

AN OPERATIONAL CONTINUED FRACTION SOLUTION
TO A SECOND ORDER EQUATION IN BANACH SPACE

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CHAPTER I

INTRODUCTION

Since the beginning of the twentieth century, mathematicians have turned with interest and vigor to the abstraction and generalization of classical mathematical concepts. In the case of mathematical analysis, the trend toward abstraction has led to the definition and investigation of abstract spaces. The technique of "geometrizing" the space, defining it to possess geometric properties such as the triangle inequality, has proved to be extremely fruitful. Among the more prominent of such abstract spaces is the Banach space, named in honor of S. Banach who contributed greatly to its development (2, pp. 500 - 506). The Banach space will be the basic concept here in our investigations.

Another concept which has proved particularly useful in abstracted form is that of the linear operation or operator. The linear operator establishes a correspondence between the elements of two Banach spaces, such that for every element of one space there corresponds a unique element of the second space. Closely allied with this concept is that of a bilinear operator. The bilinear

operator is a linear operator which establishes a correspondence between pairs of elements of one space and elements of a second space. We shall investigate both of these concepts.

A quite natural problem arising from these abstracted notions has been the question of finding solutions for equations defined in Banach space. Both linear and non-linear equations have been subject to investigation. In the case of non-linear equations, no extensive theory has been put forth; the available methods of approach are few and somewhat limited in their scope.

Here, our interest will concern the solution of a particular non-linear equation, a second order equation. Our approach to the problem will be in a manner analogous to the use of continued fractions for solving the quadratic equation in real and complex number space. Using the concepts of Banach space, linear and bilinear operators, we shall define a generalized continued fraction, called an operational continued fraction. The operational continued fraction will prove to be, under certain conditions, a solution to a second order equation in Banach space. We shall find an error bound which will be used to prove convergence of the continued fraction and will also be an aid for measuring error in numerical examples.

Chapter II will be devoted to the definition of Banach space, the linear operator, and the bilinear operator. Those properties arising from these definitions which will be necessary for Chapter III will be investigated. In Chapter III we shall define an operational continued fraction and investigate its application to a second-order equation. Sufficient conditions for an operational continued fraction solution to the equation will be given. Finally, Chapter IV will be reserved for examples of particular equations and their solution by use of the operational continued fraction.

CHAPTER II

FUNDAMENTAL DEFINITIONS AND THEOREMS

Before any attempt may be made to define an operational continued fraction, several basic concepts must be examined. As the fundament of all the investigations of this article lies the concept of a Banach space. Hence, the first section of this chapter will be devoted to the definition of this concept. The Banach space incorporates several properties which are each sufficient to characterize an abstract space, and therefore, we shall introduce a sequence of definitions concerning these related properties.

Once the Banach space has been defined, the remainder of this chapter will be centered about the concept of a transformation which establishes a correspondence between the elements of X and the elements of Y , where X and Y are Banach spaces. Such a transformation will be called an operator which maps the space X into the space Y . The correspondence between X and Y may be many-to-one or biunique, and we shall consider each of these types of correspondence.

Our special interest in the second section of this chapter will concern a particular type of operator, the linear operator. We shall consider the operations of

addition, subtraction, and multiplication for operators, and we shall define and examine the multiplicative inverse of an operator. This inverse operator will be of great importance in the chapter on the continued fraction.

Finally, in the third section we shall define a bilinear operator, an operator which maps the space of ordered pairs (x_1, x_2) of elements of X into the space Y , where X and Y are Banach spaces. The relationship between linear and bilinear operators will be noted briefly.

Section 1. Definition of a Banach space

2.1 Definition: A set L of elements is called a linear space if for the elements of L there exist two uniquely defined operations: an addition and a scalar multiplication by complex numbers such that if x_1 and x_2 are arbitrary elements of L , and α is an arbitrary complex number, x_1+x_2 and αx_1 are elements of L . For these operations the following rules hold:

1. Addition is commutative and associative.
2. If $x_1+x_2=x_1+x_3$, then $x_2=x_3$.
3. $\alpha(x_1+x_2)=\alpha x_1+\alpha x_2$.
4. If α and β are arbitrary complex numbers,
 $(\alpha\beta)x=\alpha(\beta x)$.
5. $(\alpha+\beta)x=\alpha x+\beta x$.
6. $1\cdot x=x$.

For greater simplicity of notation, we shall hereafter write "x is in X" to denote "x is an element of X" and the "sequence $\{x_n\}$ in X" to denote " x_n is in X, $n=1,2,\dots$ ".

From the use of (5) and (2), it follows that there exists a uniquely determined null element θ such that if x is in L , $0 \cdot x = \theta$ and $x + \theta = x$. Moreover, substituting $x_2 = \theta$ in (3), we have $a \cdot \theta = \theta$ for any complex number a . We introduce subtraction in L by the definitions: $(-x) = (-1)x$ and $x_1 - x_2 = x_1 + (-x_2)$. The elements of L form, therefore, relative to the addition, an abelian group (7, pp. 92 - 93).

2.2 Definition: A linear space X is said to be a normed linear space if for every element x in X there exists a finite, non-negative real number $\|x\|$, called the norm of x , with the following properties:

1. If a is any complex number, $\|ax\| = |a| \cdot \|x\|$.
2. $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ for any x_1, x_2 in X .
3. $\|x\| = 0$ if, and only if, $x = \theta$.

2.3 Definition: A sequence of elements $\{x_n\}$ in the normed linear space X is said to converge to the element x in X if $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$. In symbols, we write $x_n \rightarrow x$ as $n \rightarrow \infty$ if $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.

We note that implicit in (2.3) is the idea of geometrical distance between elements in a space: the distance between two elements x_1, x_2 in X is the real

number $\|x_1 - x_2\|$, and $\|x_1 - x_2\| = 0$ if, and only if, $x_1 = x_2$ (7, p. 94). The important concept of convergence of a sequence $\{x_n\}$ in X will necessitate our use of the following definitions:

2.4 Definition: A sequence $\{x_n\}$ in the normed linear space X will be called a Cauchy sequence if $\|x_{n+p} - x_n\| \rightarrow 0$ uniformly for $p=0, 1, 2, \dots$, as $n \rightarrow \infty$.

2.5 Definition: A normed linear space X will be called complete if every Cauchy sequence in X converges to an element x in X .

2.6 Definition: A complete normed linear space X will be called a Banach space, or a B-space (7, pp. 92 - 93, and 4, pp. 3 - 4).

We shall defer our consideration of examples of a Banach space to Chapter IV. Other examples are given by Kantorovich (4, pp. 5 - 7) and Zaanen (7, pp. 100 - 108).

The following useful theorem is immediate from (2.5) and (2.6); it will conclude this section.

2.7 Theorem: If X is a B-space, and if $\{x_n\}$ is a sequence in X , the convergence of the series of real numbers

$\sum_{k=1}^{\infty} \|x_k\|$ implies the existence of the limit

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n x_k = \sum_{k=1}^{\infty} x_k.$$

Proof: Let $x'_n = \sum_{k=1}^n x_k$, $n=1, 2, \dots$, and consider the sequence $\{x'_n\}$ in X . By (2.2) we have

$||x'_{n+p} - x'_n|| = ||\sum_{k=n+1}^{n+p} x_k|| \leq \sum_{k=n+1}^{n+p} ||x_k|| \leq \sum_{k=n+1}^{\infty} ||x_k||$. Hence, since $\lim_{n \rightarrow \infty} \sum_{k=n+1}^{\infty} ||x_k|| = 0$, $\{x'_n\}$ is a Cauchy sequence,

and there exists an element x' in X such that

$$\sum_{k=1}^n x_k = x'_n \rightarrow x' \text{ as } n \rightarrow \infty.$$

Section 2. Definition and properties of a linear operator

In the remainder of this article, the symbols X , Y , and Z shall represent Banach spaces.

2.8 Definition: The Cartesian product of X and Y , denoted $X \times Y$, is the set of ordered pairs (x, y) , where x is in X and y is in Y (7, p. 126).

2.9 Definition: An operator P which maps X into Y is a subset of $X \times Y$ such that for every x in X there exists a unique y in Y for which (x, y) is an element of P . If (x, y) is in P , we write $y = Px$.

2.10 Definition: An operator P mapping X into Y is said to be additive if for elements x_1 and x_2 in X ,

$$P(x_1 + x_2) = Px_1 + Px_2.$$

2.11 Definition: If P is an operator from X into Y , and if α is an arbitrary complex number, the operator αP is defined by $(\alpha P)x = \alpha(Px)$.

We note two facts arising from (2.10) and (2.11). If P is an additive operator, $Px = P(x + \theta) = Px + P\theta$, so that $P\theta = \theta$. From this equality we have

$$\theta = P\theta = P[x + (-x)] = Px + P(-x), \text{ so that } P(-x) = -(Px).$$

2.12 Definition: An operator P mapping X into Y is said to be continuous if for every sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ in X as $n \rightarrow \infty$, it follows that $Px_n \rightarrow Px$ in Y as $n \rightarrow \infty$.

2.13 Definition: An operator P mapping X into Y is said to be bounded if there exists a fixed non-negative real number M such that $\|Px\| \leq M \cdot \|x\|$ for all x in X . The smallest number M satisfying this inequality is called the norm of P and is denoted $\|P\|$ (7, pp. 133 - 134).

This definition of $\|P\|$ is justified by the following theorem.

2.14 Theorem: If the operator P is bounded, $\|P\|$ exists.

Proof: Since the set of numbers M such that

$\|Px\| \leq M \cdot \|x\|$ is bounded below, the set has a greatest lower bound. Let $\|P\| = \text{g.l.b. } \{M\}$. Then $\|P\| \leq M$; and for any $\delta > 0$, there exists an M' in $\{M\}$ such that $\|P\| > M' - \delta$.

Now assume there exists an x' in X such that

$\|Px'\| > \|P\| \cdot \|x'\|$. Let $\|Px'\| - \|P\| \cdot \|x'\| = \gamma$, and choose $\delta = \gamma/2 \cdot \|x'\|$, since $x' \neq 0$. We have

$$\begin{aligned} \|Px'\| - M' \cdot \|x'\| &= (\|Px'\| - \|P\| \cdot \|x'\|) - (M' \cdot \|x'\| - \|P\| \cdot \|x'\|) \\ &> \gamma - \gamma/2 = \gamma/2 > 0, \end{aligned}$$

so that $\|Px'\| > M' \cdot \|x'\|$, a contradiction. Thus,

$$\|Px\| \leq \|P\| \cdot \|x\| \text{ for all } x \text{ in } X.$$

2.15 Definition: An additive and continuous operator L mapping X into Y is said to be linear (4, p. 7).

Examples of a linear operator may be found in Chapter IV and (4, pp. 8 - 15).

The following theorem proves that a linear operator is bounded, and also that an additive and bounded operator is linear.

2.16 Theorem: An additive operator P mapping X into Y is continuous if, and only if, it is bounded (7, p. 134, and 3, p. 16).

Proof: If P is bounded, for $x_n \rightarrow x$ in X as $n \rightarrow \infty$ we have $\|Px_n - Px\| \leq \|P\| \cdot \|x_n - x\|$; hence, $\|Px_n - Px\| \rightarrow 0$ as $n \rightarrow \infty$, and P is continuous.

We observe now that by (2.10) and the note following (2.11), $P(ax) = a(Px)$ for a an arbitrary integer, positive or negative. Thus, if $\beta = a/b$ is any rational number, where a and b are integers, $P(\beta x) = aP(1/b \cdot x) = a/b P(1 \cdot x) = \beta(Px)$. Now assume that P is continuous, but not bounded; then there is a sequence of elements $x_n \neq \theta$ such that

$\|Px_n\| > 2n\|x_n\|$, $n=1,2,\dots$. Choosing β_n such that $\|x_n\| \leq \beta_n \leq 2\|x_n\|$, where β_n are rational numbers, and letting $x'_n = x_n/\beta_n$, we have $\|x'_n\| = \|x_n\|/\beta_n \leq 1$, and $\|Px'_n\| = \|Px_n\|/\beta_n > 2n\|x_n\|/\beta_n > n$. Now, letting $x''_n = x'_n/n$, we have $x''_n \rightarrow \theta$ as $n \rightarrow \infty$, and $\|Px''_n\| > 1$ for $n=1,2,\dots$. This is a contradiction to the continuity of P .

The definition(2.11) of scalar multiplication for a linear operator L and the definition(2.13) of $\|L\|$ quite naturally raise the following question: Does the set of all linear operators from X into Y possess the properties necessary to form a particular abstract space? In the following investigations we shall answer this question by defining addition of linear operators. Moreover, we shall also be able to define multiplication of linear operators. Although this question is an interesting one, we shall be more concerned with the properties arising from our investigation.

2.17 Definition: The sum $P+Q$ of two linear operators, each mapping X into Y , is defined by $(P+Q)x=Px+Qx$ (7, p. 133).

The difference of P and Q is defined by $P-Q=P+(-Q)$.

2.18 Theorem: The sum $P+Q$ of(2.17) is a linear operator mapping X into Y , and $\|P+Q\| \leq \|P\| + \|Q\|$.

Proof: The sum $P+Q$ is additive since

$$\begin{aligned} (P+Q)(x_1+x_2) &= P(x_1+x_2)+Q(x_1+x_2) \\ &= Px_1+Px_2+Qx_1+Qx_2 \\ &= (P+Q)x_1+(P+Q)x_2. \end{aligned}$$

Moreover, we have

$\|(P+Q)x\| \leq \|Px\| + \|Qx\| \leq (\|P\| + \|Q\|)\|x\|$, so that $\|P+Q\| \leq \|P\| + \|Q\|$, and $P+Q$ is bounded. From(2.16) it follows that $P+Q$ is continuous, and hence, linear.

2.19 Definition: If P is a linear operator mapping Y into Z and Q is a linear operator mapping X into Y , the product PQ is defined by $(PQ)x = P(Qx)$.

2.20 Theorem: The product PQ of (2.19) is a linear operator mapping X into Z , and $\|PQ\| \leq \|P\| \cdot \|Q\|$ (4, p. 8).

Proof: The product PQ is additive since

$$(PQ)(x_1 + x_2) = P(Qx_1 + Qx_2) = P(Qx_1) + P(Qx_2) = (PQ)x_1 + (PQ)x_2.$$

We also have $\|(PQ)x\| \leq \|P\| \cdot \|Qx\| \leq \|P\| \cdot \|Q\| \cdot \|x\|$,

so that $\|PQ\| \leq \|P\| \cdot \|Q\|$, and PQ is a bounded operator.

By (2.16) it follows that PQ is continuous, and hence, linear.

We may now answer the question concerning the set of linear operators from X into Y . Addition of linear operators is commutative and associative, and moreover, the postulates of (2.1) and (2.2) are satisfied, so that the set of all linear operators from X into Y forms a normed linear space (4, p. 155). The null operator ϕ is such that $0 \cdot L = \phi$ for any linear operator L from X into Y , and $L + \phi = L$, $L + (-L) = \phi$. We also observe that multiplication is distributive over addition.

In the solution of equations, our interest will be restricted to the operators which map X into a subset of itself. Of prime importance among these operators is the multiplicative inverse of a linear operator. We shall define the inverse operator and investigate several theorems concerning its existence and properties.

2.21 Definition: A linear operator L which maps X into a subset of X is said to be "in X ".

2.22 Definition: A linear operator L in X is said to map X onto itself if for $x_1 \neq x_2$ it follows that $Lx_1 \neq Lx_2$; that is, L defines a one-to-one correspondence from X to X .

2.23 Definition: The identity operator I in X is defined by $Ix = x$ for all x in X . If L is a linear operator, $L^0 = I$.

2.24 Definition: The inverse of a linear operator L which maps X onto X is the operator L^{-1} such that $L^{-1}L = L \cdot L^{-1} = I$ (4, p. 23).

2.25 Theorem: If L is a linear operator in X , then L^{-1} exists if, and only if, $Lx = 0$ only for $x = 0$.

Proof: Now L^{-1} exists if, and only if, $x_1 \neq x_2$ implies $Lx_1 \neq Lx_2$; since L is linear, this is equivalent to the statement that $Lx = 0$ only if $x = 0$.

2.26 Theorem: If L is a linear operator in X , then L^{-1} exists and is bounded if, and only if, there is a positive real number m such that $\|Lx\| \geq m \cdot \|x\|$ for all x in X .

Proof: Let $Lx = x'$. If L^{-1} exists and is bounded, we have $\|x\| = \|L^{-1}x'\| \leq M \cdot \|x'\|$ for all x in X , so that $\|Lx\| = \|x'\| \geq 1/M \|x\|$, where $M > 0$. Conversely, if $\|Lx\| \geq m \cdot \|x\|$ for all x in X , then $Lx = 0$ only if $x = 0$, so that L^{-1} exists by (2.25). Moreover, $\|L^{-1}x'\| = \|x\| \leq 1/m \|Lx\| = 1/m \|x'\|$ for all x' in X so that L^{-1} is bounded (7, pp. 162 - 163).

2.27 Theorem: If L is a linear operator in X and if there exists a positive real number $m > 0$ such that $\|Lx\| \geq m \cdot \|x\|$ for all x in X , then L^{-1} is a linear operator in X .

Proof: By (2.26), L^{-1} exists and is bounded. Let x_1 and x_2 be elements of X and let x_1' and x_2' be defined by $H^{-1}x_1 = x_1'$, $H^{-1}x_2 = x_2'$. We have

$$\begin{aligned} H^{-1}(x_1+x_2) &= H^{-1}(Hx_1'+Hx_2') \\ &= H^{-1}\{H(x_1'+x_2')\} \\ &= x_1'+x_2' \\ &= H^{-1}x_1+H^{-1}x_2, \end{aligned}$$

so that H^{-1} is additive. By (2.16) therefore, H^{-1} is continuous, and hence, linear.

We remark here that if L_1 and L_2 are linear operators in X with bounded inverses L_1^{-1} and L_2^{-1} , then $(L_1^{-1})^{-1} = L_1$ and $(L_1L_2)^{-1} = L_2^{-1}L_1^{-1}$. Factorization of the sum $(L_1^{-1}+L_2^{-1})$ may be accomplished as follows:
 $(L_1^{-1}+L_2^{-1}) = L_1^{-1}L_1(L_1^{-1}+L_2^{-1}) = L_1^{-1}(I+L_1L_2^{-1})$.

The following theorem and corollary will be of extreme importance in the next chapter.

2.28 Theorem: If L is a linear operator in X and if $\|L\| \leq \gamma < 1$, then the linear operator $(I-L)$ has a unique inverse $(I-L)^{-1}$, and $\|(I-L)^{-1}\| \leq 1/1-\gamma$ (4, p. 24, and 5, pp. 194 - 195).

Proof: Let x be an arbitrary element of X , and consider the limit $\lim_{n \rightarrow \infty} \sum_{k=0}^n L^k x$. By (2.7) the limit

exists in X since $\sum_{k=0}^{\infty} \|L^k x\| \leq \sum_{k=0}^{\infty} \gamma^k \|x\|$ converges.

Let $Bx = \sum_{k=0}^{\infty} L^k x$; that is, define $B = \sum_{k=0}^{\infty} L^k$. Then

$\|Bx\| \leq \|x\| / 1 - \gamma$, so that $\|B\| \leq 1 / 1 - \gamma$. Now

$(I - L)Bx = B(I - L)x = x$ so that $B = (I - L)^{-1}$. If A is any other operator such that $A(I - L) = (I - L)A = I$, then $(I - L)Ax = x$ and $B(I - L)Ax = Bx$, so that $Ax = Bx$; hence, B is unique.

2.29 Corollary: Under the hypothesis of (2.28), $(I + L)$ has a unique inverse $(I + L)^{-1}$, and $\|(I + L)^{-1}\| \leq 1 / 1 - \gamma$.

Proof: The proof is a direct consequence of (2.28), since $(I + L) = [I - (-L)]$ and $\| -L \| = \|L\|$.

Section 3. Definition and properties of a bilinear operator

In Chapter IV the bilinear operator will play a role of no less importance than that of the linear operator. A bilinear operator B is a linear operator of two places: B maps the space $X \times X$ into the space Y . Our consideration of B will need only be brief, since for any x' in X , the operator Bx' is a linear operator mapping X into Y .

2.30 Definition: The space $(X \times X) \times Y$ is the set of all ordered pairs $[(x_1, x_2), y]$, where (x_1, x_2) is in $X \times X$ and y is in Y .

2.31 Definition: A bilinear operator B is a subset of the space $(X \times X) \times Y$ such that for any (x_1, x_2) in $X \times X$, there exists a unique y in Y for which $[(x_1, x_2), y]$ is in B ,

and such that the following is true: If we write $Bx_1x_2=y$ to denote that $[(x_1, x_2), y]$ is in B , then $B(x_1+x_2)x_3=Bx_1x_3+Bx_2x_3$ and $Bx_1(x_2+x_3)=Bx_1x_2+Bx_1x_3$; and there exists a finite non-negative real number M such that $\|Bx_1x_2\| \leq M \cdot \|x_1\| \cdot \|x_2\|$ for all x_1, x_2 in X . The least constant satisfying this inequality is called the norm of B and denoted $\|B\|$ (4, pp. 155 - 156).

In addition to the examples of Chapter IV, the reader is directed to (4, pp. 156 - 158) for examples of bilinear operators.

2.32 Theorem: If B is a bilinear operator mapping $X \times X$ into Y , for any x' in X , Bx' is a linear operator mapping X into Y (4, p. 156).

Proof: It is clear that Bx' is an operator mapping X into Y . Let $Bx'=L$. Now L is additive by (2.31), and $\|Lx\| \leq (\|B\| \cdot \|x'\|) \cdot \|x\|$, so that L is bounded; hence, L is continuous, and therefore, linear.

We note only a few facts concerning B . The bilinear operator $(-B)$ is defined by $(-B)x=-L$, and $B\theta=\phi$. If B maps the space $X \times X$ into X , we say B is "in X ".

The properties of Banach spaces, linear and bilinear operators which we have considered will be sufficient for the definitions and theorems to follow. Hence, we shall tarry no longer in our investigations of these concepts.

CHAPTER III

THE OPERATIONAL CONTINUED FRACTION
AND A SECOND ORDER EQUATION

A continued fraction, as defined by Wall (6, pp. 13 - 14), is the limiting expression of the product of a sequence of linear fractional transformations. To be more specific, if $t_0(w) = b_0 + w$, $t_p(w) = \frac{a_p}{b_p + w}$, $p = 1, 2, \dots$, where a_p and b_p are complex constants, the product $t_0 \dots t_n(w)$ is defined by $t_0 t_1(w) = t_0[t_1(w)]$, $t_0 t_1 t_2(w) = t_0 t_1[t_2(w)]$, \dots . The value of the continued fraction is the limit $\lim_{n \rightarrow \infty} t_0 \dots t_n(0)$, and the continued fraction takes the following form:

$$\lim_{n \rightarrow \infty} t_0 \dots t_n(0) = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}}$$

In an analogous manner, we could define an operational continued fraction in the following way: Let $p_i(x) = (A_i + B_i x)^{-1} x_i$, $i = 1, 2, \dots$, where x and x_i are elements of X , and A_i and B_i are linear and bilinear operators in X , respectively. Now define $F_1(x) = p_1(x)$, $F_2(x) = F_1[p_2(x)] = [A_1 + B_1(A_2 + B_2 x)^{-1} x_2]^{-1} x_1$, \dots , $F_n(x) = F_{n-1}[p_n(x)]$, \dots , and call $F_\infty(\theta) = \lim_{n \rightarrow \infty} F_n(\theta)$ an

operational continued fraction.

The similarity of this operational continued fraction with that of Wall becomes more apparent if we denote $p_i(x) = (A_i + B_i x)^{-1} x_i \equiv \frac{x_i}{A_i + B_i x}$, $i=1,2,\dots$, and let $F_2(x) = [A_1 + B_1(A_2 + B_2 x)^{-1} x_2] x_1^{-1} \equiv \frac{x_1}{A_1 + B_1 \left(\frac{x_2}{A_2 + B_2 x} \right), \dots}$.

With this notation, $F_\infty(\theta) = \frac{x_1}{A_1 + B_1 \left(\frac{x_2}{A_2 + B_2 \left(\frac{x_3}{A_3 + B_3 (\dots)} \right)} \right)}$.

This representation of the operational continued fraction has been given only for purposes of comparison; the fractional form does not represent an operation of division, but is the counterpart of division in the operator space, the inverse operator upon an element of X . An immediate difficulty is the question: Under what conditions do the inverse operators exist? For a more rigorous approach to this problem, we shall adopt the following definition:

For any $n=1,2,\dots$, let $F_n^n = A_n^{-1} x_n$, and $F_n^k = (A_k + B_k F_n^{k+1})^{-1} x_k$, $k=1,2,\dots, n-1$, where x_n is an element of X , A_n is a linear operator in X , and B_n is a bilinear operator in X , for $n=1,2,\dots$. We note that for $n=1,2,\dots$, if A_n^{-1} exists and $(A_k + B_k F_n^{k+1})^{-1}$ exists, $k=1,2,\dots, n-1$, then F_n^k is an element of X .

3.1 Definition: If for $n=1,2,\dots$, the linear operators A_n^{-1} and $(A_k+B_kF_n^k)^{-1}$ exist, $k=1,2,\dots, n-1$, the limit expression $F_\infty = \lim_{n \rightarrow \infty} F_n^1$ will be called an operational continued fraction. If there exists an element x_0 in X such that $F_n^1 \rightarrow x_0$ as $n \rightarrow \infty$, the operational continued fraction will be said to converge to the element x_0 .

Using (3.1) we see that a sufficient condition for F_∞ to converge is that $\{F_n^1\}$ be a Cauchy sequence.

Let us now focus our attention upon the application of (3.1) with which we are concerned, the solution of a second order equation in Banach space. The problem may be stated as follows: Let X be a B -space, y an element of X , A a linear operator in X , and B a bilinear operator in X . Given y , A , and B we wish to find the element, or elements, x in X for which $Ax+Bxx=y$. The equality $Ax+Bxx=y$ will be called a second order equation in X , and any element x in X for which the equality holds will be called a solution to the equation.

This problem leads us to the investigation of a particular form of (3.1) in which $x_n=y$, $A_n=A$, and $B_n=B$ for $n=1,2,\dots$. We shall state conditions on y , A , and B for which F_∞ exists and converges. The element x_0 to which F_∞ converges will prove to be a solution to the equation $Ax+Bxx=y$, and we shall be able to find a sequence of real numbers $\{\delta_n\}$ so that $\|F_n^1 - x_0\| \leq \delta_n$ where $\delta_n \rightarrow 0$

as $n \rightarrow \infty$; that is we shall be able to approximate x_0 to any given degree of accuracy.

For this particular form of (3.1) we have

$$F_n^n = A^{-1}y \text{ and } F_n^k = (A + BF_n^{k+1})^{-1}y, \quad k=1,2,\dots,n-1, \text{ and } n=1,2,\dots,$$

so that F_n^1 may be defined completely by

$$F_1^1 = A^{-1}y, \quad F_n^1 = (A + BF_{n-1}^1)^{-1}y, \quad n=2,3,\dots. \quad \text{We shall write } F_n^1 \equiv F_n.$$

Now, assuming the existence of the linear operators A^{-1} and $(A + BF_n)^{-1}$, $n=1,2,\dots$, we obtain for $n=2,3,\dots$,

$$\begin{aligned} F_n &= (A + BF_{n-1})^{-1}y = [AA^{-1}(A + BF_{n-1})]^{-1}y \\ &= [A(I + A^{-1}BF_{n-1})]^{-1}y \\ &= (I + A^{-1}BF_{n-1})^{-1}A^{-1}y \\ &= (I + A^{-1}BF_{n-1})^{-1}z, \end{aligned}$$

where $z = A^{-1}y$. The following two theorems give sufficient conditions for the existence and convergence of the operational continued fraction obtained from this definition of F_n .

3.2 Theorem: If A^{-1} exists and is bounded, and if $\|A^{-1}\| \cdot \|B\| \cdot \|z\| \leq \gamma \leq 1/4$, the operational continued fraction defined by $F_1 = z$, $F_n = (I + A^{-1}BF_{n-1})^{-1}z$, $n=2,3,\dots$, exists.

Proof: We need only show the existence of $(I + A^{-1}BF_n)^{-1}$ for $n=1,2,\dots$. By (2.29) this fact is true if

$$\|A^{-1}BF_n\| < 1 \text{ for } n=1,2,\dots. \quad \text{We show by induction}$$

that $\|A^{-1}BF_n\| \leq 1/2$ for $n=1,2,\dots$. If $n=1$,
 $\|A^{-1}BF_n\| = \|A^{-1}Bz\| \leq \|A^{-1}\| \cdot \|B\| \cdot \|z\|$ so that
 $\|A^{-1}BF_n\| \leq 1/4 < 1/2$. Assume $\|A^{-1}BF_k\| \leq 1/2$. Then,
 since $A^{-1}BF_{k+1} = A^{-1}B(I+A^{-1}BF_k)^{-1}z$, we have

$$\begin{aligned} \|A^{-1}BF_{k+1}\| &= \|A^{-1}B(I+A^{-1}BF_k)^{-1}z\| \\ &\leq \|A^{-1}\| \cdot \|B\| \cdot \|(I+A^{-1}BF_k)^{-1}\| \cdot \|z\| \\ &\leq \left(\frac{1}{1-1/2}\right)\gamma = 2\gamma \leq 1/2. \end{aligned}$$

3.3 Theorem: If A^{-1} exists and is bounded, and if
 $\|A^{-1}\| \cdot \|B\| \cdot \|z\| \leq \delta < 1/4$, then there exists an element
 x_0 in X such that the operational continued fraction of
 (3.2) converges to x_0 , and $\|x_0 - F_n\| \leq \frac{(4\delta)^{n-1}}{1-4\delta} K$ where

$$K = \|F_2 - F_1\|.$$

Proof: We note that $(I+A^{-1}BF_n)(I+A^{-1}BF_n)^{-1} = I$ gives
 us $I - (I+A^{-1}BF_n)^{-1} = A^{-1}BF_n(I+A^{-1}BF_n)^{-1}$. For $n=2,3,\dots$,
 consider the element $F_{n+1} - F_n$. We have

$$\begin{aligned} F_{n+1} - F_n &= (I+A^{-1}BF_n)^{-1}z - (I+A^{-1}BF_{n-1})^{-1}z \\ &= [(I+A^{-1}BF_n)^{-1} - (I+A^{-1}BF_{n-1})^{-1}]z \\ &= (I+A^{-1}BF_n)^{-1} [I - (I+A^{-1}BF_n)(I+A^{-1}BF_{n-1})^{-1}]z \\ &= (I+A^{-1}BF_n)^{-1} [I - (I+A^{-1}BF_{n-1})^{-1} \\ &\quad - A^{-1}BF_n(I+A^{-1}BF_{n-1})^{-1}]z \\ &= (I+A^{-1}BF_n)^{-1} [A^{-1}BF_{n-1}(I+A^{-1}BF_{n-1})^{-1} \\ &\quad - A^{-1}BF_n(I+A^{-1}BF_{n-1})^{-1}]z \\ &= (I+A^{-1}BF_n)^{-1} (A^{-1}BF_{n-1} - A^{-1}BF_n)(I+A^{-1}BF_{n-1})^{-1}z \\ &= (I+A^{-1}BF_n)^{-1} A^{-1} [B(F_{n-1} - F_n)] (I+A^{-1}BF_{n-1})^{-1}z. \end{aligned}$$

Using the inequalities of (3.2) we have for $n=2,3,\dots$, that

$$\begin{aligned} \|F_{n+1}-F_n\| &\leq \|(I+A^{-1}BF_n)^{-1}\| \cdot \|A^{-1}\| \cdot \|B\| \\ &\quad \cdot \|F_{n-1}-F_n\| \cdot \|(I+A^{-1}BF_{n-1})^{-1}\| \cdot \|z\| \\ &\leq \left(\frac{1}{1-1/2}\right) \delta \left(\frac{1}{1-1/2}\right) \|F_{n-1}-F_n\| = (4\delta) \|F_n-F_{n-1}\|. \end{aligned}$$

From this inequality we obtain $\|F_{n+1}-F_n\| \leq (4\delta)^{n-1} \|F_2-F_1\|$, $n=2,3,\dots$. Now, using the fact that for $p=1,2,\dots$, $n=2,3,\dots$,

$$\|F_{n+p}-F_n\| = \left\| \sum_{k=n}^{n+p-1} (F_{k+1}-F_k) \right\|$$

$$\begin{aligned} &\leq \sum_{k=n}^{n+p-1} \|F_{k+1}-F_k\| \\ &\leq \sum_{k=n}^{n+p-1} (4\delta)^{k-1} \|F_2-F_1\|, \end{aligned}$$

it follows that

$$\|F_{n+p}-F_n\| \leq \sum_{k=n}^{\infty} (4\delta)^{k-1} \|F_2-F_1\| = \frac{(4\delta)^{k-1}}{1-4\delta} \|F_2-F_1\|, \text{ for}$$

$p=0,1,2,\dots$, and $n=2,3,\dots$. Therefore, $\|F_{n+p}-F_n\| \rightarrow 0$ uniformly for $p=0,1,\dots$, as $n \rightarrow \infty$, and $\{F_n\}$ is a Cauchy sequence, so that there exists an x_0 in X such that

$F_n \rightarrow x_0$ as $n \rightarrow \infty$. Letting $p \rightarrow \infty$ in $\|F_{n+p}-F_n\|$ we obtain

$$\|x_0-F_n\| \leq \frac{(4\delta)^{n-1}}{1-4\delta} \|F_2-F_1\|.$$

The following theorem establishes the relationship between the operational continued fraction of (3.3) and the equation $Ax+Bxx=y$.

3.4 Theorem: If the hypothesis of (3.3) holds, then x_0 is a solution of the equation $Ax+Bxx=y$.

Proof: From the proof of (3.3) we know that $(I+A^{-1}BF_n)^{-1}z=F_{n+1}$, so that $\{(I+A^{-1}BF_n)^{-1}z\}$ is a Cauchy sequence converging to x_0 . Now,

$$\begin{aligned} & \| (I+A^{-1}BF_n)^{-1}z - (I+A^{-1}Bx_0)^{-1}z \| \\ & \leq \| (I+A^{-1}BF_n)^{-1} \| \cdot \| z - (I+A^{-1}Bx_0)^{-1}z \\ & \qquad \qquad \qquad -A^{-1}BF_n(I+A^{-1}Bx_0)^{-1}z \| \\ & \leq \| (I+A^{-1}BF_n)^{-1} \| \cdot \| A^{-1}Bx_0(I+A^{-1}Bx_0)^{-1}z \\ & \qquad \qquad \qquad -A^{-1}BF_n(I+A^{-1}Bx_0)^{-1}z \| \\ & \leq \| (I+A^{-1}BF_n)^{-1} \| \cdot \| A^{-1} \| \cdot \| B \| \cdot \| x_0 - F_n \| \\ & \qquad \qquad \qquad \cdot \| (I+A^{-1}Bx_0)^{-1} \| \cdot \| z \| \\ & \leq (45) \| x_0 - F_n \|, \end{aligned}$$

and since $\|x_0 - F_n\| \rightarrow 0$ as $n \rightarrow \infty$, it follows that

$\{(I+A^{-1}BF_n)^{-1}z\}$ has the limit $(I+A^{-1}Bx_0)^{-1}z$.

Hence $x_0 = (I+A^{-1}Bx_0)^{-1}z$. This equality gives us $x_0 + A^{-1}Bx_0x_0 = z = A^{-1}y$, so that $Ax_0 + Bx_0x_0 = y$.

Applications of this theorem will be given in

Chapter IV.

CHAPTER IV

EXAMPLES

The particular method of approach given here to the solution of the equation $Ax+Bxx=y$ originated in idea from a similar approach to a very simple equation, the quadratic equation in complex number space. For this reason, it will be interesting to return to the quadratic equation as an application of the more general notions which have been investigated.

Let X be the space of complex numbers, and consider the equation $ax^2+bx+c=0$, where a, b , and c are real numbers, $a \neq 0$. If z is an arbitrary complex number, we define $||z||=|z|$. This definition is sufficient to make X a Banach space. The number b is a linear operator, and if $b \neq 0$, $b^{-1}=1/b$ and $||b^{-1}||=|1/b|$. Similarly, a is a bilinear operator and $||a||=|a|$.

Since a, b , and c are real numbers the continued fraction may represent only real roots, and by the discriminant, a necessary condition that $ax^2+bx+c=0$ have real roots is that $b^2-4ac \geq 0$. Applying (3.4) we obtain a more restrictive condition: if $b \neq 0$, and if $b^2-4|ac| > 0$, the equation has a continued fraction solution.

Let us consider now an example of a real Banach space, that is, a Banach space defined for scalar multiplication by real numbers. Let X be the space of two-dimensional vectors $x=(x_1, x_2)$, where x_1 and x_2 are real numbers. Define equality, addition, subtraction, and scalar multiplication of the vectors in the usual manner, and let $\|x\| = \max_k |x_k|$. The convergence of $x^{(k)} = (x_1^{(k)}, x_2^{(k)}) \rightarrow x$ signifies the same thing as in the space of real numbers, that $x_i^{(k)} \rightarrow x_i$ as $k \rightarrow \infty$ for $i=1,2$, so that X is complete and is hence, a Banach space.

Linear operators A in X are the linear transformations defined by the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

a_{ij} real, $i=1,2$, $j=1,2$, where $Ax = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2)$.

To calculate the norm of A we have

$$\|Ax\| = \max_k |a_{k1}x_1 + a_{k2}x_2| \leq \max_k (|a_{k1}| + |a_{k2}|) \cdot \|x\|, \text{ so that}$$

$$\|A\| \leq \max_k (|a_{k1}| + |a_{k2}|). \text{ For any } A \text{ we may find } k' \text{ so that}$$

$$|a_{k'1}| + |a_{k'2}| = \max_k (|a_{k1}| + |a_{k2}|). \text{ Let } x' = (x'_1, x'_2), \text{ where } x'_1 = 1$$

if $a_{k'i} \geq 0$, and $x'_i = -1$ if $a_{k'i} < 0$, $i=1,2$. We have

$$\|Ax'\| = |a_{k'1}x'_1 + a_{k'2}x'_2| = (|a_{k'1}| + |a_{k'2}|) \cdot \|x'\|. \text{ Hence,}$$

$$\|A\| = |a_{k'1}| + |a_{k'2}| = \max_k \{|a_{k1}| + |a_{k2}|\}.$$

We note that

$$A^{-1} = \begin{pmatrix} a_{22}/D & -a_{12}/D \\ -a_{21}/D & a_{11}/D \end{pmatrix},$$

where $D = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Bilinear operators in X are the three dimensional matrices

$$B = \left(\begin{array}{cc|cc} b_{111} & b_{112} & b_{121} & b_{122} \\ b_{211} & b_{212} & b_{221} & b_{222} \end{array} \right),$$

where Bx is the linear operator defined by

$$Bx = \begin{pmatrix} b_{111}x_1 + b_{112}x_2 & b_{121}x_1 + b_{122}x_2 \\ b_{211}x_1 + b_{212}x_2 & b_{221}x_1 + b_{222}x_2 \end{pmatrix}.$$

If $y = (y_1, y_2)$ we have

$$\begin{aligned} ||Bxy|| &= \max_k |b_{k11}x_1y_1 + b_{k12}x_2y_1 + b_{k21}x_1y_2 + b_{k22}x_2y_2| \\ &\leq \max_k (|b_{k11}| + |b_{k12}| + |b_{k21}| + |b_{k22}|) \cdot ||x|| \cdot ||y||, \text{ so} \\ \text{that } ||B|| &\leq \max_k (|b_{k11}| + |b_{k12}| + |b_{k21}| + |b_{k22}|). \end{aligned}$$

With these definitions and given

$$B = \left(\begin{array}{cc|cc} 0 & 0.2 & 0.5 & 0 \\ 0.5 & 0.1 & 0 & 0.1 \end{array} \right), \quad A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad y = (0.414, 0.426)$$

consider the equation $Ax + Bxx = y$. We have

$$A^{-1} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.25 \end{pmatrix},$$

and $(\|A^{-1}\|)^2 \cdot \|B\| \cdot \|y\| \leq (1/4) (0.7) (0.426) = \gamma$ or $\gamma = 0.07455 < 1/4$. By (3.4) the equation, therefore, has an operational continued fraction solution. We obtain $F_1 = (0.207, 0.1065)$, $F_2 = (0.19967, 0.10053)$, and $F_3 = (0.19996, 0.100076)$, where $\|F_3 - x_0\| \leq 0.00094$. Further approximation will give us the root $x_0 = (0.2, 0.1)$.

Although we shall not examine the details here, the Chandrasekhar integral equation may also be considered an example of a second order equation to be considered in terms of an operational continued fraction.

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