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Abstract approved:

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This dissertation arose out of an awareness of difficulties undergraduate linear algebra students encounter when solving linear algebra problems from non-isomorphic settings, even when the problems could be solved with matrix representations and similar procedures as problems from a more familiar setting. This mixed-methods study utilized both traditional and actor-oriented transfer paradigms. From a traditional transfer perspective, two factorial experiments demonstrated that novice undergraduate linear algebra students experienced difficulty in solving problems from non-isomorphic settings, even though the problems admitted similar matrix representations and solution-interpretation procedures.

Upon qualitative interviewing using problems similar to those from Experiments 1 and 2, from an actor-oriented transfer perspective, three case studies revealed a *scalar-variable conflict* phenomenon as a factor impeding success for the

production of a correct matrix representation or interpretation of a matrix solution, in the context of unfamiliar settings. In addition, interview evidence suggested Harel's (1999) *contextual conception* contributed as an obstacle to actor-oriented transfer due to overly-practical reliance upon proto-typical examples from familiar settings.

Demonstrating evidence of a general pattern of actor-oriented transfer as a cycle of practical-theoretical thought processes, subjects were seen to perform similar personal constructions between conceptually non-isomorphic settings with the aid of meta-cognitive interventions which seemed to ease transitions from practical to theoretical thinking. The evidence suggests subjects produced forward co-ordinations relating setting-specific contextual information from unfamiliar settings to contextual information from familiar settings in the actor-oriented transfer of a matrix representation for a novel problem setting. Likewise, in a process of actor-oriented transfer, reflective-like backward co-ordinations were seen to aid the transfer of solution interpretations from a familiar setting to correct contextual interpretations in unfamiliar non-isomorphic settings. Based on Experiments 1 and 2 and the interview findings, the *Intentional Transfer Hypothesis* was conjectured and a final treatment/non-treatment designed Experiment 3 verified the effectiveness of a meta-cognitive intervention for linear algebra problem solving in an unfamiliar, non-isomorphic setting. Additional theoretical implications, limitations of the study, and recommendations for future research are also discussed.

Transfer and Meta-Cognitive Intervention in Conceptually Non-Isomorphic Linear Algebra Problem Settings

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I understand that my dissertation will become part of the permanent collection at Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Lance D. Burger, Author

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Will is the Way … and Words are Its Key ~ P. Kennett

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Transfer and Meta-Cognitive Intervention in Conceptually Non-Isomorphic Linear Algebra Problem Settings

Chapter 1

INTRODUCTION

Problem

This research was inspired from the author's previous experience in the classroom, teaching undergraduate linear algebra as a graduate student. During that teaching experience, the author often noticed the difficulties many students seem to encounter when attempting to solve problems from new conceptual settings, even though the procedures and symbolic representations needed to solve the problems were not apparently difficult when in a familiar context. As an example, linear algebra students are usually familiar, since college or high school, with solving a system of two equations in two unknowns using a matrix and row reduction techniques. Many of the author's students, however, seemed to encounter profound difficulty in applying similar matrix representations and techniques to problems from novel conceptual contexts, such as the linear combination or linear transformation settings.

Similar to the author's anecdotal observations from teaching linear algebra, prior research in mathematics education concerns transition difficulties and cognitive inflexibilities associated with the appropriation of different linear algebra mathematical settings to the same representational formalism as a consequence of the *multiple embodiments* of linear algebra concepts (Harel (1989a, 1989b, 1990); Hillel & Mueller, 2006; DeVries & Arnon, 2004; Hillel & Sierpinska, 1994; Dias &

Artique, 1995; Dorier, 2000b). Following Harel (1989a); "the principle of *multiple embodiments* is found in the process of translating general definitions and theorems to terms of given situations" (p.49). For example, the linear independence concept may be embodied through a set of translational arrow vectors in $\Re^2$, as polynomials from the vector-space $\Re[x]$, or in a purely formal definitional sense at the vector-space level. Although the embodied objects may lie in different mathematical settings, the same representational formalism may be used to express the concept of linear independence as a defined condition on linear combinations. But, as Harel (1987) warns: "the embodiment process can have no constructive cognitive effect if the situation being embodied is not fully understood by the student" (p.31). The relationships between multiple embodiments of settings which share formal representations and computational strategies remain to be a crucial topic of interest in linear algebra mathematics education research, as well as a domain of inquiry for this dissertation.

The flexibility of abstract representations is of particular importance to the educational value of linear algebra. Students new to linear algebra will have probably encountered mathematical representation in more elementary contexts, such as the formalism of a variable representing an unknown numerical value. Linear algebra, however, takes representation to a level requiring a great deal of cognitive flexibility with new mathematical concepts and objects, as Harel (1990) explains:

> Students learn that different concepts can be represented by the same symbol. For example, that a matrix can represent a vector, linear transformation, change of bases, coefficients of a linear system, or span or row space (p. 387).

Similarly, Dorier (1998) attributes many of the difficulties students have in learning linear algebra to "having elements of knowledge which initially are not well enough distributed in different settings (geometric, analytical, logical, formal settings mainly)," thus; students are frequently seen to "lack the ability to change settings and points of view." Likewise, Hillel & Mueller (2006) express the necessity for contextual flexibility in linear algebra in the following:

> From a learning point of view, understanding a mathematical concept/object therefore entails more than an ability to operate within each context. It also requires interpreting information culled from one perspective within another perspective; a fluency in shifting points of view … a solution to a problem within one context simultaneously gives answers to problems framed in a linked context (p. 3-4).

The variety of multiple embodiments of concepts found in linear algebra are important since students require an ability to establish meaningful links between representational forms in order to "understand the necessity for representing these situations by a general concept," a competence referred to as *representational fluency* (Harel, 1987, p.30-31).

Inquiry into psychological theories of knowledge acquisition led the author to the subject of *transfer theory* (Marton, 2006). In recent years, the topic of transfer has undergone a vibrant debate between classical and newly emerging constructivist-influenced transfer paradigms (Lobato, 2006). From both classical and contemporary transfer perspectives, it is the focus of this study to investigate the application of knowledge, i.e. transfer, across conceptually non-isomorphic linear algebra settings. Following Douady (1986), a *setting* is defined as "being made of objects of some mathematical branch, of relationships between these objects, of their eventually various formulations and of mental images associated with these objects and relationships" (p.5).

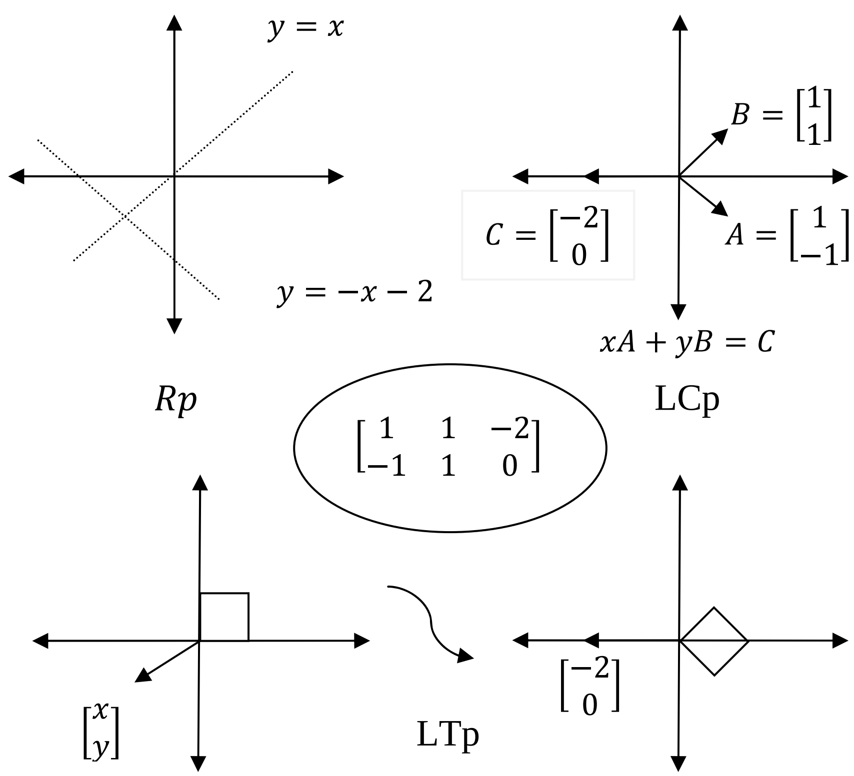


Figure 1.1 Row-picture (Rp), linear combination-picture (LCp), and linear transformation-picture (LTp) settings associated with augmented matrix (*shown in center*).

The term, *conceptually non-isomorphic settings*, refers to mathematical settings which are not mere re-labelings of each other, but may involve different definitions, mathematical objects, concept-images, and structural relationships within the settings. The linear algebra settings of interest for this study include the Cartesian row setting (Rp), the linear combination setting (LCp), and the linear transformation setting (LTp). As seen in Fig. 1.1, the matrix $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 1 & 0 \end{bmatrix}$ forms a representation having the following associated meanings: (i) in the Rp setting the matrix represents the system which solves for the intersection of two lines; (ii) in the LCp setting the matrix solves for the linear combination of vectors $A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which add to give the vector $C = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$; and (iii) in the LTp setting the matrix solves for the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ which, upon clockwise rotation of 45 degrees, would result in the vector $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$. In the next section, a brief history of linear algebra is discussed for the purpose of providing an historical account illustrating how linear algebra came to possess generalizing and unifying characteristics which render the subject deceptively simple in terms of introductory computation, yet inherently difficult for undergraduate mathematics students new to formal abstraction.

Brief History of Linear Algebra

The earliest known use of a linear equation dates back to 2000 B.C. with the Egyptians. Archaeological evidence suggests the Egyptians worked with simple, single linear equations in 1-variable using a guess-and-check, *ad hoc* technique (Hodgkin, 2005). During the Han dynasty, 206-220BCE, the Chinese discovered a method essentially similar to modern day row reduction. In the over 2000 year old Chinese work, *Nine Chapters on the Mathematical Art*, methods are described for the adding and subtracting of positive and negative numbers in *rectangular arrays* (Stillwell, 2002).

A similar method to the Chinese approach for solving a system of linear equations is not discovered in the West until the 16th century, by the French mathematician Buteo (Li & Du, 1987). The origin of the term “Gaussian elimination” derives from the German mathematician Johann Carl Friedrich Gauss (1777-1855), who used row reduction to solve *6X6* system least-square estimation problems related to celestial mechanics. Gauss’ methods of linear approximation achieved widespread fame after his accurate prediction of the location of the temporarily lost comet Ceres in 1801 (Stewart, 1995). For nearly a hundred years, Gauss’s system of elimination was used in the West *without* matrices, usually employed to compute solutions to normal least squares approximation equations in connection with astronomical and geodesic measurements.

The term *matrix*, Latin for *womb*, was first used by Sylvester in 1848. By 1855, Cayley connected the concepts of function composition and matrix multiplication, thereby laying the foundation for matrix algebra (Athloen & McLaughlin, 1987). In 1875, Froebenius gave the first modern treatment of linear dependence that was flexible and could be used on equations as well as *n-tuples*.

From 1875 to roughly 1940, the development of the vector space concept occurred. The term *vector* was originally used in various ways by mathematicians such as: Hamilton in 1845, *quaternions*; Mobius in 1827, *barycentric calculus*; Bellavitis in 1835, *calculus of equipollences*; Grassman in 1845, *calculus of extension*; and Argand in 1806, *complex numbers*. By 1900, the identification between linear transformations and matrices was established. The axiomatic vector space structure was introduced by Peano in 1888, becoming widely used only after 1930 in its application to the theory of function spaces (Banach), and ring theory stemming from Dedekind's work in algebraic number theory (Dorier, 2000a, p. 3).

Although the study of linear systems and their associated geometric interpretations were key ingredients for the eventual development of vector space theory, the theory of *determinants,* a term coined by Gauss, occupied much of the work done in linear algebra up to the first quarter of the 20th century (Dorier, 1995b). Up until 1750, it was generally assumed that a system of *n* equations in *n* unknowns always had a unique solution. In Cramer's famous paper *Introduction a l'Analyse des Courbes Algebriques*, the lack of a unique solution is seen as the vanishing of the determinant (Cramer, 1750). This work inspired Euler's analytic approach to algebra and geometry which eventually culminated into the notion of *linear dependence* as a result of his innovative solution to *Cramer's Paradox* (Euler, 1750; Dorier, 2000a).

Cramer's paradox arises when one considers both: the theorem by Maclaurin that any algebraic curve of degree d is uniquely determined by $d(d+3)/2$ points, and by Bezout's theorem, which says that any two algebraic curves of degrees m and n intersect in exactly mn points. For two deg 3 algebraic curves, there is the contradiction that the two curves must intersect in nine points, yet nine points uniquely determine *one* deg 3 algebraic curve. Moreover, for two deg 4 algebraic curves, both must intersect in 16 points, yet one deg 4 algebraic curve is uniquely determined by only 14 points! Euler solved the paradox by showing that the appearance of the contradiction was a result of dependency which could occur within the $n^2 \times n^2$ systems of equations associated with two given deg n algebraic curves' coefficients (Dorier, 1995a).

This example illustrates how the evolution of linear algebra did not take place in a smooth dialectical fashion. Usually associated with vector space theory, the previous example shows the concept of linear dependence was developed long before Peano's axiomatic theory. Historical analysis by Dorier (2000a) lends further insight into this apparently *non-linear* epistemological evolution of linear algebra:

> Vector space theory does not really constitute new material, rather a new way of 'looking at' old problems and of organizing mathematical knowledge. With the axiomatic theory, new problems in infinite dimension were solved, but the most valuable benefit came from the improvement in solving old problems. Also, a better cohesion of mathematics was reached, and differences could be overcome by putting forward similar structures between very *distinct* fields (p. 4).

Dorier labeled this "putting forward similar structures between very distinct fields" feature of linear algebra as *generalizing* and *unifying*. He went on to speculate, with his *Epistemological Hypothesis* (see Chapter Two), that although linear algebra was powerful in its formal unification of branches of mathematics such as algebra, analysis, and geometry, its generalizing and unifying characteristics contribute to the underlying difficulty contemporary undergraduates experience in the learning of a subject which condenses mathematical relationships from several mathematical settings, into the computational and formal representation systems of matrix algebra and vector space theory, respectively.

Linear Algebra Reform

In 1990, the Linear Algebra Curriculum Study Group (LACSG) was formed for the purpose of addressing perceived inadequacies in linear algebra curriculums. Largely made up of university faculty and persons from client-disciplines, the formation of the LACSG marks the beginning of what is popularly referred to as the *reform movement* in undergraduate linear algebra (Carlson et al., 1993). Based on surveys, questionnaires, and input from industry professionals who use linear algebra, the following five recommendations were made:

1) The syllabus and presentation of the first course in linear algebra must respond to the needs of client disciplines.
2) Mathematics departments should seriously consider making their first course in linear algebra a matrix-oriented course.
3) Faculty should consider the needs and interests of students as learners.
4) Faculty should be encouraged to utilize technology in the first linear algebra course.
5) At least one second course in matrix theory/linear algebra should be a high priority for every mathematics curriculum.

Since the formation of the LACSG and the subsequent publication of these recommendations in 1993, there has been considerable interest in the teaching and learning of undergraduate linear algebra. The LASCG recommendations indeed tend towards a de-emphasis on an abstract formal approach and a shift of focus towards a more applied matrix-oriented approach, a style not significantly different than most of the linear algebra texts since the 1970's. The use of technology is a distinguishing feature of linear algebra reform. The issue of how to teach linear algebra, which ultimately involves abstract conceptualization, from a concrete computational standpoint such as row reduction and systems of equations, is seen as problematic from several researchers in mathematics education (Dorier, 1998; Dubinsky, 1997; Vinner, 1997; Harel, 1987).

In reference to Cayley's identification of linear transformations with matrices, Artin (1957) succinctly expresses the apparent conflict between traditional and contemporary approaches to the teaching of linear algebra in the following:

> Mathematical education is still suffering from the enthusiasms which the discovery of this isomorphism has aroused. The result has been that geometry was eliminated and replaced by computations. Instead of the intuitive maps of a space preserving addition and multiplication by scalars (these maps have an immediate geometric meaning), matrices have been introduced (p. 13).

Teaching linear algebra in a way that promotes *meaningful learning* can be difficult. Meaningful learning refers to the ability to "relate new knowledge to what is previously known" (Ausubel, 1963). As earlier discussed, the author's own experience in teaching university level linear algebra, in which Penney's (2004) reform-oriented text was used, played a large part in the motivation for this study. Even with ample use of technology, such as MATLAB, and a matrix-oriented approach, many of the author's undergraduate linear algebra students were able to easily implement computational algorithms, yet were later seen to encounter great difficulty with problem solving involving the use of computational methods in unfamiliar settings. Perhaps Dubinsky (1997) was correct in his assertion that the "[LACSG] recommendations and some of the solutions that arise from considering them could, unwittingly, cause some teachers to fall into this trap," the *trap* being the tendency to just teach students imitative procedures, whereby they only "master the mechanics of the operations … having no understanding of what they have done" (Harel, 1989a).

In 1998, eight years after the LACSG meetings, the Park City Mathematics Institute held a series of workshops on the topic of linear algebra teaching and reform. During those workshops, it was acknowledged that most contemporary textbooks reflected changes similar to the LACSG recommendations, including the use of technology, however; participants still expressed that "many students do not master some topics and harbor serious misunderstandings about some concepts" (Day & Kalman, 1999, p.7). The consensus drawn from the PCMI workshops was that it is of the utmost importance to better understand how and why students learn, or do not learn, linear algebra. It is the purpose of this study, within a limited sub-domain of problems from linear algebra settings usually encountered in a first course, to better understand the difficulties novice students often encounter in learning a subject having the same computational and formal characteristics applicable to multiple contexts.

Research Questions

This study employs an integrated, *mixed methods* (Glaser & Strauss, 1967, p.36) approach, using both traditional and actor-oriented theoretical perspectives of transfer. Two pilot studies, three experiments, and three interview case studies, aim to better understand why beginning linear algebra students often seem unable to apply problem solving knowledge from a familiar setting to a conceptually non-isomorphic setting, even though the problems may share common solution procedures and problem representations (Lobato, 2003). In the upcoming Chapter Two, research literature in linear algebra mathematics education and transfer theory outline the development of the following questions which guide this study:

Research Question 1 (Quantitative)

> Is there evidence, from the traditional transfer perspective, that transfer is facilitated between linear algebra problems from non-isomorphic settings which share similar (matrix) representations AND solution procedures?

Research Question 2(Qualitative)

> In what ways, from the theoretical perspective of actor-oriented transfer, do novice linear algebra students commonly have difficulty with conceptually non-isomorphic problem settings, even when novel problem settings share similar problem representations and solution procedures as familiar problem settings?

Research Question 3(Mixed: Quantitative + Qualitative)

> What evidence can be found that indicates meta-cognitive intervention(s) may facilitate traditional and/or actor-oriented transfer across conceptually non-isomorphic problem settings involving novel target problems which share similar problem representations and solution procedures as more familiar problems?

Chapter 2

REVIEW OF LITERATURE

Introduction

This dissertation research was initially inspired from the personal experience of the author in the classroom, teaching undergraduate linear algebra as a graduate student during the Spring 2005 quarter at a mid-sized state university in the United States. The pedagogical design of the text used for the class, Penney's *Linear Algebra: Ideas and Applications*, was largely based on the LACSG recommendations previously discussed in Chapter One (Penney, 2004; Carlson et al., 1993). Throughout the author's teaching experience, which emphasized matrix representations and the row echelon form (REF), the author often noticed the difficulty some students encountered when attempting to solve problems from new conceptual settings, even though the procedures and representations needed to solve the problems were similar to previous problems from a more familiar setting. It is the basic purpose of this chapter to situate the initial perceptions of the author in the context of relevant research and theory for the formulation of research questions to better guide this inquiry into the learning of undergraduate linear algebra.

This chapter first discusses relevant empirical research literature in mathematics education related to undergraduate linear algebra. The next section discusses the foundations of constructivist thought which underlie the researcher's theoretical perspective. After the discussion on constructivism, the next section briefly outlines two theoretical perspectives of transfer comprising the theoretical framework guiding the methods for this study. Concluding this chapter, the related research literature on transfer is presented, which, based on the theoretical framework, generates Research Question 1. Presentations of the remaining research questions which frame this study conclude the literature review.

Mathematics Education Research in Linear Algebra

A predominant amount of previous educational research on undergraduate linear algebra focuses on difficulties associated with the appropriation of different linear algebra mathematical contexts to the same representational formalism (Harel (1989b, 1990); Hillel & Mueller, 2006; DeVries & Arnon, 2004; Hillel & Sierpinska, 1994; Dias & Artique, 1995; Dorier, 2000b). Traditionally, students taking an introductory course in linear algebra were first exposed to an axiomatic treatment of vector-spaces, eventually finishing with the topic of diagonalization of linear operators (Dorier, 1998). During the past 30 years, linear algebra courses have tended to deal less with abstract formal treatments and have begun more from the computational, matrix-oriented approaches reflected in the LACSG recommendations (Carlson et al., 1993; Uhlig, 2003). Harel (1987) referred to this pedagogical approach as *computation-to-abstraction* (p.29). Although this strategy is meant to "enable the student to learn the new language and the new reasoning gradually while moving toward more abstract material," many undergraduate linear algebra students still encounter "the obstacle of formalism" related to abstract representational characteristics inherent in the subject (Harel, 1987, p.29; Dorier, 2000a, p.28).

Research Studies in Linear Algebra

In Harel (1989b; 1990), a study was conducted to see if emphasis on a familiar geometric system would lead to a better understanding of the abstract vector space concept, rather than emphasis from only an algebraic viewpoint. Visual geometric embodiments were employed as a major part of a pedagogical strategy aimed at teaching linear algebra to high-school and university students. The instructional strategy consisted of three phases. *Visual vector-space models* comprised the first phase, wherein vector-spaces were modeled using familiar 2-d and 3-d embodiments of directed line-segments. Vector-space concepts such as “vector-space, subspace, basis, and dimensions, were translated in terms of these geometric models, keeping the concepts and their construction processes parallel to those in the theory of vector-spaces” (Harel, 1990, p.389). In the second phase of the teaching, central vector-space concepts were translated into the language of $\Re^1$, $\Re^2$, $\Re^3$, and finally $\Re^n$ using vector coordinates.

Also in the second phase, the geometric concept of dimension was explored algebraically by the study of $n \times m$ systems of equations, using matrices as flexible representational forms which could represent vectors, systems of equations, and linear transformations. In the final third phase, central vector-space concepts were explored in the abstract vector-space setting with undefined elements, yet remaining in the dimensions of 1, 2, or 3. It was the goal of the design of the third stage for students to "absorb the idea that derived results in linear algebra depend solely upon the axioms of a vector-space, not upon the definitions of specific elements" (Harel, 1989b, p.145).

The participants of the study were 56 high school students from upper level 11^{th} and 12^{th} grade classes, and in addition, 72 sophomores from two second-semester university classes. Three of the high school classes had 30, 40, and 60 hours of instruction in one year, while a fourth class received 65 hours of instruction over a period of two years. The classes were further divided into control and experimental groups with the control groups receiving standard instruction, while the experimental groups received instruction based on visual embodiments. The experimental group of university students was exposed only to phase 1 of the didactic sequence for a period of four weeks, two hours per week.

Problem 1: Let μ and β be vectors in the vector space V. We define $L_\mu = \{x_\mu : x \in \mathbb{R}\}$, and $L = \beta + L_\mu$.

a. Prove that L_μ is a subspace of V.
b. Prove that $\beta - \mu$ is a vector in L.

Problem 2: The square of numbers: $\begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$ is called a magic square because each row, each column, and each diagonal has the same total sum of entries. Let W be the set of all magic squares. Define two operations on magic squares so that W will be a vector-space over the real numbers. Justify your answer.

Figure 2.1 Questionnaire, Harel (1990).

Two questions were used in the experimental methods for the study as an instrument for determining the students' understanding of the vector-space concept (see Figure 2.1). Results indicated that the experimental high school class scored the highest, followed by the experimental university class, which had a shorter period of instruction (see Table 2.1). The lowest performing group was the high school linear algebra class receiving standard instruction. The researchers concluded that the instructional treatment had a significant positive effect for learning vector-space concepts. It was also concluded that the 'poor' performance on Problem 2 was due to students having greater difficulty with unfamiliar algebraic systems whose elements were "collections of numbers, such as matrices or functions" (Harel, 1989b, p.147).

	High school classes		University Classes	
Problem	Control	Experimental	Control	Experimental
	% correct	% correct	% correct	% correct
1	28	83	51	76
2	28	62	44	50

Table 2.1 Test results for evaluating understanding of vector-space concept.

This conclusion would eventually take form as Harel's Piaget-inspired *Concreteness Principle*, which states:

> For students to abstract a mathematical structure from a given model of that structure, the elements of that model must be conceptual entities in the student's eyes; that is to say, the student has mental procedures that can take these objects as inputs" (Dorier, 2002, p.880).

As an example, a 2×2 matrix could be viewed as a collection of numbers in an array, or thought of as an object which is an element of a vector space, such as $M_{2,2}(\Re)$, the vector space of 2×2 matrices with real entries. In order to understand problem 2, it was necessary for students to be able to objectify an array of numbers to be an object which may be operated upon with operations such as composition or functional evaluation, or to be considered as an element of a set.

Although the results of the experiment from Harel (1989b; 1990) suggest evidence substantiating the hypothesis that familiar geometric embodiments cause a better understanding of vector-space concepts than do unfamiliar algebraic embodiments, it is difficult to make a comprehensive conclusion based on such a limited scope of questions used for measuring understanding. In addition, at least one anomaly within the study would eventually prompt Harel towards further inquiry and *reappraisal* of the notion of using geometry as a concretized foundation for a computation-to-abstraction approach to teaching linear algebra (Harel, 1999). This anomaly refers to Harel's (1989b; 1990) result that during the first phase of experimental instruction, students sometimes concluded that the only space of vectors that existed were the geometric embodiments presented; namely, spaces of directed line-segments in 1, 2, and 3 dimensions. In Harel's (1990) concluding commentary, he states that:

> Visualization alone is not sufficient to foster the learning of abstract theories such as linear algebra; other factors should be included in the content presentation (p.391).

Harel (1990) went on to say that the above factors needed in conjunction with visualization must also fulfill a *Necessity Principle*, which states that:

> For students to learn, they must see an (intellectual, as opposed to social or economic) need for what they are intended to be taught" (Dorier, 2002, p.880).

Similar to the Concreteness Principle, the Necessity Principle was inspired by Piaget via his concept of *disequilibrium* found in his developmental *genetic epistemology*; which asserts that "knowledge develops as a solution to a problem" (Dorier, 2002, p.880; Piaget, 1970). Through *isomorphization*, whereby "a set of geometrical objects is associated with a set of algebraic objects;" the visualization approach of the Harel (1989b; 1990) study aimed to follow the Necessity Principle in going from geometry to the language of $\Re^n$, with the goal of motivating students to "see how one mathematical system is better suited to computational techniques" (Harel, 1987, p.31; Harel, 1990, p.387). In view of the instructional design for the Harel (1989b, 1990) study being motivated by the Necessity Principle, it is not clear why subjects would seem constrained to proto-typical, $\Re^1$, $\Re^2$, and $\Re^3$ levels of conceptualization of the vector-space concept. Before assertions regarding the causal effectiveness of the "gradual abstraction of concepts and their constructive processes along a firm visual base," could be better understood, it became necessary to take a deeper and perhaps more qualitative approach to the role of visualization in making the leap from computation-to-abstraction (Harel, 1989b, p.147-8).

Similar to Harel (1989b; 1990), Harel (1999) addressed the question of whether or not undergraduate mathematics majors could understand geometry and linear algebra from a purely axiomatic viewpoint, without appealing to embodied, intuitive conceptualizations such as spatiality. The participants for the study included 169 mathematics and engineering majors involved in six teaching experiments. A wide variety of data was collected via classroom observations, interviews, homework, exams, and quizzes. The study found that introducing geometry "before algebraic concepts have been formed" could constrain students to the level of the geometry, taking the geometry itself as the object of study, and thereby unable to progress to a level of generality necessary for understanding related abstract linear algebraic concepts (Harel, 2000, p.184). From this result, it is understandable why some subjects from the Harel (1989b; 1990) study might have remained in the geometric world of directed line-segments, unless they received the necessary *factors* in their instruction which showed them how the geometric situation was isomorphic to the algebraic one. Harel (2000) adds:

> It has been shown in my previous studies, geometry can be a very powerful tool in solidifying linear algebra concepts, but we need to consider carefully the *way* geometry is introduced and used. We, the teachers, see how the geometric situation is isomorphic to the algebraic one, and so we believe that the geometric concept can be a corridor to the more abstract algebraic concept. Unfortunately, many students do not share this important insight (p.184).

Harel (1999) defines *contextual conception* as the inability "to detach from a specific context, whether it is the context of intuitive Euclidean space in geometry or the context of $\Re^n$ in linear algebra" (p.603). His revised findings suggest that although geometric embodiments could be valuable representations for introducing abstract linear algebra concepts, it was not enough to expect students to form natural generalizations based solely on geometric examples of higher order abstract principles. Harel (1999) distinguished between the terms *intuitive axiomatic conception*, *structural conception*, and *axiomatic conception* (p.602). Intuitive axiomatic conception, the first level, was defined as conceptualization contingent on correlation to a subject's own intuition. Second level structural conception was defined as "understanding that definitions and theorems represent situations from different realizations (settings) that share a common structure determined by a permanent set of axioms," while third level axiomatizing conception was defined as the ability "to investigate the implications of varying a set of axioms, or to understand the idea of axiomatizing a certain field" (Harel, 1999, p.603).

According to Harel (1999), in order to avoid contextual conception, where "general statements are interpreted in terms of a specific context," students require *meta-mathematical* knowledge in the form of necessity, concreteness, and an understanding of the structural relationships between the contextual and the abstract-formal representations of vector-space and matrix theory (p.613). Dorier (2000a) defines meta-mathematical knowledge as:

> Information of that which constitutes mathematical knowledge concerning methods, structures, and (re)organization (of lower level competencies). Methods are defined as the procedures applicable to a set of similar problems within a given field: the methods designate that which is common to problem solving and not the technique itself (algorithm) (p.151).

The author points out that Dorier's notion of meta-knowledge and its reference to 'structure' appears closely related to Harel's (1999) conjecture that the structural conception level is a necessary attainment for learner's to "detach" from the limiting confines of the “extremely powerful concept image(s)” which may be formed from an overly intuitive geometric approach to teaching linear algebra (p.613).

Hillel & Mueller (2006) investigated the multiple roles played by a matrix in a first course at university level linear algebra. Their research was interested if subjects could link the concept of a vector $\vec{x}$ being a solution to a homogenous system, $A\vec{x} = \vec{0}$, with the following equivalent notions:

1. The components of $\vec{x}$ satisfy a set of equations.
2. $\vec{x}^t$ is a vector orthogonal to the rows of the matrix A.
3. $\vec{x}$ may express a particular linear relationship among the columns of A.
4. $\vec{x}$ belongs to the kernel subspace of the linear transformation induced by the matrix A (p. 3).

Hillel & Mueller (2006) characterized these multiple roles as manifestations of the following three modes or viewpoints:

- an *abstract* mode characterized by “the language and concepts of the generalized formal theory.”
- an *algebraic* mode relating to “using the language and concepts of the more specific theory of $\Re^n$.”
- a *geometric* mode involving “using the language and concepts of 2- and 3- dimensional space” (p.3).

Their study specifically addressed the following questions:

1) Do students possess understanding of systems of equations and solutions beyond a procedural sense? Can they interpret the homogenous system in the four different ways described above? Can they translate statements from one setting into equivalent statements in another setting?

2) Is there improvement in the negotiation, interpretation, and translation of multiple linear algebra settings as students go from completing a first course to completing a second course in linear algebra?

The methods Hillel & Mueller (2006) used to answer these research questions consisted primarily in the implementation and analysis of a questionnaire. The participants in the study were 154 students from three classes in three different universities; all having already completed a first one-semester linear algebra course. The students were from majors such as: mathematics, computer science, engineering, actuarial mathematics, physics, economics, and education. The first group of 40 subjects, G1, was mostly mathematics majors except for 10 taking the second linear algebra course as an elective. This group was a mix of 1st and 2nd semester linear algebra students. Their questionnaire was given as part of the final exam. The second group, G2, was composed of 47 subjects taking their 1st course in linear algebra, having diverse major backgrounds. They were administered their questionnaire on a voluntary basis during 15 minutes at the end of the course. The third group, G3, was composed of 67 subjects of mostly engineering, science, and mathematics education majors. This group had just completed their 1st linear algebra course and their questionnaire was also implemented as part of the final exam (see Figure 2.2).

It was the focus of the questionnaire to see if subjects could link the concept of a solution to a homogenous system with the other three interpretations of a solution involving orthogonally, linear combination, and null-space. In order to study whether these linkages occurred, it was necessary for the subjects to answer correctly on question Q.I. Twenty of the 154 subjects did not conclude that the two vectors in Q.I were solutions to the given homogenous system; hence the study was reduced to 134 total subjects. Analysis of the results of the questionnaire was based on the performance of the subjects on questions Q.II, Q.III, and Q.IV in relation to question Q.I (see Table 2.2).

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 5 \\ 0 & -3 & 6 & 2 & -9 \end{bmatrix}$$

Q.I
Consider the homogenous system of linear equations $A\vec{x} = \vec{0}$.
Given $[1,1,1,1,1]^t$ and $[-3,1,1,3,1]^t$, which of these two column vectors is a solution to the above system? Explain.

Q.II
(a) Give an example of a non-zero row vector that is orthogonal to the first row of the matrix A.
(b) Can a non-zero row vector be orthogonal to all the rows of A? If YES, give an example of such a vector; if NO, explain your reasoning.

Q.III
Are the columns of A linearly independent? If YES, explain your reasoning; if NO, give an example of a non-trivial linear combination of the columns of A which yields the zero vector.

Q.IV
Does the kernel of A contain any non-zero vectors? If NO, explain your reasoning; if YES, give an example of such vectors.

Figure 2.2 Questionnaire, Hillel & Mueller (2006).

Abbreviation	Description
L	linkage with **Q.I**
U	unsubstantiated answers (whether correct or not)
NR	No response
REF	using row reduction on A (whether it led to a correct or incorrect conclusion)
Eq	setting up equations
Z	the kinds of answers that would usually merit a mark of zero
O	others

Table 2: Characteristics of questionnaire responses of the 4 groups

	Q.II	**Q. III(a)**	**Q. III(b)**	**Q.IV**
G 1: N=35				
L	14 (40%)	4 (11%)	4 (11%)	8 (23%)
U	7 (20%)	2 (6%)	0 0	8 (23%)
Eq,	1 (3%)	0 0	1 (3%)	0 0
REF	0	17 (49%)	0 0	13 (37%)
NR	2 (6%)	1 (3%)	25 (71%)	3 (9%)
Z	10 (29%)	9* (26%)	4 (11%)	2 (6%)
O	1 (3%)	2 (6%)	1 (3%)	1 (3%)
G 2: N=38				
L	22 (58%)	7 (18%)	7 (18%)	15 (39%)
U	5 (13%)	12 (32%)	0 0	4 (9%)
Eq,	2 (5%)	1 (3%)	0 0	0 0
REF	0	7 (18%)	0 0	1 (2%)
NR	7 (18%)	5 (13%)	27 (71%)	15 (39%)
Z	2 (5%)	3 (8%)	0 0	2 (5%)
O	0 (0%)	3 (8%)	4 (9%)	1 (2%)
G 3: N=61				
L	15 (25%)	4 (7%)	3 (5%)	18 (30%)
U	11 (18%)	5 (8%)	10 (16%)	11 (18%)
Eq,	3 (5%)	7 (11%)	0 0	0
REF	0	36 (59%)	34 (58%)	13 (21%)
NR	8 (13%)	1 (1%)	4 (7%)	7 (11%)
Z	16 (26%)	5 (8%)	7 (11%)	6 (10%)
O	7* (11%)	3 (5%)	3 (5%)	6 (10%)

*These include the six students that gave the zero vector as a one which is orthogonal to all the rows.

Table 2.2 Questionnaire results, Hillel & Mueller (2006).

Using interpretive analysis of the data, as well as inspection of the questionnaire work, the researchers concluded that “most students of linear algebra are not able to link different but equivalent conceptions of the same underlying object” (Hillel & Mueller, 2006, p.10). The results suggest that linkage did improve, however, with subjects who had reached their second linear algebra course. Students from G2 had higher rates of establishing links, [58%, 18%, 39%], compared to the rest of the population, [38%. 11%, 30%], although statistical analysis was not demonstrated for determining the significance of these results (see Table 2.2).

Furthermore, Hillel & Mueller (2006) observed that in the *matrix-based course*, the roles played by a matrix A include:

1. Representation of a system of linear equations, a linear transformation, a change of basis, or an element itself of a vector space.

2. Matrix multiplication $A\vec{x}$ interpreted as dot products of rows or linear combinations of columns.

3. The matrix association with fundamental subspaces such as row space, column space, and null space; as well as relationships between range and kernel as a linear transformation.

4. The implications of elementary row operations as related to the various interpretations of the matrix A.

Hillel & Mueller (2006) concluded that it may be unrealistic for beginning students to attain conceptual understanding regarding the linking of "seemingly disparate contexts," unless it is made "an actual object of study" to understand the underlying "web of connections" associated with the multiple embodiments of the matrix representation (pp. 12-13). This conclusion finds particular accord with work by Douady (1986), which expressed the duality of mathematical concepts in the sense of how the tools of a mathematical activity may become an eventual object of study, a notion labeled the *tool/object dialectic*.

Using only one instrument in the form of a questionnaire, it is difficult to determine precise cognitive explanations as to the difficulties subjects exhibited in the Hillel & Mueller (2006) study. In addition, the researchers surmised that subjects taking the questionnaire as part of a final exam may have tried harder than subjects who were voluntarily completing the question at the end of class, as evidenced by the tendency for the G1 and G3 groups to leave no questions unanswered as opposed to the G2 group which had more than half of its subjects not answering Q. IV. Nevertheless, one explanation made by the researchers was relevant in conjunction with the admitted limitations of the questionnaire.

Based on a closer inspection of the written student work from the questionnaire, the researchers noticed that students not enrolled in a second linear algebra course usually relied on row reduction as an initial solution procedure when encountering Q1. This is not surprising based on the predominantly *computation-to-abstraction* approach usually taken in teaching a first course on linear algebra, where emphasis is commonly placed on finding the REF form for a given matrix. Since most examples of an REF approach occurred in G1 and G3, there was speculation that having the questionnaire given as part of the final exam may have encouraged a tendency towards use of familiar studied procedures.

DeVries & Arnon (2004) developed an interview questionnaire for the purpose of investigating the different meanings which might be associated with a solution to a system of equations. Their methodology involved twelve students who had just finished a 1st semester course in linear algebra. Forty-five minute interviews were conducted individually using a structured questionnaire (see Figure 2.3). In their analysis of the interviews there were 4 response types characterized. In response type 1, the subjects simply did not understand the question; ‘What does a

A. What is a solution of this equation (what does it look like)? $3x_1 + 2x_2 - x_3 + x_4 = 5$. How many solutions does it have? Is the sum of two solutions also a solution? What about a scalar multiplication?

B. What does a solution of this equation look like?

$$x_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

Which of the following might be a solution? a. $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ b. 7 c. $(1,0,1,7)$ d. $\begin{pmatrix} 2 \\ 0 \\ 1.5 \\ 7 \end{pmatrix}$

How would you check whether it is a solution?

C. Here is a homogenous system of equations $A\bar{x} = 0$. Suppose each of the vectors u and v is a solution of this system. What do you think of the vector $u + v$? Is it a solution of the system or not?
(If no answer) Would you like to use an example?
(If no answer) Would you like me to present an example?

Here is an example: $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ -2 & -3 & -5 & -3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

D. What about a non-homogeneous system? How does it differ from a homogenous system? Here is a non-homogenous system:

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ -2 & -3 & -5 & -3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Suppose each of the vectors u and v is a solution of this system. What do you think of the vector $u + v$? Is it a solution of the system as well? How can we check/prove?

E. A and B are $n \times n$ matrices of the same order. What would be a solution of such an equation: $AX = B$?

Figure 2.3 Questionnaire, DeVries & Arnon (2004).

solution look like?' Response type 2 was characterized by memorization of rules rather than being able to provide an explanation. Response type 3 involved students confusing the right hand side of a matrix system equation with the solution, i.e., for a system AX=*b*, the subjects viewed *b* as the solution; echoing results by Kieran (1981) in that type 3 errors may be due to confusion over the meaning of the equality sign in the sense of "interpreting the equality to mean: *the result is,* rather than symbolizing equivalence of both sides of the equation" (DeVries & Arnon, 2004, p.58). Response type 4 typified students who, when asked 'What a solution looked like,' attempted to solve for the solution. Subjects giving type 4 explanations appeared to be confused between the concepts of *solution* and *solving*. Instead of attempting the substitution of given potential solution candidates, or looking for features such as the proper size of a solution vector, they seemed restricted to an immediate carrying out of learned procedures for finding the REF form of a matrix by the familiar Gaussian elimination algorithm. The theoretical framework used in the DeVries & Arnon study was based on *APOS theory* (Dubinsky, 1997).

APOS, an acronym for action-process-object-schema, was influenced by Piaget's theory of cognitive development (Piaget & Garcia; 1983). According to APOS, mathematical concepts first begin in a learner's mind as an action. When a learner streamlines performance of an action such that they can verbalize it without performing it, or can predict the outcome; the learner is said to operate on a *process level* of understanding. Furthermore, when a learner can view a process as an element which can be operated on in the sense that actions are performed on objects;

they are said to operate on an *object level* of conception. As an example, Harel's previous Concreteness Principle can be restated in terms of the necessity for linear algebra students to attain an object-level of understanding. The constructions which result from the webs of interrelated objects and processes are referred to as *schemas.* In terms of the APOS framework, DeVries & Arnon (2004) concluded that type 4 subjects interpreted the *solution* to a system of equations at the *action* level of understanding, not being at a sufficient level of development to view the solution concept at the object level (De Vries & Arnon, 2004, p.59).

By confining problem solving to algorithmic procedures such as finding the REF, subjects from both the Hillel & Mueller (2006) and DeVries & Arnon (2004) studies demonstrated the propensity for beginning linear algebra students to engage in, what Sierpinska (2000) referred to as, *practical* verses *theoretical* thinking, a distinction influenced by Vygotsky's notion of *everyday concepts* verses *scientific concepts* (Sierpinska, 2000, p.211). Practical thinking is characterized by the propensity for students to view a subject, such as linear algebra, as an aggregate of "prototypical examples" interconnected and understood primarily through "goal-oriented, physical action(s)" (Sierpinska, 2000, p.212). The representations important at the practical level are the necessary symbols needed to perform computations. Theoretical thinking, according to Sierpinska (2000), produces formal systems where "semiotic representation systems become themselves an object of reflection and analysis" (p.212). In theoretical thinking, the definition of a mathematical object takes precedence over any proto-typical example, as formal definitions are necessary vehicles for conveying the various meanings and existences of generalizing mathematical relationships.

Hillel & Sierpinska (1994) examined a ‘persistent mistake’ made by undergraduate students when finding representations of linear operator matrices under different bases. The following question (Fig. 2.4) was asked as part of an activity to a group of 29 students taking a second undergraduate linear algebra course during the January 1993 term:

> Let $L:\Re^3 \to \Re^3$ be a linear operator represented by the matrix $A = matrix([[1,2,3],[3,4,5],[6,7,8]])$ relative to the basis $\{(1,1,1),(2,1,0),(0,5,6)\}$ for $\Re^3$. Find $T(1,1,1)$.

Figure 2.4 Activity question, Hillel & Sierpinska (1994).

Only 3 subjects were able to correctly answer the question, while 17 students left the question blank. Of particular interest were the following solutions:

> One student took the first column of A as the answer: “$T(1,1,1) = (1,3,6)$” and another had the same approach though she hesitated between choosing the first column and the first row and wrote: “$T(1,1,1) = (1,3,6)$ or $(1,2,3)$.” One student found the formula for a coordinate vector in the given basis, obtained, by substitution, $(1,0,0)$ as a coordinate vector for $(1,1,1)$ and claimed that this is $T(1,1,1)$ (Hillel & Sierpinska, 1994, p.69)

The researchers speculated that subjects were confused over the intended meaning of the linear operator matrix representation by interpreting the column of an operator matrix to mean the image of the corresponding basis vector as a result of simply multiplying that vector by the matrix; as opposed to meaning the image of a basis vector written as a linear combination of possibly non-standard basis vectors.

A similar question, in Figure 2.5, was asked as part of lab activities involving the use of the computer application *Maple* to a group of 29 students from the Fall 1993 term:

> Let $T:\Re^3 \to \Re^3$ be a linear operator represented by the matrix $A = matrix([[7,0,1],[0,2,0],[1,3,0]])$ relative to the basis $\{(1,3,0),(0,1,3),(0,0,1)\}$ for $\Re^3$. Find the matrix for T in the standard basis.

Figure 2.5 Lab question, Hillel & Sierpinska (1994).

There were 15 correct answers (52%), while 10 students' answers (34%), again as the previous group of students, exhibited the mistake of treating the columns of the matrix $A = \begin{pmatrix} 7 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix}$ as the images, under T, of the corresponding basis vectors. For operators under the standard basis, it is indeed true that the columns of the matrix are the images of the standard basis vectors under the operator, as written in the standard basis. In a non-standard basis, however, the columns again represent the scalar representations of the image of each basis vector, but are scalars relative to the non-standard basis. The 4 remaining solutions to the question, comprising (14%) of the subjects, were deemed nonsensical by the researchers. The Hillel & Sierpinska (1994) study revealed a consistent difficulty undergraduate linear algebra students experience when translating between different matrix representations of a linear operator. Several other iterations of the questions similar to the above were conducted on other linear algebra classes as a part of their study, producing similar consistent results. Analysis of variance was not included in their report, in terms of determining statistical significance.

Hillel & Sierpinska (1994) concluded that different levels and ways of possible thinking about representation in linear algebra are a probable source for the difficulties unveiled by their study. They based much of their theoretical analysis on Piaget & Garcia's (1983) notions of *intra*-, *inter*-, and *trans-level* thinking (p.173). Although Piaget is usually known for his stage-theory and genetic epistemology, in his later work inspired by category theory, he formulated these three universal types of knowledge construction as a generalization and reformulation of his previous work. Intra-level thought, similar to Sierpinska's practical thinking and Skemp's instrumental understanding, results from the Piagetian process known as *empirical abstraction* which abstracts the form and local properties of objects encountered in perception and thought (Sierpinska, 2000; Skemp, 1976; Piaget & Garcia, 1983). Inter-level thought results from the relationships that are constructed between the intra-type *objects*. For example, Hillel & Sierpinska (1994) characterized learning about the *properties* of matrices and linear operators as inter-level thinking about mathematical objects (p.65). Trans-level thought, similar to Sierpinska's *theoretical thinking*; Skemp's *relational understanding*; and Dorier's *meta-level cognition*; refers to thinking associated with another Piagetian process known as *reflective abstraction* (Sierpinska, 2000; Skemp, 1976; Dorier, 1995b; Piaget & Garcia, 1983).

In one sense, reflective abstraction leads to the hierarchical development seen in Dubinsky's APOS; going from actions to processes to objects to schemas.

In another sense, reflection induces a “reorganization, on a new level, of what was derived from the preceding one” (Piaget & Garcia, 1983, p.270). Through reflective abstraction, transformations of the relationships derived from inter-level thought result in trans-level constructions. Dorier (2002) explains:

> Pre-existing elements of knowledge or competencies at a lower level … need to be integrated within a process of abstraction, which means they have to be looked at critically, and their common characteristics have to be identified, and then generalized and unified (p.876).

For example, in the case of basis representations of operators, trans-level thinking produces insight into notation consisting of representational information for how the concept of basis is encoded into the matrix formalism, thus able to represent any operator in any basis. On the other hand, inter-level thinking is concerned more with the immediate “relations between matrices, vectors, and operators” written in the prototypically local standard basis (Hillel & Sierpinska, 1994, p.65).

In Hillel & Sierpinska (1994), results indicated that undergraduate linear algebra subjects operated primarily at the inter-level stage, able to work with the standard basis matrix representations of operators of which they are familiar with. Similarly, from Sierpinska (2000), it was demonstrated that beginning linear algebra students largely only seem to require inter-level thinking to understand basic linear algebra from a computational standpoint. It is not surprising that a typical computation-to-abstraction approach to the teaching of undergraduate linear algebra might only require inter-level thought to, ‘just get by,’ as Dorier’s *Fundamental Epistemological Hypothesis* suggests:

> Linear algebra is a *generalizing* and *unifying* theory. As a consequence, it is also a formal theory. It simplifies the solving of many problems, but the simplification is only visible to the specialist who can anticipate the advantage of generalization because they already know many contexts in which the new theory can be used. For a beginner, on the other hand, the simplification is not so clear as the cost of learning many new definitions and theorems seems too great with regard to the use she or he can make of the new theory in contexts in which solving systems of linear equations is usually quite sufficient (Dorier, 2000a, p.29).

Hillel & Sierpinska (1994) theorized that trans-level thinking was the missing aspect of thought causing students to remain thinking at the level of objects and relationships, instead of grasping the general implications and meanings of the matrix representations themselves, apart from familiar local properties.

Based on their evidence of the pitfalls students consistently exhibited when viewing a matrix in a limited and prototypical way, Hillel & Sierpinska (1994) proposed the necessity for trans-like thinking requiring representation itself to be an object of reflection in the learning of linear algebra. This prescription is identical in substance to the *tool/object dialectic* and *practical/theoretical thinking* constructs mentioned earlier in connection with the Hillel & Mueller (2006) and DeVries & Arnon (2004) studies; as well as the call for linear algebra *meta-knowledge* in Harel (1999). In all of the mentioned studies, it appears that the *generalizing* and *unifying* formal and conceptual characteristics of linear algebra demand a trans-level type of mathematical thinking largely unfamiliar to the typical undergraduate student.

As a final study concluding this section, Dias & Artigue (1995) examined flexibility issues regarding the Cartesian and parametrical viewpoints in linear algebra; where they distinguish between *hierarchical* and *non-hierarchical* flexibility (p.34). Hierarchical flexibility refers to the going back and forth between dual notions of "process and object" that a mathematical concept may operate (Dubinsky, 1997). For example, a linear transformation may be thought of as a process of associating vectors of a domain with vectors of a range, or a linear transformation may be thought of as an element of a vector space. The cognitive development from process to an object is referred to as "encapsulation" according to APOS terminology (Dubinsky, 1997). An important aspect of advanced mathematical understanding which 'essentially marks the progress between intra- and inter-operational systems," is *encapsulation – de-encapsulation*; inspired from Piaget's concept of *reversibility*. Reversibility refers to the ability to go from process to object and then back again from object to process in mathematical thought related to a particular concept (Piaget & Garcia, 1983, p.177). Non-hierarchical flexibility, the focus of the Dias & Artigue (1995) study, refers to the going back and forth between different settings or registers a concept may function in, an important aspect of the Cartesian and parametric viewpoints in the learning of linear algebra (Rogalski, 1996).

Recall from Chapter One, a setting was defined as "being made of objects of some mathematical branch, of relationships between these objects, of their eventually various formulations and of mental images associated with these objects and relationships" (Douady, 1986, p.5). While setting involves the function of a mathematical concept, the term *viewpoint* refers to the semiotic representation of that concept. After Duval (1993), *semiotic representations* are "productions made by the use of signs belonging to a system of representation which has its own constraints of meaning and functioning (from Dorier & Sierpinska, 2001, p.260). For example, an equation may be discussed in either algebraic or geometric settings; however, within each setting the equation is represented using Cartesian or parametric viewpoints. As an example, in the parametric viewpoint $\{(x, y, z) \in \Re^3 : x = -t - s, y = t, z = s : s, t \in \Re\}$ describes the same planar subspace as $\{(x, y, z) \in \Re^3 : x + y + z = 0\}$ does in the Cartesian viewpoint. Pertinent to this dissertation, in which I noticed my linear algebra students having a striking degree of difficulty with changing settings, the Cartesian viewpoint could be characterized as representational interpretation belonging to a *row-picture*, *(Rp)*, viewing linear systems as row equations; whereas a parametric viewpoint comprises representational interpretations characteristic more of a *column-picture*, *(Cp)*, emphasizing vectors as generators through linear combinations (Strang, 2003). It was the basic purpose of the Dias & Artigue (1995) study to determine the conditions mathematically and cognitively necessary for flexibility between these two viewpoints.

Dias & Artigue (1995) employed of a variety of methods aimed at better understanding articulation problems students commonly have between the Cartesian and parametric viewpoints in linear algebra. Of central focus for this dissertation is Dias & Artigue's (1995) empirical study which involved the analysis of an exam for 113 1st year students from the University of Lille, France. There were four questions in the linear algebra part of the exam; the following two questions are analyzed in their research report (see Fig. 2.6). For the analysis of the exam questions, Dias & Artigue (1995) identified 6 variables related to the technical and conceptual demands arising from transitions between the Cartesian and parametric viewpoints; based largely on previous epistemological, historical, and textbook analysis from earlier phases of the methodology (see Table 2.3).

In $\Re^4$ are given the following vectors; $a = (0,-1,1,0), b = (2,1,1,0), c = (0,0,3,1), d = (2,0,-1,-1)$:

(1) Give a parametrical representation and a linear equations system for $lin\{a,b,c,d\}$.

(2) Does the system: $\{2y + 2t = \alpha, -x + y = \beta, x + y + 3z - t = \chi, z - t = \delta\}$, have a solution for all $(\alpha, \beta, \chi, \delta)$? Justify your answer without any calculations.

Figure 2.6 Exam questions. Dias & Artique (1995).

For question 1, the study found 34 different procedures in use. Two of those procedures, referred to as P and Q, comprised 38% of the answers. Procedure P essentially begins incorrectly by inputting vector coefficients into the system matrix as rows instead of columns, and then proceeds to correctly solve the system using Gaussian elimination (see Table 2.4). Procedure Q is similar to procedure P, however; instead of incorrectly inputting coefficients as rows, the coefficients are incorrectly inputted into a matrix as columns. Similar to both procedures P and Q, subjects correctly applied the Gaussian method to the incorrect matrix representations.

-type of space: $\Re^4$;
- type of given representations: 4 vectors represented by their coordinates in the canonical basis and an intrinsic symbolic notation of the generated subspace;
-type of required representations: a parametrical representation and a Cartesian representation;
-space and subspace dimensions: 4 and 3.
-compulsory/potential flexibility: If the expression "find a parametrical representation" is understood as: "find a minimal parametrical representation," flexibility is strongly necessary. But, students can produce the trivial parametrical representation $\{xa + yb + zc + td = 0 \mid x, y, z, t \in \Re\}$ and solve the associated linear system $xa + yb + zc + td = v$ in order to find the condition of $\alpha - \beta - \gamma + 3\delta = 0$ for $v = (\alpha, \beta, \gamma, \delta)$, which gives directly the Cartesian representation. If so, flexibility remains necessary but is reduced.
-flexibility knowledge: here it appears tightly linked to the resolution of linear systems, more precisely to the relations made between resolution conditions/Cartesian representation, rank of the linear system/rank of vectors system, number of necessary parameters/number of necessary equations with the fundamental theorem linking these two numbers (Rank-nullity theorem). This flexibility can function at a technical level, encapsulated in some way in algorithmic processes or at a conceptual level.

Table 2.3 Flexibility table, level of tasks, Dias & Artigue (1995, p.39).

For example, to find the linear system representations for $lin\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, P and Q procedures might formulate the system matrices as; $\left[\begin{array}{cc|c}1 & 1 & 0\\0 & 0 & 0\end{array}\right]$ and $\left[\begin{array}{cc|c}1 & 0 & 0\\1 & 0 & 0\end{array}\right]$; respectively. These representations would lead to incorrect parametric representations, as opposed to forming the correct initial representation, $\left[\begin{array}{cc|c}1 & -1 & 0\\0 & 0 & 0\end{array}\right]$, and finding a parametric representation such as; $\left\{\begin{bmatrix}x\\y\end{bmatrix} \in \Re^2 : x = t, y = t : t \in \Re\right\}$.

Similar to both procedures P and Q, subjects correctly apply the Gaussian method to incorrect matrix representations.

-to write the matrix whose lines are given by the coordinates of a, b, c, d;
-to write the associated linear system: $\{-y+z=0; 2x+y+z=0; 3z+t=0; 2x-z-t=0\}$, seen as a Cartesian representation of the subspace.
-to apply the familiar Gaussian elimination method to this system. This leads to a parametrical representation depending on one variable, as there are infinite many solutions, for example: $x = z, y = z, t = 3z$.

Table 2.4 Procedure P characteristics (Dias & Artigue, 1995, p.40).

Fifty-seven of the subjects correctly used elimination to determine the correct rank of 3 for the system arising from $lin\{a, b, c, d\}$; however, most do not use this information in order to find a correct "minimal parametric representation" (Dias & Artigue, 1995, p.40).

In reference to these findings, Harel (1989b) elaborated on the tendency for linear algebra students to engage in *practical thinking*:

> Students successfully manipulate matrices and functions, but they fail to understand the meaning behind these manipulations … students can bring a matrix to reduced row-echelon form, but they do not understand the relationship between the two matrices. The students master mechanics of the operations, but they seem to have no understanding of what they have done (p.141).

Out of the 113 subjects, only 20 gave the correct solutions to question 2. Fifty-five of the subjects were able to make the semiotic transitions from the given parametric representation to the Cartesian representation, $xa + yb + zc + td = 0$, yet they were not able to exploit these symbolic transitions to obtain a correct solution; as Artigue, Chartier, & Dorier (2000) explain in the following:

> The conversion between parametric and Cartesian representations can be easily dealt with through algorithms linked to the solving of linear systems. Such a fact tends to hide the cognitive complexity of articulation processes (p.253).

In a problem such as question 2, where "conceptual flexibility was compulsory," having difficulties after making the correct semiotic transitions to obtain the correct representations lends support to the notion that "cognitive flexibility is not a purely semiotic process; that it is of a more complex nature" (Dorier, 2000a, p.253).

The researchers also noted for question 2 that, "The attachment to the Gauss technique was so important that 13 students used it explicitly and 7 students used it implicitly without respecting the instructions" (Dias & Artigue, 1995, p.40). One of the possible reasons cited for this attachment is that:

> Students often identify one type of representation exclusively though semiotic characteristics (a representation with x's and y's is obviously Cartesian) without questioning the meaning of the representation (Dias & Artigue, 1995; Dorier, 2000a, p.255).

This observation further indicates the tendency for beginning linear algebra students to engage in *practical thinking*; demonstrating an excess of computational focus on the Gaussian technique and the REF form without theoretical understanding as to the conceptual implications and meanings of the representations involved (Hillel & Sierpinska, 1994; DeVries & Arnon, 2004; Hillel & Mueller, 2006). This finding also reflects Dorier's fundamental epistemological hypothesis, which earlier in this review described how students may not see the *necessity* for understanding linear algebra concepts because many of the problems encountered at the undergraduate level may be approached with simpler algorithmic techniques such as Gaussian elimination.

In summary, the Dias & Artigue (1995) study concerning flexibility between parametric and Cartesian viewpoints, revealed evidence that on one level, a significant number of subjects were unable to obtain a correct matrix representation for problems requiring viewpoint transition, but were able to correctly use the Gaussian technique on the incorrect representation and; on another level, subjects were able to correctly make semiotic transitions to obtain correct representations, yet were unable to make useful inferences as to the meanings of the representations in co-ordination with the problem at hand. Their basic conclusion to their inquiry concerning the mathematical necessities for non-hierarchical flexibility of changing settings is that this flexibility was of a complex nature in that: “flexibility competencies cannot be left to the students’ personal effort; they have to be explicitly taken into account in the teaching process and managed in the long run” (Dias & Artigue, 1995, p.41). The author notes that conclusions of a *meta-cognitive*, interventional nature such as this were similarly reached in several of the previous studies concerning the learning of linear algebra (Hillel & Sierpinska, 1994; Dias & Artigue, 1995; Harel, 1999; Hillel & Mueller, 2006).

As a result of this review of literature in linear algebra education research, the author formulates the following portrait of the key features shared by the studies reviewed. Undergraduate linear algebra, as taught through modern approaches similar to the guidelines of the LACSG, presents itself as a subject with both computational and abstract aspects. Computationally, linear algebra initially poses no great difficulties; however; the technical feasibilities often mask the representational import of the symbolic structures, often matrices, used for computations across differing linear algebra settings. This fundamental epistemological feature of linear algebra sets the scenario for a tendency for novice linear algebra students to fall into *practical thinking*, without understanding the theoretical ramifications of their computations or the *necessity* for the formal vector-space theory (Sierpinska, 2000; Dorier, 2000a, p.29; Dorier, 2002, p.880).

In conclusion, several of the previously reviewed studies point to the general inadequacy for students to develop theoretical setting change competencies on their own, requiring didactic intervention on the meta-cognitive level to better resolve issues related to non-hierarchical flexibility. It is hypothesized in Harel (1989b, 1990) that if intervention is not made, students may become overly attached to particular examples and develop contextual conception; unable to construct more abstract, structural-procedural relationships. None of the reviewed studies indicate precisely <u>how</u> meta-cognitive interventions may relate setting-specific meanings to representational forms, as well as the setting-specific interpretations which result from computations upon those forms (Hillel & Sierpinska, 1994; Dias & Artigue, 1995; Harel, 1999; Hillel & Mueller, 2006). In addition, none of the previous studies provide specific examples of meta-cognitive interventions in an experimental setting.

Based on the review of literature in mathematics education research, the author poses the following focus questions:

Focus Question 1

> What is the nature of non-hierarchical structural-procedural relationships which may exist between problem representations, such as matrices, and common linear algebra settings in which these representations can arise in a first course?

Focus Question 2

> In which ways can meta-cognitive interventions address difficulties associated with non-hierarchical flexibility between non-isomorphic linear algebra problem settings sharing similar problem representations and solution procedures?

In order to better study the relationships between different linear algebra setting structures, procedures, and representations, this research dissertation approaches the topic of non-hierarchical flexibility from standpoint of both traditional and contemporary perspectives in transfer theory, thereby affording a rich history of experimental methods and relevant results from research in psychology in the general domain of transfer theory.

Theoretical Framework

Introduction

It is the purpose of this section of the literature review to present a brief historical background of the development of constructivism, a theory which grounds the researcher's epistemological beliefs. Unlike previous research in mathematics education concerning linear algebra, the author reframes the questions surrounding linear algebra settings, representations, and structural relationships, i.e. non-hierarchical flexibility, as an issue involving both traditional and actor-oriented transfer perspectives. After an historical presentation of constructivism spanning from Greek thought, through Kant, and ending with Piaget; this section concludes with a presentation of the traditional and actor-oriented transfer perspectives which form the theoretical framework for this dissertation.

Constructivist Origins in Greek Thought

The notions of *practical* and *theoretical* thinking, as discussed in the previous review of literature in mathematics education research, belong to a dichotomy with a long tradition in human thought by their similitude with the ideas of procedural and conceptual understandings, respectively (Sierpinska, 2000, p. 212). In *The Psychology for Mathematics Instruction*, Resnick & Ford (1981) highlight that, "the relationship between computational skill and mathematical understanding is one of the oldest concerns in the psychology of mathematics" (p. 246). Hiebert & Lefevre (1986) defined conceptual understanding as "knowledge that is rich in relationships" and procedural understanding as "the formal language or representation system, as well as the algorithms and rules for completing tasks." The ideas of procedural and conceptual understanding form dual threads of thought that have run from antiquity throughout philosophy, science, and cognitive science; to the present. This dual relationship between skill and understanding, in different terms, can be traced back to the ancient Greek notions of *epistêmê* and *technê*.

Epistêmê and technê may be translated as pure-knowledge and craft; or as pure-theory and experience-based practice, respectively (Lyons, 1963). In Plato's writings, oftentimes theory and practice are described in opposition to each other, as the following quote suggests:

> The ordinary arithmetician, surely, operates with unequal units; his 'two' may be two armies or two cows or two anythings from the smallest thing in the world to the biggest; while the philosopher will have nothing to do with him, unless he consents to make every single instance of his unit equal to every other of its infinite number of instances (Plato, 360 BCE).

In Greek thought, the technê, or craft is often characterized by its function or goal. Some technê, such as *calculation,* may have no concrete goal or byproduct. When a goal exists, however, it can be articulated in conjunction with its dually related epistemic knowledge; which is more than just knowledge of how to perform the craft, but rather knowledge that results from *reflection* upon the craft. This "intimate positive relationship between episteme and technê, as well as a fundamental contrast" foreshadows a tension between the 18^{th} century philosophies of Rationalism and Empiricism; eventually giving rise to the basic epistemological belief system in which this study was grounded … *constructivism* (Roochnik, 1993).

Constructivist theories of learning are based on the philosophical premise that "reality is a conceptual construction" (Trochim, 2007, p.19). For mathematical learning, this typically translates that learners construct their own mathematical knowledge. Constructivist theories essentially fall into two basic categories; cognitive and social. The cognitive emphasis of constructivism, characteristic of approaches by Ausubel (1963), Bruner (1990), and Piaget (1972), generally focuses on the adaptive cognitive structures of the individual. In contrast, the social constructivism of Vygotsky (1978), which influenced theories such as Lave & Wenger's (1991) *situated learning*; holds that knowledge develops as a sociogenesis of phenomena involving a complex interplay between the biological processes and social interactions which coalesce to construct an individuals' mentality.

Constructivism-Kant's Role

> Up to now it has been assumed that all our cognition must conform to the objects; but all attempts to find out something about them a priori through concepts that would extend our cognition have, on this presupposition, come to nothing. Hence let us once try whether we do not get farther with the problems of metaphysics by assuming that the objects must conform to our cognition (Kant, 1781).

Pre-Copernican realism views objects as having existence independent of a knowing subject (Gardener, 1999). Just as Copernicus' heliocentric theory places the sun at the center of our solar system instead of the earth, Immanuel Kant's own Copernican revolution in philosophy shifted the notion that man's knowledge revolved around the intuiting or reasoning on idealized objects, to the notion that objects revolved around the mind of man; i.e. the mind *constructs* knowledge. Kant's *Critique of Pure Reason* synthesized the opposition between the Empiricist thought of Hume and Locke – whom believed that knowledge derived from sensory experience, verses the purely Rationalist thought of Descartes and Leibniz – which held that innate, apriori faculties of reason unfolded the truth of a Platonically idealized world (Kant, 1781; Gardner, 1999). For Kant, holding on to purely Rationalist or Empiricist views of epistemology leads to fundamental contradictions.

In the case of Rationalism, if knowledge is due to completely innate, apriori powers of reasoning within the mind, then how could this reasoning reveal knowledge of the external world? In the case of Empiricism, if knowledge acquisition is constrained to the subjectivity of sensory experience, then how could one obtain objective universal knowledge such as mathematics? Kant's solution was to combine rationalist and empiricist philosophies via the *schema* concept; thereby laying the foundation for constructivism, as Richardson (1998) explains:

> Intellect and sensibility must work together. But the results in the mind are not simple, direct associations. When they work together the results are *schemas* - constructed representations that form our working knowledge and thoughts about the world without being direct "copies" of it (p. 87).

The schema concept thereby functions in the constructivist paradigm as the fundamental commodity of the constructive enterprise, comprising the basic apparatus for the formation of cognitive faculties such as discrimination, meaning, and understanding.

The term schema derives from the Greek word *skhema*, meaning "form or figure," and has acquired various meanings since its introduction by Kant. Bartlett (1932) first introduced the term schema in psychology in relation to his studies concerning memory recall of stories and pictures. His work concluded that memory recall involves a reconstructive process involving the formation of an abstract schema to describe the recall. Rumelhart (1980) defines *schema theories* in the following way:

> A schema theory is basically a theory about… how knowledge is presented and about how that representation facilitates the use of the knowledge in particular ways. According to schema theories, all knowledge is packaged

> into units … [called] schemata. Embedded in these packets of knowledge is … information about how this knowledge is to be used.

Finally, after Marshall (1995), a schema is "a form of knowledge that relates a set of declarative facts with a set of procedural rules.

These various formulations of the schema concept essentially convey that schemas in themselves are empty of content, being structures which retain how applicable content is co-ordinated with other relevant structures, and used. The Kantian schema, being more than just an interface between the "logical machinery" of the mind and the phenomenal world, consists of "analogical procedures … entailing principles of iteration linking *knowledge* and *action*" (Radford, 2003, p.2). Although Kant's schemas theoretically link the conceptual with the procedural, they are not relegated to experience, and therefore not appropriate elements to describe a dynamic and interactive process such as learning. Development is problematical for the static Kantian schema and hence relegated to an obscure concept called *Judgment*, which Kant described as a *faculty* of the mind that humans simply possess, as Radford (2003) elaborates:

> Knowledge is more than just a cocktail of conceptual and sensual ingredients. The sensual ingredients, Kant claimed, have to be linked to their corresponding concepts. Judgment is a "peculiar talent" that distinguishes whether something goes under a particular concept or not. For Kant, the schema is precisely a function of the faculty of Judgment (p. 2).

Constructivism-Piaget's Theory of Genetic Epistemology

> The fundamental hypothesis of *genetic epistemology* is that there is a parallelism between the progress made in the logical and rational organization of knowledge and the corresponding formative psychological processes (Piaget, 1970).

It would take the psychologist Piaget to extend Kant's schema concept in a manner capable of forming a completely structural account of intelligence and learning. While Kant's Judgment occupies a role in the mind as a *peculiar talent*, Piaget completely abandons the Rationalist tendency to separate Reason from Experience, as is alluded to in the following:

> Knowledge cannot be dissociated from its historical context and that, consequently, the history of a concept gives some indication as to its epistemic significance (Piaget & Garcia, 1983).

Where Kant inserted the concept of Judgment as the fix for lack of a mechanism for distinguishing and coordinating schemas, Piaget utilized the concept of *equilibration*; which no longer relegated understanding to Judgment, but rather provided a dynamic structural account of understanding and knowledge formation.

Inspired by the adaptive qualities of biological systems, Piaget's concept of equilibration is the central idea in his genetic epistemology. The key assumption of the theory of equilibration is that knowledge structures tend towards equilibrium. *Equilibration* is defined as the tendency for human thought to seek consistency and order in an attempt to avoid inconsistencies and discrepancies within experience (Piaget, 1970). According to Piaget, to accomplish equilibrium, schemas undergo a series of psycho-physical processes (see Table 2.5). The schema itself is not an *abstraction* drawn from experience. Experience is possible, and the empirical data become thinkable, *because* of the schema. This is why Kant's theory of knowledge does not include a theory of abstraction. What Kant needed was a theory of *subsumption*, i.e. a theory indicating how representations and perceptions were subsumed under an a priori concept. In giving up apriorism, Piaget found himself in need of a theory of abstraction (Radford, 2003, p.42).

1) ACTION: Early knowledge arises from actions upon *objects*. Eventually, actions may be carried out in thought only.
2) EQUILIBRIATION: i. ASSIMILATION-the attempt to bring new information into a schema. ii. ACCOMODATION- to alter a schema so as to fit new information into the schema which was inconsistent with the previous knowledge structure.
3) STAGES: Intelligence develops through an invariant sequence of stages. a. Sensorimotor stage: (approx. 0-2years) Development of object awareness. Object permanence. Events can occur in world independent of one's own actions. b. Preoperational: (approx. 2 -7years) Detachment of thought from physical world. Thinking is intuitive and analogical. c. Concrete Operational: (approx. 7-11 years) Operational thought free from constraints of physical world but only may reason on objects physically present. Non-abstract. Non-hypothetical. d. Formal Operational: (approx. 11-15 years) Thinking entirely free from physical world. Able to think abstractly and hypothetically.

Table 2.5 Piaget's genetic epistemology, (Richardson, 1998).

In Piaget's genetic epistemology, objects are initially abstracted from experience through a process called *empirical abstraction*. In reference to ACTION, not only does knowledge arise from actions upon objects, but through empirical abstraction, objects themselves arise through actions embedded in experience. Whereas empirical abstraction abstracts the properties of objects from the environment, *pseudo-empirical abstraction* abstracts properties obtained through the actions upon the objects of empirical abstraction. In Piaget's latter *triadic* formulation of his theory, these two forms of abstraction belong to the INTRA and INTER-phases of knowledge construction, respectively; while the TRANS-phase results from a form of abstraction called *reflective abstraction*. This form of abstraction involves abstracting properties from the performance of actions upon the actions performed between the already abstracted objects (Piaget & Garcia, 1983).

An example of reflective abstraction, in the context of this dissertation, is noticing that a particular system of equations may arise from a variety of contexts or linear algebra settings. More specifically, general principles may be reflectively drawn from actions amongst linear combinations of vectors, images of vectors under linear transformations, and vector solutions to intersections of linear subspaces. After reflection upon the commonalities existing between the actions amongst objects from the different contexts, the commonalities may become concretized, forming new mathematical objects such as augmented matrices representing systems of equations. As mentioned in the previous section, the formation of new objects as a result of reflective abstraction is referred to as *encapsulation*. Encapsulation refers to how "a physical or mental action is reconstructed or organized on a higher plane of thought, and so comes to be understood by the knower" (Beth & Piaget, 1966, p.247). Encapsulated objects being known may thus also be acted upon and new salient features reflectively drawn out, resulting in new higher-order structures or schemas which generalize lower-order structures. Another important feature of TRANS-level constructions is the *reorganization* of schemas which can result from reflective abstraction, causing stage-like transitions from old epistemological structures becoming new different structures (Piaget & Garcia, 1983). Piaget's notions of reorganization and accommodation are important concepts for this study in that they will prove central to the author's eventual theoretical integration of traditional and actor-oriented transfer perspectives with Piaget's constructivist epistemology.

Brief History of Traditional and Contemporary Transfer Paradigms

Transfer of knowledge from the educational realm to every day applications forms the foundation on which academic institutions have largely been based (Bransford & Schwartz, 2001). Thorndike first performed experiments with transfer in connection with his work concerning the applicability of a classical education, referred to as *formal discipline*, to other areas of knowledge (Thorndike & Woodworth, 1901). Thorndike challenged the notion of the *general* transfer of knowledge, dating back to Greek antiquity (Mann, 1979), which holds that a broad education composed of subjects such as Latin and Mathematics would best foster a mind which could apply itself to a variety of different real-world situations. Thorndike and Woodworth (1901) performed experiments in which subjects were trained to estimate rectangular areas, and then tested for their ability to estimate the areas of different polygons such as triangles and circles. Their experiments produce results of 80-90% failure in the estimation of areas of the differing polygons after subjects learn to estimate area for rectangles. They conclude that improvement in one mental function does not imply improvement in other functions. Hundreds of experiments similar to that of Thorndike and Woodworth (1901) were duplicated with the same basic result that a general transfer between differing tasks is difficult to attain, yet transfer is enhanced in relation to the commonality of structural elements shared between tasks (Krusch, 1994).

The notion of traditional transfer derives from Thorndike's *Theory of Identical Elements*, which states that transfer is more likely to occur when tasks share similar elements. When tasks have similar elemental structures, they are said to be *isomorphic* (Hummel & Holyoak, 1997). Gick and Holyoak (1987) point out the vagueness of Thorndike's elements and insist that task structures may be isomorphic on a number of different *levels*; such as the perceptual, categorical, procedural, and/or conceptual levels. Thorndike further distinguishes between *near transfer*, or situations which had many elements in common; *far transfer*, such as situations in a school setting verses non-school setting; *vertical transfer*, applying to tasks which are low level and transfer to higher level forms of the task or vice versa, such as learning arithmetic before learning algebra; and *negative transfer*, where elements in one situation function as obstacles to transfer in another situation (Thorndike & Woodworth, 1901).

Previous research in traditional transfer has often been criticized for its predominant emphasis on what MacKay (1969) termed, the *expert point of view*, as deciding criteria for task similarity or determination for whether or not transfer occurs (Greer & Harel, 1998; Marton, 2006, p.499; Lobato, 2006). Over the past several decades, a vibrant debate has ensued concerning the theoretical foundations of transfer theory. Most notably, Lave (1988) strongly critiqued "the culture of transfer experiments" as not taking into account the different ways subjects might cognitively go about solving problems (p. 34). Typically in transfer experiments, "the researcher defines the tasks and the expected outcomes, and transfer is concluded when what learners do matches the researcher's expectations (Lave, 1988, p. 37; from Marton, 2006, p. 503).

Lave (1988) distinguished between two epistemological views of the application of knowledge; in the *functionalist view*, explicit knowledge developed in one situation can be applied, i.e. transferred to other situations, while in the Vygotsky-influenced *practice view*; transfer results from the relationships derived from the local and situational social dynamics of the individual, as a member of a community where "the community of practice decides what counts as knowledge and what is excluded" (Marton, 2006; p. 503). Instead of the "fundamentally flawed transportation metaphor" of passively carrying over "knowledge from one situation to another once learners recognize the similarity between situations," Lave shifts the discourse on transfer towards the question of how subjects *construct relationships of similarity* between different situations, or settings (Carraher & Schliemann, 2002, p. 19; in Lobato, 2006, p.431).

Since Lave's (1988) influential critique, several constructivist-influenced theories of transfer have emerged (see Greeno, Smith, & Moore, 1993; Carraher, Nemirovsky, & Schliemann, 1995; Lobato, 2003). Of most notable in similarity to the author's own constructivist epistemological leanings is Lobato's theory of *actor-oriented transfer*. Rather than judging transfer according to normative, i.e., expert functionalist views, Lobato set about to study transfer from Mackay's (1969) *practice* point of view of the *actor*, i.e., learner. According to Lobato & Siebert (2002), "actor-oriented transfer is defined as the personal construction of relations of similarity between the activities, or how "actors" see situations as similar (p. 89).

Primarily, traditional transfer concerns problem solving execution and application; while, actor-oriented transfer emphasizes the role of previous knowledge in the construction and development of thought processes from the learner's point of view. Based on results of Experiments 1 and 2 (see Chapter Three), which were influenced by the traditional transfer perspective, interviews were conducted and analyzed according to the actor-oriented theoretical perspective. In conclusion of this section, both traditional and actor-oriented transfer perspectives comprise the theoretical framework for this study, upon a foundation of the author's constructivist epistemological beliefs. The next section presents relevant research and results in transfer theory, spanning from traditional to contemporary perspectives.

Relevant Transfer Literature

The Luchins Water Jar Experiment

An important result from experimental psychology research during the Gestalt school period, related to the idea of negative transfer, is the 'Einstellung effect' or 'set effect,' also known as the *mechanization of thought* (Luchins, 1942, p. 242). This result is of importance for this dissertation due to its similarity to Harel's (1999) *contextual conception*, where the "inability to detach from a specific context" was seen to be an impediment to transferring knowledge to another context (Harel, 1999, p.603). The mechanization of thought, or Einstellung effect was first

	Given Jars of the Following Sizes			Obtain the
Problem	A	B	C	Amount
1	29	3		20
2 Einstellung	21	127	3	100
3 Einstellung	14	163	25	99
4 Einstellung	18	43	10	5
5 Einstellung	9	42	6	21
6 Einstellung	20	59	4	31
7 Critical 1	23	49	3	20
8 Critical 2	15	39	3	18
9 Set Breaker	28	76	3	25
10 Critical 3	18	48	4	22
11 Critical 4	14	36	8	6

Table 2.6 Luchins (1942) experimental design.

postulated in Luchins (1942) as a result of a now classic experiment involving the manipulation of water jars of various volumes and an unlimited supply of water.

The goal of the experiment is to achieve a desired quantity of water using combinations of three various sized jars of water. After receiving a practice problem, the experimental group is given five problems to solve in which each of the problems were solvable by the same *Einstellung formula* (B-A-C-C). This formula means to fill the B jar with water, then pour out water from the B jar, filling the A jar, and then twice pouring out more water from the B jar to fill two C jars; for example, as seen in problem 2 of Table 2.6, 127-21-3-3=100. After the five *Einstellung* problems, the experimental subjects are given two more critical problems (C_1, C_2), solvable either by the previous formula (B-A-C-C), or a much easier and efficient formula, such as 23-3=20 in problem (7). The experimental subjects are next given a *set breaker* problem which was only solvable by an efficient non-einstellung formula. Finally, the experimental group is again given two problems, (C_3, C_4) which were solvable by the Einstellung formula or a more direct and efficient formula. The control group of subjects is given only the warm-up problem and the problems (C_1, C_2, C_3, C_4).

Results from the study, (see Table 2.7), indicate that it was very difficult for the experimental group subjects to overcome the tendency to use the learned Einstellung formula, even though there was a much more efficient method available for problems 7, 8, 10, and 11. Contrastingly, control group subjects used the direct solution formulae. The tendency for the experimental group to persist in using the learned solution, as opposed to making use of the direct formulas for the (C_1, C_2, C_3, C_4) problems, is labeled the *Einstellung effect*, an example of negative transfer. Luchins went on to conjecture that the mechanization of thought is a strong tendency resulting from a “school-learned,” overly instrumental approach to problem solving similar to the notion of *practical thinking* earlier discussed

(McKelvie, 1984; Sierpinska, 2000). Recall that practical thinking inclines linear algebra students to often consider immediate algorithmic approaches to problem solving, instead of using *theoretical thinking* characteristic of reflection upon the contextual ramifications and connections of the symbolic representations being algorithmically operated upon (Sierpinska, 2000).

Group	Einstellung Solution (percent)	Direct Solution (percent)	No Solution (percent)
Control (Children)	1	89	10
Experimental (Children)	72	24	4
Control (Adults)	0	100	0
Experimental (Adults)	74	26	0

Table 2.7 Luchins (1942) results.

With respect to the ‘mechanization of thought’ and the Luchins study, could it be the case that in analogous experimental settings to Luchins (1942), linear algebra students might demonstrate Harel’s (1999) *contextual conception* as a manifestation of Einstellung-like effects? Recall that in Harel (1989b, 1990), working ‘too much’ in the realm of low dimensional geometry is seen to sometimes function as an obstacle to transferring knowledge from familiar geometric settings to less familiar formal settings. To pursue this line of speculation in the context of focus questions 1 & 2, additional research in transfer theory was completed for the purpose of discovering more associations with results from linear algebra mathematics education research.

Transfer Research Concerning Non-Isomorphic Settings

Previous research findings, from the perspective of traditional transfer, reveal spontaneous transfer between *non-isomorphic* problem settings to be rare in the absence of hints concerning the relationships between those settings (Holyoak & Koh, 1987; Novick, 1990; Ross, 1984; Dias & Artigue, 1995, p.41). Ross (1984) discovered that when analogous probability problems were from the same setting or domain, subjects averaged 77% success in transfer for source problem to target problem performance. However, when the settings of analogous problems are different, target success dropped to 43%. Gick & Holyoak (1983), however, found that when hints were supplied as to the *relationship* between source and target problems from disparate yet analogically related settings, then procedural transfer was significantly increased. Of note, from a constructivist standpoint, the result that hints may elucidate relationships facilitating transfer between different mathematical contexts (Gick & Holyoak, 1983) seems to agree with Harel's (1999) notion of the meta-cognitive intervention, which intends to induce trans-like theoretical thinking capable of discerning relationships between differing settings. Schmid, Wirth, & Polkehn (1999) define *non-isomorphic sources* as 'problems which are not structurally identical to a target problem' (p. 128). Similar to Gick & Holyoak (1983), several studies (Reed, Ackinclose, & Voss, 1990; Spellman & Holyoak, 1996) found that even transfer between *non-isomorphic settings* was possible when retrieval and mapping information was explicitly provided.

Typically, research on transfer in problem solving concerns *procedural similarity* between source and analogous target problem structure, in which there is considerable evidence of transfer between source problems and analogous (isomorphic) target problems sharing similar procedures (Gick & Holyoak, 1980; Novick, 1990, p.128; Bassok & Holyoak, 1989; Chen & Mo, 1994). Transfer research has also shown the importance of symbolic representations, such as matrices, for 'conceptualizing the underlying structure' which even conceptually non-isomorphic settings may share (Novick & Hmelo, 1994). Research findings in transfer suggest that when *isomorphic* target and source problems share symbolic problem solving representations, then procedural transfer is facilitated (Gick, 1985; Holyoak & Koh, 1987; Reed, 1987), as Chen & Mo (2004) comment:

> Analogical comparisons among instances also facilitate the formation or refinement of a *representation* that encompasses the essential information among problems and thus enhances subsequent transfer.

Novick (1990) defines the term *representational transfer* as "transfer of a representation in the *absence* of a common solution procedure" (p. 130). Novick (1990) conducted an experiment to determine if representational transfer might occur between source and target problems in the *absence* of common solution procedures. Novick (1990) studied matrix representations due to their application to a wide variety of settings. The goal of their experiment was to determine the likelihood of a subject using a matrix representation to solve a target problem after having been exposed to a matrix representation in a previous source problem solution. The participants were 30 undergraduate university students randomly assigned to control and experimental groups. Each group was given three source problems to solve, and then a fourth target problem. The target problem was constructed to be solved using a matrix representational strategy. The source problems contained a sketch of the beginning of a solution prompting a particular representational strategy.

For the control group, all three source problems involved representational strategies that were not compatible with a matrix representation. For the experimental group, one of the source problems was a probability problem which could be solved using a matrix representation. For this particular source problem for the experimental group, after its statement, an empty 5x5 matrix was written below the problem and subjects were instructed to use it to help solve the problem. The subjects were given 5 minutes to solve each of the source problems. After solving each source problem, the subjects were instructed to write down the benefits of using the particular representational strategy they used in solving the problem. Upon completion of the source problems, the subjects were then given thirty minutes to solve a target problem involving deductive reasoning. The target deductive reasoning problem and the source probability problem from the experimental group both involved different solution procedures and were thus not analogous, yet both could be solved using matrix representations, as Novick (1990) indicated:

> The probability and logic problems are similar in that they both present situations in which there is a factorial combination of possibilities. What makes the matrix a useful representation for these diverse problems is that it provides a means of making

> accessible simultaneously all of the possible combinations. Furthermore, it provides a concise means of indicating the solver's current state of knowledge through marks placed inside the cells of the matrix (p. 131).

It was the goal of the experiment to determine if representational transfer occurred; whereby, although source and target problems for the experimental group were not analogous and required different reasoning procedures, the representational strategy used for the source problem may have transferred to the target problem causing target success in light of dissimilar reasoning procedures. The results of the experiment indicated that subjects exposed to the matrix probability problem from the experimental condition were significantly more likely to use a matrix solution for the target deductive reasoning problem as compared with the control group, who did not receive a matrix solvable source problem (75% vs. 21% respectively, $p<0.005$). These findings suggest that transfer need not occur solely along lines of procedural similarity, but also in a representational sense, not dependent on the procedural isomorphicity amongst the problems. The findings are also significant in light of the greater generality and flexibility of representations verses solution procedures, as Novick (1990) emphasizes:

> Clearly, solution strategies are more general than solution procedures; that is, they can be used with many different kinds of problems. Problem representations are more general than solution procedures in this same sense. Thus, the probability that two problems with the same representation have similar solution procedures is smaller than the probability that two problems with the same solution procedure have similar representations (p. 129).

In summary, results from Novick (1990) complement other results (Reed, Ackinclose, & Voss, 1990; Spellman & Holyoak, 1996; Holyoak & Koh 1987;

Gick, 1985; Reed, 1987) to the extent that the author puts forth the conjecture that transfer between problems from non-isomorphic settings may occur if the problems share a combination of similar procedures and representations. As a consequence of these results from transfer theory, the following research question was formulated:

Research Question 1

> Is there evidence, in the traditional transfer paradigm, that transfer is facilitated between linear algebra problems from non-isomorphic settings which share similar (matrix) representations AND solution procedures?

To address this question, in Chapter Three the author adopts the traditional transfer paradigm as one of the theoretical perspectives influencing the methods of this study for the purpose of verifying the possible extension of results from previous non-isomorphic setting transfer research, in terms of the conceptually non-isomorphic linear algebra problems of interest in this dissertation (Reed, Ackinclose, & Voss, 1990; Spellman & Holyoak, 1996; Novick, 1990; Holyoak & Koh 1987; Gick, 1985; Reed, 1987). This review of the research literature on transfer now focuses on the contemporary perspective of actor-oriented transfer for the purpose of laying the literary foundation of the theoretical perspective used in the interview portion of this study.

In Lobato & Siebert (2002), a research study was conducted for the purpose of finding evidence of actor-oriented transfer. The setting for the study was a teaching experiment conducted in summer, 1999, in which 9 students were recruited for interviews from participating 8^{th}- and 10^{th}-grade mathematics classes. All of the participants had average grades of B or C in their most recent mathematics course, had experienced difficulty in the past with algebra or pre-algebra content, and had appeared to understand the content presented during the summer teaching experiment. The subjects met for 3 hours per day for 10 days in a lab format, for the experimental course taught by Lobato and Siebert. The principle objective of the course was to aid the students in their development of quantitative reasoning concerning linear functions and slope. Interviews of the subjects were conducted on the first, fifth, and last days of the course. Of the interviewees, Lobato & Siebert (2002) conducted a case study on a particular interview subject, Terry, concerning his reasoning on the association of slope as a measure of ramp steepness, for wheelchair ramp tasks inspired by the ski ramp problems of Simon & Blume (1994) (see Fig. 2.7).

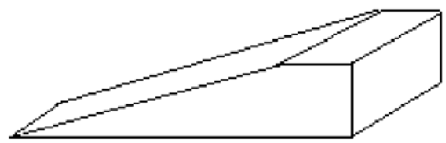
Collecting data in the form of individual interview data and videotaped classroom data, the researchers focused on Interview 3, looking for evidence of actor-oriented transfer through analyzing the interview activities to see evidence of "similarities formed by the student" based on prior experiences in the teaching experiment, as well as in the previous Interviews 1 and 2 (Lobato & Siebert, 2002, p. 94). In Interviews 1 and 2, Terry appeared to have difficulty associating the height of a ramp with its steepness, especially in the task of creating a higher ramp of the same slope, as explained in Lobato & Siebert (2003):

> The relationship Terry had found between height and length was not multiplicative in nature, that height dominated Terry's image of steepness, that height was explicit and length implicit, and that changes in length were dependent on changes in height (p. 96).

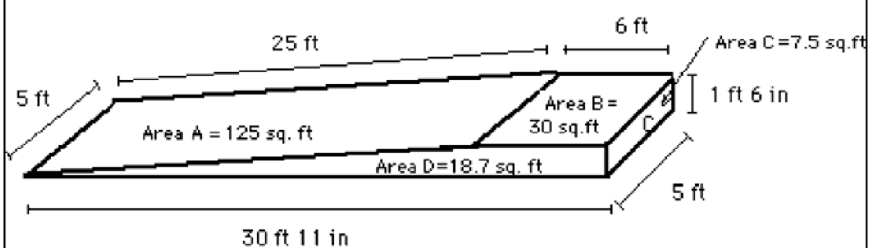
According to Terry's reasoning, the higher one goes, than the more steep the ramp must get. It was apparent to the researchers that Terry did not have the conceptual understanding that the rise and run could change together, thereby producing constant steepness.

During Interview 3, although Terry had been exposed to the slope formula, *rise over run*; during the teaching experiment, he was seen to go through a personal construction of the quantities involved with the three tasks in a different way than the methods discussed in the class, or methods based on his previous interview's misconception of the predominance of height in the concept of steepness. In his attempt during Interview 3 to solve Task 3, (see Fig. 2.7), after several unsuccessful attempts and a long pause; Terry was seen to spontaneously discover how to keep the steepness of the ramp constant, yet increase the height the ramp to 3 ft in order to reach the door. To do this, Terry utilized ratios by reasoning how much length of the ramp went with 1 ft of height, and then forming the ratios 15:2 = $7\frac{1}{2}$:1. Based on this reasoning, Terry knew that to increase the height to 3 ft, he needed to increase the length by $7\frac{1}{2}$ ft, thereby obtaining an answer of $22\frac{1}{2}$ ft. This occurrence was significant because Terry was not seen to utilize knowledge of the slope formula he had learned in class.

Task 1. Suppose you work for a company that builds wheelchair ramps. You want to measure or calculate the steepness of each ramp model that your company builds. You feel this information is important so that people will know how difficult it is to climb each ramp in a wheelchair. How would you go about determining how steep any ramp is (and what measurement would you need to take)?



Task 2. Here is a ramp that has all kinds of measurements. How can you determine the steepness of this ramp.



Task 3. Suppose a customer needs to have a wheelchair ramp that reaches to his doorstep, which is 3 ft high (point to ground and door). This particular ramp won't reach because it's only 2 ft high. How can you change the dimensions so that you have a new ramp that is the same steepness as this ramp but that reaches the door?

Figure 2.7 Interviews 1 & 3 wheelchair ramp tasks, (Lobato & Siebert, 2002).

The researcher’s then wondered if Terry’s method was based on his own constructions of similarity. After carefully looking back at the material covered during the teaching experiment, Lobato & Siebert (2002) discover in Session 8, that during the “same speed” activity, there was an animated clown problem as part of the *Simcalc Mathworld*s software where the clown walked at a constant speed of 10 cm per 4 s, and it was the goal for the students to make an animated frog walk at the same speed as the clown, however; the distances needed to be covered by the frog were different than 10cm. In order to accomplish this problem, the subjects had to realize that the speed of the clown represented a ratio, and that they needed to form an equal ratio, while having different top numbers than the clown’s ratio.

The researchers surmised that Terry most likely was influenced by this prior activity, based on the evidence of how he used the ratios to solve the task 3 wheelchair ramp problem. Marton (2006) similarly concludes:

> Nobody can argue with great certainty that the learner [Terry] actually made use of what he had learned in the clown-frog problem … Lobato & Siebert (2002) simply argued that if it was the case, then it was an example of transfer between situations thanks to the learner’s own construction of similarities between the two situations where there was no transfer in the traditional sense (p. 507).

Lobato & Siebert (2002) concluded that this example constituted a case of actor-oriented transfer because the relations of similarity between the ramp problem and the clown animation were based on Terry’s constructions. This example of research employing the actor-oriented transfer perspective, which was seen to emphasize the personal dynamic constructions of the subject, as well as the influence of the subject’s prior knowledge, constitutes an example of the type of theoretical analysis used during the interview portion of this study (Chapter Five).

Research Questions

Based on the review of the transfer literature in conjunction with the pilot and experimental results to be presented in Chapter Three, refinements of Focus Questions 1 and 2 have resulted in a synthesis of mathematics education research through both traditional and constructivist transfer perspectives, leading to the formulation of the following three research questions guiding this dissertation:

Research Question 1

> Is there evidence, from the traditional transfer perspective, that transfer is facilitated between linear algebra problems from non-isomorphic settings which share similar (matrix) representations AND solution procedures?

Research Question 2

> In what ways, from the theoretical perspective of actor-oriented transfer, do novice linear algebra students commonly have difficulty with conceptually non-isomorphic problem settings, even when novel problem settings share similar problem representations and solution procedures as familiar problem settings?

Research Question 3

> What evidence can be found that indicates meta-cognitive intervention(s) may facilitate traditional and/or actor-oriented transfer across conceptually non-isomorphic problem settings involving novel target problems which share similar problem representations and solution procedures as more familiar problems?

In conclusion of this final section of Chapter Two, literature and findings in research from traditional and actor-oriented transfer were presented which related to several results from the review of research in linear algebra mathematics education. As a whole, these relationships not only suggest correspondence to certain phenomena reported in the linear algebra mathematics education research, but also open the door to research methods and results from studies in transfer which may further address issues such as *cognitive inflexibility*, *multiple embodiments*, *contextual conception*, and *practical/theoretical thinking* in undergraduate linear algebra. Research in transfer theory led to the formulation of *Research Question 1*, which, in the perspective of traditional transfer, poses the question of whether or not transfer is facilitated between non-isomorphic linear algebra problems involving the combination of similar representations and solution procedures. Chapter Three next discusses the quantitative experimental methods used, from a traditional transfer perspective, for addressing *Research Question 1*, as well as the qualitative methods used, from an actor-oriented transfer perspective, to address the results of the traditionally-based experiments through *Research Question 2*. Chapter Three concludes with a description of the *Intentional Transfer Hypothesis*, and a discussion of the combination of literary, quantitative, and qualitative results used for experimentally addressing *Research Question 3*.

Chapter 3

METHODOLOGY

Introduction

This study, spanning five consecutive ten-week university quarters, employs a *mixed methods* design articulated in quantitative, qualitative, and mixed (quantitative + qualitative) components. Johnson & Onwuegbuzie (2004) define mixed methods research as, “a class of research where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts or language, into a single study.” The quantitative segment of this study (Experiments 1 & 2) utilizes experiments influenced by classical transfer theory in connection with three common conceptually non-isomorphic problem settings found in undergraduate linear algebra. Next, the qualitative interview phase was implemented for the purpose of gaining a deeper understanding of the data gathered in the previous experiments, through the lens of a more contemporary, *actor-oriented* transfer perspective (Lobato, 2002, p. 89). The study concludes with Experiment 3, which was inspired and designed from a combination of the previous quantitative and qualitative results.

Following Patton (2002) and Schoenfeld (2001), both quantitative and qualitative approaches are integrated to address the needs of a study consisting of quantitatively oriented experiments, qualitatively approached semi-structured interviews (Bernard, 1988), and a final quantitative experiment based on the previous quantitative and qualitative results. This chapter describes both inductive and deductive methods, involving multiple sources of measurement for the purpose of *triangulating* across different error sources to obtain a more accurate picture of the initial phenomena observed during the author's undergraduate linear algebra teaching experience (Trochim, 2007, p.19). Schoenfeld (2001) emphasized the importance of this type of approach when conducting research in education and the social sciences:

> Evidence can be misleading – what we think is general may in fact be an artifact or a function of circumstances rather than general phenomena. … One way to check for artifactual behavior is to vary the circumstances – to ask, do you see the same thing at different times, different places? Another is to seek as many sources of information as possible about the phenomenon in question, and to see whether they portray a consistent 'message' (p.234).

In this regard, it is the researcher's methodological stance for this study that both classical and contemporary views of transfer were beneficial as research tools. In the classical transfer experiment, "the researcher defines the tasks and the expected outcomes, and transfer is concluded when what the learners do matches the researcher's expectations" (Marton, 2006, p.503; Lave, 1988, p.37). For the quantitative portion of this study, the classical transfer experiment paradigm functions as a valuable benchmark for focusing research questions and testing hypotheses. Following Lobato (2003), an a*ctor-oriented transfer perspective* shifts focus from normative "cognitive models of expertise to an account of learners' personal generalizing processes within social settings that structure their generalizing experiences" (p.3). For the qualitative portion of this study, situatively cognitive approaches such as the actor-oriented perspective are used to determine and analyze the knowledge subjects indeed do transfer when problem solving, as well as the transfer related to the subjects' personal constructions (Lave, 1988).

The methods used for this research were also influenced by two pilot studies. The first pilot study, inspired by the researcher's experience teaching undergraduate linear algebra during the Spring 2005 quarter, was comprised of initial data collection and analysis which, along with the literature review, contributed to the formation of the research questions for this study. The second pilot study aimed to refine and adapt a multi-dimensional research model, based on work by (Chen & Mo, 1994), for the purpose of implementing two *factorial* experiments designed for the addressing the *Traditional Transfer Hypothesis*, (Research Question 1) (Spector, 1981). This chapter, divided into five sections, will first discuss general information about the participants in this study, which were all enrolled in a mathematics course entitled Matrix and Power Series Methods. Next, the second section discusses the design, methods, results, and influences of the two pilot studies; including a review of the influential Chen & Mo (2004) study, which inspired the Linear Algebra Model (LAM) for Experiments 1 & 2. The third section outlines the design and methods for Experiments 1 & 2, and the fourth section presents the design and methods for the interview portion of the study. Based on the results of Experiments 1 & 2 and the interviews, the *Intentional Transfer Hypothesis* was conjectured which warranted a further *treatment/non-treatment* designed Experiment 3 (Spector, 1981). The fifth section presents the *Intentional Transfer Hypothesis* in connection with the design and methods for experiment 3 (see Table 3.1 for timetable of study).

Winter 2006	Spring 2006	Fall 2006	Winter 2007	Spring 2007
Pilot Study 1	Pilot Study 2	Exp. 1a	Exp. 2a	Exp. 3a
		Exp. 1b	Exp. 2b	Exp. 3b
			Interviews	Exp. 3c

Table 3.1 Research timeline.

General Participant Information

All of the subjects in this study were university sophomores and juniors, mostly engineering and science majors, enrolled in a course which combined introductory linear algebra during the first half with infinite series content in the latter half. The author functioned as their recitation instructor. The subjects were screened as to the number of times they might have previously taken the course. Students who were not taking the course for the first time were not used in the study. There were a total of three lecturers involved during the course of the study. The subjects went to lecture three times per week for fifty-minute lectures. The recitation sections met for one hour and twenty minutes, once per week. All of the subjects received similar homework, quizzes, group work, and exams throughout the study. Also, the same textbook, *Matrix and Power Series Methods*, was used for all of the lecture and recitation sections during the entire study (Lee, 2006).

The course was organized into two midterms and a final exam. The first midterm covered the linear algebra portion of the course, while the second covered the material on infinite series. The experiments implemented during the quantitative phases of this study were uniformly carried out on the first recitation session after the first midterm, before the subjects received the results of their midterm exams. For the pilot studies and experiments 1-3, the subjects were not compensated for their participation. For the qualitative interview portion of the study, subjects received five dollars for their participation.

Pilot Study 1

Pilot study 1 involved eighty-seven undergraduates, (79 male, 8 female), enrolled in the course entitled Matrix and Power Series Methods during the Winter ’06 quarter. A *non-experimental design* was used for the purpose of gathering preliminary evidence to support speculation based upon the researcher’s prior experience teaching undergraduate linear algebra (Trochim, 2007, p.174). During the author’s teaching experience, it was observed that students appeared to have no significant difficulty with computational matrix methods, such as row reduction; however, when systems of equations and matrix representations arose in connection with problems from new linear algebra settings, such as the linear combination setting, students often appeared to have difficulty transferring previous knowledge gathered from exposure to examples and homework. Commonalities between previous examples, homework, and problems from new settings included the following General Goal Strategy, or GGS:

1. Representation of the problem with an appropriate matrix.
2. Row reduction algorithm employed as solution procedure to solve matrix.
3. Interpretation of matrix system solution in the context of a novel problem setting.

Consider the system of equations:

$$\begin{array}{rcrcrcr} x & - & 2y & + & 2z & = & 1 \\ -2x & + & 3y & - & 2z & = & -6 \\ x & - & 3y & + & 4z & = & -3 \end{array}$$

(a) Write out the augmented matrix for the system and using elementary row operations on this matrix, find all solutions to the given system.

(b) Find all solutions to the corresponding homogenous system: $A\vec{X} = \vec{0}$; where A is the coefficient matrix for the above system.

(c) Are the columns of the coefficient matrix linearly dependent or independent? Explain. If linearly dependent, find scalars c_1, c_2, and c_3 such that $c_1v_1 + c_2v_2 + c_3v_3 = \vec{0}$, where v_1, v_2, and v_3 are the columns of the coefficient matrix A of the system.

Figure 3.1 Pilot study 1 exam problem.

To initially explore the author's anecdotal observations, an exam problem, inspired by the questionnaire used in the work of Hillel & Mueller (2006), was formulated as part of a 1 hour and 50 minute final exam (see Fig. 3.1). Part (a) of the exam question consisted of a 3×4 system of equations in which the subjects were asked to solve. Part (b) asked for the solution to the *homogenous* system of equations, which could be found from part (a) by simply removing the constant vector part of the *particulate solution*, or by solving the system in part (a) with zeros down the last column (Strang, 2003; Penney, 2004). Part (c) switched settings and asked a linear combination question based on the columns of the original system matrix in part (a). Note it was possible for the subjects to resolve Part (c) based on the correct interpretation of the previous homogenous system solution, which coincided with the procedure for checking linear independence/dependence, often referred to as the *test for independence* (Lee, 2006, p.66; Penney, 2004, p.98).

During the previous month before the exam, the subjects were generally exposed to material on linear combinations and systems of equations in lecture, recitation, homework, and quizzes. In the well-attended recitation sections, the subjects saw many source exemplars of problems similar, but not identical, to the pilot question. In addition, the subjects were also aware that this material may be on the exam because they were given review sheets including this material two weeks prior. Statistical analysis of the data challenged the following null hypothesis:

> H_0 (*Pilot Study 1)*:
>
> The number of subjects scoring a correct answer on part (c) of the exam question is not significantly different than the number of students scoring a correct answer on parts (a) or (b), or there will be an increase in the number of correct answers on part (c) as compared to parts (a) or (b).

The following directional alternative hypothesis was subsequently formulated:

> H_A (*Pilot Study 1)*:
>
> The subjects would perform significantly worse on part (c) of the exam question. In addition, there will be no significant difference between parts (a) and (b) of the exam question.

Scoring

Although the exam was graded on a partial credit basis, the dichotomous data recorded for this segment of the study was comprised of the following semi-partial scoring rubric; a 1 was recorded if the problem was essentially correct, with perhaps a very minor error. A 0 was recorded if the problem was incorrect, having even moderate error. For example, if a problem part is correct all excepting a wrong minus sign in the answer, then a score of 1 was assigned. If there were more errors than that, then a 0 was assigned. This dichotomous recording of the data was implemented for the purpose of modeling a traditional transfer experiment which should indicate whether transfer occurred, or did not occur, relative to models of expert performance. The models of expert performance for successful transfer in Pilot Study 1 were the mathematically correct solutions to the research problem parts, as formulated by the author.

Results

The percentages of participants successfully solving the successive parts of the exam problem are as follows: part (a) (37%); part (b) (32%); and part (c) (17%) (see Fig. 3.2). A one-factor, within-subjects ANOVA (e.g., Myers & Well, 1991; Bevan, Denter, & Myers, 1974) reveals a main effect for "problem part" of the exam question, $F(2,258) = 4.537$, $MSE = 0.908$, $p = 0.012 < 0.05$. A post-hoc protected *Least Squares Difference* (LSD) test reveals problem solving performance was reliably lower in part (c) of the exam question, as compared to parts (a) (

$p = .004 < .05$) and (b) ($p = .028 < .05$), while parts (a) and (b) were not significantly different ($ns = .498 > .05$). In conclusion, there is strong evidence to reject the null hypothesis for Pilot study 1 and confirm the directional hypothesis that performance on the exam question significantly decreased on part (c), while there was no significant decrease between parts (a) and (b).

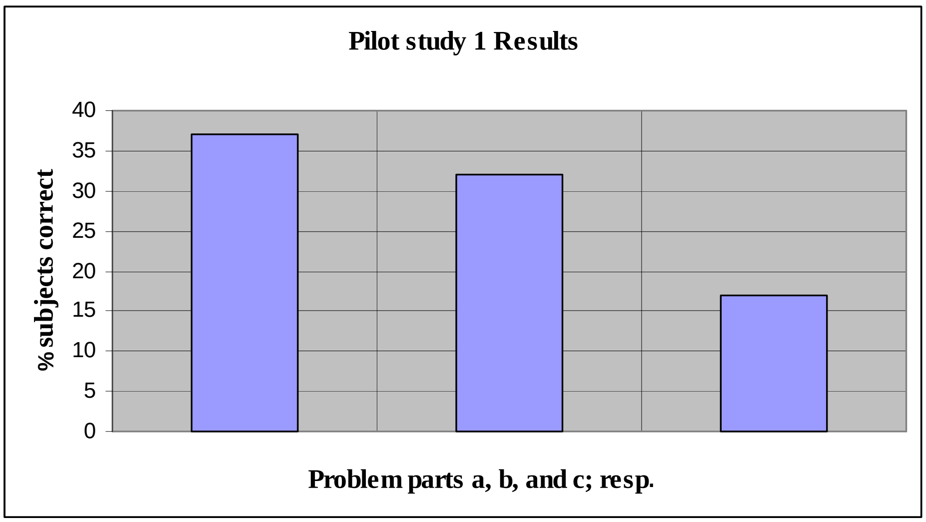


Figure 3.2 Pilot study 1 results.

Discussion

Results from the first pilot study suggest that beginning linear algebra students have difficulty working in an unfamiliar linear algebra setting, even though the problems encountered may be solved with similar matrix representations and row reduction procedures as more familiar problems. As a result of Pilot Study 1, the author conjectures that this difficulty may be due either to interpretation problems with the solution of the system of equations, or to the unfamiliar setting, or both. Statistical analysis reveals that solving a non-homogenous system as in part (a), verses solving the same homogenous system as in part (b), presents no significant difficulty. In Pilot study 1, when the initial setting was changed from a generic system of equations to a linear combination problem, statistically significant decline in problem solving performance manifested even though the linear combination problem could be solved with the same matrix representation and row-reduction procedures. Because Pilot study 1 involved only an infinite solution case, it is unclear to the author the potential effect different types of solutions might have in a context of changing settings. In view of these findings and rationale, the researcher set about to further study the potential effects of setting change and solution interpretation in a context of similar problem representations and solution procedures.

Pilot study 1 involved the use of two linear algebra settings. The first setting was given in part (a) of the exam question and consisted of a generic system of equations. A system of equations is often introduced to students geometrically as row equations represented as hyper-planes in n-space; the associated solutions being the intersections of the hyper-planes. Strang (2003) refers to this perspective as the *row-picture (Rp)* (p. 21). Since this context represents a predominant algebraic viewpoint found in most linear algebra and algebra textbooks, the row-picture will also be referred to as a *familiar setting* for the novice linear algebra students in this study (Harel, 1987, p.31).

The linear combination concept, central to linear algebra, represents a significantly new set of ideas, relationships, and formalism for the beginning student. One of the reasons the linear combination concept is central to linear algebra lies in the fact that linear combinations are the defining operations for a vector space. Again following Strang (2003), the linear combination setting is defined as the *column-picture (Cp)* setting (p. 22). As Artin's quote suggests in Chapter 1, the modern day inclination to view the historically geometric and function-oriented linear transformation concept as a computational matrix representation has perhaps contributed to the difficulties with linear algebra conceptualization for beginning students. The concept of linear transformation is also included in the settings under study in this dissertation due to the importance of linear transformations in undergraduate linear algebra, as well as due to the difficulties the researcher observed students display in problem solving related to the linear transformation concept. In conformity with other previous setting terminology, the linear transformation setting is referred to as the *linear transformation-picture (LTp)*.

The types of linear algebra problems thus addressed in this study derive from the following three conceptually non-isomorphic linear algebra settings:

1. The Row-picture setting *(Rp);*
2. The Column-picture setting *(Cp);*
3. The Linear Transformation-picture setting *(LTp).*

The *conceptual* non-isomorphicity of the above settings refers to their lack of structural similarity in the conceptual sense; as they do not differ only in surface-semantic features, such as labeling. For example, each of the three settings above has different associated geometric representations. The *Rp* setting may describe hyper-planes and their intersections, or simply refer to a generic system of equations written in variables. The *Cp* setting involves vectors in n-space under any variety of bases. The *LTp* setting consists of pre-images and images, as well as domains and ranges, of vectors under linear transformations (see Fig. 1.1, p.5). These settings exemplify the *generalizing* and *unifying* character of linear algebra as a body of mathematics which consolidated several different branches of mathematics into a unified formalism of matrix and vector space theory (Dorier, 1998). Problems from these three settings may be representationally isomorphic, however, since matrices may be used for a variety of problem solving in all three settings. According to Novick (1990), "problem representations are more general than solution procedures" (p.129). Because these three well-known linear algebra settings all share the potential of admitting problems sharing matrix representations, as well as similar procedures, e.g., matrix reduction; they fulfill the researcher's need for investigation into conceptually non-isomorphic setting transfer in introductory linear algebra.

For the inquiry into the potential effects of system solution interpretation, there are the following possibilities:

1. Unique solution.
2. Infinite solutions.
3. No solution.

These three possibilities in the solution category are clearly tied to the solving of systems of equations and matrix methods. Each solution possibility entails different concepts and geometrical interpretations in the various above three settings. For example, the unique solution is a unique point of intersection in the *Rp* setting, but in the *LTp* setting it would imply a linear transformation that is 1-1 and has an inverse. Alternatively, a unique solution for the LCp setting implies the columns of the representing matrix are linearly independent. In this way, the three solution types form relationships contributing to the overall schemas of each of the three settings. Although Pilot study 1 only involved one type of solution, the infinite solution type, the researcher decided that it was unclear how subjects might interpret other solution types in different contexts. For this reason, solution type was designated as a necessary factor for inquiry into the cause(s) of setting change difficulties. In summary, the results of Pilot study 1 indicated evidence verifying the initial speculations of the author in regards to problem solving transfer for a scenario of conceptual non-isomorphic setting change involving similar representation and solution procedures. Two categories of linear algebra settings and solution types were subsequently defined for the purpose of further investigation into conceptually non-isomorphic setting transfer difficulties.

Pilot Study 2

Pilot study 1 conjectured that difficulties subjects have with transfer across conceptually non-isomorphic settings in linear algebra were related to the solution type and setting non-isomorphicity features characteristic of several types of introductory linear algebra problems. These conceptually non-isomorphic categories or settings of problems shared the features of being solvable using matrix representations and associated matrix reduction techniques. A 2×2 *factorial design* would be an advantageous approach to studying the systematic manipulation of the factors of *setting* and *solution type*, and subsequently test for their effect and potential interaction in problem solving (Trochim, 2007, p.192; Spector, 1981). Before implementing a factorial design, however, a theoretical multi-dimensional research model would be needed which could dissociate significant procedures involved with solving linear algebra problems from the various settings and solution types of interest to this study.

The Chen & Mo, 2004 Study

Using experiments based on tasks similar to the Luchins' (1942) water jar problems previously discussed in Chapter Two, Chen & Mo (2004) conducted a quantitative study which examined transfer in analogical problem solving from a multidimensional, procedural standpoint. According to Chen & Mo (2004), it was hypothesized that successful transfer in problem solving involved a process of extracting generalizing principles from previous instances of related problems, which subsequently could enable transfer to novel target problems having similar goal structures. Furthermore, when problems shared similar goal structures differing in surface semantic features, exposure to related problems facilitated the discovery of some generalizing characteristics for conditions in which subjects were able to form abstract schemas overcoming contextual limitations.

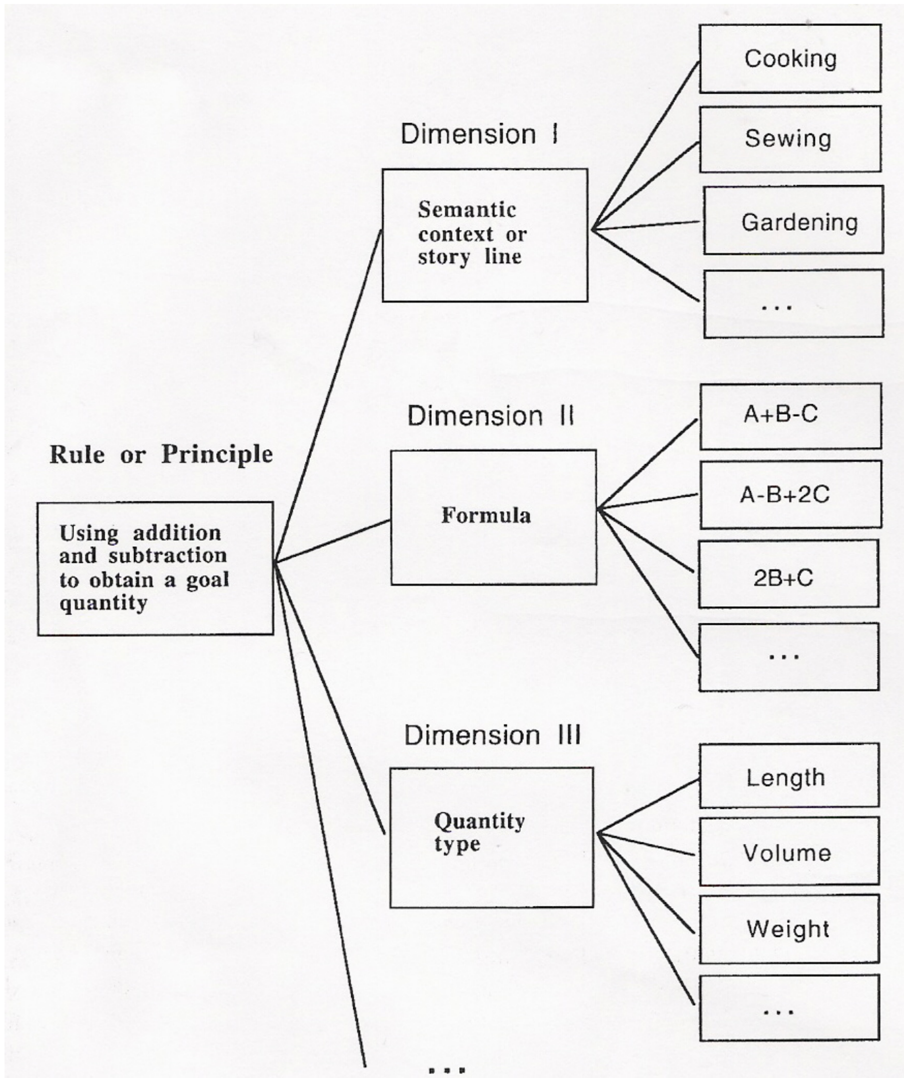


Figure 3.3 Representational hierarchy of solution abstraction.

Many studies have supported findings that the experiencing of multiple source problems which differ in surface-contextual features, yet share common problem solving procedures, enhances traditional transfer (Catrambone & Holyoak, 1989; Gick & Holyoak, 1987). Chen & Mo (2004) considered dimensions of procedural similarity or dissimilarity as central factors in their research. Their paper consisted of a systematic examination of the multidimensional procedural relations between source and target problem transfer. The Chen & Mo (2004) study was highly relevant for this study due to the author's observations that although the settings of interest for this dissertation were conceptually non-isomorphic, they contained problems which shared common and distinguishable procedural operations previously summarized as the *GGS*, or Generalized Goal Structure.

As illustrated in Fig. 3.3, their research model primarily involved the procedural dimensions of formula type and quantity type, in addition to a semantic context dimension. Recall in Luchins' original experiment, using three jars of different volumes, a subject was given an unlimited water supply in which to use combinations of the three jars to obtain a desired goal volume of water. By the formulation of the quantity type procedural dimension, the Luchins experiment was expanded to include area, length, volume, and weight problem possibilities. For each different type of quantity, the isomorphic procedural structure of achieving a particular goal quantity is the same. Having designed a model which supported multiple procedural dimensions for solving Luchins-like problems, Chen & Mo (2004) hypothesized that when source problems sharing procedural similarities were similar to target problems, then a rapid *near transfer* could be expected; along with an abstract problem solving schema that is more narrow and limited in its scope. Contrastingly, they predicted that exposure to source problems with a wide variability in procedural dimensions would lead to a slower yet superior *far transfer* for novel target problem solving performance.

Source Problems

Similar Formula, Similar Quantity

Chef Smith needed to get 18 oz of flour for his cooking but had only a balance scale and three weights of 24, 13, and 7 oz available to him. How did he obtain the correct amount (18 oz)?

24 - 13 + 7 = 18

Dissimilar Formula, Similar Quantity

Chef Smith needed to get 12 oz of flour for his cooking but had only a balance scale and three weights of 34, 13, and 9 oz available to him. How did he obtain the correct amount (12 oz)?

34 - 13 - 9 = 12

Similar Formula, Dissimilar Quantity

A farmer wanted to fill a barrel with 13 in. of water. He knew that the barrel was 18 in. high. He had two sticks, one 11 in. in length and the other, 6 in. How did he obtain the right depth of water (13 in.) without using a ruler?

18 - 11+ 6 = 13

Dissimilar Formula, Dissimilar Quantity

A farmer wanted to fill a barrel with 12 in. of water. He knew that the barrel was 34 in. high. He had two sticks, one 13 in. in length and the other, 9 in. How did he obtain the right depth of water (12 in.) without using a ruler?

34 - 13 - 9 = 12

Target Problem

A chemist needed to prepare exactly 25 lbs of coal for a special experiment. However, he did not have an accurate scale available. His only option was to use a two-pan balance scale and three weights of 31, 18, and 12 lbs, which he found in the lab. Please describe a method by which he could obtain 25 lbs of coal.

Figure 3.4 Experiment 1 source and target problems, Chen & Mo (2004).

Of particular relevance to Research Question 1 for this dissertation, Experiment 1 from Chen & Mo (2004) addressed whether similarity in quantity type and/or solution formula could facilitate or inhibit transfer between source and target problems (see Figure 3.4). In Experiment 1, after limited exposure to source problems, the subjects attempted to solve a novel target problem. Source problem similarity or dissimilarity to the target problem were the levels manipulated in a 2X2 factorial design (see Table 3.2). The quantitative data were the percentages of participants successfully solving the target problem (see Table 3.3). ANOVA was used for analysis of variance and the Student Newman-Keuls test was used for further ad hoc multiple comparisons of means; all of which concluded significance.

Table 3.2 Experiment 1 design summary.

Condition	*n*	Formula	Quantity	Problem
1	16	Similar	Similar	S22,S22;T22
2	16	Similar	Dissimilar	S21,S21;T22
3	16	Dissimilar	Similar	S12,S12;T22
4	16	Dissimilar	Dissimilar	S11,S11;T22
5	14			T22

Note: Problems sharing the same number for the same column within a condition are similar (but not identical) in the corresponding dimension. Blank cells indicate a control condition. S=source problem, T=target problem.

Results from Experiment 1 revealed that dissimilar procedures between source and target problems indeed created *obstacles* in applying source solutions, as summarized in the following:

> When the source and target problems required different operational procedures, participants experienced difficulty in using the learned solutions, even when a common general strategy was shared. Participants' representations were tied to the specific procedural features they encountered, and participants therefore applied the specific procedures directly to the target problem situation (Chen & Mo, 2004, p.588).

Similar formula, Similar quantity (88%)
Similar formula, Dissimilar quantity (44%)
Dissimilar formula, Similar quantity (50%)
Dissimilar formula, Dissimilar quantity (33%)
Control (21%)

Table 3.3 Experiment 1 results, Chen & Mo (2004).

These findings were pertinent to this dissertation due to similarities between the problems studied in the Chen & Mo (2004) study and linear algebra problems at focus for this dissertation. In both cases, problems were categorized by their relationship to non-isomorphic settings while sharing similar procedures and symbolic representations. In conclusion, the Chen & Mo (2004) study indicated that generalization of a schema to a broad range of problem solving situations may involve multiple procedural dimensions. Furthermore, the use of the multidimensional procedural model was shown to be a useful tool in the implementation of factorial experiments designed for the purpose of determining potential procedurally related einstellung effects in problem solving. The Chen & Mo (2004) multi-dimensional procedural model illustrated in Fig. 3.3 subsequently inspired the formation of the *Linear Algebra Model, (LAM),* framework which comprised the foundation for transfer Experiments 1 and 2 in this study (see Fig. 3.5).

LAM Framework

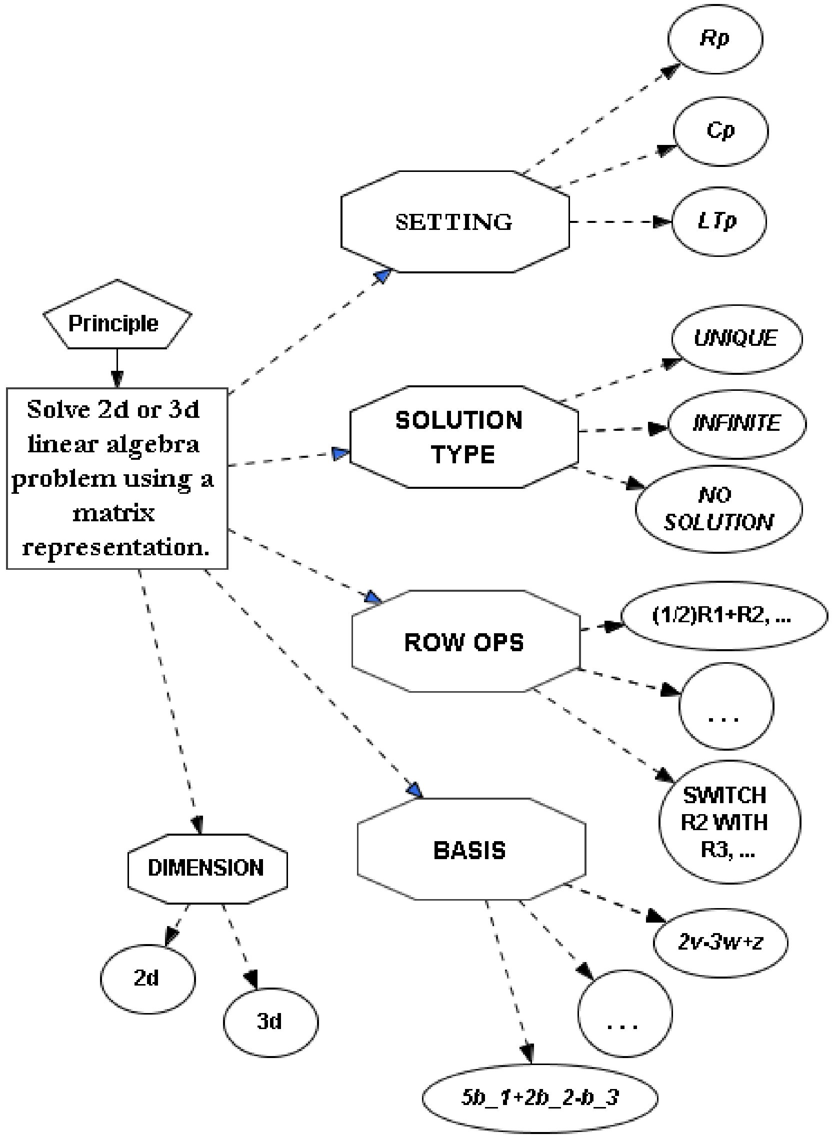


Figure 3.5 Linear Algebra Model (LAM).

Following (Chen & Mo, 2004, p.585), their multi-dimensional framework was extended for solving linear algebra problems representable by matrices and involving the three *Rp*, *Cp*, and *LTp* settings. The basic premise of the LAM-framework for matrix representable linear algebra problems allowing variability in settings and solution type, was that mathematical problem solving involves multiple procedural and representational dimensions. The following dimensions of the LAM were theorized based on the researcher's experience in teaching linear algebra: SETTING, SOLUTION TYPE, ROW OPS, BASIS, and DIMENSION (see Fig. 3.5). Note the similarity between Figures 3.3 and 3.5.

The SETTING dimension consisted of the *Rp*, *Cp*, and *LTp* settings; earlier defined in the discussion of the pilot I study. The SOLUTION TYPE dimension consisted of the unique, no solution, and infinite solution types also introduced in the pilot I discussion. The ROW OPS dimension contained the multitude of sequences and types of row operations possible when performing the row reduction algorithm on matrices (Strang, 2003). The BASIS dimension consisted of procedures related to converting a vector represented in a given basis to its basis-free matrix representation. Hillel & Sierpinska (1994) cited the fact that "two representations of a vector can both be n-tuples with no notational devices relativizing them to a basis" as a source of conceptual difficulty for linear algebra students (p.66). The DIMENSION dimension gave the dimensional setting of a given problem, thus its associated system matrix size. For this study, this dimension was variable between only 2-d or 3-d systems, which are dimensions of usual interest in terms of teaching from the standpoint of the common "familiar geometric embodiments" of undergraduate linear algebra concepts (Harel, 1989, p.57).

Similar to pilot study 1, the solutions to problems designed for this study were procedurally similar in the sense of being solvable by employing the general goal structure (GGS):

1. Representing the problem with an appropriate matrix.
2. Performing the row reduction algorithm to solve the matrix.
3. Interpreting the result of row reduction in the context of the problem setting.

The recitation instruction received by the students consisted of many examples and homework of problems which could be solved by the above procedures. According to Chen & Mo (2004), a critical factor for transfer "is the opportunity to process multiple instances of diverse problems that share a similar goal structure," thus, transfer "in human cognition involves extraction of general, essential information from individual instances that are encountered" (p. 583). The overall design of the problems for pilot study 2 were novel relative to what the subjects had previously been exposed to in their instruction, yet were solvable by the aforementioned common generalized goal structure (GGS). Pilot study 2 thus constituted a classical transfer experiment in the sense of whether or not the GGS could successfully be transferred to the novel setting after exposure to experimental source exemplars.

Confounding variables are variables which might influence the outcome of an experiment, although they were not the original variables of interest. Based on results from Novick (1990), there presented the possibility for *representational transfer* to become a confounding variable for an experiment aimed at also studying procedural transfer (p.130). With this in mind, the problems for this study were blocked as much as possible in terms of representation, meaning that the problems used in the experiments were highly conducive to being formulated in terms of matrix representations. The main justification for this assumption of blocking occurring was that matrix representations were the focus of the instruction in linear algebra problem solving for the subjects in this study. In addition, the source problem solutions given to the subjects prior to their attempt at the target problem were given in terms of matrix representations.

With matrix representation fixed in the experimental design, it was the main purpose of pilot study 2 to determine other potential confounding variables of the LAM which would might require blocking (being held fixed) for a factorial experiment not studying those variables. Based largely on the researcher's experience in teaching undergraduate linear algebra, it was conjectured that the BASIS and ROW OPS dimensions might have a confounding influence when studying the SETTING and SOLUTION TYPE dimensions. Pilot study 2 tested for significance of the BASIS and ROW OPS dimensions in terms of their effect on problem transfer under factorial manipulation. Successfully disproving the following null hypothesis would suggest evidence that the BASIS and ROW OPS dimensions were significant influences for problem solving transfer involving matrix representations and should thus be blocked in order to limit their effect:

H_0 *(Pilot Study 2)*:

> Source problems being similar or dissimilar to target problems in terms of basis representation of vectors, or types of row operations used to solve systems, presents no statistically significant difference for solving novel linear combination (Cp) problems.

Participants

A total of 100 undergraduates, (83 male, 17 female), enrolled in the Matrix and Power Series Methods course during the Spring 2006 quarter, participated in Pilot study 2 for no course credit or compensation. Within recitation sections, the subjects were randomly assigned to condition groups. This study assumed that recitation sections represented random samplings of undergraduates taking the course, *Matrix and Power Series Methods*.

Design and Methods

Pilot study 2 utilized a 2X2 factorial design (Spector, 1981). The levels of the factors for the experiment consisted of source problem similarity or dissimilarity to target problem along the BASIS or ROW OPS dimensions of the LAM framework (see Fig. 3.6). For all of the conditions except the CONTROL condition,

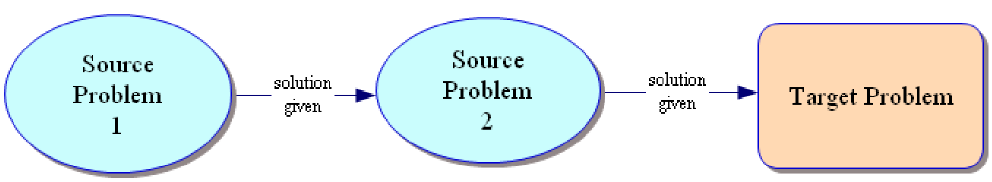


Figure 3.6 Pilot Study 2 design.

the participants were asked to solve a novel target problem after exposure to two source problems. The term *novel* refers to a problem which may seem new and unfamiliar to the subject. The source problems were similar to each other in both factors, and either similar to or dissimilar to the target problem according to which factor was manipulated for a specific condition. The target problem selected by the researcher was a linear combination problem involving three given vectors; $A=v-w$, $B=v+2w$, and $C=v+w$. The subjects were asked to find the linear combination of the vectors A and B that would equal vector C (see Appendix A). The CONTROL condition was composed solely of the target problem.

Prior to beginning the experiment, the subjects were instructed that they may not ask questions during any phase of the experiment. In addition, the students were

Condition	**n**	**Basis**	**Row-Ops**	**Problem**
1	20	Similar	Similar	S_{22}, S_{22}, T_{22}
2	20	Similar	Dissimilar	S_{21}, S_{21}, T_{22}
3	20	Dissimilar	Similar	S_{12}, S_{12}, T_{22}
4	20	Dissimilar	Dissimilar	S_{11}, S_{11}, T_{22}
5	20	---	---	T_{22}

Table 3.4 Pilot study 2 design summary.

told that no calculators or scratch paper may be used during the experiment. The subjects were given a booklet which contained the experimental problems. The subjects were also instructed not to open any part of the booklet, or turn a page, without explicit instructions to do so from the experiment administrator. Before attempting the target problem, two source problems were given which shared the general strategy of the target problem. After each source problem was completed, a solution was uniformly presented to the subjects using an overhead projector and transparencies. The subjects were given 5 minutes to solve each of the problems. According to the factorial design of the experiment (see Table 3.6), procedural relationships between the source and target problems were manipulated. Relative to the LAM framework, the problems constructed for pilot study 2 were fixed in the SETTING , SOLUTION TYPE, and DIMENSION dimensions at the values of Cp, unique solution, and dim=2 respectively. The BASIS condition was manipulated to either be similar to the target problem, i.e. having an abstract vector form, or dissimilar to the target problem, having a basis-free 2×1 column matrix form. Likewise, the ROW OPS dimension was manipulated by having row operations for the source problem matrix system that were either very similar to, or dissimilar to the row operations needed to solve the target matrix system.

Scoring

Source problems and target problem performance from each condition were scored by the author according to a dichotomous scoring rubric in which the score was a **1** if the subject indicated the correct final linear combination solution, or **0** if the subject did not indicate a correct final solution. Both source and target problems were scored. No partial solutions were considered as successful negotiation of source or target problems.

Results

The percentage of participants successfully solving the target problem in each condition was as follows: similar basis, similar row operations (90%); similar basis, dissimilar row operations (50%); dissimilar basis, similar row operations (60%); dissimilar basis, dissimilar row operations (30%); and control (15%), (see Fig. 3.7). A 2 (basis: similar vs. dissimilar) $\times$ 2 (row operation: similar vs. dissimilar) between-subjects ANOVA gave a main effect for basis similarity; $F(1,76) = 6.013$, $MSE = 1.250$, $p = 0.016 < 0.05$, and for row operation similarity, $F(1,76) = 11.785$, $MSE = 2.450$, $p = 0.001 < 0.05$. The interaction between these two factors was not significant ($p = .625 > .05$).

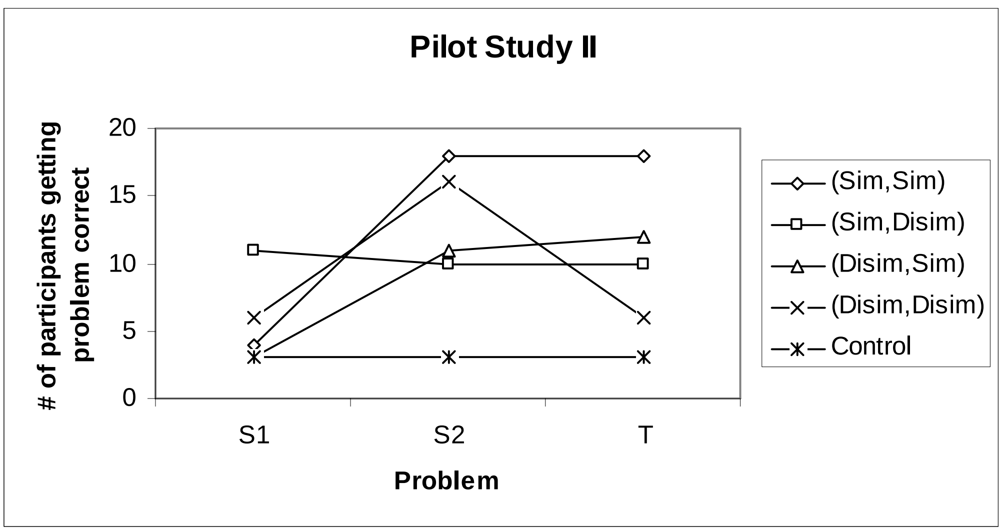


Figure 3.7 Pilot study 2 results.

A post-hoc protected *Least Squares Difference* (LSD) test revealed that target problem solving performance in the similar basis, similar row operation, condition 1, was reliably higher than in the other conditions ($ps < .05$). In addition, condition 3 performed reliably higher than condition 4. No other significant differences were obtained.

Discussion

Results from pilot study 2 indicated that similarity in basis representation and the types of row operations needed to solve a matrix system entailed significant procedural difficulty for solving problems in the Cp setting. Statistical analysis revealed strong evidence to reject the null hypothesis, H_0, that there is no statistically significant difference in solving a novel Cp problem based on source exemplars being either similar or dissimilar in initial basis representation of vectors, or types of row operations needed to solve associated matrix systems. Based on the results of this pilot study, it is concluded that the BASIS and ROW OPS dimensions should be blocked (held fixed) for the design of future experiments not focused on the BASIS and ROW OPS variables.

An additional finding from pilot study 2 related to the construct validity of the LAM framework. *Construct validity* refers to the validation of a theoretical model in terms of the "degree to which inferences can legitimately be made from the operationalizations in a study to the theoretical constructs on which those operationalizations were based" (Trochim, 2007, p.56). The researcher interpreted the finding that there was no significant *interaction* between the BASIS and ROW OPS factors as supporting the independence of the two factors and subsequent use of those categories as independent dimensions for the model (Ramsey & Schafer, 2002).

Experiments 1 & 2

It was the purpose of Experiments 1 & 2 to address the following research question resulting from the author's own teaching experience, Pilot studies 1 & 2, and the research literature on transfer (see Chapter Two):

Research Question 1

> Is there evidence, from the traditional transfer perspective, that transfer is facilitated between linear algebra problems from non-isomorphic settings which share similar (matrix) representations AND solution procedures?

This study addressed research question 1 through the context of undergraduate linear algebra based upon the three conceptually non-isomorphic settings, Rp, Cp, and LTp (see Fig. 1.1, p.5), all sharing the properties of accepting a common matrix representation for problem solving, while entailing similar solution procedures such as row reduction. Experiments 1 & 2 quantitatively sought to address the following null hypotheses:

H_0 (*Experiment 1-2/SETTING)*:

> A source problem being similar or dissimilar to a target problem in terms of its setting *(Rp)* presents no statistically significant difference for solving a novel linear combination *(Cp)*/linear transformation *(LTp)* problem.

H_0 (*Experiment 1-2/SOLUTION TYPE)*:

> A source problem being similar or dissimilar to a target problem in terms of the type of solution to its associated system of equations presents no statistically significant difference for solving a novel linear combination *(Cp)*/linear transformation *(LTp)* problem.

Participants

A total of 130 undergraduates, (102 male, 28 female) enrolled in the Matrix and Power Series Methods course during the Fall 2006 quarter, and some subjects from the previous Spring 2006 quarter, participated in two iterations of experiment 1 for no course credit or compensation. A total of 105 subjects were from the Fall 2006 quarter, while 25 subjects were randomly selected from the Spring 2006 quarter. The 25 subjects from the Spring 2006 quarter also had participated in pilot study II. Similar to experiment 1, experiment 2 had a total of 130 undergraduates, (107 male, 23 female) from the Matrix and Power Series Methods course. A total of 108 subjects were from the Winter 2007 quarter, while 22 subjects were selected from the Spring 2006 quarter. None of the subjects in experiment 2 had participated in Experiment 1, although all of the 22 subjects from the Spring 2006 quarter participated in pilot study 2.

Within recitation sections, the subjects were randomly assigned to condition groups. Subjects from the previous class were assigned to condition groups based on their availability to attend the recitation section. This study assumed that recitation sections represented random samplings of undergraduates taking the course, Matrix and Power Series Methods. The extra students from the previous recitations were recruited for the purpose of constructing a balanced design with as large a sample size as feasible. The sample size for condition groups for both experiments 1 & 2 was $n = 13$. This number was determined based on the largest number of participates the researcher could reliably organize for the experiments.

Design and Methods

Experiments 1 & 2 both employed a source-target problem, 2×2 factorial design, (see Fig. 3.8). Experiment 1 juxtaposed the familiar Rp setting with the Cp settings in the SETTING dimension of the LAM framework, as well as manipulated different solution types to the matrix systems formed via the SOLUTION TYPE dimension of the LAM framework. Experiment 2 was designed similarly to Experiment 1, except the LTp setting was used instead of the Cp setting in the manipulation of the SETTING factor. Based on the previous Pilot study 2, the basis form of the given objects was held constant. Also as a result of Pilot study 2, the types of row operations used to solve associated systems were designed to be as similar as possible relative to the standard way the author instructed the subjects to perform row reduction. Experiments 1 & 2 were each divided into two identically administrated iterations, 1A -1B and 2A-2B, respectively.

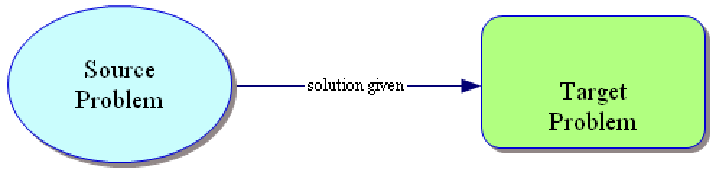


Figure 3.8 Experiments 1 & 2 design.

It was the purpose to conduct the experiments in two iterations in order to address the issue of *reliability*, which pertains to the "consistency or dependability" of the measurement instruments (Trochim, 2007, p.97). The reliability estimates for this study were based on comparisons across condition groups for the different iterations of an experiment, between subjects. For example, 65 subjects took part in Experiment 1A, while a different 65 subjects took part in Experiment 1B. *Cronbach's alpha*, a method for estimating the reliability of a measure, was implemented to check the reliability of the data (Trochim, 2007, p.92).

For Experiment 1, the target problem consisted of a linear combination problem having vectors presented with graphical 2-tuple coordinates and a unique solution. Based on Pilot Study 2, all of the conditions were designed with similar basis representations of vectors (when relevant), and row operations used to solve matrix systems (see Appendix B). Condition 1 consisted of a source problem similar to the target problem in terms of the setting and the unique type of solution obtained from solving the associated system of equations. Condition 2 was different from the target problem in terms of the source problem having no solution. Condition 3 contained a source problem situated in the Rp setting, while the solution was unique, similar to the target problem solution type. Condition 4 was designed with a source problem situated in the *Rp* setting and having no solution. The control condition was identical to the target problem. The infinite solution case was not included in this experiment since the vectors would all be in a line, and might pose a lack of novelty and consistency in the graphical setting in which the other problems were presented in. A control condition was included in both Experiments 1 and 2 to insure adequate difficulty in *baseline performance* for indicating sufficient novelty of the target problem (Chen & Mo, 1990).

Condition	**n**	**Basis**	**Row-Ops**	**Problem**
1	13	Similar	Similar	S_{22}, S_{22}, T_{22}
2	13	Similar	Dissimilar	S_{21}, S_{21}, T_{22}
3	13	Dissimilar	Similar	S_{12}, S_{12}, T_{22}
4	13	Dissimilar	Dissimilar	S_{11}, S_{11}, T_{22}
5	13	---	---	T_{22}

Table 3.5 Experiments 1 & 2 design summary.

For Experiment 2, the target problem was embedded in the *LTp* setting and involved solving for the pre-image of a vector in the range of the linear transformation represented by the given matrix A. In condition 1, a problem similar in both factorial dimensions to the target problem was given. In condition 2, the source problem was the same as in condition 1, however, the target problem changed to a no solution case. For condition 3, the setting changed to the familiar *Rp* setting for the source problem, however, both problems shared a similar unique solution. Condition 4 also involved the same source problem condition 3, situated in the *Rp* setting and having a unique solution, whereas, the condition 4 target problem was from the *LTp* setting and had no solution. Similar to Experiment 1, it was decided to not include the infinite solution case in experiment 2 in order to maintain consistency between the two experimental designs. The control problem for experiment 2 was an *LTp* setting problem with a unique solution.

Prior to beginning Experiments 1 and 2, the subjects were instructed that no questions were to be asked during any phase of the experiment. In addition, the students were told that no calculators or scratch paper may be used during the experiment. The subjects were given a booklet which contained the problem(s). Before attempting the target problem, one source problem was given which shared the general strategy of the target problem. The subjects were allowed 5 minutes to solve each of the problems. After the source problem was completed, a brief GGS-type solution was uniformly presented to the subjects using an overhead projector and transparencies. See Appendices B and C for a complete list of the problems used in Experiments 1 & 2, respectively. See Table 3.5 for the design summary of Experiments 1 & 2.

Scoring

Source problem and target problem performance from each condition of both Experiments 1 & 2 were scored by the author according to a dichotomous scoring rubric in which the score was a 1 if the subject indicated the correct final linear combination solution (Exp. 1); or correct vector or lack thereof for the linear transformation target problem (Exp. 2). A 0 was recorded if the subject did not indicate a correct final solution. Both source and target problems were scored.

Interviews

Based on results from the preceding pilot experiments and Experiments 1 and 2, within the theoretical perspective of traditional transfer (see Chapter 4), the interview portion of this study was composed from a theoretical perspective of actor-oriented transfer, for the further investigation into student difficulties in transfer in mathematics problems which may share representations and procedures, yet originate from conceptually non-isomorphic linear algebra settings. *Semi-structured* interviews (Bernard, 1988) were conducted during the Winter 2007 quarter as part of a mixed-method study investigating transfer in linear algebra problem solving. Four males and 1 female, all sophomore engineering majors, were initially recruited for the interviews based on the criteria of their willingness to participate in the interview process, as well as their non-participation in the previous pilot studies or Experiments 1 and 2. The researcher wanted the non-participation of the interview subjects in the previous pilot studies and experiments for the purpose of preserving the novelty of the interview problems, which were similar in design to the problems used in Pilot study 2, and Experiments 1 and 2, as well as create interview conditions suitable for an inquiry into actor-oriented transfer, as opposed to traditional transfer studies where “subjects are typically taught a solution, response, or principle in an initial learning task and then perform a transfer task" (Lobato & Siebert, 2002, p. 94).

The subjects were contacted through email by the researcher to solicit their participation in the interviews. All of the subjects were concurrently taking the introductory linear algebra course, *Matrix and Power Series Methods*, in addition to having the same lecture professor and using the same text, Lee (2006). The author acted as recitation instructor and conducted all recitation sections in a uniform manner in that the participants had the same type of homework, quizzes, and exams as the previous subjects in the study. The subjects were compensated five dollars each for their participation in a single, hour-long interview.

Instruments

Three interview questions were designed based on the target questions from pilot study II and Experiments 1 and 2 (see Appendix D). The interview design consisted of two consecutive problems from the *Cp* setting, and one problem involving the *LTp* setting. Recall that Pilot study 2 was implemented for the purpose of determining the significance of certain theorized dimensions of the multi-dimensional procedural model for linear algebra problem solving (LAM). The target question from Pilot study 2 formed Interview question 1, and was included in the interview design due to its novelty and *Cp* context.

The design of Interview question 2 was influenced from the target question design from Experiment 1, however, an over-determined system was given in the setting of $\Re^3$ instead of the usual $\Re^2$ setting in Experiment 1. Based on results of Experiment 1, problems 1 and 2 were designed to be non-isomorphic within the same Cp setting for the purpose of further inquiry into the SOLUTION TYPE dimension of the LAM, as well as a hypothesized representational difficulty based on Experiment 1 results associated with the Cp setting (see *Representational Correctness Hypothesis*, Chapter 4). Interview question 3 involved the LTp setting, where a given linear transformation matrix had no solution, in terms of a pre-image vector mapping into a given vector. This third interview problem was similar in design to the target problem of condition 2 from Experiment 2 (see Appendix C), and was included for further inquiry into a hypothesis generated from Experiment 2 (see *Semantic Access Hypothesis*, Chapter 4). As in the previous experiments, all of the interview problems could be solved through the successful use of the general goal structure (GGS):

1. Representation of the problem with an appropriate matrix.
2. Row reduction algorithm employed as solution procedure to solve matrix.
3. Interpretation of matrix system solution in the context of a novel problem setting.

Data Collection

The data collected from the interviews consisted of both individual interview work and videotaped data. The individual data, written on white paper, showed the work the subjects did as related to the interview problems presented to them. The videotape data came from one video camera, set up in front of both the interviewer and the subjects. The author functioned as the interviewer for all of the interviews.

Each interview lasted approximately 1 hour. All of the interviews took place in the same reserved private room in the university library. Using problems experimentally verified as belonging to novel non-isomorphic linear algebra settings (Pilot study 2, Experiments 1 and 2), it was the purpose of the interview portion of this study to investigate transfer difficulties from a non-expert, actor-oriented perspective, characterized by the subjects' "personal structuring" of phenomena, as well as "transfer distributed across social planes," hence; the interview format was largely *semi-structured*, with its adherence to the order and completion of the interview problems, and minimal interviewer interaction, unless subjects appeared stalled in problem solving (Lobato, 2003; Strauss & Corbin, 1990).

Analysis of Data

After all of the interviews had been completed, analysis of the data began with the researcher transcribing all of the interviews using video transcribing software. This software allowed the author to watch the video at various speeds while typing in the transcription. In addition to the spoken words of the interviewer and subjects, physical gesticulations were also transcribed by the researcher. Throughout this process, each of the interviews was listened to several times. Upon completion of the transcription of the data, the author conducted a line-by-line reading of the transcripts in order to obtain a descriptive analysis of the interviews (Patton, 1990). During this analysis of the transcript data, the following basic questions were asked:

- What did the subjects say which might indicate traditional transfer of previous knowledge to the interview problems?

- Did the subjects transfer knowledge relative to their own personal structuring and perception(s) of a problem, indicative of actor-oriented transfer?

- How did the subjects go about solving the problems? Did they use procedures consistent with the GGS, and LAM constructs?

- How might interviewer interaction with subjects affect traditional and/or actor-oriented forms of transfer from familiar settings or actor-constructed knowledge, to unfamiliar non-isomorphic problem settings?

The researcher used these questions as a basis for analysis of interview data for the purpose of understanding not only what knowledge the interview subjects were transferring to the interview problems, but also, in the sense of actor-oriented transfer, what the subjects believed they might be transferring during the course of solving the interview problems. Also of interest to the researcher were the possibilities of interaction between the interviewer and the subjects which might consist of "hints" or evidence of meta-cognitive information which could influence transfer (Gick & Holyoak, 1983; Holyoak & Koh, 1987; Novick, 1990, p.128).

After the descriptive analysis, it was determined that subjects: Iliana, Josh, and Dan were of particular interest based on the following criteria: (a) they demonstrated initial difficulty with the interview questions; (b) their interviews afforded a detailed investigation into the transfer of a common goal strategy across non-isomorphic linear algebra problem settings; (c) they were vocal during their interviews which allowed for a detailed investigation into transfer difficulties associated with Experiments 1 and 2; (d) each of their interviews provided potential evidence of *actor-oriented transfer*, while initially demonstrating a lack of traditional transfer (Lobato & Siebert, 2002, p.92).

"Actor-oriented transfer is found by scrutinizing a given activity for any indication of influence from previous activities and by examining how people construe situations as similar" (Lobato & Siebert, 2002, p.89). According to Lobato (2003), actor-oriented transfer is defined as "the personal construction of relations of similarity between activities, or how 'actors' see situations as similar" (p.89). Next, the researcher began writing individual case studies of the above selected interviewees. It was the purpose of writing the case studies in order to "expand and extend beyond a purely descriptive account with an analysis which proceeds in some careful, systematic way to identify key factors and relationships among them" (Wolcott, 1994, p.10). Working in this way, the researcher was attempting to develop evidence which could address the following question:

Research Question 2

> In what ways, from the theoretical perspective of actor-oriented transfer, do novice linear algebra students commonly have difficulty with conceptually non-isomorphic problem settings, even when novel problem settings share similar problem representations and solution procedures as familiar problem settings?

Experiment 3

The following *Intentional Transfer Hypothesis* (*ITH*) was conjectured based on the results of the interviews, as well as previously discussed mathematics education research concerning the use of meta-cognitive interventions in the learning of linear algebra (Harel (1989b, 1990); Hillel & Mueller, 2006; DeVries & Arnon, 2004; Hillel & Sierpinska, 1994; Dias & Artique, 1995; Dorier, 2000).

Intentional Transfer Hypothesis

> In order to coordinate the solution resulting from a process of row reduction on a matrix representative of a given problem embedded in a particular linear algebra setting, it is necessary to understand the process and/or definition(s) that motivated the initial matrix representation of the problem, in relation to its particular setting.

In combination with quantitative results from Experiments 1 and 2 confirming the rarity of transfer between non-isomorphic settings in the absence of information concerning the relationship between those settings, the qualitatively based *ITH* was formed to address the final, mixed-method, (qualitative + quantitative) component of this study:

Research Question 3

> What evidence can be found that indicates meta-cognitive intervention may facilitate traditional transfer across conceptually non-isomorphic problem settings involving novel target problems which share similar problem representations and solution procedures as more familiar problems?

Concluding the methods for this study, an Experiment 3 of treatment/non-treatment design was constructed for the purpose of testing the *ITH* and providing evidence addressing Research Question 3. Statistical analysis of the results of Experiment 3 challenged the following directional null hypothesis:

H_0 *(Experiment 3)*:

> There is no statistically significant difference between the control and experimental group performances, or the control group performed better than the experimental group on a novel linear combination problem.

Experiment 3 sought to affirm the corresponding alternative hypothesis:

> H_a *(Experiment 3)*: The experimental group performed significantly better than the control group on a novel linear combination problem.

Participants

Sixty-six subjects took part in Experiment 3 during the Spring 2007 quarter. The subjects were randomly assigned to condition groups. Within each recitation section, 11 subjects were assigned to the experimental treatment group and 11 subjects were assigned to the non-treatment group. Excess subjects not randomly assigned to a group within the recitation section did not take part in the experiment; however, they were allowed to work on the problems as did the participants. The subjects were not compensated for their participation.

Design and Methods

Experiment 3 was developed based on a “two-group” treatment, non-treatment design (see Fig.3.9), Spector (1981). Based on the results of Experiments 1, 2, and the Interviews, the author made the assumption in the design of Experiment 3 that the Rp setting constituted a familiar setting for students. In this respect, the 1-target design of Experiment 3 is still viewed as a traditionally-oriented transfer experiment from the familiar Rp setting to the unfamiliar, conceptually non-isomorphic LCp setting. The author made this design decision for the purpose of isolating the effects of the meta-cognitive treatment, and due to the predominant interview and experimental evidence suggesting that the Rp setting is, in general, a familiar setting. The non-treatment subjects were asked to temporarily leave the classroom and wait while the treatment group was given an instructional intervention by the author. During the course of the treatment presentation, no questions were to be asked or answered. The duration of the treatment was approximately five minutes. After the instructional treatment was implemented, the non-treatment subjects were let back into the classroom to attempt the target problem along with the treatment group (see Fig. 3.10).

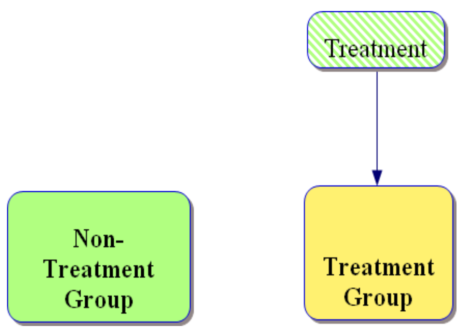
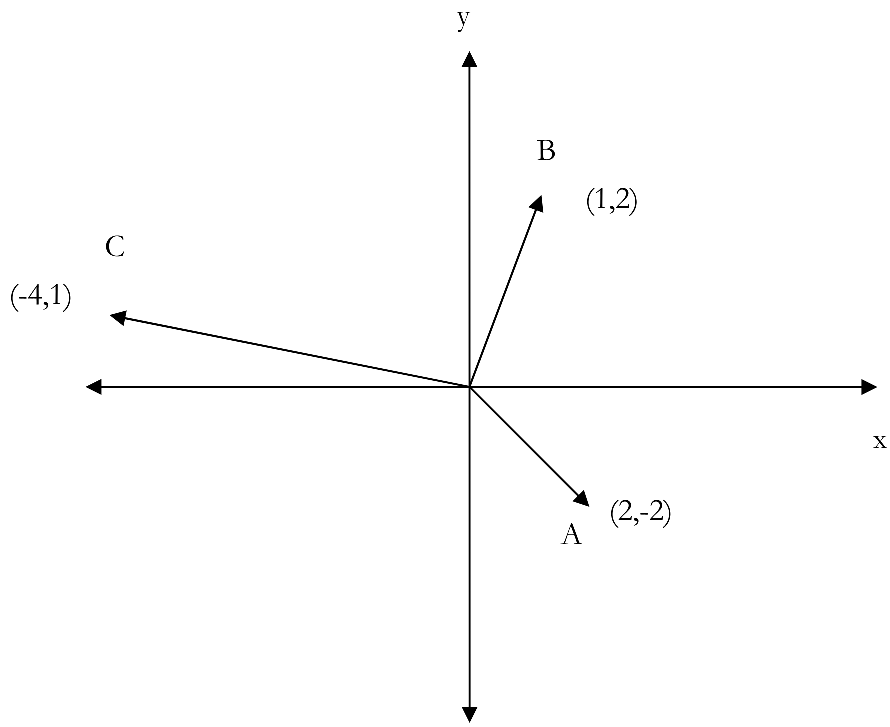


Figure 3.9 Experiment 3 design.



In $\Re^2$, find the linear combination of vectors A and B that give vector C.

Fig. 3.10 Exp. 3 target problem.

It was the purpose of the instructional treatment to function as a meta-cognitive intervention that would convey to the treatment group the intended meaning of a matrix representation in the context of the linear combination (*Cp*) setting and matrix multiplication; as contrasted with the relationship between matrix multiplication and the (*Rp*) setting (see Fig. 3.11). Recall the term *meta-cognitive* refers to instructional information which is reflective in nature and about mathematics; or in connection with this study, “what constitutes a mathematical operation: for example, information on the role of interplay of settings in problem solving, the role of questioning, examples and counter-examples, the role of identifying parameters in a mathematical question, the role of testing, etc." (Dorier et al., 2000, p.151). The reflective nature of the meta-cognitive intervention is thought to facilitate the process of reflective abstraction, thereby introducing an intervention which could "contribute to a certain imbalance, a dynamic imbalance between meta- knowledge and knowledge" (Dorier et al., 2000, p.153).

Given a matrix like $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and the vector $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, notice

that $AX = b$ could mean: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ which could

represent: $\begin{array}{rcrcl} x & + & 2y & = & 2 \\ 3x & + & 4y & = & 2 \end{array}$. But <u>also</u>, $AX = b$ could mean:

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix} = \begin{bmatrix} x \\ 3x \end{bmatrix} + \begin{bmatrix} 2y \\ 4y \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, i.e. a linear

combination of the columns of A.

Figure 3.11 Meta-cognitive intervention.

By the construction of an instructional intervention which highlights the representational aspects of matrices as "themselves an object of reflection and analysis," the researcher sought to induce trans-level thinking characteristic of theoretical thought upon the structural relationships between the LCp and Rp settings and matrix representations thereof (Sierpinska, 2000; Piaget & Garcia, 1983). It was the focus of Experiment 3 to provide preliminary evidence that such meta-cognitive instructional interventions could facilitate transfer under the conditions of a traditional transfer experiment. Furthermore, in terms of the author's blended view of constructivism and transfer outlined in Chapter Two, the design of the meta-cognitive intervention used in Experiment 3 was thought to induce the trans-level reflection necessary to construct more general and abstract problem solving schemas able to better negotiate the disequilibrium induced by the *Rp-LCp* setting change. As a consequence of the construction of more generalized schemas, the author expects subjects to exhibit classical *transfer* via the accommodation of pre-existing problem solving schemas becoming more *powerful* for the assimilation of previously conceptually unrelated settings; where "one meaning of powerful is that a system of representation has a wide and varied domain of applicability," (Goldin & Kaput, 1996, p.425).

The remainder of this dissertation is divided into four chapters. Chapter Four presents the results of Experiments 1 and 2, as well as discussions concerning the interpretations of the results. Chapter Five contains the analysis and discussion of the interview case studies. Chapter Six contains the results and discussion of Experiment 3. Chapter Seven concludes the study discussing its significance, theoretical implications, limitations, and the researcher's direction of future research.

Chapter 4

EXPERIMENTS 1 & 2 RESULTS

Recall from Chapter Three, it was the purpose of Experiments 1 and 2 to address questions surrounding several results from traditional transfer research (see Chapter Two) which suggested that transfer between non-isomorphic settings was rare in the absence of 'hints' conveying information of the relationships between the non-isomorphic settings, while transfer has been shown to occur between isomorphic settings sharing similar problem solving procedures (Reed, Ackinclose, & Voss, 1990; Spellman & Holyoak, 1996; Holyoak & Koh 1987; Gick, 1985; Reed, 1987; Chen & Mo, 1994). In addition, Novick (1990) found that representations may be transferred between non-isomorphic problem settings. Based on these results, the researcher designed two randomized, 2×2 factorial (Spector, 1981) experiments for the two-fold purpose of exploring the phenomena he observed from his teaching experience, as well as testing a combination of findings from previous transfer research to determine if transfer between conceptually non-isomorphic settings might occur when the settings shared similar procedures and problem representations, as the following research question expresses:

Research Question 1

> Is there evidence, from the traditional transfer perspective, that transfer is facilitated between linear algebra problems from non-isomorphic settings which share similar (matrix) representations AND solution procedures?

Experiments 1A-B

Experiment 1a Results

The percentage of participants successfully solving the target problem in each condition was as follows: (similar setting, similar solution type) (69%); (similar setting, dissimilar solution type) (46%); (dissimilar setting, similar solution type) (23%); (dissimilar setting, dissimilar solution type) (8%); and control (15%); see Fig. 4.1. A 2 (*setting:* similar vs. dissimilar) $\times$ 2 (*solution type*: similar vs. dissimilar) between-subjects ANOVA gave a main effect for SETTING similarity; $F(1,48) = 12.100$, $MSE = 2.327$, $p = 0.001 < 0.05$. No main effect was found for SOLUTION TYPE, $F(1,48) = 2.500$, $MSE = .481$, $p = .120 > .05$. The interaction between these two factors was not significant ($p = .753 > .05$). A post-hoc, protected *Least Squares Difference* (LSD) test revealed that target problem solving performances in the (similar setting, similar solution type) condition 1, and (similar setting, dissimilar solution type) condition 2, were reliably higher as compared with (dissimilar setting, similar solution type), condition 3. In addition, the (dissimilar setting, similar solution type), condition 3 performed reliably higher in target performance as did the (dissimilar setting, dissimilar solution type) conditions 4 (see Table 4.1).

Experiment 1b Results

The percentage of participants successfully solving the target problem in each condition was as follows: (similar setting, similar solution type) (77%); (similar setting, dissimilar solution type) (38%); (dissimilar setting, similar solution type) (31%); (dissimilar setting, dissimilar solution type) (23%); and control (8%); see Fig. 4.2. A 2 (*setting*: similar vs. dissimilar) $\times$ 2 (*solution type:* similar vs. dissimilar) between-subjects ANOVA gave a main effect for SETTING similarity; $F(1,48) = 5.647$, $MSE = 1.231$, $p = 0.022 < 0.05$. The effect of SOLUTION TYPE was not significant, $F(1,48) = 3.176$, $MSE = .692$, $p = 0.081 > 0.05$. The interaction between these two factors was not significant ($p = .241 > .05$). A post-hoc, protected *Least Squares Difference* (LSD) test revealed that target problem solving performance in the (similar setting, similar solution type) condition 1 was reliably higher than in the other conditions (ps<0.05); see Table 4.2. No other significant differences were found. Reliability analysis of experiments 1a and 1b indicated satisfactory results; Cronbach's alpha coefficient = 0.937>0.7, (see Table 4.3).

Discussion

The results indicated that when source and target problems required similar solution procedures and problem representations, yet transitioned from Rp to Cp non-isomorphic problem settings, participants experienced difficulty. The effects of similarity in setting were evident, even with the provision of a GGS-type solution given for an familiar setting source problem before subjects attempted a novel setting target problem. The results of both experiments 1a and 1b also indicated, however, that source problems being similar to or dissimilar to target problems in terms of the *type* of solution obtained from the matrix system, presented no significant difficulty for participants.

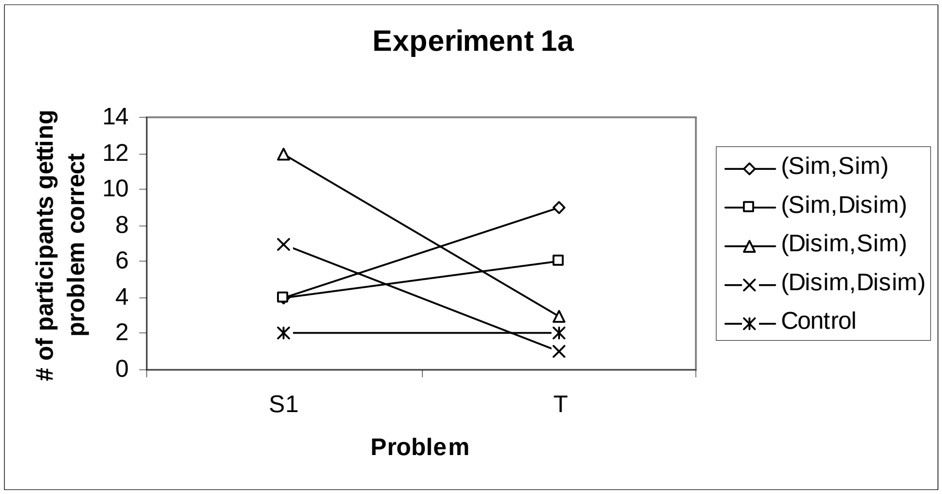


Figure 4.1 Experiment 1a results.

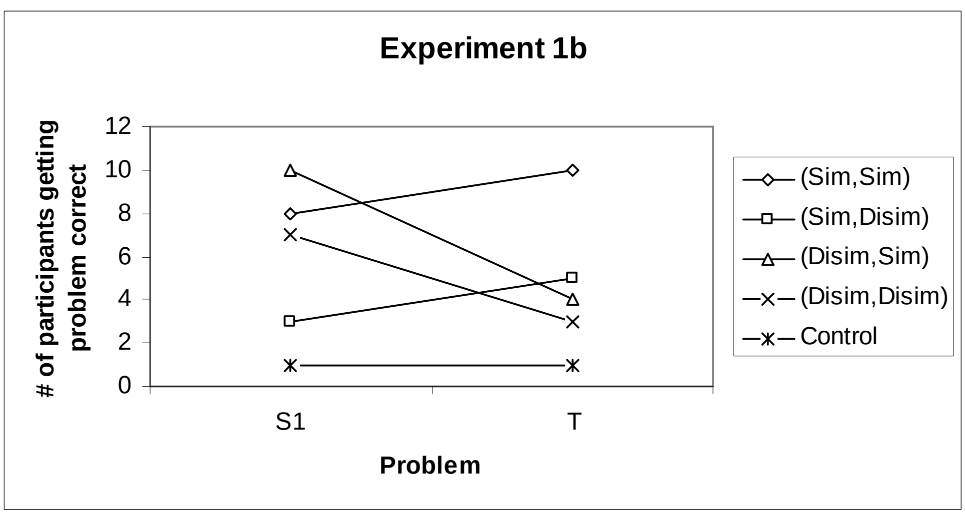


Figure 4.2 Experiment 1b results.

(I) Condition	(J) Condition	Mean Difference (I-J)	Sig.
dd	ds	-.38(*)	.034
	sd	.23	.197
	ss	.23	.197
ds	dd	.38(*)	.034
	sd	.62(*)	.001
	ss	.62(*)	.001
sd	dd	-.23	.197
	ds	-.62(*)	.001
	ss	.00	1.000
ss	dd	-.23	.197
	ds	-.62(*)	.001
	sd	.00	1.000

Table 4.1 Experiment 1a LSD post hoc analysis.

(I) Condition	(J) Condition	Mean Difference (I-J)	Sig.
dd	ds	-.08	.676
	sd	-.15	.405
	ss	-.54(*)	.005
ds	dd	.08	.676
	sd	-.08	.676
	ss	-.46(*)	.015
sd	dd	.15	.405
	ds	.08	.676
	ss	-.38(*)	.041
ss	dd	.54(*)	.005
	ds	.46(*)	.015
	sd	.38(*)	.041

Table 4.2 Experiment 1b LSD post hoc analysis.

Reliability Statistics

Cronbach's Alpha	N of Items
.937	2

Table 4.3 Experiment 1 overall reliability.

Based on the results of statistical analysis of the data, evidence was found to reject the null hypothesis that:

H_0 (*Experiment 1-SETTING)*:

> A source problem being similar or dissimilar to a target problem in terms of its setting *(Rp)* presents no statistically significant difference for solving a novel linear combination *(Cp)* problem.

Statistical analysis also indicated, however, the acceptance of the null hypothesis:

H_0 (*Experiment 1-SOLUTION TYPE)*:

> A source problem being similar or dissimilar to a target problem in terms of its type of solution to its associated system of equations presents no statistically significant difference for solving a novel linear combination *(Cp)* problem.

Upon inspection, with the exception of the relatively high performance on the source problem for the (SIM, SIM) condition in experiment 1b, the graphs shown in Figures 4.1 and 4.2 appeared qualitatively similar. The relatively high performance on the source problem for the (SIM, SIM) condition in experiment 1b was peculiar because this problem was designed to be from an *unfamiliar* setting. The unfamiliar setting in the (SIM, DISIM) condition for experiment 1b, however, exemplified the expected low success in attempts to solve the source problem from the Cp setting. Recall from the design of experiment 1 that SIM in the first factor means that the source problem is similar in setting to the novel, less familiar setting of the target problem. The likeness between the two graphs in Figures 4.1 and 4.2 was also reflected in the 0.937 Cronbach's alpha reliability estimate, which was above the social-science research accepted 0.7 tolerance.

Experiment 1 also provided some indications for substantiation of the construct validity of the LAM-framework. One reason for this was that both experiments 1a and 1b obtained negligible interaction between the SETTING and SOLUTION TYPE factors, indicating dimensional independence (Ramsey & Schafer, 2002). An explanation for this non-interaction, however, may have been the non-necessity for the SOLUTION TYPE dimension in the LAM framework, based on the evidence to accept H_0 *(Experiment 1-SOLUTION TYPE).* Other evidence substantiating the construct validity of the LAM framework included the overall observation that most of the written work in the subject experiment booklets consisted of matrix representations and row reduction solution attempts consistent with the GGS.

Upon closer inspection of the written work of the subjects, however, a major difficulty of transfer does not appear to involve the absence of access to the *basic* GGS schema, but rather the inability to formulate a contextually correct matrix representation relative to the particular setting a problem was posed. As an example, within the (SIM, DISIM) condition for both experiments 1a and 1b, representational errors of this type accounted for an averaged total of 58% of all target success failures. In the following Figures 4.3 and 4.4, subject examples are shown illustrating attempts to formulate a matrix representation for the Problem 2 (target problem) in Condition 2 of experiments 1a and 1b, respectively (see Appendix B).

Note that typical successful matrix representations for the target problem in the (SIM, DISIM) Condition 2, consisted of the following augmented matrices:

$\left[\begin{array}{cc|c} 1 & -4 & 3 \\ 2 & 2 & 1 \end{array}\right]$; or alternatively: $\left[\begin{array}{cc|c} -4 & 1 & 3 \\ 2 & 2 & 1 \end{array}\right]$. Also note the relative diversity in the representations shown in Figures 4.3 and 4.4, as compared with the matrices $\left[\begin{array}{cc|c} 1 & -4 & 3 \\ 2 & 2 & 1 \end{array}\right]$ or $\left[\begin{array}{cc|c} -4 & 1 & 3 \\ 2 & 2 & 1 \end{array}\right]$. This phenomenon was a particularly peculiar observation due to its prevalence in the (SIM, DISIM) condition; a condition in which subjects had already been exposed to a GGS procedural solution of a source problem from a similar setting, which consisted of a similar procedure for forming the correct matrix representation for the target problem. In the other conditions for experiments 1a and 1b, the combined averages of occurrences of initial matrix representational errors in unsuccessful target solution attempts were: (SIM, SIM) (43%); (DISIM, SIM) (42%); and (DISIM, DISIM) (41%).

Since the DISIM aspect of Condition 2 only involved the *type* of solution obtained from solving an existing matrix representation, the author concluded that the DISIM level in the SOLUTION TYPE dimension of the LAM should have had no adverse effect in forming the correct initial matrix representation. This raised the question of why successful *representational transfer* did not occur in the (SIM, DISIM) condition, as well as other conditions. Based on the evidence, the researcher proposed the *Representational Correctness Hypothesis*, which hypothesized that transfer difficulties for the Cp setting may involve the inability to form the correct initial matrix representation. Inquiry into possible causes for this difficulty would be further addressed during the interview portion of this study.

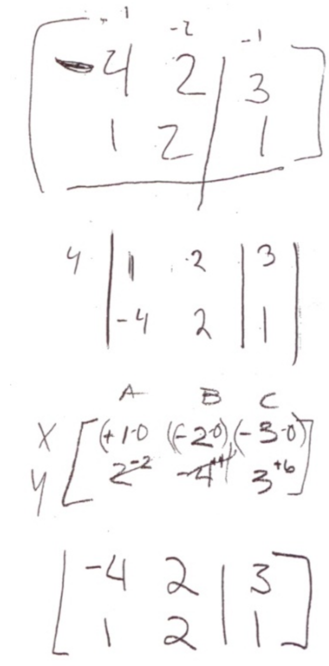


Figure 4.3 Experiment 1a; Condition 2 examples.

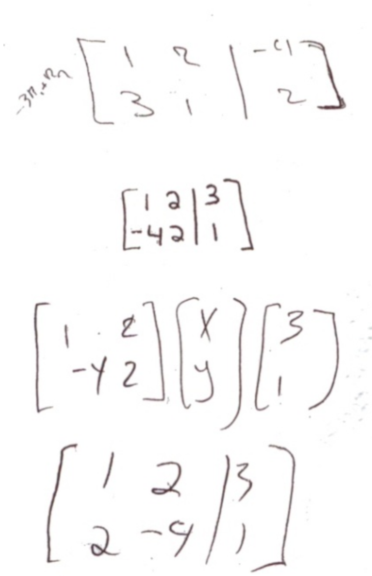


Figure 4.4 Experiment 1b; Condition 2 examples.

Experiments 2A-B

Experiment 2a Results

The percentage of participants successfully solving the target problem in each condition was as follows: (similar setting, similar solution type) (69%); (similar setting, dissimilar solution type) (38%); (dissimilar setting, similar solution type) (31%); (dissimilar setting, dissimilar solution type) (23%); and control (8%); see Fig. 4.5. A 2 (*setting:* similar vs. dissimilar) × 2 (*solution type*: similar vs. dissimilar) between-subjects ANOVA gave a main effect for SETTING similarity; $F(1,48) = 4.141$, $MSE = .942$, $p = .047 < 0.05$. No main effect was found for SOLUTION TYPE, $F(1,48) = 2.113$, $MSE = .481$, $p = .153 > .05$. The interaction between these two factors was not significant ($p = .387 > .05$). A post-hoc, protected *Least Squares Difference* (LSD) test revealed that target problem solving performance in the (similar setting, similar solution type) condition 1 was reliably higher than both conditions 3 and 4 involving the dissimilar setting factor; see Table 4.5. No other significant differences were found.

Experiment 2b Results

The percentage of participants successfully solving the target problem in each condition was as follows: (similar setting, similar solution type) (62%); (similar setting, dissimilar solution type) (46%); (dissimilar setting, similar solution type) (38%); (dissimilar setting, dissimilar solution type) (23%); and control (8%); see Fig. 4.6. A 2 (*setting:* similar vs. dissimilar) × 2 (*solution type*: similar vs.

dissimilar) between-subjects ANOVA gave a main effect for SETTING similarity; $F(1,48) = 5.731$, $MSE = 1.231$, $p = .021 < 0.05$. No main effect was found for SOLUTION TYPE, $F(1,48) = 3.224$, $MSE = .692$, $p = .079 > .05$. The interaction between these two factors was not significant ($p = .552 > .05$).

A post-hoc protected *Least Squares Difference* (LSD) test revealed that target problem solving performances in the (similar setting, similar solution type) condition 1 and (similar setting, dissimilar solution type) condition 2 factors were reliably higher than the target performance in the (dissimilar setting, dissimilar solution type) condition 4 factor. No other significant differences were found; see Table 4.5. Reliability analysis of experiments 2a and 2b indicated a satisfactory result exceeding the accepted 0.7 standard for social sciences research; (Cronbach's α coefficient = 0.971); see Table 4.6.

Discussion

The results of Experiment 2, similar to Experiment 1, revealed that when source and target problems required similar overall solution procedures and problem representations, yet went from the familiar Rp to less familiar LTp non-isomorphic problem settings respectively, then participants experienced difficulties. The effects of similarity in setting were evident, even with the provision of a GGS-type solution given for a familiar source problem before subjects had attempted a novel target problem. The results of Experiment 2 also indicated, similar to Experiment 1, that source problems being similar to or dissimilar to target problems in terms of the *type* of solution obtained from the matrix system, presented no significant difficulty for participants.

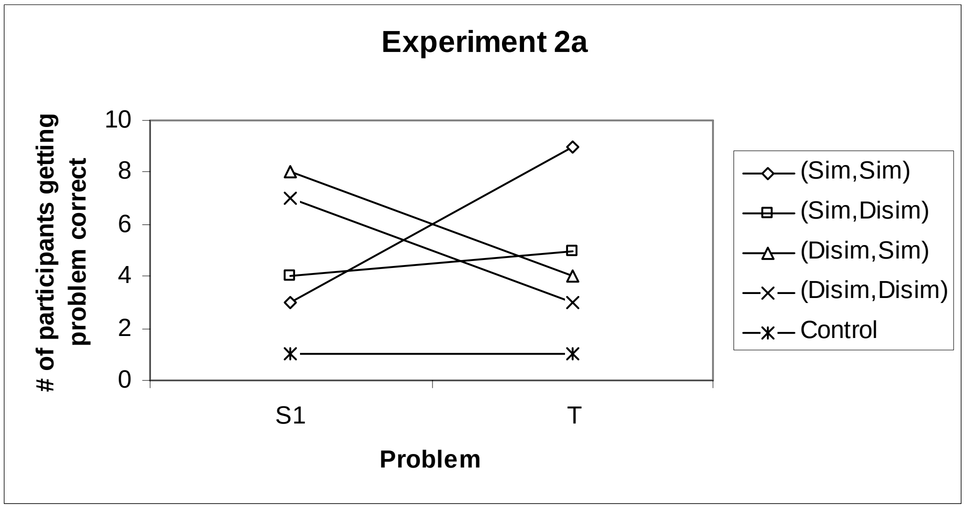


Figure 4.5 Experiment 2a results.

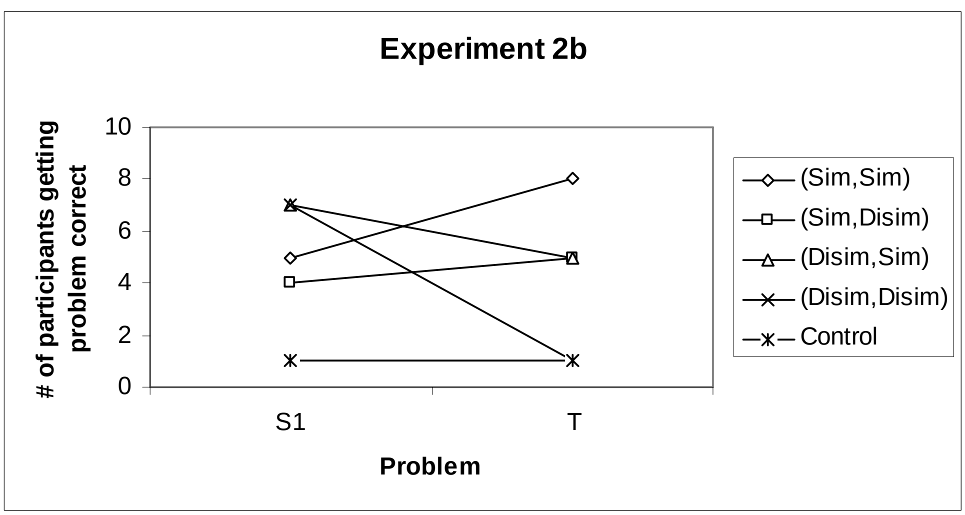


Figure 4.6 Experiment 2b results.

(I) Condition	(J) Condition	Mean Difference (I-J)	Sig.
dd	ds	-.08	.683
	sd	-.15	.415
	ss	-.46(*)	.017
ds	dd	.08	.683
	sd	-.08	.683
	ss	-.38(*)	.045
sd	dd	.15	.415
	ds	.08	.683
	ss	-.31	.107
ss	dd	.46(*)	.017
	ds	.38(*)	.045
	sd	.31	.107

Table 4.4 Experiment 2a LSD post hoc analysis.

(I) Condition	(J) Condition	Mean Difference (I-J)	Sig.
dd	ds	-.31	.097
	sd	-.38(*)	.040
	ss	-.54(*)	.005
ds	dd	.31	.097
	sd	-.08	.674
	ss	-.23	.210
sd	dd	.38(*)	.040
	ds	.08	.674
	ss	-.15	.402
ss	dd	.54(*)	.005
	ds	.23	.210
	sd	.15	.402

Table 4.5 Experiment 2b LSD post hoc analysis.

Reliability Statistics

Cronbach's Alpha	N of Items
.971	2

Table 4.6 Experiment 2 reliability.

Based on these results of the statistical analysis of the data, evidence was found to reject the null hypothesis that:

H_0 (*Experiment 2-SETTING)*:

> A source problem being similar or dissimilar to a target problem in terms of its setting *(Rp)* presents no statistically significant difference for solving a novel linear combination *(LTp)* problem.

Statistical analysis, however, indicated justification to accept the null hypothesis that:

H_0 (*Experiment 1-SOLUTION TYPE)*:

> A source problem being similar or dissimilar to a target problem in terms of its type of solution to its associated system of equations presents no statistically significant difference for solving a novel linear combination *(Cp)* problem.

Upon inspection, the graphs shown in Figures 4.5 and 4.6 appeared qualitatively similar, as also reflected in the Cronbach's alpha reliability estimate 0.971>0.7. By looking at the general trend of the graphs, it was evident that subjects did well on familiar initial source problems, and then did worse as they attempted target problems from unfamiliar settings. Alternatively, the subjects appear to have done worse on initial unfamiliar settings, but did better as they had another chance to attempt a problem in the unfamiliar setting after experiencing a solution from a same-setting source exemplar.

As in Experiment 1, it was found that the SOLUTION TYPE dimension did not play a significant role in terms of the participant performance on the target problems. Upon inspection of the experiment booklets, matrix representational errors did not stand out, as was the case in the Experiment 1, (SIM, DISIM) condition. One reason for this, the researcher inferred, was due to the matrix representation of the linear transformation being given in the statement of the target problems for experiment 2. By seeing the matrix for the transformation, the subject was almost seeing the entire augmented matrix needed to solve the problem, and need only juxtapose the transpose of the given image column vector with the already given linear transformation matrix to form a potentially successful matrix representation.

Upon further inspection of the experiment booklets, due to a predominant absence of target problem work, it became evident that subjects might not have understood the statement of the target problem in the LTp setting. Chen & Mo (2004) distinguished between *accessing processes* and *executing processes*, where access refers to semantic content and execution refers to procedural features (p.586). The researcher subsequently proposed the *Semantic Access Hypothesis*, which hypothesizes that predominant difficulties in the LTp setting may involve access to semantic content defining the LTp setting. *Semantic access* refers to setting-specific definitional meanings and properties of key objects which characterize a particular setting. For example, in the LCp setting, vector space axioms and the definition of linear combination are key items. As another example, in the LTp setting, function terminology, the definition of linearity, and the identification of a linear transformation with its associated matrix, are all key elements defining the LTp setting. Because of the Semantic Access Hypothesis (SAH), further inquiry into the LTp setting was planned for the interview portion of this study.

Summary of Results

Both Experiments 1 and 2, in obtaining a positive and a negative result for the SETTING and SOLUTION TYPE *LAM* dimensions, respectively; replicated previous findings indicating that similarity in mathematical problem solving procedures between source and target problems from similar isomorphic settings enhanced transfer and, in addition, the evidence substantiates that procedural transfer may not be one dimensional (Chen & Mo, 2004, p.588). The results of both experiments showed convincingly that procedural transfer did not occur between conceptually non-isomorphic linear algebra settings, even in the presence of hints for solutions to source problems from familiar settings, procedural similarity in solving the problems (GGS), and the presence of common matrix representations. The results of experiments 1 and 2 can be summarized as follows:

1. The negative results of Experiments 1 and 2 were not seen to extend results from (Reed, Ackinclose, & Voss, 1990; Spellman & Holyoak, 1996; Novick, 1990; Gick, 1985; Holyoak & Koh, 1987; Reed, 1987)) in regard to the traditional transfer of problem solving knowledge across conceptually non-isomorphic problem settings which share problem representational features and solution procedures.

2. The evidence duplicates prior research that transfer is rare between non-isomorphic settings without information of relationships between the settings (Holyoak & Koh, 1987; Novick, 1988; Ross, 1984; Dias & Artigue, 1995, p.41).

3. Transitions between different problems involving the *unique* and *no solution* solution types presented no statistically significant difficulties in traditional transfer.

4. A key component to successful transfer from the Rp to Cp setting may involve the ability to obtain the correct initial matrix representation. *(Representational Correctness Hypothesis)*

5. A key component to successful transfer from the Rp setting to the LTp setting may involve access to relevant semantic content information defining the LTp schema. *(Semantic Access Hypothesis)*

Recall from Chapter Three, it was the purpose of Experiments 1 and 2 to address the following research question:

Research Question 1

> Is there evidence, in the traditional transfer paradigm, that transfer is facilitated between linear algebra problems from non-isomorphic settings which share similar (matrix) representations AND solution procedures?

In conclusion of this chapter, Experiments 1 and 2 provide convergent evidence, in the classical transfer paradigm, answering the above first research question in the <u>negative</u>:

> There is no evidence that similarity in matrix representation and solution procedures facilitates transfer across non-isomorphic linear algebra problem settings.

Furthermore, these experiments generated several hypotheses as to the source of difficulties for non-isomorphic setting transfer associated with the Cp and LTp settings. Finally, the results of Experiments 1 and 2 call into question the necessity of the SOLUTION TYPE dimension in the LAM framework. Based on the limited exploration of the SOLUTION TYPE dimension in Experiments 1 and 2, however; it is the researcher's decision to design an interview problem addressing the representational difficulties posed by the infinite case solution, (see problem 2, Appendix D). The results of Experiments 1 and 2 subsequently influenced the design of an over-determined LCp setting linear combination interview problem which still maintains the unique solution characteristics of the previous experiments, yet possesses a potentially perplexing and novel infinite solution form (see problem 2, Appendix D).

C h a p t e r   5

INTERVIEW RESULTS

Introduction

Experimental results from the previous chapter, contrary to implications drawn from a combination of results from traditional transfer research (Reed, Ackinclose, & Voss, 1990; Spellman & Holyoak, 1996; Novick, 1990; Holyoak & Koh 1987; Gick, 1985; Reed, 1987), resulted in evidence supporting the notion that students often experience significant difficulty in the traditional transfer of linear algebra problem solving knowledge from familiar settings to non-isomorphic settings, even when problems from differing settings shared similar matrix representations and solution procedures. It is the purpose of the interviews to qualitatively explore questions surrounding the results of Experiments 1 and 2 from the theoretical perspective of *actor-oriented transfer* (Lobato, 2006, p.437). After preliminary coding and case analysis, as described in Chapter Three, two pivotal stages of problem solving were observed. The *representational transfer* stage consisted of the learner's obtaining of the correct matrix representation for a problem. The *setting-solution transfer* stage involved the necessity for the learner to interpret the results of a matrix solution relative to the problem setting. Of note, the author inferred that the appearance of these two particular stages of transfer, during the course of the interviews, supports the construct validity of the *Generalized Goal Structure* (GGS) design construct, due to their correspondence with (a) the formation of a matrix for the problem, and (b) the interpretation of a matrix solution after performance of the row reduction procedure (see Chapter Three).

This chapter is organized into two main parts. Part 1 presents evidence of representational transfer that subjects demonstrated as viewed from an actor-oriented perspective. Evidence from part 1 also informs the Representational Correctness (RC) and Semantic Access (SA) Hypotheses conjectured in the previous chapter. Part 2 presents evidence of how, through an actor-oriented perspective, subjects demonstrated their contextual understanding of matrix solutions relative to novel problem settings. Recall from Chapter Two, in Lobato (2006) *actor-oriented transfer* was distinguished from *traditional transfer* in that:

> Transfer from the classical approach is the application from one setting to another of a predetermined set of knowledge from the researcher's or expert's point of view; transfer from the actor-oriented perspective is the influence of learners' prior activities on their activity in novel situations (p.437).

Table 5.1 provided the researcher with a summary of useful categories for distinguishing between both traditional and actor-oriented forms of transfer (Lobato, 2003, p.20).

Upon analysis and coding of the interviews, evidence of actor-oriented transfer most notably occurred with subjects Iliana, Dan, and Josh. Based on the criteria outlined in Chapter Three, these subjects were the foci of the interview analysis. Parts 1 and 2 of this chapter will present the unfolding evidence of both actor-oriented representational and setting-solution types of transfer, as demonstrated by Iliana, Dan, and/or Josh, in a total of ten episodes. For part 1, based on the instructional design of the course material, results of experiments 1 and 2, and the design of the interview questions; it was expected that subjects would most likely transfer matrix representations in order to solve the interview problems. In view of the observations from Experiment 1 leading to the formulation of the Representational Correctness Hypothesis, the researcher anticipated subjects might experience difficulty in the immediate production of a matrix representation based solely on the *static application* of prior knowledge, (see Table 5.1). It is the specific purpose of part 1 to illustrate evidence of representational transfer from an actor-oriented perspective, as well as discuss possible sources of cognitive difficulty associated with the transferring of a correct matrix representation. Six episodes tracked evidence of representational transfer for interview subjects Iliana, Dan, and Josh, (see Table 5.2).

Part 2 of this chapter includes episodes 7 through 10, which cite interview evidence of actor-oriented transfer of the *meaning* of a matrix solution in the context of the *setting* a particular problem was in. Based on the original anecdotal evidence of the author's teaching experience which motivated this study, it was also expected that subjects might experience difficulty interpreting a matrix solution in a context other than the familiar Rp setting. Although the results of experiments 1 and 2 produced no main effects for solution type, a stage of solution-setting interpretation, from an actor-oriented perspective, was seen by the author in the interview data. In Table 5.2, an information guide lists which episodes, subjects, and problems will be presented in parts 1 and 2 of this chapter. Chapter 5 concludes with a summary of the interview findings as well as the conjecturing of the *Intentional Transfer Hypothesis*.

Dimension	Traditional Transfer	Actor-oriented Transfer
Definition	The application of knowledge learned in one situation to a new situation.	The personal construction of relations of similarity across activities.
Perspective	Observer's (expert's) opinion.	Actor's (learner's) perspective.
Research Method	Researcher's look for improved performance between learning and transfer tasks.	Researcher's look for influence of prior activity on current activity and how actors construe situations as similar.
Research Questions	Was transfer attained? What conditions facilitate transfer?	What relations of similarity are created? How are they supported by the environment?
Transfer Tasks	Paired learning and transfer tasks share structural features but differ by surface features.	Researchers acknowledge that what experts consider a surface feature may be structurally substantive for a learner.
Location of Invariance	Transfer measures a psychological phenomenon.	Transfer is distributed across mental, material, social, and cultural planes.
Transfer processes	Transfer occurs if two symbolic mental representations are identical or overlap, or if a mapping between them can be constructed.	Multiple processes, such as an attunement to affordances and constraints, assimilation, language use, and "focusing phenomena", influence transfer.
Metaphor	Static application of knowledge.	Dynamic production of "sameness"

Table 5.1 Actor-oriented and traditional transfer, (Lobato, 2003).

Episode	Subject	Problem
Part 1		
1	Iliana	1
2	Josh	1
3	Dan	1
4	Iliana	2
5	Dan	2
6	Dan	3
Part 2		
7	Iliana	1
8	Josh	2
9	Dan	2
10	Iliana	3

Table 5.2 Interview results content guide.

Part 1: Actor-oriented Representational Transfer

Episode 1: Iliana's demonstration of actor-oriented representational transfer in problem 1.

Interview problem 1

For the following vectors v and w in a vector space *V*, $A = v - w$, $B = 2v + w$,

and $C = 3v + 2w$; write the vector C as a linear combination of the vectors *A* and *B*.

Overview. Evidence will be shown that initially Iliana was unable to formulate a matrix representation for problem 1. After an interviewer-provided hint of the linear combination definition, it will be shown how Iliana demonstrated actor-oriented transfer of a matrix representation by finding a way to transfer a correct problem 1 matrix representation through her co-ordination of the results of algebraic manipulations of the linear combination equation with her previous knowledge of linear systems of equations in the Rp setting.

After finishing her reading of the problem 1, Iliana's immediate response in line [1] was to question if the *cross-product* concept might be "like" the concept of linear combination. After Iliana expressed "I do not know," in line [1], I provided assistance in the form of a hint, for recalling the definition of linear combination.

> [1] I: So, write the vector C as a combination (…) oh well then a linear combination, I could just put those in here, kind of like a cross product (…) I don't know.
>
> *<5 second pause>*
>
> [2] LB: Well, a linear combination if you recall, is a number times one vector plus another number times the other vector.
>
> [3] I: Ok, so in this case since it's a linear combination of A and B. This would be the x value for A and this would be y value for B.

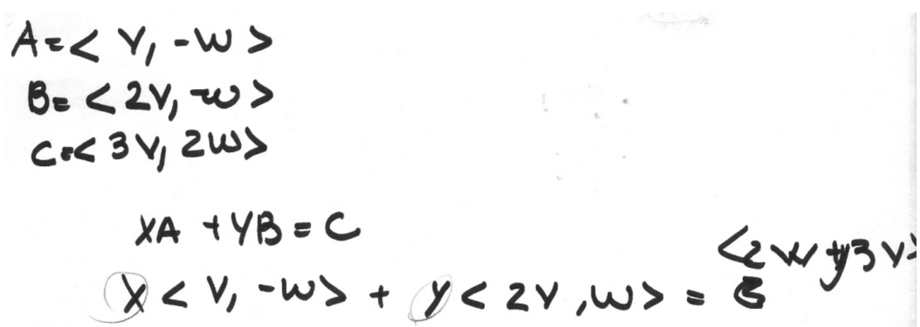


Figure 5.1 Iliana's initial representation of problem 1.

Before receiving the hint, however; Iliana applied knowledge consistent with vector notation when she wrote the given vectors as horizontal coordinate representations (see Fig. 5.1). The spontaneity of the application of this knowledge was interpreted by the researcher as indicative of Iliana's familiarity with the vector component form. The author also noted that Iliana did not write the vectors in the form of 2×1 basis-free column matrices, rather she employed notation using the arrow-like parentheses indicative of symbols used in a vector calculus class. Iliana's apparent inability to recall the linear combination definition demonstrated evidence substantiating and expanding the *Semantic Access Hypothesis* to include the LCp setting within its scope (see Chapter Four). In this respect, from a traditional transfer perspective, Iliana's ability to demonstrate successful transfer in a timed experiment might have been severely hampered without recourse to the linear combination definition.

Note that the hint in line [2] did not contain any specific directions to use any particular formalism. In line [3], Iliana was seen to readily transfer the algebraic unknowns x and y for use as representative variables of the scalars in the linear combination definition, corresponding to the expression 'number' in line [2].

$$x\langle v_1, -w\rangle + y\langle 2v, w\rangle = \langle 3v, 2w\rangle$$
$$\langle xv, -xw\rangle + \langle 2vy, yw\rangle = \langle 3v, 2w\rangle$$
$$\langle xv+2vy, -xw+yw\rangle = \langle 3v, 2w\rangle$$
$$3v = xv+2vy \qquad 2w = -xw+yw$$

Figure 5.2 Iliana's problem 1 algebraic-mode transformations.

Her distinction between scalars and vectors was evident by her substitution of the given vector expressions in for the vector variables *A*, *B*, and *C*, from the linear combination definition (see bottom two lines of Fig. 5.1). In addition, Iliana's use of horizontal, row-like representations of vectors, as well as the variables x and y, was consistent with representational structures belonging to the Cartesian row-picture setting, (Rp), defined earlier in chapter 3.

As the first sentence in line [4] indicates, Iliana immediately began performing algebraic manipulations on the linear combination equation (see Fig. 5.2). In performing these transformations, Iliana applied skills indicative of prior knowledge or experience related to vector space axioms such as: (i) addition of vectors, (ii) multiplication of vectors by a scalar, and (iii) equality of vectors in component form. Following Hillel & Mueller (2006), Iliana is said to function in an *algebraic mode*, whereby she eventually transformed the linear combination equation into the two correct vector expressions seen at the bottom of Fig. 5.2. In the following key exchange, evidence will be shown of: (i) Iliana's contextual conception of algebra as consisting of operations strictly with numbers, (ii) transfer of the linearity concept, and (iii) the representational transfer of the matrix form as a result of her personal constructions (see Fig. 5.3):

> [4] I: Well, these here are (…) numbers, so can I go ahead and multiply them in. Well, ok we have xv comma xw (…) ok, I see, so $3v$ is equal with $2vy$ (…) and $2w$ equals negative xw plus yw. Ok, so (…) well there's a common v here that I can divide by, so the 3 equals $x+2y$ (…) and dividing by w is 2 equals $-x+y$. Oh (…) there straight lines (…) linear. They're linear! Say here, y equals $3-x$ over 2, and y equals $2+x$.
>
> *<12 second pause>*
>
> [5] LB: What if you write them like this?
>
> [6] I: Oh, as a matrix?
>
> [7] LB: Why do you say that?

[8] I: Well, you see the 1 and -1's, and 2 and 1's.

[9] LB: Did you see that earlier?

[10] I: No, I did not.

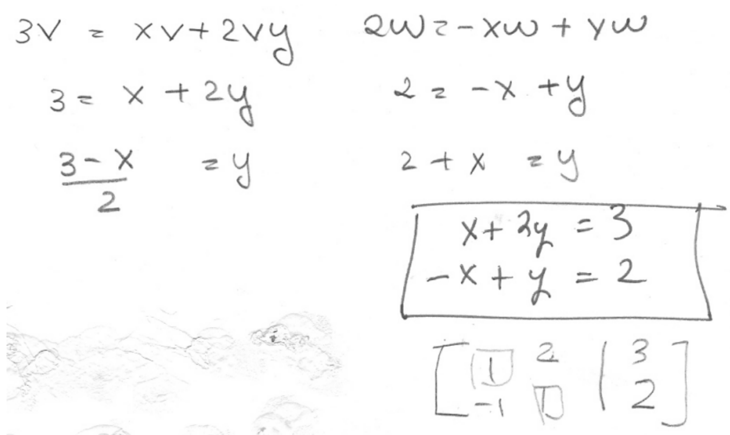


Figure 5.3 Iliana's problem 1 representational transfer.

Also in line [4], Iliana was seen to treat the vectors v and w more as numbers with her intention of 'dividing' by v and w. Although this evidence may have cast some doubt on Iliana's awareness of the vector space properties she transferred earlier when performing scalar multiplications and vector additions, it may have also been an isolated misconception which had nevertheless produced adequately correct results for Iliana in the past, as it had continued to do in this example. Based on this evidence in line [4], it was possible that earlier in Fig. 5.2, Iliana may not have had the conscious *intension* of utilizing vector space axioms, since operations on numbers and vectors, in this case, were similar enough that traditional transfer might occur without her intent awareness.

Upon recognition of the linear equations in line [4], Iliana solved each for y, putting the equations into the y-intercept form, even though she had not been asked to perform graphing (see Fig. 5.3). This occurrence reminded the author of the following passage from Artigue, Chartier, & Dorier (2000):

> Students often identify one type of representation exclusively through semiotic characteristics (a representation with x's and y's is obviously Cartesian (p. 255).

After considerable pause on the part of Iliana, in line [5] I intervened to write the equations lined up in the familiar way (see box in Fig. 5.3). In lines [6], [7], and [8], Iliana then spontaneously transferred the correct matrix representation to the problem based on her viewing the linear equations lined up, as seen in the bottom right corner of Fig. 5.3. From Lobato (2006):

> Reflective abstraction is a constructive process in which the regularities abstracted by the learner are not inherent in the situation, but rather are a result of personal structuring related to the learner's goals and prior knowledge (p.441).

The above quote highlights the evidence that while the manipulations Iliana performed while functioning in algebraic mode were not procedures necessarily needed to solve problem 1, the decision to embark on the algebraic transformations of the linear combination equation was more a part of the personal structurin*g* of Iliana's immediate practical problem solving goals. This personal structuring, initiated by Iliana's practical actions, was seen to culminate in the *co-ordination* of her more familiar linear system knowledge with less familiar knowledge concerning how matrices represent linear combination equations.

Co-ordination is defined as, 'the action of arranging, or condition of being arranged or combined, in due order or proper relation' (OED, 1989). From Piaget (1978), "reflective abstraction relates to the co-ordinations of the subject's actions" (p.162). In this respect, Iliana's co-ordination of her algebraic-mode actions with her previous Rp-setting linear systems knowledge can be seen as the functioning of reflective abstraction. Similar to the concept of meta-cognitive intervention discussed in chapter 2, Lobato (2003) defined *focusing phenomena* as, "regularities in the ways in which teachers, students, artifacts, and curriculum act together to direct students' attention toward certain mathematical properties over others" (p.19). Also from Sierpinska (2000):

> Practical thinking expresses itself in goal-oriented, physical action. Theoretical thinking signals its existence and finds expression through texts. Consequently, semiotic representation systems become themselves objects of reflection and analysis. (p. 212).

In this regard, my suggestion in line [5] to view the equations lined up, comprised a focusing, meta-cognitive intervention, as evidenced by Iliana's subsequent reflection upon the equations in line [8], where she later identified the coefficients of the equations as the numbers for a matrix, thus suspending her former *practical* algebraic manipulation in order to make *theoretical* observations and co-ordinations, thereby treating the derived equations as objects themselves to be reflected upon.

Recall from chapter three that actor-oriented transfer was defined as "the personal construction of relations of similarity between activities, or how 'actors' see situations as similar" (Lobato, 2003, p.89). Since Iliana explicitly articulated the connections between seeing her equations lined up and the identification of the equation coefficients in matrix form (lines [6] and [8]), it was concluded that as a result of her construction of a system of linear equations based on her prior knowledge of working in an algebraic mode with vectors, this prior *constructive* activity had contributed to Iliana's conception of the *connection* between a matrix representing her derived system of equations and the matrix representing a linear combination equation of vectors, thus comprising an example of actor-oriented representational transfer.

Episode 2: Josh's demonstration of actor-oriented representational transfer in problem 1

Overview. Evidence will demonstrate that although Josh was initially able to produce the correct matrix representation and compute a matrix solution, his lack of understanding of the meaning of the matrix representation resulted in a scalar-variable conflict motivating him to construct another incorrect representation. After demonstrating theoretical thinking concerning the representation itself, Josh will be shown to engage in algebraic mode actions, similar to Iliana, where through a process of actor-oriented transfer, he constructed a scenario which produced a meaningful version of his original matrix representation.

> [1] J: Vectors v and w in a vector space V, all right (…) I will just assume we have a nice, flat, Cartesian coordinates. Ok, what we need to do is write the vector *C* as a linear combination of the vectors *A* and B. So (…) that means we want some constant k times *A* plus some constant d times *B* to equal *C*. Ok (…) so I should probably organize it so that the v's add up to 3 and I want the w's to add up to 2.

Initially in problem 1, Josh was seen to spontaneously produce the linear combination definition (see Fig. 5.4). The evidence seems to indicate that Josh intended to employ a matrix representation, based on his use of the phrase, "organize it," in line [1]. This inference was substantiated with the follow-up question and answer in lines [2] and [3], respectively, where I confirmed that Josh wanted to use a matrix for the problem.

$$KA + DB = C$$

Figure 5.4 Josh’s initial linear combination representation.

In the last sentence of line [1], Josh demonstrated what the researcher referred to as a *scalar-variable conflict* concerning the meaning of the matrix form, in relation to the linear combination equation. In his statement “I should probably organize it so the v's add up to 3 and I want the w's to add up to 2,” Josh appeared to be thinking in a characteristically *algebraic mode* by indicating correctly that according to the linear combination equation, the $kv's$ and $dv's$ should add up to 3 as the $kw's$ and $dw's$ would add to 2 (see Fig. 5.5).

$$\begin{aligned} A &= v + w \\ B &= 2v + w \\ C &= 3v + 2w \end{aligned}$$

Figure 5.5 Josh’s problem 1 vector expressions.

[2] LB: So you’re building a matrix?

[3] J: Yes, because A is right here and B (…) and then v and w. I'm trying to get to (…) am I approaching this in the right direction?

[4] LB: What's bothering you?

[5] J: I am trying to think of if getting the identity matrix is the right thing to do, because as it is it looks more like I am solving more for v and w than for what these constants are. I could do row operations on this matrix (…) so we end up with 0, 1, and 5/3. I think I'll think about it some more. What's bothering me is that I am just approaching this as finding out what are v and w.

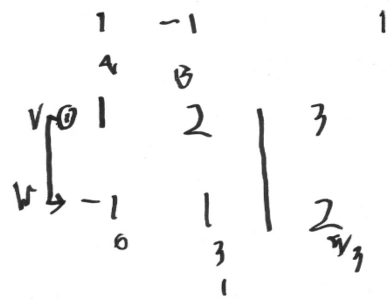


Figure 5.6 Josh’s initial matrix for problem 1.

Based on Josh’s comments in lines [3] and [4], the researcher inferred that his uncertainty was due to the horizontal alignment of the $v's$ and $w's$, as seen in the above fig. 5.6, which led him to believe that their juxtaposition along the rows implied they were the unknown variables for the system of equations represented by the matrix. In line [5], with his statement “it looks more like I am solving more for v and w than for what these constants are. w,” Josh clarified his scalar-variable conflict with the meaning of his matrix, in relation to what he was trying to solve for. Josh next went on to quickly solve the system, which demonstrated practical thinking by his reluctance to confront the issues involving his uncertainty with the meaning of his matrix, and subsequently engage the “goal-oriented, physical action” of the computation of the REF form (Sierpinska, 2000, p.212). Nevertheless, even after Josh’s finding of the REF form with his statement in line [3], “am I approaching this in the right direction?” it was evident that Josh was still unsure about his work.

$$\begin{array}{cc|c} 1 & -1 & A \\ 2 & 1 & B \\ 3 & 2 & C \end{array}$$

Figure 5.7 Josh's alternative matrix for problem 1.

After his statements in line [5], Josh proceeded to adapt his matrix representation in an apparent attempt to resolve the scalar-variable conflict. The matrix Josh produced in Fig.5.7 was similar to his vector expressions in Fig. 5.5, where he earlier had circled the column composed of the scalar- v components. After a considerable period of inactivity, I interjected the question in line [6] as a meta-cognitive intervention so that Josh might reflect on the representational meaning of his linear combination equation, in which case he revised the original linear combination equation to the form seen in Fig. 5.8. After pause, in line [7] Josh conveyed the understanding that the x and y symbols were variables to be solved for. The researcher hypothesized that Josh came to that conclusion based on the typical use of x and y as unknowns in the Rp, row-equation setting.

$$Ax + By = C$$

Figure 5.8 Josh's revised linear combination equation.

[6] LB: Instead of using k and d, can you rewrite that linear combination using x and y?

<pause 8 seconds>

[7] J: Oh, and we want to figure out what x and y are.

$$\begin{array}{cccc} & v & w & \\ x & 1 & -1 & \vert\ A \\ y & 2 & 1 & \vert\ B \\ & 3 & 2 & C \end{array}$$

Figure 5.9 Josh's updated alternative problem 1 matrix.

Josh was next seen to update Fig. 5.7 with more notation (see Fig. 5.9). In his updated matrix, Josh intentionally labeled the columns with v and w, which left him labeling the rows with x and y. The researcher inferred that Josh's statement in line [8], "I could keep going this way and get v and w but does that really get me anywhere?" indicated that he viewed the labeling of the v and w at the top of the columns as the variables to be solved for. Based on this information and the additional labeling around the matrix in Fig. 5.6, the evidence showed that for Josh, it may have seemed that the variables to be originally solved for at the beginning of problem solving were the letters representing the vectors A and B. This evidence provided an explanation for why Josh did not continue with his original matrix, nor pursue checking the correct solution he obtained earlier in line [5], because due to his scalar-variable conflict, it appeared he was not sure which unknowns the matrix was solving for.

> [8] J: Yeah, that helps a little bit (…) I could keep going this way and get v and w but does that really get me anywhere? Let's try this: 1, -1 will be A, and 2, 1 that will be B (…) so now we have a different orientation, the columns are v and w. And then we have (…) I guess this is just the

> solution row (…) so we know that C is $3v$ plus $2w$ (…) let's see if that gets us somewhere. So we want to (…) what if I did v and w in terms of A?

Seemingly unclear as to the meaning of either of the matrices he had produced, at the end of line [8], Josh suddenly changed course and began to work in an *algebraic mode* similar to Iliana in the previous episode 1 (see Fig. 5.10). In line [10], the

$$Ax + By$$

$$x(v-w) + y(2v+w) = c = 3v+2w$$

$$xv - xw + 2yv + yw = 3v + 2w$$

$$v(x+2y) + w(y-x)$$

Figure 5.10 Josh in algebraic mode.

> [9] LB: Do you mean into the linear combination equation?
>
> [10] J: Yes, so we have x times $v-w$, plus y times $2v+w$ and we know that equals C (…) all right, this is looking better. So we have $xv - xw + 2yv + yw = 3v + 2w$. I'm going to collect the v's and w's now. We have v times $x+2y$ and we have w times $y-x$. So, we know that $x+2y=3$, because that's $3v$, and then $y-x=2$, because its $2w$. So, now (…) now we can go back to this.

evidence of Josh's prior knowledge related to vector space axioms such as: (i) addition of vectors, (ii) multiplication of vectors by a scalar, and (iii) equality of vectors in component form. Upon transformation of the linear combination equation into the Cartesian-like row equations $x+2y=3$ and $y-x=2$, at the end of line [10] Josh stated, "now we can go back to this," indicating the original matrix in Fig.

5.6, as reflected in my interview notes concerning where in his written work he was pointing. Recall that Josh had derived this matrix representation and even solved it earlier in his solution attempt, yet he did not check the solution for correctness. The researcher hypothesized that Josh did not check the solution because he did not know its meaning in terms of which variables the matrix was solving for and how they would relate to the problem (*scalar-variable conflict*). When, in line [11], I questioned why his direction in problem solving changed upon seeing equations with the x's and y's, Josh indicated in line [12], his strong association with a matrix as a representation for Rp-like systems of equations, hence his predilection for using x's and y's for the unknown linear combination scalars. Similar to Iliana, Josh seemed to co-ordinate his matrix representation with his previous familiar knowledge of linear systems, thus indicating the constructive presence of reflective abstraction forming generalizing relations between, in this case, Rp and LCp schemas.

> [11] LB: I noticed when you got to two equations with the x's and y's, it seemed obvious to you what the matrix was.
>
> [12] J: Because I look at the matrix as primarily a tool for solving (…) systems of equations. Like right here I was trying to sort of make a system of equations. If I had looked at it as more of a $vx + 2yv = 3v$ and if I actually looked at it (…) I could see that it would be the same (…) and the same thing with the w's. Ok, I'm happier now!

The evidence indicating resolution of Josh's hypothesized disequilibrium due to scalar-variable conflict was evident with his statement, "Ok, I'm happier now!," at the end of line [12]. During the course of Josh's problem solving for problem 1, ample evidence was presented showing the influence of his prior algebraic and vector knowledge, which ultimately seemed to allow him to construct familiarity from that which was initially unfamiliar, and subsequently facilitate actor-oriented representational transfer for his understanding of the correct matrix form. As seen in its entirety, line [13] constitutes direct interview evidence justifying the researcher's earlier hypothesis that Josh produced his original matrix without clear understanding of its meaning due to scalar-variable conflict, however, after proceeding through practical-oriented algebraic mode computations of the linear combination equation, he transformed the problem into familiar forms that "helped [him] me see what this right here that I drew first off really meant" (see line [13]).

> [13] J: And I see where I was kind of getting hung up (…) I wanted to go and right away stick it into (…) you know (…) a matrix, and looking right here, this was really ending up being what it was. But, going through this helped me see what this right here that I drew first off really meant. Because, I was kind of trying to fumble it into that (…) that did end up being the correct way, but I just wasn't quite sure what my results were giving me.

At the beginning of solving problem 1, Josh was seen to engage in practical thought when he immediately solved his first matrix representation, even though the evidence demonstrated his uncertainty concerning the meaning of the matrix. A second demonstration of practical thought was noted when Josh began performing the algebraic manipulations which eventually transformed the linear combination equation into recognizable forms facilitating actor-oriented transfer. Highlighted by the statement, "But, going through this helped me see what this right here that I drew first off really meant," line [13] tracked Josh's own explanations for how the actions he produced, based initially in practical thought, eventually led to theoretical thinking. The researcher interpreted the evidence to indicate that Josh's theoretical thinking consisted in the apparent reflection and co-ordination of connections made between the algebraic mode transmutations and the matrix representation he had personally constructed, concluding in the production of a *personally meaningful* matrix representation. Based on this interview evidence, the researcher observed that both practical and theoretical modes of thinking comprised aspects of *multiple processes* involved with the *attunement of affordances* and *constraints*, i.e. Josh's resolution of the scalar-variable conflict, indicative of actor-oriented transfer (see Table 5.1).

Episode 3: Dan's demonstration of actor-oriented representational transfer in problem 1

Overview. Like Iliana, Dan seemed to have trouble remembering the definition of linear combination. And as in Josh's case, evidence also tracked Dan's experiencing of scalar-variable conflict, and subsequent demonstration of functioning in an algebraic mode which transformed the linear combination equation into familiar forms, promoting reflective abstraction and the co-ordination of previous knowledge consistent with the theoretical perspective of actor-oriented transfer.

> [1] D: So, this is just vector addition is what it's asking? Does $A+B=C$? (…) no, because if you do $v+2v$ that would be $3v$, but $-w$ and w would cancel and not be $2w$ (…) so, if you multiplied *A* times something, like *A* times x plus *B* would equal *C*?
>
> [2] LB: Do you think that you always only maybe need just B?
>
> *<pause 4 seconds>*

[3] D: No, so you would also need a number times *B*. So, we could have $Ax + By = C$ (…) ok, so now we would want to find what x and y are. So (…) if you put these in a matrix would that be something you could do?

After finishing reading problem 1, Dan demonstrated evidence of *practical thinking* by initially attempting to *guess-n-check* for a combination of *A* and *B* making vector *C*. To the researcher, he seemed formulated his definition of linear combination as a continuance of attempts at guess-n-check relying on a prototypical manner that left B always fixed with scalar 1. Similar to Iliana's statement "This would be the x value for A and this would be y value for B," in line [3] of Episode 1, in line [3] Dan chose to represent scalars for the linear combination equation using the familiar variables x and y, after he apparently thought about my meta-cognitive and theoretically-natured hint suggesting he think in general terms about the particular linear combination expression relevant to Problem 1 (see Fig. 5.11). In line [3], with his statement "now we would want to find what x and y are," Dan indicated an awareness of which variables needed to be solved for in his linear combination expression.

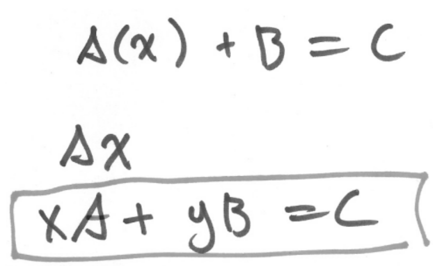


Figure 5.11 Dan's initial formulation of problem 1 linear combination.

Also in line [3], after he formulated the linear combination expression (see Fig. 5.11), Dan wrote an incorrect matrix representation of problem 1. This evidence contributed to support for both the *Semantic Access and Representational Correctness Hypotheses*, since based Experiments 1 and 2, the researcher inferred that it was doubtful whether or not Dan's limited apparent initial facility with the linear combination definition would have been sufficient to produce successful transfer along the confines of a timed transfer experiment. Dan next wrote down a matrix representation which had the given vectors written in as rows, as opposed to columns. This error is similar to the types of errors seen in the Representational Correctness Hypothesis from Chapter Four. This example contributed to evidence of how a subject might transfer the matrix representation to problem solving, although it was not an entirely correct representation.

$$\begin{array}{cc} & \begin{array}{cc} v & w \end{array} \\ \begin{array}{c} \Delta v \\ w \end{array} & \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Figure 5.12 Dan's initial matrix representation of problem 1.

Next, Dan seemed to demonstrate *scalar-variable conflict*, which also appeared in Josh's interview, as can be seen by his mislabeling of the matrix in Fig. 5.12, listing the v and w at the top of the matrix as if they were the variables in the system of equations representing the unknown linear combination scalars at the bottom of Fig. 5.11. This labeling was contradictory to his earlier reasoning in line

[3], where Dan had indicated that it was the goal to find x and y. In lines [5] and [6] below, Dan seemed to be in disequilibrium and explored the possibility that the v and w should be labeled on the side of the matrix instead of along the tops of the columns (see Fig. 5.12). Recall in the previous episode, Josh was seen to implement a similar strategy in the process of his personal reconciliation of a conflicted understanding of the matrix representation for the linear combination setting.

> [4] D: So, ok we'll do (…) I'm not sure if this is exactly right.
>
> [5] LB: Why do you think something might be wrong?
>
> [6] D: Because this column here and this column here are representing the v and the w, but here I have it in rows. If I did something more where I have the two vectors (…) so another way I would do it would be to move the vectors to the side here.
>
> [7] LB: What would you get if you multiplied what you have written?, (bottom of fig. 5.12)

Although Dan did label the v and w vectors along the side of the matrix in Fig. 5.12, he did not reformulate the matrix columns and rows. In line [7], the interviewer suggested that he multiply the left side of the matrix equation in Fig. 5.12. It was the intent of this focusing question to provide Dan with a practical computation which might lead to theoretical thinking, as was seen earlier in episode 2 with Josh. After completing this computation and demonstrating previous knowledge of matrix multiplication, Dan reflected and noticed, in the first two sentences of line [8], that the multiplied expression contradicted his earlier observations in line [1]. Dan himself refocused on the linear combination in the third sentence of line [8], and proceeded to spontaneously engage in virtually the same algebraic mode computations of the linear combination equation as Iliana and Josh (see Fig. 5.13).

> [8] D: Oh, yes (…) so you get $x - y$ and you get $2x + y$. So, this doesn't seem right (…) its got to be something different. Well I know that $xA + yB = C$ is going to be something that I need to be looking for. Ok (…) so we have $x \cdot (v - w) + y \cdot (2v + w) = 3v + 2w$ Ok, now we can multiply through to get $xv - xw + 2yv + 2yw = 3v + 2w$ (…) so now we can separate them out and kind of rearrange things and have v's next to each other and w's next to each other (…) $xv + 2yv - xw + 2yw = 3v + 2w$. And so I think a good next step would be to get things of like terms on each side. Ok, so now what else can I do? Let's see, I am looking for x and y. I see that I could put the v's on one side and the w's on the other side, but I don't see where that would get me. As I look at it, I see that whatever this is next to v has to be equal to this side next to v (…) and the same for w. So, since we have everything here and everything there, we can cancel the v's and get $x + 2y = 3$ and also for w we can get $-x + 2y = 2$. Now, we can put this into a matrix form, and solve for x and y.

$$
\begin{aligned}
&x(v-w) + y(2v+w) = 3v+2w \\
&xv - xw + y2v + yw = 3v+2w \\
&xv + y2v - xw + yw = 3v+2w \\
&v(x+2y) + w(-x+y) = 3v+2w \\
&(x+2y) + (-x+y) = 3+2 \\
&\cancel{x =}\ \boxed{\begin{matrix}(x+2y) = 3 \\ (-x+y) = 2\end{matrix}}
\end{aligned}
$$

Figure 5.13 Dan’s algebraic problem 1 vector transformations.

Also in line [8], Dan made several observations of the linear combination equations, thus demonstrating theoretical thinking concerning the meaning of the notation. With his statements in line [8], “Let's see, I am looking for x and y,” and “I see that I could put the v's on one side and the w's on the other side, but I don't see where that would get me,” Dan demonstrated understanding of which variables he was looking for, however, his second statement indicated that these algebraic computations were still characteristic of practical thinking. After making the same arithmetic error as Iliana did in episode 1 when he stated, “we can cancel the v's,” Dan does achieve the linear equations solely in terms of x and y, and only then does he recognize the correct form of the desired matrix representation at the end of line [8], creating the matrix equation seen in Fig. 5.14.

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Figure 5.14 Dan’s revised matrix representation for problem 1.

[9] LB: How did you know to put those into a matrix form and solve?

[10] D: Well, since we have two equations with two unknowns, we can just make a matrix.

As in the cases of Iliana and Josh, Dan’s algebraic mode computations demonstrated the transfer of at least procedural prior knowledge related to vector space axioms such as: (i) addition of vectors, (ii) multiplication of vectors by a scalar, and (iii) equality of vectors in component form. When questioned in line [9] as to why he knew to write the matrix expression in Fig. 5.14, after the algebraic mode computations Dan responded with the familiar language of college or high school level algebra, indicative of the Rp setting (see line [10]). Lines [11]-[16] track the questioning of Dan as to the nature of his reasoning towards the transfer of a correct matrix representation

[11] LB: If you compare to what you first wrote, what do you notice different?

[12] D: The 3 and 2 are still the same, but these two are swapped. I was kind of on the right track, but ... anyway.

[13] LB: How would you explain the issues you encountered in getting the correct matrix for this problem?

[14] D: First, you need to make a relationship of *A* and *B* to relate to *C*. That's where we got these two variables. Then we needed to take the equations we had and put them into terms of this. Then (…) I notice that at first (…) I put the vectors in as rows. But (…) after going through all of this I see that they should have went into the original matrix as columns.

[15] LB: Why is that?

[16] D: Well, these two need to be here because they both need to multiply by x. Yes, that explains how you would put it into the matrix. If I would have remembered that I would have not made this mistake, or at least been able to correct it.

In line [12], Dan indicated his awareness of the difference between his first and final matrix formulations. Next, in an example of theoretical thought, Dan revealed his understanding of the relationship between the linear combination equation and the matrix representation in line [14] in terms of "where we got these two variables." I included this case as an example of theoretical thinking because of Dan's treatment of representation as an object for reflection. In line [16], Dan further demonstrated theoretical thought and reflection, as evidenced by his statement that the vectors needed to be written as columns because the *entire* column needed to be multiplied by one scalar unknown, and not two.

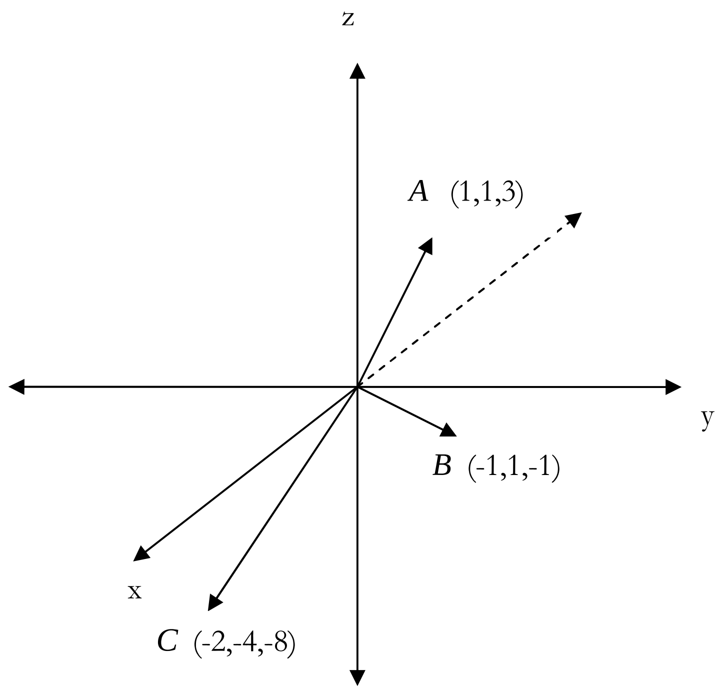
It is possible that Dan uncovered this fact through a combination of possibly three avenues. In the first avenue, his linear combination expression in Fig. 5.11

showed that the entire vector A was multiplied by x and the entire vector B was multiplied by y; in a second avenue, upon collection of the $v's$ and $w's$ in Dan's algebraic mode transformations, the x scalar multiplied both components of vector A, and the y scalar multiplied both components of vector B; and in a third avenue, when Dan saw and reflected upon the final set of linear system of equations in terms of $x's$ and $y's$, (see boxed equations in Fig. 5.13), it was evident that the columns of the vectors A and B were next to the x and y scalar variables, respectively.

In line [16], with Dan's statement "Yes, that explains how you would put it into the matrix," it seemed to the interviewer that the disequilibrium, occurring as a result of scalar-variable conflict, had reached equilibrium. As in the previous episodes with Iliana and Josh, Dan seemed to undergo a series of multiple cognitive processes, both practical and theoretical, which led to a personal construction of mathematics enabling representational assimilation of the linear combination schema, thereby accommodating the scalar-variable conflict. Episode 3 seems to be another example of actor-oriented transfer in which the subjects' own personal linear combination schema dynamically changed due to personal and environmental (interviewer) constructive leading to favorable conditions for the transfer of a correct matrix representation from previous constructed knowledge.

Episode 4: Iliana's demonstration of actor-oriented representational transfer in problem 2

Interview problem 2



In $\Re^3$, write the vector *C* as a linear combination of the vectors *A* and *B*.

Overview. Evidence will show that as in Episode 1, Iliana did not initially produce a definition of linear combination, or a matrix representation. The evidence tracks how Iliana traditionally transferred the definition of linear combination and the algebraic mode procedures from problem 1 within her overall actor-oriented transfer process, which similarly facilitated the representational transfer of a correct matrix representation for problem 2. The evidence also demonstrated a slight reduction of the multiple processes of actor-oriented transfer, as compared with her solution to problem 1.

[1] I: Oh, I'm glad you used x, y, and z (…) its not the u, v, w land! The vector *C* is here, and a combination of these two (…) I would say, when I first looked at it I would take the cross product of *A* and *B* to find *C*. That's

> the first thing I thought about, but I don’t know if it is (…) perpendicular, so I'm not sure.
> *<8 second pause>*

After reading problem 2, Iliana begins line [1] indicating a preference for notation consistent with the Rp Cartesian setting. As in Episode 1, Iliana did not immediately recollect the linear combination definition, but seemed to believe it was somehow related to the cross-product concept. In a modification of her previous attempt, in Episode 1, to assimilate the linear combination problem with the cross-product schema, Iliana revealed prior knowledge concerning the perpendicularity aspect of the cross-product schema and expressed that she was not sure if it applied to the setting of problem 2. From Sierpinska (2000), “in theoretical thinking, reasoning is based on logical and semantic connections between concepts within a system; connections between concepts are made on the basis of their relations to more general concepts of which they are special cases rather than on empirical associations” (p.211). Following Sierpinska (2000), Iliana’s thought in this instance was considered *theoretical* by the researcher because she was able to distinguish semantic aspects of the cross-product schema and realize the generality of her given problem, which did not necessarily contain a particular case of perpendicular vectors.

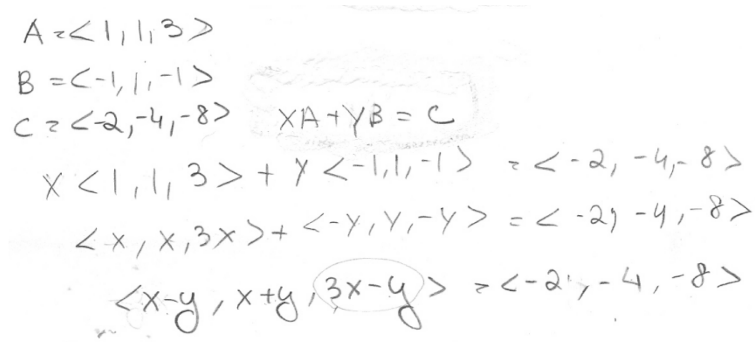


Figure 5.15 Iliana's algebraic problem 2 vector transformations.

[2] LB: Again we have that term *linear combination*.

[3] I: So, we would do the same thing as in the other problem, same question (…) but this time instead of having letters we have numbers. So we would say that (…) x times one comma one comma three plus y times minus 1 comma one comma minus one equals C which is minus 2 comma minus 4 comma minus 8. Then what we did last time, we transferred the variables in to get rid of them, and so that'd be (…) um (…) $x, x, 3x$ plus $-y, y, -y$. So last time (…) we now need to add them. So, it would be $x-y$, $x+y$, and $3x-y$ equals $\langle -2,-4,-8 \rangle$ And so next we set those equal to each other. Well, I like the matrix idea, but if we do that we would have a free variable.

After Iliana appeared stalled in problem solving, with the cross-product construct functioning as an apparent obstacle, in line [2] I reminded her of the topic of the problem, *linear combinations*. Next, the interview evidence in line [3] shows Iliana immediately recalling and traditionally transferring the definition of linear combination and *algebraic mode* procedures from problem 1, to problem 2 (see Fig. 5.15). I described this transfer from a traditional perspective in the sense of the evidence showing characteristics of, 'the application of knowledge learned in one situation to a new situation' (see Table 5.1), as shown by the following statement in line [3], “Then what we did last time, we transferred the variables in to get rid of them.” Although Iliana did not immediately produce a matrix representation, she began applying similar algebraic mode procedures which had precipitated the eventual actor-oriented representational transfer of a correct matrix representation for problem 1.

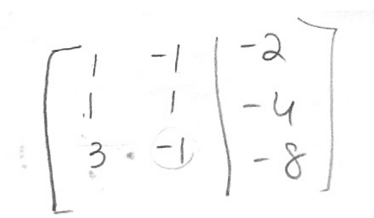


Figure 5.16 Iliana’s problem 2 matrix representation.

Recall from Episode 1, Iliana went through the algebraic mode transformations of the linear combination definition; however, she did not appear to recognize the matrix form until she reflected upon them when *lined up* (see Fig. 5.3). In problem 2, Iliana again performed the algebraic mode manipulations of the linear combination definition, and after making the statement, “Well, I like the matrix idea, but if we do that we would have a free variable” in line [3], she wrote the correct matrix form seen in Fig. 5.16. In Episode 4, although Iliana still performed the algebraic mode computations as she did in Episode 1, she did not need to see the equations lined up in order to produce the matrix. The researcher interpreted this reduction in component processes of Iliana’s actor-oriented transfer from problem 1 to problem 2 as constituting evidence of the *streamlining* of actor-oriented transfer.

By the term *streamlining*, the researcher is referring to the constructivist process in which *actions* may become consolidated into *processes*, such that when a learner can verbalize a sequence of actions without performing them, or can predict the outcome, the learner is said to operate on a *process level* of understanding (Dubinsky, 1997). That Iliana was able to traditionally transfer several processes of the actor-oriented representational transfer of problem 1 to problem 2 is consistent with the fact that problems 1 and 2 lie in the same LCp setting. Because Iliana still relied on her past transmutations of the linear combination definition into forms familiar enough for her to recognize the correct matrix representation, the researcher found the evidence in Episode 4 as indicative of actor-oriented transfer.

Episode 5: Dan's demonstration of actor-oriented representational transfer in problem 2

> *Overview. Evidence will show how the geometry of $\Re^3$ influenced Dan's production of a linear combination expression, resulting in an example of Harel's (1999) contextual conception, which induced scalar-variable conflict in terms of the number of scalar unknowns structurally arising from scalars of the linear combination equation, verses the 3-tuple representation for vectors in $\Re^3$. Evidence also tracked Dan's theoretical thinking which eventually overcame the contextual conception arising from geometric features, eventually leading to the production of a correct matrix representation for problem 2, as well as another interview example of actor-oriented representational transfer.*
>
> [1] D: The diagram (...) I know that there needs to be (...) that these two need to add up and they need to have, again an x and y, that when you multiply and add these two vectors, creates *C*. So, visually the diagram really doesn't

> give me any clue what to do or what to expect. I guess it really doesn't do very much. Now again, we need to do something like before in matrix form. Let's see if I can set it up. In our other one, we had $xA + yB = C$, but now we are in 3-space so if we put z here, I don't know if that makes sense or not, or if it would affect the answer.

In line [1], Dan stated that, "visually the diagram really doesn't give me any clue what to do or what to expect." The researcher noted Dan's use of the variables x, y, and z on the axes (see Fig. 5.17). In line [1], when Dan stated,

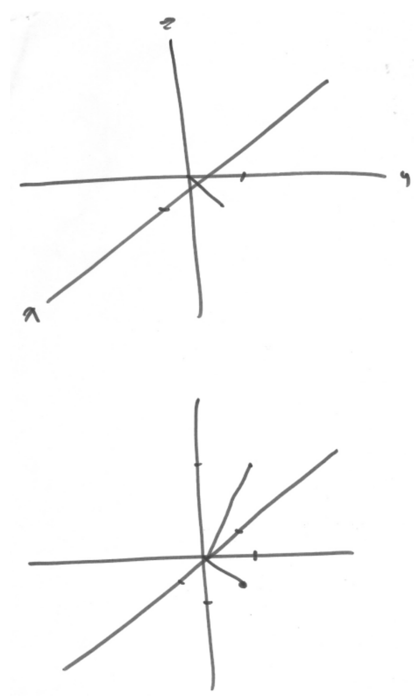


Figure 5.17 Dan's initial problem 2 sketches.

"that these two need to add up and they need to have, again an x and y, that when you multiply and add these two vectors, creates *C*," after uttering the word *again* Dan was seen to point to the linear combination definition from problem 1. Instead of traditionally transferring the same linear combination expression from problem 1 to problem 2, however; the visual geometry appeared to obstruct Dan's understanding of the problem, leading him to produce the linear combination expression described in the last sentence of line [1], and seen in Fig. 5.18. The author interpreted the role geometry seemed to play in Dan's production of the linear combination definition as an example of Harel's (1999) *contextual conception* ,being the inability "to detach from a specific context, whether it is the context of intuitive Euclidean space in geometry or the context of $\Re^n$ in linear algebra."

$$\cancel{xA + yB = zC}$$

Figure 5.18 Dan's initial formulation of linear combination in problem 2.

Furthermore, based on Sierpinska's (2000) characterization of theoretical thinking as, "definitions of concepts, comparisons between concepts and their differentiation are constructed on the basis of the relations of these concepts to more general concepts, and not, e.g., on the basis of their most common examples," the researcher also described Dan's thinking as practical since his reasoning was based more on the proto-typical structure of $\Re^3$, rather than the general structural features of the problem.

> [2] LB: What seems not right?
>
> [3] D: Ok, so I am going to use v and w instead of x and y because that is kind of confusing, so I know that there's a combination of vectors times this plus a combination of vectors times this is going to equal our final vector (…) I don't think that's quite right.
>
> [4] LB: Have you ever multiplied two vectors?

[5] D: Yeah, that's not quite right. Our vector needs to be multiplied by something, I guess a number.
[6] LB: Isn't this saying that, without the z?

[7] D: Yes, but it's confusing for me to use x and y.

[8] LB: Why is that confusing?

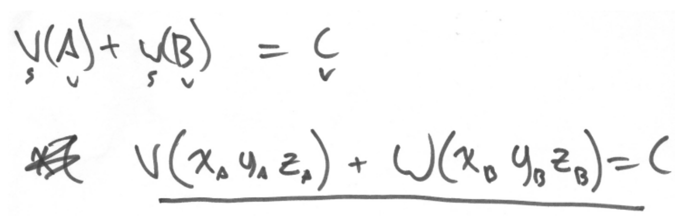


Figure 5.19 Dan's reformulation of linear combination definition for problem 2.

In line [2], after I inquired into Dan's observed pause and seeming disequilibrium, in line [3], Dan attempted a semantic clarification of the key structural features of problem 2, in terms of "I know that there's a combination of vectors times this plus a combination of vectors times this is going to equal our final vector." It seemed Dan's use of the variables v and w, typically reserved for vectors, caused him confusion concerning the linear combination definition itself (see lines [4] and [5], and fig. 5.19). In lines [3], [6] and [7], Dan communicates his contextual restriction of the variables x and y to refer to spatial variables with his statement, "Ok, so I am going to use v and w instead of x and y because that is kind of confusing." He went on to further confirm the researcher's inferences in line [9], where he seemed to insist on not using x and y to represent anything else since according to Dan, “Because I know I am in three space and have to use x, y, and z.”

> [9] D: Because I know I am in three space and have to use x, y, and z. And so I wonder, where does z go? I still don't want to use x, y, and z. I am going to use v and w as the two scalars, or it will confuse me. So now from this, it would be a good idea for me to put it into a matrix.
>
> [10] LB: Can I ask you why you put the 1, 1, 3 as a column in the matrix?
>
> [11] D: Because I know this is going to be according to x, y, and z (…) and so when we do this, I want this to be equal to x, this to be equal to z (…) uh, no that's not right. Because these are both the x's (…) so I want these two to go on this side; so when you multiply that (…) so this is not right to me.

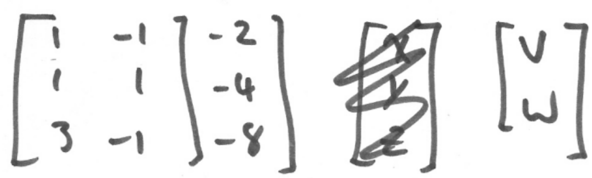


Figure 5.20 Dan’s initial matrix representation for problem 2.

As seen in Fig. 5.20, it appeared that Dan had transferred the correct matrix representation; however, in line [10] when he was questioned as to his understanding of the representation, he expressed doubt indicative to the researcher of scalar-variable conflict. As in previous episodes, the scalar-variable conflict concerned confusion between representational aspects of the vectors and scalars involved, and their relationship to the organization of the matrix. In line [9], Dan’s statement, “Because I know I am in three space and have to use x, y, and z. And so I wonder, where does z go?” demonstrated the likely conceptual source of his scalar-variable conflict in this problem, in that he seemed to confuse the unknowns of the system he was solving with the 3-tuple structure of the initial given vectors. In Fig. 20, before the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, was scribbled out, Dan noticed that the 3×2 coefficient matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 3 & -1 \end{bmatrix}$ could not be multiplied by the 3×1 column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. This seemed to conflict Dan because, as another manifestation of the geometric contextual conception, he wanted the $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to function both as the solution vector for the problem, as well as maintain consistency with the 3-tuple form of vectors in $\Re^3$.

In problem 2, unlike Iliana, Dan was not seen to repeat the strategy of employing algebraic mode computations on the linear combination equation. The researcher speculated that one reason Dan may not have tried an algebraic approach was due to notational unfamiliarity related to his use of v and w, as well as the 3-tuples, (x_A, y_A, z_A) and (x_B, y_B, z_B), in fig. 19. Recall from Episode 3, where through multiple processes of actor-oriented representational transfer, evidence was presented of Dan's theoretical thinking that for linear combinations, the vector needed to be written as a column in the matrix so that the scalars multiply their corresponding vector, and not the corresponding components of *different* vectors. In line [11], Dan's statement "Because these are both the x's (…) so I want these two to go on this side," indicated evidence of his extending theoretical thinking from

Episode 3 concerning how a matrix needs to be organized for the representation of a linear combination equation (see line [6], Episode 3).

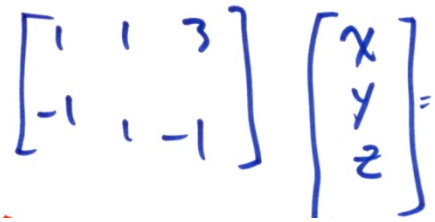


Figure 5.21 Dan's revised coefficient matrix for problem 2.

> [12] D: Ok (…) so, the way I have this written is not going to work, so let me change this. So now if we multiply these two together, these will be for x, these for y, and these for z.
>
> [13] LB: Could you multiply them?
>
> [14] D: That would be $x + x + 3y$ and then $-x + y - z$.
>
> [15] LB: What would you like them to equal?

Dan next wrote the matrix representation seen in Fig. 5.21. In the new representation, he had satisfied his apparent need to use an $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ vector from 3-space, which could also multiply the coefficient matrix. After performing the multiplication at the request of the researcher, in line [16] Dan commented "The vector *C*. So, I see there's a problem here. If I write it as a column, it does not match up, and if I write *C* as a row, it does not match up!"

> [16] D: The vector *C*. So, I see there's a problem here. If I write it as a column, it does not match up, and if I write *C* as a row, it does not match up! So, what I'm looking for is v and w. I'm getting all my variables all mixed

> up. So I am not looking for x, y, z but for v, w ;so that does work. Ok, that's where I went astray, because I noticed (…) something wasn't right here in the amount of terms.

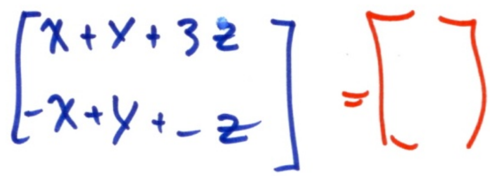


Figure 5.22 Dan's multiplication of matrix times arbitrary vector.

Demonstrating theoretical thinking by his treatment of the matrix representation as an object for semiotic reflection, as well as indicating evidence of scalar-variable conflict, in line [16] Dan stated, "So, what I'm looking for is v and w. I'm getting all my variables all mixed up. So I am not looking for x, y, z but for v, w; so that does work." What Dan was referring to with the statement, "so that does work," was noted in the video and interview transcript as a reference to his initial matrix formulation seen in Fig. 5.20, with the modification that he scribbled out the $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and indicated replacement with the column matrix, $\begin{bmatrix} v \\ w \end{bmatrix}$. At the end of line [16], Dan indicated resolution of the scalar-variable conflict, stating that the source of the conflict involved the "Ok, that's where I went astray, because I noticed (…) something wasn't right here in the amount of terms," in agreement with the researcher's characterization of this episode as demonstrative of geometric contextual conception inducing scalar-variable conflict. Upon further inquiry into this example of scalar-variable conflict, in the following exchange from lines [17] and [18], Dan communicated evidence of the abstract problem solving knowledge he had constructed during the course of developing the understanding of the correct matrix representation, where he confirmed inferences made by the researcher as to the nature of his scalar-variable conflict as a fundamental confusion between "what we are looking for," and "the dimensions of the problem," (see line [18]).

> [17] LB: Can you explain why you used x, y, and z originally?
>
> [18] D: It is not the dimensions of the problem that are important, but what we are looking for that is important. So now I will solve.

Recalling from Table 5.1, the following characteristic of actor-oriented transfer that, 'Researchers acknowledge that what experts consider a surface feature may be structurally substantive for a learner,' this episode was characteristic of actor-oriented transfer in how the 'surface feature' of the $\Re^3$ geometric setting of problem 2 was seen to become a 'structurally substantive' factor, described as *contextual conception*, obstructing Dan's initial understanding and transfer of a correct matrix representation. Also indicative of actor-oriented transfer were the instances of Dan's multiple processes of practical-theoretical thinking seen in lines [12], [14], and [16], where from his *perspective*, he was seen to *dynamically attune* his understanding of his initially correct matrix representation, *transferring* and *co-ordinating* elements of prior knowledge across from previous established schemas to unfamiliar ones (see Table 5.1). In addition, the evidence generated during this episode also demonstrated how a linear algebra student might initially produce a correct problem representation, yet not immediately possess conceptual understanding for how the representation related to a conceptually non-isomorphic setting.

Episode 6: Dan's demonstration of actor-oriented representational transfer in problem 3

Interview Problem 3

A linear transformation $T:\Re^3 \rightarrow \Re^3$ has matrix $A=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & -2\\ 1 & 2 & 0\end{bmatrix}$. Find avector $v \in \Re^3$ that satisfies $T(v)=[6,-2,4]^t$.

Overview. Evidence will demonstrate substantiation of the Semantic-Access Hypothesis based on Dan's initial insufficient surface-semantic knowledge defining the LTp schema. Similar to previous episodes, through a pattern of practical-theoretical thinking, episode 6 will also show how Dan was seen to engage in algebraic mode computations (matrix multiplication), which would lead to familiar Rp-type systems of equations, reflection upon his algebraic actions, and co-ordination of Rp and LTp knowledge constructs producing understanding of the matrix form in relation to the LTp setting, indicative of an overall process of actor-oriented representational transfer.

[1] D: What does the small t mean?

[2] LB: It means transpose, or make the vector a column.

[3] D: Oh. I have to remember how to do this. Ok, lets see (…) so, I'm not quite sure (…) did we learn that?
<7 second pause>

[4] LB: Yes. So, for $T(v)$, where A is the matrix for T, what is a natural thing to do to v to get $T(v)$ with a matrix A?

[5] D: So, use the vector and the matrix and multiply them? Oh, so we want to find a vector v so that when you multiply by this then you get this. So,

this would be where we set up an augmented matrix. I don't think I'm doing this right.

<8 second pause>

As seen in lines [1]-[5], constituting evidence substantiating the SAH, Dan required several pieces of information defining the Ltp schema in order to proceed with the production of a problem representation. After the researcher's description of the transformation *T* having a matrix *A*, in line [5] Dan stated, "So, use the vector and the matrix and multiply them?" indicating he seemed to recall that $T(v)$ was calculated by matrix multiplication of the vector v by the matrix A. After the line [5] statement, "Oh, so we want to find a vector v so that when you multiply by this then you get this," Dan was seen to physically point first to the column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and next to the vector $\begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}$, and then write what he referred to as the "augmented matrix" in Fig. 5.23. Note Dan's use of the familiar letters x, y, and z, as the column vector v, in Fig. 5.23. Even though Dan had produced a matrix representation for the problem, his statement "I don't think I'm doing this right," in line [5] indicated uncertainty as to his understanding of the matrix representation. Note Dan's original version of the augmented system appeared as: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 2 & -2 & y \\ 1 & 2 & 0 & z \end{array}\right] = \begin{bmatrix} 6 \\ 2 \\ -4 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}$$

Figure 5.23 Dan's matrix representation for problem 3.

> [6] LB: Ok, well, if we add these lines, then you have the matrix A multiplying times the $[x, y, z]^t$ vector, giving us the $[6,-2,4]^t$ vector.
>
> [7] D: Ok ... so it would be $x + y + z = 6$. Then the next row is going to be $0 + 2y - 2z = -2$, and then $x + 2y + 0 = 4$ (…) If I had seen this first, then I would set up the augmented matrix!
>
> [8] LB: Why is that?
>
> [9] D: When I see it like the equations, it is clearer and what I've always seen (…) so do you want me to solve it then?

Upon the interviewer's intervention, in line [6], suggesting Dan create lines such that the matrix *A* appears separate from the vector v, as seen in Fig. 5.23; he immediately proceeded to multiply the matrix times the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, correctly producing the equations seen in line [7]. After viewing the equations in the familiar Rp Cartesian form, Dan immediately conveyed "If I had seen this first, then I would set up the augmented matrix!" at the end of line [7]. In Dan's own words, he had just previously identified the matrix in Fig. 5.23 as the augmented matrix, yet; his comment at the end of line [7] seems to indicate that he really did not understand that that the augmented matrix represented a system of equations, until he saw the system in line [7]. In line [9], when Dan further explained: “When I see it like the equations, it is clearer and what I've always seen,” he gave further evidence suggesting his familiarity of the Rp Cartesian-like setting.

This episode provided evidence of the potential difficulty subjects may have in recalling information defining the LTp setting. Not immediately knowing notation such as the transpose, or how a linear transformation acts upon a vector with its given matrix, the researcher deemed it unlikely Dan could have produced an adequate matrix representation during a timed traditional transfer experiment, such as experiment 2. In this regard, the author submits that the evidence in Episode 6 supports the Semantic Access Hypothesis conjectured in Chapter Four. Similar to previous Episodes 3 and 5, the author characterized Dan’s thinking as practical-theoretical, as evidenced by his reflection upon the objects produced by his action of matrix multiplication, namely the equations: $\begin{matrix} x+y+z=6 \\ 2y-2z=2 \\ x+2y=-4 \end{matrix}$. This episode was also similar to Episodes 2 and 5, with Josh and Dan respectivly, in that a correct matrix representation was initially produced, but not understood until a series of *dynamic* and *personal* practical actions and theoretical reflections led to *constructions of relations of similarity*, characteristic of actor-oriented transfer (see Table 5.1).

Part 2: Actor-oriented Setting-Solution Transfer

Episode 7: Iliana's demonstration of actor-oriented setting-solution transfer in problem 1

> Problem 1
>
> For the following vectors v and w in a vector space *V*, $A = v - w$, $B = 2v + w$,
>
> and $C = 3v + 2w$; write the vector C as a linear combination of the vectors *A* and *B.*

Overview. Evidence tracked how Iliana's initial difficulty in interpreting the solution to her system of equations in the context of the LCp setting manifested psychologically as a preference for algebraic over formal forms of representation. Upon receiving a meta-cognitive hint, Iliana seemed to engage in reflective abstraction, constructing co-ordinations between her system solution and previous knowledge of the LCp schema, as well as demonstrating reversibility by her checking of the solutions relative to the LCp setting, reinforcing her personal constructions of similarity between the Rp and LCp schemas, indicative of actor-oriented transfer of the solution meaning to the problem 3 context.

After Iliana obtained a matrix representation for problem 1, she immediately began the "specialized, goal oriented, physical actions" indicative of *practical thinking*, seen in line [1], by her speedy application of matrix row reduction methods (Sierpinska, 2000, p.212). During the exchange which occurred in lines [2]-[5], Iliana seemed to be in a state of disequilibrium concerning her understanding of the meaning of the solutions she obtained, relative to the context of problem 1.

> [1] I: And the solution would be getting this to be zero and that to be zero. Here you can multiply this by negative one and add it to this to get this to be zero. Then this would have to go back as a one, and multiply this row by one third. So y is equal to five thirds. Now multiply this by negative two and add it to here. So the solution to this, you would say that x equals negative one third, and y equals five thirds.
> *<12 second pause>*

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 3 & 5 \end{array}\right] 1/3$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 5/3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \end{array}\right]$$

$$x = -1/3$$
$$y = 5/3$$

Figure 5.24 Iliana’s row-reduction in problem 1.

[2] LB: So what does this have to do with the problem?

[3] I: Um (…) truthfully, its like combining two different maths for me (…) sorry.

[4] LB: What different maths?

[5] I: Vector calculus and matrices. Because in this one, we would never have used matrices, but when I wrote it like this, I saw the matrices. Would you say that $C = \left(\frac{-1}{3}\right)v + \left(\frac{5}{3}\right)w$? Where does this go?

<9 second pause>

In lines [3] and [5], she stated: “its like combining two different maths for me,” i.e., “Vector calculus and matrices.” These statements were similar to her earlier statement in episode 4: “Oh, I'm glad you used x, y, and z (…) its not the u, v, w land!,” which to the author seemed to indicate Iliana’s familiarity with the *algebraic mode*, and her unfamiliarity with the *abstract mode*, in the sense of Hillel & Mueller’s (2006) three mathematical modes discussed in Chapter Two.

Iliana’s apparent disequilibrium associated with interpreting the results of the row reduction process seemed to be related to cognitive conflict between *abstract* and *algebraic* modes. Further evidence supporting this notion was seen with her statement in line [5], "Because in this one, we would never have used matrices, but when I wrote it like this, I saw the matrices." Videotaped evidence, as well as notes I took from her interview, recorded that her reference for "this one" referred to the following equation from Episode 1: $x\langle v,-w\rangle + y\langle 2v.w\rangle = \langle 3v,2w\rangle$, written prior to her performance of algebraic-mode procedures which were eventually seen to transform the vector expression into a *familiar* system of linear equations, as she revealed with the statement, "but when I wrote it like this, I saw the matrices" (line [5]).

> [6] LB: Can you see anywhere on the paper where there ever was an *x* and *y* originally?
>
> [7] I: Uh, yes, up here (…) we called it the linear combination of vectors A and B (…) which is what we want. So then these ones would be A and B, instead of v and w (…) sorry!

In the last two sentences of line [5], Iliana suggested that the solution fitted in as $C = \left(\frac{-1}{3}\right)v + \left(\frac{5}{3}\right)w$, where she was seen to confuse the vectors *A* and *B* with their defining vectors v and w. This error was interpreted by the researcher as a form of scalar-variable conflict since Iliana seemed confused concerning the symbolic meaning of a system matrix in relation to the equations it represented, as she indicated with her statement, "Where does this go?" (line [5]). The researcher interpreted this evidence as similar to results from DeVries & Arnon (2004), which found that many linear algebra students operate at an *action level* for the concept of the solution to a system of equations, meaning that they are mostly familiar with computing the solution step-by-step, as seen with Iliana, but have difficulty viewing the solution as an mathematical object which may be reflected upon and interpreted for different meanings in other contexts.

$$C = -\frac{1}{3}A + \frac{5}{3}B$$

Figure 5.25 Iliana checks solution.

After a 9 second pause, the author *meta-cognitively* intervened in line [6] to ask Iliana if there was any previous work she had done which might help her with interpretation of the solution to the system matrix. This intervention was considered meta-cognitive in nature because it directed Iliana to reflect upon "semiotic representations systems" in her previous work (Sierpinska, 2000, p. 212). Seen in Fig. 5.25 is Iliana's final formulation of her linear combination expression with the solutions substituted in. According to Piaget and Garcia (1983):

> New schemata thus constructed cannot remain in isolation: sooner or later the assimilatory process will lead to *reciprocal assimilations*, and the requirements of equilibration impose upon the schemata or subsystems, thus interrelated, more or less stable forms of coordination and transformations (p. 134).

Next Iliana immediately substituted in and *checked* the solutions relative to the defining linear combination equation, in which the researcher hypothesized demonstrated *reversibility* in the sense of Piaget & Garcia's (1983) *reciprocal assimilations*. Although the schemic relations between Iliana's Rp and LCp schemas may be in a formative stage, the researcher further hypothesized that Iliana's demonstration of reversibility constituted evidence of reciprocal assimilations back through co-ordinations formed between her matrix solution and linear combination equation constructs. Iliana's matching up of the solution obtained from row reduction upon her matrix representation with the linear combination equation that she had started with, constituted further evidence of reflective abstraction, as revealed by the *coordination* of her linear system solution with her linear combination equation.

A discernable similarity between the actor-oriented representational transfer of part 1, and actor-oriented setting-solution transfer of part 2 is related to the two aspects characteristic of reflective abstraction. From Dubinsky and Levin (1986):

> The first is a reflection of one or more structures onto a higher plane in which the structures function in greater generality by being applied to new aliments which can even be structures functioning on lower planes. The second is a reconstruction of these reflected structures into new structures that are distinct from the old ones, although important similarities may continue to be apparent (p. 61).

In the sense of the above differentiations, the interview evidence demonstrated that the constructive characteristics in both cases of actor-oriented representational and setting-solution transfers are of the first type described in the above quote by Dubinsky and Levin (1986). In both cases of actor-oriented transfer, co-ordinations were made which tended to expand the contextual generality of the Rp setting schema in terms of its relation to matrix representations and solutions arising from problem solving in the LCp schema.

Episode 8: Josh's demonstration of actor-oriented setting-solution transfer in problem 1

Overview. Evidence will demonstrate the actor-oriented transfer of the meaning of the solution to Josh's matrix system relative to problem 1's LCp setting. Based on the algebraic-mode actions from episode 2, Josh was seen to build upon the co-ordinative aspects of the relationship between the algebraic and formal vector expressions in problem 1, requiring no significant meta-cognitive interventions, on the part of the interviewer, for the discovery of co-ordinations between the solutions obtained from row reduction, and the LCp setting.

Similar to Iliana in the previous episode, Josh immediately began row reduction of the matrix he had produced by actor-oriented transfer in Episode 2 (see line [1]). After Josh's significant pause, it seemed that he may have mentally inserted his solution, obtained at the end of line [1], into his original linear combination vector expression, (see Fig. 5.8). Later, in line [3], my inference was confirmed as Josh reflected upon his previous work and discovered a minor copying error, saying "Oh, but. (…) you know what (…) here I wrote this on the wrong." After correcting the error and row reducing again to find the correct solutions, Josh stated, "So if we go back over here;" in which case the interview notes recorded him referring to the original linear combination expression, where he was next seen to verify that upon multiplication of the *A* and *B* vectors by their corresponding scalar solutions x and y, the vector *C* was the result.

> [1] J: Now we can start doing row reduction method (…) so we just subtract that and that ends up being zero. Then we divide both sides by 3 and that ends up being 5, with 1 there, we end up with 1/3 here. So we got 1, 0, 1, and here we have 3. We have to subtract 2/3 from that, which is 1/3, and subtract 2, and that's 0. Ok, so that would give us x is 7/3 and y is 1/3 (…)
> *<11 second pause>*
>
> [2] LB: Is there anything wrong?

> [3] J: Oh, but. (…) you know what (…) here I wrote this on the wrong (…) oops! 3 - 10/3 that's 9/3 -10/3 which is -1/3 (…) so we have -1/3 is *x* and 5/3 is *y* . So if we go back over here, we see we got 5/3 minus -1/3 which is minus a minus so it is 6/3 which is 2 which is what we want (…) and we got (…) 10/3 and -1/3 which is 9/3 which is 3.

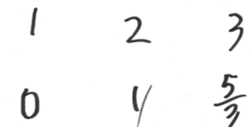


Figure 5.26 Josh row-reducing augmented matrix.

This seemingly minor error in row reduction emphasized the relevance and construct validity of ROW OPS as a significant procedural dimension in the LAM framework. In the researcher's estimation, this error may have significantly impeded Josh's successful transfer for a timed traditionally oriented experiment such as the Experiment 1 in this study. In lines [4] and [5], the following exchange indicated evidence explaining Josh's familiarity of the Rp setting:

> [4] LB: So it was difficult understanding what the results were giving you earlier?
>
> [5] J: Yeah, you know because algebra, I understand that. Well, because I've been doing that for years and years.

In line [4], my question to Josh concerned the interpretation of the solutions he had obtained from row reduction upon the system matrix. I interpreted his answer in line [5] as indicating evidence that for *him*, the interpretation of the solution in the new context of the LCp setting, was in the realm of algebra.

In reference to the Table 5.1 guidelines for actor-oriented transfer, "The personal construction of relations of similarity across activities," line [5] summarized the degree of personal similarity Josh's algebraic-mode actions, beginning in episode 2, seemed to create for him between familiar algebraic structures like Rp-type systems of equations and linear combination equations, both being representable my matrices. In conclusion of episode 8, the author found evidence that Josh created conditions of similarity and co-ordination between the LCp and Rp settings, thus rendering the non-isomorphic settings to appear isomorphic, facilitating the actor-oriented transfer of the *meaning* of the linear systems solution in the LCp setting.

Episode 9: Dan's demonstration of actor-oriented setting-solution transfer in problem 2

Overview. The evidence will show how Dan's practical actions of row reduction techniques successfully solved his system matrix, yet he encountered disequilibrium in the interpretation of the 0 0 0 row associated with a novel, over-determined linear system. Through a dynamic process of practical thought and meta-cognitively induced theoretical thinking, the evidence will track how Dan's reflective abstraction upon the structural features of the solution matrix led him to make a discovery of a key feature distinguishing the Rp and LCp settings in terms of the coordinated meaning of those settings via matrix representations.

With Dan's use of the expression, "So I guess I am going to solve it the way I always solve it, but I guess it is not square, so I can't really make a diagonal, but since we are only looking for two things I guess it is ok" in line [1], he exhibited prior knowledge of the row reduction algorithm for solving the augmented matrix system he obtained in part 1 of problem 2, episode 5. In addition, with the statement "we are only looking for two things," the researcher interpreted that Dan demonstrated reflective abstraction and theoretical thinking with his reference to the co-ordination of the matrix unknowns with the linear combination equation (see Fig. 5.27).

> [1] D: So I guess I am going to solve it the way I always solve it, but I guess it is not square, so I can't really make a diagonal, but since we are only looking for two things I guess it is ok. So since this is 1 and we need to turn this into 0, so negative row 1 plus row 2 (…) I can then get rid of that (...) and then -(-1) plus 1 will be 2 , and then 2 minus 4 will be -2. Then again lets do -3 times row 1 plus row 3, so -3 +3 is 0 (…) 3-1 is going to be 2 ... and then -3 times -2 will be 6... -8 will be -2. Now, I want to turn this into a 1. So, I can just take this one and just add it to that one.

$$\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

$-R_1+R \quad -3R_1+R_3 \quad -R_2+R_3$

$$\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$\frac{1}{2}R_2 \quad R_2+R_1$

Figure 5.27 Dan's row-reduction in problem 2.

In lines [2]-[7], through the course of his correct implementation of the Gaussian technique, Dan encountered disequilibrium in terms of assimilating the implications of the 0 0 0 row into the context of problem 2, (see Fig. 5.27). Having often treated the 0 0 0 row as a free-variable in past problem examples, as the interview evidence in lines [3], [5], and [7], indicated; the presence of the 0 0 0 row seemed to function as a form of contextual conception for Dan. With his line [7] statement, "I know (…) I have three answers and I'm looking for two things (...) I think (…) this is what I was kind of worried about," it seemed that Dan was over his previous dilemma concerning scalar-variable conflict, and clear about how many variables he was solving for, yet the obstacle of his prior experiences concerning the 0 0 0 row led him to persist in the contextual conception that the 0 0 0 row always involves a free-variable. Dan's apparent clarity in distinguishing the potential solutions in the matrix from the requirements of the original linear combination equation signals to the researcher an improvement from the previous Episode 3, and indicate evidence of the streamlining of his Episode 3 actor-oriented construction, similar to Iliana in Episode 4.

> [2] LB: Wouldn't using row 1 mess up these nice zeros you made? You have two repeated rows here (…) what do you think of that?
>
> [3] D: Ok, yes we can get rid of a row then by adding the opposite. -R2 + R3. So, this one will be left alone and these are going to all go away, so we have 0 0 0. So this is going to be our free variable. I mean (…) uh (…) since these are all zeros (…) I'm trying to think back on how it was done in class.
>
> [4] LB: I don't know if you ever did one quite like this in class.
>
> [5] D: Well, when we were doing problems like this with the solving of matrices and stuff like that I remember this is going to count for something later.
>
> [6] LB: Well go ahead and do more if you can.
>
> [7] D: So, then we will do R2 divided by 2 to make a 1 here. And then we will work backwards and turn this into a zero. So that will be just add R2 to R1, so that will go away and so that will be 1 0 0 , 0 1 0 , and that will be -3 and then -1 0. Ok, so (…) now we have our answer. So, now (…) I'm not quite sure what I have. I know (…) I have three answers and I'm looking for two things (...) I think (…) this is what I was kind of worried about. That I am not going to know what I have once I have it. So, you see what I'm saying?
>
> *<7 second pause>*

After significant pause on the part of Dan, I intervened in order to aid him in his theoretical thinking concerning the solution matrix relative to the context of problem 2. In line [8], my reference to "this" referred to the top two entries of the $\begin{matrix} -3 \\ -1 \\ 0 \end{matrix}$, third column in the final matrix of Fig. 5.27. In line [9], it appeared Dan correctly transferred the co-ordinated knowledge of the variables he originally used for his linear combination expression, with the solutions to the corresponding system matrix, (see episode 5). He also correctly represented the solution in the simple, non-redundant form seen in line [9], however; it seemed he still had not made the connection of the insignificance of the 0 0 0 row, until the meta-cognitive hint in line [10] prompting Dan to *reflect* upon the meaning and real significance of the 0 0 0 row relative to the demands of the problem.

[8] LB: What does this represent?

[9] D: It is going to represent v. So, if we put it in a column, it would be $\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$.

<6 second pause>

[10] LB: What about this last line, is it a false statement that 0v+0w=0?

[11] D: No (…) ok (…) uh yeah I think what you just said cleared it up. Because 0 v + 0 w is going to equal 0, so essentially this row is not needed. So (…) $-3-(-1)=-2$, $-3+(-1)=-4$, and $-3(3)=-9$ and $-9+1=-8$ So yeah (…) that's what I am looking for.

With Dan's response in line [11] that, "Because 0 v + 0 w is going to equal 0, so essentially this row is not needed," he seemed to extend the researcher's hint from being simply an observation, to a stronger statement of the *irrelevance* of the 0 0 0 row to the problem. The author probed Dan's understanding further in line [12], in which he seemed to respond with deeper reflective analysis of the solution matrix in terms of the problem context, as exemplified in line [13], where he indicated a key structural feature of the LCp setting, in terms of its matrix representation being viewed as columns instead of rows.

> [12] LB: It seems there is still some kind of confusion about this problem. Can you summarize in any way about that?
>
> [13] D: Ok (…) what the confusion is (…) is there are three terms in here and I know I'm looking for two things and I have three things. So it's confusing for me because where did the extra thing come from, when actually I'm only looking for these two things? I think that is the biggest confusion for me, is that I am looking for the wrong thing. I am looking at the rows here, assuming that I'm going to come out with three outcomes when I should be looking at the columns.

In reference to the previous quote from Dubinsky and Levin (1986) in episode 7:

> The first is a reflection of one or more structures onto a higher plane in which the structures function in greater generality by being applied to new aliments which can even be structures functioning on lower planes. The second is a reconstruction of these reflected structures into new structures that are distinct from the old ones, although important similarities may continue to be apparent (p. 61).

The researcher hypothesized that evidence in line [13] thus tracked the occurrence of reflective abstraction of the first type, in terms of the structural features of Dan's total LCp schema becoming more refined and reorganized, able to function in a greater generality by being able to accommodate *disequilibria*, such as the occurrence of the irrelevant 0 0 0 row in the over-determined system encountered in problem 2.

Through a dynamic process of practical algorithmic actions, the encountering of disequilibria, and meta-cognitively induced reflective-like theoretical thinking, Dan was seen to successfully assimilate and meaningfully accommodate the difficulties associated with the novelty of the over-determined system in the LCp setting. Based on the criteria distinguishing traditional transfer from actor-oriented transfer in Table 5.1, the researcher concluded episode 9 indicated clear evidence of Dan's actor-oriented transfer of the matrix solution meaning, to the context of problem 2 within the LCp setting. Furthermore, similar to the case of Iliana in Episode 4 , evidence was shown of Dan's transfer of problem solving knowledge arising from his previous Episode 3 actor-oriented transfer, having the effect of *streamlining* the actor-oriented transfer of this episode through the absence of scalar-variable conflict.

Episode 10: Iliana's demonstration of actor-oriented setting-solution transfer in problem 3

Overview. Evidence will demonstrate how practical thinking restricted to proto-typical examples led Iliana to form a scalar-variable conflicted interpretation of the solution to her problem 3 system matrix. After a meta-cognitive hint, the evidence also tracked how, through actor-oriented transfer, Iliana attuned the co-ordinations between her system-solution and LTp schemas enough to satisfy the immediate demands of the problem, even though she still retained the false contextual conception of an infinite solution.

In line [1], Iliana demonstrated her *action level* competence of the row reduction algorithm, as seen by her sentence, “Step by step, otherwise I mess up,” (see DeVries & Arnon, 2004). Seen at the bottom of Fig. 5.28, Iliana produced the row 0 0 0 11. With her statement, “And what we notice is that we do have an x and a y component, but z is a free variable, because if you were to say that a vector 0x +0y +0z =11 (…) it doesn't really work out,” Iliana conveyed a scalar-variable conflicted interpretation of the free variable and non-solution concepts. After significant pause, in line [3] I asked Iliana to write the equation associated with the third row of the partially reduced matrix, in which case; she immediately wrote an expression with a not-equals sign, underlining the $0 \cdot z$ term, (see bottom right corner of Fig. 5.28), also indicating that she was aware of the fact that $0 \neq 11$.

> [1] I: And then you go through it and find what x, y, and z equal. This one, since it already has 0 below pivot point, all we have to do is get rid of the 1 at the bottom. So we multiply this row by -1 and add it to bottom row, so this would be the same. -1 +1 is 0 ... -1+2 is 1 ... -1+0 is -1 ... and then -1 times -6 is 6, and then +4 is 10 (…) yes, I think. We've already gotten rid now of the x, now we have to get rid of these ones. One can easily do that by multiplying this row 2 by 1/2 so we can get it down to 1. So (…) this would be a 1, -1, and -1. Now that we have a second pivot point we can go back to the first one by multiplying the second row by -1 and adding it to the first row, then do the same thing to the last row. Step by step, otherwise I mess up. -1 + 1 is 0 (...) then -1 times -1 is 1 plus 1 is 2. Then -1 times -1 is 1 times -6 is -6 (...) no plus -6 is -5. And then at the bottom its -1 +1 is 0 , -1 times -1 is 1 plus -1 is 0 (…) is it?
>
> [2] I: Then -1 times -1 is 1 (…) plus 10 is 11. And what we notice is that we do have an x and a y component, but z is a free variable, because if you were to say that a vector 0x +0y +0z =11 (…) it doesn't really work out.
>
> *<9 second pause>*
>
> [3] LB: Could you please write the equation you just said?

[4] I: Ok (…) so z is a free variable (…) it could mean two things. It could either be that the problem has multiple solutions, I'm sorry, infinite solutions (…) due to the fact that z is a free variable, or no solutions.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -6 \\ 0 & 2 & -2 & -2 \\ 1 & 2 & 0 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -6 \\ 0 & 2 & -2 & -2 \\ 0 & 1 & -1 & 10 \end{array}\right] - R_2 \times \tfrac{1}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -6 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 11 \end{array}\right] \quad 0x + 0y + \underline{0z} \neq 11$$
$$0 \neq 11$$

Figure 5.28 Iliana’s row reduction for linear system, problem 3.

In line [3], the researcher injected a question of a highly meta-cognitive nature, due to its emphasis on the third, 0 0 0 11 row, “becoming an object of reflection and analysis in theoretical thinking” (Sierpinska, 2000, p. 212). It seemed to the researcher that Iliana’s prior knowledge of the meaning of the third row of the coefficient matrix, as associated with the z-variable solution, somehow over-rode the implications of an impossible equation arising from the row- reduction of the overall system. Because Iliana continued to insist that the no-solution case was related solely to the z-variable, as in line [4], the researcher considered this occurrence as exemplifying *contextual conception*.

[5] LB: If it is an infinite solution, then if we put any of those infinite solutions in for *x*, *y*, and *z* in that equation you wrote, will it be true?
<6 second pause>

> [6] I: No (…) never. So it is no solution. It does not satisfy the vector T(v) because you can find a vector in the *xy*- plane but you can't find a 3d vector since there's no solution in the *z*-axis. And so we don't have a *v* that satisfies T(v) is [-6,-2,4] transpose.

In light of the difficulties Iliana seemed to experience with her interpretation of the partially reduced system matrix for problem 3, after reasoning through the meta-cognitive hint of line [5], she seemed to settle upon the notion that ‘no solution’ was the overall proper interpretation of the row reduction process, as shown in her statement, “So it is no solution.,” (see line [6]). This evidence suggested to the researcher that failure of traditional transfer could occur due to problems of interpretation of the results of row reduction, in addition to the lack of factual semantic information defining the LTp setting, supporting the SAH. Iliana’s line [6] statement, “you can find a vector in the *xy*- plane but you can't find a 3-d vector since there's no solution in the *z*-axis,” in the view of me the ‘expert,’ demonstrated her confusion in assuming that the non-solution aspect of the system had occurred solely due to, “no solution in the *z*-axis,” when in fact, any *xy*-plane vector would still produce a non-solution when inserted into the equation $0 \cdot x + 0 \cdot y + 0 \cdot z = 0$, as was suggested in the line [5] interviewer hint.

It was the researcher’s interpretation that, in addition to contextual conception, this incident illustrated the constructive nature of Iliana’s actor-oriented transfer of the correct solution interpretation to problem 3. In the sense of Piaget’s genetic epistemology, as introduced in Chapter Two, Iliana experienced a disequilibrium in her LTp schema when she encountered the 0 0 0 11 row during her row reduction process, however; instead of completely resolving the disequilibrium after receiving meta-cognitive intervention, Iliana was seen, in line [6], to attune her LTp schema as much as possible in order to relatively resolve the immediate disequilibrium of problem 3.

Iliana's personal resolution of inconsistencies she encountered when interpreting her practically natured row reduction procedures illustrated how affordances can be made for the resolution of immediate disequilibria, even though the understanding of the subject's schema may not have attained a level of conceptual competence satisfactory to the expert. Iliana demonstrated a measure of theoretical thinking when she concluded that there was no solution, yet she still retained a false understanding that the no-solution was in the *z*-axis only, based on a form of scalar-variable conflict confusing her as to the meaning of the solution in relation to the *z*-position of the third row of the coefficient matrix. In the author's analysis, this final episode demonstrated a case of actor-oriented transfer which best revealed the constructive and regulating nature, consistent with Piaget's genetic epistemology, because of the *relative* equilibria Iliana was seen to attain as a result of her personal restructuring and co-ordination of schemas related to meta-cognitively induced reflective abstraction.

Part 3: Summary of the Results

Based on the interview component of this study, the author drew the following six results:

1. Lack of understanding of the representational meaning of a matrix in relation to the problem setting, equation scalars, and equation variables (scalar-variable conflict), created obstacles in the transfer of correct matrix representations and solution interpretations (Episodes 1, 2, 4, and 10).

2. Lack of sufficient semantic-access to setting-specific information defining both the LTp and LCp settings, posed significant obstacles to successful representational and solution-setting transfer across conceptually non-isomorphic problem settings (Episodes 1, 3, 4, 6, and 10).

3. Contextual conception, as an overly-practical reliance on proto-typical geometry and examples, was seen to function as an obstacle to representational and solution-setting transfer across conceptually non-isomorphic problem settings (Episodes 1, 5, 10).

4. The beneficial effects of meta-cognitive intervention appear to consist in the formation of co-ordinations between non-isomorphic settings, matrix representations, and corresponding matrix solution interpretations (Episodes 1-10).

5. Algebraic-mode actions characteristic of practical thinking, in combination with meta-cognitively induced reflection characteristic of theoretical thinking, characterized a general pattern of *actor-oriented transfer* related to the formation of co-ordinations between non-isomorphic settings, their representations, and corresponding solution interpretations. Furthermore: (a) Forward co-ordinations facilitating *representational transfer* were constructed from algebraic-mode transformations from definitions, to familiar Rp-like systems of equations. (b) Backward co-ordinations facilitating setting-solution transfer involved the reversibility of reflectively interpreting the solution to a system of equations back through the representational co-ordinations connecting the matrix representation to the problem setting (see Fig.5.29) (Episodes 1-10).

6. Upon successive exposure to problems from conceptually non-isomorphic settings sharing GGS procedurality and matrix representations, the evidence indicated a reduction of the multiple constructive processes characteristic of previous actor-oriented transfer, leading to the conjecture that actor-oriented transfer may become progressively streamlined, encapsulating towards *linear algebra understanding* (Episodes 4 and 9).

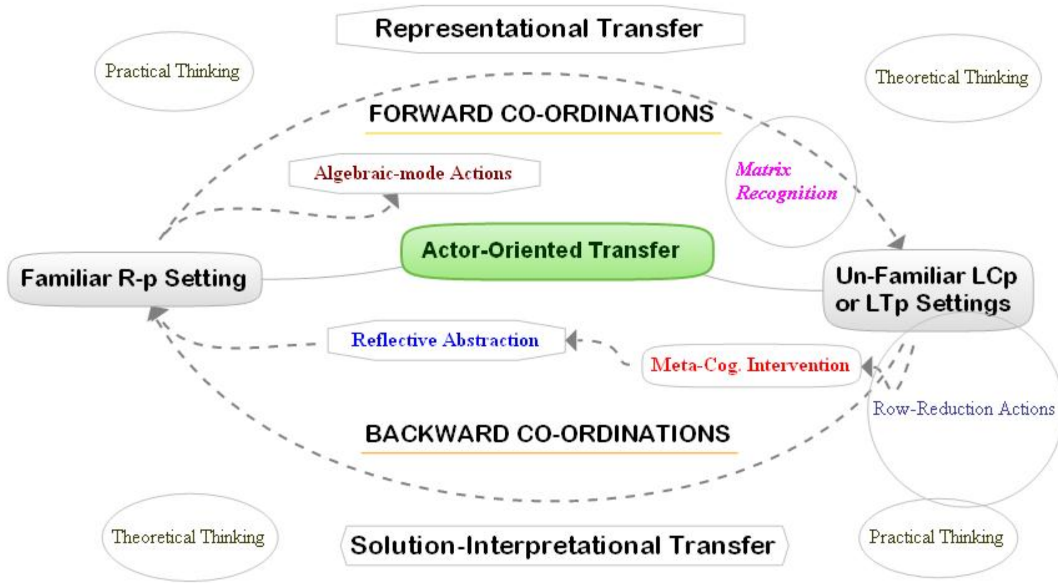


Figure 5.29 Actor-oriented transfer; Result 5.

Recall that the general purpose of the interview portion of this dissertation was to address the following research question:

Research Question 2

> In what ways, from the theoretical perspective of actor-oriented transfer, do novice linear algebra students commonly have difficulty with conceptually non-isomorphic problem settings, even when novel problem settings share similar problem representations and solution procedures as familiar problem settings?

Analysis of the interview data produced converging evidence in the form of Findings 1-6 as to the nature of difficulties expressed in Research Question 2. Based on these findings produced by the researchers' synthesis of the evidence, Findings 1, 2, and 3 suggest the following ways in which linear algebra students may encounter difficulty solving the problems at focus in this study:

1. Subjects may experience *scalar-variable conflict* by not understanding which aspects of a matrix (*columns or rows*) correspond to which scalars or variables of equations from unfamiliar settings, the matrix is intended to represent. Due to this conflict, errors may occur concerning the production of correct matrix representations and solution interpretations, thus substantiating the Representational Correctness Hypothesis (Result 1).

2. Lack of knowledge concerning key definitions related to unfamiliar settings (LCp and LTp) might prevent students from producing correct matrix representations and solution interpretations without the aid of hints (Result 2).

3. Overly-practical thinking might induce *contextual conception*, resulting in difficulties for students due to their inability to relinquish characteristics of familiar settings, such as the geometry of $\Re^2$ and $\Re^3$, and/or exhibit theoretical thinking needed to reflect on structural properties of matrices and matrix solutions in conceptually unfamiliar settings (Result 3).

In addition to addressing Research Question 2, the results also addressed the following two hypotheses generated from the experimental results of Chapter 4:

Representational Correctness Hypothesis (RCH)

A key component to successful transfer from the Rp to Cp setting may involve the ability to obtain the correct initial matrix representation.

Semantic-Access Hypothesis (SAH)

A key component to successful transfer from the Rp setting to the LTp setting may involve access to relevant semantic content information defining the LTp schema.

The evidence synthesized to produce Result 1 substantiates the Representational Correctness Hypothesis based on the discovery of ample cases of scalar-variable conflict, as well as interview examples of matrices resembling the incorrect matrices in Chapter Four which motivated the RCH (see Figures 4.3 and

4.4). The evidence which produced Result 2 was found to support and extend the Semantic-Access hypothesis to also include the LCp setting, where ample cases of lack of recall of the linear combination definition motivated this extension. In relation to Result 2, the evidence also showed that many students did not have the requisite understanding for how a linear transformation acts upon a vector, thus impeding their use of algebraic-mode actions which could construct relations of similarity of the unfamiliar LTp setting with the familiar Rp setting.

Based on Results 1, 4, and 5, the following *Intentional Transfer Hypothesis* was conjectured:

Intentional Transfer Hypothesis

> In order to coordinate the solution resulting from a process of row reduction on a matrix representative of a given problem embedded in a particular linear algebra setting, it is necessary to understand the process and/or definition(s) that motivated the initial matrix representation of the problem, in relation to its particular setting.

In order to test this hypothesis, three iterations of a treatment/non-treatment experimental design were implemented with linear algebra students, using a meta-cognitive instructional treatment influenced from interview Results 4 and 5 (see Chapter Three, Fig. 3.11). The upcoming Chapter Six will present the results of Experiment 3 concerning the possible effect(s) of meta-cognitive intervention on problem solving from the assumed familiar Rp setting to the conceptually non-isomorphic LCp setting. Interview Result 6, being of a theoretical nature, will be discussed in the *Theoretical Implications* section of Chapter Seven.

Chapter 6

EXPERIMENT 3 RESULTS

Recall from Chapter Three, it was the purpose of Experiment 3 to address the following research question:

Research Question 3

> What evidence can be found that indicates meta-cognitive intervention(s) may facilitate traditional and/or actor-oriented transfer across conceptually non-isomorphic problem settings involving novel target problems which share similar problem representations and solution procedures as more familiar problems?

Based on interview results 1, 4, and 5 which combined to suggest that subjects often lack information pertaining to the meaning of a matrix in a novel context (scalar-variable conflict), the following *Intentional Transfer Hypothesis* was conjectured, and a meta-cognitive intervention was constructed for the purpose of conveying to treatment participants the two-fold interpretation of matrix multiplication as a system of row equations belonging to the row-picture (Rp) setting, or a linear combination of the columns of the matrix, belonging to the linear combination-picture (LCp) setting (see Fig. 3.11):

Intentional Transfer Hypothesis

> In order to coordinate the solution resulting from a process of row reduction on a matrix representative of a given problem embedded in a particular linear algebra setting, it is necessary to understand the process and/or definition(s) that motivated the initial matrix representation of the problem, in relation to its particular setting.

Employing a treatment/non-treatment design, based on convergent evidence from the previous experiments and interviews, it was assumed by the researcher that the

Rp setting constituted a familiar setting for most undergraduate linear algebra students, hence, although Experiment 3 did not have a source problem, the author interpreted target success as an indication of successful traditional transfer from Rp to LCp settings. The purpose of this *sourceless* variation of a traditional transfer design was to eliminate the source problem as a *confounding variable* (Trochim, 2007) for isolating the effectiveness of the meta-cognitive intervention.

Experiment 3A Results

The percentage of participants successfully solving the Non-treatment target problem was (18%), as compared with (82%) target success for the Treatment group. Non-parametric analysis using the Fisher's exact test for 2×2 contingency tables tested for the likelihood that the data occurred by chance. The results revealed that the Non-treatment and Treatment target success percentages were significantly different, $p = 0.0089 < 0.05$, thus implying the Treatment group performed reliably higher than the Non-treatment group (see Table 6.1, Fig. 6.1).

Non-treat.	Treat.	
2	9	11
9	2	11
11	11	

P-value generated by Fisher's Test Statistic:
0.0089 estimated by Monte Carlo Simulation

Table 6.1 Exp. 3A results of Fisher's exact test.

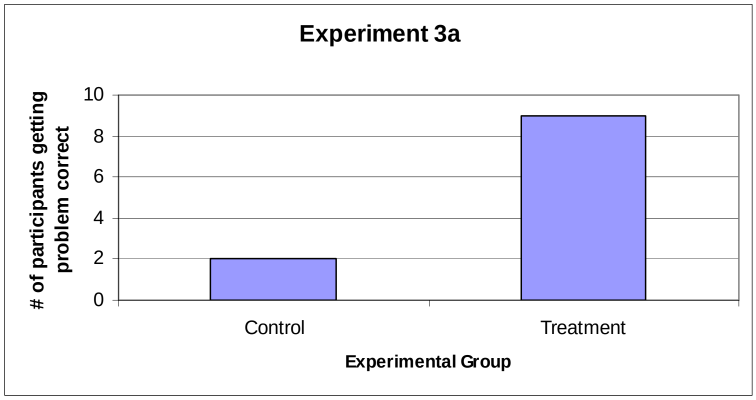


Figure 6.1 Experiment 3A results.

Experiment 3B Results

The percentage of participants successfully solving the Non-treatment target problem was (0%), as compared with (55%) target success for the Treatment group. Fisher's exact test revealed the means were significantly different, $p = 0.0124 < 0.05$, implying the Treatment group performed reliably higher than the Non-treatment group (see Table 6.2).

	Non-treat.	Treat.	
	0	6	11
	11	5	11
	11	11	

P-value generated by Fisher's Test Statistic:
0.0124 estimated by Monte Carlo Simulation

Table 6.2 Exp. 3B results of Fisher's exact test.

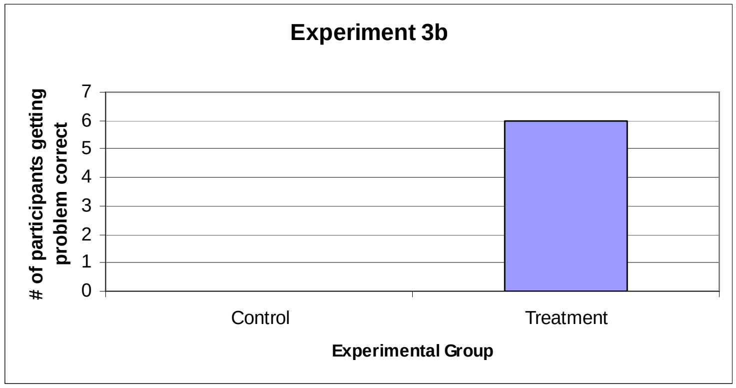


Figure 6.2 Experiment 3B results.

Experiment 3C Results

The percentage of participants successfully solving the non-treatment target problem was (18%), as compared with (73%) target success for the treatment group. Fisher's exact test revealed the means were significantly different, $p = 0.0300 < 0.05$, implying the Treatment group performed reliably higher than the Non-treatment group (see Table 6.3).

Non-Treat.	Treat.	
2	8	11
9	3	11
11	11	

P-value generated by Fisher's Test Statistic:
0.0300 estimated by Monte Carlo Simulation

Table 6.3 Exp. 3C results of Fisher's exact test.

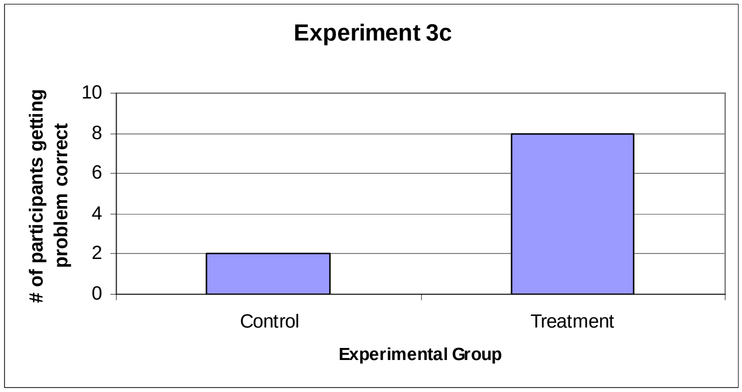


Figure 6.3 Experiment 3C results.

Reliability Statistics

Cronbach's Alpha	N of Items
.945	3

Table 6.4 Experiment 3 reliability.

Summary of Results

The results of Experiment 3 revealed that when subjects received the meta-cognitive intervention treatment (see Fig. 3.8), they performed significantly better on the target linear combination problem, as compared with subjects who did not receive the treatment. Upon inspection, the graphs shown in Figures 6.1-6.3 appeared qualitatively similar, as also reflected in the satisfactory result of the Cronbach's alpha reliability estimate $0.945 > 0.7$, which was above the social-science research accepted 0.7 tolerance (see Table 6.4). By looking at the general trend of the graphs, it was evident that post-treatment subjects performed better on the linear combination target problem, than did the non-treatment subjects.

Based on the results of the statistical analysis of the data, evidence was found to reject the null hypothesis that:

H_0 (*Experiment 3)*:

> There is no statistically significant difference between the control and experimental group performances, or the control group performed better than the experimental group on a novel linear combination problem.

Statistical analysis, however, indicated justification to accept the null hypothesis that:

> H_a (*Experiment 3)*: The experimental group performed significantly better than the control group on a novel linear combination problem.

The results of Experiment 3 provide convergent evidence, in the form of three iterations of experiments of treatment/non-treatment design, answering the above research question in the affirmative:

> Experiment 3 demonstrated statistically significant evidence that a meta-cognitive intervention could facilitate transfer across non-isomorphic problem settings, such as in a novel LCp setting target problem which shared similar problem representation and solution procedures as problems from the assumed familiar Rp setting.

Again, the meta-cognitive intervention used in Experiment 3 was designed based on interview Findings 1, 4, and 5, from the standpoint of explaining to the subjects how matrix multiplication can be interpreted as row equations or as a linear combination of the columns. In addition to verification of the *Intentional Transfer Hypothesis*, the author submits that the positive results from Experiment 3 add ‘construct validity’ to Findings 1, 4 and 5 (seen below), since they had originally motivated the formation of the *ITH*, as well as the design of the meta-cognitive intervention used in Experiment 3 (Trochim, 2007) .

Interview Findings 1, 4, and 5

1. Lack of understanding of the representational meaning of a matrix in relation to the problem setting, equation scalars, and equation variables (scalar-variable conflict), created obstacles in the transfer of correct matrix representations and solution interpretations.

4. The beneficial effects of meta-cognitive intervention appear to consist in the formation of co-ordinations between non-isomorphic settings, matrix representations, and corresponding matrix solution interpretations.

5. Algebraic-mode actions characteristic of practical thinking, in combination with meta-cognitively induced reflection characteristic of theoretical thinking, characterized a general pattern of *actor-oriented transfer* related to the formation of co-ordinations between non-isomorphic settings, their representations, and corresponding solution interpretations. Furthermore: (a) Forward co-ordinations facilitating *representational transfer* were constructed from algebraic-mode transformations from definitions, to familiar Rp-like systems of equations. (b) Backward co-ordinations facilitating setting-solution transfer involved the reversibility of reflectively interpreting the solution to a system of equations back through the representational co-ordinations connecting the matrix representation to the problem setting.

In conclusion, the *Intentional Transfer Hypothesis*, based on the information of the constructive characteristics of the actor-oriented transfer seen from the interviews, was supported. In a traditionally designed treatment/non-treatment setting, results from Experiment 3 indicated statistically significant evidence of the effectiveness of the meta-cognitive intervention, as well as arguable evidence contributing to the validity of the interview findings which helped motivate Experiment 3. In Chapter 7, a summary, synthesis, and significance of the results of this dissertation are presented. In addition, Chapter Seven discusses the implications of this work concerning the theoretical perspectives of traditional and actor-oriented transfer, the pedagogical significance, the limitations of the study, and the direction of future research.

Chapter 7

DISCUSSION

Introduction

This chapter offers a concise discussion summarizing the purpose, basic methodology, and conclusions of this study. Included in the summary, results from different components of the study are synthesized in relation to the three research questions. Finally, a discussion of this study, its limitations, pedagogical and theoretical significance, and the recommendations for future research conclude the dissertation.

Summary and Synthesis of Results

The purpose of this study was to perform a systematic mixed-methods inquiry into the nature of difficulties the author observed his linear algebra students encountering when confronted with linear algebra problems which shared similar solution procedures and matrix representations, yet derived from what has been termed, *conceptually non-isomorphic settings*. In order to find secondary evidence substantiating the anecdotal observations, the researcher conducted Pilot Study 1as part of a final exam question, and gathered evidence suggesting that undergraduate linear algebra students have difficulty in transferring knowledge from the familiar row-picture setting (Rp) to the unfamiliar linear combination-picture (LCp) setting (see Fig. 1.1, p.5), even though during the course of problem solving, identical matrix representations and calculations were applicable to both settings. As a result of Pilot study 1, the author obtained evidence that difficulties in negotiating non-isomorphic setting problems involving matrices and row-reduction likely were related to the encountering of unfamiliar settings, as well as to the interpretation of the solutions to corresponding systems of equations, such as the Linear Combination picture (LCp) and Linear Transformation picture (LTp) settings.

Recall from Chapter Two, from a traditional transfer perspective, several studies suggested that transfer between non-isomorphic settings could occur in the presence of 'hints' concerning 'retrieval and mapping information' between settings (Reed, Ackinclose, & Voss, 1990; Spellman & Holyoak, 1996; Holyoak & Koh 1987; Gick, 1985; Reed, 1987). In addition, Novick (1990) found indicating problem representations may be transferred between non-isomorphic problem settings. Due to these findings, the researcher designed two randomized, 2×2 factorial (Spector, 1981) Experiments 1 and 2, for the purpose of checking the phenomena he observed from his teaching experience against the combination of findings from previous transfer research in order to verify if transfer between non-isomorphic settings might occur when the settings shared similar procedures and problem representations.

Inspired by the multi-dimensional procedural problem solving model from work by Chen & Mo (2004), the researcher designed a similar model for the implementation of factorial experiments. With the aid of a further Pilot Study 2, needed for the fine-tuning of the researcher's Linear Algebra Model (LAM) framework, results indicated that row operations and basis representation of vectors were both significant procedural dimensions for solving linear algebra problems from different settings which shared the following Generalized Goal Structure solution procedures:

1. Representation of the problem with an appropriate matrix.

2. Row reduction algorithm employed as solution procedure to solve matrix.

3. Interpretation of matrix system solution in the context of a novel problem setting.

Also from Pilot Study 2, it was concluded that the BASIS and ROW OPS dimensions of the LAM framework should be held fixed in order to reduce the effect of confounding variables for Experiments 1 and 2, which were designed to factorially focus on the SETTING and SOLUTION TYPE dimensions of the LAM (see Fig. 3.4, Chapter Three).

The results of transfer Experiments 1 and 2 indicated that when source and target problems required similar solution interpretations and matrix representations, yet transition from the familiar Rp setting to the less familiar non-isomorphic LCp or LTp problem settings, many participants could not seem to apply the knowledge from the Rp source problem, even though the traditional transfer design permitted the subjects to see the source problem solution before attempting the target problem. The results of Experiments 1 and 2 are summarized as follows:

1. Dissimilar setting conditions from the familiar Rp setting to the less familiar non-isomorphic LCp and LTp settings, between source and target problems, created obstacles in applying source solutions.

2. Transitions between different problems involving the *unique* and *non-solution* solution types presented no significant difficulties in transfer.

3. A key component to successful transfer from the Rp to Cp setting may involve the ability to obtain the correct initial matrix representation. *(Representational Correctness Hypothesis)*

4. A key component to successful transfer from the Rp setting to the LTp setting may involve access to relevant semantic, setting-specific content defining the LTp schema. *(Semantic Access Hypothesis)*

Research Question 1

Is there evidence, in the traditional transfer paradigm, that transfer is facilitated between linear algebra problems from non-isomorphic settings which share similar (matrix) representations AND solution procedures?

In view of these findings, Research Question 1 was answered in the *negative*, i.e., the findings did not support the conjecture that a combination of similarity in problem representation and solution procedures would enhance transfer between source and target problems from conceptually non-isomorphic settings. Questions surrounding the results of Experiments 1 and 2 subsequently led the researcher to formulate the following:

Research Question 2

In what ways, from the theoretical perspective of actor-oriented transfer, do novice linear algebra students commonly have difficulty with conceptually non-isomorphic problem settings, even when novel problem settings share similar problem representations and solution procedures as familiar problem settings?

To address Research Question 2, the author conducted *semi-structured* interviews (Bernard, 1988) using problems similar to those in the previous experiments and pilot studies, for the purpose of gaining deeper insight into the difficulties experimentally shown to exist at the traditional transfer experimental design level. From the theoretical perspective of *actor-oriented transfer* (Lobato, 2003, p.18), the author analyzed three interview case studies of novice linear algebra students who were seen to experience similar difficulties as the author's linear algebra students whom motivated this study. Upon initial coding analysis of the case study data, two stages of problem solving became evident in the form of (1) actor-oriented *representational transfer* (Novick, 1990), and (2) actor-oriented *solution-interpretational* transfer. From the perspective of actor-oriented transfer, the following six results were found:

1. Lack of understanding of the representational meaning of a matrix in relation to the problem setting, equation scalars, and equation variables (scalar-variable conflict), created obstacles in the transfer of correct matrix representations and solution interpretations.

2. Lack of sufficient semantic-access to setting-specific information defining both the LTp and LCp settings, posed significant obstacles to successful representational and solution-setting transfer across conceptually non-isomorphic problem settings.

3. Contextual conception, as an overly-practical reliance on proto-typical geometry and examples, was seen to function as an obstacle to representational and solution-setting transfer across conceptually non-isomorphic problem settings.

4. The beneficial effects of meta-cognitive intervention appear to consist in the formation of co-ordinations between non-isomorphic settings, matrix representations, and corresponding matrix solution interpretations.

5. Algebraic mode actions characteristic of practical thought, in combination with meta-cognitively induced reflection characteristic of theoretical thinking, characterized a general pattern of *actor-oriented transfer* related to the formation of co-ordinations between non-isomorphic settings, their representations, and corresponding solution interpretations. Furthermore: (a) Forward co-ordinations facilitating *representational transfer* were constructed from algebraic-mode

transformations from definitions, to familiar Rp-like systems of equations. (b) Backward co-ordinations facilitating setting-solution transfer involved the reversibility of reflectively interpreting the solution to a system of equations back through the representational co-ordinations connecting the matrix representation to the problem setting (see Fig.5.29).

6. Upon successive exposure to problems from conceptually non-isomorphic settings sharing GGS procedurality and matrix representations, the evidence indicated a reduction of the multiple constructive processes characteristic of previous actor-oriented transfer, leading to the conjecture that actor-oriented transfer may become progressively streamlined, encapsulating towards *linear algebra understanding*.

Contrary partially to the results of Experiments 1 and 2 which found no significant difficulty in solution interpretation, results from the interviews produced convergent evidence of actor-oriented representational <u>and</u> setting-solution transfer, in addition to evidence suggesting the causal nature of setting-change and solution-interpretation difficulties. For example, Result 1 identified a phenomenon called *scalar-variable conflict* as a likely factor contributing to the inability for many subjects to produce a correct problem representation, or matrix solution interpretation. Result 1 was based on evidence of subjects possessing a conflicted notion of the *meaning* of a matrix representation in relation to the variable and scalar associations to equations arising from unfamiliar settings. This result agrees with Harel's (1989b) description of students' understanding of the symbolic manipulations they do in linear algebra:

> Students successfully manipulate matrices and functions, but they fail to understand the meaning behind these manipulations … students can bring a matrix to reduced row-echelon form, but they do not understand the relationship between the two matrices. The students master mechanics of

> the operations, but they seem to have no understanding of what they have done (p.141).

Written evidence of initial incorrect matrix representations by the interview subjects, seen to coincide with scalar-variable conflict, were markedly similar to matrices seen in examples of unsuccessful transfer in Experiment 1, hence supporting the conjecture of the Representational Correctness Hypothesis (see Figures 4.3 and 4.4). In addition, interview Result 2 emerged from several instances of subjects lacking sufficient setting-specific factual information necessary to proceed in problem solving, supporting the Semantic-Access Hypothesis, conjectured in Chapter Four, for <u>both</u> LCp and LTp settings.

As a consequence of Results 1-5 from the interviews, in addition to the literary educational research recommending meta-cognitive facilitation (see Chapter Two), a meta-cognitive intervention was designed from the standpoint of explaining the *meaning* of a matrix representation in connection with the linear combination (LCp) setting. It should be noted that the meta-cognitive treatment did not constitute a hint since it did not tell the subjects how to do the problem, but was more a concise reflection upon matrices as representational objects able to be understood in different ways. In Experiment 3, the final method used in this study, the meta-cognitive treatment was implemented as part of three iterations of a treatment/non-treatment designed linear combination problem similar to (LCp) problems from previous experiments and the interviews. The results of Experiment 3 indicated that the meta-cognitive treatment was effective and reliable in promoting target problem success and traditional transfer from the assumed familiar knowledge in the Rp setting. This concludes the summary and synthesis section for Chapter Seven.

Significance of Results

Traditional Transfer Perspective

The quantitative methods of this study produced four significant results in the area of traditional transfer experimentation. (i) Experiments 1 and 2 duplicated previous research by finding spontaneous transfer between *conceptually non-isomorphic* linear algebra problem settings to be rare in the absence of hints concerning the relationships between those settings (Holyoak & Koh, 1987; Novick, 1988; Ross, 1984; Dias & Artigue, 1995, p.41). Furthermore, (ii) Experiments 1 and 2 extended research in the area of traditional transfer experimentation by finding that the combination of representational similarity and solution-procedural similarity (GGS), between conceptually non-isomorphic linear algebra settings, did not enhance transfer (Novick, 1990; Reed, Ackinclose, & Voss, 1990; Spellman & Holyoak, 1996; Holyoak & Koh 1987; Gick, 1985; Reed, 1987). Also, (iii) Experiment 3 verified previous findings by showing transfer to be rare between non-isomorphic settings in the absence of hints of relationships between settings (Holyoak & Koh, 1987; Dias & Artigue, 1995). And finally, (iv) Experiment 3 extended previous work in traditional transfer theory by demonstrating transfer was facilitated between non-isomorphic settings in the presence of information concerning relationships of such non-isomorphic settings to common representations. This fourth significance is of particular subtlety in the sense that Experiment 3 shows that transfer of familiar Rp knowledge to the conceptually non-isomorphic LCp setting is facilitated not by information concerning the relationships explicitly between the settings themselves, rather, transfer is facilitated with information giving meaning to a common representation in the context of both of the non-isomorphic settings (see Fig. 7.1).

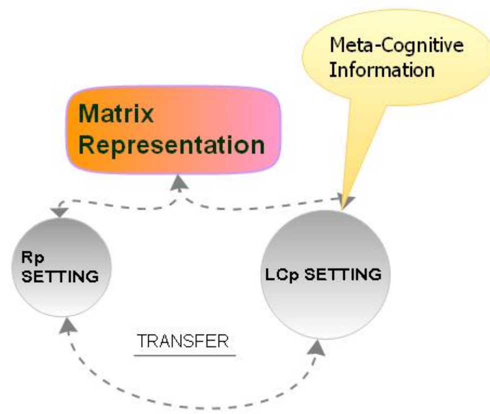


Figure 7.1 Experiment 3 transfer diagram.

Actor-Oriented Transfer Perspective

In agreement with Harel (1999), this study found interview evidence and examples of geometric contextual conception (Result 3; Interview episodes 1, 5, and 10), in addition to evidence supporting Harel's (1999) suggestion that in order to avoid contextual conception, students require meta-mathematical knowledge inducing understanding (*theoretical thinking*) concerning structural relationships between abstract and contextual representations (p.613). In regard to the efficacy of meta-cognitive interventions, interview Results 4 and 5 and Experiment 3, support Harel's (1999) assertion that meta-mathematical thinking, in the form of meta-cognitive hints initiating reflection, produced understanding of structural relationships between representations, solutions to matrix systems, and non-isomorphic settings.

Other ways in which contextual conception seemed to cause difficulty with the interview subjects' transferring of problem solving knowledge from familiar settings also involved cases of *practical thinking*, where subjects seemed unable to correctly reason and reflect on the meaning of representations or solutions due to their pre-occupation with proto-typical examples they had accumulated from prior mathematical experience. The predominance of practical thinking seen in the interviews, as characterized by the "goal-oriented, physical actions(s)" of the algebraic-mode transformations the subjects frequently demonstrated, recasts similar findings by Hillel & Mueller (2006), De Vries & Arnon (2004), and Sierpinska (2000, p. 211).

As an example, in interview Episode 5, Dan exhibits contextual conception as a form of *practical thinking* similar to research findings from the Harel (1989b; 1990) studies, which found that when a familiar geometric system was used in the teaching of linear algebra, oftentimes; the students' reasoning was restricted to the proto-typical geometry (Sierpinska, 2000). In the case of Dan, his familiar knowledge of 3-space appeared to induce *scalar-variable conflict* (Result 1), which seemed to block his ability to reason correctly concerning the meaning of the matrix representation he had originally produced in interview problem 2, in relation to the scalars and variables of the linear combination equation he was attempting to represent. In the interview analysis, every case of over-dependence on practical thinking, focused too much on computation or familiar knowledge, led to contextual conception as some form of scalar-variable conflict, comprising an obstacle to representational and/or solution interpretation transfer, and thus interpreted by the researcher as the occurrence of a form of *negative transfer*. Furthermore, interview Results 3 and 5 indicated that while practical thought was necessary in order to engage subjects to begin forming personal constructions based on their immediate prior knowledge, at some point meta-cognitive intervention was needed to suspend practical thinking and engage theoretical thinking, in order for subjects to perform reflective abstraction upon the meaning of their previous actions in relation to novel contexts (Results 4 and 5) (see Fig. 5.29).

Of central significance for this study, were its findings concerning <u>how</u> meta-cognitive interventions may affect deficits in understanding of relationships between non-isomorphic settings. From a lecture of Piaget (1968):

> I think that human knowledge is essentially active. To know is to assimilate reality into systems of transformations. To know is to transform reality in order to understand how a certain state is brought about. To my way of thinking, knowing an object means acting upon it. It means constructing systems of transformations that can be carried out on or with this object. Knowing reality means constructing systems of transformations that correspond, more or less adequately, to reality. They are more or less isomorphic to transformations of reality. The transformational structures of which knowledge consists are not copies of the transformations in reality; they are simply possible isomorphic models among which experience can enable us to choose. Knowledge, then, is a system of transformations that become progressively adequate.

In agreement with Piaget (1968), the researcher found that although practical thinking was often seen to function as a form of negative transfer, obstructing the understanding of subjects concerning their transfer of correct matrix representations or solution interpretations, it was also a necessary part of the dynamic process of actor-oriented transfer in motivating actions producing ”transformational structure(s),” as was seen in the interview data when subjects often resorted to algebraic-mode transformations, which later led to trans-level reflection (Piaget, 1968).

Along these lines, another significance of this study was the many interview instances when Piagetian ideas were exemplified in the experimental evidence. For example, according to Piaget (1978), “the correct interpretation of the physical data discovered by empirical abstraction presupposes a measure of reflective abstraction” (p. 160). Interview Results 4 and 5 exemplified this statement in that, for every instance where the interview subjects were seen to ‘correct’ their mistaken matrix representations or solution-interpretations, they did so through a general process of practical, i.e., ‘empirical,’ thinking oftentimes in the form of familiar algebraic-mode computations, in which case; either through meta-cognitive intervention or the transfer of previous meta-cognitive knowledge, they were then seen to reflect upon their previous actions in the context of their unfamiliar setting and ‘correctly interpret’ the problem in the conceptually non-isomorphic setting.

As another example of the significance of this study in demonstrating Piaget’s ideas, Results 4 and 5 describe <u>how</u> reflective abstraction enabled the *personal constructions of similarity* indicative of actor-oriented transfer (Lobato, 2003). As Results 4 and 5 report, reflective abstraction was found to consist of the formation of two kinds of co-ordinations. *Forward co-ordinations* resulted from meta-cognitive reflection upon the actions and symbols of practical thinking, typically algebraic-mode computations, which identified relationships between unfamiliar setting-specific information, such as the linear combination definition, through the transformation of such *unfamiliar* setting-specific information into *familiar* (Rp) systems of equation information. *Backward co-ordinations* similarly resulted after performing computational actions such as row reduction, i.e. practical actions, whereby upon meta-cognitive intervention, the subjects interpreted the results of the computation backwards through the forward co-ordinations earlier constructed (see Fig. 5.29).

Additional Conclusions and Inferences

The primary research questions have been answered, and while the results show several types of difficulties linear algebra students seem to encounter when attempting linear algebra problems from conceptually non-isomorphic settings; evidence showed meta-cognitive intervention to be an effective tool to encourage theoretical thinking, and subsequently negotiate conceptually non-isomorphic setting change. The remainder of this chapter will be devoted to discussing a number of additional implications that have grown out of this study. The issue of transfer is important in undergraduate linear algebra because beginning students often attempt to solve problems analogically, based on familiar examples they already understand. In this sense, the pedagogical design mentioned earlier in Harel (1989b; 1990) could be reframed as a strategy meant for students to transfer knowledge from the familiar geometry and the language of $\Re^n$, to abstract vector-space theory. In the author's experience teaching linear algebra, the difficulties students appeared to experience could similarly be expressed as difficulties related to transferring knowledge from familiar settings to representationally and procedurally related, non-isomorphic settings. Based upon the review of research in mathematics education in Chapter Two, the existence of gaps in understanding between significantly different linear algebra settings indeed appears related to the absence of certain meta-knowledge and reflective insight into representational and procedural commonalities which may exist between such settings.

Pedagogical Significance

The foremost pedagogical significance of this study relates to the use of the meta-cognitive intervention as a pedagogical tool for inducing reflective abstraction, seen in the interview evidence to aid in the 'personal constructions of similarity' indicative of actor-oriented transfer, as well as in Experiment 3 in the traditional transfer of knowledge from the familiar Rp setting to the non-isomorphic LCp setting (Lobato, 2003). Recall Chapter One discussed how linear algebra evolved from trans-structural relationships between entirely different systems of mathematics, as opposed to inter-structural relationships between objects of *a* mathematical system (Dorier, 2000; Hillel & Sierpinska, 1994). The formation of forward and backward co-ordinations caused by meta-cognitive intervention signal the beginning of trans-like constructions fundamentally similar to the histrionic trans-constructions characteristic in the *psychogenesis* of linear algebra (Piaget & Garcia, 1983). In this regard, it is thought by the researcher that the tool of the meta-cognitive intervention, designed in a manner consistent with Experiment 3 of this study, has the potential to aid in the student's cultivation of theoretical thinking necessary to overcome the epistemic limitations of learning linear algebra described in Dorier's (2000) *Fundamental Epistemological Hypothesis* (see Chapter Two, p.35).

Theoretical Implications

Genetic Transfer Theory

Inspired by the results of this study in connection with the evidence found of actor-oriented transfer in Chapter Five (Results 5 and 6), the author integrated the traditional and actor-oriented transfer perspectives with the language and concepts of Piaget's *Genetic Epistemology* for the purpose of building a unified constructivist picture of transfer in general (Piaget, 1972). From the Latin, the term *transfer* may be translated as:

> L. *transferre* "bear across, carry over, transfer, translate," from trans- "across" + ferre "to carry" (OED, 1989).

In reference to the meaning of *transferre* to imply a transport metaphor or epistemological "carrying over," recall Lave's paradigm shift which redefined transfer as ‘constructions of similarity’, as opposed to the actual transferring of some *thing* (Lobato, 2006). Byrnes (1996) similarly employed the transport metaphor to define *transfer* as the ability to take learning from one particular context and extend that learning to other contexts.

From the point of view of Piaget’s *Genetic Epistemology*, another way to view the transport metaphor, in the sense of the above characterization of transfer from Byrnes (1996), begins with the question: ‘How does a learner know they are in a different context?’ The author’s answer to this question, consistent with Piaget’s theory of *Genetic Epistemology*, is that the learner knows she is in a different context because she is epistemologically uncomfortable and her pertinent knowledge schemas are in a state of disequilibrium. Furthermore, in reference to Lave’s paradigm shift, *nothing* is transferred in the traditional sense, rather; the non-local character of an unfamiliar situation or *structure* is rendered local by the subject. The following two properties characterize the author’s *Theory of Genetic Transfer*:

A. What is carried over and transferred from one particular context to another are the previous underlying structures of pre-existing *schemas* which may undergo modifications, co-ordinations, and reorganizations to create new structures, defining transfer (traditional or actor-oriented) when said new structures assimilate and accommodate *relative* disequilibria.

B. Transfer appears traditional when normative disequilibria are accommodated, and transfer appears actor-oriented when personally relative disequilibria are accommodated.

In Property A, similar to Lave (1988), the author replaces the transport metaphor usually associated with classical transfer, with the dynamic constructive transformations of underlying schemic structure consistent with Piaget's *Genetic Epistemology*. Property B is regarded as the *unification property* in the sense that both concepts of traditional and actor-oriented transfer are unified by the concept of disequilibrium from Piaget's *Genetic Epistemology*. The term *relative* relates to the subjects' awareness of disequilibria of either expert or personal determination.

Evidence supporting the above theoretical connections of actor-oriented transfer with Piaget's Genetic Epistemology can be found in Lobato (2003):

> [The actor-oriented approach] emphasizes learners' personal perceptions of affordances in a way that is both akin to Piaget's (1977) notion of generalizing assimilation, yet also accounts for the structuring roles of artifacts and the social structuring of language and actions (p.19).

Similar to actor-oriented transfer, in the author's *Genetic Transfer T*heory, <u>what</u> is transferred becomes more a question of, <u>how</u> have schemas equilibrated in order to assimilate and accommodate problems from new settings? The key distinction separating traditional and actor-oriented transfer perspectives in the author's Genetic Transfer Theory concern the status of the learner's relationship to disequilibria. Disequilibria may be normative and/or external to the subject, or disequilibria may be phenomenological and internal to the subject. From the actor-oriented transfer perspective the subject encounters internal disequilibria which she may understand more than the external disequilibria imposed by normative expert standards of problem solving performance. In other words, it is understandable why the actor-oriented transfer perspective concerns the personal constructions of the learner, whereas from a traditional transfer perspective; a subject may seem to hit a brick wall and be unable to begin performing problem solving actions because, quite possibly, she is cognitively unaware of the normative disequilibria (key problem features) understood by the expert, which must be cognized by the subject in order to begin the process of building accommodating structures.

Other evidence justifying associations between schemas as modifiable structures and transfer can be found in work by Wason (1983), where a phenomenon called the *realism effect*, similar to Harel's (1999) *contextual conception*, showed that subjects were usually limited to reasoning within the "content-specific schemas" in which a problem occurred (Greer & Harel, 1998, p.8). More evidence for the correspondence between Piaget's developmental psychology and transfer can be found in work by Jeeves & Greer (1983). Their work, based on the two-group structure defined as $(x,x)\to x$, $(x,y)=(y,x)\to y$, and $(y,y)\to x$; found that subjects began to demonstrate "structurally mediated transfer" at around age 11, between the 2-group and isomorphic contexts differing in only surface symbolic features (Greer & Harel, 1998, p.8). Note this result is in agreement with the border between Piaget's concrete and formal operative stages, signaling the emergence of abstract thought (see Table 2.5).

In Result 6 from the interview portion of this study, as subjects repeatedly solved problems from conceptually non-isomorphic settings, the evidence showed a subtle decrease in the degree of personal constructions necessary in order to facilitate actor-oriented transfer. The author characterized this phenomenon as the occurrence of transfer, in the traditional sense, of prior problem solving knowledge from *within* the multiple constructive processes of actor-oriented transfer. It is the researcher's theoretical conjecture that the appearance of this transfer, streamlining the formation of forward and backward co-ordinations within the general process of actor-oriented transfer, constitutes evidence of *reversibility* and potential *encapsulation*, marking the progress between inter-operational and trans-operational constructive phases, or in other terms, practical verses theoretical thinking (Piaget and Garcia, 1983, p. 177; Sierpinska, 2000; Dubinsky, 1997).

In Result 6, I characterized the encapsulation of the actor-oriented transfer process as one to, "eventually encapsulate <u>towards</u> *linear algebra understanding*." What the author meant by the expression *linear algebra understanding* was in reference to the flexibility gained by a learner from developing greater and greater co-ordinations between the conceptually non-isomorphic settings of interest for this study, a competence earlier referred to by the terms *representational fluency* and *meaningful learning* (Lesh, 1999; Ausubel, 1963). From Gray & Tall (1994), the term *procept* refers to the duality of a process and a concept, in the sense that an operation such as addition involves the *process* of adding two numbers, 2+4, and it also consists in the *object* or concept of (2+4), which may have further processes performed on it, such as (2+4)+1. Gray & Tall (1994) also explained the relationship of the procept to encapsulation in that:

> The cognitive process of forming a (static) conceptual entity from a (dynamic) process has variously been called *encapsulation* (after Piaget) (p. 3).

Recall from Chapter Two, in *APOS* theory, a learner was characterized as achieving a *process level* of an action when they could verbalize the procedure without performing it, or could predict the outcome (Dubinsky, 1997). The author theorizes that in actor-oriented transfer, as the learner streamlines the co-ordinations between non-isomorphic settings to a point where they no longer need to go through as many personal constructions, then greater understanding has taken place concerning the learner's knowledge of the relationships between the conceptually non-isomorphic settings. What remains unclear to the researcher, in the context of Gray & Tall's (1994) *procept*, as well as Dubinsky's (1997) APOS theory, is the question of: To what do the multiple processes of actor-oriented transfer encapsulate? It is thought by the researcher that the encapsulation of multiple processes on schemic objects may encapsulate to a new constructivist object – the *meta-strategy*.

Limitations of the Study

The first limitations of this study involved admitted weaknesses in Pilot Study 2, and Experiments 1 & 2, concerning the lack of opportunity to perform 3×3 factorial experiments which manipulated all the defined settings against all of the possible solution types. Due to basic *power* requirements (Ramsey & Schafer, 2002) for the numbers of subjects needed for experimentation and analysis in relation to the limited number of subjects available, this study followed Chen & Mo (2004) in the design of factorial experiments consisting of just two-level, similarity vs. dissimilarity of source problem to target problem, designs. As in the Chen & Mo (2004) study, decisions were necessarily made by the researcher as to the selection of which settings or solution types were to be used in the experimental designs.

A second limitation to quantitative component of this study involved the way in which subjects were randomly assigned to condition groups for the experiments. Due to schedule requirements, experimental conditions were relegated to individual classrooms; hence randomization only occurred in terms of assignment to 1 of 2 condition groups per implementation. Also, in relation to both quantitative and qualitative components of this study, the data consisted almost exclusively of the collected data, with no attempt on the part of the researcher to collect personal data of the students or data related to their study habits, class attendance, or past performance in mathematics.

A third overall limitation to this study possibly may have related to the structure of the *Matrix and Power Series Methods* course, which consisted of a lecture format along with a recitation session. It could have been the case that the lecture and recitation format was significantly different enough from the traditionally oriented transfer experiments and the semi-structured interviews, that the subjects may have felt uncomfortable displaying their knowledge in the experimental and qualitative settings of this study. In addition, having the recitation instructor also function as the interviewer may have placed unforeseen feelings of expectation to do well, upon the subjects, thus possibly affecting their performance.

Fourth, this study employed an Experiment 3, which tested the effectiveness of a meta-cognitive intervention for the solving of a novel linear combination problem. Without conducting interviews for each subject, it is hard to assign conclusive validity concerning the effectiveness of the intervention on problem solving. In addition, it is difficult to generalize about the role of meta-cognitive intervention in problem solving across non-isomorphic settings, unless more cases are tested, within different contexts.

Recommendations for Future Research

As a result of this study, several additional research questions have been raised. First, in regards to the theoretical question raised by the author concerning the nature of the encapsulation of the actor-oriented transfer process, it would be interesting to conduct another similar study, but incorporate more interviews into the methodology. By having a series of interviews, a better profile of the dynamics of actor-oriented transfer might be found, thus indicating a more clarified and consistent trend, rather than the subtle trend of streamlining detected in this study. Another interesting scenario would be to have students work together, to see the effects of social interaction in the context of actor-oriented transfer. Some additional questions that might arise in connection with groups are: Could it be the case that groups of students construct group-personal constructions of phenomena? How do groups of students reconcile disequilibria in understanding, as compared with single individuals, when encountering non-isomorphic linear algebra problem settings?

Another area of future research derives the aforementioned limitation of this study related to the quantitative experiments, in which the researcher intends to revisit the topic of this study to perform 3×3 factorial experiments capable of testing the entire spectrum of settings verses solution types. It is thought that if this were done, the results of the quantitative Experiments 1 and 2 might align with interview findings, suggesting that SOLUTION TYPE is a significant dimension of the LAM, affecting procedural transfer. In addition to the expansion of the quantitative methods of this study, the researcher also intends to perform studies on similar topics in mathematics dealing with non-isomorphic setting change in order to explore the potential universality of the general pattern of actor-oriented transfer described in Findings 5 and 6, as well as investigate the open question regarding the encapsulation of the component processes in actor-oriented transfer in relation to the author's expanded *Theory of Genetic Transfer*, theorized as a Piaget and Lobato-inspired constructivist unification of both traditional and actor-oriented transfer perspectives.

Closing Remarks

Although research in undergraduate linear algebra has occurred for the past 25 years since the formation of the LACSG, in many respects, this research is in its infancy. As the 1998 PCMI workshops and this research have implied, many of the recommendations, such as use of technology and a matrix-oriented syllabi, have not apparently resulted in novice students' better learning of linear algebra. As recommended by the PCMI, this dissertation embarked upon a study concerning students' efforts to learn certain topics in linear algebra. The results of this study point to fundamental problems concerning students' representational fluency in regards to formalism and context, or setting. The results also point to methods which might shift students' thinking to be more aligned with the very structure of linear algebra itself, as a generalizing and unifying branch of mathematics. How and when meta-cognitive interventions are used in connection with this research is an unanswered question in general. Work needs to be completed applying the key results of this study to other topics in linear algebra, as well as other areas of mathematics. In this regard, this research represents a first step in bridging and unifying classical and contemporary perspectives of transfer theory to further explore the role of meta-cognitive intervention in negotiating conceptually non-isomorphic mathematical settings.

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APPENDIX A

PILOT STUDY 2 PROBLEMS

Condition 1; (Similar, Similar)

Problem 1 (S_{22})

For the following vectors v and w in a vector space *V*, *A=v-w* , *B=2v+w*,

and *C=3v+2w*; write the vector *C* as a linear combination of the vectors *A* and *B*.

Problem 2 (S_{22})

For the following vectors v and w in a vector space *V*, *A=v-w* , *B=v+2w*,

and *C=4v+3w* ; write the vector *C* as a linear combination of the vectors *A* and *B*.

Problem 3 (T_{22})

For the following vectors v and w in a vector space *V*, *A=v-w* , *B=v+2w*,

and *C=v+w* ; write the vector *C* as a linear combination of the vectors *A* and *B*.

Condition 2; (Similar, Dissimilar)

Problem 1 (S_{21})

For the following vectors *v* and *w* in a vector space *V*, *A=v+w* , *B=v-w*

and *C=-v+5w* ; write the vector *C* as a linear combination of the vectors *A* and *B*.

Problem 2 (S_{21})

For the following vectors v and w in a vector space *V*, *A=v-w* , *B=v+2w*, and *C=v+w* ; write the vector *C* as a linear combination of the vectors *A* and *B*.

Problem 3 (T_{22})

For the following vectors v and w in a vector space *V*, *A=2v-w* , *B=v+2w*, and *C=v* ; write the vector *C* as a linear combination of the vectors *A* and *B*.

Condition 3; (Dissimilar, Similar)

Problem 1 (S_{12})

For the following vectors in $\Re^2$, $A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$; write the vector *C* as a linear combination of the vectors *A* and *B*.

Problem 2 (S_{12})

For the following vectors in $\Re^2$, $A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $C = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$; write the vector *C* as a linear combination of the vectors *A* and *B*.

Problem 3 (T_{22})

For the following vectors v and w in a vector space *V*, *A=v-w* , *B=v+2w* , and *C=3v+w* ; write the vector C as a linear combination of the vectors *A* and *B*.

Condition 4; (Dissimilar, Dissimilar)

Problem 1 (S_{11})

For the following vectors in $\Re^2$, $A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $C = \begin{bmatrix} -4 \\ -5 \end{bmatrix}$; write the vector *C* as a linear combination of the vectors *A* and *B*.

Problem 2 (S_{11})

For the following vectors in $\Re^2$, $A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,and $C = \begin{bmatrix} 2 \\ -11 \end{bmatrix}$;

write the vector *C* as a linear combination of the vectors *A* and *B*.

Problem 3 (T_{22})

For the following vectors v and w in a vector space *V*, *A=2v-w* , *B=w+2v*,

and *C=w+4v* ; write the vector *C* as a linear combination of the vectors *A* and *B*.

Condition 5; Control (T_{22})

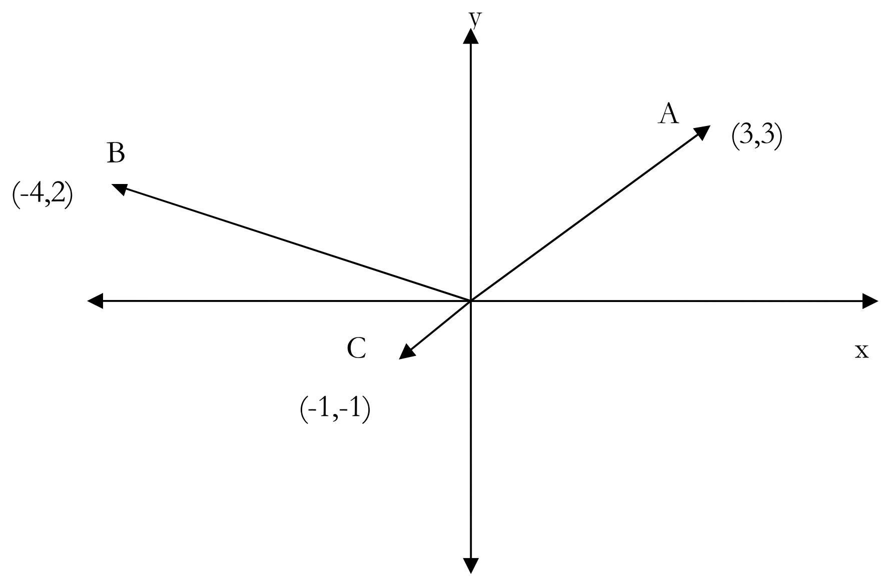
For the following vectors v and w in a vector space *V*, *A=v-w* , *B=v+2w*,

and *C=3v+w* ; write the vector *C* as a linear combination of the vectors *A* and *B*.

APPENDIX B

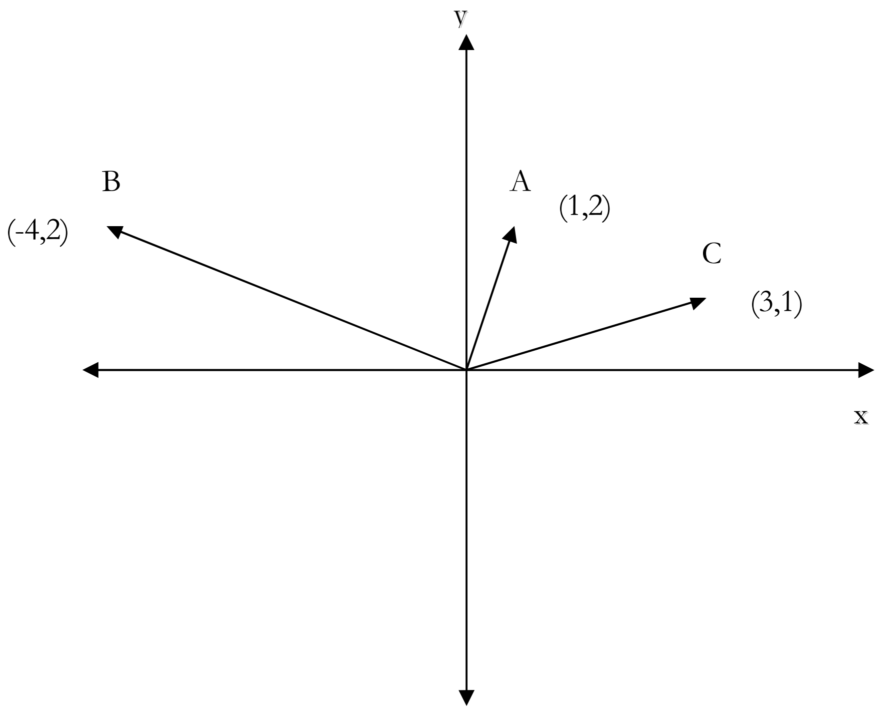
EXPERIMENT 1 PROBLEMS

Condition 1; (Similar, Similar)



Problem 1 (S_{22})

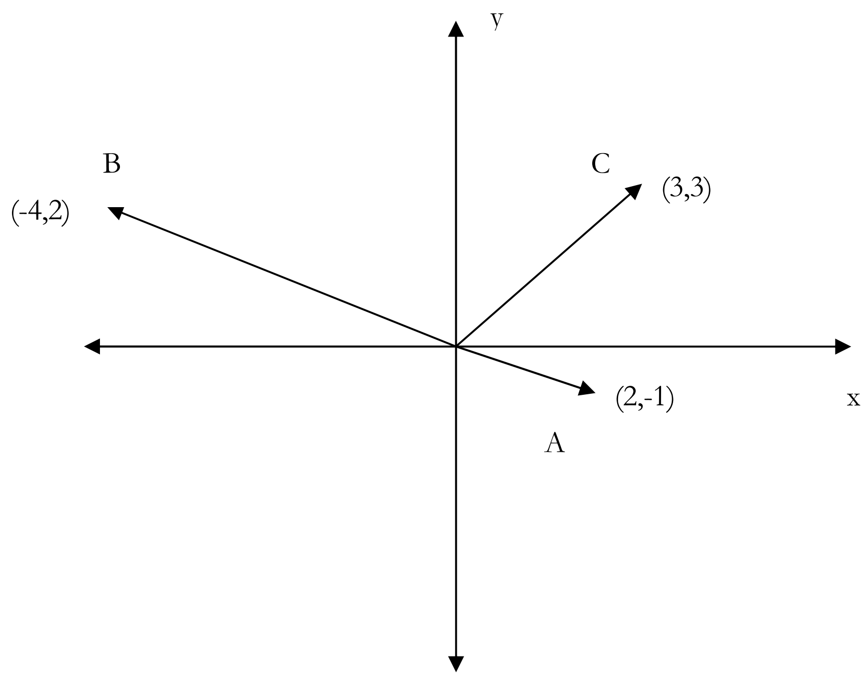
In $\Re^2$, find the linear combination of vectors A and B that give vector C.



Problem 2 (T_{22})

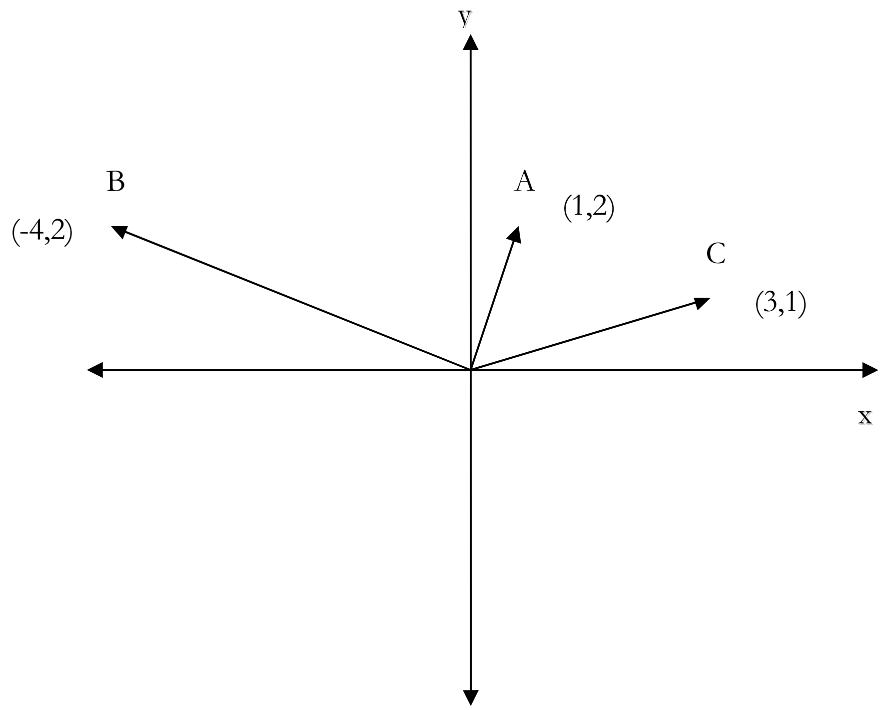
In $\Re^2$, find the linear combination of vectors A and B that give vector C.

Condition 2; (Similar, Dissimilar)



Problem 1 (S_{21})

In $\Re^2$, find the linear combination of vectors A and B that give vector C.



Problem 2 (T_{22})

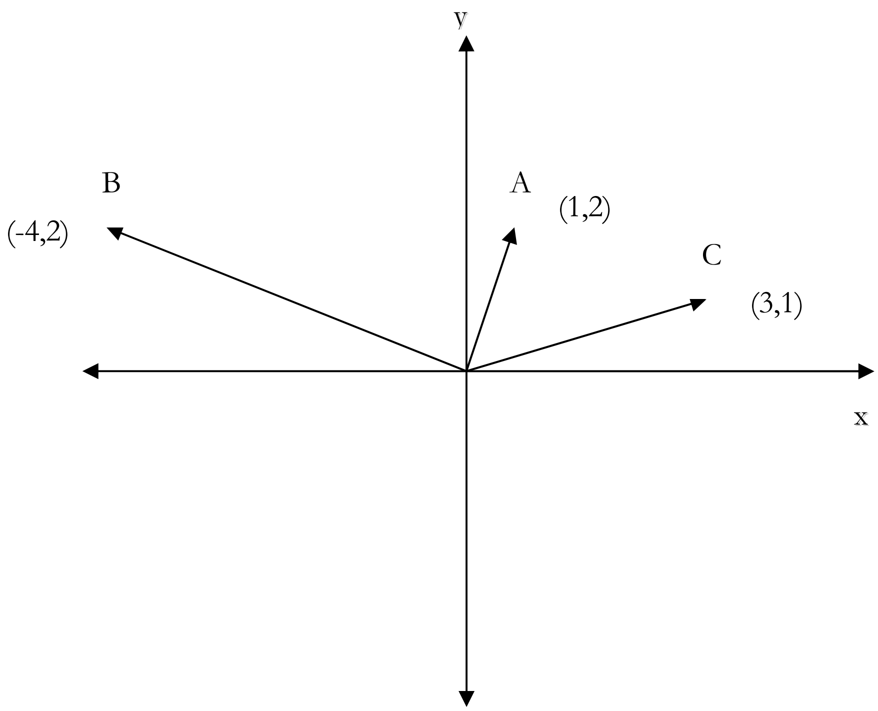
In $\Re^2$, find the linear combination of vectors A and B that give vector C.

Condition 3; (Dissimilar, Similar)

Problem 1 (S_{12})

Find the intersection of the following two lines in $\Re^2$.

$$\begin{aligned} x \quad - \quad y \quad &= \quad 1 \\ 3x \quad + \quad 2y \quad &= \quad -2 \end{aligned}$$



Problem 2 (T_{22})

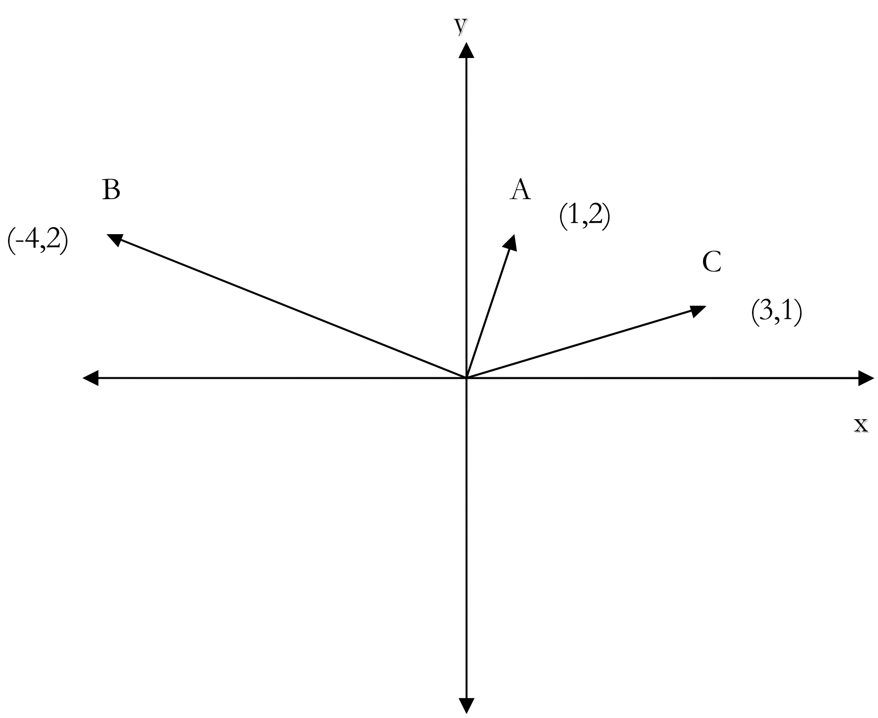
In $\Re^2$, find the linear combination of vectors A and B that give vector C.

Condition 4; (Dissimilar, Dissimilar)

Problem 1 (S_{11})

Find the intersection of the following two lines in $\Re^2$.

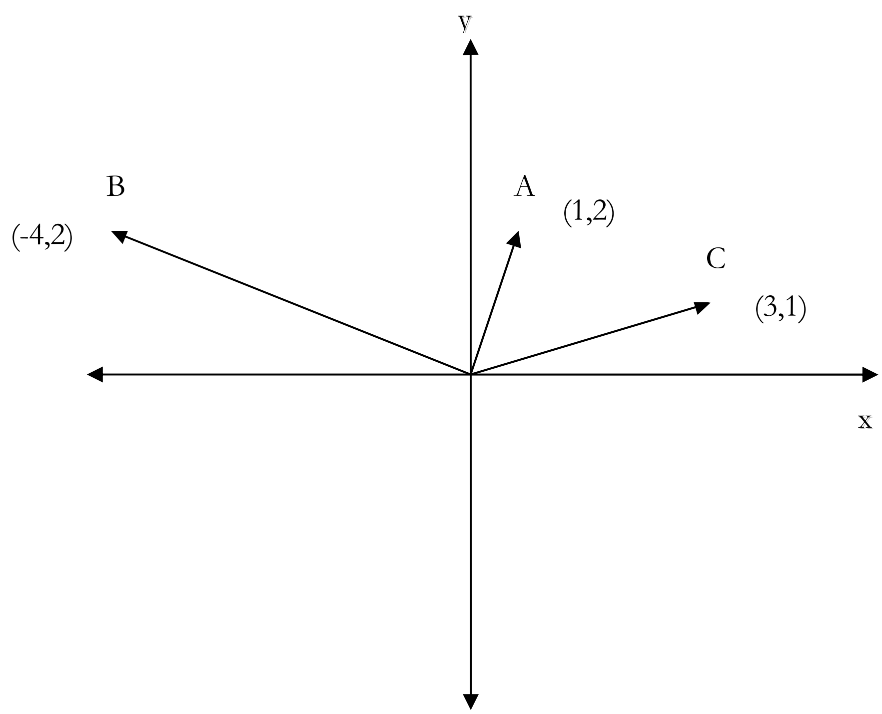
$$\begin{aligned} -x &- y &= 1 \\ 2x &+ 2y &= 2 \end{aligned}$$



Problem 2 (T_{22})

In $\Re^2$, find the linear combination of vectors A and B that give vector C .

Condition 5; Control (T_{22})



Problem (T_{22})

In $\Re^2$, find the linear combination of vectors A and B that give vector C.

APPENDIX C

EXPERIMENT 2 PROBLEMS

Condition 1; (Similar, Similar)

Problem 1 (S_{22})

A linear transformation $T : \Re^3 \rightarrow \Re^3$ has matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 4 & -2 \\ 3 & 0 & 1 \end{bmatrix}$. Find a vector

$v \in \Re^3$ that satisfies $T(v) = [-4,9,2]^t$.

Problem 2 (T_{22})

A linear transformation $T : \Re^3 \rightarrow \Re^3$ has matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix}$. Find a

vector $v \in \Re^3$ that satisfies $T(v) = [1,-5,-4]^t$.

Condition 2; (Similar, Dissimilar)

Problem 1 (S_{21})

A linear transformation $T:\Re^3 \to \Re^3$ has matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 4 & -2 \\ 3 & 0 & 1 \end{bmatrix}$. Find a vector

$v \in \Re^3$ that satisfies $T(v) = [-4,9,2]^t$.

Problem 2 (T_{22})

A linear transformation $T:\Re^3 \to \Re^3$ has matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -3 & 3 \end{bmatrix}$. Find a

vector $v \in \Re^3$ that satisfies $T(v) = [2,4,-2]^t$.

Condition 3; (Dissimilar, Similar)

Problem 1 (S_{12})

Find the intersection of the following three planes in $\Re^3$.

$$\begin{array}{ccccccc} x & - & y & + & z & = & 2 \\ x & + & y & - & z & = & 0 \\ 2x & + & 2y & + & z & = & 6 \end{array}$$

Problem 2 (T_{22})

A linear transformation $T : \Re^3 \to \Re^3$ has matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix}$. Find a vector $v \in \Re^3$ that satisfies $T(v) = [1,-5,-4]^t$.

Condition 4; (Dissimilar, Dissimilar)

Problem 1 (S_{11})

Find the intersection of the following three planes in $\Re^3$.

$$\begin{array}{ccccccc} x & - & y & + & z & = & 2 \\ x & + & y & - & z & = & 0 \\ 2x & + & 2y & + & z & = & 6 \end{array}$$

Problem 2 (T_{22})

A linear transformation $T:\Re^3 \rightarrow \Re^3$ has matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -3 & 3 \end{bmatrix}$. Find a vector $v \in \Re^3$ that satisfies $T(v) = [2,-2,2]^t$.

Condition 5; Control (T_{22})

A linear transformation $T:\Re^3 \rightarrow \Re^3$ has matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$. Find a vector $v \in \Re^3$ that satisfies $T(v) = [4,-2,6]^t$.

APPENDIX D

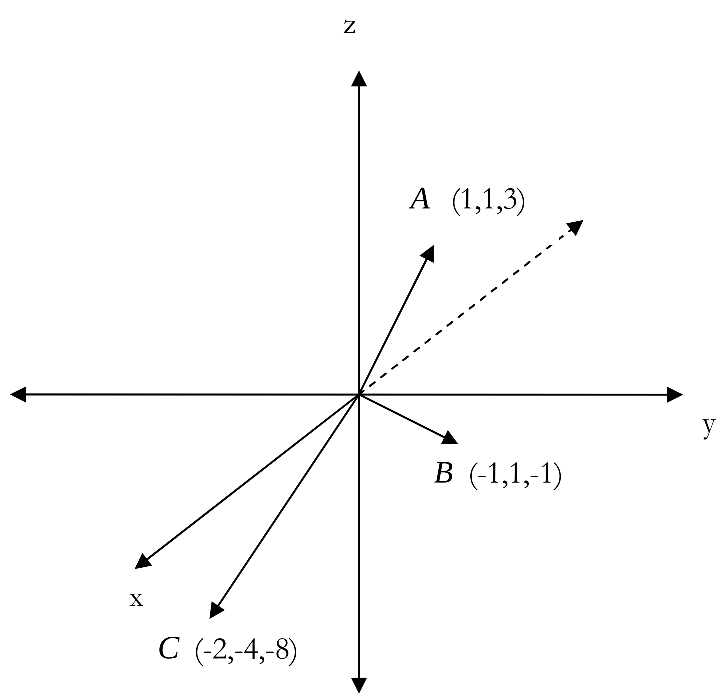
INTERVIEW PROBLEMS

Problem 1

For the following vectors v and w in a vector space *V*, $A = v - w$, $B = 2v + w$,

and $C = 3v + 2w$; write the vector C as a linear combination of the vectors *A* and *B*.

Problem 2



In $\Re^3$, write the vector *C* as a linear combination of the vectors *A* and *B*.

Problem 3

A linear transformation $T : \Re^3 \to \Re^3$ has matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 1 & 2 & 0 \end{bmatrix}$. Find a vector $v \in \Re^3$ that satisfies $T(v) = [6,-2,4]^t$.

APPENDIX E

EXPERIMENT 3 TREATMENT & TARGET PROBLEM

Experimental Treatment

Given a matrix like $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and the vector $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, notice

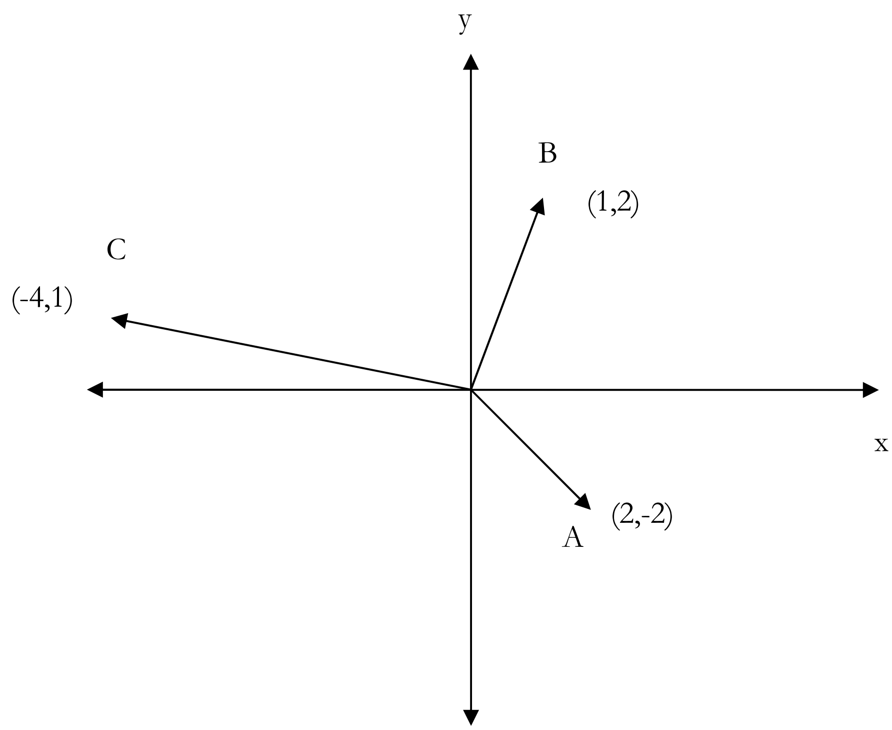
that $AX = b$ could mean: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ which could

represent: $\begin{array}{rcrcr} x & + & 2y & = & 2 \\ 3x & + & 4y & = & 2 \end{array}$. But also, $AX = b$ could mean:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix} = \begin{bmatrix} x \\ 3x \end{bmatrix} + \begin{bmatrix} 2y \\ 4y \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$ i.e. a linear

combination of the columns of A.

Treatment & Non-Treatment Target Problem



In $\Re^2$, find the linear combination of vectors A and B that give vector C.