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ABSTRACT

The numerical integration of differential equations is important to the computer, the engineer, and the scientist. In many problems, the differential equation is such that the solution cannot be obtained in terms of known functions. In such cases methods which yield an approximation to the solution are required.

The solution $y\ (x, \alpha, \beta)$, of the differential equation $y'' = f(x,y,y')$ contains two arbitrary constants, α and β. These constants are determined uniquely when the solution is required to pass through a given point and to have a given slope at that point. The point and slope must be such that $f(x,y,y')$ has no singularity for the required values. These constants are also determined, but not always uniquely, when there are two given points through which the solution is to pass. The numerical processes available for the solution of the second order differential equation are those for which the boundary conditions relate to one point.

This thesis is a discussion and comparison of the methods which can be applied to obtain an approximation to the solution of the second order differential equation, $y'' = f(x,y,y')$ when the boundary conditions relate to two points. The first part is a comparison of three methods in which the solution is obtained by successive approximations. The second part is a discussion of a step-by-step method by which the approximation is obtained by assuming conditions relating to one point and then correcting the assumption so that the approximation satisfies the required condition at the second point. The third part is a discussion of the convergence of the approximation to the solution.

NUMERICAL SOLUTION OF
THE SECOND ORDER
DIFFERENTIAL EQUATION

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NUMERICAL SOLUTION OF THE SECOND ORDER DIFFERENTIAL EQUATION

INTRODUCTION

The numerical integration of differential equations is important to the computer, the engineer, and the scientist. In many problems, the differential equation is such that the solution cannot be obtained in terms of known functions. In such cases methods which yield an approximation to the solution are required.

The solution, $y = (x, \alpha, \beta)$, of the differential equation $y'' = f(x, y, y')$ contains two arbitrary constants, α and β. These constants are determined uniquely when the solution is required to pass through a given point, and to have a given slope at that point. The point and slope must be such that $f(x, y, y')$ has no singularity for the required values. These constants are also determined, but not always uniquely, when there are given two points through which the solution is to pass. The numerical processes available for the solution of the second order differential equation are those for which the boundary conditions relate to one point.

This thesis is a discussion and comparison of the methods which can be applied to obtain an approximation to the solution of the second order differential equation,

$y'' = f(x, y, y')$ when the boundary conditions relate to two points. The first part is a comparison of three methods in which the solution is obtained by successive approximations. The second part is a discussion of a step-by-step method by which the approximation is obtained by assuming conditions relating to one point and then correcting the assumption so that the approximation satisfies the required condition at the second point. The third part is a discussion of the convergence of the approximation to the solution.

I. SUCCESSIVE APPROXIMATIONS

1. BRIDGER'S METHOD

C.A. Bridger* has devised a method by which one can obtain the approximate solution to

(1) $$y'' = f(x, y, y'),$$

when the boundary conditions relate to both ends of the range, viz.,

(2) $$x = x_0, \quad y = y_0; \qquad x = x_1, \quad y = y_1.$$

The method consists in dividing the range into 2^r equal subintervals. The approximation to the second derivative is found for these 2^{r+1} values of x in the range; by two successive integrations by Simpson's Rule, an approximation to the solution is found. To illustrate the method, let us use the second order differential equation

$$y'' = \frac{1 + y'^2}{y},$$

subject to the conditions

$$x = 0, \quad y = 1; \quad x = 1, \quad y = 2.$$

For the first approximation, an ordinate y_1, is assumed at the midpoint of the range and also three slopes y'_0, y'_1, y'_2 at the first end point, the midpoint, and the second end point of the range respectively.

*C.A. Bridger, On the Numerical Integration of the Second Order Differential Equation with Assigned End Points. A thesis, Oregon State College, August, 1936.

The usual assumption is to take the slopes as equal to the slope of the line joining the two end points and the ordinate y as lying on this line. For the example this is put into the table

x	y'	y
0	1	1
0.5	1	1.5
1.0	1	2.0

These values for y and y'_n are substituted into the differential equation, obtaining three second derivatives

$$y''_0 = 2, \qquad y''_1 = 1.33 \qquad y''_2 = 1.00 .$$

The assumed slopes and ordinate are refined by the following set of formulas:

$$\text{(i)} \qquad y_1 \quad \frac{(y_0 + y_2)}{2} - \frac{h^2}{24}(y''_0 + 10y'' + y''_2)$$

$$\text{(ii)} \qquad y'_0 \quad \frac{(y_2 - y_0)}{2h} - \frac{h}{3}(y''_0 + 2y''_1)$$

$$\text{(iii)} \qquad y'_1 \quad \frac{y_2 - y_0}{2h} + \frac{h}{12}(y''_0 - y''_2)$$

$$\text{(iv)} \qquad y'_2 \quad \frac{y_2 - y_0}{2h} + \frac{h}{3}(2y''_1 + y''_2), \quad \text{where,}$$

at this stage $h = \frac{x_2 - x_0}{2.}$ For actual computation the work is arranged in the table

x	y''	y'	y
x_0	y''_0	y'_0	y_0
x_1	y''_1	y'_1	y_1
x_2	y''_2	y'_2	y_2

Since this is the first approximation, two decimal accuracy will suffice. In the example the table is

x	y"	y'	y
0	2.00	0.22	1.00
0.5	1.33	1.04	1.32
1.0	1.00	1.83	2.00

Using these values for y and y' the second derivative is again calculated.

For the second approximation the range is now divided into four equal subintervals. There are three second derivatives calculated; for the end points and the midpoint of the range. To obtain two more second derivatives, we interpolate, using two formulas derived from Lagrange's Interpolation formula in which interpolated values are to be midway between two consecutive given second derivatives:

(v) $$y''_1 = \frac{1}{8}\left(3y''_0 + 6y''_2 - y''_4 \right)$$

(vi) $$y''_3 = \frac{1}{8}\left(-y''_0 + 6y''_2 + 3y''_4 \right) .$$

Five second derivatives are now available. At this point, a five column table is used:

	I	II	III	IV
x	y"	$y' - y'_0$	y'	y

The entries in column II are calculated by

(vii) $$y'_1 - y'_0 = \frac{h}{24}\left(9y''_0 + 19y''_1 - 5y''_2 + y''_3 \right) \text{ and}$$

(viii) $$y'_{i+2} - y'_i = \frac{h}{3}\left(y''_{i+2} + 4y''_{i+1} + y''_i \right) .$$

Now (viii) is Simpson's rule. Adding $(y'_{i+2} - y'_i)$ to $(y'_i - y'_o)$, the quantity $(y'_{i+2} - y'_o)$ is obtained. By advancing (viii) one step at a time the entries in column II are calculated. The initial slope c.e. the first entry in column III is calculated from the equation

$$(ix) \qquad (a-x_o)\, y'_o = (y_a - y_o) - \frac{h}{3}\Big\{ 4\Big[(y'_1 - y'_o) + \text{------} + (y'_i - y'_o) + \text{-----}\Big] + 2\Big[(y'_2 - y'_o) + \text{------} + (y'_k - y'_o) + \text{-----}\Big] + (y'_a - y'_o)\Big\},$$

where $i = 1,3,5$-----$a - 1$, and $k = 2,4,6$----- $a - 2$. The subscript a denotes the values at the second end of the range. The remaining entries in column III are obtained by adding algebraically y'_o to the entries in column II. The entries in column IV, or the ordinates are calculated by formulae similar to (vii) and (viii),

$$(x) \qquad y_1 - y_o = \frac{h}{24}\,(\,9y'_o + 19y'_1 - 5y'_2 \quad y'_3\,)$$

$$(xi) \qquad y_{i+2} - y_i = \frac{h}{3}\,(\,y'_{i+2} + 4y'_{i+1} + y'_i\,) \quad .$$

The work for the example appears in the table:

x	y"	y' - y'_o	y'	y
0	1.0484		0.3035	1.0000
0.25	1.2981	0.2929	0.5964	1.1112
0.50	1.5679	0.6507	0.9542	1.3036
0.75	1.8579	1.0785	1.3820	1.5941
1.00	2.1680	1.5713	1.8748	1.9999

From these values for y', y'' is calculated; and the process from (vii) to (xi) inclusive is used to refine our values.

x	y''	$y' - y'_0$	y'	y
0	1.1094		0.3465	1.0000
0.25	1.1728	0.2791	0.6156	1.1189
0.50	1.4504	0.6042	0.9507	1.3133
0.75	1.7809	1.0027	1.3492	1.5994
1.000	2.2465	1.5059	1.8514	1.9997

To obtain further approximations, the subintervals are divided in half, The necessary second derivatives needed are calculated by interpolation formulae

$$y''_1 = \frac{1}{16}\left(5y''_0 + 15y''_2 - 5y''_4 \quad y''_6 \right)$$

$$\text{(xii)} \qquad y''_3 = \frac{1}{16}\left(-y''_0 + 9y''_2 + 9y''_4 - y''_6 \right)$$

$$y''_5 = \frac{1}{16}\left(y''_0 - 5y''_2 + 15y''_4 - 5y''_6 \right) .$$

The last interpolation formula is advanced one step at a time until the end of the interval is reached. The integration is carried out using formulae (vii) to (xi) inclusive. After every interpolation a refinement is made to correct for variation due to the interpolation. The following table carries the work out to 16 ordinates.

x	y"	y'-y₀'	y'	y
0	1.0916		0.2744	1.0000
.125	1.1979	0.1437	0.3181	1.0312
.25	1.2689	0.2980	0.6724	1.0925
.375	1.3455	0.5522	0.8266	1.1810
.50	1.4552	0.6357	0.9101	1.2962
.625	1.6077	0.9178	1.1922	1.4168
.75	1.7994	1.0390	1.3134	1.5877
.825	2.0108	1.3685	1.6429	1.7539
1.000	2.2074	1.5411	1.8155	1.9918
0	1.0753		0.3229	1.0000
.125	1.0689	0.1290	0.4519	1.0483
.25	1.3291	0.2783	0.6012	1.1138
.375	1.4253	0.4634	0.7863	1.2001
.50	1.4105	0.6300	0.9529	1.3096
.625	1.7107	0.8292	1.1521	1.4397
.75	1.7163	1.0454	1.3683	1.5983
.825	2.1090	1.2744	1.5973	1.7827
1.000	2.1480	1.5579	1.8808	1.9999
0	1.1043		0.3299	1.0000
.0625	1.1279	.0680	0.3979	1.0227
.1250	1.1487	.1409	0.4708	1.0498
.1875	1.1782	.2118	0.5417	1.0815
.2500	1.2223	.2885	0.6184	1.1176
.3125	1.2810	.3649	0.6948	1.1588
.3750	1.3484	.4488	0.7787	1.2046
.4375	1.4129	.5339	0.8638	1.2561
.5000	1.4569	.6250	0.9549	1.3127
.5625	1.5308	.7166	1.0465	1.3754
.6250	1.6165	.8166	1.1465	1.4436
.6875	1.7063	.9187	1.2486	1.5187
.7500	1.7971	1.0299	1.3598	1.5998
.8125	1.8903	1.1434	1.4733	1.6887
.8750	1.9921	1.2664	1.5963	1.7841
.9375	2.1131	1.3928	1.7227	1.8873
1.000	2.2687	1.5312	1.8611	1.9997

2. A MODIFICATION OF BRIDGER'S METHOD

A modification of the above process is to divide the range into an even number of equal subintervals. The range of these subintervals is to be sufficiently small. The approximation to the solution is obtained by repeated use of formulae (vii) to (xi). This method does away with the interpolation formulae and the formulae used in making the first approximation. This process is repeated until repeated use produces no change to the required numbers of decimal points in the result. In this process the first approximation for calculations of the second derivative is obtained by passing a straight line through the two end points.

In applying this method to $y'' = \frac{1+y'^2}{y}$ with boundary condition $x = 0$, $y = 1$; $x = 1$, $y = 2$ and with ten subintervals, we have the results listed in the table.

x	y"	$y'-y_0'$	y'	y
0			1	1.00
.1			1	1.10
.2			1	1.20
.3			1	1.30
.4			1	1.40
.5			1	1.50
.6			1	1.60
.7			1	1.70
.8			1	1.80
.9			1	1.90
1.0			1	2.00
0	2.00		.2293	1.0000
.1	1.82	.1909	.4202	1.0332
.2	1.66	.3647	.5940	1.0837
.3	1.54	.5212	.7505	1.1514
.4	1.43	.6730	.9023	1.2336
.5	1.33	.8075	1.0368	1.3313
.6	1.25	.9396	1.1689	1.4408
.7	1.18	1.0578	1.2871	1.5646
.8	1.11	1.1756	1.4049	1.6982
.9	1.05	1.2801	1.5094	1.8451
1.0	1.00	1.3859	1.6152	2.0001
0	1.0526		.3131	1.0000
.1	1.1388	.1093	.4224	1.0367
.2	1.2483	.2285	.5416	1.0848
.3	1.3577	.3590	.6721	1.1754
.4	1.4706	.5002	.8133	1.2196
.5	1.5586	.6523	.9654	1.3384
.6	1.6424	.8118	1.1349	1.4133
.7	1.6979	.9798	1.2929	1.5650
.8	1.7511	1.1513	1.4644	1.6723
.9	1.7768	1.3291	1.6422	1.8581
1.0	1.8044	1.5067	1.8198	2.0007

x	y"	y'-y_0'	y'	y
0	1.1088		.3282	1.0000
.1	1.1520	.1130	.4412	1.0384
.2	1.2064	.2308	.5590	1.0884
.3	1.2746	.3547	.6829	1.1504
.4	1.3587	.4863	.8145	1.2252
.5	1.4586	.6269	.9551	1.3136
.6	1.5713	.7784	1.1066	1.4165
.7	1.6990	.9417	1.2699	1.5353
.8	1.8537	1.1201	1.4483	1.6710
.9	2.0183	1.3128	1.6410	1.8254
1.0	2.2231	1.5251	1.8533	1.9999
0	1.0980		.3319	1.0000
.1	1.1367	.1115	.4434	1.0387
.2	1.1922	.2279	.5598	1.0888
.3	1.2351	.3495	.6814	1.1508
.4	1.3623	.4777	.8096	1.2254
.5	1.4435	.6204	.9523	1.3132
.6	1.6189	.7695	1.1014	1.4161
.7	1.7071	.9413	1.2732	1.5342
.8	1.8803	1.1138	1.4457	1.6708
.9	1.9896	1.3152	1.6471.	1.8243
1.0	2.1558	1.5136	1.8455	2.0001
0	1.1110		.3240	1.0000
.1	1.1520	.1130	.4370	1.0380
.2	1.2063	.2308	.5548	1.0876
.3	1.2724	.3622	.6862	1.1494
.4	1.3509	.4857	.8097	1.2240
.5	1.4521	.6331	.9571	1.3121
.6	1.5628	.7764	1.1004	1.4153
.7	1.7084	.9468	1.2708	1.5331
.8	1.8494	1.1179	1.4419	1.6695
.9	2.0353	1.3282	1.6522	1.8226
1.0	2.2029	1.5244	1.8484	1.9995
0	1.1105		.3299	1.0000
.1	1.1474	.1128	.4427	1.0382
.2	1.2025	.2299	.5598	1.0887
.3	1.2797	.3530	.6829	1.1504
.4	1.3526	.4857	.8156	1.2256
.5	1.4603	.6247	.9546	1.3137
.6	1.5621	.7776	1.1075	1.4170
.7	1.7057	.9385	1.2684	1.5355
.8	1.8443	1.1186	1.4485	1.6713
.9	2.0464	1.3095	1.6384	1.8255
1.0	2.2083	1.5265	1.8564	1.9999

This modification of Bridger's Process seemed to be much faster. In the example we know at the second approximation the solution to one decimal place, for all the required values of x. The speed of this modification is due to two things perhaps, the choice of the length of the subinterval and also the differential equation itself.

The chief advantage of this modification is that fewer formulas need to be remembered. The disadvantage here is the same as in Bridger's Process, namely that so far the error term of the r^{th} approximation has not been determined. The only check is that after a few approximations the successive values obtained for y, y', and y'' do not change to the required number of significant figures.

3. A THIRD METHOD OF SUCCESSIVE APPROXIMATION

The method developed in this section is exceedingly easy to use when a calculating machine is available, and can be used to good advantage even when a calculating machine is not available. Of the three methods tested using successive approximations, this seemed to be the fastest.

If the function $y = f(x)$ has a continuous fifth derivative in the neighborhood of $x = x_0$, then $f(x)$ can be expanded by the extended theorem of the mean

$$(1) \quad f(x) = f(x_0) + f'(x_0)(x-x_0) + \text{------} + \frac{f^{(4)}(x_0)}{4!}(x-x_0)^4$$

$$\int_{x_0}^{x} \frac{(x-s)^4}{4!} f^{(5)}(s)\, ds.$$

The integral

$$\int_{x_0}^{x} \frac{(x-s)^4}{4!} f^{(5)}(s)\, ds$$

is the remainder term. If we make a change of variable by transforming the origin to $x = x_0$ and let $\underline{h}$ be the new independent variable, the (1) becomes

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2}f''(0) + \frac{h^3}{6}f'''(0)$$

$$(2) \qquad + \frac{h^4}{24}f^{(4)}(0) + \int_0^h \frac{(h-s)^4}{24} f^{(5)}(0)\, ds.$$

Equation (2) yields the further equations

$$(3) \quad hf'(h) = hf'(0) + h^2 f''(0) + \frac{h^3}{2}f'''(0) + \frac{h^4}{6}f\,(0)$$

$$+ \int_0^h \frac{h\,(h-s)^3}{6} f^{(5)}(s)\,ds,$$

(4)

$$h^2 f''(h) = h^2 f''(0) + h^3 f''(0) + \frac{h^4}{2} f\ (0) + \int_0^h \frac{h^2(h-s)}{2} f\ (s)ds$$

Multiplying (2) by 12 and (3) by -6 and adding to (4) we have

$$12\, f(h) - 6\, hf'(h) + h^2 f''(h) = 12\, f(0) - 6\, hf'(0)$$

(5)

$$+ h^2 f''(0) + \int_0^h f^{(5)}(s)\ .\ \frac{(h-s)^2 s^2}{2} ds$$

If $f\ (h)$ exists and is continuous in the interval from $h=0$ to $h=h$, the integral in (5) can be written

$$\int_0^h f^{(5)}(s) \frac{(h-s)^2 s^2}{2} ds = \frac{f^{(5)}(\xi)}{2} \int_0^h s^2 (h-s)^2\, ds$$

$$= \frac{h^5 f^{(5)}(\xi)}{60}, \quad 0 \leqq \xi \leqq h,$$

as $s^2(h-s)^2$ is positive for all values of s and h. Equation (5) may be rewritten in the following form

$$f(h) - f(0) = \frac{h}{2}\Big[f'(h) + f'(0)\Big] - \frac{h^2}{12}\Big[f''(h) - f''(0)\Big]$$

(6)

$$+ \frac{h}{720} f^{(5)}(\xi).$$

If $f\ (h)$ is replaced by y_{i+1} and $f(0)$ by y_i, equation (6) becomes

(7) $$y_{i+1} - y_i\ \frac{h}{2}\Big[y'_{i+1} + y'_i\Big] - \frac{h^2}{12}\ y''_{i+1}\ - y''_i\quad \frac{h^5}{720} y^{(5)}(\xi).$$

If equation (7) is used as a formula of integration, its error is found to be about 1/8 the error of Simpson's rule.

For use in integrating second order differential equations, we have the two formulas

$$(8)\qquad y_{i+1} - y_i = \frac{h}{2}\left[y'_{i+1} + y'_i\right] - \frac{h^2}{12}\left[y''_{i+1} - y''_i\right], \text{ and}$$

$$y'_{i+1} - y'_i = \frac{h}{2}\left[y''_{i+1} + y''_i\right] - \frac{h^2}{12}\left[y''_{i+1} - y''_i\right].$$

The only difficulty with these formulas is that many times the third derivative is very difficult to calculate. If the third derivative is easy to compute and if h is taken small enough, these equations (8) give a very rapid method for computing the approximate solution of $y'' = f(x,y,y')$.

To use (8) in the calculation of the approximation to the solution of $y'' = f(x,y,y')$ passing through the end points, $x = x_0$, $y = y_0$; $x = x_n$, $y = y_n$, the range of integration is divided into a number of equal subintervals of length h. The integration is carried out step-by-step over each subinterval.

The initial slope is given by

$$(9)\qquad y'_0 = \frac{y_n - y_0}{\ell} - \frac{1}{\ell} = \int_0^{\ell} (\ell - t) \; y''(t) \; dt,$$

where ℓ $x_n - x_0$. The formula is derived in a later section. The integral in (9) evaluated by the following considerations, is

$$r(x) = \int_0^x (\ell - t) \; y''(t) \; dt.$$ Then it follows

$$r'(x) = (\ell - x)\, y''(x)$$

and $$r''(x) = (\ell - x)\, y'''(x) - y''(x) \quad .$$

If we apply (7) to $r(x)$ for each subinterval, then

$$r_{i+1} - r_i = \frac{h}{2}(r'_{i+1} + r'_i) - \frac{h^2}{12}(r''_{i+1} - r''_i) \quad .$$ Hence

$$r(\ell) \int_0^\ell (\ell - t)\, dt = h \sum_{i=1}^{n-1} r'_i \quad \frac{h}{2}(r'_n + r'_o) - \frac{h^2}{12}(r''_n - r''_o).$$

But $r'_n = 0$, $r'_o = \ell y''$, $r''_n = -y''_n$, $r''_o = \ell y'''_o - y''_o$.

Therefore

$$(11) \quad \int_0^\ell (\ell - t) y'' dt = h \sum_{i=1}^{n-1} (\ell - x_i) y'' - \frac{h}{2} y'' - \frac{h^2}{12}(y''_o - y''_n)$$
$$- \frac{h}{12} \ell y''_o \, .$$

Substituting (11) in (9) we find that

$$(12) \quad y'_o = \frac{y_n - y_o}{\ell} - \frac{h}{\ell} \sum_{i=1}^{n-1} (\ell - x_i)\, y''_i - \frac{h}{2} y''_o + \frac{h^2}{12\ell}(y''_o - y''_n)$$
$$+ \frac{h^2}{12} y'''_o \, .$$

For the first approximation of y'' and y'' we take

$$y \quad \frac{y_n - y_o}{x_n - x_o}(x - x_o) + y_o$$

which is the straight line passing through the two points, and calculate y'' and y''. With these values of y'' and y''' we now by the use of (8) and (12) calculate the first approximation to the solution. With this first approximation, we again calculate the second and third derivatives and integrate using (8) and (12). The process is repeated until further integrations produce no change in the approximations.

Let us apply this to the familiar example, the solution is required for $y'' = \frac{1 + y'^2}{y}$ passing through (0,1) and (1,2). Now $y''' = \frac{y'y''}{y}$.

The work is arranged in the following manner.

TABLE I

x	1 y	2 y'	3 y''	4 y'''
.0	1.0	1	2.00	2.00
.1	1.1	1	1.82	1.65
.2	1.2	1	1.66	1.38
.3	1.3	1	1.54	1.18
.4	1.4	1	1.43	1.02
.5	1.5	1	1.33	0.89
.6	1.6	1	1.25	0.78
.7	1.7	1	1.18	0.69
.8	1.8	1	1.11	0.61
.9	1.9	1	1.05	0.55
1.0	2.0	1	1.00	0.50

x	5 Δy''	6 Δy'''	7 y'/2	8 y''/2	9 (1-x) y''
.0				1.000	2.000
.1	-.18	-0.35		.910	1.638
.2	-.16	-0.27		.830	1.328
.3	-.12	-0.20		.770	1.078
.4	-.11	-0.16		.715	.858
.5	-.10	-0.13		.665	.665
.6	-.08	-0.11		.625	.500
.7	-.07	-0.09		.590	.354
.8	-.07	-0.08		.555	.222
.9	-.06	-0.06		.525	.105
1.0	-.05	-0.05		.500	0

Using equations (8) and (12) and Table I, we calculate the entries in column 2 of Table II. Column 7 in Table II is calculated from column 2 in Table II. Again using equation (8) and Table I, column 1 in Table II is calculated. These operations are repeated to obtain the successive tables which are the approximations to the solution.

TABLE II

x	1 y	2 y'	3 y"	4 y"'
.0	1.0000	.2276	1.0518	.2280
.1	1.0324	.4189	1.140	.3610
.2	1.0830	.5931	1.250	.6846
.3	1.1504	.7533	1.370	.8940
.4	1.2331	.9019	1.470	1.0740
.5	1.3301	1.0400	1.565	1.2150
.6	1.4405	1.1691	1.640	1.3330
.7	1.5634	1.2906	1.700	1.4100
.8	1.6981	1.4051	1.750	1.4420
.9	1.8439	1.5131	1.783	1.4650
1.0	2.0003	1.6156	1.805	1.4600

x	5 Δy"	6 Δy"'	7 y'/2	8 y"/2	9 (1-x)y"
.0	.089		.113	.525	1.051
.1	.089	.133	.209	.570	1.026
.2	.110	.323	.296	.625	1.000
.3	.120	.210	.376	.685	.959
.4	.100	.180	.450	.735	.882
.5	.095	.141	.520	.782	.782
.6	.075	.118	.584	.820	.656
.7	.060	.077	.645	.850	.510
.8	.050	.032	.702	.875	.350
.9	.033	.023	.756	.891	.178
1.0	.022	-.005	.807	.902	0

For the second approximation we have:

x	1 y	2 y'	3 y"	4 y"'
.0	1.0000	.3136	1.0982	.3438
.1	1.0367	.4231	1.1380	.4548
.2	1.0849	.5426	1.1930	.5960
.3	1.1456	.6736	1.2700	.7950
.4	1.2199	.8156	1.3650	.9170
.5	1.3089	.9673	1.4700	1.0870
.6	1.4135	1.1275	1.6050	1.2780
.7	1.5340	1.2945	1.7420	1.4710
.8	1.6715	1.4670	1.8850	1.6550
.9	1.8268	1.6436	2.0220	1.8250
1.0	2.0000	1.8229	2.1540	1.9590

x	5 Δy"	6 Δy"'	7 y/2	8 y"/2	9 (1-x)y"
.0	.0		.156	.549	1.0982
.1	.0398	.1110	.211	.569	1.0242
.2	.0550	.1412	.271	.596	.9544
.3	.0770	.1990	.336	.635	.8890
.4	.0950	.1620	.407	.682	.8190
.5	.1050	.1700	.483	.735	.7350
.6	.1350	.1910	.563	.802	.6420
.7	.1370	.1930	.642	.871	.5226
.8	.1430	.1840	.733	.942	.3770
.9	.1370	.1700	.821	1.011	.2022
1.0	.1320	.1340	.911	1.077	0

For the third approximation we have:

x	1 y	2 y'	3 y"	4 y'''
.0	1.0000	.3280	1.10725	.36317
.1	1.0388	.4398	1.14884	.48638
.2	1.0885	.5563	1.20300	.61481
.3	1.1501	.6794	1.27074	.75066
.4	1.2244	.8111	1.35403	.89697
.5	1.3124	.9528	1.45369	1.05537
.6	1.4152	1.1065	1.57175	1.22890
.7	1.5340	1.2738	1.70962	1.41963
.8	1.6702	1.4551	1.86643	1.62605
.9	1.8253	1.6504	2.04011	1.84462
1.0	2.0006	1.8592	2.22831	2.07143

x	5 Δy"	6 Δy'''	7 y'/2	8 y"/2	9 (1-x)y"
.0			.169	.5536	1.1072
.1	.04159	.12321	.219	.5744	1.0339
.2	.05416	.12843	.278	.6015	0.9624
.3	.06774	.13565	.339	.6353	0.8895
.4	.08329	.14631	.405	.6770	0.8124
.5	.09966	.15840	.476	.7268	0.7268
.6	.11806	.17353	.553	.7858	0.6287
.7	.13805	.19073	.636	.8548	0.5128
.8	.15681	.20642	.727	.9332	0.3732
.9	.17368	.21857	.825	1.0200	0.2040
1.0	.18820	.22681	.929	1.1141	0

For the fourth approximation we have:

x	1 y	2 y'	3 y''	4 y'''
.0	1.00000	.3296	1.1086	.3640
.1	1.0385	.4423	1.1520	.4699
.2	1.0885	.5597	1.2070	.6222
.3	1.1505	.6833	1.2690	.7480
.4	1.2252	.8144	1.3570	.9030
.5	1.3135	.9547	1.4560	1.0410
.6	1.4168	1.1058	1.5700	1.2240
.7	1.5358	1.2607	1.6999	1.4080
.8	1.6715	1.4483	1.8480	1.5950
.9	1.8259	1.6435	2.0210	1.8220
1.0	2.0007	1.8568	2.2230	2.0660

x	5 $\Delta y''$	6 $\Delta y'''$	7 $y'/2$	8 $y''/2$	9 $(1-x)y''$
.0			.164	.5543	1.1086
.1	.0434	.1059	.221	.5760	1.0368
.2	.0540	.1523	.279	.6035	.9656
.3	.0620	.1258	.341	.6345	.8883
.4	.0880	.1550	.407	.6785	.8142
.5	.0990	.1580	.477	.7280	.7280
.6	.1140	.1670	.557	.7850	.6280
.7	.1299	.1830	.634	.8499	.5097
.8	.1481	.1870	.724	.9240	.3696
.9	.1730	.2270	.821	1.0105	.2021
1.0	.2020	.2440	.928	1.1115	0

For the fifth approximation we have:

x	1 y	2 y'	3 y'/2
.0	1.0000	.3297	.164
.1	1.0385	.4427	.221
.2	1.0886	.5605	.280
.3	1.1508	.6841	.342
.4	1.2256	.8152	.407
.5	1.3139	.9557	.477
.6	1.4168	1.1069	.553
.7	1.5355	1.2702	.635
.8	1.6712	1.4474	.723
.9	1.8254	1.6407	.820
1.0	1.9999	1.8527	.926

If the calculated y_n does not check with the given y_n, a mistake has been made in the calculation. If a mistake has been made it is well to check first, the value for y_0', as it is the most important for the determination of the ordinates.

II. STEP-BY-STEP INTEGRATION

1. MILNE METHOD

The most convenient method for the numerical solution of differential equations is the step-by-step method devised by W.E. Milne.* This method requires two calculations. First, predicted values are calculated. This is followed by a calculation which checks the predicted values.

The method may be applied to the solution of second order differential equations. Suppose that four values of the solution to

(1) $$y'' = f(x,y,y')$$

are known:

(2)

x	y	y'	y''
x_0	y_0	y'_0	y''_0
x_1	y_1	y'_1	y''_1
x_2	y_2	y'_2	y''_2
x_3	y_3	y'_3	y''_3

The predicted value for y' is calculated from

(4) $$y'_4 = y'_0 + \frac{4h}{3}\left\{2y''_3 - y''_2 + 2y''_1\right\} .$$

The predicted value for y is calculated from

(5) $$y_4 = y_2 + \frac{h}{3}\left\{y'_4 + 4y'_3 + y'_2\right\} .$$

Now y'' is computed; with this value of y'', y' is checked by

$$y'_4 = y'_2 + \frac{h}{3}\left\{y''_4 + 4y''_3 + y''_2\right\} .$$

*W.E. Milne, On the Numerical Integration of Ordinary Differential Equation, Am. Math. Monthly, vol.33, 1926 pp. 455-460.

If there is a discrepancy ϵ between the predicted and checked values of y'_4, and if $\epsilon/29$ will not effect the value of y'_4 to the number of decimals required, it is assumed that the predicted value is correct to the required accuracy. If ϵ is too large the calculation is repeated until no corrections are necessary.

To obtain the starting values to the solution of the second order differential equation $y'' = f(x,y,y')$ subject to $x = x_0$, $y = y_0$, $y' = y'_0$. A process of successive approximation is used. Two additional slopes and two additional ordinates are calculated by

$$
(7)\qquad
\begin{aligned}
y_1 &= y_0 + hy' + \frac{h}{2} y''_0 + \frac{h^3}{6} y''_0 \\
y_{-1} &= y_0 - hy' + \frac{h}{2} y''_0 - \frac{h^3}{6} y''_0 \\
y'_1 &= y'_0 + hy'' + \frac{h}{2} y'''_0 \\
y'_{-1} &= y'_0 - hy'' + \frac{h}{2} y''_0 .
\end{aligned}
$$

With these values of y_{-1}, y'_{-1}, and y_1, y'_1, the second derivatives y''_{-1} and y''_1 are calculated. The values for the slopes are refined by the following formulas:

$$
(8)\qquad
\begin{aligned}
y'_1 &= y'_0 + \frac{h}{24}(7y''_1 + 16y''_0 + y''_{-1}) + \frac{h^2 y'''_0}{4} \\
y'_{-1} &= y'_0 - \frac{h}{24}(7y''_{-1} + 16y''_0 + y''_1) + \frac{h^2 y'''_0}{4} .
\end{aligned}
$$

With these newly calculated slopes, the ordinates are checked by

$$
(9) \qquad
\begin{aligned}
y_1 &= y_0 + \frac{h}{24}\left(7y'_1 + 16y'_0 + y'_{-1} \right) + \frac{h^2 y''_0}{4} \\
y_{-1} &= y_0 - \frac{h}{24}\left(7y'_{-1} + 16y'_0 + y'_1 \right) + \frac{h^2 y''_0}{4} .
\end{aligned}
$$

At this stage the solution is known for one backward point and one forward point. To approximate the second forward point, the formulas

$$
(10) \qquad
\begin{aligned}
y'_2 &= y'_0 + \frac{2h}{3}\left(5y''_1 - y''_0 - y''_{-1} \right) - 2h\, y'''_0 \\
y_2 &= y_0 + \frac{h}{3}\left(y'_2 + 4y'_1 + y'_0 \right)
\end{aligned}
$$

are used. From these values y''_2 is calculated. Now the second forward solution is checked by

$$
(11) \qquad
\begin{aligned}
y'_2 &= y'_0 + \frac{h}{3}\left(y''_2 + 4y''_1 + y''_0 \right) \quad \text{and} \\
y_2 &= y_0 + \frac{h}{3}\left(y'_2 + 4y'_1 + y'_0 \right) .
\end{aligned}
$$

If the checks have a large discrepancy, (11) is repeated until further repitition produces no change to the required number of decimals.

Let us now solve $y'' = \dfrac{1 + y'^2}{y}$ with $x = 0$, $y = 1$, $y' = 0$, with $h = \dfrac{1}{10}$.

x	y	y'		y"
-.1	1.00500	-0.10017		1.00501
0	1.00000	0		0
.1	1.00500	0.10017		1.00501
.2	1.02006	0.20134		1.02007
.3	1.04533	0.30453	15	1.04535
.4	1.08109	0.41076	-1	1.08108
.5	1.12762	0.52112	-9	1.12765
.6	1.18547	0.63664	2	1.18545
.7	1.25516	0.75861	12	1.25521
.8	1.33744	0.88810	3	1.33742
.9	1.43308	1.02654	-2	1.43312
1.0	1.54309	1.17520	0	1.54309

2. METHOD OF VARIATIONS

The method described above is excellent when the initial conditions all relate to one point. When the initial conditions relate to two points, the initial slope must be chosen so that the solution will pass through the two points.

Let

(1) $$y'' = f(x,y,y')$$

be the given equation; let $x = x_0$, $y = y_0$ be the initial point. Let $\bar{y}$ be a solution obatined by a step-by-step process, using initial conditions

(2) $$x = x_0, \quad y = y_0, \quad y' = y_0' \; .$$

Thus we have the table

(3)

x	y	y'	y''
x_0	$\bar{y}_0$	$\bar{y}_0'$	$\bar{y}_0''$
x_1	$\bar{y}_1$	$\bar{y}_1'$	$\bar{y}_1''$
x_2	$\bar{y}_2$	$\bar{y}_2'$	$\bar{y}_2''$
. . .	. . .	. . .	. . .
x_n	$\bar{y}_n$	$\bar{y}_n'$	$\bar{y}_n''$

A solution is desired for $x = x_0$, $y = y_0$, $y' = y_0' + \delta y'$, where $\delta y'$ is small. Then $y = \bar{y} + \delta y$, $y' = y' + \delta y'$, and from (1) we have as the principal infinitesimal

(4) $$\delta y'' = \left(\frac{\partial \bar{f}}{\partial y'}\right)\delta y' + \left(\frac{\partial f}{\partial y}\right)\delta y, \text{ where}$$

$\left(\frac{\partial \bar{f}}{\partial y}\right), \left(\frac{\partial f}{\partial y}\right)$, is found by differentiating (1) and then substituting the known values of x, $\bar{y}$, $\bar{y}'$ from the table above.

Now (4) is a linear homogeneous differential equation of the second order in δy. A solution is desired for the initial condition

$$\delta y = 0, \quad \delta y' = 1 \quad \text{at} \quad x = x_0 .$$

Since it is linear and homogeneous, then if y is a solution of (4), $C\delta y$ is a solution.

The desired end point is (x_n, b). The end point of the table (3) is x_n, y_n. To determine the correction, we set

$$\bar{y}_n + C\delta y = b,$$

whence

$$C = \frac{b - \bar{y}_n}{\delta y_n} .$$

Since the initial slope for δy was 1, the initial slope for $C\delta y$ will be C. Hence the corrected slope is

$$y_0' = \bar{y}_0' + \frac{b - \bar{y}_n}{\delta y_n} . \tag{5}$$

Example. Apply the method of variations to

$$y'' = \frac{1 + y'^2}{y} ,$$

subject to $x = 0$, $y = 1$ and $x = 1$, $y = 2$.

From the solution in the preceding section, the table for $\frac{\partial f}{\partial y} = -\frac{1 + y'^2}{y}$ and $\frac{\partial f}{\partial y'} = \frac{2y'}{y}$ is constructed.

x	$\frac{\partial f}{\partial y}$	$\frac{\partial f}{\partial y'}$
0	-1.0000	0
.1	-1.0000	0.19934
.2	-1.0000	0.39476
.3	-1.0000	0.58265
.4	-1.0000	0.75991
.5	-1.0000	0.92428
.6	-1.0000	1.07407
.7	-1.0000	1.20876
.8	-1.0000	1.32806
.9	-1.0000	1.43263
1.0	-1.0000	1.52318

By the use of this table, the solution to

$$\delta y'' = \left(\frac{\partial f}{\partial y'}\right) \delta y' + \left(\frac{\partial f}{\partial y}\right) \delta y$$

is constructed by Milne's Method.

x	δy	$\delta y'$	ϵ	$\delta y''$
-.1	-.10016	1.00500		-.10018
0	0	1.00000		0
.1	.10016	1.00500		.10018
.2	.20133	1.02015		.20138
.3	.30453	1.04534	0	.30454
.4	.41075	1.08115	-8	.41083
.5	.521111	1.12758	7	.52110
.6	.63688	1.18554	0	.63647
.7	.75860	1.25510	4	.75856
.8	.88831	1.33710		.88745
.9	1.02648	1.43282		1.02622
1.0	, 1.17536	1.54268		1.17514

$\delta y_n = 1.17536$

From 5 the corrected slope is $y_0' = .38874$.

The solution by Milne's Method is

x	y	y'	y"
-.1	0.96681	.27566	1.11292
0	1.00000	0.38874	1.5112
.1	1.04469	0.50616	1.20246
.2	1.10144	0.62970	1.26790
.3	1.17086	0.76023	1.34768
.4	1.25405	0.89964	1.44281
.5	1.35113	1.04943	1.55520
.6	1.46433	1.21122	1.68477
.7	1.59384	1.38704	1.83448
.8	1.74232	1.58026	2.00722
.9	1.91042	1.78930	2.19930
1.0	2.10093	2.02097	2.42003

The slope here is evidently too large. A second application of the process of variation gives the corrected slope to be $y'_0 = .33027$. The solution is tabulated.

x	y	y'	y"
0	1.0000	0.3303	1.1091
.1	1.0386	0.4433	1.1520
.2	1.0888	0.5612	1.2077
.3	1.1510	0.6853	1.2768
.4	1.2261	0.8170	1.3599
.5	1.3147	0.9577	1.4583
.6	1.4180	1.1092	1.5729
.7	1.5369	1.2730	1.7082
.8	1.6731	1.4517	1.8573
.9	1.8277	1.6451	2.0278
1.0	2.0028	1.8578	2.2226

The step-by-step process is perhaps the most accurate, at least the approximate error can be determined. The disadvantage is that it is time consuming.

III. CONVERGENCE

The numerical integration of second order differential equations with assigned end points is a process of successive approximations.

Consider the equation

(1) $$y'' = f(x,y,y');$$

let the solution be required to satisfy the conditions

(2) $$x = 0, \quad y = a; \quad x = \ell, \quad y = b.$$

These boundary conditions may be considered as quite general. If there are any other boundary conditions, say

$$x = x_0, \; y = y_0; \quad x = x_1, \quad y = y_1,$$

by means of the transformations

$$X = \left(\frac{x-x_0}{x_1-x_0}\right)\ell, \qquad Y = a + \left(\frac{y-y_0}{y_1-y_0}\right)b$$

we have the boundary condition (1). Now

$$\frac{dy}{dx} \quad \frac{dY}{dX} \left(\frac{y_1-y_0}{x_1-x_0}\right) \frac{\ell}{b} \qquad \text{and}$$

$$\frac{d^2y}{dx^2} \quad \frac{d^2Y}{dX^2} \cdot \frac{y_1-y_0}{(x_1-x_0)^2} \cdot \frac{\ell}{b} \qquad \text{are to be substituted}$$

in equation (1).

The differential equation (1) can be transformed into the integral equation

(3) $$y = A \quad Bx + \int_0^x (x-t)\, f(t,\, y(t),\, y'(t) \quad dt.$$

Now A and B must be determined so that (3) satisfies (2).

$$A = a \quad \text{and}$$

$$b = a \quad B\ell + \int_0^{\ell} (\ell - t)\, f(t),\, y(t),\, y'(t) \quad dt$$

.

whence

$$B \quad \frac{b-a}{\ell} - \frac{1}{\ell}\int_0^{\ell} (\ell - t)\, f(t),\, y(t),\, y'(t)\, dt$$

Thus equation (3) becomes

(4)

$$y = a + \frac{b-a}{\ell}\, x - \frac{x}{\ell}\int_0^{\ell} (\ell - t)\, f(t,\, y(t),\, y(t)\, dt$$

$$+ \int_0^{x} (x-t)\, f(t,\, y(t),\, y'(t)\quad dt$$

which satisfies both (1) and (2).

Let $y = y_0(x)$ be a continuous function which satisfies condition (2). Now substitute into equation (4) and we have

(5) $$y_1 = a + \frac{b-a}{\ell}\, x - \frac{x}{\ell}\int_0^{\ell} (\ell - t)\, \bar{f}_0 dt + \int_0^{x} (x-t)\, \bar{f}_0 dt,$$

where

$$\bar{f}_0 = f(t,\; y_0(t),\; y'_0(t)$$

and

$$y'_1 = \frac{b-a}{\ell} - \frac{1}{\ell}\int_0^{\ell} (\ell - t)\, \bar{f}_0 dt + \int_0^{x} \bar{f}_0 dt.$$

From (5) and (6) we determine

$\bar{f} = f(x,\; y'_1(x),\; y_1(x)\quad;$ Now substituting f in (4)

we find

$$y_2 = a + \frac{b-a}{\ell}\, x - \frac{x}{\ell}\int_0^{\ell} (\ell - t)\, \bar{f}_1 dt \int_0^{x} (x-t)\, \bar{f}_1 dt$$

and

$$y'_2 = \frac{b-a}{\ell} - \frac{1}{\ell}\int_0^{\ell} (\ell - t)\, \bar{f}_1\, dt + \int_0^{x} \bar{f}_1 dt.$$

By means of similar substitutions we find a sequence of functions

(7) $$y_{r+1} = a \quad \frac{b-a}{\ell}\, x - \frac{x}{\ell}\int_0^{\ell} (\ell - t)\, \bar{f}_r dt$$

$$+\int_0^x (x-t)\,\bar{f}_r\,dt$$

$$y'_{r+1} = \frac{b-a}{\ell} - \frac{1}{\ell}\int_0^\ell (\ell-t)\,\bar{f}_r\,dt + \int_0^x \bar{f}_r\,dt.$$

where

$$f_r = f(t,\ y_{r-1}(t),\ y'_{r-1}(t))\ .$$

Now let us examine the sequence

$$(8)\qquad \begin{aligned} \epsilon_1 &= y_1 - y_0, & \epsilon_2 &= y_2 - y_1 & \text{---------}\ \epsilon_{r+1} &= y_{r+1} - y_r \\ \epsilon'_1 &= y'_1 - y'_0, & \epsilon'_2 &= y'_2 - y'_1 & \text{------}\ \epsilon'_{r+1} &= y'_{r+1} - y'_r \end{aligned}$$

$$(9)\qquad \epsilon_{r+1} = -\frac{x}{\ell}\int_0^\ell (\ell-t)(\bar{f}_r - \bar{f}_{r-1})\,dt + \int_0^x (x-t)(\bar{f}_r - \bar{f}_{r-1})\,dt$$

$$\epsilon'_{r+1} = -\frac{1}{\ell}\int_0^\ell (\ell-t)(\bar{f}_r - \bar{f}_{r-1})\,dt + \int_0^x (\bar{f}_r - \bar{f}_{r-1})\,dt$$

If f is continuous over the range y_{r-1} to y_r and y'_{r-1} to y'_r for each value of x in the interval $0 \leqq x \leqq \ell$, and if $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ exist and are finite in this x, y'_r and y_r region, then from the law of the mean

(10) $\quad f_r - f_{-1} = \dot{\bar{f}}_y (y_r - y_{r-1}) + \dot{\bar{f}}_{y'}(y'_r - y'_{r-1})$, where $\dot{\bar{f}}_y$ and $\dot{f}_{y'}$ denote the values of the partials taken at

$$\begin{aligned} y &= y_{r-1} + \theta(y_r - y_{r-1}) \\ y' &= y'_{r-1} + \theta(y'_r - y'_{r-1})\ , \end{aligned} \qquad 0 \leqq \theta \leqq 1.$$

Let M_r denote the maximum absolute value of $\dot{\bar{f}}_y$ and N_r denote the maximum absolute value of $\dot{\bar{f}}'_{y'}$ when x varies in the range $0 \leqq x \leqq \ell$. The equality (10) becomes the inequality

$$(11) \qquad |f_r - f_{r-1}| \leqq M_r|\epsilon_r| + N_r|\epsilon_r'| .$$

Thus

$$|\epsilon_{r+1}| \leqq \frac{x}{\ell}\int_0^\ell (\ell - t)\,(M_r|\epsilon_r| + N_r|\epsilon_r'|)\,dt$$

$$+ \int_0^x (x-t)\,(M_r|\epsilon_r| + N_r|\epsilon_r'|)\,dt \qquad .$$

Let e_r denote the maximum absolute value of ϵ_r in the interval $0 \leqq x \leqq \ell$ and e_r' denote the maximum absolute value of ϵ_r' in the same interval.

$$(11\text{-a}) \quad |\epsilon_{r+1}'| \leqq \frac{x}{\ell}\int_0^\ell (\ell - t)\,(M_r e_r + N_r e_r')\,dt$$

$$\int_0^x (x-t)\,(M_r e_r + N_r e_r')\,dt$$

If we integrate the inequality (11-a), we have

$$|e_{r+1}| \leqq \frac{x}{\ell}\left(\ell^2 - \frac{\ell^2}{2}\right)(M_r e_r + N_r e_r')$$

$$+ \left(x^2 - \frac{x^2}{2}\right)(M_r e_r + N_r e_r')$$

$$|e_{r+1}| \leqq (M_r e_r + N_r e_r')\left(\frac{x\ell}{2} + \frac{x^2}{2}\right) .$$

Hence

$$(12) \quad e_{r+1} < (M_r e_r + N_r e_r')\,\ell^2 .$$

In a similar manner we find that

$$|\epsilon_{r+1}'| \leqq (M_r e_r + N_r e_r')\,\frac{1}{\ell}\int_0^\ell (\ell - t)\,dt$$

$$+ \quad (M_r e_r + N_r e_r')\int_0^x dt,$$

$$|\epsilon_{r+1}'| \leqq \frac{\ell}{2}(M_r e_r + N_r e_r') + x(M_r e_r + N_r e_r') .$$

(13)

$$\therefore \; e_{r+1}' \leqq \frac{3\ell}{2}(M_r e_r + N_r e_r') .$$

Now let both $\left|\frac{\partial f}{\partial y}\right|$ and $\left|\frac{\partial f}{\partial y'}\right|$ be less than Q a positive constant, whence (12) and (13) become

(14) $e_{r+1} < Q\ell^2(e_r + e_r')$

and

(15) $e'_{r+1} \quad \frac{3\ell Q}{2}(e_r + e'_r)$.

Now adding (14) and (15) we have

(16) $e_{r+1} + e'_{r+1} < \ell\, Q(\ell + 3/2)\,(e_r + e'_r)$ or
$e_r + e'_r < \ell\, Q(\ell + 3/2)\,(e_{r-1} + e'_{r-1})$

From (14) and (16) we see that

$$e_{r+1} < \frac{\ell^3 Q^2}{2}(2\ell + 3)\,(e_{r-1} + e'_{r-1}),$$

$$e_{r+1} < \frac{\ell^4 Q^3}{2}(2\ell + 3)\,(e_{r-2} + e'_{r-2}),$$

$$e_{r+1} < \frac{\ell^5 Q^4}{2}(2\ell + 3)\,(e_{r-3} + e'_{r-3}),$$

(17) $\vdots$

$$e_{r+1} < \frac{\ell^{r+1} Q^r}{2}(2\ell + 3)^{r-1}\,(e_1 + e'_1)$$

.

In a similar manner, it can be shown that

(18) $e'_{r+1} < \frac{3\ell^r Q^r}{2}(2\ell + 3)^{r-1}\left\{e_1 + e'_1\right\}$.

Now

$$y_{r+1} = y_0 + (y_1 - y_0) + (y_2 - y_1) \quad \text{-------}(y_{r+1} - y_r).$$

Since $|y_{r+1} - y_r| < e_{r+1}$, y_{r+1} converges to a limit as r increases indefinitely, provided that the series of positive constants

(19) $e_1 + e_2 + e_3 + \text{----------------} + e_{r+1}$

converges. If the ratio $R = \frac{e_{r+1}}{e_r}$ becomes and remains less than 1, for r greater than ρ , however large ρ may be, the series (19) converges. Since

$$e_{r+1} < \frac{\ell^{r+1} Q^r}{2^{r-1}} (2\ell + 3)^{r-1} (e_1 + e_1'),$$

the ratio R will be less than 1, according as

(20) $$\frac{\ell Q}{2}(2\ell + 3) < 1.$$

Now (20) is satisfied when

$$Q\ell^2 + \frac{3\ell Q}{2} < 1, \text{ or when}$$

(21) $$\ell < \frac{-3Q + \sqrt{9Q^2 + 16}}{2Q}$$

Thus the first sequence of (17) will converge uniformly to a limit, and this limit will be a continuous function.

Let us consider the second sequence of (17). In this case

$$y'_{r+1} = y'_0 \quad (y'_1 - y'_2) \quad (y'_3 - y'_2) \text{ --------} (y'_{r+1} - y'_r).$$

As $|y'_{r+1} - y'_r| < e'_{r+1}$, the series of positive constants is

(22) $$e'_1 + e'_2 + e'_3 \text{ ---------- } e'_r \, .$$

This series converges if $\frac{e'_{r+1}}{e'_r}$ becomes and remains less than 1. From (18) we see that $\frac{e'_{r+1}}{e'_r}$ remains less than 1 if

(20) $$\frac{\ell Q}{2}(2\ell + 3) < 1.$$

This is the same condition which insures the convergence of the first sequence.

We have shown that the sequences (7) do converge. The problem now is to show that these sequences actually do satisfy the differential equation (1).

Since

(7) $$y_{r+1} = a + \frac{b-a}{\ell} x - \frac{x}{\ell}\int_0^{\ell} (\ell - t)\, \bar{f}_r\, dt + \int_0^{x} (x-t)\, \bar{f}_r\, dt,$$

then upon differentiating twice with respect to x,

(23) $$y''_{r+1} = f_r \;.$$

Subtracting the quantity f″ from both sides of (20),

we have

$$y''_{r+1} - f_{r+1} = f_r - f_{r+1},$$

whence from (11),

(24) $$|y''_{r+1} - f_{r+1}| < Q\,(e_r + e'_r).$$

From (17) and (18), (21) becomes

(25) $$|y''_{r+1} - f_{r+1}| < \frac{\ell^{r-1} Q^{r-1} (2\ell + 3)^{r-1}}{2^{r-1}} (e_1 + e'_1).$$

The right hand side of (22) can be made less ,

$0 \leqq \eta \leqq$ 1, however small η may be, provided that r is taken large enough and furthermore, ℓ must satisfy (20),

(20) $$\frac{\ell\, Q\,(2\ell + 3)}{2} < 1.$$

Hence the sequences (7) satisfy the differential equation if ℓ satisfies condition (20).

In many practical cases the integration can be carried out over a much larger range than that given by (20). The integration can be carried out, however, over a range at least as great as that given by (20).

IV. CONCLUSION

Of the methods investigated in obtaining the approximation to the solution of $y'' = f(x,y,y')$ passing through two end points, the speediest and simplest was the third method developed in part I. The disadvantage is that although the results obtained do apparently converge, there is no way at the present time to check the limit of the error. The most accurate method is that developed by W.E. Milne, along with the process of variations.

In using a step-by-step method, to solve the second order differential equation with assigned end points, the first guess of the initial slope is important. To make a good first guess, we use the process of successive approximations to obtain an initial slope. With this initial slope, the differential equation is solved by means of a step-by-step process, and incidentally the results obtained by successive approximations are checked. If the solution obtained by a step-by-step process and this initial slope misses the second end point, it can be corrected by the process of variations.

In studying any second order differential equation, it would be well to make a rough graphical solution and then proceed to the numerical processes. The graph would give some indication as to the total range of integration permissible, and also indicate the best length of subinterval to take.

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