

GEOMETRICAL ASPECTS OF CERTAIN FIRST ORDER  
DIFFERENTIAL EQUATIONS

by

ARTHUR D WIRSHUP

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APPROVED:

[REDACTED]

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Professor of Mathematics

In Charge of Major

[REDACTED]

---

Head of Department of Mathematics

[REDACTED]

---

Chairman of School Graduate Committee

[REDACTED]

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Dean of Graduate School

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INTRODUCTION

The known geometry of the integral curves of various types of ordinary differential equations in two variables is a very sizeable body of material. Even the geometry of the integral curves of the elementary standard types of differential equations of the first order and first degree is considerable. This paper is accordingly limited chiefly to a study of some of the geometry of the integral curves of just two of the elementary types, the linear and the homogeneous first order differential equations. These two types are selected not only because of their own intrinsic interest but because the geometry of their integral curves throws special light on the behavior, in the neighborhood of both ordinary and singular points, of the solutions of the general first order first degree differential equation.

## LINEAR DIFFERENTIAL EQUATIONS

### 1. DIRECTION FIELD OF A LINEAR DIFFERENTIAL EQUATION

We first establish three theorems which will lead to a method of rapidly constructing the direction field of a first order linear differential equation.

**THEOREM 1.1** If we draw any two particular integral curves  $C_1$  and  $C_2$  of the linear differential equation

$$(1.1) \quad y' + a(x)y = f(x),$$

then the general integral curve of (1.1) is the locus of points which divide in a constant ratio the segments of the ordinates intercepted by  $C_1$  and  $C_2$  [6, pp.27-29].

If  $y = y_1(x)$  and  $y = y_2(x)$  are distinct particular solutions of (1.1) then it is known that

$$(1.2) \quad y = c\{y_2(x) - y_1(x)\} + y_1(x)$$

is the general solution of (1.1). If we designate the curves  $y = y_1(x)$  and  $y = y_2(x)$  by  $C_1$  and  $C_2$ , and the curve (1.2) by  $C_3$ , and if any ordinate cuts the curves  $C_1$ ,  $C_2$ ,  $C_3$  in  $P_1$ ,  $P_2$ ,  $P_3$ , then

$$P_1P_3/P_3P_2 = c/(1-c).$$

**THEOREM 1.2** Draw the line  $x = p$  and at the points of intersection of this line with the integral curves of

$$y' + a(x)y = f(x)$$

draw tangents to all the integral curves. All these

tangents either pass through the same point (which depends on p), or they are parallel [3, p.15; 4, p.306; 6, pp.27-29; 7, p.306].

Let  $C_1, C_2, C_3$ , be any three integral curves of the differential equation and let  $P_1, P_2, P_3; Q_1, Q_2, Q_3$ , be the points of intersection of these three curves with  $x = p$  and  $x = q$  respectively (see Figure 1). Then by theorem 1.1,

$$P_1 P_3 / P_3 P_2 = Q_1 Q_3 / Q_3 Q_2$$

and the chords  $P_1 Q_1, P_2 Q_2, P_3 Q_3$  are either concurrent or parallel. If we let  $q$  approach  $p$ , the chords approach as limiting positions the tangents to the curves at the points  $P_1, P_2, P_3$ . Hence these tangents are either concurrent or parallel, and the theorem is established.

**THEOREM 1.3** The coordinates of the point of concurrency of the tangents in theorem 1.2 are, if  $a(p) \neq 0$ ,

$$x = \{1 + pa(p)\} / a(p), \quad y = f(p) / a(p).$$

If  $a(p) = 0$  the tangents are parallel.

The coordinates of  $P_1$  and  $P_2$  are  $(p, y_1(p))$  and  $(p, y_2(p))$ , and the slopes of the tangents to  $C_1$  and  $C_2$  at  $P_1$  and  $P_2$  are

$$y_1'(p) = f(p) - a(p)y_1(p), \quad y_2'(p) = f(p) - a(p)y_2(p).$$

The equations of the two tangents then are

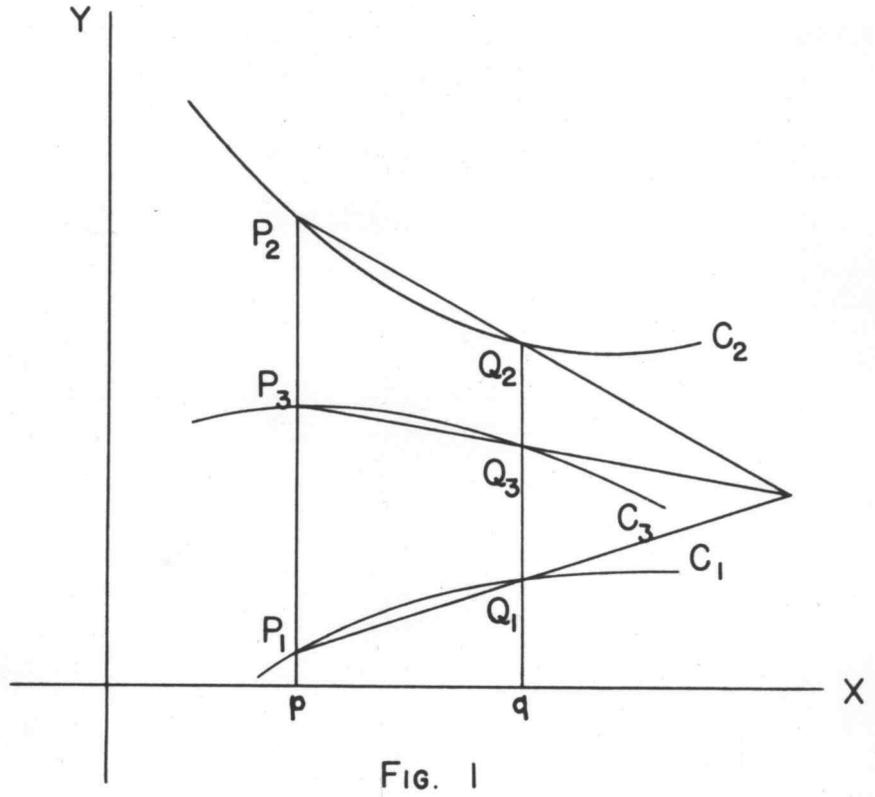


FIG. 1

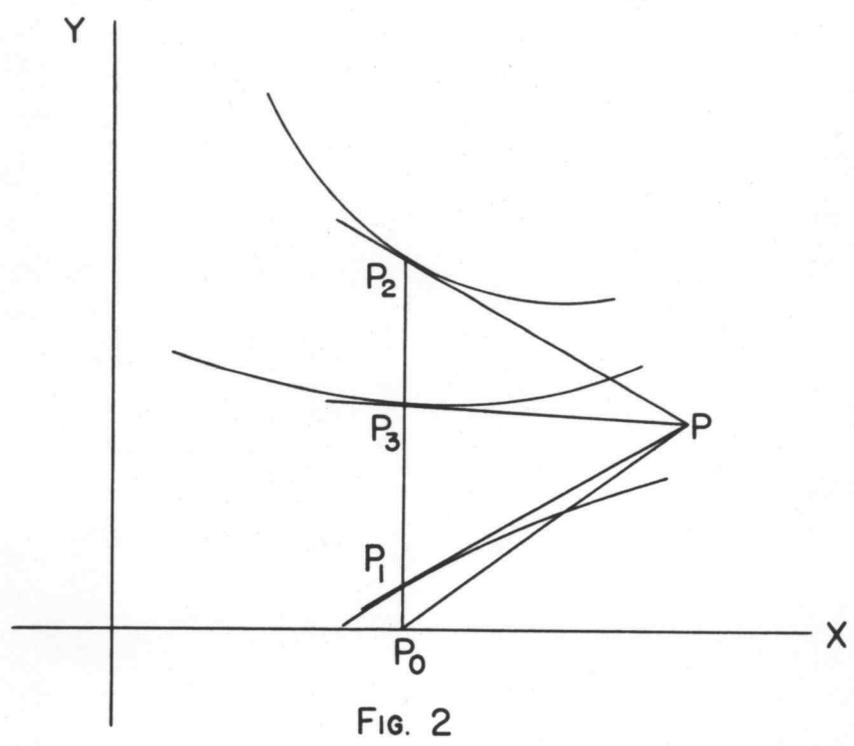


FIG. 2

$$y - y_1(p) = [f(p) - a(p)y_1(p)] (x-p),$$

$$y - y_2(p) = [f(p) - a(p)y_2(p)] (x-p).$$

Solving these simultaneously for  $x$  and  $y$  we find the desired results.

Note: Denote the point of concurrency of the tangents in theorem 1.2 by  $P$  and let  $P_0$  be the point  $(p, 0)$  (see Figure 2). Then it is easy to show that  $1/a(p)$  is the distance of  $P$  from the line  $x = p$  and that  $f(p)$  is the slope of the line  $P_0P$ . We thus have geometrical interpretations of  $a(p)$  and  $f(p)$ . If  $a(x) = k$ , a non-zero constant, then  $P$  is always the fixed distance  $1/k$  away from the line  $x = p$ .

Theorem 1.3 furnishes an excellent method for rapidly drawing the direction field, or linear element diagram, for a given first order linear differential equation [1, p.55]. To find the linear elements at points along a selected ordinate  $x = p$  we merely locate the associated point  $P$  by means of its coordinates as given by theorem 1.3. The linear elements at points along  $x = p$  are all directed toward the point  $P$ . Figures 3 and 4 illustrate the process for the direction fields of the linear differential equations  $y' + y/x = 3x$  and  $y' + y/x = 2$ . The direction field of the second equation illustrates the following interesting theorem concerning

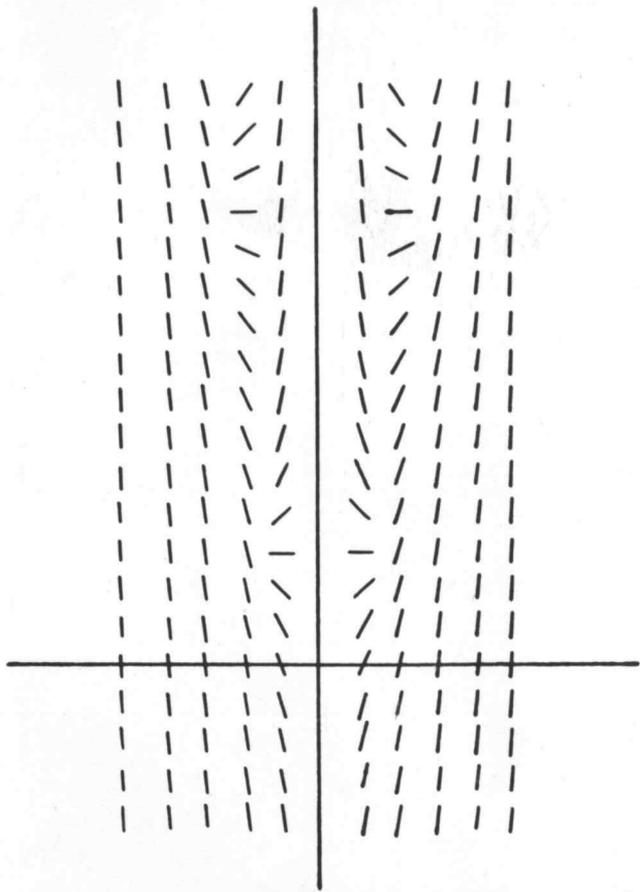


FIG. 3

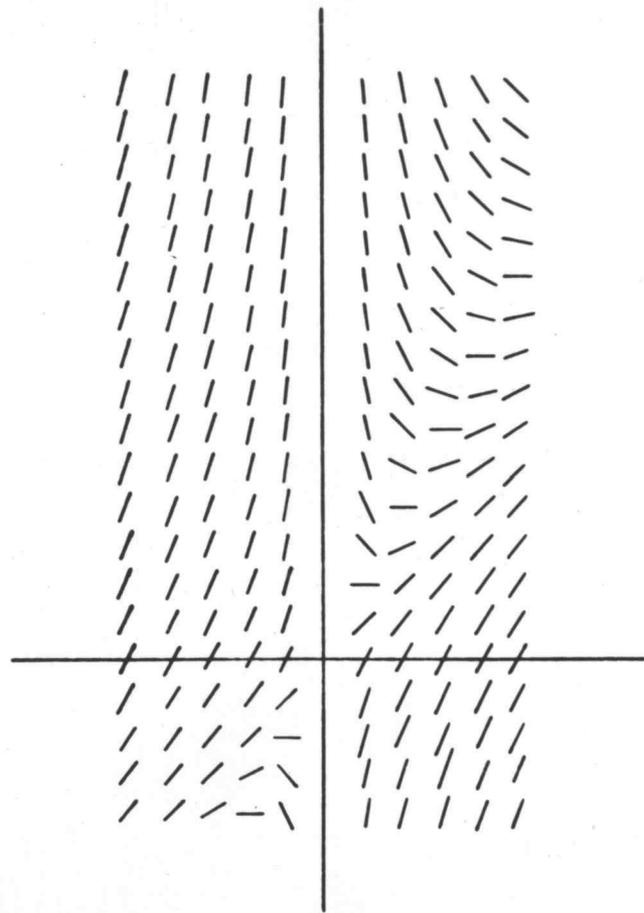


FIG. 4

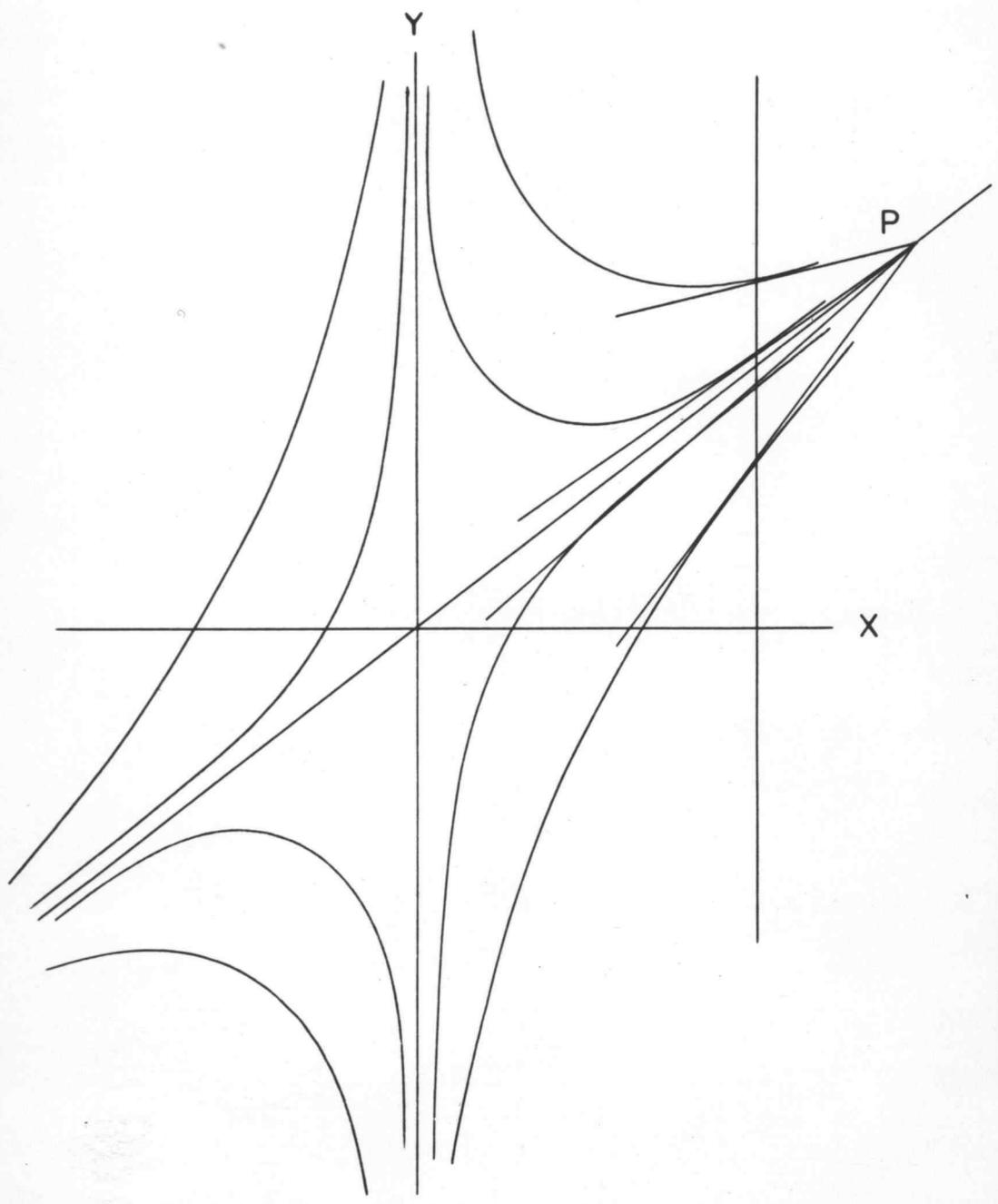


FIG. 5

a certain family of hyperbolas (see Figure 5).

**THEOREM 1.4** Given the family of hyperbolas having the same asymptotes. The points of contact of the tangents drawn to the hyperbolas from a point on one of the asymptotes lie on a line parallel to the other asymptote.

Take one asymptote of the family along the  $y$ -axis and let  $y = nx$  be the other asymptote. Then the family of hyperbolas is given by

$$x(y-nx) = c,$$

where  $c$  is an arbitrary parameter. The differential equation of this family of hyperbolas is found to be the linear equation

$$y' + y/x = 2n.$$

The coordinates of the point  $P$  of theorem 1.3 are then  $(2p, 2pn)$ , and  $P$  lies on the asymptote  $y = nx$ . This proves the theorem.

## 2. THE EQUATION LINEAR IN X AND Y.

Although the differential equation considered in this section is not necessarily linear, its integral curves satisfy a theorem somewhat analogous to theorem 1.1. Moreover, the important special linear equations considered in section 3 are also of this type. We first make a definition.

DEFINITION 2.1 The curves of equal slope of a first order differential equation are the loci of the points at which the tangents to the integral curves have the same slope.

We now have the obvious theorem:

THEOREM 2.1 The curves of equal slope of the differential equation

$$y = xf(y') + g(y')$$

are straight lines.

The analogue of theorem 1.1 is the following theorem:

THEOREM 2.2 If we draw any two particular integral curves  $C_1$  and  $C_2$  of the differential equation

$$(2.1) \quad y = xf(y') + g(y'),$$

then the general integral curve of this equation is the

locus of points which divide in a constant ratio the segments of the lines of equal slope intercepted by  $C_1$  and  $C_2$  [6, pp.27-29].

Taking differentials of both sides of (2.1) we find

$$y'dx = dx f(y') + x f'(y') dy' + g'(y') dy',$$

or,

$$(2.2) \quad dx/dy' - x f'(y')/(y' - f(y')) = g'(y')/(y' - f(y')),$$

a linear equation in  $x$  and  $dx/dy'$ . Let the general solution of (2.2) be

$$x = F(y', c),$$

where  $c$  is the arbitrary constant of integration. If we regard  $y'$  as a parameter then the general solution of (2.1) is given by

$$x = F(y', c), \quad y = F(y', c) f(y') + g(y').$$

Two particular solutions of (2.1) are obtained by setting  $c = c_1$  and  $c_2$ , giving

$$x_1 = F(y', c_1), \quad y_1 = x_1 f(y') + g(y'),$$

$$x_2 = F(y', c_2), \quad y_2 = x_2 f(y') + g(y').$$

But, since (2.2) is linear, its general solution in terms of the two particular solutions  $x_1 = F(y', c_1)$  and  $x_2 = F(y', c_2)$  is

$$x = c(x_2 - x_1) + x_1.$$

It follows that the general solution of (2.1) is given by

$$x = c(x_2 - x_1) + x_1,$$

and

$$\begin{aligned}y &= [c(x_2 - x_1) + x_1] f(y') + g(y') \\&= c[x_2 f(y') + g(y') - x_1 f(y') - g(y')] + x_1 f(y') + g(y') \\&= c(y_2 - y_1) + y_1,\end{aligned}$$

which shows that the segment  $P_1 P_2$  joining the points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  is divided by the point  $P(x, y)$  in a constant ratio independent of  $y'$ . This proves the theorem. An illustration of theorem 2.2 is given in Figure 6.

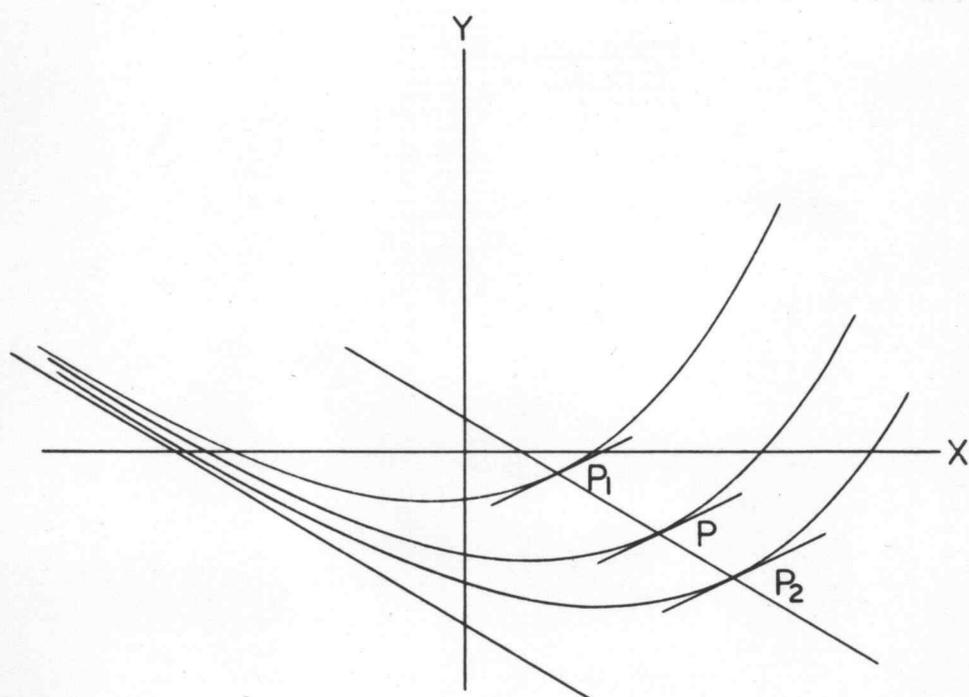


FIG. 6

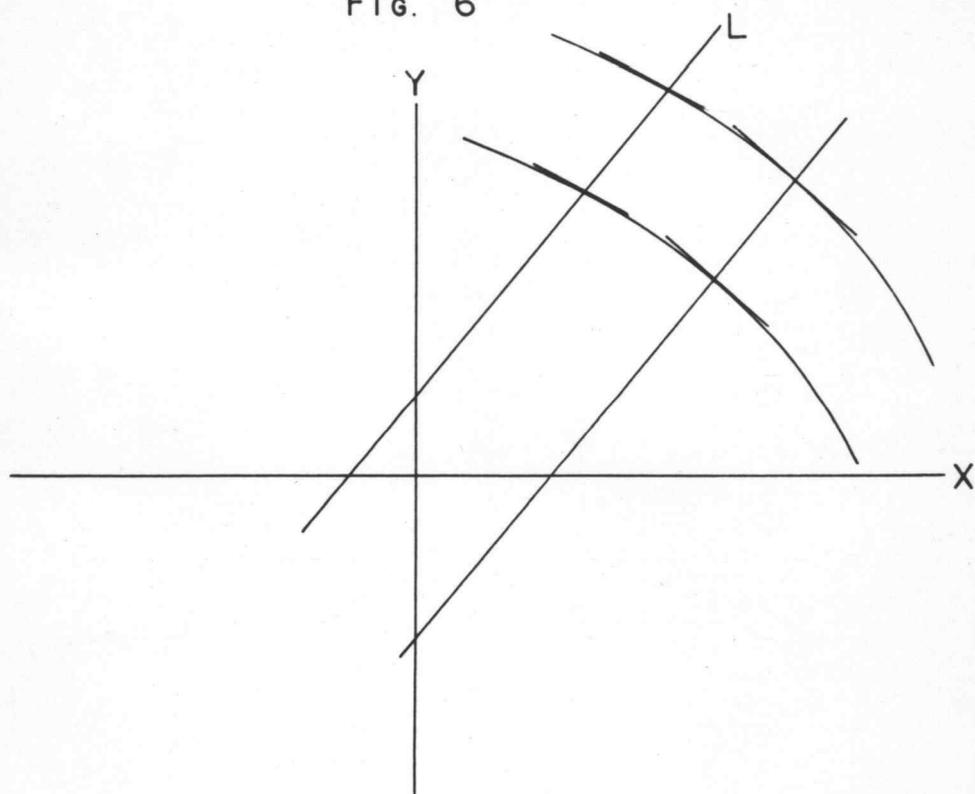


FIG. 7

### 3. THE SPECIAL LINEAR EQUATION $y' = nx + my + q$ .

Of particular importance is the special linear equation

$$(3.1) \quad y' = nx + my + q,$$

where  $n, m, q$  are constants. This equation is also of the type considered in section 2. For convenience we shall let  $L$  denote the line

$$nx + my + q = 0.$$

It will be noted that the lines of equal slope of (3.1) are the lines parallel to  $L$  (see Figure 7). This immediately gives us the following theorem.

**THEOREM 3.1** Any translation of the plane parallel to  $L$  carries an integral curve of (3.1) into another integral curve of (3.1) [ 2, p.40].

An illustration of theorem 3.1 is given in Figure 6.

**THEOREM 3.2** The slope at any point of an integral curve of (3.1) is proportional to its distance from  $L$  [ 2, p.41].

For we have

$$y' = kh(x,y)$$

where  $k = (n^2 + m^2)^{1/2}$ , and  $h(x,y) = (nx + my + q) / (n^2 + m^2)^{1/2}$ .

Hence  $h(x,y)$  is the distance of the point  $(x,y)$  from the

line  $L$  and the theorem is established.

The differential equation (3.1) is important because of the light it throws on the behavior of the solution of the general equation

$$(3.2) \quad M(x,y)dx + N(x,y)dy = 0$$

at ordinary points where  $M(x,y)$  and  $N(x,y)$  are not both zero. Thus, if  $M(x,y) \neq 0$  at  $(x_0, y_0)$  we may write (3.2) in the form

$$dy/dx = f(x,y)$$

in a neighborhood of  $(x_0, y_0)$ . If the second partial derivatives of  $f(x,y)$  are continuous at  $(x_0, y_0)$  we may represent  $f(x,y)$  approximately by the first few terms of its Taylor's expansion,

$$f(x,y) = f(x_0, y_0) + \partial f / \partial x |_0 (x-x_0) + \partial f / \partial y |_0 (y-y_0),$$

the error being of the order of smallness of the square of the distance of  $(x,y)$  from  $(x_0, y_0)$ . Setting

$$n = \partial f / \partial x |_0, \quad m = \partial f / \partial y |_0, \quad q = f(x_0, y_0) - nx_0 - my_0,$$

the equation takes the form

$$y' = nx + my + q.$$

Thus, in the neighborhood of  $(x_0, y_0)$ , the integral curves of (3.2) have approximately the properties of the integral curves of (3.1).

## HOMOGENEOUS DIFFERENTIAL EQUATIONS.

## 4. HOMOTHETIC FAMILY OF CURVES

In this section we shall consider the so-called homogeneous differential equation

$$(4.1) \quad M(x,y)dx + N(x,y)dy = 0,$$

where  $M(x,y)$  and  $N(x,y)$  are homogeneous functions of the same degree of homogeneity. The following theorem characterizes a family of curves having the origin as center of homothety.

**THEOREM 4.1** The integral curves of (4.1) constitute a homothetic family of curves having the origin as center of homothety, and conversely, the differential equation of a family of curves having the origin as center of homothety is of the form (4.1) (Figure 8) 2, [pp.44-45].

Since  $M(x,y)$  and  $N(x,y)$  in (4.1) are homogeneous functions of the same degree of homogeneity it follows that

$$M(x,y)/N(x,y) = M(1,y/x)/N(1,y/x) = f(y/x)$$

and (4.1) can be written in the form

$$dy/dx = -f(y/x).$$

Thus the slope  $dy/dx$  has the same value at all points on any line  $y/x = \text{constant}$ . It follows that the integral

curves form a homothetic family having the origin as center of homothety, and the direct part of the theorem is established.

The converse part of the theorem is easily established by reversing the above argument.

**THEOREM 4.2** A family of oblique (or orthogonal) trajectories of a family of homothetic curves is another family of homothetic curves having the same center of homothety.

Take the center of homothety of the given family at the origin. Then the differential equation of the family is of the form

$$y' = f(v), \quad v = y/x.$$

Hence the differential equation of the oblique trajectories is

$$(4.2) \quad (y' - \tan \theta)/(1 + y' \tan \theta) = f(v),$$

where  $\theta$  is the angle between the two families of curves.

Solving (4.2) for  $y'$  we find

$$y' = [f(v) + \tan \theta]/[1 - f(v) \tan \theta] = F(v).$$

Since this is a homogeneous differential equation, the family of trajectories is also a homothetic family of curves having the origin as center of homothety.

5. THE EQUATION WHERE  $M(x,y)$  AND  $N(x,y)$  ARE LINEAR.

In this section we shall consider the differential equation  $M(x,y)dx + N(x,y)dy = 0$ , where  $M(x,y)$  and  $N(x,y)$  are linear functions of  $x$  and  $y$ . Thus, let

$$M \equiv a_1x + b_1y + c_1, \quad N \equiv a_2x + b_2y + c_2.$$

We shall call

$$D \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

the determinant of the differential equation. If  $D \neq 0$  the lines  $M = 0$  and  $N = 0$  intersect in a point  $Q$ , whose coordinates we shall denote by  $(h,k)$ . This section relies heavily on [2, pp.34-52].

**THEOREM 5.1** If  $D \neq 0$  the translation  $x = x_1 + h$ ,  $y = y_1 + k$  reduces the differential equation

$$(5.1) \quad (a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$$

to the homogeneous differential equation

$$(5.2) \quad (a_1x_1 + b_1y_1)dx_1 + (a_2x_1 + b_2y_1)dy_1 = 0.$$

Substituting  $x = x_1 + h$ ,  $y = y_1 + k$  in (5.1) we get

$$(a_1x_1 + b_1y_1 + a_1h + b_1k + c_1)dx_1 + (a_2x_1 + b_2y_1 + a_2h + b_2k + c_2)dy_1 = 0,$$

which, because of our choice for  $h$  and  $k$ , reduces to (5.2).

**THEOREM 5.2** If  $D \neq 0$  and if the tangent to an integral curve of (5.1) at  $P$  does not pass through  $Q$ , then  $Q$  and the curve near  $P$  lie on opposite sides of the tangent if  $D < 0$ , and on the same side if  $D > 0$ .

Suppose the origin has been shifted to the point  $Q$ . Then, by theorem 5.1, the differential equation takes the form

$$y' = - \frac{a_1 x + b_1 y}{a_2 x + b_2 y} .$$

Differentiating we find

$$\begin{aligned} y'' &= \frac{-(a_2 x + b_2 y)(a_1 + b_1 y') + (a_1 x + b_1 y)(a_2 + b_2 y')}{(a_2 x + b_2 y)^2} \\ &= \frac{-(a_1 b_2 - a_2 b_1)(y - xy')}{(a_2 x + b_2 y)^2} \\ &= \frac{-D(y - xy')}{(a_2 x + b_2 y)^2} . \end{aligned}$$

Hence  $y''$  and  $y - xy'$ , if not zero, have the same or opposite signs according as  $D \leq 0$ .

Let  $M$  be the foot of the perpendicular from  $P$  on the  $x$ -axis as shown in Figures 9a and 9b, and let the tangent to the curve at  $P$  cut the  $y$ -axis in the point  $N$ . Through the origin  $O$  draw  $OR$  parallel to  $NP$  to cut  $MP$  in  $R$ . Then

$$ON = MP - MR = y - xy' .$$

It follows that the origin lies below or above the tangent at  $P$  according as  $y - xy' \geq 0$ . But the curve, in the

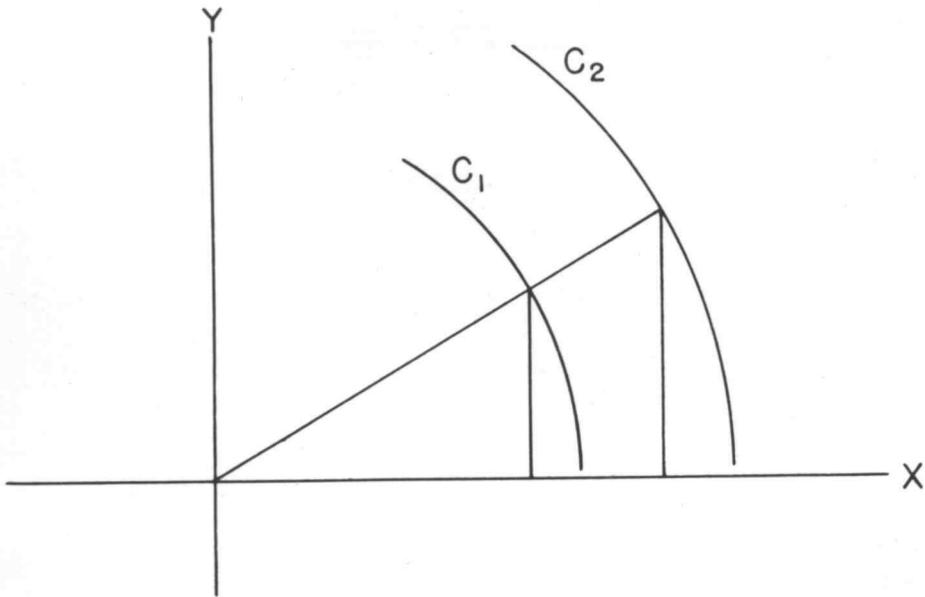


FIG. 8

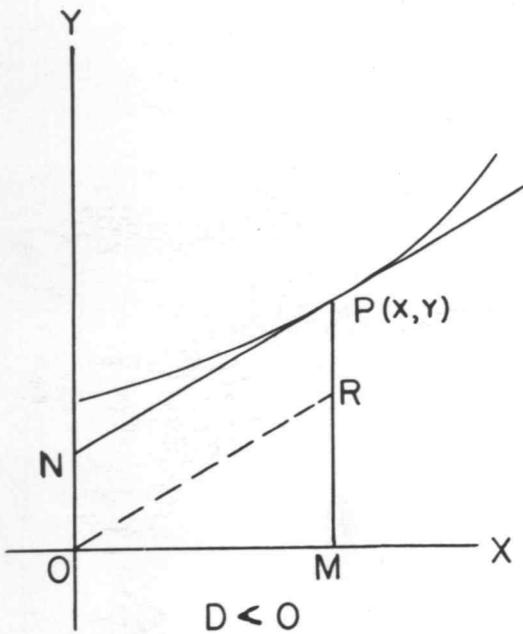


FIG. 9A

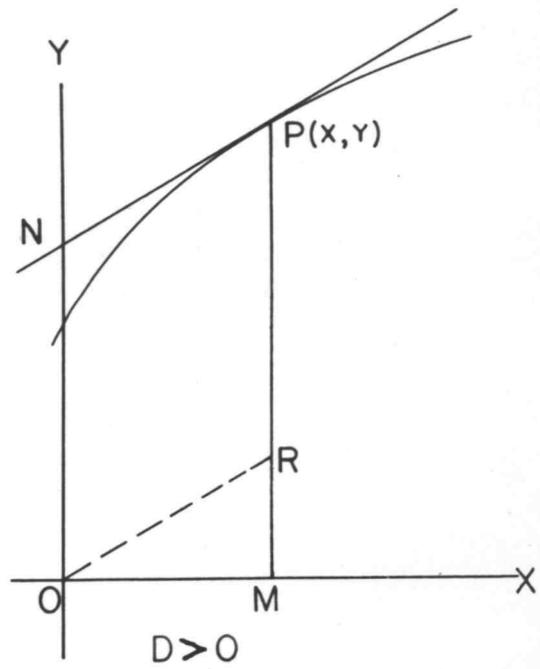


FIG. 9B

neighborhood of  $P$ , lies above or below the tangent at  $P$  according as  $y'' \geq 0$ . The theorem now follows.

Theorem 5.2 is illustrated by the family of homothetic central conics

$$(5.3) \quad ax^2 + 2bxy + cy^2 = k,$$

where  $k$  is a parameter. The origin is the center of homothety of the family. The differential equation of the family is found to be

$$\frac{dy}{dx} = - \frac{ax - by}{bx - cy},$$

which is of the form (5.2). Here we have

$$D = ac - b^2.$$

But the family (5.3) is a family of ellipses or hyperbolas according as  $ac - b^2 = D \geq 0$ . This example gives us the following theorem.

**THEOREM 5.3** The integral curves of equation (5.2) constitute a family of homothetic central conics, with the origin as center of homothety, if  $b_1 = a_2$ . The conics are ellipses or hyperbolas according as

$$D = a_1 b_2 - a_2 b_1 = a_1 b_2 - a_2^2 \geq 0.$$

**THEOREM 5.4** The differential equation of a family of oblique trajectories of the integral curves of equation (5.2) is an equation of the same form as (5.2), and the determinants of the two equations have

the same sign.

Let  $\theta$  be the angle of intersection of the two families. Then the differential equation of the trajectories is

$$\frac{y' - \tan \theta}{1 + y' \tan \theta} = - \frac{a_1 x + b_1 y}{a_2 x + b_2 y},$$

or, setting  $\tan \theta = t$ ,

$$(5.4) \quad y' = \frac{(ta_2 - a_1)x + (tb_2 - b_1)y}{(a_2 + ta_1)x + (b_2 + tb_1)y}.$$

Letting  $D_t$  be the determinant of this equation we have

$$\begin{aligned} D_t &= (a_1 - ta_2)(b_2 + tb_1) - (b_1 - tb_2)(a_2 + ta_1) \\ &= t^2(a_1 b_2 - a_2 b_1) + (a_1 b_2 - a_2 b_1) \\ &= D(t^2 + 1) \end{aligned}$$

It follows that  $D$  and  $D_t$  have the same sign.

THEOREM 5.5 Among the oblique trajectories of the integral curves of equation (5.2), with  $b_1 \neq a_2$ , is a unique family of central conics. These conics are ellipses or hyperbolas according as  $D \geq 0$ .

Applying theorem 5.3 to the differential equation (5.4), we see that the oblique trajectories will be a family of conics if

$$b_1 - tb_2 = a_2 + ta_1,$$

or

$$(b_2 + a_1)t = b_1 - a_2.$$

If  $b_2 + a_1 \neq 0$ , then

$$\theta = \tan^{-1} \left( \frac{b_1 - a_2}{b_2 + a_1} \right).$$

If  $b_2 + a_1 = 0$ ,

$$\theta = \pi/2.$$

Since  $D$  and  $D_t$  are of like sign, the conics are ellipses or hyperbolas according as  $D \geq 0$ .

**THEOREM 5.6** Differential equation (5.2) has two, one, or no straight line solutions according as

$$\Delta = (a_2 - b_1)^2 - 4D \geq 0,$$

unless  $a_1 = b_2 = 0$  and  $a_2 = -b_1$ , in which case all the solutions are straight lines through the origin.

Straight line solutions occur when and only when  $y - xy' = 0$ . But we find

$$(5.5) \quad y - xy' = \frac{b_2 y^2 + (a_2 + b_1)xy + a_1 x^2}{a_2 x + b_2 y}.$$

This certainly will vanish if  $a_1 = b_2 = (a_2 + b_1) = 0$ , in which case the differential equation reduces to

$$y' = y/x,$$

whose solutions are all the straight lines through the origin. For the general case the vanishing of  $y - xy'$  depends upon the discriminant,  $\Delta$ , of the numerator on the right side of equation (5.5). We find

$$(5.6) \quad \Delta = (a_2 + b_1)^2 - 4a_1 b_2 = (a_2 - b_1)^2 + 4(a_2 b_1 - a_1 b_2) = (a_2 - b_1)^2 - 4D.$$

If  $\Delta > 0$  the numerator in (5.5) breaks up into two real distinct linear factors, say  $(ax + by)(cx + dy)$ . In this case, then, there are the two straight line solutions

$$(5.7) \quad ax + by = 0, \quad cx + dy = 0$$

of equation (5.2). If  $\Delta = 0$  the factors of the numerator in (5.5) are equal and there is a single straight line solution of (5.2). If  $\Delta < 0$  the numerator in (5.5) does not factor into real factors and there is no straight line solution of (5.2).

The above theorems enable us to picture the behavior of the integral curves of equation (5.2) in the neighborhood of the origin  $O$ , or those of equation (5.1) in the neighborhood of point  $Q$ . Several cases arise.

I.  $D < 0$ . In this case we see, from (5.6), that  $\Delta > 0$ . The lines (5.7) exist and the integral curves, while not usually hyperbolas, have the general appearance of the family of hyperbolas in Figure 5.

II.  $D > 0$ ,  $t = 0$ . Since  $t = 0$  we have  $b_1 = a_2$ , and the integral curves constitute a homothetic family of ellipses having the origin as center of homothety.

III.  $D > 0$ ,  $t \neq 0$ ,  $\Delta > 0$ . The straight lines (5.7) exist and the integral curves cut a homothetic family of ellipses about  $O$  under a constant angle.

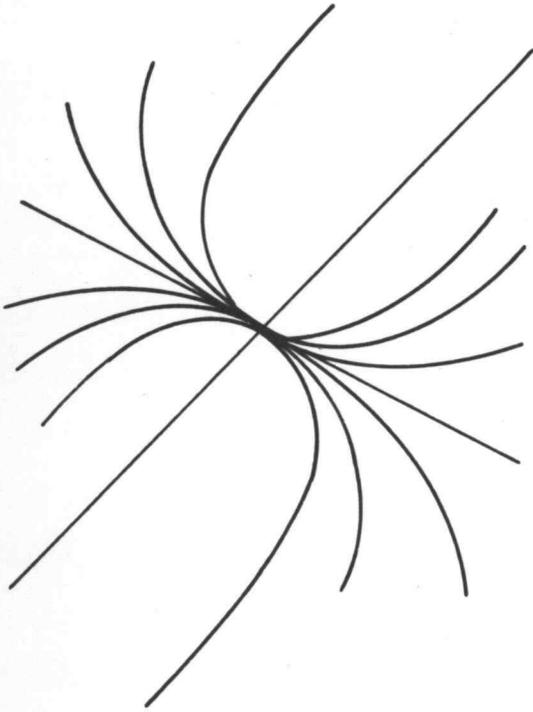


FIG. 10

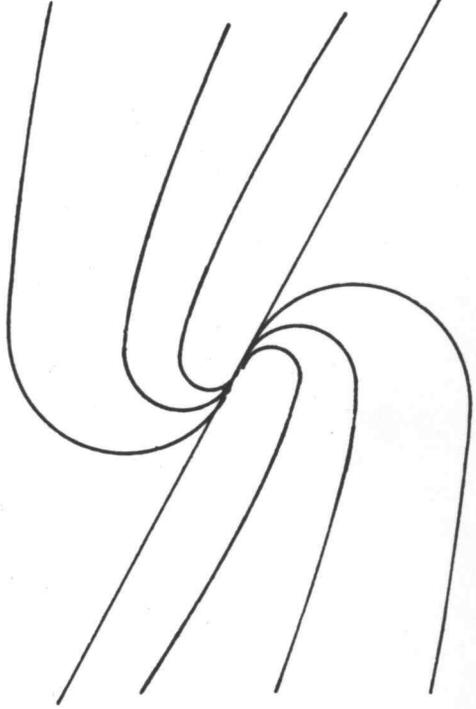


FIG. 11

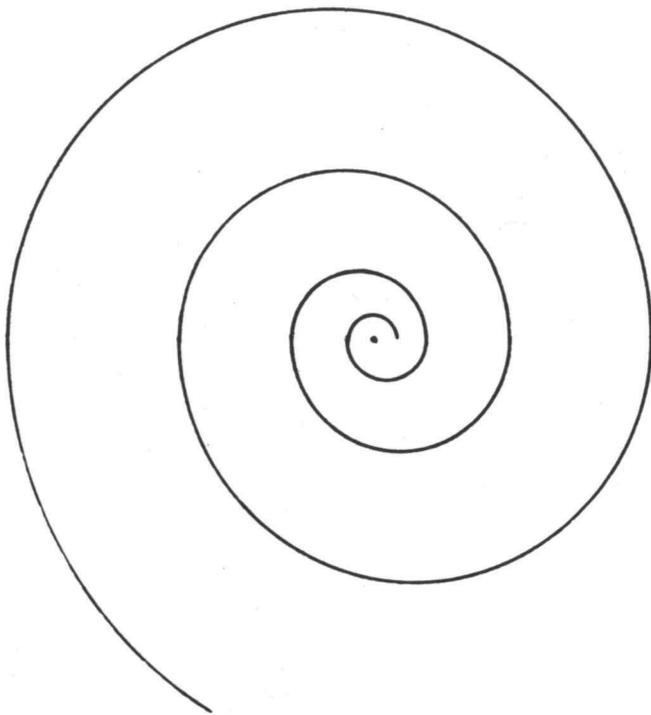


FIG. 12

The integral curves have the appearance indicated in Figure 10.

IV.  $D > 0$ ,  $t \neq 0$ ,  $\Delta = 0$ . There is only one straight line solution. The integral curves again cut a homothetic family of ellipses about 0 under a constant angle and have the appearance indicated in Figure 11.

V.  $D > 0$ ,  $t \neq 0$ ,  $\Delta < 0$ . There is no straight line solution. The integral curves cut a homothetic family of ellipses about 0 under a constant angle, and have the appearance indicated in Figure 12.

Some of the above cases are illustrated in [5, p.9] and [8, pp.186-197].

Equation (5.2) is important because of the light it often throws on the behavior of the solutions of the general equation

$$M(x,y)dx + N(x,y)dy = 0$$

in the neighborhood of an exceptional point  $(x_0, y_0)$  where both  $M(x,y)$  and  $N(x,y)$  vanish. If the second partial derivatives of  $M$  and  $N$  are continuous at  $(x_0, y_0)$  and do not all vanish at  $(x_0, y_0)$  we may represent  $M$  and  $N$  approximately by the first few terms of their Taylor's expansions,

$$\partial M / \partial x |_0 (x - x_0) + \partial M / \partial y |_0 (y - y_0),$$

$$\partial N / \partial x |_0 (x - x_0) + \partial N / \partial y |_0 (y - y_0).$$

Shifting the origin to the exceptional point  $(x_0, y_0)$ , these expressions take the form

$$a_1x + b_1y, \quad a_2x + b_2y,$$

and the differential equation is approximated by equation (5.2).

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