the *operator-sum representation*, and it is quite useful, particularly when computing the Holevo quantity. Consider a system $\mathcal{S}$ with states located in an associated Hilbert space $\mathcal{H}_{\mathcal{S}}$, and an environment $\mathcal{E}$ with an associated Hilbert space $\mathcal{H}_{\mathcal{E}}$. To study the dynamics of the combined system we use the joint Hamiltonian operator which acts on states in the joint Hilbert space $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$:

$$\mathbf{H} = \mathbf{H}_{\mathcal{S}} \otimes I_{\mathcal{E}} + I_{\mathcal{S}} \otimes \mathbf{H}_{\mathcal{E}} + \mathbf{H}_{\mathcal{SE}} \tag{A.1}$$

where $\mathbf{H}_{\mathcal{S}}$ is the system Hamiltonian acting on $\mathcal{H}_{\mathcal{S}}$, $\mathbf{H}_{\mathcal{E}}$ is the environment Hamiltonian acting on $\mathcal{H}_{\mathcal{E}}$, and $\mathbf{H}_{\mathcal{SE}}$ is the interaction Hamiltonian which we write as

$$\mathbf{H}_{\mathcal{SE}} = \sum_k \Pi_k^{\mathcal{S}} \otimes \Upsilon_k^{\mathcal{E}} \tag{A.2}$$

where $\Pi_k^{\mathcal{S}}$ is a Hermitian operator acting on the kth subsystem of the entire system, and $\Upsilon_k^{\mathcal{E}}$ is a Hermitian operator that acts on the kth subsystem of the environment. The time evolution of the

density operator for the combined system is given in the familar way:

$$\rho(t) = U(t)\rho(0)U^\dagger(t) \tag{A.3}$$

where $U(t) = \exp(-it\mathbf{H})$. As usual, we assume that the system and environment are not initially entangled allowing us to write the initial state of the combined system $\rho_\mathcal{S}$ as a product state $\rho(0) = \rho_\mathcal{S}(0) \otimes \rho_\mathcal{E}(0)$. Thus,

$$\rho(t) = U(t)(\rho_\mathcal{S}(0) \otimes \rho_\mathcal{E}(0))U^\dagger(t) \tag{A.4}$$

This coupling of the system exemplifies decoherence. To see this, let's consider the density operator of the system after the system and environment have interacted for a finite time t:

$$\rho_\mathcal{S}(t) = \mathrm{tr}_\mathcal{E}(U(t)(\rho_\mathcal{S}(0) \otimes \rho_\mathcal{E}(0))U^\dagger(t)) \tag{A.5}$$

Now writing the state of the environment as $\rho_\mathcal{E}(0) = \sum_i a_i|i\rangle\langle i|$, the above becomes

$$\rho_\mathcal{S}(t) = \sum_k \langle k| \left(U(t)\rho_\mathcal{S}(0) \otimes \sum_i a_i|i\rangle\langle i|U^\dagger(t) \right) |k\rangle = \sum_{i,k} \sqrt{a_i}\langle k|U(t)|i\rangle\rho_\mathcal{S}(0)\sqrt{a_i}\langle i|U^\dagger(t)|k\rangle$$

$$\equiv \sum_\mu M_\mu \rho_\mathcal{S}(0) M_\mu^\dagger \tag{A.6}$$

where

$$M_\mu = \sqrt{a_i}\langle k|U(t)|i\rangle \tag{A.7}$$

is a called a *superoperator* or a *Kraus* operator. Using the fact that $U(t)$ is a unitary operator, it's easy to check that the superoperator satisfies

$$\sum_\mu M_\mu^\dagger M_\mu = I_\mathcal{S} \tag{A.8}$$

From here we see that

$$\mathrm{tr}\left(\sum_\mu M_\mu \rho_\mathcal{S}(0) M_\mu^\dagger \right) = \sum_\mu \mathrm{tr}\left(M_\mu \rho_\mathcal{S}(0) M_\mu^\dagger \right) = \mathrm{tr}\left(\sum_\mu M_\mu^\dagger M_\mu \rho_\mathcal{S}(0) \right) = 1$$

as expected. In this sense superoperators are to be viewed as a linear map from density operators to density operators.

The importance of the operator-sum representation and superoperators lies in the fact that it

hands us with a formalism to witness decoherence in a more elegant way. Notice that if there is only one term in the operator sum we can describe it's evolution unitarily. On the other hand, if the sum involves two or more terms, the evolution of the system cannot be described with a unitary operator (although the joint system and environment still obeys unitary evolution); the initially uncorrelated system and environment have become entangled, exemplifying decoherence. In this way, the operator-sum formalism provides a framework that demonstrates the effect of the environment as a sequence of transformations on the reduced density operator of the system, formally describing how the reduced density matrix, $\rho_{\mathcal{S}}(t)$, evolves in time. Despite its elegance, the use of Kraus operators is typically not very practical (finding the Kraus operators involves diagonalizing the entire Hamiltonian of the composite system), and so we do not make use of the formalism here.

A.2 Schmidt's Decomposition Theorem

Schmidt's decomposition theorem is as follows: Let two systems A and B be endowed with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Then, an arbitrary pure state $|\psi\rangle$ of the composite system AB can always be written in diagonal form

$$|\psi\rangle = \sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle \tag{A.9}$$

where $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ are orthonormal (Schmidt) bases of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively and where the λ_i satisfy

$$\sum_i |\lambda_i|^2 = 1 \tag{A.10}$$

We can in fact choose $\lambda_i = \sqrt{p_i}$ so that [20]

$$|\psi\rangle = \sum_i \sqrt{p_i} |a_i\rangle \otimes |b_i\rangle \tag{A.11}$$

which is unique if and only if the coefficients $\sqrt{p_i}$ are all different. Before we replicate the proof of the theorem, we first note a corollary: Given the composite system as described above, the reduced density matrices ρ_A and ρ_B have the same von Neumann entropy, i.e., $H_A = H_B$. The proof is direct and straightforward. By Schmidt's decomposition theorem we have

$$\rho_A = \mathrm{tr}_B \, |\psi\rangle\langle\psi| = \sum_k \langle b_k| \left(\sum_{i,j} \sqrt{p_i p_j} |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j| \right) |b_k\rangle$$

$$= \sum_i p_i |a_i\rangle\langle a_i| \tag{A.12}$$

Likewise,

$$\rho_B = \sum_i p_i |b_i\rangle\langle b_i| \tag{A.13}$$

Thus, ρ_A and ρ_B, while diagonal in their respective bases, share the same eigenvalue spectrum, and hence, have the same entropy

$$H(\rho_A) = H(\rho_B) \tag{A.14}$$

Therefore, using the Schmidt theorem we can confirm the bound of the mutual information between a system and fragment of the environment. As the fragment size approachs the size of the environment, $\sharp\mathcal{F} = \sharp\mathcal{E}$, we have that

$$I(\mathcal{S} : \mathcal{E}) = H_{\mathcal{S}} + H_{\mathcal{E}} - H_{\mathcal{SE}} = 2H_{\mathcal{S}} \tag{A.15}$$

since $H_{\mathcal{SE}} = 0$ and where we used the corollary above.

Let's now prove Schmidt's decomposition theorem. Let A and B be subsystems of a composite system AB endowed with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Here we assume that the state spaces have the same dimension (an assumption one can in fact relax, however we won't do so here). Also let $\{|j\rangle\}$ and $\{|k\rangle\}$ be any fixed orthonormal bases for A and B respectively. Then, any generic state vector of the joint Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$|\psi\rangle = \sum_{j,k} m_{jk} |j\rangle \otimes |k\rangle \tag{A.16}$$

for some matrix M of complex numbers m_{jk}. Using the *singular value decomposition* theorem, we may rewrite the matrix of coefficients M as $M = UDV$, where D is a diagonal matrix with non-negative elements and U and V are unitary matrices. Then, using $m_{jk} = \sum_i u_{ji} d_{ii} v_{ik}$, the state vector may be written as

$$|\psi\rangle = \sum_{i,j,k} u_{ji} d_{ii} v_{ik} |j\rangle \otimes |k\rangle \tag{A.17}$$

Let us now define

$$|a_i\rangle \equiv \sum_j u_{ji} |j\rangle \qquad |b_i\rangle \equiv \sum_k v_{ik} |k\rangle \tag{A.18}$$

and also $\lambda_i \equiv d_{ii}$. Indeed, $\{|a_i\rangle\}$ forms an orthonormal set (to check, use the fact that U is a unitary

operator and that $\{|j\rangle\}$ is orthonormal). Likewise for $\{|b_i\rangle\}$. Upon substitution we find

$$|\psi\rangle = \sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle \tag{A.19}$$

and thus completing the proof of the theorem (for the case of Hilbert spaces of the same size).

Appendix B

Calculating the Density Matrices $\rho_{\mathcal{S}}(t), \rho_{\mathcal{SF}}(t)$, and $\rho_{\mathcal{F}}(t)$

In this appendix we give the calculational details of each of the density matrices $\rho_{\mathcal{S}}(t), \rho_{\mathcal{SF}}(t)$, and $\rho_{\mathcal{F}}(t)$ in order to calculate the mutual information $I(\mathcal{S}:\mathcal{F})$.

B.1 Calculating $\rho_{\mathcal{S}}(t)$

We start by noting that density matrices time evolve as

$$\rho(t) = U(t)\rho(0)U^{\dagger}(t) \tag{B.1}$$

In our specific case we have that

$$\rho(t) = U(t)\left(\rho_{\mathcal{S}}(0)\otimes\rho_r(0)^{\otimes\sharp\mathcal{E}}\right)U^{\dagger}(t) \tag{B.2}$$

with

$$U(t) = \exp\left[-it\left(\Delta\sigma_{\mathcal{S}}^{x} + \frac{1}{2}\sigma_{\mathcal{S}}^{z}\sum_{k=1}^{\sharp\mathcal{E}}\sigma_k^{z}\right)\right] \tag{B.3}$$

Then to find $\rho_{\mathcal{S}}(t)$ we simply trace over the environment states,

$$\rho_{\mathcal{S}}(t) = \mathrm{tr}_{\mathcal{E}}\,\rho(t) = \sum_{i_1,\ldots,i_{\mathcal{E}}\in\{+1,-1\}}\langle i_1\ldots i_{\mathcal{E}}|\rho(t)|i_1\ldots i_{\mathcal{E}}\rangle \tag{B.4}$$

where the range for the i_k's are denoted as the eigenvalues of σ_k^z, which are either ± 1. From the context it should also be clear what the associated states are. In order to calculate this trace we must know how each environment state acts on the unitary $U(t)$. We have

$$\exp\left[it\left(\Delta\sigma_{\mathcal{S}}^x + \frac{1}{2}\sigma_{\mathcal{S}}^z\sum_{k=1}^{\mathcal{E}}\sigma_k^z\right)\right]|i_1...i_{\mathcal{E}}\rangle = \exp\left[it\left(\Delta\sigma_{\mathcal{S}}^x + \frac{1}{2}\sigma_{\mathcal{S}}^z\sum_{k=1}^{\mathcal{E}} i_k\right)\right]|i_1...i_{\mathcal{E}}\rangle \tag{B.5}$$

where we used

$$\exp[B + C\otimes A]|(\psi\rangle_{\mathcal{S}}\otimes|a\rangle_{\mathcal{E}}) = \exp[B + aC]|(\psi\rangle_{\mathcal{S}}\otimes|a\rangle_{\mathcal{E}}) \tag{B.6}$$

where $[B, C] \neq 0$, $[A, B] = [A, C] = 0$, and $A|a\rangle_{\mathcal{E}} = a|a\rangle_{\mathcal{E}}$. We gain a similar expression for the Hermitian conjugate, leaving us with

$$\rho_{\mathcal{S}}(t) = \sum_{i_1,...,i_{\mathcal{E}}\in\{+1,-1\}} U_k(t)\rho_{\mathcal{S}}(0)\langle i_1...i_{\mathcal{E}}|\rho_r^{\otimes\sharp\mathcal{E}}|i_1...i_{\mathcal{E}}\rangle U_k^\dagger(t) \tag{B.7}$$

$$= \sum_{i_1,...,i_{\mathcal{E}}\in\{+1,-1\}} U_k(t)\rho_{\mathcal{S}}(0)\prod_{k=1}^{\sharp\mathcal{E}}\langle i_k|\rho_r|i_k\rangle U_k^\dagger(t) \tag{B.8}$$

where

$$U_k(t) = \exp\left[-it\left(\Delta\sigma_{\mathcal{S}}^x + \frac{1}{2}\sigma_{\mathcal{S}}^z\sum_{k=1}^{\sharp\mathcal{E}} i_k\right)\right] \tag{B.9}$$

Since $\langle i_k|\rho_r|i_k\rangle = r_{00}$ or r_{11} depending on the value of i_k, we have for n r_{11}'s there are $\mathcal{E} - n$ r_{00}'s (that is, n of the i_k's are equal to -1). This also tells us that

$$\sum_{k=1}^{\sharp\mathcal{E}} i_k = (\sharp\mathcal{E} - n)(+1) + (n)(-1) = \sharp\mathcal{E} - 2n$$

Putting all of this together we find

$$\rho_{\mathcal{S}}(t) = \sum_{n=0}^{\sharp\mathcal{E}}\binom{\sharp\mathcal{E}}{n} r_{00}^{\sharp\mathcal{E}-n} r_{11}^n U_{0,n}(t)\rho_S(0)U_{0,n}^\dagger(t) \tag{B.10}$$

where we define $U_{0,X}(t)$ to be

$$U_{0,X}(t) = \exp\left[-it\left(\Delta\sigma_{\mathcal{S}}^x + \frac{1}{2}(\sharp\mathcal{E} - X)\sigma_{\mathcal{S}}^z\right)\right] \tag{B.11}$$

with, in this case, $X = 2n$, matching our expressions from before. Immediately one realizes that $\rho_{\mathcal{S}}(t)$ is simply a 2×2 matrix, and, without serious difficulty, can be diagonalized, a necessary

calculation for computing the entropy $\mathbf{H}_{\mathcal{S}}(t)$, and hence the mutual information.

B.2 Calculating $\rho_{\mathcal{SF}}(t)$

To find $\rho_{\mathcal{SF}}(t)$ a very similar idea as above is applied, however this time we use

$$\rho_{\mathcal{SF}}(t) = \mathrm{tr}_{\mathcal{E}/\mathcal{F}}\,\rho(t) \tag{B.12}$$

That is, we trace over the environment states, minus some fraction of the fragments

$$\rho_{\mathcal{SF}}(t) = tr_{\mathcal{E}/\mathcal{F}}\rho(t) = \sum_{i_1,\ldots,i_{\mathcal{E}/\mathcal{F}}\in\{+1,-1\}} \langle i_1 \ldots i_{\mathcal{E}/\mathcal{F}}|\rho(t)|i_1\ldots i_{\mathcal{E}/\mathcal{F}}\rangle \tag{B.13}$$

Applying a similar procedure as done for $\rho_{\mathcal{S}}(t)$ we have that

$$\rho_{\mathcal{SF}}(t) = \sum_{i_1,\ldots,i_{\mathcal{E}/\mathcal{F}}\in\{+1,-1\}} U_{\sharp\mathcal{F},n}(t)\rho_{\mathcal{S}}(0)\otimes\rho_r^{\otimes^{\sharp\mathcal{F}}}\prod_{k=1}^{\sharp\mathcal{E}/\mathcal{F}}\langle i_k|\rho_r|i_k\rangle U^{\dagger}_{\sharp\mathcal{F},n}(t) \tag{B.14}$$

Then, using the same combinatorial argument, we arrive to

$$\rho_{\mathcal{SF}}(t) = \sum_{n=0}^{\sharp\mathcal{E}/\mathcal{F}}\binom{\sharp\mathcal{E}/\mathcal{F}}{n} r_{00}^{\sharp\mathcal{E}/\mathcal{F}-n} r_{11}^{n} U_{\sharp\mathcal{F},2n}(t)\rho_S(0)\otimes\rho_r^{\otimes^{\sharp\mathcal{F}}}U^{\dagger}_{\sharp\mathcal{F},2n}(t) \tag{B.15}$$

where

$$U_{\sharp\mathcal{F},2n}(t) = \exp\left[-it\left(\mathbf{H}_{\mathcal{S}} + \frac{1}{2}(^{\sharp}\mathcal{E}/\mathcal{F}-2n)\sigma^z_{\mathcal{S}} + \frac{1}{2}\sigma^z_{\mathcal{S}}\sum_{k\in\mathcal{F}}\sigma^z_k\right)\right] \tag{B.16}$$

are unitaries.

Upon comparison of the above form of $\rho_{\mathcal{SF}}(t)$ to $\rho_{\mathcal{S}}(t)$, we recognize that $\rho_{\mathcal{SF}}(t)$ is far more difficult to diagonalize as we are no longer dealing with a 2×2 matrix. For this reason it behooves us to apply a technique which places $\rho_{\mathcal{SF}}(t)$ in a form which can then be diagonalized efficienctly to obtain the entropy of $\mathcal{SF}$. For details on this technique, see appendix C. Using this diagonalization procedure one finds

$$\rho_S(0)\otimes\rho_r^{\otimes^{\sharp\mathcal{F}}} \to \sum_{j=0}^{\sharp\mathcal{F}/2}\sum_{m,m'=-j}^{j} p_{jmm'}\rho_S(0)\otimes|jm\rangle\langle jm'|\otimes I_{B_j} \tag{B.17}$$

Hence,

$$\rho_{\mathcal{SF}}(t) = \sum_{n=0}^{\sharp\mathcal{E}/\mathcal{F}} \binom{\sharp\mathcal{E}/\mathcal{F}}{n} r_{00}^{\sharp\mathcal{E}/\mathcal{F}-n} r_{11}^{n}$$

$$\times \left[\sum_{j=0}^{\mathcal{F}/2} \sum_{m,m'=-j}^{j} p_{jmm'} U_{\sharp\mathcal{F},2n}(t)(\rho_{\mathcal{S}}(0) \otimes |jm\rangle\langle jm'|) U_{\sharp\mathcal{F},2n}(-t) \otimes I_{B_j} \right] \tag{B.18}$$

where the $p_{jmm'}$ are expansion coefficients corresponding to a specific angular momentum state $|jm\rangle$. Substitution leads to

$$\rho_{\mathcal{SF}}(t) = \sum_{n=0}^{\sharp\mathcal{E}/\mathcal{F}} \binom{\sharp\mathcal{E}/\mathcal{F}}{n} r_{00}^{\sharp\mathcal{E}/\mathcal{F}-n} r_{11}^{n}$$

$$\times \left[\sum_{j=0}^{\sharp\mathcal{F}/2} \sum_{m,m'=-j}^{j} p_{jmm'} U_{\sharp\mathcal{F},2n-2m}(t) \rho_{\mathcal{S}}(0) U_{\sharp\mathcal{F},2n-2m'}(-t) \otimes |jm\rangle\langle jm'| \otimes I_{B_j} \right] \tag{B.19}$$

where we used $\sum_{k\in\mathcal{F}} \sigma_k^z |jm\rangle = 2m|jm\rangle$ (recall $S_z|jm\rangle = m|jm\rangle$ and $S_z = 2\sigma_z$), and where we have defined

$$U_{\sharp\mathcal{F},2n-2m}(t) = \exp\left[-it \left(\Delta\sigma_{\mathcal{S}}^x + \frac{1}{2}(\sharp\mathcal{E}/\mathcal{F} - 2n + 2m)\sigma_{\mathcal{S}}^z \right) \right] \tag{B.20}$$

Now, when written in this form the calculation becomes a matter of computing $U_{\mathcal{S},j,m}\rho_{\mathcal{S}}(0)U_{\mathcal{S},j,m'}$, which is straightforward.

Even more, in this form, finding $\rho_{\mathcal{F}}(t)$ is also a straightforward calculation: simply trace out the states of the system since

$$\rho_{\mathcal{F}}(t) = \mathrm{tr}_{\mathcal{S}}\, \rho_{\mathcal{SF}}(t) \tag{B.21}$$

allowing for $\rho_{\mathcal{F}}(t)$ to be diagonalized in a more efficient way to obtain the entropy of $\mathcal{F}$. To see how these computations are implemented, see appendix D.

Appendix C

Diagonalizing $\rho_{\mathcal{SF}}(t)$: The Rotation Technique

In this appendix we outline the general procedure for diagonalizing $\rho_{\mathcal{SF}}(t)$. This technique follows previous work [15], and is also motivated by [33]. We start with $\rho_{\mathcal{SF}}(t)$ given above before the decomposition. First note that at $t = 0$ the state becomes slightly simpler:

$$\rho_{\mathcal{SF}}(0) = \sum_{n=0}^{\sharp\mathcal{E}/\mathcal{F}} \binom{\sharp\mathcal{E}/\mathcal{F}}{n} r_{00}^{\sharp\mathcal{E}/\mathcal{F}-n} r_{11}^{n} \rho_{\mathcal{S}}(0) \otimes \rho_r^{\otimes^{\sharp\mathcal{F}}} = \rho_{\mathcal{S}}(0) \otimes \rho_r^{\otimes^{\sharp\mathcal{F}}} \tag{C.1}$$

Now, our strategy is as follows: we first rewrite $\rho_r^{\otimes^{\sharp\mathcal{F}}}$ into direct sums of total spin states in such a way that the density matrix becomes block diagonal. This in effect makes calculating the entropy computationally feasible. We first make a unitary transformation to diagonalize ρ_r

$$\rho_r = \begin{pmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{pmatrix} \tag{C.2}$$

Alternatively this diagonalization procedure can be understood as rotating the density matrix with a Wigner D matrix $R(\alpha, \beta, \gamma)$ [15, 33]

$$R(\alpha, \beta, \gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \tag{C.3}$$

where J_x, J_y, and J_z are the components of the angular momentum (in this case these reduce to the usual Pauli matrices σ_x, σ_y, and σ_z). From this point of view, the representation of the density

matrix ρ_r changes from σ_z to $\sigma_{\vec{n}}$, where $\vec{n}$ is the Bloch spin vector.

The second step in this procedure is to decompose $\rho_r^{\otimes \sharp \mathcal{F}}$ into unitary and symmetric components using *Schur-Weyl duality*. This allows us to rewrite $\rho_r^{\otimes \sharp \mathcal{F}}$ into direct sums of total spin states by utilizing the Clebsch-Gordan coefficients, in which the representation then moves from $\sigma_{\vec{n}}$ to $|j,m\rangle_{\vec{n}}$. The third step involves moving from $|j,m\rangle_{\vec{n}}$ to $|j,m\rangle_z$ through the use of an inverse Wigner D matrix

$$R(-\gamma,-\beta,-\alpha) \equiv R^{-1}(\alpha,\beta,\gamma)$$

Once this is achieved, we tensor product in $\rho_{\mathcal{S}}(0)$ and evolve the total state using the unitaries $U_{\sharp\mathcal{F},2n}(t)$, yielding the desired expression.

Let us further proceed with the details of this procedure. It starts with determining the rotation angles α,β,γ. The Wigner D matrix is a square matrix with dimension $2j+1$ and a general element [15]

$$D^j_{m,m'} = \langle jm'|R(\alpha,\beta,\gamma)|jm\rangle = e^{-im'\alpha} d^j_{m',m}(\beta) e^{-im\gamma} \tag{C.4}$$

where

$$d^j_{m,m'} = \langle jm'|e^{-i\beta J_y}|jm\rangle$$

$$= [(j+m')!(j-m')!(j+m)!(j-m)!]^{1/2} \sum_{s=max(0,m-m')}^{min(j+m,j-m')} \frac{(-1)^{m'-m+s}}{(j+m-s)!s!(m'-m+s)!(j-m'-s)!}$$

$$\times (\cos(\beta/2))^{2j+m-m'-2s} (\sin(\beta/2))^{m'-m+2s} \tag{C.5}$$

The *Euler angles* are found from the unitary matrix that diagonalizes ρ_r. It's easy to work out that this unitary is simply

$$U_r = \begin{bmatrix} \frac{-r_{01}}{\sqrt{|r_{01}|^2+(r_{00}-\lambda_+)^2}}, & \frac{r_{00}-\lambda_+}{\sqrt{|r_{01}|^2+(r_{00}-\lambda_+)^2}} \\ \frac{r_{11}-\lambda_-}{\sqrt{|r_{10}|^2+(r_{11}-\lambda_+)^2}}, & \frac{-r_{10}}{\sqrt{|r_{01}|^2+(r_{00}-\lambda_+)^2}} \end{bmatrix} \tag{C.6}$$

where

$$\lambda_\pm = \frac{1}{2}\left(1 \pm \sqrt{(r_{00}-r_{11})^2 + 4r_{01}r_{10}}\right) \tag{C.7}$$

This unitary can be identified as the Wigner D matrix

$$D^{1/2}(\alpha,\beta,\gamma) = \begin{bmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2), & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}\sin(\beta/2), & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{bmatrix} \tag{C.8}$$

where the Euler angles are given by

$$\alpha = \gamma = 0 \tag{C.9}$$

$$\sin(\beta/2) = -\frac{r_{00} - \lambda_+}{\sqrt{|r_{01}|^2 + (r_{00} - \lambda_+)^2}} \qquad \cos(\beta/2) = \frac{-r_{01}}{\sqrt{|r_{01}|^2 + (r_{00} - \lambda_+)^2}} \tag{C.10}$$

It follows then that given $U_r \rho_r U_r^\dagger \to \text{Diag}[\lambda_+, \lambda_-]$,

$$\rho_r^{\otimes \sharp\mathcal{F}} \to \text{Diag}[\lambda_+, \lambda_-]^{\otimes \sharp\mathcal{F}} \tag{C.11}$$

If one then uses the Clebsch-Gordan coefficients, we recast the above as the direct sum of the total spin states [15],

$$\text{Diag}[\lambda_+, \lambda_-]^{\otimes \sharp\mathcal{F}} \to \bigoplus_{j=0}^{\sharp\mathcal{F}/2} \left(M_j^{\oplus B_j} \right) \tag{C.12}$$

where

$$M_j = \text{Diag}\left[\lambda_+^{\sharp\mathcal{F}/2+j} \lambda_-^{\sharp\mathcal{F}/2-j}, \lambda_+^{\sharp\mathcal{F}/2+j-1} \lambda_-^{\sharp\mathcal{F}/2-j+1}, ..., \lambda_+^{\sharp\mathcal{F}/2-j} \lambda_-^{\sharp\mathcal{F}/2+j} \right] \tag{C.13}$$

and B_j is the following combination of binomial coefficients:

$$B_j = \binom{\sharp\mathcal{F}}{\sharp\mathcal{F}/2 - j} - \binom{\sharp\mathcal{F}}{\sharp\mathcal{F}/2 - j - 1} \tag{C.14}$$

As noted before, currently the basis of $\rho_r^{\otimes \sharp\mathcal{F}}$ is $|jm\rangle_{\vec{n}}$. In order to make this expression more useful we must transform this basis into $|jm\rangle_z$, which can be done using the inverse Wigner D matrix mentioned before:

$$\bigoplus_{j=0}^{\sharp\mathcal{F}/2} \left(M_j^{\oplus B_j} \right) \to \bigoplus_{j=0}^{\sharp\mathcal{F}/2} \left[e^{-i(-\gamma)J_z} e^{-i(-\beta)J_y} e^{-i(-\alpha)J_z} M_j e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \right]^{\oplus B_j} \tag{C.15}$$

Ultimately, this allows us write $\rho_{\mathcal{SF}}(t)$ in a block diagonal form in the basis $|jm\rangle_z$. This can be diagonalized more efficiently, allowing for an easier time computing the entropy of $\mathcal{SF}$ and $\mathcal{F}$, and therefore the mutual information $I(\mathcal{S} : \mathcal{F})$. From a computational standpoint, use of the Wigner D matrices transforms a problem of size $2^{\sharp\mathcal{E}} \to \approx \sharp\mathcal{E}^4$, i.e., moving from a problem solved in exponential time to a problem solved in polynomial time (recall that the computational complexity of diagonalizing a $d \times d$ matrix is proportional to d^3, and $d \approx \sharp\mathcal{E}$). Below is a plot comparing a couple of mutual information curves computed using a direct approach, and using the rotation technique for small environment sizes, indicating this computing technique is valid in regions which can be checked directly.

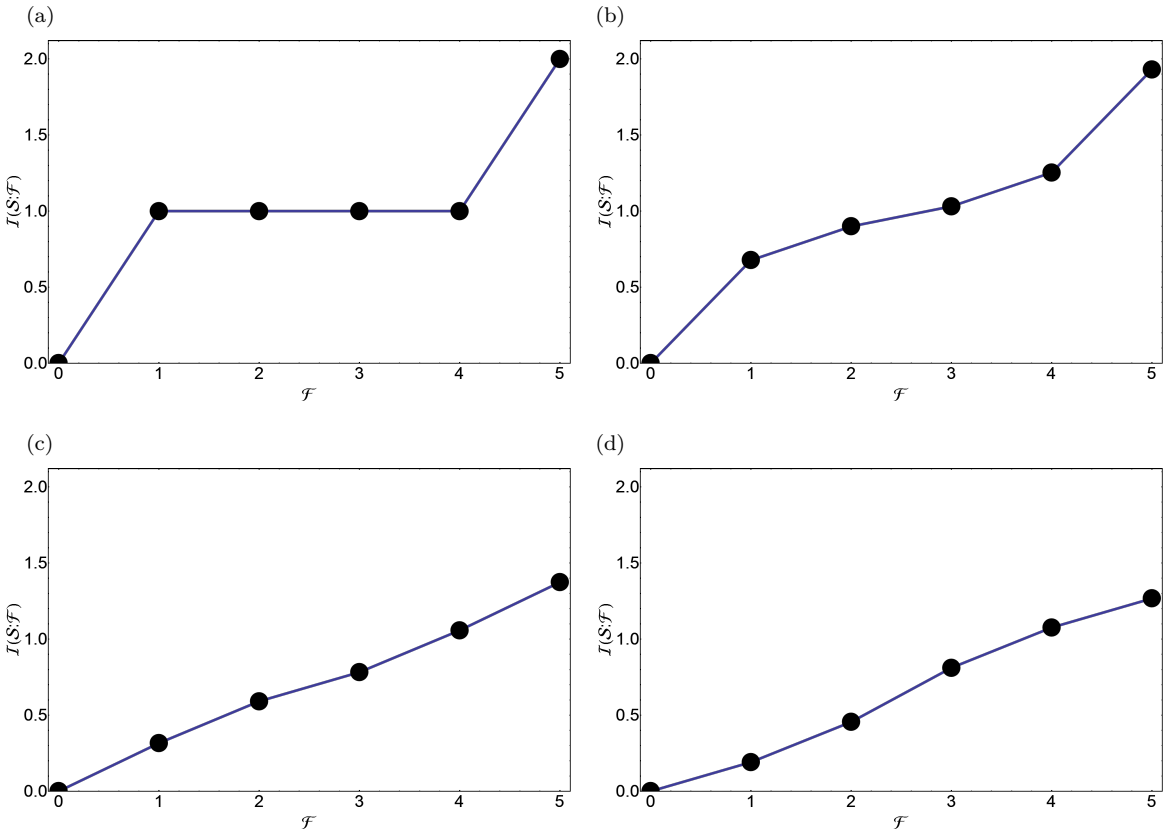

Figure C.1: A comparison plot of the mutual information $\mathcal{I}(\mathcal{S} : \mathcal{F})$ of a system $\mathcal{S}$ vs. fragment size $\sharp\mathcal{F}$ generated by the using a "direct" approach computation (blue curve, with the black dot plot markers) and the "rotating technique" (purple curve). In each case $r_{00} = s_{00} = 1/2$, and $\sharp\mathcal{E} = 5$, with initially pure $\mathcal{S}$, $H_{\mathcal{S}}(0) = 0$. (a, b, c, and d) The mutual information when $\Delta = 0$, $t = \pi/2$, $\Delta = 1/2$, $t = \pi/2$, $\Delta = 1$, $t = \pi/2$, and $\Delta = 1$ at $t = 2\pi/3$. respectively. We notice that in each case curves generated by the rotation technique match almost exactly to those generate by the direct approach, indicating that the rotation technique is valid.

Appendix D

Implementation of the Rotation Technique

Here we describe how the rotation technique is implemented. We start by noting that Schur-Weyl decomposition can equivalently be written as

$$\rho_{\mathcal{S}}(0) \otimes \rho_r^{\otimes \sharp\mathcal{F}} \to \sum_{j=0}^{\sharp\mathcal{F}/2} \sum_{m,m'=-j}^{j} p_{jmm'} \rho_{\mathcal{S}}(0) \otimes |jm\rangle\langle jm'| \otimes I_{B_j} \tag{D.1}$$

with $p_{jmm'}$ being expansion coefficients. Using this form allows us to write the time evolved state of the system and fragment as

$$\rho_{\mathcal{SF}}(t) = \sum_{n=0}^{\sharp\mathcal{E}/\mathcal{F}} \binom{\sharp\mathcal{E}/\mathcal{F}}{n} r_{00}^{\sharp\mathcal{E}/\mathcal{F}-n} r_{11}^{n} \left[\sum_{j=0}^{\sharp\mathcal{F}/2} \sum_{m,m'=-j}^{j} p_{jmm'} U_{\sharp\mathcal{F},2n-2m}(t) \rho_{\mathcal{S}}(0) U_{\sharp\mathcal{F},2n-2m'}(-t) \otimes |jm\rangle\langle jm'| \otimes I_{B_j} \right] \tag{D.2}$$

where we have defined

$$U_{\mathcal{S},j,m}(t) = \exp\left[-it \left(\Delta \sigma_{\mathcal{S}}^{x} + \frac{1}{2} (\sharp\mathcal{E}/\mathcal{F} - 2n + 2m) \sigma_{\mathcal{S}}^{z} \right) \right] \tag{D.3}$$

For a variety of reasons, we use Mathematica as our computing tool and implement the rotation technique by writing out explicitly what the tensor product is doing. To see this, for simplicity let $j = \sharp\mathcal{F}/2$, with $\sharp\mathcal{F} = 2$. Therefore $m, m' = -1, 0, 1$. Therefore the above expression translates to (without the sum over n, and, for now, without the expansion coefficients $p_{jmm'}$, and where we only

label the unitaries by m and m'):

$$\begin{pmatrix} U_{-1}(t)\rho_{\mathcal{S}}(0) \\ U_0(t)\rho_{\mathcal{S}}(0) \\ U_1(t)\rho_{\mathcal{S}}(0) \end{pmatrix} \begin{pmatrix} U_{-1}^\dagger(t) & U_0^\dagger(t) & U_1^\dagger(t) \end{pmatrix}$$

$$= \begin{pmatrix} U_{-1}(t)\rho_{\mathcal{S}}(0)U_{-1}^\dagger(t) & U_{-1}(t)\rho_{\mathcal{S}}(0)U_0^\dagger(t) & U_{-1}(t)\rho_{\mathcal{S}}(0)U_1^\dagger(t) \\ U_0(t)\rho_{\mathcal{S}}(0)U_{-1}^\dagger(t) & U_0(t)\rho_{\mathcal{S}}(0)U_0^\dagger(t) & U_0(t)\rho_{\mathcal{S}}(0)U_1^\dagger(t) \\ U_1(t)\rho_{\mathcal{S}}(0)U_{-1}^\dagger(t) & U_1(t)\rho_{\mathcal{S}}(0)U_0^\dagger(t) & U_1(t)\rho_{\mathcal{S}}(0)U_1^\dagger(t) \end{pmatrix} \tag{D.4}$$

To put in the expansion coefficients $p_{jmm'}$, make a separate matrix of coefficients, multiplying them into the above matrix of matrices, element by element, i.e., using the *Schur product*. That is,

$$P_{jmm'} = \begin{pmatrix} p_{j-1-1} & p_{j-10} & p_{j-11} \\ p_{j0-1} & p_{j00} & p_{j01} \\ p_{j01} & p_{j10} & p_{j11} \end{pmatrix} \tag{D.5}$$

and hence (roughly, as an explicit value of j is needed)

$$\sum_{j=0}^{\natural\mathcal{F}/2} \sum_{m,m'=-j}^{j} p_{jmm'} U_{\natural\mathcal{F},2n-2m}(t)\rho_{\mathcal{S}}(0)U_{\natural\mathcal{F},2n-2m'}(-t) \otimes |jm\rangle\langle jm'| \otimes I_{B_j}$$

$$\to \begin{pmatrix} p_{j-1-1}U_{-1}(t)\rho_{\mathcal{S}}(0)U_{-1}^\dagger(t) & p_{j-10}U_{-1}(t)\rho_{\mathcal{S}}(0)U_0^\dagger(t) & p_{j-11}U_{-1}(t)\rho_{\mathcal{S}}(0)U_1^\dagger(t) \\ p_{j0-1}U_0(t)\rho_{\mathcal{S}}(0)U_{-1}^\dagger(t) & p_{j00}U_0(t)\rho_{\mathcal{S}}(0)U_0^\dagger(t) & p_{j01}U_0(t)\rho_{\mathcal{S}}(0)U_1^\dagger(t) \\ p_{j1-1}U_1(t)\rho_{\mathcal{S}}(0)U_{-1}^\dagger(t) & p_{j10}U_1(t)\rho_{\mathcal{S}}(0)U_0^\dagger(t) & p_{j11}U_1(t)\rho_{\mathcal{S}}(0)U_1^\dagger(t) \end{pmatrix} \otimes I_{B_j} \tag{D.6}$$

From here it is a simple matter of taking the sum over n, yielding the full expression for $\rho_{\mathcal{SF}}(t)$. This expression is much better than before for the sake of coding in Mathematica as now the sums over m and m' are being dealt with simultaneously with the angular momentum states.

Written in this form makes computing $\rho_{\mathcal{F}}(t)$ a straightforward task, as the basic structure is similar to the above, where the object being summed over becomes

$$\begin{pmatrix} p_{j-1-1}\,\mathrm{tr}[U_{-1}(t)\rho_{\mathcal{S}}(0)U_{-1}^\dagger(t)] & p_{j-10}\,\mathrm{tr}[U_{-1}(t)\rho_{\mathcal{S}}(0)U_0^\dagger(t)] & p_{j-11}\,\mathrm{tr}[U_{-1}(t)\rho_{\mathcal{S}}(0)U_1^\dagger(t)] \\ p_{j0-1}\,\mathrm{tr}[U_0(t)\rho_{\mathcal{S}}(0)U_{-1}^\dagger(t)] & p_{j00}\,\mathrm{tr}[U_0(t)\rho_{\mathcal{S}}(0)U_0^\dagger(t)] & p_{j01}\,\mathrm{tr}[U_0(t)\rho_{\mathcal{S}}(0)U_1^\dagger(t)] \\ p_{j1-1}\,\mathrm{tr}[U_1(t)\rho_{\mathcal{S}}(0)U_{-1}^\dagger(t)] & p_{j10}\,\mathrm{tr}[U_1(t)\rho_{\mathcal{S}}(0)U_0^\dagger(t)] & p_{j11}\,\mathrm{tr}[U_1(t)\rho_{\mathcal{S}}(0)U_1^\dagger(t)] \end{pmatrix} \otimes I_{B_j} \tag{D.7}$$

The real challenge therefore is numerically computing the state $\rho_{\mathcal{SF}}(t)$. We start by rewriting the

matrix of matrices as an outer product of matrices as shown. To get a column of matrices one may simply use the **Table** command in Mathematica to generate each of the unitary matrices and then use the **apply@@** command to merge the table of matrices into a single column of matrices. To input the expansion coefficients, one simply generates a table of these coefficients, to be multiplied into the matrix of matrices using the '$*$' command, equivalent to the Schur product. The rest of the computation is then simply a matter of multiplying lists in Mathematica.

Once $\rho_{\mathcal{SF}}(t)$ has been computed, calculating $\rho_{\mathcal{F}}(t)$ is even simpler as now the elements in the row and column constituting the outer product are scalars. Hence, Mathematica's table command is sufficient in finding the matrix representation of $\rho_{\mathcal{F}}(t)$, which is all that is necessary to complete the computation of the objects of interest, $I(\mathcal{S}:\mathcal{F}), \chi(\Pi_s:\mathcal{F})$, and $\mathcal{D}(\Pi_s,\mathcal{F})$.