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Relation of the angular momentum of surface modes to the position of their power-flow center

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Abstract: We show that the value of the total angular momentum (AM) carried by a surface mode can be interpreted as representing the transverse position of the center or balance point of the power flow through the mode. Especially in the lossless cases, the value of the Abraham AM per unit power (multiplied by the square of the speed of light in vacuum) is exactly the same as the transverse position of this power-flow center. However, the Minkowski counterpart becomes proportional to that position with a coefficient in the form of \(1 + \eta\), where \(\eta\) is determined mainly by the constitutive parameters of media.

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References and links
1. Introduction

Metamaterials have captured the interest of many researchers in optics and photonics fields because the real parts of their complex constitutive parameters (permittivity \( \varepsilon \) and permeability \( \mu \)) and/or refractive indexes can be negative. This unique property has brought out groundbreaking applications such as superlenses [1], invisibility cloaks [2, 3], ‘trapped rainbow’ storage of light [4], and so-called lasing ‘spasers’ [5] (for a comprehensive review, see [6, 7]). Metamaterials have also been studied extensively in the context of optical waveguides because near the interface between two media having opposite signs of constitutive parameters, there can exist a new class of waveguide mode, a surface mode.

Usually, surface modes are thought to propagate only in their longitudinal directions, i.e., along the medium interfaces. However, as was shown in [8], the actual power guided through a surface mode is transported in a vortex-like way (see Fig. 1). That is, some of the optical power originally moving forward in one layer comprising the interface is transported transversely to the other layer and made to flow longitudinally again in a backward direction. This feature makes us investigate the angular momentum (AM) of light propagating in a form of surface mode.

Recently, several groups have reported their studies on the spin AM of surface modes [9–13] or similar (1+1)D waves [14]. As is well known, the spin AM of an optical beam induces the rotation of particles around their own axes which are parallel to the propagating direction of the beam [15, 16]. However, the spin AM of a surface mode is a transverse one, i.e., its direction is normal to the propagating axis of the mode. In [13], one of the present authors proved that this transverse spin AM results from the rotation of the electric field component of the surface mode. In this paper, we extend our study to the (total) AM and suggest a way of interpreting it as a quantity related to the transverse position of the center or balance point of the power flow through the surface mode. As will be shown later, this interpretation enables us to estimate the qualitative features of AM quite easily. In addition, the differences between the Abraham and Minkowski versions of AM are identified in the proportional coefficients between the AM and the position of the power-flow center.

2. Mathematical expressions

Let us consider a material interface (with a normal vector \( \hat{z} \)), which supports a surface mode along the \(+x\) direction [see Fig. 1(a)]. In the case of TM polarization, the magnetic field of this mode can be written as \( h = \frac{1}{2}(\phi^t(z)\cos(\beta't x - \omega t) - \phi^t(z)\sin(\beta't x - \omega t)) \exp(-\beta't x) \), where \( \phi \) is the angular frequency of light in vacuum and \( \phi(z) = \phi^t + i\phi^r \) denotes the complex transverse profile of the surface mode whose propagation constant has been written as \( \beta = \beta^t + i\beta^r \). \( \beta^t \) becomes positive and negative in forward and backward modes, respectively [17, 18], while \( \beta^t \) is always nonnegative. The above mode field becomes that of the electric component (e) if we assume TE-polarized configuration. Following [8], we can write the (time-averaged) linear
but also experimental evidences in favor of either of them (see [10] and the references therein).

momentum density are respectively given by

\[ \mathbf{p} = \frac{e^{-2i\beta' x}}{2\omega} \left\{ \mathbf{\hat{x}}(\beta' \xi' + \beta_i \xi_i) |\mathbf{\phi}|^2 + \mathbf{\hat{z}} \left[ \xi_i (\phi' \phi'' - \phi'' \phi'') - \xi_i (\phi' \phi' + \phi'' \phi') \right] \right\}, \quad (1) \]

where the prime indicates differentiation with respect to \( z \).

For the past hundred years, extensive efforts have gone into the resolution of the so-called Abraham-Minkowski debate on the exact form of the linear momentum of light in a medium [19–21]. From the Abraham’s point of view, the momentum of light becomes inversely proportional to the refractive index of the medium, while Minkowski argued that it should be proportional to the refractive index. With the electromagnetic quantities, their forms of the linear momentum density are respectively given by \( \mathbf{g}^A = \mathbf{e} \times \mathbf{b}/c^2 \) and \( \mathbf{g}^M = \mathbf{d} \times \mathbf{b} \), where \( \mathbf{d} \) and \( \mathbf{b} \) are the electric displacement and the magnetic flux density, respectively. We have not only theories but also experimental evidences in favor of either of them (see [10] and the references therein). In Eq. (1), this difference between \( \mathbf{g}^A \) and \( \mathbf{g}^M \) appears in the actual form of \( \xi = (\xi' + i\xi_i) \). With the Abraham form, \( \xi \) is written as \( \xi_E^A = \mu/|\mathbf{d}|^2c^2 \) and \( \xi_M^A = \varepsilon/|\mathbf{b}|^2c^2 \) for TE and TM modes, respectively (\( c \) is the speed of light in vacuum), while it becomes \( \xi_E^M = \varepsilon \) and \( \xi_M^M = \mu \) when we take the Minkowski form.

The AM density \( \mathbf{j} \) with respect to a reference point \( \mathbf{r}_0 = (x_0, z_0) \) is then given by \( (\mathbf{r} - \mathbf{r}_0) \times \mathbf{g} \), resulting in

\[ \mathbf{j} = \frac{e^{-2i\beta' x}}{2\omega} \mathbf{\hat{y}} \left\{ (z - z_0)(\beta' \xi' + \beta_i \xi_i) |\mathbf{\phi}|^2 \\
- (x - x_0) \left[ \xi_i (\phi' \phi'' - \phi'' \phi') - \xi_i (\phi' \phi' + \phi'' \phi') \right] \right\}, \quad (2) \]

The AM per unit length along the propagation direction (hereafter will be referred to as simply AM) can be defined by

\[ \mathbf{J}(x) = \int \mathbf{j} \, dz = \frac{e^{-2i\beta' x}}{2\omega} \mathbf{\hat{y}} \left[ \Lambda - (x - x_0) \Gamma \right], \quad (3) \]
where \( \Lambda = \int (z - z_0)(\beta' \xi' + \beta^2 \xi^2)|\phi|^2 \, dz \) and \( \Gamma = \int [\xi'((\phi' \phi'^* - \phi^* \phi') - \xi(\phi' \phi'^* + \phi^* \phi'))] \, dz \). \( \Gamma \) results from the waveguide loss, its integrand vanishing in the lossless medium. If we put the relative permittivity and relative permeability of the medium as \( \varepsilon_r \) and \( \mu_r \), the complex decay constant \( (\kappa = \kappa' + i \kappa'') \) can be written as \( \kappa^2 = \beta^2 - \varepsilon_r \mu_r (\omega/c)^2 \). Writing the mode profile \( \phi(z) \) as \( \phi(z < 0) = \exp(\kappa_1 z) \) and \( \phi(z \geq 0) = \exp(-\kappa_2 z) \) (hereafter the subscript 1 and 2 indicate respectively the left and right layers) and after some manipulation, we can obtain

\[
\Lambda = \frac{1}{2} \left[ -(\beta' \xi_1' + \beta^2 \xi_1^2) \left( \frac{1}{2(\kappa_1')^2} + \frac{z_0}{\kappa_1^2} \right) + (\beta' \xi_2' + \beta^2 \xi_2^2) \left( \frac{1}{2(\kappa_2')^2} - \frac{z_0}{\kappa_2^2} \right) \right],
\]

(4)

\[
\Gamma = \frac{1}{2} \left( \frac{\xi_1' \kappa_1'}{\kappa_1^2} - \xi_1' - \frac{\xi_2' \kappa_2'}{\kappa_2^2} + \xi_2' \right).
\]

(5)

3. Interpretation

3.1. Abraham AM

Let us look into the Abraham form specifically. In this case, we have \( \mathbf{g}^A = s/c^2 \), where \( s \) denotes the Poynting vector [10]. Then, Eq. (3) can be rewritten as

\[
c^2 \mathbf{J}^A(x) = \hat{s} \left[ \int (z - z_0)s_z \, dz - (x - x_0) \int s_z \, dz \right].
\]

(6)

By normalizing Eq. (6), we can obtain the Abraham AM of the surface mode per unit power (\( \mathbf{J}_0^A \)) as

\[
c^2 \mathbf{J}_0^A(x) = \hat{s} \left[ \frac{\int (z - z_0)s_z \, dz}{\int s_z \, dz} - (x - x_0) \tan \Phi \right],
\]

(7)

where \( \Phi \) is a constant expressing the direction of overall power flow \( \mathbf{S} = \int s \, dz \), given by

\[
\tan \Phi = \frac{\Gamma}{\int (\beta' \xi' + \beta^2 \xi^2)|\phi|^2 \, dz},
\]

(8)

In Eq. (7), we can find that the first term on its right side is the expectation value \( \langle z - z_0 \rangle \) with respect to the distribution of optical power flow. It thus represents the position of the center or balance point of the power flow. It is independent of the longitudinal coordinate \( x \) as can be easily seen in Eq. (3) (the exponential decay term is canceled out by the normalization). The second term of Eq. (7) actually amounts to the transversal shift of the overall power flow gains during the propagation distance of \( (x - x_0) \) along the longitudinal axis [see Fig. 2(a)]. However, it should be remembered that although the overall power flows obliquely, the (transverse) position of the power-flow center is conserved (this is quite evident from the translational symmetry of the waveguide). Therefore, the right-side terms of Eq. (7) correspond to the transverse coordinate of a point at the reference longitudinal coordinate \( x_0 \) from which the overall power flows obliquely and reaches the point at \( (s, \langle z - z_0 \rangle) \). This coordinate is what the Abraham AM of a unit-power surface mode at a longitudinal coordinate \( x \) stands for (multiplied by \( c^2 \)), although its direction is not along the \( z \) axis but along the \( y \) axis.

Let us apply this geometrical interpretation to actual examples. We considered two surface modes guided through interfaces between a negative-index metamaterial (NIM; \( z < 0 \)) and silica (\( z > 0 \)) and between \( \varepsilon \)-negative (ENG; \( z < 0 \)) and \( \mu \)-negative (MNG; \( z > 0 \)) media. In both cases, the constitutive parameters of adopted materials (see the caption) are such that \( \kappa_2' \) is smaller than \( \kappa_2^2 \), and thus the major power flow occurs in the right layer as in Fig. 1. We note that the selected values of \( \varepsilon_r \) and \( \mu_r \) remain within the range that can be actually implemented using current technologies (see [22] for an example).
Fig. 2. (a) Geometrical interpretation of Eq. (7). It is notable that the direction of \( S(x) \), i.e., \( \Phi \) remains constant even though its magnitude is dependent on \( x \) in lossy waveguides. Please note that \( \Phi < 0 \) in this figure. (b) Abraham AMs of surface modes per unit power at the wavelength \( \lambda = 1550 \) nm. \( L_p \) denotes the propagation length of the surface mode. Solid and dotted lines correspond, respectively, to the mode at the NIM (\( \varepsilon_r = -2.2 + 0.5i \) and \( \mu_r = -0.8 \))-silica interface and the mode at the ENG (\( \varepsilon_r = -2 + 0.5i \) and \( \mu_r = 1 \))-MNG (\( \varepsilon_r = 1.2 \) and \( \mu_r = -0.5 + 0.1i \)) interface. We took \( r_0 = (0,0) \) where \( x = 0 \) denotes the launching position of light to the waveguide (see the inset).

In the implicit assumption of forward power transport, the major forward power can flow either in the left or right layer. When it flows in the right layer, resulting in a minor backward power flow in the left layer, we can easily discern that the power-flow center lies in the right layer or \( \langle z \rangle > 0 \). This enables us to identify that \( J^A_0(0) \) with respect to \( r_0 = (0,0) \) is positive, i.e., its direction is along the +\( y \) direction. Similarly, when the major forward power flows in the left layer, we have \( \langle z \rangle < 0 \) and thus \( J^A_0(0) \) becomes along the \(-y\) direction. In our two examples, since the major power flows occur in the right layers, we can presume that \( J^A_0(0) \) is positive in both cases. We can confirm this in Fig. 2(b) in which we plotted the numerical values of Abraham AMs per unit power carried by respective surface modes.

Then, the remaining detail is whether \( J^A_0(x) \) increases or decreases during propagation. It is dependent on the sign of \( \Phi \) or the transverse direction of \( S \). To estimate this, let us compare Figs. 1(b) and 1(c), especially the two filled (blue) arrows which represent the instantaneous power flow along the transverse direction. If the medium is lossless, then the amounts of these two power flows are the same. Therefore, if we time-average them, the overall power flow along the transverse direction vanishes, and \( \Gamma \) (which is proportional to \( \int s_z dz \)) and \( \Phi \) become zero as well. That is, \( J^A_0(x) \) remains constant during propagation. When the medium is lossy, however, the amounts of those two power flows become different: that of Fig. 1(b) gets larger than that of Fig. 1(c) due to the propagation loss. Therefore, in our examples, the overall power flow is tilted along the \(-z\) direction. We thus have \( \Phi < 0 \) and \( J^A_0(x) \) increases during propagation. This feature can also be found in Fig. 2(b).

3.2. Minkowski AM

In the case of the Minkowski form, the situation becomes somewhat complex. For the simple physical intuition, let us restrict our discussion to the lossless cases where \( \mathbf{g}^M = (\varepsilon_r \mu_r) \mathbf{S}/c^2 \) [10].
Then, the Minkowski AM of the surface mode per unit power is given by

\[ c^2 J^M_0(x) = \hat{y} \left[ \frac{\langle z - z_0 \rangle \varepsilon_r \mu_r \bar{s}_x}{\bar{s}_x} \right] dz \int_{s_x} \frac{d z}{s_x} \]  

(9)

If we set \( \bar{s}_x = (\varepsilon_r \mu_r - 1) s_x \), Eq. (9) becomes

\[ c^2 J^M_0(x) = \hat{y} \left[ \langle z - z_0 \rangle + (\varepsilon_r \mu_r - 1) \frac{\langle z - z_0 \rangle \bar{s}_x}{\bar{s}_x} \right] \int_{\bar{s}_x} \frac{d z}{\bar{s}_x} \]  

(10)

where \( \langle \cdots \rangle_{\bar{s}_x} \) denotes the expectation value with respect to \( \bar{s}_x \) instead of \( s_x \). If we put \( \langle z - z_0 \rangle_{\bar{s}_x} = \gamma (z - z_0) \), Eq. (10) can be simply written as

\[ c^2 J^M_0(x) = \hat{y} (1 + \eta) \langle z - z_0 \rangle, \]  

(11)

where \( \eta = \gamma (\varepsilon_r \mu_r - 1) \). Equation (11) clearly shows that the Minkowski AM of a unit-power surface mode (multiplied by \( c^2 \)) is proportional to the position of the power-flow center. The proportional coefficient is of the form of \( 1 + \eta \) where \( \eta \) is determined by the constitutive parameters of the media. This characteristic reminds us of the usual understanding that the Minkowski momentum includes both contributions of light and matter while only that of the field comprises the Abraham momentum [13, 21]. If we note that \( \langle z - z_0 \rangle \) is determined by the mode profile, we can regard it as representing the AM contained in the light field. Only this comprises the Abraham AM. However, in the Minkowski counterpart, we can find additional term which is proportional to \( \eta \). If \( \langle \varepsilon_r \mu_r \rangle = 1 \), i.e., the media are effectively free space, \( \eta \) vanishes and the Minkowski AM reduces to the Abraham AM. This fact indicates that the additional \( \eta \)-proportional term in Eq. (11) is the momentum contribution of the media.

To see the effect of the \( \eta \)-related term more clearly, we reconsidered the surface modes in Fig. 2(b) (however, neglecting material losses) and compared their Abraham and Minkowski AMs per unit power in Fig. 3. When the media comprising the interface are an NIM and silica, since \( \varepsilon_r \mu_r > 1 \) everywhere in this example, we have \( \langle \varepsilon_r \mu_r \rangle - 1 > 0 \) and thus the Minkowski AM becomes larger than the Abraham AM. However, interestingly, when the
media are ENG and MNG ones, \((\epsilon \mu) - 1\) can be negative, making the direction of the
Minkowski AM opposite to that of the Abraham AM. That is, although the directions of \(J^0\)
(or those of the AMs contained in the light field) are identical, those of \(J^0\) can be different
due to the different properties or responses of the media comprising the interface. [By deriving
\(\langle \epsilon \mu \rangle = (1 - f)\epsilon_1\mu_1 + f\epsilon_2\mu_2\) where \(f = -\sigma_{12}/(1 - \sigma_{12}^2)\) (\(\sigma_{12} = \mu_1/\mu_2\) and \(\epsilon_1/\epsilon_2\) for TE
and TM modes, respectively), and checking two cases of when \(\sigma_{12}^2 < 1\) and \(\sigma_{12}^2 > 1\) (which re-
quire respectively \(\epsilon_1\mu_1 > \epsilon_2\mu_2\) and \(\epsilon_1\mu_1 < \epsilon_2\mu_2\) for the existence of surface modes), we
can easily show that \(\langle \epsilon \mu \rangle\) always becomes larger than the larger value of \(\epsilon_1\mu_1\) and \(\epsilon_2\mu_2\).
Therefore, we always have \(\langle \epsilon \mu \rangle - 1 > 0\) when \(\epsilon \mu > 1\) in both layers. However, even if
\(\epsilon \mu < 0\) in both layers, it is not always true that we can have \(\langle \epsilon \mu \rangle < 0\) or \(\langle \epsilon \mu \rangle - 1 < 0\).

In the above discussion, we have assumed \(\gamma > 0\) without any proof. Here, let us check its
validity by re-viewing the power flow characteristics via surface modes from the perspective
of \(s_x\). In this point of view, the original forward or backward features of two layers (from
the viewpoint of \(s_x\)) can be changed by the sign of \((\epsilon \mu) - 1\). In addition, the relative magnitude
of \((\epsilon_1\mu_1 - 1)\) and \((\epsilon_2\mu_2 - 1)\) can reverse the major or minor power flow characteristics of
those layers. Among these, what is more important is the latter, i.e., \(|(\epsilon_1\mu_1 - 1)|/(\epsilon_2\mu_2 - 1)|\).

Let us restrict our attention to the case of \((z) > 0\), i.e., when the major power flow occurs in
the right layer (the other case can be dealt with quite similarly). This characteristic is maintained
in the new viewpoint of \(s_x\) when \(|\epsilon_1\mu_1 - 1| \leq |\epsilon_2\mu_2 - 1|\) regardless of any changes in the
forward or backward features of the left and right layers (the first of our two examples in Fig.
3, i.e., the surface mode at the NIM-silica interface belongs to this case). This entails that
\((z)_{s_x}\) is positive and thus \(\gamma > 0\). However, if \(|\epsilon_1\mu_1 - 1| > |\epsilon_2\mu_2 - 1|\), the major power flow
characteristic of the right layer can or cannot be preserved, depending on the actual situations.
After some straightforward calculation, we can obtain the relationship between the amounts of
power flows in the left and right layers as \(|S_{s_2}| = \sigma_{12}^2|S_{s_1}|\), where \(S_{s_1} = \int_0^\infty s_x dz\) and \(S_{s_2} = \int_0^\infty s_x dz\). We can thus presume that if \(|\epsilon_2\mu_2 - 1| > |\epsilon_1\mu_1 - 1| < \sigma_{12}^2|\epsilon_2\mu_2 - 1|\), the major
power flow characteristic of the right layer can be still retained, resulting in \((z)_{s_x} > 0\) and
\(\gamma > 0\). The mode at the ENG-MNG interface in Fig. 3 belongs to this case, and we numerically
confirmed that \(\gamma\) is positive.

4. Conclusion

In this paper, we provided a way of interpreting the AM carried by a surface mode: it can be
seen as a quantity closely associated with the position of the power-flow center through the
mode. Neglecting a correction term which results from the propagation loss, the Abraham AM
of the surface mode (per unit power and multiplied by \(c^2\)) is the same as the position of the
power-flow center. Its Minkowski counterpart, however, becomes proportional to that position
with a coefficient in the form of \(1 + \eta\) in lossless cases, where the additional \(\eta\)-involved term
was shown to represent the momentum contribution of the media. We hope that these results
can provide a clear picture on the meaning of the AM carried by surface modes.

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