THE GENERATION OF SERIES WITH HIGHER TRANSCENDENTAL FUNCTIONS BY MEANS OF INTEGRAL TRANSFORMS

by

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TABLE OF CONTENTS

Chapter		Page No.
	INTRODUCTION	1
1	SERIES DERIVED FROM JACOBI'S EXPANSIONS	4
2	SERIES DERIVED FROM GEGENBAUER'S EXPANSIONS	30
	BIBLIOGRAPHY	46

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THE GENERATION OF SERIES WITH HIGHER TRANSCENDENTAL FUNCTIONS BY MEANS OF INTEGRAL TRANSFORMS

INTRODUCTION

Poisson's formula (11, p. 60)

$$(I)\quad a^{1/2}\left(\tfrac{1}{2} f(0) + \sum_{n=1}^{\infty} f(na)\right) = (2b/\pi)^{1/2}\left(\tfrac{1}{2} g_c(0) + \sum_{n=1}^{\infty} g_c(nb)\right) \qquad \text{where} \quad ab = 2\pi$$

admits the transformation of a certain infinite series into another one. In (I)

$$(II)\quad g_c(y) = \int_0^{\infty} f(x)\cos(xy)\,dx$$

is the Fourier cosine transform of $f(x)$. (About the conditions for the validity of (I) see the reference above.) Poisson's summation formula seems to be the first attempt to connect the sum of infinite series with the Fourier transform of the function involved in the summation. The application, for instance of (I) to $f(x) = e^{-\frac{1}{2}x^2}$ (with $g_c(y) = \tfrac{1}{2}(2\pi)^{1/2} e^{-\frac{1}{2}y^2}$) leads to

$$b^{1/2}\left(\tfrac{1}{2} + \sum_{n=1}^{\infty} e^{-n^2b^2/2}\right) = a^{1/2}\left(\tfrac{1}{2} + \sum_{n=1}^{\infty} e^{-n^2a^2/2}\right)$$

$$(ab = 2\pi)$$

which is a special case of the transformation formula of Jacobi's theta function. The merit of this formula lies

in the fact that the left hand side series converges very slowly for values of b which are close to zero, while the right hand side series converges very rapidly. Of special importance is the case when $f(x)$ is such, that its Fourier cosine transform $g_c(y)$ is identically zero for $y > a$. In this case the series on the right hand side of (I) terminates and the left hand side series can be summed. (For an exhaustive number of such cases see (8, p. 26). Numerous contributions concerning the transformation of infinite series along lines as suggested by Poisson's formula have been given. Those which involve the summation and transformation of infinite series whose terms are certain higher transcendental functions are of particular interest here as for instance (1, p. 236), (2), (5), (6), (11).

Another method to obtain series expansions of the character mentioned above in a closed form consists of the use of a series representation of a given function $g(x)$, (" key series"). Let

(III) $g(x) = \sum a_n u_n(x)$

If this series expression for $g(x)$ is such, that its integral transform $T\{g(x)\}$ can be obtained by applying the same transform term by term to the right hand side of (III) one obtains

(IIII) $T\{g(x)\} = \sum a_n T\{u_n(x)\}$.

If now both, $T\{g(x)\}$ and $T\{u_n(x)\}$ are known the series in (IIII) is summed. This principle has been used (8) for the case of certain trigonometric " key series" in conjunction with the finite Fourier and the finite Hankel transform. Methods were also used by which both sides of a series of the type (III) were made subject to fractional differentiation and integration (the right hand side term by term) (13). But this is equivalent with making (III) subject to an integral transform of the Riemann-Liouville or Weyl type (4, p. 181).

The material displayed in this thesis is based upon well known expansion theorems and the Fourier and Hankel transform.

In 1. we shall use Jacobi's series expansions involving Bessel functions.

In 2. extensions of these due to Gegenbauer are used.

Questions as to the convergence of the series obtained in this procedure can be answered by the aid of a list of " asymptotic formulas" given in an appendix.

1. SERIES DERIVED FROM JACOBI'S EXPANSIONS

We use here two well known results from the theory of Bessel's functions (3, vol. 2, p. 7)

$$(1)\quad \cos(x\cos\varphi) = \sum_{n=0}^{\infty} (-1)^n \varepsilon_n J_{2n}(x) \cos(2n\varphi),$$

$$(2)\quad \sin(x\cos\varphi) = 2\sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x) \cos((2n+1)\varphi),$$

$$(\varepsilon_0 = 1,\ \varepsilon_n = 2,\ n=1, 2, 3, \cdots).$$

The function $J_\mu(x)$ denotes the Bessel function of order μ and argument x (3, Ch. 7). The series above are absolutely convergent for any arbitrary (also complex) value of x and φ. We introduce now the abbreviations

$$(3)\quad g_c(y) = \int_0^\infty f(x)\cos(xy)dx$$

$$(4)\quad g_s(y) = \int_0^\infty f(x)\sin(xy)dx$$

$$(5)\quad h_\mu(y) = \int_0^\infty f(x)J_\mu(xy)dx.$$

These properties are the Fourier cosine, the Fourier sine and the Hankel transform, respectively, of a function $f(x)$, (7), (4).

Multiplying each side of (1) and (2) with a suitable function $f(x)$ we obtain integrating term by term

between the limits 0 and ∞, using the abbreviations (3), (4), (5):

$$(6) \quad \sum (-1)^n \varepsilon_n \cos(2n\varphi) h_{2n}(1) = g_c(\cos\varphi)$$

$$(7) \quad 2 \sum (-1)^n \cos((2n+1)\varphi) h_{2n+1}(1) = g_s(\cos\varphi)$$

provided that all occurring properties exist and that the integration can be performed term by term. It is obvvious that the left hand side infinite series are the Fourier series expansions of $g_c(\cos\varphi)$ and $g_s(\cos\varphi)$ respectively. Since by (3) $g_c(y)$ is an even function of y and $g_s(y)$ by (4) is an odd function of y it follows that $g_c(\cos\varphi)$ is an even periodic function with the period π, while $g_s(\cos\varphi)$ is an odd periodic function with the period 2π. It is interesting that the coefficients in the above expansions for the Fourier cosine and Fourier sine transforms are the corresponding Hankel transforms for the parameter unity. The following part gives a list of series involving higher functions using (6) and (7). It is clear that only such functions $f(x)$ can be admitted for the use of (6) and (7) for which $g_c(y)$, $h_\mu(y)$ or $g_s(y)$, $h_\mu(y)$ respectively are known. Existing tables for Fourier transforms (7), and Hankel transforms (4) were used.

EXAMPLES

1) $f(x) = (a^2 + x^2)^{-\frac{1}{2}}$

$g_c(y) = K_0(ay)$ ... (7, p. 3)

$g_s(y) = \frac{1}{2}\pi\,(I_0(ay) - L_0(ay))$ ... (7, p. 113)

$h_\mu(y) = I_{\frac{1}{2}\mu}(\frac{1}{2}ay)\, K_{\frac{1}{2}\mu}(\frac{1}{2}ay)$ ... (4, p. 23).

Inserting these into (6) and (7) we get:

a) $K_0(a\cos\varphi) = \Sigma\,(-1)^n \varepsilon_n \cos(2n\varphi)\, I_n(\frac{1}{2}a)\, K_n(\frac{1}{2}a)$

b) $\frac{1}{2}\pi(I_0(a\cos\varphi) - L_0(a\cos\varphi))$

$= 2\,\Sigma\,(-1)^n \cos((2n+1)\varphi)\, I_{n+\frac{1}{2}}(\frac{1}{2}a)\, K_{n+\frac{1}{2}}(\frac{1}{2}a)$

which converge by appendix: (1) and (2).

2) $f(x) = (a - x)^{-\frac{1}{2}}$, $0 < x < a$

$= 0$, $0 < a < x$

$g_c(y) = (\frac{2\pi}{y})^{\frac{1}{2}}(\cos(ay)\, C(ay) + \sin(ay)\, S(ay))$

... (7, p. 3)

$g_s(y) = (\frac{2\pi}{y})^{\frac{1}{2}}(\sin(ay)\, C(ay) - \cos(ay)\, S(ay))$

... (7, p. 111)

$h_\mu(y) = \pi(\frac{1}{2}a)^{\frac{1}{2}}\, J_{\frac{1}{2}\mu-\frac{1}{4}}(\frac{1}{2}ay)\, J_{\frac{1}{2}\mu+\frac{1}{4}}(\frac{1}{2}ay)$ ... (4, p. 193).

Inserting these into (6) and (7) we get:

a) $2(\pi a \cos\varphi)^{-\frac{1}{2}}(\cos(a\cos\varphi)\, C(a\cos\varphi) + \sin(a\cos\varphi)$

$S(a\cos\varphi)) = \Sigma\, (-1)^n \varepsilon_n \cos(2n\varphi)\, J_{n-\frac{1}{4}}(\frac{1}{2}a)\, J_{n+\frac{1}{4}}(\frac{1}{2}a), a > 0$

b) $(a\pi\cos\varphi)^{-\frac{1}{2}}(\sin(a\cos\varphi)\, C(a\cos\varphi) - \cos(a\cos\varphi)$

$S(a\cos\varphi)) = \Sigma\, (-1)^n \cos((2n+1)\varphi) J_{n+\frac{1}{4}}(\frac{1}{2}a) J_{n+\frac{3}{4}}(\frac{1}{2}a), a>0$

which converge by appendix: (3).

3) $f(x) = (a^2-x^2)^{-\frac{1}{2}}$, $x < a$

$= 0$, $a < x$

$g_c(y) = \frac{1}{2}\pi\, J_0(ay)$ ... (7, p. 4)

$g_s(y) = \frac{1}{2}\pi\, H_0(ay)$ ... (7, p. 113)

$h_\mu(y) = \frac{1}{2}\pi(J_{\frac{1}{2}\mu}(\frac{1}{2}ay))^2$ ... (4, p. 24).

Inserting these into (6) and (7) we get:

a) $J_0(a\cos\varphi) = \Sigma\, (-1)^n \varepsilon_n \cos(2n\varphi)(J_n(\frac{1}{2}a))^2$

$H_0(a\cos\varphi) = 2\,\Sigma\, (-1)^n \cos((2n+1)\varphi)(J_{n+\frac{1}{2}}(\frac{1}{2}a))^2$

which converge by appendix: (3).

4) $f(x) = 0 \quad , \; x<a$

$= (x^2-a^2)^{-\frac{1}{2}} \quad , \; a<x$

$g_c(y) = -\frac{1}{2}\pi\, Y_0(ay)$ ... (7, p. 4)

$g_s(y) = \frac{1}{2}\pi\, J_0(ay)$ ... (7, p. 113)

$h_\mu(y) = -\frac{1}{2}\pi\, J_{\frac{1}{2}\mu}(\frac{1}{2}ay)\, Y_{\frac{1}{2}\mu}(\frac{1}{2}ay)$ ... (4, p. 24).

Inserting these into (6) and (7) we get:

a) $Y_0(a\cos\varphi) = \Sigma\, (-1)^n\, \varepsilon_n \cos(2n\varphi)\, J_n(\frac{1}{2}a)\, Y_n(\frac{1}{2}a)$

b) $J_0(a\cos\varphi) = \Sigma\, (-1)^{n+1} \cos((2n+1)\varphi) J_{n+\frac{1}{2}}(\frac{1}{2}a) Y_{n+\frac{1}{2}}(\frac{1}{2}a)$

which converge by appendix: (3) and (4).

5) $f(x) = (a^2+x^2)^{-\frac{1}{2}}((a^2+x^2)^{\frac{1}{2}} + x)^\tau$

$g_c(y) = -\pi \csc(\pi\tau)\, a^\tau\, (\frac{1}{2}\, \mathbb{J}_{-\tau}(iay) + \frac{1}{2}\, \mathbb{J}_{-\tau}(-iay)$

$- \cos(\frac{1}{2}\pi\tau)\, I_{-\tau}(ay))$ ... (7, p. 7)

$g_s(y) = -\pi \csc(\pi\tau)\, a^\tau\, (\frac{1}{2}i\, \mathbb{J}_{-\tau}(iay) - \frac{1}{2}i\, \mathbb{J}_{-\tau}(-iay)$

$- \sin(\frac{1}{2}\pi\tau)\, I_{-\tau}(ay))$ ... (7, p. 118)

$h_\mu(y) = a^\tau\, I_{\frac{1}{2}\mu-\frac{1}{2}\tau}(\frac{1}{2}ay)\, K_{\frac{1}{2}\mu+\frac{1}{2}\tau}(\frac{1}{2}ay)$ ... (4, p. 26)

$\text{Re}\ \tau < 1, \quad \text{Re}\ \mu > -1.$

Inserting these into (6) and (7) we get:

a) $\pi\csc(\pi\tau)(\frac{1}{2}J_{-\tau}(ia\cos\varphi)+\frac{1}{2}J_{-\tau}(-ia\cos\varphi)-\cos(\frac{1}{2}\pi\tau)I_{-\tau}(a\cos\varphi))$

$= \Sigma\ (-1)^{n+1}\ \varepsilon_n\cos(2n\varphi)I_{n-\frac{1}{2}\tau}(\frac{1}{2}a)K_{n+\frac{1}{2}\tau}(\frac{1}{2}a)$, $\mathrm{Re}\ \tau < 1$

b) $\pi\csc(\pi\tau)(\frac{1}{2}iJ_{-\tau}(ia\cos\varphi)-\frac{1}{2}iJ_{-\tau}(-ia\cos\varphi)-\sin(\frac{1}{2}\pi\tau)$

$I_{-\tau}(a\cos\varphi)) = 2\Sigma\ (-1)^{n+1}\ \cos((2n+1)\varphi)\ I_{n+\frac{1}{2}-\frac{1}{2}\tau}(\frac{1}{2}a)$

$K_{n+\frac{1}{2}+\frac{1}{2}\tau}(\frac{1}{2}a)$, $\quad \mathrm{Re}\ \tau < 1$,

which converge by appendix: (1) and (2).

6) $f(x) = (a^2+x^2)^{-\frac{1}{2}}((a^2+x^2)^{\frac{1}{2}} + x)^{\tau}-((a^2+x^2)^{\frac{1}{2}}-x)^{\tau}$

$g_s(y) = 2a^{\tau}\sin(\frac{1}{2}\pi\tau)\ K_{\tau}(ay)$ ... (7, p. 119)

$h_{\mu}(y) = a^{\tau}(I_{\frac{1}{2}\mu-\frac{1}{2}\tau}(\frac{1}{2}ay)K_{\frac{1}{2}\mu+\frac{1}{2}\tau}(\frac{1}{2}ay)-I_{\frac{1}{2}\mu+\frac{1}{2}\tau}(\frac{1}{2}ay)$

$K_{\frac{1}{2}\mu-\frac{1}{2}\tau}(\frac{1}{2}ay))$, $-1< \mathrm{Re}\ \tau < 1$, $\mathrm{Re}\mu>0$, ...(4, p. 26).

Inserting these into (7) we get:

a) $2\sin(\frac{1}{2}\pi\tau)K_{\tau}(a\cos\varphi) =2\Sigma(-1)^n\cos((2n+1)\varphi)(I_{n+\frac{1}{2}-\frac{1}{2}\tau}(\frac{1}{2}a)$

$K_{n+\frac{1}{2}+\frac{1}{2}\tau}(\frac{1}{2}a)-I_{n+\frac{1}{2}+\frac{1}{2}\tau}(\frac{1}{2}a)K_{n+\frac{1}{2}-\frac{1}{2}\tau}(\frac{1}{2}a))$, $-1< \mathrm{Re}\ \tau<1$,

which converges by appendix: (1) and (2).

7) $f(x) = (a^2+x^2)^{-\frac{1}{2}}((a^2+x^2)^{\frac{1}{2}} + x)^{\tau} + ((a^2+x^2)^{\frac{1}{2}} - x)^{\tau})$

$g_c(y) = 2a^{\tau}\cos(\frac{1}{2}\pi\tau)\ K_{\tau}(ay)$ ... (7, p. 8)

$h_{\mu}(y) = a^{\tau}(I_{\frac{1}{2}\mu-\frac{1}{2}\tau}(\frac{1}{2}ay)K_{\frac{1}{2}\mu+\frac{1}{2}\tau}(\frac{1}{2}ay)+I_{\frac{1}{2}\mu+\frac{1}{2}\tau}(\frac{1}{2}ay)$

$K_{\frac{1}{2}\mu-\frac{1}{2}\tau}(\frac{1}{2}ay))$, $-1< \mathrm{Re}\tau< 1$, $\mathrm{Re}\ \mu > 0$, ...(4, p. 26)

Inserting these into (6) we get:

a) $2\cos(\frac{1}{2}\pi\tau)K_\tau(a\cos\varphi) = \Sigma\,(-1)^n\,\varepsilon_n\cos(2n\varphi)$

$(I_{n-\frac{1}{2}\tau}(\frac{1}{2}a)K_{n+\frac{1}{2}\tau}(\frac{1}{2}a) + I_{n+\frac{1}{2}\tau}(\frac{1}{2}a)K_{n-\frac{1}{2}\tau}(\frac{1}{2}a))$

which converges by appendix: (1) and (2).

8) $f(x) = (a^2-x^2)^{-\frac{1}{2}}(x+i(a^2-x^2)^{\frac{1}{2}})^\tau+(x-i(a^2-x^2)^{\frac{1}{2}})^\tau,\ x<a$

$= 0,\ a<x$

$g_c(y) = \frac{1}{2}\pi a^\tau\sec(\frac{1}{2}\pi\tau)(J_\tau(ay)+ J_{-\tau}(ay))$ ... (7, p. 8)

$g_s(y) = \frac{1}{2}\pi a^\tau\csc(\frac{1}{2}\pi\tau)(J_\tau(ay)- J_{-\tau}(ay))$ ...(7, p. 119)

$h_\mu(y) = \pi a^\tau\, J_{\frac{1}{2}\mu+\frac{1}{2}\tau}(\frac{1}{2}ay)J_{\frac{1}{2}\mu-\frac{1}{2}\tau}(\frac{1}{2}ay)$ ... (4, p. 27)

$\mathrm{Re}\,(\mu+\tau) > -1.$

Inserting these into (6) and (7) we get:

a) $\frac{1}{2}\sec(\frac{1}{2}\pi\tau)(J_\tau(a\cos\varphi)+ J_{-\tau}(a\cos\varphi))$

$= \Sigma\,(-1)^n\,\varepsilon_n\cos(2n\varphi)\,J_{n+\frac{1}{2}\tau}(\frac{1}{2}a)\,J_{n-\frac{1}{2}\tau}(\frac{1}{2}a)$

b) $\frac{1}{2}\csc(\frac{1}{2}\pi\tau)(J_\tau(a\cos\varphi)-J_{-\tau}(a\cos\varphi))$

$= 2\,\Sigma\,(-1)^n\cos((2n+1)\varphi)J_{n+\frac{1}{2}+\frac{1}{2}\tau}(\frac{1}{2}a)\,J_{n+\frac{1}{2}-\frac{1}{2}\tau}(\frac{1}{2}a)$

which converge by appendix: (3).

9) $f(x) = 0,\quad x<a$

$= (x^2-a^2)^{-\frac{1}{2}}((x+(x^2-a^2)^{\frac{1}{2}})^\tau+(x-(x^2-a^2)^{\frac{1}{2}})^\tau),\ a<x$

$g_c(y) = -\pi a^\tau(\sin(\frac{1}{2}\pi\tau)J_\tau(ay)+\cos(\frac{1}{2}\pi\tau)Y_\tau(ay))$...(7, p. 9)

$g_s(y) = \pi a^\tau(\cos(\frac{1}{2}\pi\tau)J_\tau(ay)-\sin(\frac{1}{2}\pi\tau)Y_\tau(ay))$...(7, p. 120)

$$h_\mu(y) = -\tfrac{1}{2}\pi a^\tau \left(J_{\frac{1}{2}\tau+\frac{1}{2}\mu}(\tfrac{1}{2}ay)Y_{\frac{1}{2}\mu-\frac{1}{2}\tau}(\tfrac{1}{2}ay)+J_{\frac{1}{2}\mu-\frac{1}{2}\tau}(\tfrac{1}{2}ay)\right.$$

$$\left.Y_{\frac{1}{2}\mu-\frac{1}{2}\tau}(\tfrac{1}{2}ay)\right), \quad -1< \mathrm{Re}\,\tau < 1 , \quad \ldots (4, \text{ p. } 27)$$

Inserting these into (6) and (7) we get:

a) $\sin(\tfrac{1}{2}\pi\tau)\, J_\tau(a\cos\varphi) + \cos(\tfrac{1}{2}\pi\tau)\, Y_\tau(a\cos\varphi) = \tfrac{1}{2}\Sigma(-1)^n$

$\varepsilon_n \cos(2n\varphi)(J_{n+\frac{1}{2}\tau}(\tfrac{1}{2}a)Y_{n-\frac{1}{2}\tau}(\tfrac{1}{2}a) + J_{n-\frac{1}{2}\tau}(\tfrac{1}{2}a)Y_{n+\frac{1}{2}\tau}(\tfrac{1}{2}a))$

b) $\cos(\tfrac{1}{2}\pi\tau)\, J_\tau(a\cos\varphi) - \sin(\tfrac{1}{2}\pi\tau)\, Y_\tau(a\cos\varphi)$

$= \Sigma\,(-1)^{n+1}\cos((2n+1)\varphi)(J_{n+\frac{1}{2}+\frac{1}{2}\tau}(\tfrac{1}{2}a)\; Y_{n+\frac{1}{2}-\frac{1}{2}\tau}(\tfrac{1}{2}a)$

$+ J_{n+\frac{1}{2}-\frac{1}{2}\tau}(\tfrac{1}{2}a)\; Y_{n+\frac{1}{2}+\frac{1}{2}\tau}(\tfrac{1}{2}a))$

which converge by appendix: (3) and (4).

10) $f(x) = x^{\tau-1}\, e^{-ax}$

$g_c(y) = \Gamma(\tau)(a^2+y^2)^{-\frac{1}{2}\tau}\cos(\tau\arctan(\frac{y}{a}))$ ... (7, p. 10)

$g_s(y) = \Gamma(\tau)(a^2+y^2)^{-\frac{1}{2}\tau}\sin(\tau\arctan(\frac{y}{a}))$ ... (7, p. 121)

$h_\mu(y) = (a^2+y^2)^{-\frac{1}{2}\tau}\Gamma(\tau+\mu)P^{-\mu}_{\tau-1}(a(a^2+y^2)^{-\frac{1}{2}})$ ... (4, p. 29)

$\mathrm{Re}\,\tau > 0,\ \mathrm{Re}(\mu+\tau) > 0,\ \mathrm{Re}\,a > 0.$

Inserting these into (6) and (7) we get:

a) $\Gamma(\tau)(a^2+\cos^2\varphi)^{-\frac{1}{2}\tau}\cos(\tau\arctan(\frac{\cos\varphi}{a})) = \Sigma(-1)^n\varepsilon_n\cos(2n\varphi)$

$(a^2+1)^{-\frac{1}{2}\tau}\Gamma(\tau+2n)P^{-2n}_{\tau-1}(a(a^2+1)^{-\frac{1}{2}}),\ a > 0,\ \mathrm{Re}\,\tau > 0$

b) $\Gamma(\tau)(a^2+\cos^2\varphi)^{-\frac{1}{2}\tau}\sin(\tau\arctan(\frac{\cos\varphi}{a})) = 2\,\Sigma\,(-1)^n\cos((2n+1)\varphi)$

$(a^2+1)^{-\frac{1}{2}\tau}\Gamma(\tau+2n+1)P_{\tau-1}^{-2n-1}(a(a^2+1)^{-\frac{1}{2}})$, $a>0$, Re $\tau > 0$

which converge by appendix: (6).

11) $f(x) = e^{-ax^2}$

$g_c(y) = \frac{1}{2}(\frac{\pi}{a})^{\frac{1}{2}} \exp(-y^2(4a)^{-1})$ ... (7, p. 11)

$g_s(y) = -\frac{1}{2}i(\frac{\pi}{a})^{\frac{1}{2}}\exp(-y^2(4a)^{-1})\mathrm{Erf}(\frac{1}{2}iya^{-\frac{1}{2}})$ ...(7, p. 122)

$h_\mu(y) = \frac{1}{2}(\frac{\pi}{a})^{\frac{1}{2}}\exp(\frac{-y^2}{8a})J_{\frac{1}{2}\mu}(\frac{y^2}{8a})$, Re $\mu > -1$...(4, p. 29).

Inserting these into (6) and (7) we get: a)

$\exp(\frac{a}{4}(\frac{1}{2}-\cos^2\varphi) = \Sigma\ (-1)^n\ \varepsilon_n\ \cos(2n\varphi)\ J_n(\frac{1}{8a})$

b) $i\ \exp(\frac{a}{4}(\frac{1}{2} - \cos^2\varphi)\mathrm{Erf}(\frac{1}{2}iya^{-\frac{1}{2}})$

$= \Sigma\ (-1)^{n+1}\ \cos((2n+1)\varphi)\ J_{n+\frac{1}{2}}(\frac{1}{8a})$

which converge by appendix: (3).

12) $f(x) = x^\eta \exp(-ax^2)$

$g_c(y) = \frac{1}{2}a^{-\frac{1}{2}\eta-\frac{1}{2}}\Gamma(\frac{1}{2}+\frac{1}{2}\eta)\,{}_1F_1(\frac{1}{2}+\frac{1}{2}\eta;\frac{1}{2};-\frac{y^2}{4a})$ ... (7, p. 12)

$g_s(y) = \frac{1}{2}a^{-\frac{1}{2}\eta-1}\Gamma(1+\frac{1}{2}\eta)\,{}_1F_1(1+\frac{1}{2}\eta;\frac{3}{2};-\frac{y^2}{4a})$ ... (7, p. 123)

$h_\mu(y) = (\Gamma(\mu+1))^{-1}\Gamma(\frac{1}{2}\mu+\frac{1}{2}\eta+\frac{1}{2})a^{-\frac{1}{2}\eta}y^{-1}\exp(-\frac{y^2}{8a})$

$M_{\frac{1}{2}\eta,\frac{1}{2}\mu}(\frac{y^2}{4a})$, Re $\eta > -1$, Re$(\mu+\eta)>-1$...(4, p. 30).

Inserting these into (6) and (7) we get:

a) $\frac{1}{2}a^{-\frac{1}{2}}e^{\frac{1}{8a}}\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)\,{}_1F_1(\frac{1}{2}+\frac{1}{2}\alpha;\frac{1}{2};\ -\frac{\cos^2\varphi}{4a})$

$= \Sigma\ (-1)^n\varepsilon_n\cos(2n\varphi)\ \frac{\Gamma(n+\frac{1}{2}\alpha+\frac{1}{2})}{(2n)!}\ M_{\frac{1}{2}\alpha,n+\frac{1}{2}}((4a)^{-1}),$

b) $(4a)^{-1}e^{\frac{1}{8a}}\Gamma(1+\frac{1}{2}\alpha)\,{}_1F_1(1+\frac{1}{2}\alpha;\frac{3}{2};-\frac{\cos^2\varphi}{4a})$

$= \Sigma\ (-1)^n\cos((2n+1)\varphi)\ \frac{\Gamma(\frac{1}{2}\alpha+n+1)}{(2n+1)!}\ M_{\frac{1}{2}\alpha,n+\frac{1}{2}}((4a)^{-1}),$

which converge by appendix: (7).

13) $f(x) = x^{-1}e^{-\frac{a}{x}}$

$g_c(y) = K_0(2(iay)^{\frac{1}{2}})+K_0(2(-iay)^{\frac{1}{2}})$ ... (7, p. 12)

$g_s(y) = iK_0(2(iay)^{\frac{1}{2}})-iK_0(2(-iay)^{\frac{1}{2}})$ ... (7, p. 124)

$h_\eta(y) = 2J_\eta((2ay)^{\frac{1}{2}})K_\eta((2ay)^{\frac{1}{2}})$ ... (4, p. 30).

Inserting these into (6) and (7) we get:

a) $K_0(2(ia\cos\varphi)^{\frac{1}{2}})+K_0(2(-ia\cos\varphi)^{\frac{1}{2}})$

$= 2\Sigma\ (-1)^n\varepsilon_n\cos(2n\varphi)\ J_{\underline{2n}}((2a)^{\frac{1}{2}})K_{\underline{2n}}((2a)^{\frac{1}{2}})$

b) $i(K_0(2(ia\cos\varphi)^{\frac{1}{2}})-K_0(2(-ia\cos\varphi)^{\frac{1}{2}}))$

$= 4\Sigma\ (-1)^n\cos((2n+1)\varphi)J_{2n+1}((2a)^{\frac{1}{2}})K_{2n+1}((2a)^{\frac{1}{2}})$

which converge by appendix: (1) and (3).

14) $f(x) = x^{-1}e^{-ax-bx^{-1}}$

$g_c(y) = K_0(2b^{1/2}(a+iy)^{1/2})+K_0(2b^{1/2}(a-iy)^{1/2})\cdots$(7, p. 13)

$g_s(y) = iK_0(2b^{1/2}(a+iy)^{1/2})-iK_0(2b^{1/2}(a-iy)^{1/2})\cdots$(7, p. 124)

$h_\mu(y) = 2J_\mu((2b)^{1/2}((a^2+y^2)^{1/2}-a)^{1/2})K_\mu((2b)^{1/2}((a^2+y^2)^{1/2}+a)^{1/2})$

$\cdots$ (4, p. 30)

Inserting these into (6) and (7) we obtain:

a) $K_0(2b^{1/2}(a+i\cos\varphi)^{1/2})+K_0(2b^{1/2}(a-i\cos\varphi)^{1/2}) = 2\Sigma(-1)^n\varepsilon_n$

$\cos(2n\varphi)J_{2n}((2b)^{1/2}((a^2+1)^{1/2}-a)^{1/2})K_{2n}((2b)^{1/2}((a^2+1)^{1/2}+a)^{1/2})$

b) $iK_0(2b^{1/2}(a+i\cos\varphi)^{1/2})-iK_0(2b^{1/2}(a-i\cos\varphi)^{1/2})$

$= 4\Sigma(-1)^n\cos((2n+1)\varphi)J_{2n+1}((2b)^{1/2}((a^2+1)^{1/2}-a)^{1/2})$

$K_{2n+1}((2b)^{1/2}((a^2+1)^{1/2}+a)^{1/2})$

which converge by appendix: (1) and (3).

15) $f(x) = x^{-1/2}e^{-ax^{1/2}}$

$g_c(y) = (2\pi y^{-1})^{1/2}(\cos(\frac{a^2}{4y})(1/2-S(\frac{a^2}{4y}))-(1/2-C(\frac{a^2}{4y}))\sin(\frac{a^2}{4y}))$

$\cdots$ (7, p. 13)

$g_s(y) = (\frac{2\pi}{y})^{1/2}(\cos(\frac{a^2}{4y})(1/2-C(\frac{a^2}{4y}))+\sin(\frac{a^2}{4y})(1/2-S(\frac{a^2}{4y})))$

$\cdots$ (7, p. 124).

$$h_\mu(y) = (\tfrac{2}{\pi})^{1/2}\Gamma(\tfrac{1}{2}+\mu)y^{-1/2}D_{-\mu-1/2}((\tfrac{1}{2}ia^2y^{-1})^{1/2})D_{-\mu-1/2}((-\tfrac{1}{2}ia^2y^{-1})^{1/2}),$$

$$\text{Re } \mu > -\tfrac{1}{2} \cdots \text{(4, p. 30)}.$$

Inserting these into (6) and (7) we get:

a) $$\pi(\cos\varphi)^{-1/2}(\cos(\frac{a^2}{4\cos\varphi})(\tfrac{1}{2}-S(\frac{a^2}{4\cos\varphi}))-\sin(\frac{a^2}{4\cos\varphi})$$

$$(\tfrac{1}{2}-C(\frac{a^2}{4\cos\varphi}))) = \Sigma\ (-1)^n\varepsilon_n\cos(2n\varphi)\Gamma(\tfrac{1}{2}+2n)$$

$$D_{-2n-1/2}((\tfrac{1}{2}ia^2)^{1/2})D_{-2n-1/2}((-\tfrac{1}{2}ia^2)^{1/2})$$

b) $$\pi(\cos\varphi)^{-1/2}(\cos(\frac{a^2}{a\cos\varphi})(\tfrac{1}{2}-C(\frac{a^2}{4\cos\varphi}))+\sin(\frac{a^2}{4\cos\varphi})$$

$$(\tfrac{1}{2}-S(\frac{a^2}{4\cos\varphi}))) = 2\Sigma\ (-1)^n\cos((2n+1)\varphi)\Gamma(\tfrac{3}{2}+2n)$$

$$D_{-2n-\frac{3}{2}}((\tfrac{1}{2}ia^2)^{1/2})D_{-2n-\frac{3}{2}}((-\tfrac{1}{2}ia^2)^{1/2})$$

which converge by appendix: (8).

16) $$f(x) = (a^2+x^2)^{-1/2}e^{-b(a^2+x^2)^{1/2}}$$

$$g_c(y) = K_0(a(b^2+y^2)^{1/2}) \qquad \cdots \text{(7, p. 13)}$$

$$h_\mu(y) = I_{1/2\mu}(\tfrac{1}{2}b((a^2+y^2)^{1/2}-a))K_{1/2\mu}(\tfrac{1}{2}b((a^2+y^2)^{1/2}+a))$$

$$\cdots \text{(4, p. 30)}$$

Inserting these into (6) we obtain:

a) $K_0(a(b^2+\cos^2\varphi)^{1/2})$

$= \Sigma\ (-1)^n \varepsilon_n \cos(2n\varphi) I_n(\tfrac{1}{2}b((a^2+1)^{1/2}-a)) K_n(\tfrac{1}{2}b((a^2+1)^{1/2}+a))$

which converges by appendix: (1) and (2).

17) $f(x) = \sin(ax^2)$

$$g_c(y) = \frac{1}{4}\left(\frac{2\pi}{a}\right)^{1/2}\left(\cos\left(\frac{y^2}{4a}\right) - \sin\left(\frac{y^2}{4a}\right)\right) \quad \cdots \text{ (7, p. 23)}$$

$$g_s(y) = \left(\frac{\pi}{2a}\right)^{1/2}\left(\cos\left(\frac{y^2}{4a}\right)C\left(\frac{y^2}{4a}\right) + \sin\left(\frac{y^2}{4a}\right)S\left(\frac{y^2}{4a}\right)\right)$$

$\cdots$ (7, p. 135)

$$h_\mu(y) = -\tfrac{1}{2}\left(\frac{\pi}{a}\right)^{1/2}\sin\left(\frac{y^2}{8a} - \frac{\pi}{4} - \frac{\pi\mu}{4}\right)J_{\frac{1}{2}\mu}\left(\frac{y^2}{8a}\right), \qquad \text{Re } \mu > -3$$

$\cdots$ (4, p. 34).

Inserting these into (6) and (7) we get:

a) $2^{-1/2}\left(\cos\left(\frac{\cos^2\varphi}{4a}\right) - \sin\left(\frac{\cos^2\varphi}{4a}\right)\right)$

$= \Sigma\ (-1)^{n+1}\varepsilon\ \cos(2n\varphi)\sin\left(\frac{1}{8a} - \frac{\pi}{4} - \frac{\pi n}{2}\right)J_n\left(\frac{1}{8a}\right)$

b) $2^{-1/2}\left(\cos\left(\frac{\cos^2\varphi}{4a}\right)C\left(\frac{\cos^2\varphi}{4a}\right) + \sin\left(\frac{\cos^2\varphi}{4a}\right)S\left(\frac{\cos^2\varphi}{4a}\right)\right)$

$= \Sigma\ (-1)^{n+1}\ \cos((2n+1)\varphi)\ \sin\left(\frac{1}{8a} - \frac{\pi}{2} - \frac{n\pi}{2}\right)J_{n+\frac{1}{2}}\left(\frac{1}{8a}\right)$

which converge by appendix: (3).

18) $f(x) = \cos(ax^2)$

$$g_c(y) = \frac{1}{4}\left(\frac{2\pi}{a}\right)^{1/2}\left(\cos\left(\frac{y^2}{4a}\right)+\sin\left(\frac{y^2}{4a}\right)\right) \quad \cdots \text{ (7, p. 24)}$$

$$g_s(y) = \left(\frac{\pi}{2a}\right)^{1/2}\left(\sin\left(\frac{y^2}{2a}\right)C\left(\frac{y^2}{4a}\right)-\cos\left(\frac{y^2}{4a}\right)S\left(\frac{y^2}{4a}\right)\right)$$

$$\cdots \text{ (7, p. 136)}$$

$$h_\mu(y) = \tfrac{1}{2}\left(\frac{\pi}{a}\right)^{1/2}\cos\left(\frac{y}{8a}-\frac{\pi}{4}-\frac{\pi\mu}{4}\right)J_{\frac{1}{2}\mu}\left(\frac{y^2}{8a}\right) \quad \cdots \text{ (4, p. 38)}$$

$$\text{Re } \mu > -1$$

Inserting these into (6) and (7) we get:

a) $$2^{-1/2}\left(\cos\left(\frac{\cos^2\varphi}{4a}\right) + \sin\left(\frac{\cos^2\varphi}{4a}\right)\right)$$

$$= \Sigma\ (-1)^n \varepsilon_n \cos(2n\varphi)\ \cos\left(\frac{1}{8}-\frac{\pi}{4}-\frac{\pi n}{2}\right)J_n\left(\frac{1}{8a}\right)$$

b) $$2^{-1/2}\left(\sin\left(\frac{\cos^2\varphi}{4a}\right)C\left(\frac{\cos^2\varphi}{4a}\right)-\cos\left(\frac{\cos^2\varphi}{4a}\right)S\left(\frac{\cos^2\varphi}{4a}\right)\right)$$

$$= \Sigma\ (-1)^n\ \cos((2n+1)\varphi)\cos\left(\frac{1}{8a}-\frac{\pi}{2}-\frac{\pi n}{2}\right)J_{n+\frac{1}{2}}\left(\frac{1}{8a}\right)$$

which converge by appendix: (3).

19) $f(x) = (a^2+x^2)^{-1/2}\sin(b(a^2+x^2)^{1/2})$

$$g_c(y) = \frac{\pi}{2}\ J_0(a(b^2-y^2)^{1/2}) \ , \qquad y > b$$

$$= 0 \qquad , \qquad y < b \quad \cdots \text{ (7, p. 27)}$$

$$h_\mu(y) = \frac{\pi}{2}\ J_{\frac{1}{2}\mu}(\tfrac{1}{2}a(b-(b^2-y^2)^{1/2}))J_{-\frac{1}{2}\mu}(\tfrac{1}{2}a(b+(b^2-y^2)^{1/2})) \ ,$$

$$\text{Re } \mu > -1, \quad y<b \qquad \cdots \text{ (4, p. 35)}$$

Inserting these into (6) we get:

a) $J_0(a(b^2-\cos^2\varphi)^{\frac{1}{2}}) = \Sigma\ (-1)^n\varepsilon_n\cos(2n\varphi)J_n(\frac{1}{2}a(b-(b^2-1)^{\frac{1}{2}}))$

$J_{-n}(\frac{1}{2}a(b+(b^2-1)^{\frac{1}{2}}))$, $\quad b>1$

which converges by appendix: (3), since $J_{-n}(x) = (-1)^nJ_n(x)$

20) $f(x) = (a^2+x^2)^{-\frac{1}{2}}\cos(b(a^2+x^2)^{\frac{1}{2}})$

$g_c(y) = -\frac{1}{2}\pi Y_0(a(b^2-y^2)^{\frac{1}{2}})$, $\quad y<b$

$= K_0(a(y^2-b^2)^{\frac{1}{2}})$, $\quad y>b$ ...(7, p. 28)

$h\mu(y) = -\frac{1}{2}\pi J_{\frac{1}{2}\mu}(\frac{1}{2}a(b-(b^2-y^2)^{\frac{1}{2}}))Y_{-\frac{1}{2}\mu}(\frac{1}{2}a(b+(b^2-y^2)^{\frac{1}{2}}))$,

$\mathrm{Re}\ \mu > -1, \quad y<b$ ... (4, p. 39)

Inserting these into (6) we get:

a) $Y_0(a(b^2-\cos^2\varphi)^{\frac{1}{2}}) = \Sigma\ (-1)^n\varepsilon_n\cos(2n\varphi)$

$J_n(\frac{1}{2}a(b-(b^2-1)^{\frac{1}{2}}))Y_{-n}(\frac{1}{2}a(b+(b^2-1)^{\frac{1}{2}}))$, $\quad b>1$

which converges by appendix: (3), (4), and (5), since

$Y_{-n}(x) = (-1)^nY_n(x)$.

21) $f(x) = (a^2-x^2)^{-\frac{1}{2}}\cos(b(a^2-x^2)^{\frac{1}{2}})$, $0<x<a$

$= 0$, $a<x$

$g_c(y) = \frac{1}{2}\pi J_0(a(b^2+y^2)^{\frac{1}{2}})$ ... (7, p. 30)

$h_\mu(y) = \frac{1}{2}\pi J_{\frac{1}{2}\mu}(\frac{1}{2}a((b^2+y^2)^{\frac{1}{2}}-b))J_{\frac{1}{2}\mu}(\frac{1}{2}a((b^2+y^2)^{\frac{1}{2}}+b))$,

$\text{Re}\ \mu > -1$ ... (4, p. 39)

Inserting these into (6) we get:

a) $J_0(a(b^2+\cos^2\varphi)^{\frac{1}{2}}) = \Sigma\ (-1)^n J_n(\frac{1}{2}a((b^2+1)^{\frac{1}{2}}-b))$

$J_n(\frac{1}{2}a((b^2+1)^{\frac{1}{2}}+b))\varepsilon_n\cos(2n\varphi)$

which converges by appendix: (3).

22) $f(x) = (a^2-x^2)^{-\frac{1}{2}}\cos(\tau\ \arccos(xa^{-1}))$, $x<a$

$= 0$, $x>a$

$g_c(y) = \frac{\pi}{4}\sec(\frac{1}{2}\pi\tau)(J_\tau(ay)+J_{-\tau}(ay))$ ... (7, p. 33)

$g_s(y) = \frac{\pi}{4}\csc(\frac{1}{2}\pi\tau)(J_\tau(ay)-J_{-\tau}(ay))$ ... (7, p. 144)

$h_\mu(y) = \frac{\pi}{2}J_{\frac{1}{2}\mu+\frac{1}{2}\tau}(\frac{1}{2}ay)J_{\frac{1}{2}\mu-\frac{1}{2}\tau}(\frac{1}{2}ay)$, $\text{Re}(\mu)>-1$

... (4, p. 41)

Inserting these into (6) and (7) we get:

a) $\frac{1}{2}\sec(\frac{1}{2}\pi\tau)(J_{\tau}(a\cos\varphi)+J_{-\tau}(a\cos\varphi))$

$= \Sigma\ (-1)^n\ \varepsilon_n \cos(2n\varphi)J_{n+\frac{1}{2}\tau}(\frac{1}{2}a)\ J_{n-\frac{1}{2}\tau}(\frac{1}{2}a)$

b) $\frac{1}{4}\csc(\frac{1}{2}\pi\tau)(J_{\tau}(a\cos\varphi)-J_{-\tau}(a\cos\varphi)) = \Sigma(-1)^n\cos((2n+1)\varphi)$

which converge by appendix: (3).

23) $f(x) = x^{\tau}J_{\tau}(ax)$

$g_c(y) = -\pi^{-\frac{1}{2}}\Gamma(\frac{1}{2}+\tau)(2a)^{\tau}\sin(\pi\tau)(y^2-a^2)^{-\tau-\frac{1}{2}},\ y > a$

$= \pi^{-\frac{1}{2}}\Gamma(\frac{1}{2}+\tau)(2a)^{\tau}(a^2-y^2)^{-\tau-\frac{1}{2}},\ y<a,$ ...(7, p. 62)

$g_s(y) = \pi^{\frac{1}{2}}(\Gamma(\frac{1}{2}-\tau))^{-1}(2a)^{\tau}(y^2-a^2)^{-\tau-\frac{1}{2}},\ y > a$

$= 0\ ,\ \ y < a,$ ... (7, p. 166)

$h_{\mu}(y) = y^{-1}2^{\tau}\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\tau)(\Gamma(\frac{1}{2}+\frac{1}{2}\mu-\tau))^{-1}(y^2-a^2)^{-\frac{1}{2}\tau}$

$P^{-\tau}_{\frac{1}{2}\mu-\frac{1}{2}}(1-\frac{2a^2}{y^2}),\quad y > a$

$= 2^{\tau+1}\pi^{-1}\cos(\frac{1}{2}\pi\mu)y^{-1}(a^2-y^2)^{-\frac{1}{2}\tau}Q^{\tau}_{\frac{1}{2}\mu-\frac{1}{2}}(\frac{2a^2}{y^2}-1)$

$e^{-i\pi\tau},\ y<a,\ -\frac{1}{2}<\mathrm{Re}(\tau)<\frac{1}{2},\ \mathrm{Re}(2\tau+\mu)>0$..(4, p. 48).

Inserting these into (6) and (7) we get:

a) $\pi^{\frac{1}{2}}2^{-\tau-1}(a^2-1)^{\frac{1}{2}\tau}(a^2-\cos^2\varphi)^{-\tau-\frac{1}{2}}\Gamma(\frac{1}{2}+\tau)(2a)^{\tau}\ e^{i\tau\pi}$

$= \Sigma\ \varepsilon_n\cos(2n\varphi)\ Q^{\tau}_{n-\frac{1}{2}}(2a^2-1),\ 0<\mathrm{Re}(\tau)<\frac{1}{2},\ \ a > 1$

b) $\pi^{\frac{1}{2}}(\Gamma(\frac{1}{2}-\tau))^{-1}a^{\tau}(1-a^2)^{-\frac{\tau}{2}}(\cos^2\varphi-a^2)^{-\tau-\frac{1}{2}} = 2\Sigma\ (-1)^n$

$\cos((2n+1)\varphi)\Gamma(n+1+\tau)(\Gamma(n+1-\tau))^{-1}P_n^{-\tau}(1-2a^2)$,

$0 < a < \cos\varphi\ ,\quad -\frac{1}{2} < \mathrm{Re}\ \tau < \frac{1}{2}$

which converge respectively by appendix: (10) and (9).

24) $f(x) = x^{-\tau}J_{\tau}(ax)$

$g_c(y) = \pi^{\frac{1}{2}}(2a)^{-\tau}(\Gamma(\frac{1}{2}+\tau))^{-1}(a^2-y^2)^{\tau-\frac{1}{2}}\ ,\quad y < a$

$= 0\qquad ,y > a\ ,\qquad$ ...(7,p. 62)

$h_{\mu}(y) = 2^{1-\tau}(y\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\tau)\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\tau))^{-1}e^{-i\pi\tau}(a^2-y^2)^{\frac{1}{2}\tau}$

$Q^{\tau}_{\frac{1}{2}\mu-\frac{1}{2}}(\frac{2a^2}{y^2}-1)\ ,\quad y < a$

$= 2^{-\tau}y^{-1}(y^2-a^2)^{\frac{1}{2}\tau}P^{-\tau}_{\frac{1}{2}\mu-\frac{1}{2}}(1-\frac{2a^2}{y^2})\ ,\quad y > a,$

$\mathrm{Re}\ \tau > -\frac{1}{2}\ ,\quad \mathrm{Re}\ \mu > -1\ ,$ ... (4, p. 48)

Inserting these into (6) we get:

a) $\pi^{\frac{1}{2}}(a^2-\cos^2\varphi)^{\tau-\frac{1}{2}}(2a^{\tau}\Gamma(\frac{1}{2}+\tau)(a^2-1)^{\frac{1}{2}\tau})^{-1}e^{i\pi\tau} = \Sigma\ (-1)^n\varepsilon_n$

$\cos(2n\varphi)Q^{\tau}_{n-\frac{1}{2}}(2a^2-1)(\Gamma(\frac{1}{2}+n+\tau)\Gamma(\frac{1}{2}-n+\tau))^{-1}$

which, using the relation $\Gamma(Z)\Gamma(1-Z) = \pi(\csc(\pi Z))$ and and the elementary relation for the quotient of two gamma functions of large arguments, converges by appendix: (10).

25) $f(x) = (J_o(ax))^2$

$$g_c(y) = (\pi a)^{-1} K\left(\left(1-\frac{y^2}{4a^2}\right)^{\frac{1}{2}}\right) , \quad y < 2a$$

$$= 0 , \quad y > 2a , \ldots (7, \text{ p. } 64)$$

$$g_s(y) = (\pi a)^{-1} K\left(\frac{y}{2a}\right) , \quad y < 2a$$

$$= 2(\pi y)^{-1} K\left(\frac{2a}{y}\right) , \quad y > 2a , \ldots (7, \text{ p. } 169)$$

$$h_\mu(y) = y^{-1}\left(P_{\frac{1}{2}\mu-\frac{1}{2}}\left(\left(1-\frac{4a^2}{y^2}\right)^{\frac{1}{2}}\right)\right)^2, \quad y>2a , \ \mathrm{Re}\ \mu > -1$$

$$\ldots (4, \text{ p. } 51)$$

Inserting these into (6) and (7) we get:

a) $$(\pi a)^{-1} K\left(\left(1-\frac{\cos^2\varphi}{4a^2}\right)^{\frac{1}{2}}\right) = \Sigma\ (-1)^n \varepsilon_n \cos(2n\varphi)\left(P_{n-\frac{1}{2}}\left((1-4a^2)^{\frac{1}{2}}\right)\right)^2$$

$$0 < \cos\varphi < 2a < 1$$

b) $$(\pi a)^{-1} K\left(\frac{\cos\varphi}{2a}\right) = 2\Sigma\ (-1)^n \cos((2n+1)\varphi)\left(P_n\left((1-4a^2)^{\frac{1}{2}}\right)\right)^2,$$

$$0 < \cos\varphi < 2a < 1$$

b)' $$2(\pi\cos\varphi)^{-1} K\left(\frac{2a}{\cos\varphi}\right) = 2\Sigma\ (-1)^n \cos((2n+1)\varphi)\left(P_n\left((1-4a^2)^{\frac{1}{2}}\right)\right)^2$$

which converge by appendix: (9), since $P_n = P_n^o$.

26) $f(x) = (J_\tau(ax))^2$

$$g_c(y) = -(\pi a)^{-1}\sin(\pi\tau)Q_{\tau-\frac{1}{2}}(\frac{y^2}{2a^2}-1) \ , \quad y>2a$$

$$= (\pi a)^{-1}Q_{\tau-\frac{1}{2}}(1-\frac{y^2}{2a^2}) \ , \quad y<2a \quad \ldots (7, \text{ p. } 65)$$

$$h_\mu(y) = \Gamma(\tfrac{1}{2}\mu+\tau+\tfrac{1}{2})(y\Gamma(\tfrac{1}{2}\mu-\tau+\tfrac{1}{2}))^{-1}(P^{-\tau}_{-\frac{1}{2}+\frac{1}{2}\mu}(1-\frac{4a^2}{y^2})^{\frac{1}{2}})^2,$$

$$y > 2a \ , \ \mathrm{Re}\,\tau > -\tfrac{1}{2} \ , \ \mathrm{Re}(2\tau+\mu) > -1$$

$$\ldots (4, \text{ p. } 52)$$

Inserting these into (6) we get:

a) $(\pi a)^{-1}Q_{\tau-\frac{1}{2}}(1-\frac{\cos^2\varphi}{2a^2}) = \Sigma\ (-1)^n\varepsilon_n\cos(2n\varphi)\Gamma(n+\tau+\frac{1}{2})$

$(\Gamma(n-\tau+\frac{1}{2}))^{-1}(P^{-\tau}_{-\frac{1}{2}+n}(1-4a^2)^{\frac{1}{2}})^2, \quad 0<\cos\varphi<2a<1$

a)' $-(\pi a)^{-1}\sin(\pi\tau)Q_{\tau-\frac{1}{2}}(\frac{\cos^2\varphi}{2a^2}-1) = \Sigma\ (-1)^n\varepsilon_n\cos(2n\varphi)$

$\Gamma(n+\tau+\frac{1}{2})(\Gamma(n-\tau+\frac{1}{2}))^{-1}(P^{-\tau}_{-\frac{1}{2}+n}(1-4a^2)^{\frac{1}{2}})^2, \quad 1>\cos\varphi>2a>0$

which converges by appendix: (9), and the asymptotics fo r the ratio of two gamma functions of large arguments.

27) $f(x) = J_\tau(ax)J_{-\tau}(ax)$

$$g_c(y) = \tfrac{1}{2}a^{-1}P_{\tau-\frac{1}{2}}(\frac{y^2}{2a^2}-1) \ , \quad y<2a$$

$$= 0 \ , \quad y>2a \quad \ldots (7, \text{ p. } 65)$$

$$h_\mu(y) = \frac{1}{y}P^{\tau}_{\frac{1}{2}\mu-\frac{1}{2}}((1-\frac{4a^2}{y^2})^{\frac{1}{2}})P^{-\tau}_{\frac{1}{2}\mu-\frac{1}{2}}((1-\frac{4a^2}{y^2})^{\frac{1}{2}}) \ , \quad y>2a$$

$$\ldots (4, \text{ p. } 54)$$

Inserting these into (6) we get:

a) $\frac{1}{2a} P_{\tau-\frac{1}{2}}(\frac{\cos^2\varphi}{2a^2} - 1) = \Sigma (-1)^n \varepsilon_n \cos(2n\varphi) P^{\tau}_{n-\frac{1}{2}}((1-4a^2)^{\frac{1}{2}})$

$P^{-\tau}_{n-\frac{1}{2}}((1-4a^2)^{\frac{1}{2}})$, $\cos\varphi<2a<1$

which converges by appendix: (9).

28) $f(x) = x^{\tau}K_{\tau}(ax)$

$g_c(y) = \frac{1}{2}\pi^{\frac{1}{2}}(2a)^{\tau}\,\Gamma(\frac{1}{2}+\tau)(a^2+y^2)^{-\tau-\frac{1}{2}}$ ... (7, p. 87)

$\mathrm{Re}\tau>-\frac{1}{2}$

$g_s(y) = 2^{\tau}a^{-\tau-2}\Gamma(1+\tau)y\,{}_2F_1(1+\tau,1;\frac{3}{2};-\frac{y^2}{a^2})$ ... (7, p. 184)

$\mathrm{Re}\tau>-1$

$h_{\mu}(y) = 2^{\tau}y^{-1}(a^2+y^2)^{-\frac{1}{2}\tau}\, e^{-i\pi\tau}\, Q^{\tau}_{\frac{1}{2}\mu-\frac{1}{2}}(1+\frac{2a^2}{y^2})$

$\mathrm{Re}(2\tau+\mu)>-1$ ... (4, p. 63)

Inserting these into (6) and (7) we get:

a) $\frac{(\pi)^{\frac{1}{2}}}{2}(a^2+1)^{\frac{1}{2}\tau}e^{i\pi\tau}a^{\tau}\Gamma(\frac{1}{2}+\tau)(a^2+\cos^2\varphi)^{-\tau-\frac{1}{2}} = \Sigma (-1)^n$

$\varepsilon_n\cos(2n\varphi)\, Q^{\tau}_{n-\frac{1}{2}}(1+2a^2)$, $\mathrm{Re}\tau>-\frac{1}{2}$

b) $\frac{(a^2+1)^{\frac{1}{2}\tau}\, e^{i\pi\tau}}{2a^{\tau+2}}\Gamma(1+\tau)\cos\varphi\ {}_2F_1(1+\tau,1;\frac{3}{2};-\frac{\cos^2\varphi}{a^2}) =$

$\Sigma (-1)^n \cos((2n+1)\varphi)\, Q^{\tau}_{n-\frac{1}{2}}(1+2a^2)$, $\mathrm{Re}\tau>-\frac{1}{2}$

which converge by appendix: (10).

29) $f(x) = I_0(ax)K_0(ax)$

$$g_c(y) = (y^2+4a^2)^{-\frac{1}{2}}K(2a(y^2+4a^2)^{-\frac{1}{2}}) \quad \ldots (7, \text{ p. } 88)$$

$$g_s(y) = (y^2+4a^2)^{-\frac{1}{2}}K(y(y^2+4a^2)^{-\frac{1}{2}}) \quad \ldots (7, \text{ p. } 184)$$

$$h_\mu(y) = P_{\frac{1}{2}\mu-\frac{1}{2}}((1+4a^2y^{-2})^{\frac{1}{2}})\, Q_{\frac{1}{2}\mu-\frac{1}{2}}((1+4a^2y^{-2})^{\frac{1}{2}}),$$

$$\text{Re } \mu > -1 \quad \ldots (4, \text{ p. } 65)$$

Inserting these into (6) and (7) we get:

a) $$(\cos^2\varphi+4a^2)^{-\frac{1}{2}}K(2a(\cos^2\varphi+4a^2)^{-\frac{1}{2}}) = \Sigma\,(-1)^n\varepsilon_n\cos(2n\varphi)$$

$$P_{n-\frac{1}{2}}((1+4a^2)^{\frac{1}{2}})\, Q_{n-\frac{1}{2}}((1+4a^2)^{\frac{1}{2}})$$

b) $$(\cos^2\varphi+4a^2)^{-\frac{1}{2}}\, K(\cos\varphi(\cos^2\varphi+4a^2)^{-\frac{1}{2}}) = 2\Sigma\,(-1)^n$$

$$\cos((2n+1)\varphi)\, P_n((1+4a^2)^{\frac{1}{2}})\, Q_n((1+4a^2)^{\frac{1}{2}})$$

which converge by appendix: (10) and (11), since the Legendre functions with upper index zero are written without an upper index.

30) $f(x) = I_\tau(ax)K_\tau(ax)$

$$g_c(y) = \tfrac{1}{2}a^{-1}Q_{\tau-\frac{1}{2}}(1+\tfrac{y^2}{2a^2}) \quad \ldots (9)$$

$$h_\mu(y) = \Gamma(\tfrac{1}{2}+\tau+\tfrac{1}{2}\mu)(\Gamma(\tfrac{1}{2}-\tau+\tfrac{1}{2}\mu))^{-1} P^{-\tau}_{\frac{1}{2}\mu-\frac{1}{2}}((1+\frac{4a^2}{y^2})^{\frac{1}{2}})e^{i\pi\tau}$$

$$Q^{-\tau}_{\frac{1}{2}\mu-\frac{1}{2}}((1+\frac{4a^2}{y^2})^{\frac{1}{2}}),\ \mathrm{Re}\ \tau > -\tfrac{1}{2},\ \mathrm{Re}(2\tau+\mu) > -1$$

... (4, p. 65)

Inserting these into (6) we get:

a) $$(2a)^{-1}e^{-i\pi\tau}\, Q_{\tau-\frac{1}{2}}(1+\frac{\cos^2\varphi}{2a^2}) = \Sigma\,(-1)^n\varepsilon_n\cos(2n\varphi)$$

$$\Gamma(\tfrac{1}{2}+\tau+n)(\Gamma(\tfrac{1}{2}+n-\tau))^{-1}P^{-\tau}_{n-\frac{1}{2}}((1+4a^2)^{\frac{1}{2}})\, Q^{-\tau}_{n-\frac{1}{2}}((1+4a^2)^{\frac{1}{2}})$$

$$\mathrm{Re}\ \tau > -\tfrac{1}{2}$$

which converges by appendix: (10) and (11).

31) $$f(x) = (K_\tau(ax))^2$$

$$g_c(y) = \pi^2(4a)^{-1}\sec(\pi\tau)P_{\tau-\frac{1}{2}}(1+\frac{y^2}{2a^2}) \quad \ldots (7, \text{ p. } 88)$$

$$h_\mu(y) = \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tau)(\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu-\tau))^{-1}(e^{i\pi\tau}\, Q^{-\tau}_{\frac{1}{2}\mu-\frac{1}{2}}((1+\frac{4a^2}{y^2})^{\frac{1}{2}}))^2$$

$$-\tfrac{1}{2} < \mathrm{Re}\ \tau < \tfrac{1}{2},\ \mathrm{Re}(\mu\pm 2\tau) > -1 \ \ldots (4, \text{p. } 66)$$

Inserting these into (6) we get:

a) $$\frac{\pi^2}{4a}\sec(\pi\tau)\, P_{\tau-\frac{1}{2}}(1+\frac{\cos^2\varphi}{2a^2}) = e^{2i\pi\tau}\Sigma\,(-1)^n\varepsilon_n\cos(2n\varphi)$$

$$\Gamma(\tfrac{1}{2}+n+\tau)(\Gamma(\tfrac{1}{2}+n-\tau))^{-1}(Q^{-\tau}_{n-\frac{1}{2}}((1+4a^2)^{\frac{1}{2}}))^2, \quad -\tfrac{1}{2} < \mathrm{Re}\ \tau < \tfrac{1}{2}$$

which converges by appendix: (10) and the asymptotics of the quotient of two gamma functions.

32) $f(x) = J_\tau(2(ax)^{\frac{1}{2}})K_\tau(2(ax)^{\frac{1}{2}})$

$$g_c(y) = \frac{\pi}{2}\csc(\pi\tau)y^{-1}\left(\sin\left(\frac{\pi\tau}{2}\right)I_\tau\left(\frac{2a}{y}\right)+\frac{i}{2}J_\tau\left(\frac{2ia}{y}\right)-\frac{i}{2}J_\tau\left(-\frac{2ia}{y}\right)\right)$$

:::(7, p. 92)

$$g_s(y) = \frac{\pi}{2}\csc(\pi\tau)y^{-1}\left(\tfrac{1}{2}J_\tau\left(\frac{2ia}{y}\right)+\tfrac{1}{2}J_\tau\left(-\frac{2ia}{y}\right)-\cos\left(\frac{\pi\tau}{2}\right)I_\tau\left(\frac{2a}{y}\right)\right)$$

...(7, p. 188)

$$h_\mu(y) = (4a)^{-1}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\tau+\tfrac{1}{2}\mu)(\Gamma(1+\tau))^{-1}W_{-\frac{1}{2}\mu,\frac{1}{2}\tau}\left(\frac{2a}{y}\right)$$

$$M_{\frac{1}{2}\mu,\frac{1}{2}\tau}\left(\frac{2a}{y}\right), \quad \mathrm{Re}(\tau,\mu)>-1, \ \mathrm{Re}(\mu+\tau)>-1$$

...(4, p. 71)

Inserting these into (6) and (7) we get:

a) $$2\pi a\frac{\csc(\pi\tau)}{\cos\varphi}\left(\sin\left(\tfrac{1}{2}\pi\tau\right)I_\tau\left(\frac{2a}{\cos\varphi}\right)+\tfrac{1}{2}iJ_\tau\left(\frac{2ia}{\cos\varphi}\right)-\tfrac{1}{2}iJ_\tau\left(-\frac{2ia}{\cos\varphi}\right)\right)=$$

$$+\Sigma\,(-1)^n\varepsilon_n\cos(2n\varphi)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\tau+n)(\Gamma(1+\tau))^{-1}W_{-n,\frac{1}{2}\tau}(2a)$$

$$M_{n,\frac{1}{2}\tau}(2a), \qquad \mathrm{Re}\,\tau > -1$$

b) $$\pi a\frac{\csc(\pi\tau)}{\cos\varphi}\left(\tfrac{1}{2}J_\tau\left(\frac{2ia}{\cos\varphi}\right)+\tfrac{1}{2}J_\tau\left(\frac{-2ia}{\cos\varphi}\right)-\cos\left(\frac{\pi\tau}{2}\right)I_\tau\left(\frac{2a}{\cos\varphi}\right)\right) = \Sigma\,(\varepsilon_n)$$

$$(-1)^n\cos((2n+1)\varphi)\Gamma(1+\tfrac{1}{2}\tau+n)(\Gamma(1+\tau))^{-1}W_{-n-\frac{1}{2},\frac{1}{2}\tau}(2a)$$

$$M_{n+\frac{1}{2},\frac{1}{2}\tau}(2a), \quad \mathrm{Re}\,\tau > -1$$

which converge by appendix: (12) and (13).

33) $f(x) = (K_0(ax))^2$

$$g_c(y) = 2\pi(y^2+4a^2)^{-\frac{1}{2}}K(y(y^2+4a^2)^{-\frac{1}{2}}) \cdots (7, \text{ p. } 88)$$

$$h_\mu(y) = (Q_{\frac{1}{2}\mu-\frac{1}{2}}((1+\frac{4a^2}{y^2})^{\frac{1}{2}}))^2, \text{ Re } \mu > -1 \cdots (4, \text{ p. } 66)$$

Inserting these into (6) we get:

a) $$2\pi(\cos^2\varphi+4a^2)^{-\frac{1}{2}}K(y(y^2+4a^2)^{-\frac{1}{2}}) = \Sigma\ (-1)^n\varepsilon_n\cos(2n\varphi)$$

$$(Q_{n-\frac{1}{2}}((1+4a^2)^{\frac{1}{2}}))^2$$

which converges by appendix: (10).

34) $f(x) = K_\tau(a(ix)^{\frac{1}{2}})K_\tau(a(-ix)^{\frac{1}{2}})$

$$g_c(y) = \pi(4y)^{-1}\sec(\tfrac{1}{2}\pi\tau)S_{0,\tau}(\tfrac{1}{2}a^2y^{-1}) \cdots (7, \text{ p. } 93)$$

$$g_s(y) = \pi\tau(4y)^{-1}\csc(\tfrac{1}{2}\pi\tau)S_{-1,\tau}(\tfrac{1}{2}a^2y^{-1}) \cdots (7, \text{ p. } 189)$$

$$h_\mu(y) = \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\tau+\tfrac{1}{2}\mu)\frac{1}{2a^2}\ \Gamma(\tfrac{1}{2}-\tfrac{1}{2}\tau+\tfrac{1}{2}\mu)\ W_{=\frac{1}{2}\mu,\frac{1}{2}\tau}(\frac{ia^2}{2y})$$

$$W_{-\frac{1}{2}\mu,\frac{1}{2}\tau}(-\frac{ia^2}{2y}), \quad -1 < \text{Re } \tau < 1, \text{ Re } \mu > -1,$$

$$\text{Re}(\mu+\tau) > -1 \qquad \cdots (9)$$

Inserting these into (6) and (7) we get:

a) $$\pi a^2(2\cos\varphi)^{-1}\sec(\tfrac{1}{2}\pi\tau)S_{0,\tau}(\frac{a^2}{2\cos\varphi}) = \Sigma\ (-1)^n\varepsilon_n\cos(2n\varphi)$$

$$\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\tau+n)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\tau+n)W_{-n,\frac{1}{2}\tau}(\tfrac{1}{2}ia^2)W_{-n,\frac{1}{2}\tau}(-\tfrac{1}{2}ia^2)\ ,$$

$$-1 < \text{Re } \tau < 1$$

b) $$\pi\tau a^2(4\cos\varphi)^{-1}\csc(\tfrac{1}{2}\pi\tau)S_{-1,\tau}\left(\frac{a^2}{2\cos\varphi}\right) = \Sigma\,(-1)^n\cos((2n+1)\varphi)$$

$$\Gamma(1+\tfrac{1}{2}\tau+n)\Gamma(1-\tfrac{1}{2}\tau+n)W_{-\frac{1}{2}-n,\frac{1}{2}\tau}(\tfrac{1}{2}ia^2)W_{-\frac{1}{2}-n,\frac{1}{2}\tau}(-\tfrac{1}{2}ia^2),$$

$$-1 < \mathrm{Re}\ \tau < 1$$

which converge by appendix: (12).

2. SERIES DERIVED FROM GEGENBAUER'S EXPANSIONS

The expansions (1) and (2) can be generalized. A generalization has been given by Gegenbauer (3, p. 102)

(8) $x^{\nu} \cos(x\cos\varphi) = 2^{\nu}\Gamma(\nu)\Sigma(-1)^{n}(\nu+2n)C_{2n}^{\nu}(\cos\varphi)J_{\nu+2n}(x)$

$\nu \neq M = 0, -1, -2, \cdots$

(9) $x^{\nu} \sin(x\cos\varphi) = 2^{\nu}\Gamma(\nu)\Sigma(-1)^{n}(\nu+2n+1)C_{2n+1}^{\nu}(\cos\varphi)$

$J_{\nu+2n+1}(x)$.

Here $C_m^{\nu}(z)$ denotes the mth Gegenbauer polynomial (3, vol. 1, p. 175) (3, vol. 2, p. 174). Jacobi's formulas can be obtained from (8), (9) using the formula

(10) $$\lim_{\nu \to 0}\Gamma(\nu)C_n^{\nu}(\cos\varphi) = 2n^{-1}\cos(n\varphi) \qquad n \neq 0 .$$

Furthermore for the case $\nu = \frac{1}{2}$ (3, vol. 1, p. 175) it reduces to

(11) $$C_n^{1/2}(z) = P_n(z)$$

where $P_n(z)$ is the nth Legendre polynomial (3, vol. 2, p. 178).

We introduce now the abbreviations

(12) $G_c(y) = \int_0^{\infty} x^{\nu} f(x)\cos(xy)dx$, Fourier cosine transform of $x^{\nu} f(x)$

(13) $G_s(y) = \int_0^\infty x^\nu f(x)\sin(xy)dx$, Fourier sine transform of $x^\nu f(x)$

(14) $h_\mu(y) = \int_0^\infty f(x)J_\mu(xy)dx$, Hankel transform of $f(x)$.

Now one obtains from (8) and (9) by the same procedure as used in chapter 1.

(15) $2^\nu\Gamma(\nu)\Sigma(-1)^n(\nu+2n)\, C^\nu_{2n}(\cos\varphi)n_{\nu+2n}(1) = G_c(\cos\varphi)$

(16) $2^\nu\Gamma(\nu)\Sigma(-1)^n(\nu+2n+1)C^\nu_{2n+1}(\cos\varphi)n_{\nu+2n+1}(1) = G_s(\cos\varphi)$.

In this case the right hand side periodic functions $G_c(\cos\varphi)$ and $G_s(\cos\varphi)$ are not expanded into Fourier series but into series after Gegenbauer polynomials. If one puts $\nu = \frac{1}{2}$ and observes (11) the Fourier transforms are expanded into series after Legendre Polynomials.

Examples

1) $f(x) = x^{\alpha}$

$$G_c(y) = -\sin(\tfrac{1}{2}\pi(\alpha+\nu))\Gamma(1+\alpha+\nu)y^{-\nu-\alpha-1}, \quad -1 < \mathrm{Re}(\alpha+\nu) < 0$$

... (7, p. 5)

$$G_s(y) = \cos(\tfrac{1}{2}\pi(\alpha+\nu))\Gamma(1+\alpha+\nu)y^{-\nu-\alpha-1}, \quad -2 < \mathrm{Re}(\alpha+\nu) < 0$$

... (7, p. 116)

$$h_{\mu}(y) = 2^{\alpha}\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}\alpha+\tfrac{1}{2})(y^{\alpha+1}\Gamma(\tfrac{1}{2}\mu-\tfrac{1}{2}\alpha+\tfrac{1}{2}))^{-1},$$

$$-1-\mathrm{Re}\,\mu < \mathrm{Re}\,\alpha < \tfrac{1}{2}, \qquad \text{... (4, p. 22)}$$

Inserting these into (15) and (16) we get:

a) $$2^{\nu+\alpha}\Gamma(\nu)\,\Sigma\,(-1)^n(\nu+2n)C^{\nu}_{2n}(\cos\varphi)\Gamma(n+\tfrac{1}{2}\nu+\tfrac{1}{2}\alpha+\tfrac{1}{2})$$

$$(\Gamma(n+\tfrac{1}{2}\nu+\tfrac{1}{2}-\tfrac{1}{2}\alpha))^{-1} = -\sin(\tfrac{1}{2}\pi(\alpha+\nu))\Gamma(1+\alpha+\nu)(\cos\varphi)^{-\nu-\alpha-1},$$

$$\mathrm{Re}\,\alpha < \tfrac{1}{2}, \quad -1 < \mathrm{Re}(\alpha+\nu) < 0$$

b) $$2^{\nu+\alpha}\Gamma(\nu)\,\Sigma\,(-1)^n(\nu+2n+1)C^{\nu}_{2n+1}(\cos\varphi)\Gamma(n+1+\tfrac{1}{2}\nu+\tfrac{1}{2}\alpha)$$

$$(\Gamma(n+1+\tfrac{1}{2}\nu-\tfrac{1}{2}\alpha))^{-1} = \cos(\tfrac{1}{2}\pi(\alpha+\nu))\Gamma(1+\alpha+\nu)(\cos\varphi)^{-\nu-\alpha-1},$$

$$-1 < \mathrm{Re}(\alpha+\nu) < 0, \quad \mathrm{Re}\,\alpha < \tfrac{1}{2}$$

which converge under the stated restrictions by appendix: (14) and the asymptotics for the ratio of two gamma functions.

2) $f(x) = (a+x)^{-1}$

$G_c(y) = (2a)^{\nu}(\pi ay)^{\frac{1}{2}}(\Gamma(\frac{1}{2}+\frac{1}{2}\nu)(\Gamma(-\frac{1}{2}\nu))^{-1}S_{-\nu-\frac{1}{2},\frac{1}{2}}(ay)-$

$2\Gamma(1+\frac{1}{2}\nu)(\Gamma(-\frac{1}{2}-\frac{1}{2}\nu))^{-1}S_{-\nu-\frac{3}{2},\frac{1}{2}}(ay))$, $-1 < \text{Re } \nu < 1$

... (7, p. 6)

$G_s(y) = (2a)^{\nu}(\pi ay)^{\frac{1}{2}}(\Gamma(1+\frac{1}{2}\nu)(\Gamma(\frac{1}{2}-\frac{1}{2}\nu))^{-1}S_{-\nu-\frac{1}{2},\frac{1}{2}}(ay)-$

$2\Gamma(\frac{1}{2}+\frac{1}{2}\nu)(\Gamma(-\frac{1}{2}\nu))^{-1}S_{-\nu-\frac{3}{2},\frac{1}{2}}(ay))$, $-2 < \text{Re } \nu < 1$

... (7, p. 116)

$h_\mu(y) = \csc(\pi\mu)(J_\mu(ay)-J_\mu(ay))$, $\text{Re } \mu > -1$

... (4, p. 23)

Inserting these into (15) and (16) we get:

a) $2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n)C^{\nu}_{2n}(\cos\varphi)\csc(\pi(\nu+2n))(J_{\nu+2n}(a)-$

$J_{\nu+2n}(a)) = (2a)^{\nu}(\pi a\ \cos\varphi)^{\frac{1}{2}}(\Gamma(\frac{1}{2}+\frac{1}{2}\nu)(\Gamma(-\frac{1}{2}\nu))^{-1}$

$S_{-\nu-\frac{1}{2},\frac{1}{2}}(a\ \cos\varphi)-2\Gamma(1+\frac{1}{2}\nu)(\Gamma(-\frac{1}{2}-\frac{1}{2}\nu))^{-1}S_{-\nu-\frac{3}{2},\frac{1}{2}}(a\ \cos\varphi))$,

$-1 < \text{Re } \nu < 1$

b) $2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n+1)C^{\nu}_{2n+1}(\cos\varphi)\csc(\pi(\nu+2n+1))$

$(\mathbb{J}_{\nu+2n+1}(a)-J_{\nu+2n+1}(a)) = (2a)^{\nu}(\pi a\cos\varphi)^{\frac{1}{2}}(\Gamma(1+\frac{1}{2}\nu)$

$(\Gamma(\frac{1}{2}-\frac{1}{2}\nu))^{-1}S_{-\nu-\frac{1}{2},\frac{1}{2}}(a\cos\varphi)-2\Gamma(\frac{3}{2}+\frac{1}{2}\nu)(\Gamma(-\frac{1}{2}\nu))^{-1}$

$S_{-\nu-\frac{3}{2},\frac{1}{2}}(a\cos\varphi))$, $-2 < \mathrm{Re}\ \nu < 1$

which converge under the imposed restrictions by

appendix: (3), (14) and (15).

3) $f(x) = (a^2+x^2)^{-\frac{1}{2}}$

$G_c(y) = \frac{1}{2}a^{\nu}\pi^{-\frac{1}{2}}\Gamma(\frac{1}{2}+\frac{1}{2}\nu)\Gamma(-\frac{1}{2}\nu)\,{}_1F_2(\frac{1}{2}+\frac{1}{2}\nu;1+\frac{1}{2}\nu,\frac{1}{2};\frac{a^2y^2}{4})+$

$+\pi^{\frac{1}{2}}2^{\nu-1}y^{-\nu}\Gamma(\frac{1}{2}\nu)(\Gamma(\frac{1}{2}-\frac{1}{2}\nu))^{-1}\,{}_1F_2(\frac{1}{2};\frac{1}{2}-\frac{1}{2}\nu,1-\frac{1}{2}\nu;\frac{a^2y^2}{4})$

$-1 < \mathrm{Re}\ \nu < 1$ ... (7, p. 10)

$G_s(y) = \frac{1}{2}ya^{\nu+1}\pi^{-\frac{1}{2}}\Gamma(1+\frac{1}{2}\nu)\Gamma(-\frac{1}{2}-\frac{1}{2}\nu)\,{}_1F_2(1+\frac{1}{2}\nu;\frac{3}{2}+\frac{1}{2}\nu,\frac{3}{2};\frac{a^2y^2}{4})$

$+\pi^{\frac{1}{2}}2^{\nu-1}\Gamma(\frac{1}{2}+\frac{1}{2}\nu)(\Gamma(1-\frac{1}{2}\nu))^{-1}y^{-\nu}$

${}_1F_2(\frac{1}{2};\frac{1}{2}-\frac{1}{2}\nu,1-\frac{1}{2}\nu;\frac{a^2y^2}{4})$, $-2<\mathrm{Re}\ \nu<1$, ...(7, p. 120)

$h_{\mu}(y) = I_{\frac{1}{2}\mu}(\frac{ay}{2})K_{\frac{1}{2}\mu}(\frac{ay}{2})$, $\mathrm{Re}\ \mu > -1$, ... (4, p. 23)

Inserting these into (15) and (16) we get:

a) $2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n)\ C_{2n}^{\nu}(\cos\varphi)\ I_{n+\frac{\nu}{2}}(\frac{a}{2})\ K_{n+\frac{\nu}{2}}(\frac{a}{2})$

$= \frac{a^{\nu}\pi^{-\frac{1}{2}}}{2}\ \Gamma\ (\frac{1+\nu}{2})\Gamma(-\ \frac{\nu}{2}),{}_1F_2(\frac{1}{2}+\frac{\nu}{2};\ 1+\frac{\nu}{2},\ \frac{1}{2};\ \frac{a^2\cos^2\varphi}{4})$

$+\ \pi^{\frac{1}{2}}\ \frac{2^{\nu-1}}{(\cos\varphi)^{\nu}}\ \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{1}{2}-\ \frac{\nu}{2})}\ {}_1F_2(\frac{1}{2};\frac{1}{2}-\ \frac{\nu}{2},1-\ \frac{\nu}{2};\frac{a^2\cos^2\varphi}{4},\ -1 < \mathrm{Re}\,\nu < 1$

b) $2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n+1)C_{2n+1}^{\nu}(\cos\varphi)I_{n+\frac{1}{2}\nu+\frac{1}{2}}(\frac{a}{2})K_{n+\frac{1}{2}\nu+\frac{1}{2}}(\frac{a}{2})$

$= \frac{\cos\varphi}{2}\ a^{\nu+1}\pi^{-\frac{1}{2}}\Gamma(1+\frac{\nu}{2})\Gamma(\frac{-1-\nu}{2})\,{}_1F_2(1+\frac{\nu}{2};\frac{3}{2}+\frac{\nu}{2},\frac{3}{2};\frac{a^2\cos^2\varphi}{4})$

$+\ \pi^{\frac{1}{2}}2^{\nu-1}\Gamma(\frac{1}{2}+\frac{\nu}{2})\ (\Gamma(1-\frac{\nu}{2}))^{-1}(\cos\varphi)^{-\nu}$

${}_1F_2(\frac{1}{2};\frac{1}{2}\ -\ \frac{\nu}{2},\ 1\ -\ \frac{\nu}{2};\ \frac{a^2\cos^2\varphi}{2})\ ,\qquad -2 < \mathrm{Re}\,\nu < 1$

which converge by appendix: (1), (2), (14).

4) $f(x) = (a^2-x^2)^{-\frac{1}{2}}\ ,\qquad x < a$

$= 0\qquad\quad ,\qquad x > a$

$G_c(y) = \frac{1}{2}a^{\nu}\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}+\frac{1}{2}\nu)(\Gamma(1+\frac{1}{2}\nu))^{-1}\,{}_1F_2(\frac{1}{2}+\frac{1}{2}\nu;\frac{1}{2},1+\frac{1}{2}\nu;-\frac{a^2y^2}{4}),$

$\mathrm{Re}\ \nu > -1 \quad \cdots$ (7, p. 9)

$G_s(y) = \frac{1}{2}ya^{\nu+1}\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}\nu+1)(\Gamma(\frac{3}{2}+\frac{1}{2}\nu))^{-1}$

${}_1F_2(1+\frac{1}{2}\nu;\frac{3}{2}+\frac{1}{2}\nu,\frac{3}{2};-\frac{a^2y^2}{4}),\mathrm{Re}\ \nu>-2\cdots$(7, p. 119)

$h_{\mu}(y) = \frac{1}{2}\pi(J_{\frac{1}{2}\mu}(\frac{1}{2}ay))^2\ ,\ \ \mathrm{Re}\ \mu > -1\ \cdots\ $ (4, p. 24).

Inserting these into (15) and (16) we get:

a) $2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^{n}(\nu+2n)C_{2n}^{\nu}(\cos\varphi)(J_{n+\frac{1}{2}\nu}(\frac{1}{2}a))^{2} = a^{\nu}\pi^{-\frac{1}{2}}$

$\Gamma(\frac{1}{2}+\frac{1}{2}\nu)(\Gamma(1+\frac{1}{2}\nu))^{-1}{}_{1}F_{2}(\frac{1}{2}+\frac{1}{2}\nu;\frac{1}{2},1+\frac{1}{2}\nu;-\frac{a^{2}\cos^{2}\varphi}{4})$, Re $\nu>-1$

b) $\pi^{\frac{1}{2}}2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^{n}(\nu+2n+1)C_{2n+1}^{\nu}(\cos\varphi)(J_{n+\frac{1}{2}\nu+\frac{1}{2}}(\frac{1}{2}a))^{2} =$

$a^{\nu+1}\cos\varphi\Gamma(\frac{1}{2}\nu+1)(\Gamma(\frac{3}{2}+\frac{1}{2}\nu))^{-1}{}_{1}F_{2}(1+\frac{1}{2}\nu;\frac{3}{2}+\frac{1}{2}\nu,\frac{3}{2};-\frac{a^{2}\cos^{2}\varphi}{4})$,

Re $\nu > -2$

which converge by appendix: (3) and (14).

5) $f(x) = x^{\eta}e^{-ax}$

$G_{c}(y) = \Gamma(\eta+\nu+1)(a^{2}+y^{2})^{-\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{2}}\cos((\eta+\nu+1)\arctan(\frac{y}{a}))$

Re $(\eta+\nu) > -1$ ... (7, p. 10)

$G_{s}(y) = \Gamma(\eta+\nu+1)(a^{2}+y^{2})^{-\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{2}}\sin((\eta+\nu+1)\arctan(\frac{y}{a}))$

Re $(\eta+\nu) > -1$ ... (7, p. 121)

$h_{\mu}(y) = (a^{2}+y^{2})^{-\frac{1}{2}\eta-\frac{1}{2}}\Gamma(\eta+1+\mu)P_{\eta}^{-\mu}(a(a^{2}+y^{2})^{-\frac{1}{2}})$,

Re $(\eta+\mu) > -1$ ... (4, p. 29)

Inserting these into (15) and (16) we get:

a) $2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^{n}(\nu+2n)C_{2n}^{\nu}(\cos\varphi)(a^{2}+1)^{-\frac{1}{2}-\frac{1}{2}\eta}\Gamma(\eta+1+\nu+2n)$

$P_{\eta}^{-2n-\nu}(a(a^{2}+1)^{-\frac{1}{2}}) = \Gamma(\eta+\nu+1)(a^{2}+\cos^{2}\varphi)^{-\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{2}}$

$\cos((\eta+\nu+1)\arctan(\frac{\cos\varphi}{a}))$, Re $(\eta+\nu) > -1$

b) $2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^{n}(\nu+2n+1)C_{2n+1}^{\nu}(\cos\varphi)(a^{2}+1)^{-\frac{1}{2}-\frac{1}{2}\eta}$

$\Gamma(\eta+2+\nu+2n)P_{\eta}^{-2n-\nu-1}(a(a^{2}+1)^{-\frac{1}{2}}) = \Gamma(\eta+\nu+1)$

$(a^{2}+\cos^{2}\varphi)^{-\frac{1}{2}-\frac{1}{2}\eta-\frac{1}{2}\nu}\sin((\eta+\nu+1)\arctan(\frac{\cos\varphi}{a}))$,

$$\mathrm{Re}\ (\eta+\nu) > -2$$

which converge by appendix: (6) and (14) when $a > 0$.

6) $f(x) = e^{-ax^{2}}$

$G_{c}(y) = \frac{a^{-\frac{1}{2}-\frac{\nu}{2}}}{2}\Gamma(\frac{1}{2}+\frac{\nu}{2})\ {}_{1}F_{1}(\frac{1}{2}+\frac{\nu}{2};\frac{1}{2};-\frac{y^{2}}{4a})$, $\mathrm{Re}(\nu) > -1$

... (7, p. 12)

$G_{s}(y) = \frac{ya^{-\frac{\nu}{2}-1}}{2}\Gamma(1+\frac{\nu}{2})\ {}_{1}F_{1}\ (1+\frac{1}{2}\nu;\frac{3}{2};\frac{-y^{2}}{4a})$, $\mathrm{Re}\ (\nu) > -2$

... (7, p. 123)

$h_{\mu}(y) = \frac{1}{2}(\frac{\pi}{a})^{\frac{1}{2}}\ e^{-\frac{y^{2}}{2}(8a)^{-1}}\ I_{\frac{\mu}{2}}(\frac{y^{2}}{8a})$, $\mathrm{Re}_{\mu} > -1$ ... (4, p. 29)

Inserting these into (15) and (16) we get

a) $2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^{n}(\nu+2n)\ C_{2n}^{\nu}(\cos\varphi)\ I_{n+\frac{1}{2}\nu}(\frac{1}{8a}) = a^{-\frac{1}{2}\nu}\pi^{-\frac{1}{2}}$

$e^{\frac{1}{8a}}\ \Gamma(\frac{1}{2}+\frac{\nu}{2})\ {}_{1}F_{1}(\frac{1}{2}+\frac{\nu}{2};\frac{1}{2};-\frac{\cos^{2}\varphi}{4a})$, $\mathrm{Re}\ (\nu) > -1$

b) $2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^{n}(\nu+2n+1)\ C_{2n+1}^{\nu}(\cos\varphi)\ I_{n+\frac{1}{2}\nu+\frac{1}{2}}(\frac{1}{8a}) =$

$\frac{a^{-\frac{\nu}{2}-\frac{1}{2}}}{\pi^{\frac{1}{2}}}(\cos\varphi)e^{\frac{1}{8a}}\Gamma(1+\frac{\nu}{2})\ {}_{1}F_{1}(1+\frac{\nu}{2};\frac{3}{2};\frac{\cos^{2}\varphi}{4a})$, $\mathrm{Re}\ (\nu) > -2$

which converge by appendix: (2) and (14).

7) $f(x) = x^{\eta}e^{-ax^2}$

$$G_c(y) = \tfrac{1}{2}a^{-\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\eta}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\eta)\,{}_1F_1(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\eta;\tfrac{1}{2};-\frac{y^2}{4a}),$$

$$\mathrm{Re}\,(\eta+\nu) > -1\ ,\ \ldots\ (7,\ \text{p. } 12)$$

$$G_s(y) = \tfrac{1}{2}ya^{-1-\frac{1}{2}\nu-\frac{1}{2}\eta}\Gamma(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\eta)\,{}_1F_1(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\eta;\tfrac{3}{2};-\frac{y^2}{4a})$$

$$\mathrm{Re}\,(\eta+\nu) > -2\ ,\ \ldots\ (7,\ \text{p. } 123)$$

$$h_\mu(y) = y^{-1}a^{-\frac{1}{2}\eta}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\eta+\tfrac{1}{2}\mu)(\Gamma(1+\mu))^{-1}e^{-\frac{y^2}{8a}}M_{\frac{1}{2}\eta,\frac{1}{2}\mu}(\frac{y^2}{4a}),$$

$$\mathrm{Re}\,(\eta+\mu) > -1\ ,\ \ldots\ (4,\ \text{p. } 30)$$

Inserting these into (15) and (16) we get:

a) $$2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n)C^{\nu}_{2n}(\cos\varphi)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\eta+\tfrac{1}{2}\nu+n)$$

$$(\Gamma(1+\nu+2n))^{-1}e^{-\frac{1}{8a}}M_{\frac{1}{2}\eta,n+\frac{1}{2}\nu}(\frac{1}{4a}) = \tfrac{1}{2}a^{-\frac{1}{2}-\frac{1}{2}\nu}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\eta)$$

$${}_1F_1(\frac{1+\nu+\eta}{2};\tfrac{1}{2};-\frac{\cos^2\varphi}{4a}),\quad \mathrm{Re}\,(\eta+\nu) > -1$$

b) $$2^{\nu}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n+1)C^{\nu}_{2n+1}(\cos\varphi)\Gamma(n+1+\tfrac{1}{2}\nu+\tfrac{1}{2}\eta)$$

$$(\Gamma(2+2n+\nu))^{-1}e^{-\frac{1}{8a}}M_{\frac{1}{2}\eta,n+\frac{1}{2}\nu+\frac{1}{2}}(\frac{1}{4a}) = \tfrac{1}{2}\cos\varphi\ a^{-1-\frac{1}{2}\nu}$$

$$\Gamma(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\eta)\,{}_1F_1(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\eta;\tfrac{3}{2};-\frac{\cos^2\varphi}{4a}),\quad \mathrm{Re}\,(\eta+\nu) > -2$$

which converge by appendix: (7) and (14) and the asymptotics of the gamma function.

8) $f(x) = x^{-\frac{1}{2}}e^{-ax^{\frac{1}{2}}}$

$$G_c(y) = \Gamma(2\nu+1)(2y)^{-\nu-\frac{1}{2}}(\exp(-i(\tfrac{1}{2}\pi(\nu+\tfrac{1}{2})+\frac{a^2}{8y}))D_{-2\nu-1}(\tfrac{1}{2}a$$

$$y^{-\frac{1}{2}}(1-i))+\exp(i(\tfrac{1}{2}\pi(\nu+\tfrac{1}{2})+\frac{a^2}{8y}))D_{-2\nu-1}((1+i)\frac{ay^{-\frac{1}{2}}}{2})),$$

$$\text{Re } \nu > -\tfrac{1}{2} \text{ , ... (7, p. 13)}$$

$$G_s(y) = i(2y)^{-\nu-\frac{1}{2}}\Gamma(2\nu+1)(\exp(-i(\tfrac{1}{2}\pi(\nu+\tfrac{1}{2})+\frac{a^2}{8y}))$$

$$D_{-2\nu-1}(\tfrac{1}{2}ay^{-\frac{1}{2}}(1-i))-\exp(i(\tfrac{1}{2}\pi(\nu+\tfrac{1}{2})+\frac{a^2}{8y}))$$

$$D_{-2\nu-1}(\tfrac{1}{2}ay^{-\frac{1}{2}}(1+i))), \text{ Re } \nu>-\tfrac{3}{2} \text{ , ... (7, p. 125)}$$

$$h_\mu(y) = (\tfrac{1}{2}\pi)^{-\frac{1}{2}}\Gamma(\mu+\tfrac{1}{2})D_{-\mu-\frac{1}{2}}((\tfrac{1}{2}ia^2y^{-1})^{\frac{1}{2}})D_{-\mu-\frac{1}{2}}((-\frac{ia^2}{2y})^{\frac{1}{2}})$$

$$\text{Re } \mu > -\tfrac{1}{2}, \text{ ... (4, p. 30)}$$

Inserting these into (15) and (16) we get:

a) $2^\nu\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n)C^\nu_{2n}(\cos\varphi)\Gamma(\nu+2n+\tfrac{1}{2})D_{-\nu-2n-\frac{1}{2}}$

$((\tfrac{1}{2}ia^2)^{\frac{1}{2}})\ D_{-\nu-2n-\frac{1}{2}}((-\tfrac{i}{2}\ a^2)^{\frac{1}{2}}) = (\tfrac{\pi}{2})^{\frac{1}{2}}(2\cos\varphi)^{-\nu-\frac{1}{2}}$

$\Gamma(2\nu+1)\ (\exp(-i(\frac{\pi}{2}(\nu+\tfrac{1}{2})+\frac{a^2}{8\cos\varphi})\ D_{-2\nu-1}(\frac{a}{2(\cos\varphi)^{\frac{1}{2}}}(1-i))+$

$e^{i(\frac{1}{2}\pi(\nu+\frac{1}{2})+a^2(8\cos\varphi)^{-1})}\ D_{-2\nu-1}((1+i)\frac{a}{2(\cos\varphi)^{\frac{1}{2}}})),$

$$\text{Re } \nu > -\tfrac{1}{2}$$

b) $2^\nu\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n+1)\ C^\nu_{2n+1}(\cos\varphi)\Gamma(\nu+2n+\tfrac{3}{2})$

$D_{-2n-\nu-\frac{3}{2}}((\tfrac{1}{2}ia^2)^{\frac{1}{2}})\ D_{-2n-\nu-\frac{3}{2}}((-\frac{ia^2}{2})^{\frac{1}{2}}) = i(\frac{\pi}{2})^{\frac{1}{2}}$

$$(2\cos\varphi)^{-\nu-\frac{1}{2}}\Gamma(2\nu+1)(e^{-i(\frac{1}{2}\pi(\nu+\frac{1}{2})+a^2(8\cos\varphi)^{-1})}$$

$$D_{-2\nu-1}(\frac{a}{2(\cos\varphi)})^{\frac{1}{2}}(1-i))-e^{i(\frac{1}{2}\pi(\nu+\frac{1}{2})+a^2(8\cos\varphi)^{-1})}$$

$$D_{-2\nu-1}(\tfrac{1}{2}a(\cos\varphi)^{-\frac{1}{2}}(1+i)),\quad \text{Re } \nu > -\tfrac{3}{2}$$

which converge by appendix: (14) and (16) and the asymptotics of the gamma function.

9) $f(x) = x^{-1}\, e^{-\frac{a}{x}-bx}$

$$G_c(y) = a^{\frac{1}{2}\nu}(b+iy)^{-\frac{1}{2}\nu}\, K_\nu(2(ab+iay)^{\frac{1}{2}})+(b-iy)^{-\frac{1}{2}\nu}$$

$$K_\nu(2(ab-iay)^{\frac{1}{2}})) \qquad \ldots (7, \text{ p. } 13)$$

$$G_s(y) = ia^{\frac{1}{2}\nu}((b+iy)^{-\frac{1}{2}\nu}K_\nu(2(ab+iay)^{\frac{1}{2}})-(b-iy)^{-\frac{1}{2}\nu}$$

$$K_\nu(2(ab-iay)^{\frac{1}{2}})) \qquad \ldots (7, \text{ p. } 124)$$

$$h_\mu(y) = 2J_\mu((2a)^{\frac{1}{2}}((b^2+y^2)^{\frac{1}{2}}-b)^{\frac{1}{2}})K_\mu((2a)^{\frac{1}{2}}((b^2+y^2)^{\frac{1}{2}}+b)^{\frac{1}{2}})$$

$$\ldots (4, \text{ p. } 30)$$

Inserting these into (15) and (16) we get:

a) $$2^{\nu+1}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n)C^\nu_{2n}(\cos\varphi)J_{2n+\nu}((2a)^{\frac{1}{2}}((b^2+1)^{\frac{1}{2}}-b)^{\frac{1}{2}})$$

$$K_{2n+\nu}((2a)^{\frac{1}{2}}((b^2+1)^{\frac{1}{2}}+b)^{\frac{1}{2}}) = a^{\frac{1}{2}\nu}((b+i\cos\varphi)^{-\frac{1}{2}\nu}$$

$$K_\nu(2(ab+i\cos\varphi)^{\frac{1}{2}})+(b-i\cos\varphi)^{-\frac{1}{2}\nu}K_\nu(2(ab-ia\cos\varphi)^{\frac{1}{2}}))$$

b) $2^{\nu+1}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n+1)C^{\nu}_{2n+1}(\cos\varphi)J_{2n+\nu+1}((2a)^{\frac{1}{2}}$

$((b^2+1)^{\frac{1}{2}}-b)^{\frac{1}{2}})K_{2n+\nu+1}((2a)^{\frac{1}{2}}((b^2+1)^{\frac{1}{2}}+b)^{\frac{1}{2}}) = ia^{\frac{1}{2}\nu}$

$((b+i\cos\varphi)^{-\frac{1}{2}\nu}K_{\nu}(2(ab+ia\cos\varphi)^{\frac{1}{2}})-(b-i\cos\varphi)^{-\frac{1}{2}\nu}$

$K_{\nu}(2(ab-iay)^{\frac{1}{2}}))$

which converge by appendix: (1), (3) and (14).

10) $f(x) = x^{\eta}J_{\eta}(ax)$

$G_c(y) = (2\pi a)^{-\frac{1}{2}}\cos(\tfrac{1}{2}\pi\nu)\Gamma(1+2\eta+\nu)\Gamma(1+\nu)(a^2-y^2)^{-\frac{\eta}{2}-\frac{\nu}{2}-\frac{1}{4}}$

$(P^{-\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(\frac{y}{a})+P^{-\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(-\frac{y}{a}))$, $y < a$

$= -(\tfrac{1}{2}\pi a)^{-\frac{1}{2}}\sin(\frac{\pi}{2}(\nu+2\eta))(y^2-a^2)^{-\frac{\eta}{2}-\frac{\nu}{2}-\frac{1}{4}}e^{-i\pi(\eta+\nu+\frac{1}{2})}$

$Q^{\eta+\nu+\frac{1}{2}}_{\eta-\frac{1}{2}}(\frac{y}{a})$, $y > a$ ... (9)

$G_s(y) = (2\pi a)^{-\frac{1}{2}}\sin(\tfrac{1}{2}\pi\nu)\Gamma(1+\nu+2\eta)\Gamma(1+\nu)(a^2-y^2)^{-\frac{\eta}{2}-\frac{\nu}{2}-\frac{1}{4}}$

$(P^{-\nu-\eta-\frac{1}{2}}_{\eta-\frac{1}{2}}(\frac{y}{a})-P^{-\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(-\frac{y}{a}))$, $y < a$

$= (\tfrac{1}{2}\pi a)^{-\frac{1}{2}}\cos(\tfrac{1}{2}\pi(\nu+2\eta))(y^2-a^2)^{-\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{4}}$

$e^{-i\pi(\eta+\nu+\frac{1}{2})}Q^{\eta+\nu+\frac{1}{2}}_{\eta-\frac{1}{2}}(\frac{y}{a})$, $y > a$, ... (9)

$h_{\mu}(y) = 2^{\eta+1}\cos(\frac{\pi\mu}{2})(\pi y(a^2-y^2)^{\frac{\eta}{2}}e^{i\pi\eta})^{-1}Q^{\eta}_{\frac{1}{2}\mu-\frac{1}{2}}(\frac{2a^2}{y^2}-1)$, $y<a$

$= 2^{\eta}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+\eta)(y(y^2-a^2)^{\frac{1}{2}\eta}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu-\eta))^{-1}$

$P^{-\eta}_{\frac{1}{2}\mu-\frac{1}{2}}(1-\frac{2a^2}{y^2})$, $y>a$, $-\frac{1}{2}<\mathrm{Re}\ \eta<1$, ... (4, p. 48)

Inserting these into (15) and (16) we get:

a) $2^{\nu+\eta+1}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n)C^{\nu}_{2n}(\cos\varphi)\cos(\tfrac{1}{2}\pi(\nu+2n))$

$Q^{\eta}_{n+\frac{1}{2}\nu-\frac{1}{2}}(2a^2-1) = (a^2-1)^{\frac{1}{2}\eta}e^{i\pi\eta}(\frac{\pi}{2a})^{\frac{1}{2}}\cos(\tfrac{1}{2}\pi\nu)\Gamma(1+2\eta+\nu)$

$\Gamma(1+\nu)(a^2-\cos^2\varphi)^{-\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{4}}(P^{-\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(\frac{\cos\varphi}{a})+$

$P^{-\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(-\frac{\cos\varphi}{a})\ ,\quad 1 < a$

b) $2^{\nu+\eta+1}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n+1)C^{\nu}_{2n+1}(\cos\varphi)\cos(\tfrac{1}{2}\pi(\nu+2n+1))$

$Q^{\eta}_{n+\frac{1}{2}\nu}(2a^2-1) = (a^2-1)^{\frac{1}{2}\eta}e^{i\pi\eta}(\frac{\pi}{2a})^{\frac{1}{2}}\sin(\tfrac{1}{2}\pi\nu)\Gamma(1+\nu+2\eta)$

$\Gamma(1+\nu)(a^2+\cos^2\varphi)^{-\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{4}}(P^{-\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(\frac{\cos\varphi}{a})-$

$P^{-\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(-\frac{\cos\varphi}{a}))\ ,\quad a > 1$

which converge by appendix: (10) and (14).

11) $f(x) = x^{-\eta}J_{\eta}(ax)$

$G_c(y) = (2\pi a)^{-\frac{1}{2}}\cos(\tfrac{1}{2}\pi(\nu-2\eta))\Gamma(1+\nu-2\eta)\Gamma(1+\nu)$

$(a^2-y^2)^{\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{4}}(P^{\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(\frac{y}{a})+P^{\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(-\frac{y}{a}))$, $y<a$

$= -(\tfrac{1}{2}\pi a)^{-\frac{1}{2}}\sin(\tfrac{1}{2}\pi\nu)(y^2-a^2)^{\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{2}}e^{-i\pi(\nu-\eta+\frac{1}{2})}$

$Q^{\nu-\eta+\frac{1}{2}}_{\eta-\frac{1}{2}}(\frac{y}{a})$, $y > a$, $-1 <$ Re $\nu<\frac{1}{2}+$Re η...(7, p. 63)

$$G_s(y) = (2\pi a)^{-\frac{1}{2}}\sin(\tfrac{1}{2}\pi(\nu-2\eta))\Gamma(1+\nu-2\eta)\Gamma(1+\nu)$$

$$(a^2-y^2)^{\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{4}}(P^{\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(\tfrac{y}{a})-P^{\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(-\tfrac{y}{a})),\ y<a$$

$$= (\tfrac{1}{2}\pi a)^{-\frac{1}{2}}\cos(\tfrac{1}{2}\pi\nu)(y^2-a^2)^{\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{4}}e^{-i\pi(\nu-\eta+\frac{1}{2})}$$

$$Q^{\nu-\eta+\frac{1}{2}}_{\eta-\frac{1}{2}}(\tfrac{y}{a}),\ y>a,\ -2<\text{Re }\nu<\tfrac{1}{2}+\text{Re }\eta$$

... (7, p. 167)

$$h_\mu(y) = 2^{1-\eta}(y\Gamma(\tfrac{1}{2}+\eta+\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}+\eta-\tfrac{1}{2}\mu))^{-1}(a^2-y^2)^{\frac{1}{2}\eta}e^{-i\pi\eta}$$

$$Q^{\eta}_{\frac{1}{2}\mu-\frac{1}{2}}(2a^2y^{-2}-1),\ y<a \quad \ldots (4, \text{p. } 48)$$

$$= 2^{-\eta}y^{-1}(y^2-a^2)^{\frac{1}{2}\eta}P^{-\eta}_{\frac{1}{2}\mu-\frac{1}{2}}(1-2a^2y^{-2}),\ y>a,\ \text{Re}\eta,\mu>-1.$$

Inserting these into (15) and (16) we get:

a) $$2^{\nu+1-\eta}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n)C^\nu_{2n}(\cos\varphi)(\Gamma(\tfrac{1}{2}+\eta+n+\tfrac{1}{2}\nu)$$

$$\Gamma(\tfrac{1}{2}+\eta-n-\tfrac{1}{2}\nu))^{-1}(a^2-1)^{\frac{1}{2}\eta}e^{-i\pi\eta}Q^{\eta}_{n+\frac{1}{2}\nu-\frac{1}{2}}(2a^2-1) = (2\pi a)^{-\frac{1}{2}}$$

$$\cos(\tfrac{1}{2}\pi(\nu-2\eta))\Gamma(1+\nu-2\eta)\Gamma(1+\nu)(a^2-\cos^2\varphi)^{\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{4}}$$

$$(P^{\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(\tfrac{\cos\varphi}{a})+P^{\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(-\tfrac{\cos\varphi}{a})),\quad a>1,\ -1<\text{Re }\nu<\tfrac{1}{2}+\text{Re }\eta$$

b) $$2^{\nu+1-\eta}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n+1)C^\nu_{2n+1}(\cos\varphi)(\Gamma(1+\eta+n+\tfrac{1}{2}\nu)$$

$$\Gamma(\eta-n-\tfrac{1}{2}\nu)e^{i\pi\eta})^{-1}(a^2-1)^{\frac{1}{2}\eta}Q^{\eta}_{n+\frac{1}{2}\nu}(2a^2-1) = (2\pi a)^{-\frac{1}{2}}$$

$$\sin(\tfrac{1}{2}\pi(\nu-2\eta))\Gamma(1+\nu-2\eta)\Gamma(1+\nu)(a^2-\cos^2\varphi)^{\frac{1}{2}\eta-\frac{1}{2}\nu-\frac{1}{4}}$$

$$(P^{\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(\tfrac{\cos\varphi}{a})-P^{\eta-\nu-\frac{1}{2}}_{\eta-\frac{1}{2}}(-\tfrac{\cos\varphi}{a})),\quad a>1,\ -2<\text{Re }\nu<\tfrac{1}{2}+\text{Re }\eta$$

which converge by the asymptotics of the gamma function and appendix: (10) and (14).

12) $f(x) = x^{\eta}K_{\eta}(ax)$

$$G_c(y) = \frac{\pi}{4}\sec(\tfrac{1}{2}\pi\nu)\Gamma(1+\nu+2\eta)(a^2+y^2)^{-\frac{1}{2}\nu-\frac{1}{2}\eta-\frac{1}{2}}$$

$$(P^{-\eta}((1+a^2y^{-2})^{-\frac{1}{2}})+P^{-\eta}(-(1+a^2y^{-2})^{-\frac{1}{2}})),$$

$$\text{Re } \nu > -1,\ \text{Re}(\nu+2\eta)>-1, \quad \ldots (9)$$

$$G_s(y) = \frac{\pi}{4}\csc(\tfrac{1}{2}\pi\nu)\Gamma(1+\nu+2\eta)(a^2+y^2)^{-\frac{1}{2}\nu-\frac{1}{2}\eta-\frac{1}{2}}$$

$$(P^{-\eta}_{\nu+\eta}((1+a^2y^{-2})^{-\frac{1}{2}})-P^{-\eta}_{\nu+\eta}(-(1+a^2y^{-2})^{-\frac{1}{2}})),$$

$$\text{Re } \nu>-2,\ \text{Re}(\nu+2\eta)>-2, \quad \ldots (9)$$

$$h_{\mu}(y) = 2^{\eta}(y(a^2+y^2)^{\frac{1}{2}\eta}e^{i\pi\eta})^{-1}Q^{\eta}_{\frac{1}{2}\mu-\frac{1}{2}}(1+2a^2y^{-2}),$$

$$\text{Re } \mu>-1,\ \text{Re}(\mu+2\eta)>-1, \quad \ldots (4, \text{p. } 63).$$

Inserting these into (15) and (16) we get:

a) $$2^{\nu+\eta}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n)\ C^{\nu}_{2n}(\cos\varphi)(a^2+1)^{-\frac{\eta}{2}}\ e^{-i\pi\eta}$$

$$Q^{\eta}_{n+\frac{\nu}{2}-\frac{1}{2}}(1+2a^2) = \frac{\pi}{4}\sec(\tfrac{\pi}{2}\nu)\Gamma(1+\nu+2\eta)(a^2+\cos^2\varphi)^{-\frac{\nu}{2}-\frac{\eta}{2}-\frac{1}{2}}$$

$$(P^{-\eta}_{\nu+\eta}((1+\frac{a^2}{\cos^2\varphi})^{-\frac{1}{2}}) + P^{-\eta}_{\nu+\eta}(-(1+\frac{a^2}{\cos^2\varphi})^{-\frac{1}{2}})),\ \text{Re } \nu>-1,$$

$$\text{Re}(\nu+2\eta)>-1$$

b) $2^{\nu+\eta}\Gamma(\nu)\ \Sigma\ (-1)^n(\nu+2n+1)C^{\nu}_{2n+1}(\cos\varphi)(a^2+1)^{-\frac{\eta}{2}}e^{-i\pi\eta}$

$$Q^{\eta}_{n+\frac{1}{2}\nu}(1+2a^2) = \frac{\pi}{4}\csc(\tfrac{1}{2}\pi\nu)\Gamma(1+\nu+2\eta)(a^2+\cos^2\varphi)^{-\frac{\nu}{2}-\frac{\eta}{2}-\frac{1}{2}}$$

$$(P^{-\eta}_{\nu+\eta}((1+\frac{a^2}{\cos^2\varphi})^{-\frac{1}{2}})-P^{-\eta}_{\nu+\eta}(-(1+\frac{a^2}{\cos^2\varphi})^{-\frac{1}{2}}))\ ,\ \mathrm{Re}\ \nu > -2,$$

$$\mathrm{Re}\ (\nu+2\eta) > -2$$

which converge by appendix: (10) and (14).

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APPENDIX

The following list gives the asymptotic behavior of the special functions used in this thesis.

For A bounded and for large μ ($\mu \to +\infty$):

(1) $K_\mu(x) \sim A(\mu^2+x^2)^{-\frac{1}{4}} e^{-\mu}(\frac{2\mu}{x})^\mu, \quad 0<x<\mu$

(2) $I_\mu(x) \sim A(\frac{2\mu}{xe})^{-\mu}\mu^{-\frac{1}{2}}, \quad 0<x$

(3) $J_\mu(x) \sim A\mu^{-\frac{1}{2}}(\frac{2\mu}{xe})^{-\mu}, \quad 0<x$

(4) $Y_\mu(x) \sim A\mu^{\mu-\frac{1}{2}}(\frac{2}{xe})^\mu, \quad 0<x<\mu$

(5) $I\eta(\mu) \sim A\mu^{-\frac{1}{2}} e^\mu,$

(6) $P_\eta^{-\mu}(\cos\varphi) \sim Ae^\mu \mu^{\mu+\frac{1}{2}}(\cot\frac{\varphi}{2})^{-\mu}, \quad 0<\varphi<\pi$

$\Gamma(\mu) \sim A(\frac{\mu}{e})^{\mu-\frac{1}{2}}, \quad \frac{\Gamma(\mu+b)}{\Gamma(\mu+a)} \sim \mu^{b-a},$

(7) $M_{\eta,\mu}(x) \sim Ax^\mu \mu^{\frac{1}{2}},$

(8) $D_{-\mu}(x) \sim Ae^{-\frac{1}{2}\mu\log(\mu)+\frac{1}{2}\mu-x\mu^{\frac{1}{2}}},$

(9) $P_\mu^\eta(\cos\varphi) \sim \frac{\Gamma(\mu+\eta+1)}{\Gamma(\mu+\frac{3}{2})} A, \quad 0<\varphi<\pi$

(10) $Q_{\mu}^{\eta}(z) \sim A\mu^{\eta-\frac{1}{2}}(z-(z^2-1)^{\frac{1}{2}})^{\mu+\frac{1}{2}}$, Re $z > 1$

(11) $P_{\mu}^{\eta}(z) \sim A(z+(z^2-1)^{\frac{1}{2}})^{\mu}\mu^{\eta-\frac{1}{2}}$, $z > 1$

(12) $W_{-\mu,\eta}(z) \sim A\exp(i\mu\pi+2i(-\mu z)^{\frac{1}{2}})\mu^{-\mu-\frac{1}{4}}\, e^{\mu}$, $|z|$ bounded away from zero, $0 < \arg(-\mu) < 3\pi$, $\pi \geq |\arg(z)-\arg(-\mu)|$

(13) $M_{\mu,\eta}(z) \sim A(\mu)^{\frac{1}{4}-\eta}\, e^{2|\mu z|^{\frac{1}{2}}}$, $z > 0$

(14) $C_n^{\eta}(z) \sim An^{\eta-1}$, $|z| < 1$, n is a large integer.

(15) $J_{\mu}(z) \sim A\mu^{-1}$, $|z| < 2$

(16) $D_{-\mu}(z) \sim A \exp(-\frac{\mu}{2}\log(\mu)+\frac{\mu}{2}-z\mu^{\frac{1}{2}})$, $|z|$ is bounded.

A list of the notation used in this thesis follows

$C_{\mu}^{\eta}(x)$ Gegenbauer polynomials , μ an integer

$D_{\mu}(x)$ Parabolic cylinder function

${}_pF_q(a_1,\ldots, a_p;b_1,\ldots, b_q;z)$ Generalized hypergeometric series

$H_{\mu}(z)$ Struve's function

$I_{\mu}(z)$ Modified Bessel function of the first kind

$J_{\mu}(z)$ Bessel function of the first kind

$\mathbb{J}_\mu(z)$ Anger's function

$K_\mu(z)$ modified Bessel function of the third kind

$L_\mu(z)$ modified Struve function

$M_{\eta,\mu}(x)$ Whittaker's function

$P_\eta^\mu(z)$ Associated Legendre function

$P_\eta^\mu(x)$ Legendre function on the cut

$Q_\eta^\mu(z)$ Associated Legendre function of the second kind

$Q_\eta^\mu(x)$ Legendre function of the second kind on the cut

$S(x), C(x)$ Fresnel integrals

$W_{\eta,\mu}(x)$ Whittaker's function

$Y_\mu(z)$ Bessel function of the second kind