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A Laplace vector transform is defined in a manner analogous to the usual scalar definition and its application to vector derivatives is illustrated in Section 2.

In Section 3, the linear vector differential equation with constant dyadic coefficients is solved in the operational form. The solution depends upon obtaining the reciprocal Ψ^{-1} of a dyadic Ψ which is a function of the transform operator p. The reciprocal is found for a special form of this dyadic which applied to a large class of linear differential equations with constant coefficients.

Application is made in Section 4 of the relationships obtained in Section 3 to solve a first order linear differential equation which contains a cross product in the first derivative. A simple second order differential equation of the motion of a projectile subject to various initial conditions is then treated in a similar manner and a complete solution obtained in terms of the initial vector velocities and the gravitational vector. More difficult equations of second order that involve the vector product with one or more of its terms is then treated and operational solutions expressed. It is noted in closing that although any combination of one or two terms in R(t) and its time derivatives with the cross product may possess a solution, a combination of all three terms with the cross product does not since the dyadic Ψ^{-1} fails to exist and no solution can be found.

THE OPERATIONAL SOLUTION
OF VECTOR DIFFERENTIAL EQUATIONS

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THE OPERATIONAL SOLUTION OF VECTOR DIFFERENTIAL EQUATIONS

1. INTRODUCTION

Physicists and mathematicians have long found the vector notation particularly advantageous in expressing differential equations governing natural phenomena. Very little has been done on the solution of vector equations completely in vector form without recourse to systems of coordinates. The facility of vector, or more generally, polyadic algebra has not been fully exploited in the solution of such problems.

It is the purpose of this paper to apply operational methods directly to solving a limited class of vector differential equations. For simplicity of notation we will let capital Latin letters represent vectors, capital Greek letters represent dyadics, and lower-case Latin letters represent scalars.

Section 2 defines a vector transform analogous to the Laplace Transform and obtains the transform of derivatives of a vector function in terms of the transform of the function. Sections 3 and 4 illustrate the use of the Laplace vector transformation on linear vector differential equations and its applications to several problems. A short table of related transforms appear in Section 5.

2. THE VECTOR TRANSFORM

A vector function $F(t)$ defined for all positive values of the parameter t, multiplied by pe^{-pt} and integrated with respect to t over the range from zero to infinity describes a vector function $G(p)$, i.e.

$$(2.1) \qquad \mathbf{L}\, F(t) = p \int_0^\infty e^{-pt} F(t)\, dt = G(p)$$

The new function, $G(p)$ is the Laplace Vector transform of $F(t)$. The transformation is designated by the bold-faced L. The inverse transform in indicated by L^{-1}. The transform pairs, $F(t)$ and $G(p)$ are tabulated in a manner similar to that of tables of integrals.

Rules and definitions which apply to the scalar transform also apply to the vector transform. The treatment by Churchill (4, pp. 2-60) or Pipes (5, pp. 110-155) and associated theorems and definitions may be applied directly by the substitution of the vector transform pair $F(t)$ and $G(p)$ for the scalar transform pair $f(t)$ and $g(p)$.

From the definition of the vector transform

$$L\,F'(t) = p\int_0^\infty e^{-pt}\,F'(t)\,dt.$$

Integrating by parts,

(2.2) $$L\,F'(t) = pF(t)\,e^{-pt}\Big]_0^\infty + p^2\int_0^\infty e^{-pt}F(t)\,dt.$$

Hence

(2.3) $$L\,F'(t) = p\,L\,F - pF(0) = p\,G(p) - pF(0).$$

Similarly,

(2.4) $$L\,F''(t) = L\,H' = p\,L\,H - pH(0),$$

$$= p^2G(p) - p^2F(0) - pF'(0).$$

Repeated applications of the foregoing procedure yields the formula

(2.5) $$L\,F^{(n)}(t) = p^n\,L\,F - \sum_{k=0}^{n-1} F^{(k)}(0)\,p^{n-k}.$$

We note that

(2.6) $$L\,F^{(n)}\cdot A = \left[p^nG(p) - \sum_{k=0}^{n-1} F^{(k)}(0)\,p^{n-k}\right]\cdot A$$

also,

(2.7) $$L\,F^{(n)} \times A = \left[p^nG(p) - \sum_{k=0}^{n-1} F^{(k)}(0)\,p^{n-k}\right] \times A$$

3. THE LINEAR VECTOR EQUATION

It is well known that any linear vector function may be written in the notation of Gibbs (1, p. 266) as

$$(3.1) \qquad F \cdot \Phi = H$$

where Φ is a dyadic. All the possible forms linear in a vector F, namely

$$(3.2) \qquad aF,\ F \times A,\ F \cdot A\, B, \text{ and } \Phi \cdot F$$

may be expressed in this form. Consequently, it is obvious that any linear vector differential equation with constant coefficients may be expressed in the form

$$(3.3) \qquad R \cdot \Phi_0 + R' \cdot \Phi_1 + R'' \cdot \Phi_2 + \ldots\ldots + R^{(n)} \cdot \Phi_n = F(t)$$

where Φ_0, Φ_1, Φ_2 .... Φ_n are constant dyadics, and

$$(3.4) \qquad R' = dR/dt,\ R'' = d^2R/dt^2, \text{ etc.}$$

We seek the solution of (3.3) subject to the initial conditions

$$(3.5) \qquad R(0) = R_0,\ R'(0) = R'_0,\ \ldots.,\ R^{(n)}(0) = R_0^{(n)}.$$

Employing the vector transform as defined in Section 2, to equation (3.3) we obtain

$$(3.6)\quad G(p)\cdot\Phi_0+\left[pG(p)-pR_0\right]\cdot\Phi_1+\left[p^2G(p)-p^2R_0-pR_0'\right]\cdot\Phi_2$$
$$+\ldots+\left[p^nG(p)-\sum_{k=0}^{n-1}R^{(k)}(0)\,p^{n-k}\right]\cdot\Phi_n = L\,F$$

Solving for the transformed variable G(p) the operational solution may be expressed as

$$(3.7)\quad G(p)=\Big\{pR_0\cdot\Phi_1+\left[p^2R_0+pR_0'\right]\cdot\Phi_2+\left[p^3R_0+p^2R_0'+pR_0''\right]\cdot\Phi_3$$
$$+\ldots+\left[\sum_{k=0}^{n-1}R^{(k)}(0)p^{n-k}\right]\cdot\Phi_n+LF\Big\}\cdot\Psi^{-1}$$

where

$$(3.8)\qquad \Psi=\Phi_0+p\Phi_1+p^2\Phi_2+\ldots+p^n\Phi_n$$

and Ψ^{-1} is the reciprocal of Ψ, provided such a reciprocal exists.

Let us find the reciprocal of a dyadic of form

$$(3.9)\qquad Y = uI + vI\mathrm{x}A$$

Let E_i be a set of unit orthogonal vectors, then

$$(3.10)\quad E^i\cdot E_j=\delta^i_j,\ E_i=E^i,\ \text{and } I=E_iE_i,\quad (i=1,2,3),$$

where repeated indices indicates summation on those indices. We have (1, pp. 288-297)

$$(3.11)\quad Y = uE^iE_i + vE^iE_i\mathrm{x}A = E^i(uE_i + vE_i\mathrm{x}A) = E^iN_i.$$

The reciprocal of Y is then given by (1, p. 290)

$$(3.12) \qquad Y^{-1} = N^i E_i, \quad \text{where} \quad N_i = uE_i + vE_i \times A$$

and N^i is the reciprocal set to N_i. Such a set always exists provided the vectors N_i are independent. We shall so assume. The reciprocal set N^i is given by

$$(3.13) \qquad N^1 = \frac{N_2 \times N_3}{N_2 \times N_3 \cdot N_1}, \quad N^2 = \frac{N_3 \times N_1}{N_3 \times N_1 \cdot N_2}, \quad N^3 = \frac{N_1 \times N_2}{N_1 \times N_2 \cdot N_3}$$

or from (3.12)

$$(3.14) \qquad N^1 = \frac{(uE_2 + vE_2 \times A) \times (uE_3 + vE_3 \times A)}{(uE_2 + vE_2 \times A) \times (uE_3 + vE_3 \times A) \cdot (uE_1 + vE_1 \times A)}$$

with similar expressions for N^2 and N^3. The expanded form of the numerator is

$$\begin{aligned}
(3.15) \qquad & u^2 E_2 \times E_3 + uvE_2 \cdot (AE_3 - E_3A) + uvE_3 \cdot (E_2A - AE_2) - v^2 E_2 \times A \cdot E_3 A \\
&= u^2 E_2 \times E_3 + uvA \times (E_3 \times E_2) - v^2 E_2 \times A \cdot E_3 A \\
&= u^2 E_1 + uvE_1 \times A + v^2 E_1 \cdot AA = E_1 \cdot (uY + v^2 AA)
\end{aligned}$$

The denominator is expanded as

$$\begin{aligned}
(3.16) \qquad & E_1 \cdot (uY + v^2 AA) \cdot (uE_1 + vE_1 \times A) \\
&= u^3 + uv^2 (E_1 \cdot A)^2 + uv^2 (E_1 \times A) \cdot (E_1 \times A) \\
&= u^3 + uv^2 A \cdot A .
\end{aligned}$$

Applying the same expansions for N^2 and N^3 we find that

$$(3.17) \qquad N^i = \frac{E_i\cdot(uY + v^2AA)}{u^3 + uv^2A\cdot A}, \quad (i=1,2,3).$$

With the aid of (3.12) and (3.10)

$$(3.18) \qquad Y^{-1} = \frac{uY + v^2AA}{u^3 + uv^2A\cdot A}$$

or applying (3.9) we have

$$(3.19) \qquad Y^{-1} = \frac{u^2I + uvIxA + v^2AA}{u(u^2 + v^2A\cdot A)}$$

as the reciprocal of

$$(3.20) \qquad Y = uI + vIxA$$

Let us now consider the differential equation of the form

$$(3.21) \qquad a_oR+a_1R'+a_2R''+\ldots+a_{q-1}R^{(q-1)}+a_qR^{(q)}xA_q$$
$$+a_{q+1}R^{(q+1)}xA_{q+1}+\ldots+a_nR^{(n)}xA_n = F(t).$$

This may be written as

$$(3.22) \qquad a_oR\cdot I+a_1R'\cdot I+a_2R''\cdot I+\ldots+a_{q-1}R^{(q-1)}\cdot I+a_qR^{(q)}\cdot IxA_q$$
$$+a_{q+1}R^{(q+1)}\cdot IxA_{q+1}+\ldots+a_nR^{(n)}\cdot IxA_n = F(t).$$

It is of the form (3.3) where

$$(3.23) \qquad \Phi_o = a_oI, \ \Phi_1 = a_1I, \ \ldots, \ \Phi_{q-1} = a_{q-1}I,$$
$$\Phi_q = a_qIxA_q, \ \Phi_{q+1} = a_{q+1}IxA_{q+1}, \ldots, \ \Phi_n = a_nIxA_n.$$

From (3.8) we write

$$\Psi = \left(\sum_{i=0}^{q-1} a_i p^i\right) I + IxA \tag{3.24}$$

This is the form of the dyadic Y of (3.6) and (3.20) where

$$u = \sum_{i=0}^{q-1} a_i p^i, \quad v = 1, \quad A = \sum_{i=q}^{n} a_i p^i A_i \tag{3.25}$$

Consequently, by (3.19)

$$\Psi^{-1} = \frac{\left(\sum_{i=0}^{q-1} a_i p^i\right)^2 I + \left(\sum_{i=0}^{q-1} a_i p^i\right) IxA + AA}{\left(\sum_{i=0}^{q-1} a_i p^i\right)\left[\left(\sum_{i=0}^{q-1} a_i p^i\right)^2 + A\cdot A\right]} \tag{3.26}$$

where A is defined by (3.25). For the operational solution we obtain from (3.7)

(3.27)

$$G(p) = \left\{ \sum_{j=1}^{q-1}\sum_{k=0}^{j-1} a_j p^{j-k} R^{(k)} + \sum_{j=q}^{n}\sum_{k=0}^{j-1} a_j p^{j-k} R^{(k)} xA_j + LF \right\}\cdot \Psi^{-1}$$

with Ψ^{-1} defined by (3.26).

4. APPLICATIONS

We now apply the method of Section 3 to develop the complete solutions of certain problems. Consider the solution of the linear differential equation of first order.

$$R'xA + mR = A \sin t, \tag{4.1}$$

subject to the initial conditions $R(0) = R_0$. We write this

$$R' \cdot IxA + mR \cdot I = A \sin t. \tag{4.2}$$

With the aid of (3.25) and (3.26)

$$I = mI + pIxA, \quad I^{-1} = \frac{m^2 I + pmIxA + p^2 AA}{m^3 + mp^2 A \cdot A}. \tag{4.3}$$

Applying (3.27) to equation (4.3) and carrying out the vector multiplication, the operational solution is

$$G(p) = \frac{pmR_0 xA}{(p^2 a^2 + m^2)} + \frac{A}{(p^2 + 1)}. \tag{4.4}$$

From the tables of transforms we now have the complete solution,

$$R(t) = R_0 xA \frac{\sin mt/a}{a} + A \sin t. \tag{4.5}$$

As an example of a simple second order differential equation let it be required to find a solution to the vector equation of the motion of a projectile fired with a muzzle velocity V_0 from a moving platform having a velocity U. The forces on the projectile are the gravitational force mG and the air resistance W, which we will assume proportional to the velocity, $W = -kR'$. Consequently, if the position vector of the projectile referred to its initial position at the time of firing is denoted by R_0, the equation of motion is

$$R'' + hR' = G \tag{4.6}$$

With $R_0' = U + V_0$, $R(0) = R_0$, and $h = k/m$. Then by specializing (3.26) we have

$$a_0 = 0,\ a_1 = h,\ a_2 = 1,\ u = p^2 + ph,\ v = 0, \tag{4.7}$$

$$Y^{-1} = \frac{I}{p^2 + ph}\ . \tag{4.8}$$

From (3.27)

$$G(p) = \frac{phR_0 + p^2R_0 + pR_0' + G}{p^2 + ph} \tag{4.9}$$

From the tables of Section 5, we have at once the complete solution

$$(4.10)\qquad R(t) = R_o + \frac{m^2(1-e^{-kt/m})(kU+kV_o-mG)}{mk^2} + \frac{mGt}{k}$$

Consider a complete solution for an equation of the type

$$(4.11)\qquad R'' + R'xA + m^2R = B,$$

subject to the initial conditions $R(0) = R_o$, $R'(0) = R_o'$. Referring to equations (3.25) and (3.26), we find $a_o = m^2$, $a_1 = 1$, $a_2 = 1$. Hence

$$\Psi = (p^2 + m^2)I + pIxA,$$

$$\Psi^{-1} = \frac{(p^2+m^2)^2I + (p^2+m^2)pIxA + p^2AA}{(p^2+m^2)\left[(p^2+m^2)^2 + p^2a^2\right]}$$

The transform solution is obtained from (3.27) as

$$(4.12)\qquad G(p) = (p^2R_o + pR_oxA + pR_o' + LB)\cdot\Psi^{-1}$$

$$= \frac{(p^2R_o + pR_oxA + pR_o' + LB)(p^2+m^2)}{\left[(p^2+m^2)^2 + p^2a^2\right]}$$

$$+ \frac{p^3R_oxA + p^2R_o'xA + pLBxA}{\left[(p^2+m^2)^2 + p^2a^2\right]}$$

$$+ \frac{p^4R_o\cdot AA + p^3R_o'\cdot AA + LB\cdot AA}{(p^2+m^2)\left[(p^2+m^2)^2 + p^2a^2\right]}$$

To evaluate the inverse transform $G^{-1}(p)$ it may be necessary, in many cases, to refer to extensive tables such as those of Doetsch (5, pp. 3-184). In the absence of such tables it is possible to evaluate many forms that are similar to (4.12) by utilizing the Heaviside Expansion Formula (3, p. 134). If by proper choice of the point of reference we may let R_o be a null vector, six terms will be dropped from the operational solution. The solution, R(t), is written

$$(4.13)\quad R(t) = R_o'\left[\frac{v \sin vt - u \sin ut}{v^2 - u^2}\right]$$

$$+ (R_o'xA + LB)\left[\frac{\cos ut - \cos vt}{v^2 - u^2}\right]$$

$$+ (m^2R_o' + LBxA)\left[\frac{u \sin vt - v \sin ut}{v^2 - u^2}\right]$$

$$+ m^2LB\left[\frac{u^2 \cos vt - v^2 \cos ut}{u^2v^2(v^2-u^2)}\right]$$

$$+ LB\cdot AA\left[\frac{(\cos ut)/u^2}{(m^2-u^2)(u^2-v^2)} - \frac{(\cos vt)/v^2}{(m^2-v^2)(u^2-v^2)} - \frac{(\cos mt)/m^2}{(m^2-u^2)(m^2-v^2)}\right]$$

$$+ R_o'\cdot AA\left[\frac{u \sin ut}{(m^2-u^2)(u^2-v^2)} - \frac{v \sin vt}{(m^2-v^2)(u^2-v^2)} - \frac{m \sin mt}{(m^2-u^2)(m^2-v^2)}\right]$$

where

$$u^2 = \frac{2m^2+a^2+a\sqrt{4m^2+a^2}}{2}, \quad v^2 = \frac{2m^2+a^2-a\sqrt{4m^2+a^2}}{2}$$

Consider next the case where the second ordered derivative in (3.21) occurs in a vector product.

$$R''xA + nR' + mR = F(t)$$

The operational solution may be found by applying (3.24) and (3.26) in the usual manner,

$$\Psi = (pn + m)I + p^2IxA \ , \tag{4.14}$$

$$\Psi^{-1} = \frac{(pn + m)^2I + p^2(pn + m)IxA + p^4AA}{(pn + m)\left[(pn + m)^2 + p^4A\cdot A\right]},$$

From which, by (3.27) we write the operational solution

$$G(p) = \left[p^2R_oxA + pR_o'xA + pnR_o + LF\right]\cdot\Psi^{-1} \ . \tag{4.15}$$

For the equation

$$R'' + mR' + nRxA = F(t) \tag{4.16}$$

we have

$$\Psi = (p^2 + mp)I + nIxA$$

$$\Psi^{-1} = \frac{(p^2 + mp)^2I + n(p^2 + mp)IxA + n^2AA}{(p^2 + mp)\left[(p^2 + mp)^2 + n^2A\cdot A\right]},$$

and by the usual procedure

$$G(p) = \left[(p^2 + mp)R_o + pR_o' + LF\right]\cdot\Psi^{-1}. \tag{4.17}$$

Consider the case where all terms of the left member of (3.21) involve a vector product. For example,

$$R''xA_2 + mR'xA_1 + nRxA_0 = F(t). \tag{4.18}$$

From equation (3.20) it is evident that $u = 0$. Since u occurs as a factor in the denominator of Ψ^{-1} the reciprocal is not defined and no solution exists.

5. TABLE OF TRANSFORM PAIRS

G(p)	F(t)	G(p)	F(t)
p^n	$\frac{t^{-n}}{\Gamma(1 - n)}$	$\frac{ap}{(p^2 + a^2)}$	Sin at
$\frac{1}{(p + a)}$	$\frac{1 - e^{-at}}{a}$	$\frac{ap}{(p^2 - a^2)}$	Sinh at
$\frac{1}{p(p + a)}$	$\frac{t}{a} - \frac{1 - e^{-at}}{a^2}$	$\frac{p^2}{(p^2 + a^2)}$	Cos at
$\frac{p}{(p + a)}$	e^{-at}	$\frac{p^2}{(p^2 - a^2)}$	Cosh at
$\frac{pa}{(p + b)^2 + a^2}$	e^{-bt} sin at $a^2 > 0$	$\frac{p^2}{\left[p^2+(b+a)^2\right]\left[p^2+(b-a)^2\right]}$	$\frac{1}{2ab}$ sin at sin bt

G(p)	F(t)
$\dfrac{p^n}{(p^2+b^2)^2+p^2a^2}$	$\dfrac{d^{(n)}}{dt^{(n)}}\left[\dfrac{u^2\cos vt - v^2\cos ut}{u^2v^2(v^2-u^2)}\right]$
$\dfrac{p^n}{(p^2+c^2)\left[(p^2+b^2)^2+p^2a^2\right]}$	$\dfrac{d^{(n)}}{dt^{(n)}}\left[\dfrac{\text{Cos } ut}{(c^2-u^2)(u^2-v^2)u^2} - \dfrac{\text{Cos } vt}{(c^2-v^2)(u^2-v^2)v^2} - \dfrac{\text{Cos } ct}{(c^2-u^2)(c^2-v^2)c^2}\right]$ $u^2 = \dfrac{2b^2+a^2+a\sqrt{4b^2+a^2}}{2}, \quad v^2 = \dfrac{2b^2+a^2-a\sqrt{4b^2+a^2}}{2}$ $n = 0, 1, 2, 3, 4.$
$\dfrac{p^n}{(p^2+ap)^2+b^2c^2}$	$\dfrac{d^{(n)}}{dt^{(n)}}\left\{\dfrac{w+e^{-ut}\left[u\sin wt + w\cos wt\right]}{w(u^2+w^2)(v^2+w^2)} + \dfrac{v+e^{-ut}\left[u\sin vt + v\cos vt\right]}{v(u^2+v^2)(v^2+w^2)}\right\}$ $u = \dfrac{a}{2}, \quad v^2 = \dfrac{4ibc - a^2}{4}, \quad w^2 = \dfrac{4ibc + a^2}{4}$ $n = 0, 1, 2, 3, 4.$

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