THESIS

on

A STUDY OF THE DIFFERENTIAL EQUATION

$$\frac{d^2w}{dz^2} + z^n w = 0.$$

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INTRODUCTION

It is the purpose of this paper to make a study of the equation

$$(1) \qquad w'' + z^n w = 0.$$

In view of the fact that it is rather simply related to Bessel's Equation, this thesis will be in part a direct study of (1) and in part a study of (1) through the relation to Bessel's Equation.

The first part of this thesis is devoted to the solutions of that equation when w is a function of the complex variable z. This will include a discussion of the existence and character of the solutions for any n and the derivations of the solutions, when n equals a positive integer, by means of power series and asymptotic series. The radius of convergence of the power series will also be discussed.

After that, it seems necessary to limit ourselves to a study of (1) when n is equal to two and z is replaced by the real variable x. With these changes made, the independent solutions of (1) are then studied. That will include the formulas for, and the calculations of, the solutions from $x = 0$ to $x = 10$ at intervals of two tenths, and the formulas for the roots of the two independent solutions of

$$(25) \qquad y'' + x^2 y = 0.$$

These latter equations are based on the asymptotic solutions derived earlier in the paper.

The results of the study of (25) to that point are then presented graphically in the succeeding section.

The remainder of the thesis is then occupied by the derivation of the orthogonality relations for $C_2(k_m x)$ and for $S_2(k_m x)$, and with their applications to the theoretical representation of a function of x in a series of $C_2(k_m x)$ or of $S_2(k_m x)$ terms.

THE EXISTENCE AND ANALYTIC CHARACTER OF THE SOLUTION OF

$$(1) \quad w'' + z^n w = 0.$$

It is proved elsewhere that if w is a function of the complex variable z and if p and p are analytic functions of z at a

$$(2) \qquad w'' = p_1 w' + p_2 w$$

admits an analytic solution

$$(3) \qquad w = b_0 + b_1 (z-a) + b_2 (z-a)^2 + \cdots\cdots$$

which together with its first derivative is uniquely determined by the conditions that w and w' shall have the assigned values

$$w = A, \; w' = B \text{ at } z = a.$$

Furthermore, this solution is valid within a circle about a extending to the nearest singular point of p_1 or of p_2.

Since in equation (1) $p_1 = 0$ and $p_2 = -z^n$, it is clear that for all positive values of n, p_2 will have no singular points in the finite plane; but that for n equal to a negative number, $z = 0$ is the one and only finite point where $-z^n$ is non-analytic.

Another theorem given and proved by Pierpont, states that if (3) is a solution of (2), all of its analytical continuations are also solutions of (2).

Upon the basis of those theorems we may conclude that:

(a) If n is positive, (1) has a solution (3) which is analytic throughout the finite plane.

(b) If n is negative, (1) has a solution (3) which is analytic in any finite region not including the point $z = 0$.

The question of the singlevaluedness of the solutions (3) will be discussed after the relations to Bessel's Equations have been derived.

THE SOLUTION OF (1) BY MEANS OF A POWER SERIES IF N IS A POSITIVE INTEGER

If n were negative we would have to keep a unequal to zero. This would lead to much difficulty in the determination of the coefficients in the power series. For that reason, it seems more practical to consider only the case in which n is positive, leaving the other cases merely mentioned as far as the actual power series are concerned.

Let us assume that

$$(4) \qquad w = a_0 + a_1 z + a_2 z^2 + \ldots\ldots\ldots\ldots$$

Putting this value of w into (1) we have

$$\begin{aligned} &2a_2 + 2.3\ a_3 z + 3.4\ a_4 z^2 + \ldots\ldots\ldots \\ &+ \left[a_0 + (n+1)(n+2)\ a_{n+2}\right] z^n + \left[a_1 + (n+2)(n+3)\ a_{n+3}\right] z^{n+1} \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots + \left[a_n + (n+m+1)(n+m+2)\ a_{n+m+2}\right] z^{n+m} \equiv 0. \end{aligned}$$

In order that this identity be true, the coefficients of the like powers of z must be equal to zero. Then it is clear that

$$a_2,\ a_3 \ldots\ldots\ldots\ldots a_{n+1}\ ,\ \text{all} = 0;$$

that $\qquad a_0 + (n+1)(n+2)\ a_{n+2} = 0,$

and that $\qquad a_1 + (n+2)(n+3)\ a_{n+3} = 0.$

Therefore, $\qquad a_{n+2} = - \dfrac{1}{(n+1)(n+2)}\ a_0,$

and $\qquad a_{n+3} = - \dfrac{1}{(n+2)(n+3)}\ a_1\ .$

Since a_{n+4} contains a_2 as a factor, it is equal to zero. For similar reasons, the other a's up to a_{2n+4} all vanish.

From the fact that

$$a_{n+2} + (2n+3)(2n+4)\ a_{2n+4} = 0$$

and

$$a_{n+3} + (2n+4)(2n+5)\ a_{2n+5} = 0,$$

we conclude that

$$a_{2n+4} = \frac{1}{(n+1)\ldots\ldots\ldots(2n+4)}\, a_0\,,$$

and

$$a_{2n+5} = \frac{}{(n+2)\ldots\ldots\ldots(2n+5)}\, a_1\,.$$

continuing that reasoning, we observe that the general coefficients are

$$a_{mn+2m} = (-1)^m \frac{1}{(n+1)\ldots\ldots\ldots\ldots\ldots(mn+2m)} \cdot a_0\,,$$

and

$$a_{mn+2m+1} = (-1)^m \frac{1}{(n+2)\ldots\ldots\ldots\ldots\ldots(mn+2m+1)} \cdot a_1\,.$$

Putting these coefficients into (4) we have the following solution of (1) by means of a power series if n equals a positive number:

$$\text{(5)} \qquad w = a_0\left[1 - \frac{1}{(n+1)(n+2)} z^{n+2} + \frac{1}{(n+1)\ldots\ldots(2n+4)} z^{2n+4}\ldots\ldots\ldots\right]$$
$$+ a_1\left[z - \frac{1}{(n+2)(n+3)} z^{n+3} + \frac{1}{(n+2)(\quad)\ldots\ldots(2n+5)} z^{2n+5}\ldots\ldots\right].$$

Since (5) is holomorphic for all finite values of z, it has an infinite radius of convergence.

THE RELATION BETWEEN (1) AND BESSEL'S EQUATION

A. Derivation of the relation

We may rewrite (1) as

(6) $$\frac{d(w')}{dz} + z^n w = 0$$

If we change the independent variable by a relation $s = f(z)$, we find after suitable rearrangements that

(7) $$\left(\frac{ds}{dz}\right)^2 \frac{d^2w}{ds^2} + \frac{d^2s}{dz^2}\frac{dw}{ds} + z^n w = 0.$$

If in (7) we now set

(8) $$s = \frac{2}{n+2} z^{\frac{n+2}{2}},$$

it becomes

(9) $$sw'' + \frac{n}{n+2} w' + sw = 0.$$

We can then make the substitution

(10) $$w = s^{\frac{1}{n+2}} v$$

and (9) becomes

(11) $$s^2 v'' + sv' + \left(s^2 - \frac{1}{(n+2)^2}\right)v = 0$$

This is Bessel's Equation.

Using (8) and (10) and the solutions of Bessel's Equation we see that

(12) $$w = z^{\frac{1}{2}}\left[AJ_{\frac{1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right) + BJ_{\frac{-1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right)\right]$$

is the completesolution of (1) if n is an integer other than -1, -2, or -3. But if n is such that $1/(n+2)$ is integral, the complete solution of (1) is

(13) $$W = z^{\frac{1}{2}}\left[AJ_{\frac{1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right) + BK_{\frac{1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right)\right].$$

A case of particular interest later in this study is that in which n = 2. If that be true,

$$(14) \qquad w = z^{\frac{1}{2}}\left[AJ_{\frac{1}{4}}(z^2/2) + BJ_{-\frac{1}{4}}(z^2/2)\right] .$$

Following is a short table of corresponding values for m and n based on the fact that

$$m = \pm \frac{1}{n+2}$$

Table I:

n	m	n	m
$+\infty$	0	0	1/2
.	.	.	.
.	.	.	.
5	+1/7	1/4	4/9
4	1/6	1/3	3/7
3	1/5	1/2	2/5
2	1/4	0	1/2
1	1/3	-1/2	2/3
0	∞	-1/3	3/5
-1	1	-1/4	4/7
-2	1/2	-1/5	5/9
-3	1	-1/6	
-4	1/2	.	.
-5	1/3	.	.
.	.	.	.
.	.	.	.
$-\infty$	0	0	-1/2

B. The solution of (1) when n = -2

If n = -2, (1) becomes

(15) $\quad z^2 w'' + w = 0.$

This type of equation is readily solved by assuming that

$$w = z^m$$

and putting that value back into (15). That gives

$$m(m-1)\, z^m + z^m = 0$$

or

$$m^2 - m + 1 = 0,$$

from which we find that $m_1 = 1/2 + (\sqrt{3}/2)\cdot i$, and $m_2 = 1/2 - \sqrt{3}\, i/2$.

Therefore

$$w = z^{\frac{1}{2}}\left(C_1 z^{\frac{\sqrt{3}}{2}i} + C_2 z^{-\frac{\sqrt{3}}{2}i}\right)$$

or

$$w = z^{\frac{1}{2}}\left[C_3 \cos\left(\frac{\sqrt{3}}{2}\log z\right) + C_4 \sin\left(\frac{\sqrt{3}}{2}\log z\right)\right].$$

This solution is multiple-valued every where and non-analytic at the origin due to the presence of log z.

C. The single-valuedness of the solutions of equation (1)

If we put the power series for

$J_{\frac{\pm 1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right)$ and for $K_{\frac{\pm 1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right)$ into (12) and (13), we can tell something about the values of n for which the solutions of (1) are single-valued.

After some simplification of the series we find that

$$(16) \quad J_{\frac{1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right) = \frac{(n+2)^{\frac{n+1}{n+2}} z^{\frac{1}{2}}}{\Gamma\left(\frac{1}{n+2}\right)}\left[1 - \frac{\frac{z^{n+2}}{n+2}}{n+3} + \frac{\left(\frac{z^{n+2}}{n+2}\right)^2}{2!\,(n+3)(2n+5)} - \frac{\left(\frac{z^{n+2}}{n+2}\right)^3}{3!\,(n+3)(2n+5)(3n+7)} + \cdots\cdots\right]$$

and that

$$J_{-\frac{1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right) = \frac{-(n+2)^{\frac{n+3}{n+2}} z^{-\frac{1}{2}}}{\Gamma\left(-\frac{1}{n+2}\right)} \left[1 - \frac{\left(\frac{z^{n+2}}{n+2}\right)}{n+1} + \frac{\left(\frac{z^{n+2}}{n+2}\right)^2}{2!\,(n+2)(2n+3)} - \frac{\left(\frac{z^{n+2}}{n+2}\right)^3}{3!\,(n+1)(2n+3)(3n+5)}\right].$$

Equation (12) now is

$$(17) \qquad w = \frac{Az}{(n+3)\,\Gamma\left(\frac{1}{n+2}\right)}\left[(n+3) - \frac{z^{n+2}}{n+2} + \frac{\left(\frac{z}{n+2}\right)^2}{2!\,(2n+5)} \cdots\cdots\right] + \frac{B}{(n+1)\,\Gamma\left(\frac{n+1}{n+2}\right)}\left[(n+1) - \frac{z^{n+2}}{n+2} + \frac{\left(\frac{z^{n+2}}{n+2}\right)^2}{2!\,(2n+3)} - \cdots\cdots\right].$$

Aslong as the powers of z in (17) remain integral, w will be a single-valued function of z. From (17) we observe that the question is whether (n+2), and hence n, is fractional or integral, remembering that 1/(n+2) must be a fraction or (17) is not the complete solution of (1).

We can conclude, then, that if n is any integer other than -1, -2, or -3, (1) will have a complete solution (16) which is single-valued and analytic in the regions designated in section II, for positive and for negative n's.

The cases in which n = -1, or -3, come under (13) and consequently give multiple-valued solutions of (1).

The statements thus far do not preclude the possibility that (1) may have particular solutions which are single-valued. From (16) we learn that

$$(18) \qquad w = A\, z^{\frac{1}{2}}\, J_{\pm\frac{1}{n+2}}\left(\frac{2}{n+2} z^{\frac{n+2}{2}}\right)$$

is single-valued for n equal to any integer or zero. However, this only adds single-valued solutions when $n = -1$ or -3.

D. The solution for (1) when $n = -4$

This particular case is studied individually because it is readily solved.

Equation (1) now becomes

(1a) $$w'' + z^{-4}w = 0.$$

If we set $z = 1/s$, (1a) becomes

(1b) $$\frac{d^2w}{ds^2} + \frac{2}{s}\frac{dw}{ds} + w = 0,$$

which has as a solution

$$w = \frac{A \sin s + B \cos s}{s}$$

Substituting back the value $s = 1/z$, we find as the solution of (1a)

$$w = Az \sin 1/z + Bz \cos 1/z.$$

THE SOLUTION OF (1) BY MEANS OF AN ASYMPTOTIC SERIES

Since the power series for w would be very impractical to use with very large values of z and since for such values a reasonable degree of accuracy may be had from the asymptotic series, it is profitable to represent w by such a series.

In order to do this, we wish to express (1) in a form that will come as close as possible to the form

$$w'' + w = 0,$$

which has a solution

$$w = A \cos z + B \sin z.$$

We can then assume asymptotic series for each of A and B; and as soon as the coefficients of these series are determined, we will have expressed w as an asymptotic series.

Let us start with the equation

$$(11) \qquad s^2 v'' + sv' + \left[s^2 - \frac{1}{(n+2)^2}\right] v = 0.$$

If in this equation we put

$$v = A(s) \cdot q(s),$$

we get

$$(19) \qquad q'' + \left(\frac{2A'}{A} + \frac{1}{s}\right) q' + \left[\frac{A''}{A} + \frac{A'}{As} + \frac{1}{s^2}\left(s^2 - \frac{1}{(n+s)^2}\right)\right] q = 0.$$

We want the q' term to be missing so we set

$$\frac{2A'}{A} + \frac{1}{\sqrt{s}} = 0.$$

Consequently,

$$A = \frac{1}{s},$$

and (19) now becomes

$$(20) \qquad q'' + \left(1 - \frac{4m^2 - 1}{4s^2}\right) q = 0,$$

in which we have set $m = \frac{1}{n+2}$.

For very large values of s, this equation becomes approximately the same as

$$q'' + q = 0.$$

Now let us assume that

(21) $$q = A \cos s + B \sin s,$$

where $A = a_0 + \frac{a_1}{s} + \frac{a_2}{s^2} + \ldots\ldots$, and $B = b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \ldots\ldots$.

If we let $k = \frac{4m^2-1}{4}$ and put the above values of q into (20) we have

(22) $$\left(A'' + 2B' - \frac{kA}{s^2}\right)\cos s + \left(B'' - 2A' - \frac{kB}{s^2}\right)\sin s = 0,$$

from which after putting in the series for A and B, we get

$$\text{I.}\quad \begin{aligned} &\frac{2a_1}{s^3} + \frac{2.3a_2}{s^4} + \frac{3.4a_3}{s^5} + \ldots\ldots \\ &- \frac{2b}{s^2} - \frac{2.2b}{s^3} - \frac{2.3b}{s^4} \ldots\ldots \\ &- \frac{ka_0}{s^2} - \frac{ka_1}{s^3} - \frac{ka_2}{s^4} - \ldots\ldots \quad = 0, \end{aligned}$$

and

$$\text{II.}\quad \begin{aligned} &\frac{2b_1}{s^3} + \frac{2.3b_2}{s^4} + \frac{3.4b_3}{s^5} + \ldots\ldots \\ &+ \frac{2a_1}{s^2} + \frac{2.2a_2}{s^3} + \frac{2.3a_3}{s^4} + \ldots\ldots \\ &- \frac{kb_0}{s^2} - \frac{kb_1}{s^3} - \frac{kb_2}{s^4} \ldots\ldots = 0. \end{aligned}$$

As previously noted, if these equalities are to be true, the coefficients of corresponding powers of s must equal zero.

Then from I:

$$-2b_1 - ka_0 = 0;$$

so $$b_1 = -\frac{k}{2} a_0.$$

$$2a_1 - 2.2b_2 - ka_1 = 0;$$

and from II:

$$2a_1 - kb_0 = 0;$$

so $$a_1 = \frac{k}{2} b_0.$$

$$2b_1 + 2.2a_2 - kb_1 = 0;$$

so $b_2 = \frac{k(2-k)}{2^3} b_0$. so $a_2 = \frac{k(2-k)}{2^3} a_0$.

....

....

Continuing in that manner we find that

$$a_n = \frac{k(2-k)(2.3-k)\ldots\ldots\ldots\ldots\left[n(n-1)-k\right]}{2^n\, n!} \cdot (a_0 \text{ or } b_0).$$

It is multiplied by a_0 when n is even and by b_0 when n is odd. The n here is not the n of (1), but is the number of the term corresponding to the subscript in a_n. The k is the only part involving the n of (1).

Similarly,

$$b_n = \frac{k(2-k)\ldots\ldots\ldots\left[n(n-1)-k\right]}{2^n\, n!} \cdot (b_0 \text{ or } a_0).$$

where b_0 is the multiplier when n is even and a_0 is used when n is odd. The above statements regarding n also apply here.

The signs go in pairs as

$$+ + - - + + - - + + - - \ldots\ldots\ldots\ldots .$$

If we put these coefficients into (21) we see that

$$(23)\quad q(s) = a_0 \cos(s) - \frac{k}{2s}\sin s + \frac{k(2-k)}{2^2 . 2!\, s^2}\cos s + \frac{k(2-k)(2.3-k)}{2^3 .\, 3!\, s^3}\sin s \ldots$$
$$+ b_0 \sin s + \frac{k}{2s}\cos s + \frac{k(2-k)}{2^2\, 2!\, s^2}\sin s - \ldots\ldots .$$

Using the relation between w and q, we find that the solution of (1) as given by an asymptotic series is

$$(24)\quad w = s^{\frac{-n}{2(n+2)}}\left[a_0\left(\cos s - \frac{k}{2s}\sin s + \ldots\ldots\ldots\right) + b_0\left(\sin s + \frac{k}{2s}\cos s \ldots\right)\right]$$

This equation can be put into terms of w and z by means of (8), but this is impractical until n has a specific value.

THE SPECIAL CASE IN WHICH N EQUALS TWO AND Z IS REAL

In order to have something definite to study and to keep the discussion within the scope of this thesis, we have devoted the principal part of the remainder of this paper to equation (1) with n equal to two and z a real variable, x. We have then

$$(25) \qquad y'' + x^2 y = 0.$$

From (5) the solution of (25) by means of a power series is

$$(26) \qquad y = a_0\left[1 - \frac{x^4}{3.4} + \frac{x^8}{3.4.7.8} \cdots\cdots\right] + a_1\left[x - \frac{x^5}{4.5} + \frac{x^9}{4.5.8.9} \cdots\cdots\cdots\right].$$

We have chosen to call the series coefficient of a_0, $C_2(x)$ and the other series $S_2(x)$ because they have certain features of the cosine and sine se series respectively. The subscript denotes the value of n in (1). Then

$$y = a_0 C_2(x) + a_1 S_2(x).$$

A. The Numerical Values of $C_2(x)$ and $S_2(x)$.

It is comparatively convenient to calculate $C_2(x)$ and $S_2(x)$ for small x's but beyond x = 3 the work becomes excessively laborious and the asymptotic series becomes sufficiently accurate for most purposes; so it is advisable to use the power series out to x equals three and then the asymptotic series beyond that point.

From the Existence Theorem given in Section II, we may deduce the fact that one and only one solution of (25) passes through x equals three with the values

$$y = C_2(3) \quad \text{and } y' = C_2'(3)$$

as given by (26) or as computed from the relations to Bessel's Functions. That is, by an appropriate determination of the arbitrary constants we can uniquely join the power series for $C_2(x)$ and its asymptotic series together at x = 3. The asymptotic series will then uniquely represent $C_2(x)$ beyond x equals three.

It is then obvious that the same can be done for $S_2(x)$.

At this point we will again capitalize upon the reliability of the $J_m(s)$ functions. When n is equal to two, a solution of (25) is

$$(27)\quad y = A\sqrt{x}\, J_{-\frac{1}{4}}\left(\frac{x^2}{2}\right) + B\sqrt{x}\, J_{\frac{1}{4}}\left(\frac{x^2}{2}\right).$$

considering these Bessel's Functions singly, we find that

$$J_{-\frac{1}{4}}\left(\frac{x^2}{2}\right) = \frac{\sqrt{2}}{\sqrt{x}\,\Gamma(\frac{3}{4})}\left(1 - \frac{x^4}{3.4} + \frac{x^8}{3.4.7.8}\right) = \frac{\sqrt{2}}{\sqrt{x}\,\Gamma(\frac{3}{4})}\, C_2(x);$$

and that

$$J_{\frac{1}{4}}\left(\frac{x^2}{2}\right) = \frac{1}{\sqrt{2x}\,\Gamma(\frac{5}{4})}\left[x - \frac{x^5}{4.5} + \frac{x^9}{4.5.8.9} \cdots\right] = \frac{1}{\sqrt{2x}\,\Gamma(\frac{5}{4})} \cdot S_2(x).$$

Therefore

$$(28)\qquad C_2(x) = \frac{\sqrt{x}\,\Gamma(\frac{3}{4})}{\sqrt{2}} \cdot J_{-\frac{1}{4}}\left(\frac{x^2}{2}\right),$$

and

$$(29)\qquad S_2(x) = \sqrt{2x}\,\Gamma(\tfrac{5}{4})\, J_{\frac{1}{4}}\left(\frac{x^2}{2}\right).$$

Formulas (28) and (29) give

$$(30)\quad C_2(4) = .124783, \quad S_2(4) = .624593.$$

These will be used later in checking the accuracy of our formulas.

In asymptotic form

$$J_{-\frac{1}{4}}\left(\frac{x^2}{2}\right) \sim \frac{2}{\sqrt{\pi}\,x}\left[\cos\left(\frac{x^2}{2} - \frac{\pi}{8}\right) \cdot \sum_{m=0}^{\infty} (-)^m \frac{\Gamma(2m+\frac{1}{4})}{(2m)!\,\Gamma(\frac{1}{4}-2m)\, x^{4m}} - \sin\left(\frac{x^2}{2} - \frac{\pi}{8}\right) \cdot \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(2m+\frac{5}{4})}{(2m+1)!\,\Gamma(-\frac{3}{4}-2m)\, x^{4m+2}}\right],$$

and

$$J_{\frac{1}{4}}\left(\frac{x^2}{2}\right) \sim \frac{2}{\sqrt{\pi}\,x}\left[\cos\left(\frac{x^2}{2} - \frac{3\pi}{8}\right) \cdot \sum_{m=0}^{\infty} (-)^m \frac{\Gamma(2m+\frac{3}{4})}{(2m)!\,\Gamma(\frac{3}{4}-2m)\, x^{4m}} - \sin\left(\frac{x^2}{2} - \frac{3\pi}{8}\right) \cdot \sum_{m=0}^{\infty} (-)^m \frac{\Gamma(2m+\frac{7}{4})}{(2m+1)!\,\Gamma(-\frac{1}{4}-2m)\, x^{4m+2}}\right].$$

For very large values of x we may consider that

$$(31)\quad J_{-\frac{1}{4}}\left(\frac{x^2}{2}\right) \sim \frac{2}{\sqrt{\pi}\,x}\left(\cos\frac{\pi}{8}\cos\frac{x^2}{2} + \sin\frac{\pi}{8}\sin\frac{x^2}{2}\right)$$

and that

$$(32)\quad J_{\frac{1}{4}}\left(\frac{x^2}{2}\right) \sim \frac{2}{\sqrt{\pi}\,x}\left(\cos\frac{3\pi}{8}\cos\frac{x^2}{2} + \sin\frac{3\pi}{8}\sin\frac{x^2}{2}\right)$$

for purposes of comparison.

From (8) and the fact that $k = \frac{4m^2-1}{4}$, we see that for n = 2, $S = \frac{x^2}{2}$ and k = - 3/16. Putting these values into (24) and replacing w by y, z by x, we obtain the asymptotic formula for y. In exact form, it is

$$(33) \quad y_a(x) = \frac{2^{\frac{1}{4}}}{\sqrt{x}}\left[a_0\left(\cos\frac{x^2}{2} + \frac{3}{16x^2}\sin\frac{x^2}{2} - \frac{\frac{3}{16}\left(2+\frac{3}{16}\right)}{2!\,x^4}\cos\frac{x^2}{2}\right.\right.$$

$$\left.\left. - \frac{\frac{3}{16}\left(2+\frac{3}{16}\right)\left(3.4+\frac{3}{16}\right)}{3!\,x^6}\sin\frac{x^2}{2} + \cdots\right) + b_0\left(\sin\frac{x^2}{2} - \frac{3}{16x^2}\cos\frac{x^2}{2} + \frac{\frac{3}{16}\left(2+\frac{3}{16}\right)}{2!\,x^4}\sin\frac{x^2}{2}\right)\right].$$

For convenience in later comtutation it is desirable to change the coefficients in these series to decimals. We then have

$$(34) \quad y_a(x) = \frac{2^{\frac{1}{4}}}{\sqrt{x}}\left[a_0\left(\cos\frac{x^2}{2} + \frac{.187500}{x^2}\sin\frac{x^2}{2} - \frac{.205078}{x^4}\cos\frac{x^2}{2}\right.\right.$$

$$\left.\left. - \frac{.422973}{x^6}\sin\frac{x^2}{2} + \cdots\right) + b_0\left(\sin\frac{x^2}{2} - \frac{.187500}{x^2}\cos\frac{x^2}{2}\cdots\right)\right].$$

As with $J_{\pm\frac{1}{4}}\left(\frac{x^2}{2}\right)$, it is evident that for very large values of x

$$(35) \quad y_a(x) \sim \frac{2^{\frac{1}{4}}}{\sqrt{x}}\left(a_0\cos\frac{x^2}{2} + b_0\sin\frac{x^2}{2}\right).$$

Since both $C_2(x)$ and $S_2(x)$ are solutions of (25), equation (28) expresses each of these solutions asymptotically if the arbitrary constants a_0 and b_0, are properly chosen in each instance. For that purpose we will use (28), (29), (31) (32), and (35).

Let us call the asymptotic series for $C_2(x)$, $y_{ac}(x)$ and that for $S_2(x)$ $y_{as}(x)$. For a_0 and b_0 properly chosen

(35) must $= y_{ac}(x)$,

which by (28) and (31) will equal

$$\frac{\sqrt{2}\,\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi x}}\left(\cos\frac{\pi}{8}\cos\frac{x^2}{2} + \sin\frac{\pi}{8}\sin\frac{x^2}{2}\right).$$

Therefore

$$\frac{2^{\frac{1}{4}}}{\sqrt{x}}\left(a_0\cos\frac{x^2}{2} + b_0\sin\frac{x^2}{2}\right) \equiv \frac{\sqrt{2}\,\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi x}}\left(\cos\frac{\pi}{8}\cos\frac{x^2}{2} + \sin\frac{\pi}{8}\sin\frac{x^2}{2}\right).$$

In order that this identity be satisfied, it is necessary that

$$\frac{a_0 2^{\frac{1}{4}}}{\sqrt{x}} = \frac{\sqrt{2}\,\Gamma(\frac{3}{4})}{\sqrt{\pi x}} \cos\frac{\pi}{8},$$

and that

$$\frac{b_0 2^{\frac{1}{4}}}{\sqrt{x}} = \frac{\sqrt{2}\,\Gamma(\frac{3}{4})}{\sqrt{\pi x}} \sin\frac{\pi}{8}.$$

Therefore (for $C_2(x)$),

$a_0 = .75959506$ and $b_0 = .31463406.$

Then, from (33) or (34) we can get the asymptotic expression for $C_2(x)$.

In a similar manner we find that for $S_2(x)$

$$a_0' = \frac{2^{\frac{5}{4}}\,\Gamma(\frac{5}{4})}{\sqrt{\pi}} \cos\frac{3\pi}{8} = .46544956,$$

and that

$$b_0' = \frac{2^{\frac{5}{4}}\,\Gamma(\frac{5}{4})}{\sqrt{\pi}} \sin\frac{3\pi}{8} = 1.1236965;$$

so as above we may obtain $y_{as}(x)$ from either (33) or (34).

If we let the quantity

$$\cos\frac{x^2}{2} + \frac{.187500}{x^2}\sin\frac{x^2}{2} \;------\; = A$$

and similarly,

$$\sin\frac{x^2}{2} - \frac{.187500}{x^2}\cos\frac{x^2}{2} \;-----\; = B,$$

we can get the following condensed expression for $y_{ac}(x)$ and $y_{as}(x)$.

(36) $$y_{ac}(x) = \frac{.903316}{\sqrt{x}}A + \frac{.374165}{\sqrt{x}}B,$$

and (37)

$$y_{as}(x) = \frac{.553515}{\sqrt{x}}A + \frac{1.336308}{\sqrt{x}}B.$$

By these formulas

$y_{ac}(4) = .124791$

and $y_{as}(4) = .624601.$

These differ from the values given in (30) by .000008 in both instances. Upon this basis, we would seem justified in assuming that the asymptotic formulas are correct to at least four places of decimals.

The following table of the values of $C_2(x)$ and $S_2(x)$ for x = 0 to x = 10 was computed from the series (25) out to x = 3 and then by mean means of (36) and (37) from that point on.

Table II:

x	$C_2(x)$	$S_2(x)$	x	$C_2(x)$	$S_2(x)$
0	1.00000	.00000	5.2	.36115	.61735
.2	.99987	.19998	5.4	-.01838	.42003
.4	.99787	.39949	5.6	-.38648	-.21441
.6	.98923	.59612	5.8	-.30659	-.59847
.8	.96612	.78371	6.0	.12681	-.26320
1.0	.91814	.95069	6.2	.39129	+.38019
1.2	.83350	1.07912	6.4	.12602	.46124
1.4	.70120	1.14509	6.6	-.28212	-.07979
1.6	.51469	1.12148	6.8	-.26817	-.52583
1.8	.27670	.98405	7.0	.16715	-.11584
2.0	.00439	.72115	7.2	.32388	.48251
2.2	-.26714	.34593	7.4	-.08551	.23155
2.4	-.48625	-.09287	7.6	-.32863	-.42181
2.6	-.59427	-.50787	7.8	.05275	-.26265
2.8	-.54481	-.78603	8.0	.33955	.42292
3.0	-.32978	-.82264	8.2	-.07473	.22701
3.2	-.00168	-.57253	8.4	-.28984	-.40073
3.4	+.27562	-.12764	8.6	.14651	-.15649
3.6	.50343	.41426	8.8	.23831	.43553
3.8	.44296	.75548	9.0	-.21203	-.02101
4.0	.12479	.62460	9.2	-.13837	-.40932

x	$C_2(x)$	$S_2(x)$	x	$C_2(x)$	$S_2(x)$
4.2	-.25439	.15575	9.4	.28974	.23739
4.4	-.46087	-.41083	9.6	-.03707	.27759
4.6	-.33200	-.67324	9.8	-.27184	-.43912
4.8	.05496	-.40525	10.0	.24424	.05718
5.0	.39041	.20271			

As previously pointed out and as will be more fully demonstrated later these values may be expected to be correct to at least four places of decimals.

B. The roots of $C_2(x)$ and $S_2(x)$.

1. Theorems on the number and distribution of the roots of $C_2(x)$ and $S_2(x)$.

Theorem 1. The solutions $C_2(x)$ and $S_2(x)$ each have an infinite number of roots.

In the proof of this statement, let us consider the two equations

(a) $$C_2'' + x^2 C_2 = 0$$

and

(b) $$y_1'' + k^2 y_1 = 0$$

where k is any constant. A solution of (b) is

(c) $$y_1 = A \sin k(x-d)$$

in which d may be assigned any value. Since (c) has a root wherever $x = d$, we can put a root of that equation at any preassigned point on the axis of x. By the Existence Theorem which we have been using, (a) has a root. Both equations now have a root at $z = d$.

If we multiply (a) by y_1, (b) by C_2 and subtract (b)' from (a)' we get

(d) $$y_1 C_2'' - C_2 y_1'' + (x^2 - k^2)\, y_1 C_2 = 0.$$

But $$y_1 C_2'' - C_2 y_1'' = \frac{d}{dx}\,(y_1 C_2' - C_2 y_1');$$

therefore when we integrate (d) over any limits a to b, we have

(e) $$\Big(y_1 C_2' - C_2\, y_1' \Big)\Big]_a^b + \int_a^b (x^2 - k^2)\, y_1 C_2\, dx = 0.$$

Equation (b) will have its next root when

$$\sin k(x-p) = 0 \text{ again;}$$

that is, when

$$k(x-p) = \pi.$$

Hence, $x = p + \pi/k$ at the next root after $x = p$. Now let $a = p$ and $b = p + \pi/k$, and (e) becomes

(f) $$\left(y_1 C_2' - C_2\, y_1' \right)\Big]_p^{p+\frac{\pi}{k}} + \int_p^{p+\frac{\pi}{k}} (x^2 - k^2)\, y_1 \cdot C_2\, dx = 0.$$

In view of the fact that y vanishes at both limits and C_2' is zero when $x = p$, we may transpose the integral in (f) and obtain

(g) $$\left(y_1' C_2 \right)_{\text{at } p+\frac{\pi}{k}} = \int_p^{p+\frac{\pi}{k}} (x^2 - k^2)\, y_1 \cdot C_2\, dx.$$

From (c),

$$y_1' = Ak \cos k(x-p);$$

so that $x = p + \pi/k$, $y_1' = -Ak$.

Then the left side of (g) has a sign which is opposite to that of $C_2(x)$.

Since k is any constant it can be taken equal to p. Then between $x = p$ and $x = p + \pi/k$

$x^2 > k^2$; so $(x^2 - k^2)$ is positive. The sine is positive as the angle goes from 0 to π; therefore y_1 is positive in the interval between $x = p$ and $x = p + \pi/k$. Then the sign of the right-hand side of (g) is the same as the sign of $C_2(x)$. But we would then have a positive quantity equal to a negative quantity unless $c_2(x)$ crosses the x-axis in the given interval. We can then say that at least one root of $C_2(x)$ lies in that interval.

Let us call this second root of $C_2(x)$, p'. It is evident that we could now start with p' as we did with p and prove the existence of at least one root of $C_2(x)$ between p' and $p' + \pi/k'$. Since there is no limit to the number of times that this process might be repeated, theoretically, $C_2(x)$ has an infinite number of roots.

A very similar argument obviously applies to $S_2(x)$ since none of the special properties of $C_2(x)$ have been employed in the above proof. Therefore the theorem is true as stated.

Theorem 2. Between two consecutive roots of $C_2(x)$ lies one and only one root of $S_2(x)$; and conversely.

Using the Wronskian determinant for $C_2(x)$ and $S_2(x)$ we see that

$$\begin{vmatrix} C_2 & S_2 \\ C_2' & S_2' \end{vmatrix} = k.$$

Then

$$C_2 \cdot S_2' - S_2 \cdot C_2' = k.$$

Let a and b be two consecutive roots of $C_2(x)$. Then at a

$$k = - C_2' \cdot S_2$$

and at b

$$k = - C_2' \cdot S_2 .$$

But in order that the continuous curve, $C_2(x)$, cross the axis at a and at b, C_2' must change signs at least once between those points. But from equation (a) in the proof of Theorem 1, we see that

$$C_2'' = -x^2 C_2 ;$$

hence C_2'' does not change sign between consecutive roots of C_2. Then there is one and only one maximum or minimum of C_2 between a and b. We may then conclude that C_2' changes signs once and only once between a and b. We may then conclude since

$$k = - C_1' S_2$$

at both a and b, that S_2 must have changed signs once and only once in that interval. Hence the first part of the theorem is proved.

It is obvious that the converse follows from the same argument.

2. The formulas for the roots of $C_2(x)$ and $S_2(x)$.

The following development is a parallel of Stokes' Method of calculating the zeros of $J_m(s)$.

If in the asymptotic formula (38) we let

$$b_0 = C \sin z$$

and

$$a_0 = C \cos z,$$

we see that

$$(38) \quad y_a(x) = \frac{2^{\frac{1}{4}}}{\sqrt{x}} C \left[\left(1 - \frac{.205078}{x^4} + \cdots\cdots\right) \cos\left(\frac{x^2}{2} - z\right) + \left(\frac{.187506}{x^2} - \frac{.422973}{x^6} + \cdots\cdots\right) \sin\left(\frac{x^2}{2} - z\right)\right].$$

If in this equation we set $x = 3$, we obtain

$$(39) \qquad y_a(3) = -C(.158016 \cos z + .666625 \sin z).$$

We can now differentiate (38) and with (39) have two equations between which we can solve for x and C, for; we can get the values of $y_a(3)$ and $y_a'(3)$ from the appropriate power series. We find that

$$(40) \qquad y_a'(3) = C\,(2.034405 \cos z - .366852 \sin z).$$

Again recalling the Existence Theorem used in section II, we see that with C and z properly chosen,

(41) $C_2(3) = y_a(3)$ and $C'_2(3) = y'_a(3)$.

From the power series for $C_2(x)$, we discover that

$$C_2(3) = -.329780 \qquad \text{and } C'_2(3) = 1.429957.$$

We can now use these values and equations (39) and (40) and (41) to obtain the values of C and z. We would have

(42) $.329780 = C(.158016 \cos z + .666625 \sin z)$

and

(43) $1.429957 = C(2.034405 \cos z - .366852 \sin z)$.

From 42,

$$C = \frac{.329780}{.158016 \cos z + .666625 \sin z};$$

so from this fact and (43) we see that

$$\tan z = .414206.$$

Consequently, $z = .392693 = .124998\pi$, and $C = .822209$.

After a few simplifications, equation (38) becomes

(44) $$y_{ac}(x) = \frac{.977777}{\sqrt{x}}\left[\left(1 - \frac{.205078}{x^4} + \frac{1.288749}{x^8} - \frac{26.179224}{x^{12}} + \cdots\right)\cos\left(\frac{x^2}{2} - .124998\pi\right) + \left(\frac{.187500}{x^2} - \frac{.422973}{x^6} + \cdots\right)\sin\left(\frac{x^2}{2} - .124998\pi\right)\right].$$

If we set

$$\left(1 - \frac{.205078}{x^4} + \cdots\cdots\right) = A; \quad \left(\frac{.187506}{x^2} - \frac{.422972}{x^6} + \cdots\right) = B,$$

and let $A = M\cos t$, and $B = M \sin t$, we can reduce (44) to

(45) $$y_{ac}(x) = \frac{.977777}{\sqrt{x}} M\cos\left(\frac{x^2}{2} - .124998\pi - t\right).$$

This can be verified by expanding

$$\cos\left[\left(\frac{x^2}{2} - .124998 \quad\right) - t\right], \text{ where } t = \tan^{-1}\frac{B}{A}.$$

It is clear that (45) vanishes when $\frac{x^2}{2} - .124998\pi - t = K\pi - \frac{\pi}{2}$, K being any positive integer. Then let us set

$$\frac{x^2}{2} = K\pi - \frac{\pi}{2} + .124998\pi + t,$$

or

(46) $$\frac{x^2}{2} = \pi(K - .375002) + \tan^{-1}\frac{B}{A}.$$

By ordinary division, we find that

$$\frac{B}{A} = \frac{.187500}{x^2} - \frac{384521}{x^6} + \frac{4.882827}{x^{10}} \cdots\cdots\cdots$$

If we now set

$$\psi = \pi(K - .375002)$$

and use Gregory's Series on $\tan^{-1}\frac{B}{A}$, we obtain

(47) $$\frac{x^2}{2} = \psi + \frac{.187500}{x^2} - \frac{.384521}{x^6} + \text{----}$$

$$-\frac{1}{3}\left(\frac{.006592}{x^6} - \frac{.040555}{x^{10}} + \frac{.542727}{x^{14}}\cdots\right) + \frac{1}{5}\left(\frac{.000232}{x^{10}} - \frac{.002377}{x^{14}} + \cdots\right)$$

------------.

At this point, let us assume that

(48) $$\frac{x^2}{2} = \psi + \frac{a}{\psi} + \frac{b}{\psi^3} + \frac{c}{\psi^5} + \text{--------}$$

where a, b, c, ...are constants to be determined. We will get from (47) an identity involving powers of $\frac{1}{\psi}$. Since it is an identity we can set the coefficients of corresponding powers of $\frac{1}{\psi}$ equal to each other and solve for a, b, c, ...

In the process, we find that

$$\frac{1}{x^2} = \frac{1}{2}\left(\frac{1}{\psi} - \frac{a}{\psi^3} + \frac{a_2 b}{\psi^5} \text{----}\right)$$

$$\frac{1}{x^6} = \frac{1}{8}\left(\frac{1}{\psi^3} - \frac{3a}{\psi^5} + \frac{5a^2 - 2b}{\psi^7} \text{---}\right)$$

----------------.

Using (47) and (48) and putting the above values into the resulting identity, we obtain the following values for a, b, c, ...

a = .093750

b = -. 057128

$c = .172787$

$d = -1.280655$

....

More of these coefficients of $\frac{1}{\Psi}$ are not needed here.

We can now use (48) and the values of ψ in terms of K and π to get a formula for $\frac{x^2}{2}$. It is

$$(49)\quad \frac{x^2}{2} = K\pi - 1.178103 + \frac{.093750}{(K-.375002)\pi} - \frac{.057128}{(K-.375002)^3\pi^3} + \cdots\cdots$$

with the values of $\frac{1}{\pi}$, $\frac{1}{\pi^3}$, ... put in in decimal form, we have from (49) a fairly convenient formula for the roots of $C_2(x)$ as follows:

$$(50)\quad \frac{x^2}{2} = K\pi - 1.178103 + \frac{.029842}{(K-375002)} - \frac{.001842}{(K-.375002)^3}$$

$$+ \frac{.000565}{(K-.375002)^5} - \frac{.000424}{(K-.375002)^7} + \cdots\cdots.$$

In an exactly analogous manner, we can derive a formula for the roots of $S_2(x)$ by using

$$S_2(3) = -.822643$$

and
$$S_2'(3) = .534730.$$

Since the determination of a, b, c, ... did not depend at all on z or C, these constants will be the same ones previously derived.

The final form of the equation analogous to (50) for the roots of $S_2(x)$ is

$$(51)\quad \frac{x^2}{2} = K\pi - .392715 + \frac{.029842}{(K-.125005)}$$

$$- \frac{.001842}{(K-.125005)^3} + \frac{.000565}{(K-.125005)^5} \cdots\cdots.$$

When using such formulas as (50) and (51), there is naturally the question of accuracy. To determine this, we have checked our work in three ways: first, by means of the relations to $J_{\pm\frac{1}{4}}(\frac{x^2}{2})$; second, by showing how closely $y_{ac}(r_1)$ and $y_{as}(r_2)$ come to equaling zero if r_1 and r_2 are roots computed from (50) and (51) respectively; and finally, by means of the Wronskian determinant for $C_2(x)$ and $S_2(x)$.

We have previously shown following equation (37) that the formulas for y_{ac} and y_{as} are accurate to at least four decimals when checked by means of the relations of $C_2(x)$ and $S_2(x)$ to $J_{-\frac{1}{4}}(\frac{x^2}{2})$ and $J_{\frac{1}{4}}(\frac{x^2}{2})$ respectively.

As computed from (50) the fifteenth root of $C_2(x)$ is

$$x = 9.58622.$$

By means of (36) we find that

$$y_{ac}(9.58622) = -.0000045,$$

which is correct to at least four places of decimals.

From (51) we find that the fifteenth root of $S_2(x)$ is

$$x = 9.66780;$$

and from (37) we learn that

$$y_{as}(9.66780) = .00000093,$$

which is correct to five places of decimals.

As a third check we will recall that in the proof of Theorem 2 in this section we derived from the Wronskian determinant the relation

$$(52) \qquad C_2 \cdot S' - S_2 \cdot C_2' \equiv k.$$

We wish to show first that

$$k \equiv 1.$$

From the power series in (26), we observe that

at $x = 0$, $C_2 = 1$, $S_2 = 0$, $C_2' = 0$, and $S_2' = 1$.

Then (52) becomes

$$(1).(1) - (0).(0) = k;$$

therefore $k = 1$. We can now say that

$$C_2 \cdot S_2' - C_2' S_2 = 1 \tag{53}$$

for all values of x.

Applying (53) we find that

$$C_2(3) \cdot S_2'(3) - S_2(3) \cdot C_2'(3) = 1.000001$$

and

$$C_2(4) \cdot S_2'(4) - S_2(4) \cdot C_2'(4) = .999985.$$

In summary, we have shown: first, that as compared through the Bessel functions, the asymptotic formulas (36) and (37) represent $C_2(x)$ and $S_2(x)$ respectively, correctly to at least four places of decimals; second, that the formulas for the roots of $C_2(x)$ and $S_2(x)$ based on $y_{ac}(x)$ and $y_{as}(x)$ are correct to at least four places of decimals; and third, that the values of $C_2(3)$ and $C_2'(3)$ ---that upon which we based both the use of the Existence Theorem in getting our asymptotic formulas and the derivation of the formulas for the roots of $C_2(x)$ and $S_2(x)$---- are correct to five places of decimals. Therefore we may reasonably state that our calculations are correct to four places of decimals. This obviously does not bar the possibility of an error in the mechanical calculations involved in this study.

Following is a table of the first seventeen roots of each of $C_2(x)$ and $S_2(x)$ to the right of the origin:

K	K^{th} root of $C_2(x)$	K^{th} root of $S_2(x)$
1	2.00480	2.35849
2	3.20095	3.43690
3	4.06398	4.25265
4	4.77419	4.93585
5	5.39190	5.53559
6	5.94588	6.07650

7	6.45253	6.57306
8.	6.92222	7.03546
9	7.36202	7.46793
10	7.77701	7.87734
11	8.10795	8.26650
12	8.54676	8.63816
13	8.90674	8.99448
14	9.25272	9.33721
15	9.58622	9.66780
16	9.90851	9.98746
17	10.22064	10.29720

C. The graphs of $C_2(x)$ and $S_2(x)$

It is hoped that the graphs given on the succeeding sheet will be of some aid in pointing out the facts that:

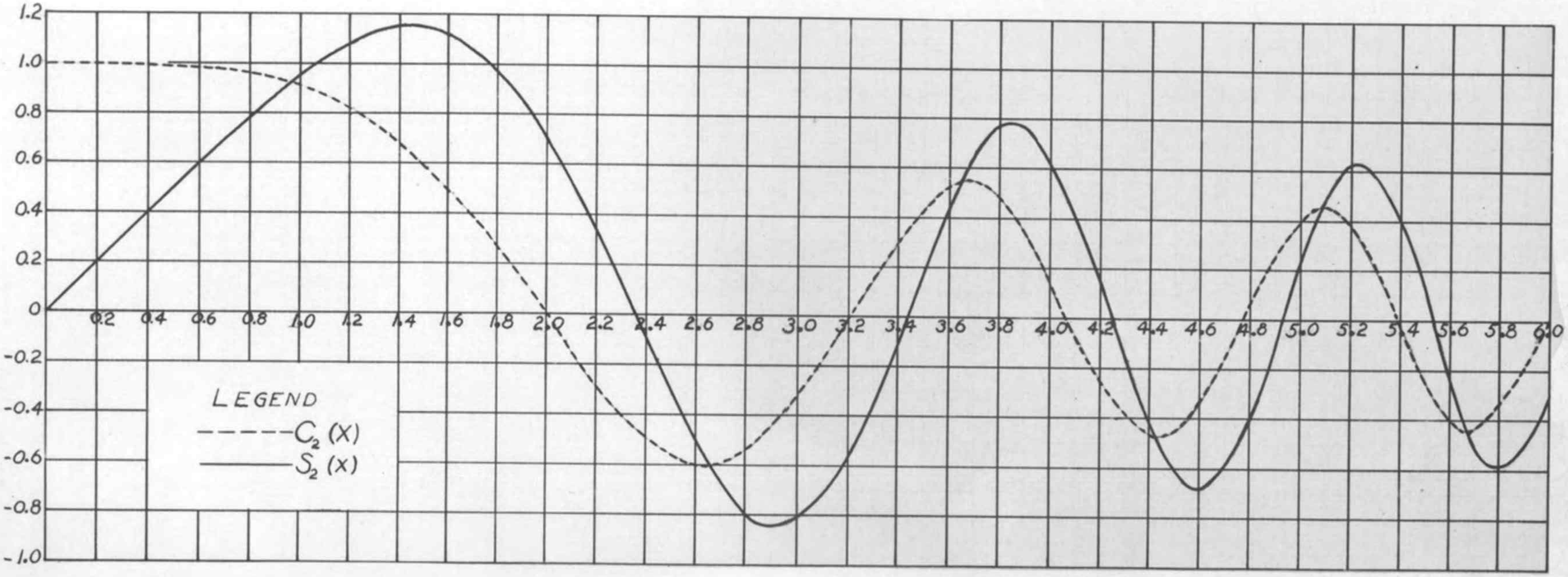
(1) There is an infinite number of roots of both solutions.

(2) The amplitudes decrease as x increases while the slopes increase as x increases.

(3) The curves are symmetrical with respect to the x-axis; and,

(4) Between consecutive roots of one solution lies one and only one root of the other.

These graphs were plotted from Tables II and III.


1.2
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4 2.6 2.8 3.0 3.2 3.4 3.6 3.8 4.0 4.2 4.4 4.6 4.8 5.0 5.2 5.4 5.6 5.8 6.0
LEGEND
$C_2(X)$
$S_2(X)$

THE ORTHOGONALITY PROPERTY OF THE SOLUTIONS OF (1)

If $w(z)$ is a solution of

$$(1) \qquad w'' + z^n w = 0,$$

$w_m(k_m z)$ is a solution for

$$(54) \qquad w_m'' + k_m^{n+2} z^n w_m = 0.$$

Consider the two following particular forms of (53):

$$(55) \qquad w_1'' + k_1^{n+2} z^n w_1 = 0$$ and

$$(56) \qquad w_2'' + k_2^{n+2} z^n w_2 = 0.$$

If now we multiply (54) by w, (56) by w and subtract (56)' from (55)' we get

$$(57) \qquad w_2 w_1'' - w_1 w_2'' + z^n(k_1^{n+2} - k_2^{n+2}) w_1 w_2 = 0.$$

But $w_2 w_1'' - w_1 w_2'' = \frac{d(w_2 w_1' - w_1 w_2')}{dz}$;

therefore if a and b are such that k_1 and k_2 can be so chosen that $k_1 a$ and $k_1 b$ are roots of w_1, and $k_2 a$ and $k_2 b$ are roots of w_2, then

$$\int_a^b (w_2 w_1'' - w_1 w_2'')dz = 0.$$

Consequently, we see from (57) that

$$\left(k_1^{n+2} - k_2^{n+2}\right) \int_a^b z^n \, \omega_1(k_1 z) \cdot \omega_2(k_2 z)\, dz = 0.$$

It is evident that if m equal any integer and q equal any different integer, we have the following expression of the orthogonal relation between the non-independent solutions of (1):

$$(58) \qquad \int_a^b z^n \, \omega_m(k_m z) \cdot \omega_q(k_q z)\, dz = 0.$$

If n equals two and z is real we have

$$(59) \qquad \int_a^b x^2 \, C_2(k_m x) \cdot C_2(k_q x)\, dx = 0$$

and

$$(60) \qquad \int_c^d x^2 \int_2 (k_m x) \cdot \int_2 (k_q x)\, dx = 0$$

where in both equations $m \neq q$.

THE REPRESENTATION OF A FUNCTION OF X BY MEANS OF A

SERIES OF $C_2(k_m x)$ OR OF $S_2(k_m x)$ TERMS

The basis for this section is the orthogonality relationship just derived. We will use (59) and (60) to determine the coefficients of a series of $C_2(k_m x)$ or of $S_2(k_m x)$ terms assumed to represent f(x).

A. The representation of a function of x by means of a $C_2(k_m x)$ series.

Let us assume that

(61) $$f(x) = a_1 C_2(k_1 x) + a_2 C_2(k_2 x) + \cdots\cdots\cdots\cdots$$

If we multiply (60) through by $x^2 C_2(k_m x)dx$ and integrate from a to b, we have as a result of (59)

(62) $$\int_a^b x^2 C_2(k_m x) f(x)\,dx = a_m \int_a^b x^2 C_2^2(k_m x)\,dx.$$

From (28), $$C_2(k_m x) = \frac{\sqrt{k_m x}\,\Gamma(\frac{3}{4})}{\sqrt{2}}\, J_{-\frac{1}{4}}\left(\frac{x^2}{2}\right);$$

therefore,

(63) $$\int_a^b x^2 C_2(k_m x) f(x)\,dx = a_m k_m \Gamma(\tfrac{3}{4})^2 \int_a^b x^3 J_{-\frac{1}{4}}^2\left(\frac{k_m^2 x^2}{2}\right)dx.$$

Turning to Bessel's Functions, we find that

(64) $$\int_a^b s J_n^2(s)\,ds = \left(\frac{s^2}{2}\left[J_n^2(s) - J_{n+1}(s)^2\right] - ns\, J_n(s)\cdot J_{n+1}(s)\right)\Bigg]_a^b. \quad (1)$$

If we let $s = \frac{k_m^2 x^2}{2}$, n = -1/4, and use (28) again we see that the right hand side of (63) reduces to

(65) $$\frac{k_m x^4}{8} \cdot J_{\frac{3}{4}}^2\left(\frac{k_m^2 x^2}{2}\right)\Bigg]_a^b$$

Again using the relation

$$s = \frac{k_m^2 x^2}{2},$$

we find that

(66) $$\int_a^b s \cdot J_{-\frac{1}{4}}^2(s)\,ds = \frac{k_m^4}{2}\int_a^b x^3 J_{-\frac{1}{4}}^2\left(\frac{k_m^2 x^2}{2}\right)dx.$$

Hence, by combining the facts stated in equations (63), (64),,(65), and (66) we learn that

(67) $$\int_a^b x^2 C_2(k_m x) f(x)\,dx = \frac{a_m k_m \Gamma^2(\frac{3}{4})}{2}\left(\frac{x^4}{4} J_{\frac{3}{4}}^2\left(\frac{k_m^2 x^2}{2}\right)\right)\Bigg]_a^b.$$

Consequently

Since at $x = 0$, $C_2'(x) = 0$, the lower limit in (68) can be taken as zero and the orthogonality property still holds.

If we let r_m denote a root of $c_2(x)$, k_m must be so chosen that

$$k_m b = r_m;$$

so

$$k_m = r_m / b$$

where b is the length of the interval chosen. Then equation (68) with $a = 0$ gives the coefficients required in (61) for a representation of a $f(x)$ by means of a series of $C_2(k_m x)$ terms.

B. The representation of a function of x by means of a $S_2(k_m x)$ series.

In a manner exactly analogous to that employed in part A, we assume a series

$$f(x) = b_1 S_2(k_1 x) + b_2 S_2(k_2 x) + \dots\dots \tag{69}$$

Using the relation (29) and proceeding as before, we finally obtain the following:

$$b_m = \frac{\int_0^d x^2 \, S_2(k_m x) \, f(x)\, dx}{2 k_m \Gamma^2(\frac{5}{4}) \int_0^d x^3 J_{\frac{1}{4}}^2 \left(\frac{k_m^2 x^2}{2}\right) dx}, \tag{70}$$

which reduces to

$$b_m = \frac{2}{k_m \Gamma^2(\frac{5}{4}) \left[x^4 J_{\frac{5}{4}}^2 \left(\frac{k_m^2 x^2}{2}\right)\right]_0^d} \int_0^d x^2 \, S_2(k_m x) f(x)\, dx. \tag{71}$$

We can also take the lower limit as zero here since $S(0) = 0$, thus preserving the orthogonality relation.

In this equation as in (67) it is necessary that

$$k_m = R_m / d$$

where R_m is a root of $S_2(x)$ and d is the length of interval chosen.

When the coefficients from (71) are put into (69), we then have f(x) theoretically represented by a series of $S_2(k_m x)$ terms.

That the series (61) and (69) converge to the value f(x) follows from their relations to similar $J_n(s)$ series which have been shown elsewhere to be convergent. The analytical proof of their convergence lies beyond the scope of this thesis.

The r's and R's needed to determine that k_m are given in Table III out as far as the seventeenth root. The length of interval taken would depend on f(x) and on the problem involved.

However, with so little work done on $C_2(x)$ and $S_2(x)$ as yet,

$$\int_a^b x^2\, C_2(k_m x)\, f(x)\, dx$$

or

$$\int_0^d x^2\, S_2(k_m x)\, f(x)\, dx$$

would be quite difficult to integrate for most functions of x, or it might be practically impossible to integrate either of them.

It would be interesting to attempt to apply the results of this study to physical problems involving equations (1) or (25), and to carry this study farther on that basis, but that lies beyond the limits of this thesis.

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