THE EXTENSION OF GIBBS' VECTOR PRODUCTS TO N-SPACE

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THE EXTENSION OF GIBBS' VECTOR PRODUCTS TO N-SPACE

1. INTRODUCTION

The purpose of this thesis is to extend some of Gibbs' well-known products from 3-space to n-space. The cross product and the double dot product of n-space have been defined by I. M. Hostetter (1, p. 192). These products will be reproduced here as they are an essential part of the proofs that will be given.

For simplicity we will let capital English letters represent vectors and small Greek letters represent scalars.

The single dot, scalar, or inner product of two vectors in n-space has been defined as

\[ A_1 \cdot A_2 = \left| A_1 \right| \left| A_2 \right| \cos(A_1, A_2) \]

where \( \left| A_1 \right| \) and \( \left| A_2 \right| \) are the magnitudes of \( A_1 \) and \( A_2 \), and the \( \cos(A_1, A_2) \) refers to the angle between the two vectors.

The double dot product of Gibbs' has been extended to mean (3, p. 96)

\[ A_1 A_2 \ldots A_n : B_1 B_2 \ldots B_m \]

\[ = A_1 \cdot B_1 A_2 \cdot B_2 \ldots A_n \cdot B_n B_{n+1} \ldots B_m \quad m \quad n. \]
This product obeys the distributive, associative and commutative laws.

In 3-space, the cross product of two vectors, \( A_1, A_2 \), is defined by

\[
(1.1) \quad 3(A_1 \times A_2) = |A_1| |A_2| \sin (A_1, A_2) N
\]

where \( N \) is a unit vector orthogonal to \( A_1 \) and \( A_2 \) and the superscript to the left of the parenthesis designates the vector space in which the cross product is taken.

In 2-space the product is obviously not a vector, since there is no vector perpendicular to \( A_1 \) and \( A_2 \), and so we define it simply as the scalar \((2, p. 198)\)

\[
(1.2) \quad 2(A_1 \times A_2) = |A_1| |A_2| \sin (A_1, A_2).
\]

From \((1.1)\) and \((1.2)\) it is evident that

\[
3(A_1 \times A_2) = A_3
\]

where

\[
1(A_3) = 2(A_1 \times A_2)
\]

where \(1(A_3)\) is the norm of \( A_3 \). We generalize to 4-space. Let \( A_3 \) and \( A_4 \) be any two independent vectors in the space complementary to the space of \( A_1 \) and \( A_2 \). That is, two vectors such that

\[
A_1 \cdot A_j = 0, \quad A_3 \times A_4 \neq 0, \quad (i = 1, 2; \quad j = 3, 4).
\]
Then the cross product in 4-space is defined as (2, p. 198)

\[ (1.3) \quad \epsilon(A_1 \times A_2) = A_3 A_4 - A_4 A_3 = \epsilon_{1213} A_1 A_3 A_4 \]

where

\[ \epsilon(A_3 \times A_4) = \epsilon(A_1 \times A_2), \quad (1 = 1, 2, 3, 4.) \]

Here repeated indices indicate summation on the indices, and \( \epsilon_{1213} \) has the usual connotation of the tensor analysis.

Now, if \( A_3, A_4 \) and \( A_5 \) are three vectors in 5-space satisfying the conditions

\[ A_i \cdot A_j = 0, \quad \epsilon(A_3 \times A_3, A_5) \neq 0, \]

\[ (i = 1, 2; j = 3, 4, 5.) \]

Then by definition

\[ \epsilon(A_1 \times A_2) = \epsilon_{121345} A_1 A_3 A_4 A_5, \]

\[ \epsilon(A_3 \times A_4, A_5) = \epsilon(A_1 \times A_2). \]

Likewise, since the cross product in 4-space has been defined by (1.3), we have

\[ \epsilon(A_1 \times A_2) = \epsilon_{1213456} A_1 A_3 \cdots A_6, \]
where
\[ k(A_{3}xA_{4}:A_{5}A_{6}) = 2(A_{1}xA_{2}). \]

And thus by the inductive process we can define the cross product in n-space by the equation

\[ n(A_{1}xA_{2}) = \varepsilon_{123\ldots n} A_{1}\ldots A_{n} \]

where
\[ n-2(A_{3}xA_{4}:A_{5}\ldots A_{n}) = 2(A_{1}xA_{2}), \]
\[ A_{i}.A_{j} = 0, \quad (i = 1, 2; \quad j = 3, \ldots, n). \]

We will find it convenient to adopt a determinant form for the right member of (1.4), that is

\[ \varepsilon_{123\ldots n} A_{1}\ldots A_{n} = \begin{vmatrix} A_{3} & A_{3} & \cdots & A_{3} \\ A_{4} & A_{4} & \cdots & A_{4} \\ A_{5} & A_{5} & \cdots & A_{5} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n} & A_{n} & \cdots & A_{n} \end{vmatrix} \]

In determinants of this form it is important to note that the order of the columns must be preserved, and that the repeating of rows will make the determinant vanish while the repeating of columns does not.
2. THE EXTENSION OF GIBBS' TRIPLE VECTOR PRODUCT

It is well known that Gibbs' triple vector product

\[3(A_1 \times A_2) \times A_3 = A_3 \cdot A_1 A_2 - A_3 \cdot A_2 A_1.\]

An obvious extension to n-space is the vector

\[n(A_1 \times A_2 : A_3 \ldots A_{n-1}) \times B_1 : B_2 \ldots B_{n-2}.\]

It is noted that (2.1) can be written in dyadic form

\[3(A_1 \times A_2) \times A_3 = A_3 \cdot \begin{vmatrix} A_1 & A_1 \\ A_2 & A_2 \end{vmatrix}\]

which suggests the following theorem.

THEOREM I: If \(A_1, A_2, \ldots, A_{n-1}, B_1\) is any set of non-null vectors in n-space, then

\[n(A_1 \times A_2 : A_3 \ldots A_{n-1}) \times B_1 = (-1)^{n+1} B_1 \cdot \begin{vmatrix} A_1 & A_1 & \ldots & A_1 \\ A_2 & A_2 & \ldots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-1} & \ldots & A_{n-1} \end{vmatrix}\]

We will first prove Theorem I is true when we choose the vectors \(A_1\) as the elements of a set of mutually orthogonal unit vectors, i.e., vectors \(e_1, e_2, \ldots, e_{n-1}\) such that
(2.3) \[ e_i \cdot e_j = \delta_{ij}, \ (i,j = 1, 2, \ldots, n). \]

That is, we will prove that

\[
(2.4) \quad n(e_{i1} e_{i2} e_{i3} \ldots e_{in-1}) e_{ik} = (-1)^{n+1} e_{ik}.
\]

We first assume the \( e_i \) in the parenthesis are different. Without losing any generality we may assume they are in the order \( e_1, e_2, \ldots, e_{n-1} \). It follows as a consequence of the definition of the cross product that

\[
\begin{vmatrix}
e_{i1} & e_{i1} & \cdots & e_{i1} \\
e_{i2} & e_{i2} & \cdots & e_{i2} \\
\vdots & \vdots & \ddots & \vdots \\
e_{in-1} & e_{in-1} & \cdots & e_{in-1}
\end{vmatrix}
\]

\[= (-1)^{n+1} e_{ik}.\]

If \( e_k = e_n \) both members of (2.4) are found to be zero. Likewise, if there is a repeated vector in the parenthesis of (2.4), the left member is zero by (2.3) and the right member vanishes, since two rows of the determinant in (2.4) are alike. The only other possible choice of unit vectors in (2.4) is that in which vectors in the parenthesis are different and the vector \( e_k = e_p \) where \( e_p \) is one of the vectors in the parenthesis. In this case we have
But this determinant is the right member of (2.4) if 
\( e_k = e_p \). Thus Theorem I holds if the \( A_i \) are mutually orthogonal unit vectors.

To prove the general theorem we write

\[ A_i = \langle^a_i e_a, \quad B = \langle^b e_b, \quad (a, b, i = 1, 2, \ldots, n-1) \]

Indicating the left member of (2.2) by \( \Gamma \), we have, by (2.4)

\[ \Gamma = \langle^{a_1}_1 e_{a_1} \rangle^{a_2}_2 \ldots \langle^{a_{n-1}}_{n-1} e_{a_{n-1}} \rangle^b (e_{a_1} e_{a_2} : e_{a_3} \ldots e_{a_{n-1}}) e_b \]

\[ = (-1)^{n+1} \langle^{a_1}_1 e_{a_1} \rangle^{a_2}_2 \ldots \langle^{a_{n-1}}_{n-1} e_{a_{n-1}} \rangle^b e_b. \]
As both the scalars and vectors are summed on the $a_i$, the vectors $e_i$ and the scalars $\lambda_{i_1}^{a_i}$ can be interchanged, yielding

$$\prod = (-1)^{n+1} \beta_{b_1 b_2 \ldots b_n} \begin{vmatrix} \lambda_1^{a_1} & \lambda_1^{a_2} & \cdots & \lambda_1^{a_n} \\ \lambda_2^{a_1} & \lambda_2^{a_2} & \cdots & \lambda_2^{a_n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n-1}^{a_1} & \lambda_{n-1}^{a_2} & \cdots & \lambda_{n-1}^{a_n} \\ \lambda_{n-1}^{a_1} & \lambda_{n-1}^{a_2} & \cdots & \lambda_{n-1}^{a_n} \end{vmatrix} e_{a_1} e_{a_2} \cdots e_{a_{n-1}}$$

And thus

$$(2.6) \quad \prod = (-1)^{n+1} \beta_{b_1 b_2 \ldots b_n} \epsilon^{i_1 \cdots i_{n-1}} \lambda_{i_1}^{a_1} \cdots \lambda_{i_{n-1}}^{a_{n-1}} e_{a_1} e_{a_2} \cdots e_{a_{n-1}} = (-1)^{n+1} \beta_{b_1 b_2 \ldots b_n} \epsilon^{i_1 \cdots i_{n-1}} A_{i_1} \cdots A_{i_{n-1}}$$

$(i = 1, \ldots, n)$

which was to be proved.
3. MULTIPLE INNER PRODUCTS OF OUTER PRODUCTS

The well-known products of Gibbs

\[ 3(A_1A_2A_3)(B_1B_2B_3) = |A_iB_j| \quad (1, j = 1, 2, 3) \]

and

\[ 3(A_1A_2)(B_1B_2) = |A_iB_j| \quad (1, j = 1, 2) \]

may be generalized for n-space. We will prove the following theorem.

**THEOREM II.** Let \( A_1, A_2, \ldots, A_p \) and \( B_1, B_2, \ldots, B_p \) be any two sets of \( p \) vectors in the same n-space. \( p \leq n \). Then

\[ (3.1) \quad n(A_1A_2A_3\ldots A_p)(B_1B_2B_3\ldots B_p) = (n-p)! |A_iB_j|, \]

\[ (1, j = 1, 2, \ldots, p), \]

where if \( p = n \) it is understood that the double dot product indicates scalar multiplication of the two scalar products.

First we prove a lemma.

**Lemma:** If \( A_1, A_2, \ldots, A_p \) are \( p \) vectors in n-space, then

\[ (3.2) \quad n(A_1A_2A_3\ldots A_p) = \varepsilon_12\ldots p_{p+1}\ldots n A_{i_{p+1}} \ldots A_{i_n} \]

where

\[ p(A_1A_2A_3\ldots A_p) = n-p(A_{p+1}A_{p+2}A_{p+3}\ldots A_n) \]
and $A_{p+3}, A_{p+4}, \ldots, A_n$ can be any set of mutually orthogonal unit vectors in the n-p space independent of $A_1, A_2, \ldots, A_{p+2}$.

Proof: From the definition of a cross-product in 2-space, it is evident that we can so choose $A_{p+1}$ and $A_{p+2}$ that

$$(3.3) \quad 2(A_{p+1} \times A_{p+2}) = p(A_1 \times A_2; A_3 \ldots A_p)$$

Our problem now reduces to proving

$$(3.4) \quad 2(A_{p+1} \times A_{p+2}) = n-p(A_{p+1} \times A_{p+2}; e_{p+3} \ldots e_n) = \Delta$$

for any selection of the $e_{p+3}, e_{p+4}, \ldots, e_n$ which satisfy the conditions

$$A_1 \cdot e_\kappa = 0, \quad (i = 1, 2, \ldots, p+2; \quad \kappa = p+3, \ldots, n).$$

With the use of (1.2) the right member of (3.4) can be expanded in the form

$$(3.5) \quad \Delta = \epsilon_{p+1, p+2, 1_{p+3} \ldots 1_{n}} A_1 p_{p+3} A_1 p_{p+4} \ldots A_1 e_{p+3} e_{p+4} \ldots e_n$$

where

$$2(A_{p+1} \times A_{p+2}) = n-p-2(A_{p+3} \times A_{p+4}; A_{p+5} \ldots A_n);$$

$$A_{p+\kappa} \cdot A_{p+\beta} = 0 \quad \kappa = 1, 2; \quad \beta = 3, \ldots, n$$
Since the $e_{p+3}, e_{p+4}, \ldots, e_n$ are a basis for the vector space of $A_{p+3} \ldots A_n$ we can let

\[(3.6) \quad A_i = \beta_i^j e^j, \quad (i,j = p+3, p+4, \ldots, n),\]

which when substituted in (3.5) gives

\[\Delta = \epsilon_{p+1, p+2, i_{p+3} \ldots i_n} \beta_{i_{p+3} j_{p+3}} \ldots \beta_{i_n j_n} e_{p+3} \ldots e_n \]

Or, since $e_1 \cdot e^j = \delta_{ij}$,

\[\Delta = |\beta_{ij}|\]

Now, from (3.6), (3.5) and (3.3) we have

\[|\beta_{ij}| = n-p-2(A_{p+3}x_{A_{p+4}}A_{p+5} \ldots A_n) = 2(A_{p+1}x_{A_{p+2}}) = p(A_1 x_{A_2} A_3 \ldots A_p),\]

which completes the proof of the lemma.

With the use of the lemma we may write the left member of (3.1) in the following manner

\[(3.7) \quad \Theta \equiv \epsilon_{1,2 \ldots p_{p+1} \ldots i_n} A_{p+1} A_{p+2} \ldots A_n : B_1 B_2 \ldots B_p\]

where
\[ P(A_1 x A_2; A_3 \ldots A_p) = n-p(A_{p+1} x A_{p+2}^2: A_{p+3} \ldots A_n) \]

Now we may change the cross in a scalar product to any position we wish and not change the value of the product. If the factors are interchanged, the sign of the product alone is changed. The sign of the new product will be determined by the number of inversions in pairs necessary to restore the vectors to the original order. An even number of inversions does not change the sign, while an odd number does change the sign (2, p. 195). Thus in (3.7) we may move the cross from between \( B_1 \) and \( B_2 \) to between \( A_{p+1} \) and \( A_{p+2} \) in each polyad of the polyadic and rearrange each polyad into the natural order, yielding

\[ \Theta = (n-p)! A_{p+1} x A_{p+2}^2: A_{p+3}^3 A_{p+4} \ldots A_n : B_1 B_2 \ldots B_p, \]

where the vectors in the expansion of \( A_{p+1} x A_{p+2}^2: A_{p+3}^3 \ldots A_n \) are in the space of \( A_1, A_2, \ldots, A_p \). From the lemma we can now write

\[ \Theta = (n-p)! \epsilon_{i_1 \ldots i_{p+1}} \ldots n^{A_1} \ldots A_i^p : B_1 B_2 \ldots B_p, \]

\[ P(A_1 x A_2; A_3 \ldots A_p) = n-p(A_{p+1} x A_{p+2}^2: A_{p+3} \ldots A_n) \]

or
\[\Theta = (n-p) \left| A_i \cdot B_j \right| \quad 1, j = 1, 2, \ldots, p\]

We have need of another approach when \( p = n \). We can assume that either the \( A_i \) or the \( B_j \) are independent, say the \( B_j \). Then there exists a set of reciprocal vectors (2, p. 195) \( B^j \) such that \( B_i \cdot B^j = \delta_i^j \). Then we can write

\[A_1 = A_1 \cdot B_j B^j \quad (1, j = 1, 2, \ldots, n)\]

Multiplying the vectors of the left members and those of the right members we obtain

\[A_1 x A_2 : A_3 \ldots A_n = A_1 \cdot B_{j_1} A_2 \cdot B_{j_2} \ldots A_n \cdot B_{j_n} (B_1 x B^2 : B^3 \ldots B^n)\]

\[\ldots B_j^n = \left| A_1 \cdot B_j \right| (B_1 x B^2 : B^3 \ldots B^n)\]

Since the products \( B_1 x B^2 : B^3 \ldots B^n \) and \( B_1 x B_2 : B_3 \ldots B_n \) are reciprocals (2, p. 198) it follows that

\[ (3.7) \quad (A_1 x A_2 : A_3 \ldots A_n) : (B_1 x B_2 : B_3 \ldots B_n) = \left| A_1 \cdot B_j \right| , \]

\[(1, j = 1, 2, \ldots, n),\]

which completes the proof.
4. THE DIVERGENCE OF A SKEW-SYMMETRIC POLYADIC

Gibbs has shown (1, p. 297) that for any skew-symmetric dyadic $E$ in 3-space that
\[
\nabla \cdot E = \frac{-\nabla \times \Phi_v}{2}
\]
where $\Phi_v$ is the vector of $\Phi$. We shall generalize this to show that

**THEOREM III:** The divergence $\nabla \cdot \Phi$ of any completely skew-symmetric polyadic $\Phi$ of class $n-1$ can be written
\[
\nabla \cdot \Phi = (-1)^n \frac{\nabla \times \Phi_v}{(n-1)!},
\]
where $\Phi_v$ is the vector formed by placing the cross between the first and second files of $\Phi$ and the double dot between the second and third files.

**Proof:** Since $\Phi$ is completely skew we can write
\[
(4.1) \quad \Phi = \epsilon_{i_1 \cdots 1_{n-1}} A_{i_1 \cdots A_{i_{n-1}}} \quad i = 1, 2, \ldots, n-1.
\]
We wish to show that
\[
\nabla \cdot \Phi = (-1)^n \frac{\nabla \times \Phi_v}{(n-1)!}
\]
where
(4.2) $\Phi_v = \varepsilon_{i_1 \ldots i_{n-1}} A_{i_1} x A_{i_2} : A_{i_3} \ldots A_{i_{n-1}}$

$= (n-1)! \ A_{i_1} x A_{i_2} : A_{i_3} \ldots A_{i_{n-1}}$

From (4.1) we can write

(4.3) $\nabla \cdot \Phi = \nabla \cdot (\varepsilon_{i_1 \ldots i_{n-1}} A_{i_1} A_{i_2} \ldots A_{i_{n-1}}),$

$(i = 1, 2, \ldots, n-1),$

where $\nabla$ operates successively on the $A_i$.

Let

$\nabla A_i = B_j C_i^j, \quad (i, j = 1, 2, \ldots, n-1).$

Then we have

(4.4) $\nabla \cdot A_i = B_j C_i^j$ and $\nabla x A_i = B_j x C_i^j$

We may now write (4.3) as

$$\nabla \cdot \Phi = B_j \cdot \begin{vmatrix} C_1^j & C_1^j & \ldots & C_1^j \\ A_2 & A_2 & \ldots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-1} & \ldots & A_{n-1} \end{vmatrix} + B_j \cdot \begin{vmatrix} A_1 & A_1 & \ldots & A_1 \\ C_2 & C_2 & \ldots & C_2 \\ A_3 & A_3 & \ldots & A_3 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-1} & \ldots & A_{n-1} \end{vmatrix} + \ldots$$
By Theorem I

\[ \nabla \cdot \Phi = (-1)^{n+1}(c_1^j x A_2 : A_3 \cdots A_{n-1}) x B_j \]

\[ + (-1)^{n+1}(A_1 + c_2^j : A_3 \cdots A_{n-1}) x B_j + \cdots \]

\[ + (-1)^{n+1}(A_1 x A_2 : A_3 \cdots A_{n-2} c_{n-1}^j) x B_j. \]

Hence

\[ \nabla \cdot \Phi = (-1)^n B_j x (c_1^j x A_2 : A_3 \cdots A_{n-1}) \]

\[ + (-1)^n B_j x (A_1 x c_2^j : A_3 \cdots A_{n-1}) + \cdots \]

\[ + (-1)^n B_j x (A_1 x A_2 : A_3 \cdots A_{n-2} c_{n-1}^j). \]

By (4.4) and (4.2) we can now write

\[ \nabla \cdot \Phi = (-1)^n \nabla x (A_1 x A_2 : A_3 \cdots A_{n-1}) = (-1)^n \nabla x \frac{\Phi_y}{(n-1)!} \]

which was to be proved.
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