AN ABSTRACT OF THE THESIS OF

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Title:

A Study of the Differential Equation

$$(1 - x^2) \frac{d^2z}{dx^2} - 3x \frac{dz}{dx} + (m^2 - 1)z = 0$$

Abstract Approved: Redacted for privacy
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The purpose of this paper is an investigation of the properties of the differential equation

$$(1) \qquad (1 - x^2) \frac{d^2z}{dx^2} - 3x \frac{dz}{dx} + (m^2 - 1)z = 0.$$

Many of the problems met in Mathematical Physics give rise to a type of homogeneous linear differential equation of the second order having variable coefficients which are usually rather simple functions. Some very well-known equations in this class are Legendre's and Bessel's equations about which whole books have been written. The polynomial solutions of Legendre's equation are called zonal harmonics while those of Bessel's equation are called cylindrical harmonics.

The equation treated in this paper bears an outward resemblance to that of Legendre, although no direct relation has been discovered. It has a set of polynomial solutions as does Legendre's equation. They both admit solutions of the type $z = a_0 p_m(x) + a_1 q_m(x)$, where $p_m(x)$ and $q_m(x)$ are power series and the coefficients a_0 and a_1 are arbitrary.

The polynomial solutions are obtained by giving to the m, positive

integral values. They are designated $M_n(x)$ and correspond to those similarly obtained and called $P_n(x)$ in the case of Legendre's equation.

The polynomials $M_n(x)$ satisfy the relations

$$\int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}} M_m(x) M_n(x)\,dx = \begin{cases} 0 \text{ if } n \neq m, \\ \dfrac{\pi}{2n^2} \text{ if } n = m, \end{cases}$$

while in the case of Legendre's polynomials, $P_n(x)$ satisfies

$$\int_{-1}^{+1} P_m(x) P_n(x)\,dx = \begin{cases} 0 \text{ if } n \neq m, \\ \dfrac{2}{2n + 1} \text{ if } n = m. \end{cases}$$

The process used by Rodrigues in developing his formula for the polynomials of Legendre, in the case of equation (1) yields a formula for those solutions which have an infinite number of terms.

$$z\,(x) = C \frac{d^m}{dx^m} (1 - x^2)^{m-\frac{1}{2}}.$$

By use of the following formulas $M_n(x)$ may be immediately obtained, simply by varying the m of the expressions

$$M_m(x) = \frac{(-1)^{\frac{m-2}{2}}}{m} \frac{\sin m(\sin\ x)}{\sqrt{1 - x^2}}, \text{ for m an even integer,}$$

$$M_m(x) = \frac{(-1)^{\frac{m-1}{2}}}{m} \frac{\cos m(\sin\ x)}{\sqrt{1 - x^2}}, \text{ for m an odd integer.}$$

A resemblance between the generating function found by Legendre and that obtained for equation (1) is evident

$$(1 - 2xS + S^2)^{-\frac{1}{2}} = P_0 + P_1 S + P_2 S^2 \ldots\ldots\ldots\ldots\ldots$$

and

$$(1 - 2xr + r^2)^{-1} = M_1 + 2M_2 r + 3M_3 r^2 \ldots\ldots\ldots\ldots$$

The recursion formulas are

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x),$$

and

$$(n + 1)M_{n+1}(x) = 2nxM_n(x) - (n - 1)M_{n-1}(x).$$

The zeros of both sets of polynomials are all real and all lie in the interval -1 to +1.

A given function can be formally expanded in terms of either set of polynomials. In the case of $M(x)$, the convergence of the resultant series resolves into a consideration of the convergence of a Fourier Series. This has been established by Dirichlet.

There is material for further investigation in this problem. The relation of the equation to the hypergeometric equation, the properties of those solutions which do not terminate with integral values of m, the asymptotic formulas involved, and other topics have not been touched upon.

A STUDY OF THE DIFFERENTIAL EQUATION

$$(1 - x^2)\frac{d^2z}{dx^2} - 3x\frac{dz}{dx} + (m^2 - 1)z = 0$$

by

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$$(1 - x^2)\frac{d^2z}{dx^2} - 3x\frac{dz}{dx} + (m^2 - 1)z = 0$$

INTRODUCTION

I. Ordinary linear differential equations of the second order have received a greater amount of attention from investigators than has any other type of differential equation. This distinction is due to the special properties of linear equations, which make them a fruitful and enticing field of study for their own sake, and also because of the important role which they play in the field of mathematical physics. By means of ordinary linear differential equations of the second order it is possible to construct solutions of many important boundary problems associated with the following basic partial differential equations in the field of mathematical physics:

1. Laplace's Equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

2. The Wave Equation

$$c^2\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) = \frac{\partial^2 V}{\partial t^2}.$$

3. The Heat Equation

$$k\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) = \frac{\partial V}{\partial t}.$$

4. The Equation of Telegraphy

$$LK\frac{\partial^2 V}{\partial t^2} + KR\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$$

II. Certain differential equations of the second order have become especially well-known and have been the object of a vast amount of detailed investigation. These are

1. Bessel's Equation

$$x^2\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + (x^2 - m^2)y = 0.$$

2. Lame's Equation

$$(x^2-b^2)(x^2-c^2)\frac{d^2z}{dx^2} + x(x^2-b^2+x^2-c^2)\frac{dz}{dx} - \left[m(m+1)x^2 - (b^2+c^2)p\right]z = 0.$$

3. Hermite's Equation

$$\frac{d}{dx}\left(e^{-\frac{x^2}{2}}\frac{dy}{dx}\right) + e^{-\frac{x^2}{2}}\,y = 0.$$

4. Legendre's Equation

$$(1 - x^2)\,\frac{d^2z}{dx^2} - 2x\,\frac{dz}{dx} + M(M + 1)z = 0.$$

III. It is the object of this thesis to present a somewhat detailed study of the properties of the following homogeneous linear differential equation of the second order:

$$(1 - x^2)\,\frac{d^2z}{dx^2} - 3x\,\frac{dz}{dx} + (m^2 - 1)z = 0.$$

So far as the writer is aware, this equation is not particularly well-known. It obviously bears a marked resemblance to the equation of Legendre, and we may therefore anticipate considerable similarity in the properties, which is indeed found to be the case. As in the solution of Legendre's equation it is found that we get two linearly independent solutions whose sum is $Z = a_0p_m(x) + a_1q_m(x)$, where $p_m(x)$ and $q\ (x)$ are two series, one in terms of the even powers of x and the other in odd powers, both converbent in the interval -1 to +1. When m is a positive integer, one of the solutions reduces to a polynomial. Thus a set of polynomials results which we call $M_n(x)$ and which are analogous to the $P_n(x)$ of Legendre. The generating functions for the two bear a resemblance, $\frac{1}{(1 - 2xS + S^2)^{\frac{1}{2}}}$ giving $P_n(x)$ as the coefficients of S in this expansion and $\frac{1}{1 - 2xr + r^2}$ giving similarly $M_n(x)$ as the coefficient of r in this expansion. All of the roots of both $P_n(x)$ and $M_m(x)$ are real and all lie in the interval -1 to +1. The other solution in each case is not a polynomial. The process yielding a formula for the polynomials in the case of Legendre's solutions, gives a formula which generates the non-terminating forms in our solution. The polynomials, $M_n(x)$, are given by a trigonometric formula. The relations existing between the polynomials $P_n(x)$ and $M_n(x)$ respectively are

$$(n + 1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x),$$

$$(n + 1)M_{n+1}(x) = 2nxM_n(x) - (n - 1)M_{n-1}(x).$$

A given function can be formally expanded in terms of either set of polynomials. In the case of $M_n(x)$, the convergence of the resultant series resolves into a consideration of the convergence of a Fourier Series. This, Dirichlet has established.

There is material for further investigation in this problem. The relation of the equation to the hypergeometric equation, the properties of those solutions which do not terminate with integral values of m, the asymptotic formulas involved, and other topics have not been touched upon.

1. THE CANONICAL FORM

In order to reduce the equation

$$(1) \qquad (1 - x^2)\frac{d^2z}{dx^2} - 3x\frac{dz}{dx} + (m^2 - 1)z = 0$$

to the form

$$(2) \qquad \frac{d}{dx}\left(P\frac{dz}{dx}\right) + (k\lambda - q)\, z = 0,$$

we first divide each term of (1) by the coefficient of its first term, the result being

$$\frac{d^2z}{dx^2} - \frac{3x}{1 - x^2}\frac{dz}{dx} + \frac{(m^2 - 1)}{(1 - x^2)}\, z = 0.$$

We now compare this with the expanded form of (2) or

$$\frac{d^2z}{dx^2} + \frac{P'}{P}\frac{dz}{dx} + \frac{(k\lambda - q)}{P} z = 0$$

and discover that the following relation must exist between the coefficients if the two equations are to be similar

$$(3) \qquad \frac{P'}{P} = \frac{-3x}{1 - x^2}\,.$$

The solution of this is

$$\log P = \log (1 - x^2) + \tfrac{1}{2}\log (1 - x^2)$$

or

$$P = C(1 - x^2)^{\frac{3}{2}}$$

and from this determination we may now write (1) in the canonical form

$$(4) \qquad \frac{d}{dx}\left[(1 - x^2)^{\frac{3}{2}}\frac{dz}{dx}\right] + (m^2 - 1)(1 - x^2)^{\frac{1}{2}}\, z = 0.$$

2. SOLUTIONS IN POWER SERIES

Since from (4) we have

$$P = (1 - x^2)^{\frac{3}{2}}$$

$$\lambda = (m^2 - 1)$$

$$k = (1 - x^2)^{\frac{1}{2}}$$

$$q = 0$$

we may assume a solution in power series (Pierpont, Functions of a Complex Variable. Page 459, 17.) of the form

(5) $$z = \sum a_n x^n.$$

We next substitute this assumed solution in equation (1), the result being

$$\sum \left[(n + 1)(n + 2)a_{n+2} - \left\{(n + 1)^2 - m^2\right\} a_n \right] x^n = 0$$

whence

$$\left[(n + 1)(n + 2)a_{n+2} - \left\{(n + 1)^2 - m^2\right\} \right] a_n = 0$$

for $n = 0, 1, 2, 3, \ldots\ldots$.

The following relation therefore exists between the coefficients

(6) $$a_{n+2} = \frac{(n + 1)^2 - m^2}{(n + 1)(n + 2)}\, a_n.$$

Then for n an even integer

$$a_0 = a_0$$

$n = 0$ $$a_2 = \frac{1^2 - m^2}{2!}\, a_0$$

$n = 2$ $$a_4 = \frac{3^2 - m^2}{3 \cdot 4}\, a_2 = \frac{(1^2 - m^2)(3^2 - m^2)}{4!}\, a_0$$

$$m = 4 \qquad a_6 = \frac{5^2 - m^2}{5 \cdot 6} a_4 = \frac{(1^2 - m^2)(3^2 - m^2)(5^2 - m^2)}{6!} a_0$$

etc.,

and for n an odd integer

$$a_1 = a_1$$

$$n = 1 \qquad a_3 = \frac{2^2 - m^2}{2 \cdot 3} a_1 = \frac{2^2 - m^2}{2 \cdot 3} a_1$$

$$n = 3 \qquad a_5 = \frac{4^2 - m^2}{4 \cdot 5} a_3 = \frac{(2^2 - m^2)(4^2 - m^2)}{5!} a_1$$

$$n = 5 \qquad a_7 = \frac{6^2 - m^2}{6 \cdot 7} a_5 = \frac{(2^2 - m^2)(4^2 - m^2)(6^2 - m^2)}{7!} a_1$$

etc.,

We now substitute these values of the coefficients in the assumed solution

$$z = \sum a_n x^n$$

obtaining

$$(7) \qquad z = a_0 \Big[1 + \frac{1^2 - m^2}{2!} x^2 + \frac{(1^2 - m^2)(3^2 - m^2)}{4!} x^4 + + \frac{(1^2 - m^2)(3^2 - m^2)(5^2 - m^2)}{6!} x^6 \ldots\ldots + \frac{(1^2 - m^2)(3^2 - m^2) \ldots\ldots \{(n-1)^2 - m^2\}}{n!} x^n + \frac{(1^2 - m^2)(3^2 - m^2) \ldots\ldots \{(n-1)^2 - m^2\}\{(n+1)^2 - m^2\}}{(n+2)!} x^{n+2} \ldots\ldots \Big]$$

$$+ a_1 \Big[x + \frac{2^2 - m^2}{3!} x^3 + \frac{(2^2 - m^2)(4^2 - m^2)}{5!} x^5 + + \frac{(2^2 - m^2)(4^2 - m^2)(6^2 - m^2)}{7!} x^7 \ldots\ldots + \frac{(2^2 - m^2)(4^2 - m^2) \ldots\ldots \{(n-1)^2 - m^2\}\{(n+1)^2 - m^2\}}{(n+2)!} x^{n+2} \ldots\ldots \Big]$$

If we call the first series $p_m(x)$ and the second $q_m(x)$, the solution of the given equation in the form of a power series is:

$$z = a_0 p_m(x) + a_1 q_m(x), \tag{8}$$

a_0 and a_1 being the two arbitrary constants required by the order of the equation (1).

3. CONVERGENCE OF THE POWER SERIES

Applying the ratio test

$$\rho = \lim_{n\to\infty} \frac{a_{n+2}}{a_n} x^2 = x^2 \lim_{n\to\infty} \frac{a_{n+2}}{a_n} = x^2 \cdot L$$

which signifies convergence if $\rho < 1$, and divergence if $\rho > 1$. We test each series separately, taking $p_m(x)$ first:

$$\rho = \lim_{n\to\infty} \frac{\dfrac{(1^2 - m^2)(3^2 - m^2) \;....\; (n-1)^2 - m^2 \quad (n+1)^2 - m^2}{(n+2)!}}{\dfrac{(1^2 - m^2)(3^2 - m^2) \;.....\; (n-1)^2 - m^2}{n!}}$$

$$\rho = \lim_{n\to\infty} \frac{(n+1)^2 - m^2}{(n+2)(n+1)} = \lim_{m\to\infty} \frac{1 - \frac{2}{n} + \frac{1}{n^2} - \frac{m^2}{n^2}}{1 + \frac{3}{n^2} + \frac{2}{n^2}} = 1.$$

So $\rho = x^2 L = x^2(1)$ and in order that this be < 1, x must lie in the interval

$$-\frac{1}{|L|} < x^2 < \frac{1}{|L|} \quad \text{or} \quad -1 < x^2 < 1.$$

Hence $p_m(x)$ is convergent within the interval $-1 < x^2 < 1$.

We find by identical reasonin that $q_m(x)$ has the same property. This radius of convergence we might have predicted since the coefficient of the highest order derivative in (1) has a singularity at $x = \pm 1$.

4. THE TERMINATING SOLUTIONS

The series terminate for positive integral values of m, $p_m(x)$ for odd integers and $q_m(x)$ for even integers. This never happens simultaneously so we have one term which becomes a finite polynomial while the other remains an infinite series for every integral value of m. The first few are given below

m	$p_m(x)$	$q_m(x)$
0	$(1 - x^2)^{-\frac{1}{2}}$	....
1	1	$x(1 - x^2)^{-\frac{1}{2}}$
2	$(1 - 2x^2)(1 - x^2)^{-\frac{1}{2}}$	x
3	$(1 - 4x^2)$	$\frac{1}{3}(3x - 4x^3)(1 - x^2)^{-\frac{1}{2}}$
4	$21(4x^2 + 3)(1 - x^2)^{-\frac{1}{2}}$	$(x - 2x^3)$
5	$1 - 12x^2 + 16x^4$	....
6	....	....
7	....	....

5. THE POLYNOMIAL SOLUTIONS

We so determine the terminating forms that $z = 1$ when $x = 1$ and designate the resultant polynomials as $M_n(x)$:

$$
\begin{aligned}
M_1(x) &= 1 \\
M_2(x) &= x \\
M_3(x) &= \frac{4x^2 - 1}{3} \\
M_4(x) &= 2x^3 - x \\
M_5(x) &= \frac{16x^4 - 12x^2 + 1}{5} \\
M_6(x) &= \frac{16x^5 - 16x^3 + 3x}{3} \qquad (9) \\
M_7(x) &= \frac{64x^6 - 80x^4 + 24x^2 - 1}{7} \\
M_8(x) &= 16x^7 - 24x^5 + 10x^3 - x \\
M_9(x) &= \frac{256x^8 - 448x^6 + 240x^4 - 40x^2 + 1}{9} \\
M_{10}(x) &= \frac{256x^9 - 512x^7 + 336x^5 - 80x^3 + 5x}{5}
\end{aligned}
$$

6. THE FORMULA FOR THE SOLUTIONS CONTAINING AN INFINITE NUMBER OF TERMS

If we take

$$0. \qquad (1 - x^2)\frac{d^2z_0}{dx^2} - 3kx\frac{dz_0}{dx} + qz_0 = 0$$

and differentiate it with respect to x, the result is

$$(1 - x^2)\frac{d^3z_0}{dx^3} - (2 + 3k)x\frac{d^2z_0}{dx^2} + (q - 3k)\frac{dz_0}{dx} = 0.$$

In this we set

$$\frac{dz_0}{dx} = z_1$$

getting

$$1. \qquad (1 - x^2)\frac{d^2z_1}{dx^2} - (2 + 3k)x\frac{dz_1}{dx} + (q - 3k)z_1 = 0$$

which we differentiate with respect to x and then collect terms, the result being

$$(1 - x^2)\frac{d^3z_1}{dx^3} - (2 + 2 + 3k)x\frac{d^2z_1}{dx^2} + (q - 6k - 2)\frac{dz_1}{dx} = 0.$$

Here again we set

$$\frac{dz_1}{dx} = z_2$$

$$2. \qquad (1 - x^2)\frac{d^2z_2}{dx^2} - (4 + 3k)x\frac{dz_2}{dx} + (q - 6k - 2)z_2 = 0.$$

The nth such differentiation and substitution results in

$$n. \qquad (1 - x^2)\frac{d^2z_n}{dx^2} - (2n + 3k)x\frac{dz_n}{dx} + (q - 3nk - n(n - 1))z_m = 0.$$

In order to make the original equation conform to the form n we now must equate coefficients of like terms in z:

$$(2n + 3k) = 3$$

$$k = \frac{3 - 2n}{3}.$$

Using this value of k after equating the last coefficients the result becomes

$$q - 3n(\frac{3 - 2n}{3}) - n(n - 1) = m^2 - 1$$

$$q + n(n - 2) = m^2 - 1.$$

If we next suppose $q = 0$ and $n(n - 2) = m^2 - 1$, then equation n is exactly the original equation (1). When $n(n - 2) = m^2 - 1$, we have the value $m + 1$ for n, and hence

$$k = \frac{3 - 2n}{3} = \frac{3 - 2(m + 1)}{3} = \frac{1 - 2m}{3}.$$

Upon substitution of these values in 0, we obtain the followind differential equation

$$(1 - x^2)\frac{d^2z_0}{dx^2} - 3(\frac{1 - 2m}{3})x\frac{dz_0}{dx} = 0$$

or

$$(1 - x^2)\frac{d^2z_0}{dx^2} - (1 - 2m)x\frac{dz_0}{dx} = 0.$$

Starting with this and differentiating n times we arrive at the equation (1). We may better write the last equation in the form

(10) $$(1 - x^2)\frac{dz_1}{dx} + (1 - 2m)xz_1 = 0$$

which can be solved as follows:

$$\int\frac{dz_1}{z_1} = -\tfrac{1}{2}\int\frac{-2x}{1 - x^2}\,dx + m\int\frac{-2x}{1 - x^2}\,dx$$

$$z_1 = C(1 - x^2)^{-\frac{1}{2}}(1 - x^2)^m = C(1 - x^2)^{m-\frac{1}{2}} .$$

The final form is then

$$(11) \qquad z_m = C \frac{d^m}{dx^m} (1 - x^2)^{m-\frac{1}{2}}$$

a formula which may be used to write out the solutions which possess an infinite number of terms.

The process whereby this formula was obtained is that used by Rodrigues to derive his formula for the Legendre Polynomials which are the solutions having a finite number of terms:

$$P_m(x) = \frac{1}{2^m \, m!} \frac{d^m}{dx^m} (x^2 - 1)^m.$$

7, FORMULAS GIVING THE POLYNOMIALS $M_n(x)$

We transform the equation (1) to the form $P \frac{d}{dx}(P \frac{dy}{dx}) + m^2y = 0$ by the substitution

$$z = \frac{y}{(1 - x^2)^{\frac{1}{2}}}$$

the result being

$$(1 - x^2)^{\frac{1}{2}} \frac{d}{dx}\left[(1 - x^2)^{\frac{1}{2}} \frac{dy}{dx}\right] + m^2y = 0.$$

Let us now make the change $\frac{P}{dx} = \frac{1}{dS}$ which in our case is

$$\frac{(1 - x^2)^{\frac{1}{2}}}{dx} = \frac{1}{dS}$$

and we obtain

$$\frac{d}{dS}(\frac{dy}{dS}) + m^2y = 0, \quad \text{or} \quad \frac{d^2y}{dS^2} + m^2y = 0,$$

the solution of which is

$$y = A \sin mS + B \cos mS.$$

Also we have the relation

$$\frac{dx}{(1 - x^2)^{\frac{1}{2}}} = dS$$

from the above substitution, which gives

$$S = \sin^{-1}x,$$

making the final formula

$$(12) \qquad z = \frac{A \sin m(\sin^{-1}x) + B \cos m(\sin^{-1}x)}{\sqrt{1 - x^2}}.$$

We notice in the following computations for m an odd integer and for m an even integer m even

2	$z_2 = M_2(x)$	if	$A = \frac{1}{2}$,	$B = 0$
4	$z_4 = M_4(x)$	if	$A = -\frac{1}{4}$,	$B = 0$
6	$z_6 = M_6(x)$	if	$A = \frac{1}{6}$,	$B = 0$
8	$z_8 = M_8(x)$	if	$A = -\frac{1}{8}$,	$B = 0$
10	$z_{10} = M_{10}(x)$	if	$A = \frac{1}{10}$,	$B = 0$
etc.	....			

n odd

1	$z_1 = M_1(x)$	if	$A = 0$,	$B = 1$
3	$z_3 = M_3(x)$	if	$A = 0$,	$B = -\frac{1}{3}$
5	$z_5 = M_5(x)$	if	$A = 0$,	$B = \frac{1}{5}$
7	$z_7 = M_7(x)$	if	$A = 0$,	$B = -\frac{1}{7}$
9	$z_9 = M_9(x)$	if	$A = 0$,	$B = \frac{1}{9}$
etc.	....			

arising in the use of the formula (12) that the constants A and B must be determined in such manner that for m even

$$M_m(x) = \frac{(-1)^{\frac{m-2}{2}}}{m} \frac{\sin m(\sin^{-1}x)}{\sqrt{1 - x^2}} \tag{13a}$$

and for m odd

$$M_m(x) = \frac{(-1)^{\frac{m-1}{2}}}{m} \frac{\cos m(\sin^{-1}x)}{\sqrt{1 - x^2}} \tag{13b}$$

m being always a positive integer.

8. THE CASE WHERE M IS NOT AN INTEGER

If the formulas (13a) and (13b) which were derived for the case where m is an integer, satisfy the given differential equation (1), then they will give the solution when m is not an integer as well. In the latter case the solutions will not, however, be finite at +1 and -1.

Upon substitution of the two formulas (13) in (1) we find the equation satisfied and hence they are valid where m has any value, whether integral or fractional, real or complex.

9. THE GENERATING FUNCTION

If in formulas (13) we set $\phi = \sin^{-1}x$, we find that

$$M_m = \frac{\sin m\phi}{\cos \phi} = M_{2n} = \frac{\sin 2n\phi}{\cos \phi}\text{, m being an even integer}$$

and

$$M = \frac{\cos m\phi}{\cos \phi} = M_{2n+1} = \frac{\cos (2n+1)\phi}{\cos \phi}\text{, m being an odd}$$

integer.

Let us expand the function $\frac{1}{1 + iz}$:

$$\frac{1}{1 + iz} = 1 - iz + (iz)^2 - (iz)^3 + (iz)^4 - (iz)^5 \ldots\ldots\ldots\ldots$$

When we set $z = re^{i\phi}$ in this the result becomes:

$$\begin{aligned}
&= 1 - ire^{i\phi} + (ire^{i\phi})^2 - (ire^{i\phi})^3 + (ire^{i\phi})^4 - (ire^{i\phi})^5 \ldots \\
&= 1 - ire^{i\phi} - r^2e^{2i\phi} + ir^3e^{3i\phi} + r^4e^{4i\phi} - ir^5e^{5i\phi} \ldots\ldots \\
&= 1 - ir(\cos \phi + i \sin \phi) - r^2(\cos 2\phi + i \sin 2\phi) \\
&\quad + ir^3(\cos 3\phi + i \sin 3\phi) + r^4(\cos 4\phi + i \sin 4\phi) \ldots\ldots \\
&= 1 - r(i \cos \phi - \sin \phi) - r^2(\cos 2\phi + i \sin 2\phi) \\
&\quad + r^3(i \cos 3\phi - \sin 3\phi) + r^4(\cos 4\phi + i \sin 4\phi) \ldots\ldots \\
&= 1 + r \sin \phi - r^2\cos 2\phi - r^3\sin 3\phi + r^4\cos 4\phi \ldots\ldots \\
&\quad + i(-r \cos \phi - r^2\sin 2\phi + r^3\cos 3\phi + r^4\sin 4\phi \ldots\ldots)
\end{aligned}$$

Also we know that

$$\frac{1}{1 + iz} = \frac{1}{1 + ir(\cos \phi + i \sin \phi)} = \frac{1}{(1 - r \sin \phi) + ir \cos \phi}$$

$$= \frac{1 + r \sin \phi - ir \cos \phi}{1 - 2r \sin \phi + r^2}.$$

Now when we equate real parts from these two expressions:

$$\frac{1 + r \sin \phi}{1 - 2r \sin \phi + r^2} = 1 + r \sin \phi - r^2 \cos 2\phi - r^3 \sin 3\phi + r^4 \cos 4\phi \ldots.$$

and similarly imaginary parts:

$$\frac{- r \cos \phi}{1 - 2r \sin \phi + r^2} = - r \cos \phi - r^2 \sin 2\phi + r^3 \cos 3\phi + r^4 \sin 4\phi \ldots\ldots$$

In our case we make use of the latter of these two, dividing through first by $(-r \cos \phi)$ and obtaining:

$$\frac{1}{1 - 2r \sin \phi + r^2} = 1 + r \frac{\sin 2\phi}{\cos \phi} - r^2 \frac{\cos 3\phi}{\cos \phi} - r^3 \frac{\sin 4\phi}{\cos \phi} - r^4 \frac{\cos 5\phi}{\cos \phi} \ldots$$

Upon setting $\sin \phi = x$, in this we arrive at the following:

$$(14) \quad \frac{1}{1 - 2xr + r^2} = 1 + (2x)r + (4x^2 - 1)r^2 + (8x^3 - 4x)r^3 + (1 - 12x^2 + 16x^4)r^4 + (32x^5 - 32x^3 + 6x)r^5 + (64x^6 - 80x^4 + 24x^2 - 1)r^6 + (128x^7 - 192x^5 + 80x^3 - 8x)r^7 + (256x^8 - 448x^6 + 240x^4 - 40x^2 + 1)r^8 + (512x^9 - 1024x^7 + 672x^5 - 80x^3 + 10x)r^9 \ldots\ldots$$

$$= M_1(x) + 2M_2(x)\cdot r + 3M_3(x)\cdot r^2 + 4M_4(x)\cdot r^3 + 5M_5(x)\cdot r^4 + 6M_6(x)\cdot r^5 + 7M_7(x)\cdot r^6 \ldots\ldots$$

Thus we find that $\frac{1}{1 - 2xr + r^2}$ is the generating function for the polynomials $M_n(x)$.

10. THE RECURSION FORMULA

We again consider the expression (14). Let us now differentiate both sides of the equation with respect to r:

$$-\frac{2(r - x)}{(1 - 2xr + r^2)^2} = 2\ M_2 + 6\ M_3 r + 12\ M_4 r^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(n - 1)(n - 2)\ M_{n-1} r^{n-3} + n(n - 1)\ M_n r^{n-2}$$
$$+ n(n + 1)\ M_{n+1} r^{n-1} + (n + 1)(n + 2)\ M_{n+2} r^n \ldots\ldots$$

We next multiply this through by $(1 - 2xr + r^2)$ and then substitute in the left side the value of $\frac{1}{1 - 2xr + r^2}$ from (14). After this is done and after both sides of the resultant equation have been expanded, we then equate the coefficients of the like powers of r. For the general term this give

$$(n + 2)\ M_{n+2}(x) = 2x\ (n + 1)\ M_{n+1}(x) - n\ M_n(x)$$

and when in this we retard the subscripts by one we get

$$(15) \qquad (n + 1)\ M_{n+1}(x) = 2x\ (n)\ M_n(x) - (n - 1)\ M_{n-1}(x)$$

which is a simple and very usable recursion formula giving each $M(x)$ in terms of the two preceding it.

11. THE ZEROS OF $M_n(x)$

To investigate the zeros of the polynomials M (x) we consider forms (13). When we make the substitution $\sin^{-1}x = X$, (13a) becomes

(a) $$M_m(x) = \frac{(-1)^{\frac{m-2}{2}}}{m} \frac{\sin mX}{\cos X}, \text{ m being an even integer,}$$

and (13b) becomes

(b) $$M_m(x) = \frac{(-1)^{\frac{m-1}{2}}}{m} \frac{\cos mX}{\cos X}, \text{ m being an odd integer.}$$

From (a), $M_m(x)$ for m an even integer will have zeros when

$$\sin mX = 0$$

or when

(16) $$x = \sin \frac{n\pi}{m}, \; n = 0, 1, 2, 3, \ldots\ldots$$

and from b), M (x) for m an odd integer will have zeros when

$$\cos mX = 0$$

or when

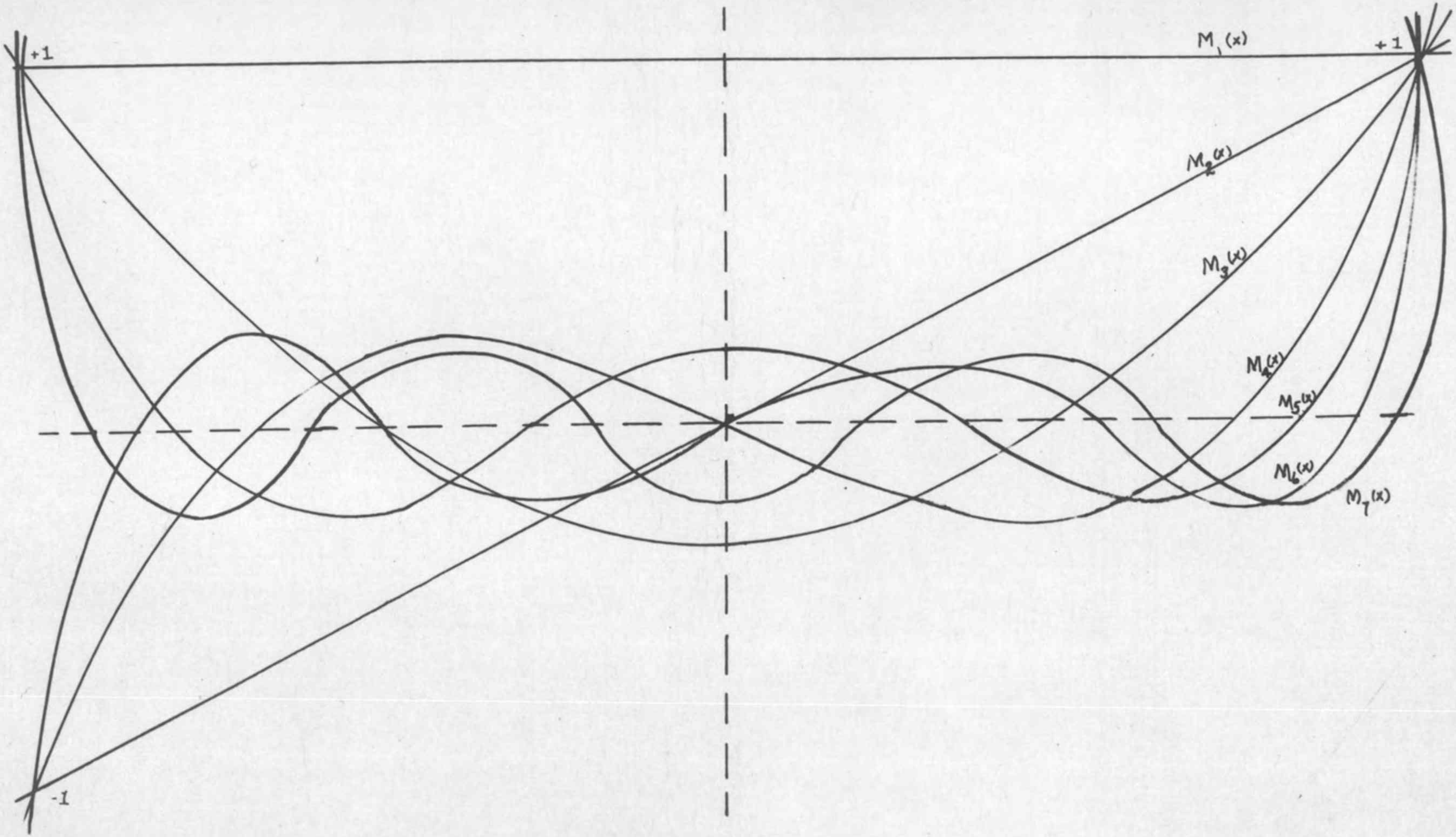
(17) $$x = \sin \frac{(2n+1)\pi}{2m}, \; n = 0, 1, 2, 3, \ldots\ldots$$

When we apply these formulas, remembering that the +1 and -1 values resulting are not valid because of the cosine term in the denominator, the following zeros are found:

m = 1	gives no zeros, since a constant
m = 2	0
m = 3	$\pm \frac{1}{2}$
m = 4	$0, \pm \frac{1}{2}\sqrt{2}$
m = 5	$\pm .3+, \pm .8+$

$m = 6$	$0, \pm \frac{1}{2}, \pm \frac{1}{2}\sqrt{3}$
....	....
$m = n$	gives $(n-1)$ zeros.

We see that $M_n(x)$ has then $(n-1)$ zeros, and since this agrees with the degree of the polynomial in each case we are certain that we have found all there are. It is evident since our formulas involve only sines and cosines that these roots all lie in the interval -1 to $+1$.



THE CURVES $Z = M_n(x)$

12. THE PROPERTY OF ORTHOGONALITY

Substituting the two solutions $M_m(x)$ and $M_n(x)$ in equation (1) we have

$$\frac{d}{dx}\left[(1 - x^2)^{3/2}\frac{dM_m}{dx}\right] + (m^2 - 1)(1 - x^2)^{\frac{1}{2}} M_m = 0$$

$$\frac{d}{dx}\left[(1 - x^2)^{3/2}\frac{dM_n}{dx}\right] + (n^2 - 1)(1 - x^2)^{\frac{1}{2}} M_n = 0.$$

If we multiply the first through by M_m, and the second by M_m, the following equations result

$$M_n \frac{d}{dx}\left[(1 - x^2)^{3/2}\frac{dM_m}{dx}\right] + (m^2 - 1)(1 - x^2)^{\frac{1}{2}} M_m M_n = 0$$

$$M_m \frac{d}{dx}\left[(1 - x^2)^{3/2}\frac{dM_n}{dx}\right] + (m^2 - 1)(1 - x^2)^{\frac{1}{2}} M_m M_n = 0.$$

We next subtract these two equations, multiply the results by dx and integrate each term between the limits -1 and +1:

$$\int_{-1}^{+1} M_n \frac{d}{dx}\left[(1 - x^2)^{3/2}\frac{dM_m}{dx}\right] - M_m \frac{d}{dx}\left[(1 - x^2)^{3/2}\frac{dM_n}{dx}\right] dx$$

$$+ \int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}}\left[(m^2 - 1 - n^2 + 1)\right] M_m M_n\, dx = 0$$

and

$$\left\{(1 - x^2)^{3/2}\left[M_n \frac{dM_m}{dx} - M_m \frac{dM_n}{dx}\right]\right\}_{-1}^{+1} + (m^2 - n^2)\int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}} M_m M_n\, dx = 0.$$

This leaves

$$(m^2 - n^2)\int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}} M_m(x)\, M_n(x)\, dx = 0$$

from which we see that for $m \neq n$ the following is true:

$$(18) \qquad \int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}} M_m(x)\, M_n(x)\, dx = 0$$

This is the property of orthogonality and is true only for those solutions which terminate.

13. THE VALUE OF THE INTEGRAL OF $M_n^2(x)$

We have established the fact that

$$(14) \qquad (1 - 2xr + r^2)^{-1} = M_1 + 2\,M_2 r + 3\,M_3 r^2 + 4\,M_4 r^3 \cdots\cdots\cdots (n + 1)\,M_{n+1}\,r^n \cdots\cdots\cdots$$

Let us square both sides of this equation and multiply through by $(1 - x^2)^{\frac{1}{2}}dx$ and integrate from -1 to +1:

$$\int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}}(1 - 2xr + r^2)^{-2}\,dx = \int_{-1}^{+1} M_1^2\,(1 - x^2)^{\frac{1}{2}}\,dx$$

$$+ 4\,r^2\int_{-1}^{+1} M_2^2\,(1 - x^2)^{\frac{1}{2}}\,dx + 9\,r^4\int_{-1}^{+1} M_3^2(1 - x^2)^{\frac{1}{2}}\,dx$$

$$\cdots\cdots\cdots(n + 1)^2\,r^{2n}\int_{-1}^{+1} M_{n+1}^2(1 - x^2)^{\frac{1}{2}}\,dx \cdots\cdots$$

+ (a group of other terms which are of the form

$$\int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}}\,M_m\,M_n\,dx,$$

and which are each equal to zero as we have found by the property of orthogonality).

When we perform the integration indicated on the left side of the above equation, the result is, after the limits have been substituted

$$\frac{\pi}{2}\left(\frac{1}{1 - r^2}\right) = \frac{\pi}{2}(1 + r^2 + r^4 + r^6 + \cdots\cdots\cdots\cdots\, r^{2n} \cdots\cdots).$$

Now when we equate the coefficients of the like powers of r from this expression and from the righthand side of the above equation, the result for the general term is

$$(n + 1)^2 \int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}} M_{n+1}^2(x)\, dx = \frac{\pi}{2}$$

whence solving for the integral and retarding the n by one

$$\int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}} M_n^2 (x)\, dx = \frac{\pi}{2n^2} \, . \tag{19}$$

14. THE EXPANSION OF A FUNCTION IN TERMS OF $M_n(x)$

It is desired to represent the function $f(x)$ in terms of the M's. We therefore assume that

$$(20) \qquad f(x) = A_1M_1 + A_2M_2 + A_3M_3 + A_4M_4 + A_5M_5 + \cdots\cdots\cdots A_nM_n \cdots\cdots$$

and then attempt to determine the coefficients, A_n.

First we multiply the equation through by $M\ (x)(1 - x^2)^{\frac{1}{2}}dx$ and integrate each term from -1 to +1:

$$\int_{-1}^{+1} f(x)(1 - x^2)^{\frac{1}{2}} M_n(x)\, dx = A_1\int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}} M_1(x)\, M_n(x)\, dx$$

$$+ A_2\int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}} M_2(x)\, M_n(x)\, dx \cdots\cdots\cdots A_n\int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}} M_n^2(x) dx$$

$$\cdots\cdots\cdots\cdots\cdots$$

By the property of orthogonality (18) all the terms on the righthand side of this equality are zero except the last which in (19) we have found equals

$$A_n\left(\frac{\pi}{2n^2}\right).$$

Upon solving this remaining equation for A_n we have as the formula for the general coefficient

$$(21) \qquad A_n = \frac{2n^2}{\pi}\int_{-1}^{+1} f(x)(1 - x^2)^{\frac{1}{2}} M_n(x)\, dx$$

which we may now substitute back into the assumed expansion above, in order to write out the expression for $f(x)$ in terms of $M_n(x)$:

$$(20) \qquad f(x) = \frac{2}{\pi}\int_{-1}^{+1} f(S)(1 - S^2)^{\frac{1}{2}} M_1(S)dS\, M_1(x)$$

$$+ \frac{2\cdot 2^2}{\pi}\int_{-1}^{+1} f(S)(1 - S^2)^{\frac{1}{2}} M_2(S)dS\, M_2(x) + \frac{2\cdot 3^2}{\pi}\int_{-1}^{+1} f(S)(1-S^2)^{\frac{1}{2}} M_3(S)dS\, M_3(x)$$

$$\cdots\cdots\cdots + \frac{2n^2}{\pi}\int_{-1}^{+1} f(S)(1 - S^2)^{\frac{1}{2}} M_n(S)dS\, M_n(x) \cdots\cdots\cdots$$

15. THE DIFFERENCE BETWEEN F(X) AND THE SUM OF THE FIRST N TERMS OF THE SERIES IN $M_n(x)$

We shall show that the difference between the function f(x) and the sum of the first n terms of its expansion in terms of $M_n(x)$ may be expressed as an integral.

Let us assume that the sum of the first n terms of the series in $M_n(x)$ is

$$S_n(x) = A_1M_1(x) + A_2M_2(x) \ldots\ldots\ldots\ldots\ldots\ldots A_nM_n(x) \ldots\ldots \tag{22}$$

$$= \sum_{k=1}^{n} A_kM_k(x) = \sum_{k=1}^{n} \frac{2k^2}{\pi}\int_{-1}^{+1} f(S)(1-S^2)^{\frac{1}{2}}M_k(S)dS\, M_k(x)$$

$M_k(x)$ is constant with respect to S so we may insert it under the radical sign:

$$= \frac{2}{\pi}\sum_{k=1}^{n}\int_{-1}^{+1} k^2 f(S)(1-S^2)^{\frac{1}{2}} M_k(S)\, M_k(x)\, dS$$

$$= \frac{2}{\pi}\int_{-1}^{+1} f(S)(1-S^2)^{\frac{1}{2}} \sum_{k=1}^{n} k^2 M_k(x)\, M_k(S)\, dS$$

$$= \frac{2}{\pi}\int_{-1}^{+1} f(S)(1-S^2)^{\frac{1}{2}} K_n(x,S)\, dS$$

where $K_n(x,S) = M_1(x)\, M_1(S) + 4M_2(x)\, M_2(S) + 9M_3(x)\, M_3(S) \ldots\ldots\ldots\ldots$

$$n^2\, M_n(x)\, M_n(S) ,$$

an expression having n terms. We now express $K_n(x,S)$ in two terms. From the recursion formula (15)

$$(n+1)\, M_{n+1}(x) = 2x\,(n)\, M_n(x) - (n-1)\, M_{n-1}(x)$$

for successive values of n we obtain

$$2M_2(x) = 2xM_1(x)$$
$$3M_3(x) = 4xM_2(x) - M_1(x)$$
$$4M_4(x) = 6xM_3(x) - 2M_2(x)$$
$$5M_5(x) = 8xM_4(x) - 3M_3(x)$$
$$6M_6(x) = 10xM_5(x) - 4M_4(x)$$
$$7M_7(x) = 12xM_6(x) - 5M_5(x)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(n + 1)M_{n+1}(x) = 2n\,(x)\,M_n(x) - (n - 1)M_{n-1}(x)$$

each of which we now multiply through respectively by $nM_n(S)$:

$$2M_1(S)M_2(x) = 2x\,M_1(S)M_1(x)$$
$$2\cdot 3M_2(S)M_3(x) = 2\cdot 4x\,M_2(S)M_2(x) - 1\cdot 2M_2(S)M_1(x)$$
$$3\cdot 4M_3(S)M_4(x) = 3\cdot 6x\,M_3(S)M_3(x) - 2\cdot 3M_3(S)M_2(x)$$
$$4\cdot 5M_4(S)M_5(x) = 4\cdot 8x\,M_4(S)M_4(x) - 3\cdot 4M_4(S)M_3(x)$$
$$5\cdot 6M_5(S)M_6(x) = 5\cdot 10x\,M_5(S)M_5(x) - 4\cdot 5M_5(S)M_4(x)$$
$$6\cdot 7M_6(S)M_7(x) = 6\cdot 12x\,M_6(S)M_6(x) - 5\cdot 6M_6(S)M_5(x)$$
$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots\cdots$$
$$n(n + 1)M_n(S)M_{n+1}(x) = 2\;2nx\,M_n(S)M_n(x) - n(n - 1)M_n(S)M_{n-1}(x)$$

We sum these now and our result, since

$$2x\,M_1(S)M_1(x) + 2\,M_2(S)M_2(x) + 3\,M_3(S)M_3(x) + 4\,M_4(S)M_4(x) \cdots\cdots$$
$$\cdots\cdots\cdots\; n\,M_n(S)M_n(x) = 2x\,K_n(x,S)$$

becomes

$$\text{(a)}\ \sum_{k=1}^{n} k(k + 1)M_{k+1}(x)M_k(S) = 2xK_n(x,S) - \sum_{k=1}^{n-1} k(k + 1)M_k(x)M_{k+1}(S)$$

As a next step we multiply each term respectively by $nM_n(x)$ where before we used $nM_n(S)$ as a multiplier, the present result being:

(b) $\sum_{k=1}^{n} k(k+1)M_{k+1}(S)M_k(x) = 2SK_n(x,S) - \sum_{k=1}^{n-1} k(k+1)M_k(S)M_{k+1}(x)$.

We see that

$$2xK_n(x,S) = \sum_{k=1}^{n} k(k+1)M_{k+1}(x)M_k(S) + \sum_{k=1}^{n-1} k(k+1)M_k(x)M_{k+1}(S)$$

$$= \sum_{k=1}^{n-1} k(k+1)M_{k+1}(x)M_k(S) + \sum_{k=1}^{n-1} k(k+1)M_k(x)M_{k+1}(S)$$

$$+ n(n+1)M_{n+1}(x)M_n(S),$$

and

$$2SK_n(x,S) = \sum_{k=1}^{n-1} k(k+1)M_{k+1}(S)M_k(x) + \sum_{k=1}^{n-1} k(k+1)M_{k+1}(x)M_k(S)$$

$$+n(n+1)M_n(x)M_{n+1}(S).$$

Upon subtracting (b) from (a) we obtain

$$2(x - S)K_n(x,S) = n(n+1)\left[M_{n+1}(x)M_n(S) - M_n(x)M_{n+1}(S)\right]$$

$$K_n(x,S) = \frac{n(n+1)\left[M_{n+1}(x)M_n(S) - M_n(x)M_{n+1}(S)\right]}{2(x - S)},$$

and therefore

(c) $$S_n(x) = \frac{2}{\pi}\int_{-1}^{+1} f(S)(1 - S^2)^{\frac{1}{2}} K_n(x,S)\, dS$$

$$= \frac{2}{\pi}\int_{-1}^{+1} f(S)(1 - S^2)^{\frac{1}{2}} \frac{n(n+1)\left[M_{n+1}(x)M_n(S) - M_n(x)M_{n+1}(S)\right]}{2(x - S)}\, dS$$

$$= \frac{n(n+1)}{\pi}\int_{-1}^{+1} f(S)(1 - S^2)^{\frac{1}{2}}\left[\frac{M_{n+1}(x)M_n(S) - M_n(x)M_{n+1}(S)}{x - S}\right] dS.$$

We hope to determine how great the difference is between the function and the sum of the first n terms of the series.

Again

$$S_n(x) = A_1M_1(x) + A_2M_2(x) + A_3M_3(x) + A_4M_4(x) + \ldots\ldots A_nM_n(x) \ldots$$

is the sum of the first n terms of the series. If $f(x) = C =$ a constant then

$$A_1 = \frac{2}{\pi}\int_{-1}^{+1} C(1 - x^2)^{\frac{1}{2}} M_n(x)\,dx$$

$$= \frac{2}{\pi}\Big[\ C\ \ x(1 - x^2)^{\frac{1}{2}} + \sin^{-1}x\ \Big]_{-1}^{+1} = C.$$

And all of the other coefficients

$$A_n = \frac{2n^2}{\pi}\int_{-1}^{+1} C(1 - x^2)^{\frac{1}{2}} M_n(x)\,dx$$

$$= \frac{2n^2}{\pi}\cdot C\int_{-1}^{+1}(1 - x^2)^{\frac{1}{2}} M_n(x)M_1(x)\,dx = 0$$

by the property of orthogonality. So

$$f(x) = C + 0 + 0 + 0 + 0 + 0 + 0 + \ldots\ldots..$$

Now if $f(S)$ be replaced by C, then the sum of the first n terms is equal to C or

$$C = \frac{n(n+1)}{\pi}\int_{-1}^{+1} f(x)(1 - x^2)^{\frac{1}{2}}\left[\frac{M_{n+1}(x)M_n(S) - M_n(x)M_{n+1}(S)}{x - S}\right] dS.$$

Since C is any constant, it may be chosen to equal $f(x)$, which is independent of our variable S. This will give

$$\text{(d)}\quad f(x) = \frac{n(n+1)}{\pi}\int_{-1}^{+1} f(x)(1 - S^2)^{\frac{1}{2}}\left[\frac{M_{n+1}(x)M_n(S) - M_n(x)M_{n+1}(S)}{x - S}\right] dS$$

We now subtract (c) from (d) and obtain as the difference between the function and the sum of the first n terms of the series in $M_n(x)$ the following integral:

$$f(x) - S_n(x) = \frac{n(n+1)}{\pi}\int_{-1}^{+1} \frac{f(x) - f(S)}{x - S}(1-S^2)^{\frac{1}{2}}\left[M_{n+1}(x)M_n(S) - M_n(x)M_{n+1}(S)\right]dS$$

which is similar to the expression obtained in the case of the Fourier expansion.

16. CONVERGENCE OF THE SERIES IN M

Let us consider the polynomials in the form

$$M_m = M_{2n} = \frac{(-1)^{n-1}}{2n} \frac{\sin 2n\theta}{\cos \theta}, \text{ for m an even integer}$$

and

$$M_m = M_{2n+1} = \frac{(-1)^n}{2n+1} \frac{\cos (2n+1)\theta}{\cos \theta}, \text{ for m an odd integer.}$$

Now if we expand the function $f(\sin \theta)$ in terms of these polynomials we obtain an expression

$$f(\sin \theta) = a_1M_1 + a_3M_3 + a_5M_5 + a_7M_7 + \cdots\cdots\cdots\; a_{2n+1}M_{2n+1}\cdots\cdots$$
$$+ b_2M_2 + b_4M_4 + b_6M_6 + \cdots\cdots\cdots\; b_{2n}M_{2n} \cdots\cdots$$

which, when we multiply throughout by the $\cos \theta$ appearing in every denominator of the righthand side, becomes

$$\cos \theta\, f(\sin \theta) = \text{an expression involving} \sum \frac{(-1)^n}{2n+1} a_{2n+1} \cos(2n+1)\theta$$
$$\text{and} \sum \frac{(-1)^{n-1}}{2n} a_{2n} \sin 2n\theta, \text{ n being any positive integer.}$$

We now employ the method used for determining the Fourier coefficients to determine our a's and b's in this expansion. The equation is first multiplied throughout by the coefficient of a_{2n+1} and integrated over the interval -1 to +1, term by term, then in a similar manner by the coefficient of b_{2n} and again integrated over the same interval. In this work we make use of the property of orthogonality and of the value of the integral of the square of M, from (19). To get these in usable form in this connection we set $x = \sin \theta$, $dx = \cos \theta\, d\theta$, $\sqrt{1 - x^2} = \cos \theta$, and for the M's, the values at the top of page 33 are used. In which case, the property of orthogonality becomes

$$\int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}} M_m(x)M_n(x)\, dx = \text{either } \frac{1}{mn}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos m\theta \cos n\theta\, d\theta$$

$$\text{or } \frac{1}{mn}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos m\theta \sin n\theta\, d\theta,$$

and (19) becomes

$$\int_{-1}^{+1} (1 - x^2)^{\frac{1}{2}} M_n^2(x)\, dx = \text{either } \frac{1}{m^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 m\theta\, d\theta = \frac{\pi}{2}\left(\frac{1}{2n+1}\right)^2$$

$$\text{or } \frac{1}{m^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 m\theta\, d\theta = \frac{\pi}{2}\left(\frac{1}{2n}\right)^2.$$

Using these values after the above indicated multiplications and integrations, we have remaining two equations, the first of which will solve for a_{2n+1} and the second for b_{2n}, giving us the following values for the general coefficients in our expansion:

$$a_{2n+1} = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, f(\sin\theta) \cos(2n+1)\theta\, d\theta$$

$$b_{2n} = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, f(\sin\theta) \sin(2n)\theta\, d\theta.$$

Let us now substitute these values for the a's and b's in the expansion for $\cos\theta\, f(\sin\theta)$, and then in the resulting expression set $\theta = 2x$, $d\theta = 2\, dx$, and $\cos\theta\, f(\sin\theta) = F(x)$, and we find that our series takes the form of a Fourier series:

$$F(x) = 2^2 \Bigg[\frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cos 2x \, dx \cdot \cos 2x$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cos 6x \, dx \cdot \cos 6x$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cos 10x \, dx \cdot \cos 10x$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cos 2(2n+1)x \, dx \cdot \cos 2(2n+1)x$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \sin 4x \, dx \cdot \sin 4x$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \sin 8x \, dx \cdot \sin 8x$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \sin 12x \, dx \cdot \sin 12x$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \sin 2(2n)x \, dx \cdot \sin 2(2n)x \cdots\cdots \Bigg]$$

The convergence of the Fourier Series to the value of the function F(x) has been rigorously established by Dirichlet (Carslaw, Fourier Series and Integrals, Page 210 and following). Hence the convergence of our series is likewise proved from his conulusions.

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