The stability of fluids in porous media subject to various physical and geometrical conditions is studied here. Criteria for the onset of convective motions are given in terms of critical temperature gradients or heat sources. The general method used involves the linearization of the field equations about the hydrostatic solutions and the reduction of the linear equations to an appropriate linear eigenvalue problem for certain positive operators. Variational methods are then used to approximate the minimum eigenvalues which yield the stability criteria.
STABILITY OF CONVECTIVE FLOWS IN POROUS MEDIA

By

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1.1 Thermal Convection

Consider a horizontal layer of a homogeneous fluid at rest with initial zero temperature. Increase the temperature uniformly at the lower side very slowly so that the temperature gradient is constant throughout. The heating causes the bottom portion of this layer to expand, making it less dense. This creates buoyancy forces and a density inversion which leave the fluid at an unstable equilibrium. Any disturbance of this unstable equilibrium will cause an upward movement in the fluid which is resisted by the frictional forces due to the viscosity of the fluid. When the temperature gradient is smaller than a certain critical value, the frictional forces slow down the upward movement considerably so that any temperature disturbances caused by this movement die out because of heat conduction. However, if the temperature gradient is increased beyond that critical value the upward movement becomes so rapid that the temperature differences cannot be equalized by conduction. The disturbance will then persist and portions of the fluid will continue to move upward causing convective motions in
the fluid. Experiments have shown that convective motions occur in individual cells in the fluid.

1.2 Basic Equations

Consider an infinite slab of a porous medium saturated with a homogeneous incompressible fluid. Under the ideal conditions where we have steady state motions in the absence of body forces, the flow is described by

$$\nabla p + \frac{\nu}{K} \mathbf{q} = 0,$$

$$\nabla \cdot \mathbf{q} = 0,$$

where

- $p = p(\mathbf{x}, t)$ pressure distribution
- $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$ mass flux density,

and $\nu, K$ are the kinematic viscosity and the permeability of the medium. The mass flux density $\mathbf{q}$ is defined as

$$\mathbf{q} = \rho \mathbf{u}$$

where $\rho$ is the density of the fluid and $\mathbf{u}$ is the velocity distribution. Equation 1.1 is empirical and is known as Darcy's law, Polubarinova-Kochina (1962). Equation 1.1 holds with the assumption that the flow is laminar and slow enough so that $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is negligible compared to the frictional forces.

In the presence of external forces and time dependent motions the behaviour of the fluid is then described by the following set of equations:
generalized Darcy's law (Polybarinova-Kochina 1962)

\[ \rho_c \ddot{u}_t = -\nabla p - \frac{\nu}{K} \rho_c \dot{u} - \rho g \dot{e} \]  

1.3

equation of thermal conduction

\[ T_t + \mathbf{u} \cdot \nabla T = \kappa \Delta T \]  

1.4

equation of continuity

\[ \nabla \cdot (\rho_c \mathbf{u}) = 0 \]  

1.5

simplified equation of state

\[ \rho = \rho_c (1 - \alpha T) \]  

1.6

where \( \nabla \) and \( \Delta \) are the usual gradient and Laplacian operators. We denote by \( T \) the temperature, \( g \) the gravitational constant, \( \dot{e} = (0,0,1) \), \( \kappa \) the thermometric conductivity of the medium, \( \rho_c \) the density at \( T = 0 \) and \( \alpha \) the coefficient of volume expansion of the fluid. We allow the permeability \( K \) to be a function of the vertical variable. Equations 1.3-1.6 are valid under the following assumptions:

(a) The motion is laminar and slow enough so that \( \mathbf{u} \cdot \nabla \mathbf{u} \) can be neglected

(b) The medium is saturated with the fluid and gravity is the only external force present

(c) The thermal conductivity of the medium and its
density $\rho_s$ are constant, i.e. $\kappa = \frac{k}{\rho_s C_v}$
is constant, where $C_v$ is the specific heat at constant volume and $k$ is its heat conductivity.

(d) The Boussinesq approximation is valid, (i.e.) density is constant except when it appears in the body force $\rho \ddot{g}$.

When written in component form, equations 1.3-1.6 represent a set of six equations for the six physical variables (3 velocity, temperature, density, pressure). The boundary conditions which accompany equations 1.3-1.6 will of course depend on the physical setting. However, the most general conditions for the temperature and the velocity are

(i) temperature:

perfectly conducting boundary, constant $T$

perfectly insulating boundary, i.e.

$$\frac{\partial T}{\partial n} = 0 \text{ there, } n \text{ is the outward normal.}$$

Radiation condition on the boundary, i.e.

$$c \frac{\partial T}{\partial n} = -T \text{ there.}$$

(ii) velocity:

we will always consider the boundary to be rigid and impervious, i.e. $\dot{u} = 0 \text{ there.}$

We recall that the study of convection involves the
examination of the behavior of small disturbances imposed on the equilibrium state. This is accomplished by linearizing equations 1.3-1.6 about the steady state solution, then studying the linear equations. This is a linear perturbation theory which has been found to yield good results, (Chandrasekhar, 1961).

1.3 The Perturbation Equations

Consider the problem defined by equations 1.3-1.6 and assume that the boundary is composed of two perfectly conducting parallel walls such that the lower wall is at \( z = 0 \) and the upper wall at \( z = d \). The boundary conditions are:

\[
\begin{align*}
\hat{u} &= 0, \quad T = A, \text{ constant at } z = 0 \\
\hat{u} &= 0, \quad T = 0 \quad \text{ at } z = d.
\end{align*}
\]

1.7

For the equilibrium state, i.e. \( \hat{u} = 0, \frac{\partial}{\partial t} = 0 \) and \( T = T(z) \), equations 1.3-1.6 admit the following solution set:

\[
\begin{align*}
\hat{u}_0 &= 0, \quad T_0 = \frac{A}{d}(d-z) = \beta(d-z) \\
p_0 &= (1+\alpha\beta z), \quad p_0 = p(0) - \gamma p_c(z+\frac{1}{2}\alpha\beta z^2)
\end{align*}
\]

1.8

1.9

where \( p(0) \) is the pressure at \( z = 0 \) and the temperature gradient \( \frac{dT_0}{dz} = \beta \).

Introduce small field perturbations on the
equilibrium state such that
\[ \ddot{u} = \ddot{u}_0 + \ddot{u}, \quad T = T_0 + \theta, \quad p = p_0 + \Pi \quad 1.10 \]
where \( \ddot{u}, \theta \) and \( \Pi \) are in general functions of position and time. They are assumed to be small enough so that all quadratic terms involving them and their derivatives are to be neglected, (Chandrasekhar (1961)). Substitute equations 1.8-1.10 into equations 1.3-1.7 and transform all dependent variables to the non-dimensional coordinate system
\( (\frac{X}{a}, \frac{Y}{a}, \frac{Z}{a}) \) by the use of the following transformations:
\[ K \rightarrow \overline{K}, \quad \ddot{u} + \frac{K}{a} \ddot{u}, \quad \theta + \beta d \theta, \quad \Pi + \frac{\rho \alpha v K}{K} \Pi \]
where \( \overline{K} \) is a reference permeability of the medium.

The linearized non-dimensional equations are:
\[ 0 = - \nabla \Pi - \frac{\ddot{u}}{K} + \lambda \theta \ddot{\theta} \quad 1.11 \]
\[ w = - \Delta \theta \quad 1.12 \]
\[ \nabla \cdot \ddot{u} = 0 \quad 1.13 \]
where \( \lambda = \frac{\overline{K} \alpha \beta d^2}{v K} \). The steady state case is considered in order to examine the marginal or neutral stability problem (Lapwood (1948)), Pellew and Southwell (1940)). The boundary conditions which accompany equations 1.11-1.13 are:
\[ \ddot{u} = 0, \quad \theta = 0 \text{ for } \zeta = 0,1 \quad 1.14 \]
where \( \zeta = \frac{Z}{a} \) is the new non-dimensional variable in the vertical direction.
It is worth noting here that equations 1.11-1.13 hold true for all convection problems in a horizontal layer of fluid in porous media if we keep in mind that

$$\beta = \frac{dT}{dz}$$

does not have to be constant and can be a function of the vertical variable.
CHAPTER II

CONVECTION AS AN EIGENVALUE PROBLEM

The perturbation equations 1.10-1.12 describe an eigenvalue problem for the velocity \( \dot{u} \), temperature \( \theta \) and pressure \( \Pi \) with an eigenvalue \( \lambda \). Physically, convection sets in whenever fluid from the bottom moves upward overcoming the frictional forces and the effects of conduction. The mathematical model implies that the onset of instability corresponds to the first velocity eigenvector and the first eigenvalue \( \lambda_1 = \frac{Kag^2d^2}{\nu\kappa} \). This determines the critical value of the temperature gradient at which convection occurs and reduces the convection problem to finding minimum eigenvalues of certain operators. This is not a simple task since the partial differential equations involved are rather difficult to solve. Using two dimensional Fourier transform we can reduce these equations to a more manageable set of ordinary differential equations. Applying this to the perturbation equations 1.10-1.12 we can generally reduce them to

\[ Mu = \lambda u \]

where \( M \) is a linear ordinary differential operator on the
Hilbert space $L^2[0,1]$ with domain $D(M)$ such that any $u \in D(M)$ is real and satisfies all the smoothness and boundary conditions associated with $M$. Exact solutions of 2.1 are often elusive but a variational technique can be applied to find the minimum eigenvalue. The method used involves expressing $\lambda$ as the ratio of two quadratic integrals, Chandrasekhar (1961, pp. 27). The form of these integrals depends solely on the nature of $M$, i.e. the physics of the situation. It will be shown in Chapters IV and V that, for the cases on hand, whenever $M$ is symmetric it is positive, thus the inner product $(u, Mu) = \int_0^1 u M u \, d\zeta$ is always positive ($u \neq 0$). Taking the inner product of $u$ with both sides of 2.1 and solving for $\lambda$ we get

$$\lambda = \frac{\int_0^1 u M u \, d\zeta}{\int_0^1 u^2 \, d\zeta} > 0.$$  

We want to show that the above ratio has stationary properties iff the right hand side represents the true eigenvalues of equation 2.1. Consider a small variation in $u, \delta u$ which satisfies the boundary and smoothness conditions on $u$. The corresponding change in $\lambda$ is

$$\delta \lambda = \frac{\int_0^1 (\delta u M u + u M \delta u) \, d\zeta}{\int_0^1 u^2 \, d\zeta} - \frac{\int_0^1 u M u \, d\zeta}{\left(\int_0^1 u^2 \, d\zeta\right)^2} \cdot \int_0^1 2u \delta u \, d\zeta.$$
but since \( \lambda = \frac{\int_0^1 uMud\zeta}{\int_0^1 u^2d\zeta} \) and \( M \) is symmetric i.e.

\[
\int_0^1 uM\delta u d\zeta = \int_0^1 \delta uMud\zeta \quad \Rightarrow \quad \delta \lambda = 2 \frac{\int_0^1 \delta u(Mu-\lambda u)d\zeta}{\int_0^1 u^2d\zeta},
\]

for a stationary property \( \delta \lambda = 0 \) hence \( Mu = \lambda u \). The converse is obviously true. The ratio \( \frac{\int_0^1 uMud\zeta}{\int_0^1 u^2d\zeta} \) is known as the Rayleigh Quotient. The eigenvalues of such operators \( M \) are bounded below and form a monotone increasing set, (Collatz 1966). This implies that the Rayleigh Quotient yields the minimum eigenvalue

\[ \lambda_1 > 0. \]

Since \( u \) is not generally available, we approximate the minimum eigenvalue \( \lambda_1 \). This involves choosing a trial function \( v(\zeta) \in L^2[0,1] \) such that:

1. \( v \) satisfies the boundary conditions for \( M \)
2. \( v \) satisfies the smoothness conditions for \( M \)
3. \( v(\zeta) \neq 0 \) for \( 0 < \zeta < 1 \).

Thus the Rayleigh Quotient outlined above implies:

\[ \lambda_1 = \frac{\int_0^1 vMvd\zeta}{\int_0^1 v^2d\zeta}, \]

where \( \lambda_1 \) is always being approximated from above,
and Naylor (1966).

Consider the case when $M$ is not symmetric. In order to represent $\lambda$ as the ratio of two positive integrals we attempt to split $M$ into two positive symmetric operators $T$ and $S$ such that $T$ is invertible, $M = TS$ and $T$ and $S$ are non-commuting. If such a factorization is possible, we introduce the auxiliary variable $\psi(\zeta)$ such that

$$T\psi = u$$

where $\psi$ can be determined from $u$ uniquely. Since $T$ is invertible and $S$ and $T$ are positive we have

$$\int_0^1 u S u d\zeta > 0$$

$$\int_0^1 \psi T \psi d\zeta > 0 ,$$

and equation 2.1 can be written

$$Su = \lambda \psi ,$$

2.2

hence

$$\lambda = \frac{\int_0^1 u S u d\zeta}{\int_0^1 \psi T \psi d\zeta} > 0 .$$

To examine the stationary properties of the above ratio, consider a small variation in $u$, $\delta u$ which satisfies the boundary and smoothness conditions on $u$ and proceeding as before we get:
\[ \delta \lambda = \frac{\int_0^1 \psi T \psi d\zeta \int_0^1 [u \delta u + \delta u u] d\zeta - \int_0^1 u \delta u d\zeta \int_0^1 [\delta \psi T \psi + \psi \delta \psi] d\zeta}{[\int_0^1 \psi T \psi d\zeta]^2} \]

but \( \lambda = \frac{\int_0^1 u \delta u d\zeta}{\int_0^1 \psi T \psi d\zeta} \) and \( T \delta \psi = \delta u \) and since \( S \) and \( T \)

are symmetric we have

\[ \int_0^1 u \delta u d\zeta = \int_0^1 \delta u u \delta u d\zeta \]

\[ \int_0^1 \delta \psi T \psi d\zeta = \int_0^1 \psi T \delta \psi d\zeta \]

\[ \implies \delta \lambda = \frac{2\int_0^1 \delta u [S u - \lambda \psi] d\zeta}{\int_0^1 \psi T \psi d\zeta} . \]

Thus the ratio \( \frac{\int_0^1 u \delta u d\zeta}{\int_0^1 \psi T \psi d\zeta} \) has stationary properties iff \( \lambda \)

is the eigenvalue set of 2.2. This procedure actually computes the minimum eigenvalue. The proof of this for a particular problem is given in Appendix I. Similar proofs for different cases can be easily given.
CHAPTER III

SOLUTION OF A CONVECTION PROBLEM
FOR A FLUID IN PORES

3.1 Mathematical Formulation of the Problem

The onset of convection in an infinite horizontal layer of porous medium saturated with an incompressible fluid under a constant temperature gradient was first examined by Lapwood (1948). We will give a refinement of the method of solution and the results obtained. The mathematical equations have already been given in section 3, Chapter I, the non-dimensional perturbation equations for neutral stability are

\begin{align*}
0 &= -\nabla p - \tilde{u} + \lambda \theta \tilde{e} & 3.1 \\
\omega &= -\Delta \theta & 3.2 \\
\nabla \cdot \tilde{u} &= 0 & 3.3
\end{align*}

Where we assume that the permeability is constant, and

$$\lambda = \frac{Kag\beta d^2}{\nu k}.$$  

Assuming that the upper and lower boundaries are perfect conductors, the boundary conditions are:

$$\tilde{u} = 0, \quad \theta = 0 \quad \text{for} \quad \zeta = 0,1. \quad 3.4$$

To simplify the eigenvalue problem defined by 3.1-3.4 we first take the divergence of 3.1 and using 3.3
we get

$$0 = - \Delta p + \lambda \frac{\partial \theta}{\partial \zeta} \quad 3.5$$

Combining 3.2 with the third component of 3.1 we get:

$$0 = - \frac{\partial P}{\partial \zeta} + \Delta \theta + \lambda \theta . \quad 3.6$$

Eliminating $p$ between 3.5 and 3.6 we get

$$\Delta^2 \theta = - \lambda \Delta \theta \quad 3.7$$

where $\Delta^2$ is the biharmonic operator and

$$\Delta^2 = \Delta - \frac{\partial^2}{\partial \zeta^2} .$$

Using equation 3.2 the boundary conditions 3.4 become

$$\theta = 0, \quad \Delta \theta = 0 \quad \text{at} \quad \zeta = 0,1 \quad 3.8$$

Assume $\theta$ to be separable of the form

$$\theta = F\left(\frac{X}{d}, \frac{Y}{d}\right) Z(\zeta) \quad 3.9$$

where $F$ satisfies

$$(\Delta^2 + a^2)F = 0 , \quad 3.10$$

We assume that convection occurs in a rectangular cell, then:

$$F = \sin lx \sin my$$

and

$$a^2 = d^2 (l^2 + m^2) . \quad 3.11$$
Substitution of equations 3.9-3.11 into 3.7-3.8

\[(D^2-a^2)^2 Z = \lambda a^2 Z, \quad D = \frac{\partial}{\partial \zeta} \quad 3.12\]

and the boundary conditions:

\[\theta = 0, \quad (D^2-a^2)\theta = 0 \text{ at } \zeta = 0,1 \quad 3.13\]

3.2 The Principle of Exchange of Stability

The mathematical formulation given in the previous section corresponds to what we called the state of neutral stability, (i.e. time independent behaviour). This is based on the principle of exchange of stability which states that at the onset of convection a steady-state pattern of motion should prevail. That this principle holds here is not obvious. In order to prove it is applicable here consider the general equations 1.1-1.4 and simplify along the same steps used in the previous section we get:

\[\left(\frac{\partial}{\partial \zeta} + \frac{\nu}{K}\right)\left(\frac{\partial}{\partial \zeta} - \kappa\Delta\right)\Delta \theta = \frac{g\alpha A}{d} \Delta \theta .\]

Assume a solution of the form

\[\theta = e^{st} \sin \lambda x \sin m y \sin \frac{n\pi z}{d}\]

where \(\lambda, m, n\) are integers and \(s\) can be complex. Hence we have the consistency condition:

\[(s + \frac{\nu}{K})(s+\kappa\frac{a^2+n^2\pi^2}{d^2})(a^2+n^2\pi^2) = \frac{g\alpha A}{d} a^2\]
or \[(s+a_1)(s+a_2) = a_3\]

where \[a^2 = (k^2+m^2)d^2, \quad \alpha_1 = \frac{\nu}{K}, \quad \alpha_2 = \frac{a^2+n^2\pi^2}{d^2}, \quad \alpha_3 = \frac{ga_3}{d(a^2+n^2\pi^2)}\].

Note that \(\alpha_1 > 0, \alpha_2 > 0\) and \(\alpha_3\) has the sign of \(A(\alpha > 0)\). Thus the roots of the above quadratic equation \(s_1, s_2\) are given by

\[-\frac{1}{2}(\alpha_1 + \alpha_2) \pm \frac{1}{2} \sqrt{(\alpha_1 - \alpha_2)^2 + 4\alpha_3}\].

When these roots are complex they have negative real part, hence \(|e^{st}| \to 0\) as \(t \to \infty\). This implies that the disturbance dies down and convection currents do not set in.

When \(s_1\) and \(s_2\) are real (i.e. \((\alpha_1 - \alpha_2)^2 + 4\alpha_3 > 0\)) they are monotone increasing functions of \(\alpha_3\) with ranges \(R_1\) and \(R_2\) such that

\[0 \in R_i, \quad i = 1, 2\]
\[R_1 \cap R^- \neq \emptyset\]
\[R_1 \cap R^+ \neq \emptyset\],

depending on the choice of \(\alpha_3\) (i.e. the temperature gradient \(\frac{A}{d}\) and the wave number \(a\)). When \(s_1, s_2 < 0\), \(e^{st} \to 0\) and the equilibrium is stable while for \(s_1, s_2 > 0\) \(e^{st} \to \infty\) and the equilibrium is unstable and convection currents develop. Thus the stable and
unstable states are separated by a neutral or marginal stability state when $s_1, s_2$ are zero (i.e. $\frac{\partial}{\partial t} = 0$).

This establishes the principle of exchange of stability. The physical interpretation of the above analysis is given in Appendix I.

3.3 A Variational Principle

The eigenvalue problem given by equations 3.12-3.13 can be treated using the first variational principle developed in Chapter II (Rayleigh Quotient). First we have to express the eigenvalues $\lambda$ as the ratio of two positive integrals then find the minimum of that ratio which corresponds to the first eigenvalue $\lambda_1$.

Take the inner product of $Z$ with both sides of equation 3.12:

$$\int_0^1 Z(D^2-a^2)^2 Z d\xi = \lambda a^2 \int_0^1 Z^2 d\xi$$  \hspace{1cm} 3.14

integrating the left hand side of the above equation and using the boundary conditions 3.13 we get:

$$\int_0^1 [(D^2 Z)^2 + 2a^2 (DZ)^2 + a^4 Z^2] d\xi = \lambda a^2 \int_0^1 Z^2 d\xi.$$  

This automatically implies that $\lambda$ is positive ($Z \neq 0$). Take $Z = \sin \pi \xi$ which satisfies the conditions given on page 10. Since $\sin \pi \xi$ is actually the first characteristic solution of 3.12 we can find the exact value of $\lambda_1$. 
From 3.14
\[ \lambda = \frac{\int_0^1 z (D^2 - a^2)^2 z \, dz}{a^2 \int_0^1 z^2 \, dz} \]

\[ \lambda_1 = \frac{(\pi^2 + a^2)^2 \int_0^1 \sin^2 \pi \, d \zeta}{a^2 \int_0^1 \sin^2 \pi \zeta \, d \zeta} \]

\[ \lambda_1 = (\pi^2 + a^2)^2 / a^2 \]

3.4 Exact Solution for the Layer Heated from Below.

The general solution of 3.12 is

\[ Z = A_1 \exp(\gamma \zeta) + A_2 \exp(-\gamma \zeta) + A_3 \exp(\delta \zeta) + A_4 \exp(-\delta \zeta) \]

where \( \gamma^2 = a(a + \Lambda) \), \( \delta^2 = a(a - \Lambda) \), \( \Lambda = \Lambda^2 \), \( \Lambda > 0 \),

the boundary conditions 3.13 give us the following matrix equation

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
e^\gamma & e^{-\gamma} & e^{\delta} & e^{-\delta} \\
e^\gamma & e^{-\gamma} & -e^{\delta} & -e^{-\delta}
\end{pmatrix} \begin{pmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4
\end{pmatrix} = 0
\]

which reduces to

\[ \sinh \gamma \sinh \delta = 0 \]

3.14

Since \( \Lambda \) is always positive, equation 3.14 implies that \( \delta = n \pi \), \( n = 0, 1, 2, \cdots \)

Hence \( \Lambda = a + \frac{n^2 \pi^2}{a} \), it is minimum for \( n = 1 \), \( a = \pi \) (the case \( n = 0 \) being discarded). A question naturally arises here, would \( \lambda_1 \) change if we superpose
horizontal waves such that
\[
\theta = (F_1 \pm F_2)Z
\]
where
\[
F_1 = \sin a_1 x \sin a_2 y
\]
\[
F_2 = \sin b_1 x \sin b_2 y
\].

To answer this question we proceed to solve the problem along the same lines used for the single wave case we find:

\[
(\Delta_2 + a^2)F_1 = 0 , \quad a^2 = a_1^2 + a_2^2
\]

\[
(\Delta_2 + b^2)F_2 = 0 . \quad b^2 = b_1^2 + b_2^2
\]

proceeding with the solution we get

\[
\lambda_1^* = \frac{(a^2 \pm b^2 + \pi^2)^2}{a^2 \pm b^2}
\]

for \( a = \pi \), the single wave case we get:

\[
\lambda_1^* = \frac{(2\pi^2 \pm b^2)^2}{\pi^2 \pm b^2}
\]

which implies that for \( b \neq 0 \) \( \lambda_1 < \lambda_1^* \).

Hence the single wave assumption gives us the absolute minimum for \( \lambda_1^* \) which is found to be \( \lambda_1^* = 2\pi \).

The critical temperature gradient is:
\[ \beta = \frac{4\pi^2 \kappa \nu}{K \alpha d^2 g} \]

Equation 3.10 reduces to

\[ Z = C \sin \pi \xi, \]  
where \( C \) is an arbitrary constant, hence

\[ \theta = C \sin \frac{\pi n}{d} \sin \xi \sin \eta \]  
\[ w = \frac{1}{2} K \alpha C \sin \frac{\pi n}{d} \sin \xi \]  
\[ u = \frac{K \alpha d \xi C}{2\pi} \cos \frac{\pi n}{d} \sin \xi \sin \eta \]  
\[ v = \frac{K \alpha d \eta C}{2\pi} \cos \frac{\pi n}{d} \sin \xi \cos \eta. \]

In this solution, the layer is divided into cells

of length \( \frac{2\pi}{\lambda}, \) width \( \frac{2\pi}{m} \) and height \( d \) where

\[ \lambda^2 + m^2 = \frac{\pi^2}{d^2}. \]

Existing work on the convection of fluids in pores is on the convection of a fluid in an infinite tube of porous medium under temperature and density gradients, Wooding (1959, 1960, 1963).
CHAPTER IV

RESULTS FOR OTHER SLAB-TYPE PROBLEMS

4.1 Convection of a Self-heating Fluid in a Porous Medium

Consider an infinite layer of a homogeneous self-heating fluid in a porous medium with constant permeability bounded by two parallel planes $Z = 0$ and $Z = d$. The field equations describing the time independent convective motions in such a fluid are:

$$ 0 = - \nabla p - \frac{V}{K} \rho_c \ddot{u} - \rho g \ddot{e}, \quad (4.1) $$

$$ 0 = \text{div} \ddot{u}, \quad (4.2) $$

$$ \ddot{u} \cdot \nabla T = \kappa \Delta T + A, \quad (4.3) $$

$$ \rho = \rho_c (1 - \alpha T). \quad (4.4) $$

where $A$ is a constant heat source density and all other symbols are defined in section 1.1. Also the restrictions on the flow given in 1.1 hold here. Assume that the lower boundary is a perfect insulator and the upper one is a perfect conductor with $T = 0$. The boundary conditions are:

$$ \text{at } Z = 0 \quad \frac{\partial T}{\partial Z} = 0, \quad \ddot{u} = 0 \quad (4.5) $$

$$ Z = d \quad T = 0, \quad \ddot{u} = 0 $$

For the equilibrium state (i.e. $\ddot{u} = 0$, $T = T(z)$),
equations 4.1-4.5 admit the following solution set:

\[
\begin{align*}
\tilde{u}_0 &= 0 \\
T_0 &= \frac{A}{2\kappa} (d^2-z^2) \\
\rho_0 &= \rho_c (1-\alpha T_0) \\
\frac{\partial \rho_0}{\partial z} &= -g \rho_0 .
\end{align*}
\]

Introduce small perturbations on the equilibrium state such that

\[
\tilde{u} = \tilde{u}_0 + \tilde{u}, \quad T = T_0 + \Theta, \quad p = p_0 + \Pi
\]

where quadratic quantities in \( \tilde{u}, \Theta \) and \( \Pi \) can be neglected. Substitute 4.6-4.7 into 4.1-4.4 and simplify, we obtain:

\[
0 = -\nabla \Pi - \frac{\nu \rho_c}{\kappa} \tilde{u} + \rho_c g \alpha \Theta ,
\]

\[-\frac{A}{\kappa} zw = \kappa \Delta \Theta ,
\]

\[\text{div} \tilde{u} = 0 .\]

We introduce non-dimensional independent and dependent variables defined by the following transformations:

\[
(x,y,z) \rightarrow d(x_1,x_2,\zeta),
\]

\[
\tilde{u} \rightarrow \frac{K}{d} \tilde{u},
\]

\[
\Theta \rightarrow \frac{Ad^2}{\kappa} \Theta ,
\]

\[
\Pi \rightarrow \frac{\nu \rho_c \kappa}{K} \Pi .
\]
The non-dimensional field equations are:

\[ 0 = - \nabla \Pi - \tilde{\omega} + \lambda \theta , \quad 4.8 \]
\[ -\zeta \omega = \Delta \theta , \quad 4.9 \]
\[ \text{div} \, \tilde{\omega} = 0 \quad 4.10 \]

where \( \lambda = \frac{g \alpha A K d^3}{\nu \kappa^2} \).

Combining 4.10 and 4.8 we get:

\[ 0 = - \Delta \Pi + \lambda \frac{\partial \theta}{\partial \zeta} \quad 4.11 \]

Next we eliminate \( \Pi \) between 4.11 and third component of 4.8 obtaining

\[ 0 = - \Delta w + \lambda \frac{\partial^2 \theta}{\partial \zeta^2} \quad 4.12 \]

The equations of the convection problem of a self-heating fluid in porous media are equations 4.9 and 4.12 with the boundary conditions

\[ \begin{aligned}
\zeta &= 0 \, , \quad w = 0 \, , \quad \frac{\partial \theta}{\partial \zeta} = 0 \\
\zeta &= 1 \, , \quad w = 0 \quad \theta = 0 ,
\end{aligned} \quad 4.13 \]

Assume that \( w \) and \( \theta \) are separable and of the form

\[ \begin{aligned}
\theta &= F(x_1, x_2) \, \Theta(\zeta) \\
w &= F(x_1, x_2) \, W(\zeta)
\end{aligned} \quad 4.14 \]

Where \( F \) satisfies the equation \((\Delta_2 + a^2) \, F = 0\) and the
nature of \( a \) is explained below. Substituting equation 4.14 in equations 4.9 and 4.12 and simplifying we get:

\[
(D^2-a^2)W = -\lambda a^2\Theta, \quad 4.15
\]
\[
(D^2-a^2)\Theta = -\zeta W, \quad D = \frac{d}{d\zeta}. \quad 4.16
\]

The assumption \((\Delta +a^2)F=0\) is equivalent to taking the 2-dimensional Fourier transform of equations 4.9-4.12 where \( a^2=y_1^2+y_2^2, y_i \) are the transform coordinates. Physically this is equivalent to analysing the disturbance into 2-dimensional periodic waves with wave number \( a \) which describes the horizontal scale of the convective motions, Appendix III. To see the positivity of \( \lambda \) take the inner product of \( \Theta \) with both sides of equation 4.16 and use equation 4.15 we get

\[
\lambda a^2\int_0^\prime \Theta(a^2-D^2)\Theta d\zeta = \int_0^\prime \zeta W(a^2-D^2)Wd\zeta
\]

integrate the above equation by parts and use the boundary conditions 4.13 we arrive at:

\[
\lambda a^2\int_0^\prime [(D^2+\Theta^2)\Theta]d\zeta = \int_0^\prime [\zeta(DW)^2+a^2W^2]d\zeta,
\]

it is obvious from the above equation that \( \lambda \) is always positive \((\Theta, W \neq 0)\).

The second variational principle introduced in Chapter II is equivalent to minimizing

\[
\int_0^\prime [\zeta(DW)^2 + a^2W^2]d\zeta
\]

for variations of \( W \) which preserve the consistency of
\[ \int_0^1 [(D\Theta)^2 + a^2 \Theta^2] d\zeta. \]

Introduce \( \lambda a^2 \) as a Lagrange multiplier, we can equally minimize

\[ J = \int_0^1 [\zeta(DW)^2 + a^2 W^2] d\zeta - \lambda a^2 \int_0^1 [(D\Theta)^2 + a^2 \Theta^2] d\zeta \]

or alternatively

\[ J = \int_0^1 \zeta W(a^2 - D^2) W - \lambda a^2 \int_0^1 \Theta(a^2 - D^2) \Theta d\zeta. \quad 4.17 \]

Let a trial function \( W \) be represented by a Fourier series

\[ W = \sum_{n=0}^{n} A_m \sin m\pi \zeta \]

where \( A_m \), the Fourier coefficients, serve as variational parameters, Chandrasekhar (1961, pp. 53).

Equation 4.16 becomes

\[ (D^2 - a^2) \Theta = -\zeta \sum_{n=0}^{n} A_m \sin m\pi \zeta, \quad 4.18 \]

the linearity of the above equation implies that \( \Theta \) can be represented by the sum

\[ \Theta = \sum_{n=0}^{n} A_m \Theta_m \quad 4.19 \]

where \( \Theta_m \) satisfy the equation

\[ (D^2 - a^2) \Theta_m = -\zeta \sin m\pi \zeta. \quad 4.20 \]
Equation 4.19 can actually be deduced from equation 4.18 by simply solving the latter subject to the boundary conditions 4.13. The general solution of equation 4.20 subject to the boundary conditions

\[ \zeta = 0, \quad D\Theta = 0, \quad \zeta = 1 \quad \Theta = 0, \]

is given by:

\[ \Theta_m = B_m \cos h a \zeta + C_m \sin m \pi \zeta \]

\[ + \frac{2m \cos m \pi \zeta}{(m^2 \pi^2 + a^2)^2} \cos h a \]

where

\[ B_m = \frac{2m \pi (-1)^m}{(m^2 \pi^2 + a^2)^2} \cos h a \]

\[ C_m = \frac{1}{(m^2 \pi^2 + a^2)} \]

We substitute \( W \) and \( \Theta \) in the right hand side of 4.17 and performing the integration we get

\[ J = \frac{1}{2} \sum_m \frac{A_m}{\gamma_m} - \lambda a^2 \sum_m \sum_n A_n (n/m)A_m, \quad \text{4.21} \]

where \( \gamma_m = \frac{1}{(m^2 \pi^2 + a^2)} \)

and \( (n/m) = \frac{2mn \pi^2 \gamma^2 m (-1)^{m+1}}{\cos h a} \) +

\[ \frac{1}{2} \left[ \frac{(-1)^{m-n-1}}{(m-n)^2 \pi^2} + \frac{(-1)^{n+m-1}}{(n+m)^2 \pi^2} \right] \gamma_m \]

if \( m \neq n \)
and

\[(n/m) = \frac{2m^2 \pi^2 (-1)^{m+1}}{\cos \alpha} \gamma_m + \frac{1}{4} \gamma_m\]

if \(m = n\).

Since we want to minimize equation 4.21 w.r.t. \(A_m\), we find \(\frac{\partial J}{\partial A_m}\) and set it equal to zero we get:

\[\sum_m \left( \frac{\delta_{mn}}{2a^2 \gamma_m \lambda} - (n/m) \right) A_m = 0, \quad 4.22\]

where \(\delta_{mn}\) is the familiar Kronecker delta.

Equation 4.22 represents an infinite set of linear homogeneous equations for \(A_m\). For a non-zero solution to exist the determinant of the system must vanish, hence

\[\left| \frac{\delta_{mn}}{2a^2 \gamma_m \lambda} - (n/m) \right| = 0. \quad 4.23\]

the roots of the characteristic equation 4.23 can be approximated by considering finite \(n \times n\) determinants \(n = 0, 1, 2, \cdots\). Then finding the minimum \(\lambda\) w.r.t. the wave number \(a^2\). Chandrasekhar (1961, pp. 55) has found that convergence is quite rapid for similar cases.

When \(m = n = 1\) we obtain the first order approximation

\[\frac{(\pi^2 + a^2)}{2a^2 \lambda} = \frac{8\pi^2 + (\pi^2 + a^2)^2 \cos \alpha}{4(\pi^2 + a^2)^4 \cos \alpha}\]
which implies:

\[ \lambda = \frac{2(\pi^2+a^2)^4 \cosh a}{a^2 [8\pi^2 +(\pi^2+a^2)^2 \cosh a]} \]

Numerical results for the second order approximation yield that \( \lambda \) is least when the wave number \( a = 2.1 \) and is

\[ \lambda \approx 500. \]

This implies that convection sets in for:

\[ A > 500 \frac{\nu k^2}{g\alpha k d^3} \]

where \( A \) is the internal heat source density.

Note that the second variational principle given in Chapter II has been used here. To see this we write equations 4.15-4.16 in the combined form:

\[ (D^2 - a^2)^2 \theta = \lambda a^2 \zeta \theta \]

which defines the eigenvalue problem for the physical problem on hand.

Hence equations 4.15 and 4.16 represents the splitting of the operator \( (D^2-a^2)^2 \) into two separate self-adjoint positive operator:

\((-D^2+a^2) \) with B.C. \( \theta = 0 \) at \( \zeta = 0, 1 \)

\((-D^2+a^2) \) with B.C. \( \frac{\partial \theta}{\partial \zeta} = 0 \) at \( \zeta = 0 \)

and \( \theta = 0 \) at \( \zeta = 1. \) Kato (1966, pp. 274).
4.2 Convection of a Fluid in Porous Media with Variable Permeability.

Consider an infinite horizontal layer of a porous medium with variable permeability saturated with a homogeneous incompressible fluid heated from below. We assume that the permeability is a function of the vertical variable only. The time independent convection equations are:

\[ 0 = -\nabla p - \frac{\nabla \rho}{K} \hat{u} - \rho g \hat{e} \]

\[ \text{div } \tilde{u} = 0 \]

\[ u \cdot \nabla T = \kappa \Delta T \]

\[ \rho = \rho_C (1 - \alpha T) . \]

The boundary conditions are:

- at \( z = 0 \) \( \tilde{u} = 0 \), \( T = A \), constant
- \( z = 1 \) \( \tilde{u} = 0 \), \( T = 0 \).

The perturbation equations, linearized about the equilibrium state solution, were found in non-dimensional form in section 1.3 and are given as equations 1.11-1.14. To reduce these equations into a simple eigenvalue problem we let \( q(\zeta) = \frac{1}{K(\zeta)} \) and write them in component form:

\[ 0 = -\frac{\partial \Pi}{\partial x_1} - q_u \quad 4.2.1 \]

\[ 0 = -\frac{\partial \Pi}{\partial x_2} - q_v \quad 4.2.2 \]
where

\[ \angle = \frac{\angle \alpha \beta d^2}{\nu \kappa} . \]

We assume that \( \Pi, \tilde{u} \) and \( \theta \) are separable and are of the form

\[
\Pi = F(x_1, x_2) G(\zeta)
\]

\[
\tilde{u} = F(x_1, x_2)(U(\zeta), V(\zeta), W(\zeta))
\]

\[
\theta = F(x_1, x_2) \Theta(\zeta),
\]

such that \( F(x_1, x_2) = \exp[-i(x_1 y_1 + x_2 y_2)] \), where \( y_1, y_2 \) are the components of the wave vector in the \( x_1 \) and \( x_2 \) directions respectively. Substitute 4.26 into 4.2.1-4.2.5:

\[ 0 = -iy_1 F G - q F U \] 4.2.7

\[ 0 = -iy_2 F G - q F V \] 4.2.8

\[ 0 = -F \frac{\partial G}{\partial \zeta} - q F W + \lambda F \Theta \] 4.2.9

\[ 0 = iy_1 F U + iy_2 F V + F \frac{\partial W}{\partial \zeta} \] 4.2.10

\[ -F W = -a^2 F \Theta + F \frac{\partial^2 \Theta}{\partial \zeta^2} \] 4.2.11

where \( a^2 = y_1^2 + y_2^2 \), \( y_1, y_2 \) are the transform variables.

Multiply equation 4.2.7 by \( iy_1 \) and 4.2.8 by \( iy_2 \) then add:
0 = a^2FG - q(i\, F_U + i\, F_V) \quad 4.2.12

By use of 4.2.10, equation 4.2.12 can be reduced to

0 = a^2FG + qF \frac{\partial W}{\partial \zeta} \quad 4.2.13

Since \( F(y_1, y_2) \neq 0 \), equations 4.2.9, 4.2.11 and 4.2.13 can be reduced to:

\[-(D(qDW) - a^2qW) = \lambda a^2 \theta \quad 4.2.14\]

\[-(D^2 - a^2) \theta = W \quad 4.2.15\]

where \( D = \frac{d}{d\zeta} \).

The boundary conditions are

\[ \theta, W = 0 \text{ at } \zeta = 0, 1. \quad 4.2.16\]

To prove the positivity of \( \lambda \) we take the inner-product of \( W \) with both sides of 4.2.14 and use 4.2.15 we get

\[ \int W[a^2qW - D(qDW)]d\zeta = \lambda a^2 \int_0^1 [a^2 - D^2] \theta d\zeta. \]

Integrating the above equation by parts and using the boundary conditions 4.2.16 we get:

\[ \int_0^1 q[(DW)^2 + a^2W^2]d\zeta = \lambda a^2 \int_0^1 [(D\theta)^2 + a^2\theta^2]d\zeta, \]

since \( q = \frac{1}{K} \) is always positive, \( \lambda \) is always positive \((\theta, W \neq 0)\). So far, we have not made any assumptions regarding the nature of the permeability function \( K \). We
will consider here 2 physical situations:

(1) the medium is composed of 2 layers of constant permeability $K_1$ and $K_2$

(2) the medium is composed of $n$ layers of constant permeabilities $K_1, \ldots, K_n$.

Physically, the first case describes the situation when the medium is a simple two-layer filter bed while the second describes the more general and complex case of an $n$-layer filter bed. The underlying assumptions here are that across any interphase between two adjacent regions, the pressure and the component of the velocity normal to the interphase are continuous.

When combined, equations 4.14-4.15 yield

$$q(D^2-a^2)\theta + DqD(D^2-a^2)\theta = \lambda a^2\theta$$

or

$$(D^2-a^2)\theta - \frac{DK}{K} D(D^2-a^2)\theta = \lambda a^2K\theta.$$ 4.2.17

Write 4.2.17 as

$$M\theta = \lambda N\theta$$ 4.2.18

where

$$M = (D^2-a^2)^2 - \frac{DK}{K} D(D^2-a^2)$$

and

$$N = a^2K.$$ 4.2.18

To find the minimum eigenvalue of 4.2.18, we shall
introduce a different variational principle. Let \( u \) be a trial function which satisfies all the smoothness and boundary conditions of the problem, and let \( J = \mu - \lambda \nu \), we want to minimize \( J \) in order to find a best approximation for \( \lambda \). Using least squares we have

\[
F(\lambda) = \int_0^L J^2 d\zeta + \min
\]

or

\[
\int_0^L (\mu - \lambda \nu)^2 d\zeta + \min
\]

which implies that \( \frac{dF}{d\lambda} = 0 \). Hence

\[
-\int_0^L 2(\mu - \lambda \nu) \nu d\zeta = 0
\]

or

\[
\lambda = \frac{\int_0^L \mu \nu d\zeta}{\int_0^L (\nu)^2 d\zeta}
\]

\[
= \int_0^L u(K(D^2-a^2)^2-DKD(D^2-a^2)) u d\zeta
\]

\[
= \frac{a^2 \int_0^L K^2 u^2 d\zeta}{a^2 \int_0^L K^2 u^2 d\zeta}
\]

Consider first the case where the filter bed has 2 layers with permeabilities \( K_1 \) and \( K_2 \) separated by the plane \( \zeta = h \). A suitable test function is

\[
u = \sin \pi \zeta.
\]

The permeability function can be written as \( K(\zeta) = K_1 H(h-\zeta) + K_2 H(\zeta-h) \) where \( 0 < h < 1 \), if \( h = 0, 1 \), then we are back to the case considered in Chapter III.
where $H$ is the Heaveside step function. For this case

\[ \lambda_1 \text{ can be approximated as:} \]

\[
\lambda_1 = \frac{(\pi^2 + a^2) C + (\pi^2 + a^2) C}{a^2 C^2}.
\]

where $C_1 = \frac{1}{2} \left( (K_2 - K_1) \left( \frac{1}{2\pi} \sin 2\pi h - h \right) + K_2 \right)$,

$C_2 = \frac{\pi}{2} (K_2 - K_1) \sin 2\pi h$,

$C_3 = \frac{1}{2} \left( (K_2 - K_1) \left( \frac{\sin 2\pi h}{2\pi} - h \right) + K_2 \right)$.

The smallest $\lambda_1$ over all wave numbers $a$ is:

\[ \lambda_1 = \frac{(2C + C_1) \sqrt{4C^2 + 4C^2 - 2C^2}}{2C \left[ (2C_1 - C) + \sqrt{4C^2 + C^2 - 2\pi^2 C_1} \right]} \]  \hspace{1cm} 4.2.19

when \( (\pi^2 + a^2) = \frac{(2C - C_1) + \sqrt{4C^2 + C^2}}{2C_1} \)  \hspace{1cm} 4.2.20

For the n-layered case the permeability function is given by

\[ K(\zeta) = \sum_{i=1}^{n} (K_i - K_{i-1}) H(\zeta_i - \zeta) \]

for $\zeta_{i-1} < \zeta < \zeta_i$,

where we define $K_0 = 0$ and $\zeta_0 = 0$. The minimum eigenvalue $\lambda_1$ and the wave number $a$ are given by equations 4.2.19 and 4.2.20 where we redefine $C_i$ to be:

\[ C_1 = \frac{1}{2} \sum_{i=1}^{n} (K_i - K_{i-1}) (\zeta_i - \zeta_{i-1}) - \frac{1}{\pi} \sin\pi(\zeta_i + \zeta_{i-1}) \cos\pi(\zeta_i - \zeta_{i-1}) \]
\[ C_2 = \frac{n}{2} \sum_{i=1}^{n} (K_i - K_{i-1}) \sin 2\pi \zeta_i \]

\[ C_3 = \frac{1}{2} \sum_{i=1}^{n} (K_i^2 - K_{i-1}^2) (\zeta_i - \zeta_{i-1} - \frac{1}{n} \sin \pi (\zeta_i + \zeta_{i-1}) \cos \pi (\zeta_i - \zeta_{i-1})) . \]

4.3 Convection of a Fluid in a Porous Medium with Radiation Boundary Conditions.

Consider the problem treated in section 4.1 in the absence of internal heat sources where the lower boundary is a perfect conductor and radiation boundary conditions are imposed at the upper boundary.

The field equations for time independent behaviour are:

\[ 0 = -\nabla p - \frac{\nu}{k} \rho c \ddot{u} - \rho g \dot{e} , \]

\[ \text{div } \ddot{u} = 0 , \]

\[ \ddot{u} \cdot \nabla T = \kappa \Delta T , \]

\[ \rho = \rho_c (1 - \alpha T) . \]

The boundary conditions are

\[ \text{at } z = 0 \quad \ddot{u} = 0 , \quad T = A \]

\[ \text{at } z = d \quad \ddot{u} = 0 \quad c \frac{\partial T}{\partial z} = T . \]
In a state of equilibrium \((\ddot{u} = 0, \ T = T(z))\) the above equations admit the solution set:

\[ T_0 = \frac{A}{c-d} z + A = -\beta z + A, \quad \beta = \frac{A}{d-c} \]

\[ \rho_0 = \rho_c (1 - \alpha T_0) \]

\[ \frac{\partial \rho_0}{\partial z} = -g \rho_0. \]

Introduce small perturbations on the equilibrium solution set and proceeding as outlined in section 1.3 we get the following linearized equations:

\[ 0 = -\nabla - \frac{\nu}{K} \rho_c \ddot{u} + \alpha g \theta \ddot{\theta}, \]

\[ \text{div} \ddot{u} = 0, \]

\[ -\beta \omega = \kappa \Delta \theta. \]

Using the transformations:

\[ (x, y, z) + d (x_1, y_2, \zeta) \]

\[ w + \frac{Kw}{d} \]

\[ \theta + \beta d \theta \]

\[ \Pi + \frac{\nu K}{K} \Pi \]

the above equations can be reduced to

\[ \Delta^2 \theta = -\lambda \Delta \theta \]

4.3.1

where \( \lambda = \frac{Kagd^2}{\nu K}. \)
Assuming $\theta$ is separable and of the same form given in section 4.2, equation 4.3.1 is reduced to

$$(D^2-a^2)^2\theta = \lambda a^2 \theta,$$

where $a$ is the wave number.

The boundary conditions are:

at $\zeta = 0 \quad \theta = 0$, \quad (D^2-a^2)\theta = 0$

at $\zeta = 1 \quad hD\theta = 0$, \quad (D^2-a^2)\theta = 0$, $h = \frac{c}{d}$.

Equation 4.3.2 admits a general solution of the form

$$\theta = A_1 \exp(\gamma \zeta) + A_2 \exp(-\gamma \zeta) + A_3 \exp(\delta \zeta) + A_4 \exp(-\delta \zeta),$$

where $\gamma^2 = a(a+\lambda)$, \quad $\gamma^2 = a(a-\lambda)$.

The boundary conditions give

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ e^\gamma & e^{-\gamma} & -e^\delta & e^{-\delta} \\ (h\gamma-1)e^\gamma & -(h\gamma+1)e^{-\gamma} & (h\delta-1)e^{-\delta} & -(h\delta+1)e^{-\delta} \end{vmatrix} = 0$$

which yields $h[\delta \coth \delta + \gamma \coth \gamma] - 2 = 0$. 4.3.3

The above equation can be written:

$$h[\delta \cos h \delta \sinh \gamma + \gamma \cosh \delta \sinh \delta] - 2 \sinh \delta \sinh \gamma = 0,$$

for $h = 0$ i.e. the upper boundary is a perfect conductor we get $\sinh \delta \sinh \gamma = 0$ which was obtained in Chapter III.

For particular values of $h$ the transcendental equation
4.3.3 can be solved numerically. This is done by trial and error in the following fashion: for a particular value of $a$ we find $\lambda_1$ by use of Newton's method, then repeat the process for a whole range of values for $a$. From the set of $\lambda_1$ values we obtain, we choose the smallest. This yields the critical temperature gradient which gives the stability criterion.

An alternative approach to this problem is by use of the method of spectral perturbation, Kato (1966). For this we expand $\lambda$ and $\theta$ in the following fashion

$$
\lambda = \lambda_0 + h\lambda_1 + h^2\lambda_2 + \cdots
$$

$$
\theta = \theta_0 + h\theta_1 + h^2\theta_2 + \cdots
$$

where $h$ is assumed to be small and $\lambda_0$ and $\theta_0$ are the solutions of the case treated in Chapter III. Substituting in equation 3.3.2 we get

$$
[(D^2-a^2)^2 - \lambda_0 a^2] \theta_0 = 0
$$

$$
[(D^2-a^2)^2 - \lambda_0 a^2] \theta_1 = a^2 \lambda_1 \theta_0
$$

3.3.4

The boundary conditions become:

at $\zeta = 0$ \hspace{1cm} $\theta_i = 0$ \hspace{1cm} $(D^2-a^2) \theta_i = 0$, \hspace{1cm} $i = 0,1,2,\cdots$

at $\zeta = 1$ \hspace{1cm} $\theta_0 = 0$ \hspace{1cm} $D\theta_i = \theta_{i+1}$ \hspace{1cm} $(D^2-a^2) \theta_i = 0$, $i=0,1,2,\cdots$

Since the operator $[(D^2-a^2)^2 - \lambda_0 a^2]$ is self adjoint we have

$$
<\theta_0, [(D^2-a^2)^2 - \lambda_0 a^2] \theta_i> = 0, \hspace{1cm} i = 0,1,2,\cdots
$$
If we consider equation 3.3.4 with the corresponding boundary conditions for $\theta_1$, we have

$$[(D^2-a^2)^2-\lambda_0 a^2] \theta_1 = a^2 \lambda_0 \theta_1$$

at $\zeta = 0 \quad \theta_1 = 0 \quad (D^2-a^2) \theta_1 = 0$ \hspace{2cm} 3.3.5

at $\zeta = 1 \quad -\theta_1 = D\theta_0 \quad (D^2-a^2) \theta_1 = 0$.

Introduce the transformation

$$\theta_1 = w - (a^2 \zeta^3 + \frac{b-a^2}{b} \zeta) D\theta_0 \bigg|_1$$

and substitute for $\theta_1$ in 3.3.5 we get

$$(D^2-a^2)^2 w - \lambda_0 a^2 w = - (\lambda_0 a^2 - a^4) (a^2 \zeta^3 + \frac{b-a^2}{b} \zeta) D\theta_0 \bigg|_1 + a^2 \lambda_0 \theta_1$$

$$- 2a^4 \zeta D\theta_0 \bigg|_1.$$

Taking the inner product of $\theta_0$ with the above equations and recalling that $\theta_0 = \sin \pi \zeta$, $\lambda_0 = \frac{(a^2+\pi^2)^2}{a^2}$ we get

$$\lambda_1 \approx \frac{2(2a^2+\pi^2)(\pi^2-a^2)-2a^4}{a^2}$$

thus

$$\lambda \approx \lambda_1 + h\lambda_1$$

$$\approx \frac{(a^2+\pi^2)^2 - 2h[(2a^2+\pi^2)(\pi^2 a^2)-a^4]}{a^2}$$

$$\approx 4\pi^2 + 2h\pi^2 \quad \text{when} \quad a \approx \pi.$$
CHAPTER V

RESULTS FOR FINITE BODIES OF FLUID

5.1 Convection in a Rectangular Body of Fluid in a Porous Medium.

Consider a region of porous medium with constant permeability bounded by the planes

\[ x = 0 , \quad x = a \]
\[ y = 0 , \quad y = b \]
\[ z = 0 , \quad z = c , \]

and filled with a homogeneous incompressible fluid. Assume that the fluid is heated at \( z = 0 \) by a constant source \( A \) and the boundary \( z = c \) is a perfect conductor while all other sides are perfect insulators. The equilibrium state solution set is:

\[ T_0(z) = \beta(c-z), \quad \beta = \frac{A}{d}, \quad \rho_0 = \rho_c(1-\alpha T_0), \quad \frac{\partial \rho_0}{\partial z} = -g \rho_0, \]

and the perturbation equation is equation 3.7 which can be written:

\[ \Delta^2 \theta = -\frac{K a g \beta}{\nu k} \Delta \theta . \]

5.1

The boundary conditions are:
Based on the above solution we can assume a Fourier type series solution to 5.1 as:

\[ \theta(x, y, z) = \cos \frac{n\pi x}{2a} \cos \frac{m\pi y}{2b} \sin \frac{l\pi z}{c} \quad \text{5.3} \]

A solution to 5.1 which satisfies the boundary conditions 5.2 is

\[ \theta(x, y, z) = \cos \frac{n\pi x}{2a} \cos \frac{m\pi y}{2b} \sin \frac{l\pi z}{c} \quad \text{5.3} \]

Inserted in 5.1 we obtain the dispersion equation

\[ 4\pi^2 (k^2 + \frac{C^2}{4ab} \gamma^2_{mn})^2 = \lambda \gamma^2_{mn} \]

where \( m, n, l \) are positive integers and

\[ \gamma^2_{mn} = \frac{n^2 b}{a} + \frac{m^2 a}{b} \]

and

\[ \lambda = \frac{K\alpha c^4}{\nu k ab} \]

Based on the above solution we can assume a Fourier type series solution to 5.1 as:

\[ \theta(x, y, z) = \sum A_{mnl} \cos \frac{n\pi x}{2a} \cos \frac{m\pi y}{2b} \sin \frac{l\pi z}{c} \]

The standard procedure used in chapters III and IV to obtain the minimum eigenvalue \( \lambda \) with respect to all wave numbers \( \gamma \) fails here. But we can on physical grounds approximate the minimum eigenvalue by assuming that the above series has a dominant term characterized by the indices \((m, n, l)\). This reduces the dispersion equation to
\[ \lambda = \frac{4\pi^2 (1 + \frac{c^2}{4ab} \gamma_{mn}^2)}{\gamma_{mn}^2} \]

which we minimize with respect to \( \gamma \). For practical reasons we consider only the case when \( a = b \), we have then:

\[ \lambda = \frac{4\pi^2 (1 + \frac{c^2}{4a^2} \gamma_{mn}^2)}{\gamma_{mn}^2} \]

where \( \gamma_{mn}^2 = m^2 + n^2 \).

Minimizing the above expression for \( \gamma \) we get \( \gamma_{mn} = \frac{2a}{c} \) for a minimum, i.e. \( \gamma_{mn}^2 = \frac{4a^2}{c^2} = m^2 + n^2 \).

We can easily find \( m, n \) as the integers which yield the closest approximation to \( 4 \frac{a^2}{c^2} \) such that

\[ 4 \frac{a^2}{c^2} \approx m^2 + n^2. \]

Note that the above procedure is equivalent to using the first variational principle given in Chapter II with the trial function given by 5.3 with \( \ell = 1 \). To see this we express \( \lambda \) as the ratio of two positive integrals (Chapter III);
\[
\lambda = \frac{\iiint_\Omega \Delta^2 \theta \, dx \, dy \, dz}{\iiint_\Omega \Delta \theta \, dx \, dy \, dz}
\]

\[
\begin{align*}
\lambda & = \frac{4\pi^2 (1 + \frac{c^2}{4ab} \gamma_{mn}^2) \iiint_\Omega \left[ \cos \frac{n\pi x}{2a} \cos \frac{m\pi y}{2b} \sin \frac{\pi z}{c} \right]^2 \, dx \, dy \, dz}{\gamma_{mn}^2 \iiint_\Omega \left[ \cos \frac{n\pi x}{2a} \cos \frac{m\pi y}{2b} \sin \frac{\pi z}{c} \right]^2 \, dx \, dy \, dz} \\
& = \frac{4\pi^2 (1 + \frac{c^2}{4ab} \gamma_{mn}^2)^2}{\gamma_{mn}^2}.
\end{align*}
\]

The above expression for \( \lambda \) is identical with that obtained earlier.
CHAPTER VI

FINITE ELEMENT TECHNIQUE

The basic mathematical model used in the previous chapters involves the linearization of the field equations about the steady state solutions for each particular flow problem subject to whatever physical conditions we have. This method is due to Rayleigh (Chandrasekhar, 1961) and has been the basis for much of the existing work on hydrodynamic stability. Even though this method yields criteria for stability in terms of critical values of relevant physical quantities (i.e. temperature, density, viscosity gradients, etc.), it fails to yield any information about the fields when convection takes place.

Simpler method developed by Bodvarsson (classnotes 1966) for convection problems yields in addition to an estimate of the critical temperature gradient on estimate of the field quantities during convection. This method is a simplification of the Rayleigh method obtained by assuming that there is no heat flow in the horizontal direction. Furthermore, the geometry of the problem is simplified by studying an individual convection cell. In its simplest version this method can be based on a model involving a vertical cylinder
of a porous medium saturated with a fluid. For simplicity we assume that the base of the cylinder is circular with unit area and height $d$. The wall of the cylinder is a perfect insulator and appropriate boundary conditions are imposed on the top and the bottom of the cylinder. A vertical pipe is placed along the wall and it is connected with the interior of the cylinder by holes at the top and at the base. The fluid contained in the cylinder can circulate through the pipe. The convection problem is now whether there will be an upflow of the fluid in the cylinder and return through the pipe. The pipe represents the fluid surrounding the convection cell and we assume therefore that a normal temperature field, e.g. a constant temperature gradient is maintained in the pipe. We will use this model to reexamine some of the cases treated in Chapters III and IV.

We start by building the general mathematical model. The time independent field equations are:

$$\nabla \rho + \frac{\mu}{K} \tilde{q} = 0$$  \hspace{1cm} (6.1)

$$\text{div} \tilde{q} = 0$$  \hspace{1cm} (6.2)

$$\text{sq} \nabla T = \kappa \Delta T$$  \hspace{1cm} (6.3)

where $s$ is the specific heat of the fluid and all other symbols are as defined before. We assume that $q, \rho, u, s$ and $K$ are all constant. Also we assume that there is no heat flow in the horizontal direction. Thus we can
combine 6.1 and 6.2 to get

\[ \nabla^2 \rho = 0 \quad 6.4 \]

and 6.3 reduces to

\[ \frac{\kappa}{\kappa} \frac{dT}{dz} = \kappa \frac{d^2 T}{dz^2} \quad 6.5 \]

Due to the thermal expansion of the fluid, the temperature distribution produces a total buoyancy driving the flow \( q \) of the amount

\[ H = \alpha \gamma \int_0^d T dz - \alpha \gamma \int_0^d T_0 dz \quad 6.6 \]

where \( T_0 \) is the steady state temperature distribution in the return pipe and \( \alpha \) is coefficient of thermal expansion and \( \gamma \) the specific weight of the fluid.

First we consider the simple case where the bottom of the cylinder is uniformly heated from below and the top is maintained at temperature \( T = 0 \). The boundary conditions are

at \( z = 0 \) \hspace{1cm} T = A, \hspace{0.5cm} \text{constant} \\

at \( z = d \) \hspace{1cm} T = 0 \hspace{0.5cm} \text{.} \\

Equation 6.5 has the solution

\[ T = A \left[ \frac{\exp \left( \frac{s q d}{\kappa} \right) - \exp \left( \frac{s q z}{\kappa} \right)}{\exp \left( \frac{s q d}{\kappa} \right) - 1} \right] . \]

Thus \( H \) is found to be

\[ H = \alpha \gamma Ad f(b) \quad 6.7 \]

where \( b = \frac{sqd}{\kappa} \)
and \[ f(b) = \frac{1}{1-e^{-b}} + \frac{1-e^{b}}{b} - \frac{1}{2} \].

From 6.1 we can find \( q \) as

\[ q = \frac{\rho KH}{\mu} \]

using 6.6 we get

\[ q = \frac{\rho \gamma K}{\mu} Af(b) , \]

multiplying the above equation by \( \frac{d}{K} \) we get

\[ b = Nf(b) \]

where

\[ N = \frac{\rho \gamma KAsd}{\kappa \mu} . \]

From the graph of \( f(b) \) we can conclude that 6.7 has solutions for \( N > 16 \). Hence the condition for convection is that

\[ A > \frac{16\kappa\mu}{\alpha \gamma K \rho} \]

which is similar to the results obtained in Chapter III.

Next we treat the case of the many layered medium. The assumptions given in Chapter III regarding this case still hold here, however, we emphasize further that the permeabilities of any 2 adjunct regions cannot be arbitrarily different to avoid a separation of the convective flow. We will treat the \( n \)-layered problem with the same physical conditions as the previous problem, i.e. the medium is heated from below and is kept at temperature \( T = 0 \) above. In order to make this problem manageable, we make the analogy between this physical
setting (i.e. n-layer of various permeabilities with a flow perpendicular to all interfaces) and the case of an electrical current moving through n resistors mounted in series. This analogie can lead us to the idea of lumping permeabilities in a fashion analogous to the electrical case. Thus we can write for the resultant permeability

\[ K_r \]

\[ \frac{1}{K_r} = \frac{\sum \frac{n L_i}{K_i}}{\sum \frac{n L_i}{1}} \]

where \( L_i, K_i \) are the individual thickness and permeability of the \( i^{th} \) layer. Proceeding exactly as the simple one layer case treated before we arrive at the condition for convection

\[ A > 16Kv \sum \frac{n L_i}{K_i} / \text{d}y (\sum \frac{n L_i}{1}) \text{sd} . \]

For the case of internally heated fluids, we use a more refined method developed by Lowell and Bodvarsson (to be published). Consider the model of a two-dimensional convection cell of wavelength \( \lambda \) depicted in Figure 1. The homogeneous, self-heated fluid is confined between rigid horizontal planes separated by a distance \( d \) with the upper surface at \( T = 0 \). The walls of the cell are assumed rigid and thermally insulated. It is further assumed that horizontal heat conduction can be neglected.
and that the flow takes place around rigid, thermally insulating cores. At the lower surface we employ the condition of zero heat flux, $\frac{dT}{dz} = 0$.

To arrive at the model we want to use, we assume that the convective flow is uniform about the core and the flow velocity is constant over the width $\tau$. The working model for calculating the temperature then becomes that shown in Figure 3. The cell has been cut along the dotted line (Figure 2), and the flow channel has been stretched out as a strip. The bottom of the channel has been folded into the plane $x = L/2$. Heat losses through the upper surface of the cell now take place through the one-dimensional heat transport equation which we have to solve is:

$$\frac{d^2T}{dx^2} - \frac{sg}{k} \frac{dT}{dx} = - \rho s A$$

the B.C. are

$$T(0) = T(L) = 0.$$  

This yields

$$T(x) = \frac{\rho s A a}{k q} \left[ x - L \left( \frac{\exp\left(\frac{ux}{a}\right) - 1}{\exp\left(\frac{uL}{a}\right) - 1} \right) \right].$$

The driving head is

$$H = \int_0^{L/2} T(x) dx - \int_{L/2}^L T(x) dx$$

Substituting for $T$ and proceeding as previously (Lowell and Bodvarsson) we obtain:

$$\lambda = \frac{guAsKd^3}{\sqrt{k^2}} > 200.$$
CHAPTER VII

APPLICATIONS IN THE EARTH SCIENCES

There is a number of important applications of hydrodynamic stability theory in the earth sciences. The theory of convective stability for viscous fluids has been applied quite extensively to the problem of convection in the earth's mantle (Knopoff, 1964).

The theory of convective stability in porous media has mainly two applications. First, in the case of geothermal systems and second in the study of convection of the molten interstitial phase of the earth's mantle mainly in the so-called low-velocity zone. Both of the above mentioned problems are receiving growing attention. Some of the work on geothermal systems has been carried out by Wooding (1959, 1960, 1963), who has applied the theory to geothermal systems in New Zealand. Geothermal systems in New Zealand are specially amenable to a treatment by this theory since the country rock there is composed mainly of porous sediments.

The results obtained in this work supplement in many ways Wooding's work on geomthermal systems. The imposing of a radiation condition (see Chapter IV above) at the top of the connecting layer makes it possible to take the
presence of a non-permeable overburden into account which act as a lid on the geothermal system. As a matter of fact, probably all geothermal systems in nature are covered by a thin, more or less non-permeable cap rock. Moreover, the treatment of finite dimension systems is of considerable interest.

The interstitial fluid in the asthenosphere of the earth's mantle is a self-heating fluid due to radioactive elements. There are strong indications (Stacy, 1969) that there are convective movements in this fluid. Since the fluid moves through narrow intercrystalline spaces, the movement is probably governed by Darcy's law and the above results with regard to convection in layers of self-heating fluids are probably applicable to convection in the asthenosphere.
BIBLIOGRAPHY


APPENDICES
APPENDIX I

Consider the problem treated in Chapter III. The convection equations are

\[ (-D^2+a^2)\theta = w \]
\[ (-D^2+a^2)w = \lambda a^2 \theta \]

with B.C. \( w = 0, \quad \theta = 0 \) at \( \zeta = 0,1 \).

Let \( \theta = \sum_{j} A_j \theta_j \) where \( \theta_j \) are orthonormal functions and

\[ A_j = \int_{0}^{1} \theta \theta_j d\zeta . \]

Thus

\[ 1 = \int_{0}^{1} \theta^2 d\zeta = \int_{0}^{1} \Sigma A_i \theta_i \Sigma A_j \theta_j d\zeta = \sum_{i=1}^{\infty} A_i^2 , \quad A-1 \]

and

\[ \int_{0}^{1} \theta (-D^2+a^2)wd\zeta = \int_{0}^{1} \lambda a^2 \theta^2 d\zeta = a^2 \int \Sigma \lambda_i A_i \theta_i \Sigma A_j \theta_j d\zeta \]

\[ = a^2 \sum_{i=1}^{\infty} \lambda_i A_i^2 \]

but

\[ \int_{0}^{1} \theta (-D^2+a^2)wd\zeta = \int_{0}^{1} w (-D^2+a^2) \theta d\zeta = \int_{0}^{1} w^2 d\zeta > 0 , \quad \text{thus} \]

\[ \int_{0}^{1} w^2 d\zeta = a^2 \sum_{i=1}^{\infty} \lambda_i A_i^2 . \]

From A-1 we get

\[ a^2 \lambda_1 = a^2 \sum_{i=1}^{\infty} \lambda_i A_i^2 , \]
thus \[ \int_0^1 w^2 d\zeta - a^2 \lambda_1 = a^2 \sum A_i^2 (\lambda_i - \lambda_1). \]

Since \( \lambda_i \geq \lambda_1 \) \( \forall i \) we have

\[ \int_0^1 w^2 d\zeta - a^2 \lambda_1 \geq 0 \]

\[ \implies a^2 \lambda_1 \leq \int_0^1 w^2 d\zeta, \] where equality holds for \( A_i = 0, \quad i > 1. \)

The inequality \( a^2 \lambda_1 \leq \int_0^1 w^2 d\zeta \) implies that \( \int_0^1 w^2 d\zeta \)

has a true minimum when \( w \) belongs to \( \lambda_1. \)
APPENDIX II

We refer back to the physical model described in Chapter I. A slab of porous medium saturated with a homogeneous incompressible fluid is heated from below. When the heating is adiabatic, the temperature profile is linear and any upward movement due to the buoyancy force which results is dissipated by conduction. In the analytical model this corresponds to the case when the roots $s_1, s_2$ are negative, i.e. $\beta_3 < 0$ or $\beta_3 > 0$ and small enough. However, as the heating is increased the upward movement (disturbance) becomes rapid and grows unchecked, i.e. $\beta_3 \gg 0$. Thus as the heating is increased the fluid goes through a period of equilibrium up to the time when the temperature gradient reaches a certain critical value beyond which convection motions set in.
APPENDIX III

Any disturbance applied to the system can be expanded in a Fourier series or integral with spectral values being the possible wave numbers. Whenever this expansion contains the particular wave number $a_c$ which corresponds to the temperature gradient at which equilibrium breaks down, this disturbance will induce convective movement in the fluid.
FIGURE 1

\[ T = 0 \]

\[ \lambda/2 \]

\[ T = T_o \text{ or } dT/dz = 0 \]
FIGURE 2

\[ T = 0 \]

\[ \lambda / 2 \]

\[ T = T_0 \text{ or } \frac{dT}{dz} = 0 \]