Fingered Flow in Two Dimensions
2. Predicting Finger Moisture Profile

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An exact integral solution for the moisture profile in growing fingers in sandy soil is derived from Richards' equation. The solution provides moisture content along a finger as a function of position and time and provides applicable results, including the calculation of the asymptotic matric potential of a growing finger and a method of obtaining the unsaturated conductivity in a single experiment. The solution is verified experimentally through comparison with measurements of matric potential and moisture content using high-speed tensiometers.

Introduction

Previous investigations of fingered flow have focused primarily on the initiation of the wetting front instability which leads to fingered flow [*Hill and Parlange*, 1972; *Philip*, 1975; *Parlange and Hill*, 1976; *Diment and Watson*, 1982]. The object of this study is to obtain an experimentally verified quantitative description of the physical structure of growing instabilities. The experimental aspect of this study requires rapid point measurements of matric potential, making use of the tensiometers introduced in the companion paper.

Observed Structure of Growing Instabilities

In experiments with continuous infiltration into a two-dimensional two-layer sand system, the tip velocity has been observed to be constant in a given finger [*Glass et al.*, 1989a]. In addition, Figures 3–6 of *Glass et al.* [1989b] strongly suggest that the moisture profile of a growing finger translates vertically with the tip velocity, as illustrated in Figure 1.

In recent studies in homogeneous soil, *Selker* [1991] measured finger growth continuously in time for five fingers in a single experiment under steady infiltration in 20–30 sand. The coefficients of determination for a fit of a straight line to a plot of finger tip position versus time were found to be between 0.999 and 0.994 for five fingers, supporting the assumption of a well-defined, constant tip velocity (Table 1). Figure 2 shows the growth history of a typical finger from the experiments of *Selker* [1991] (finger 2 in Table 1). The growing finger's moisture profile starts with a region of high moisture content, required for the finger to overcome the water entry pressure for the sand, and falls to a significantly lower moisture content moving up from the tip. Figure 2 confirms the observation that the moisture profile translates, without structural change, at the finger's tip velocity (Figure 1). Hence the vertical translation of the moisture profile

appears to be a reasonable description of the progression of water content in a growing finger.

Derivation of the Governing Equation

We start with Richards' [1931] equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] - \frac{\partial K(\theta)}{\partial z} \tag{1}$$

Here θ is the volumetric moisture content, $D(\theta)$ is the soil water diffusivity, z is the vertical distance from the elevation at which instability develops (positive downward), $K(\theta)$ is the hydraulic conductivity, and t is the time since the tip of the instability became apparent (i.e., $z > 0$). Analytical solutions of (1) have been difficult to obtain since both diffusivity and conductivity are strongly nonlinear functions of moisture content.

On the basis of the observed structure in growing fingers, we seek a similarity solution for (1) of the form $\theta(z - vt)$, where θ is taken to be the average moisture content in a horizontal section of the finger and v is the constant velocity of flow at all points in the finger. Defining $\eta = z - vt$, (1) becomes an ordinary differential equation

$$\frac{d}{d\eta}\left[D(\theta)\frac{d\theta}{d\eta}\right] - \frac{dK(\theta)}{d\eta} = -v\frac{d\theta}{d\eta} \tag{2}$$

Integrating (2) once yields

$$-v\theta = D(\theta)\frac{d\theta}{d\eta} - K(\theta) + C \tag{3}$$

where C is a constant of integration. The constant is evaluated by imposing the boundary conditions for infiltration into dry soil, so that $\eta = \infty$, $\theta = 0$, $K = 0$, and $d\theta/d\eta = 0$, which yield $C = 0$. Equation (3) reduces to

$$K(\theta) - D(\theta)\frac{d\theta}{d\eta} = \theta v \tag{4}$$

For any value of η, $\theta(\eta)v$ is the flux at η, which we refer to as $q(\eta)$. It is useful to observe that by employing the definition of diffusivity, $Kdh = Dd\theta$, (4) may be rewritten
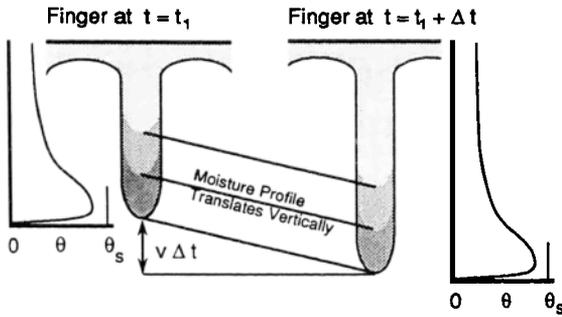
Fig. 1. Sketch of observed translation during finger growth. The tip of the finger, in which the moisture content is at the water entry value, is followed by a draining tail portion. The vertical moisture profile translates with a constant velocity.

$$\frac{K(\theta)}{\theta} = \frac{v}{1 - (dh/d\eta)} \tag{5}$$

which gives a relationship for conductivity as a function of the gradient of matric potential and is employed later to estimate K/θ.

While the velocity of the growing finger is constant, the moisture content along the length of the finger is not (Figure 1). The flux into the finger $q(0, t)$ is $\theta_0 v$, where θ_0 is the moisture content at the top of the finger:

$$q(z, t) = q(0, t) \frac{\theta}{\theta_0} \tag{6}$$

In the past [e.g., *Glass et al.*, 1989a] the system flux and finger flux have been taken as constant. Although this is true asymptotically, for a constant v the flux in each finger must decrease initially with decreasing moisture content in the upper region. Hence we allow for the flux into each finger to be a function of time. Substituting for η in (4) and integrating over θ yields

$$vt - z = \int_{\theta(z,t)}^{\theta_{we}} \frac{D(\theta)\, d\theta}{K(\theta) - v\theta} \tag{7}$$

or, in terms of diffusivity, integrating over pressure,

$$vt - z = \int_{h(z,t)}^{h_{we}} \frac{dh}{1 - v\theta/K} \tag{8}$$

where $\theta(z, t)$ and $h(z, t)$ are the moisture content and pressure at the time and position of interest, and θ_{we} and h_{we}

TABLE 1. The Velocities of the Five Fingers Grown in 20–30 Sand

Finger	Velocity, cm s^{-1}	r^{2}*
1	0.314	0.998
2	0.369	0.999
3	0.441	0.999
4	0.381	0.994
5	0.404	0.997

Velocities reported by *Selker* [1991]. The consistency of the velocity is verified by calculating the r^2 value of the fit between a straight line and the time-versus-displacement data for each finger.
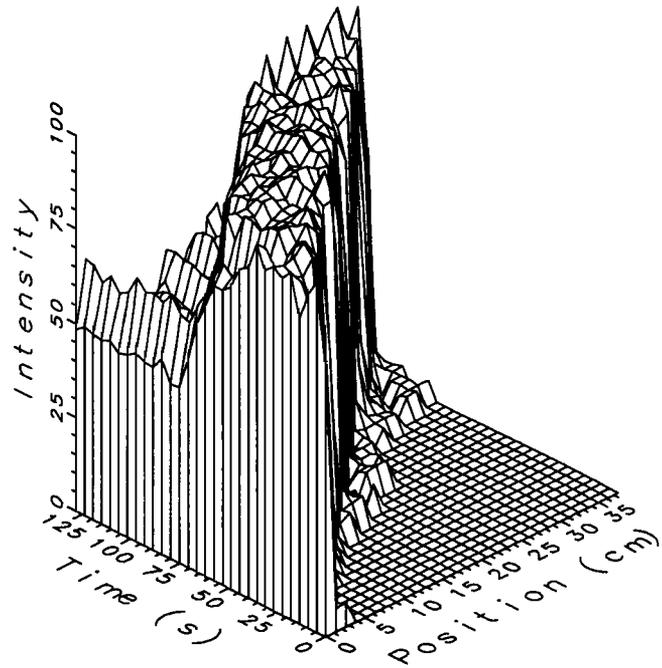*Fit of finger tip position versus time.

Fig. 2. The moisture distribution of a developing finger in 20–30 sand. Moisture content is represented here as normalized light intensity on the vertical axis, as a function of position along the center of the finger and time. These data were collected from a continuous video record of transmitted light.

are the moisture content and pressure at the advancing front of the finger (at $\eta = 0$), referred to as the water entry moisture content and water entry pressure of the soil. The variables θ_{we} and h_{we} may be mild functions of the initial moisture content and dynamic factors; however, in the fingered flow experiments carried out by *Baker and Hillel* [1990], the correlation between soil grain size and variation in measured h_{we} was found to have coefficient of determination of 0.9855 [*Baker and Hillel*, 1990; *Selker et al.*, 1991]. Given that $D(\theta)$, $K(h)$, and $K(\theta)$ are single-valued and
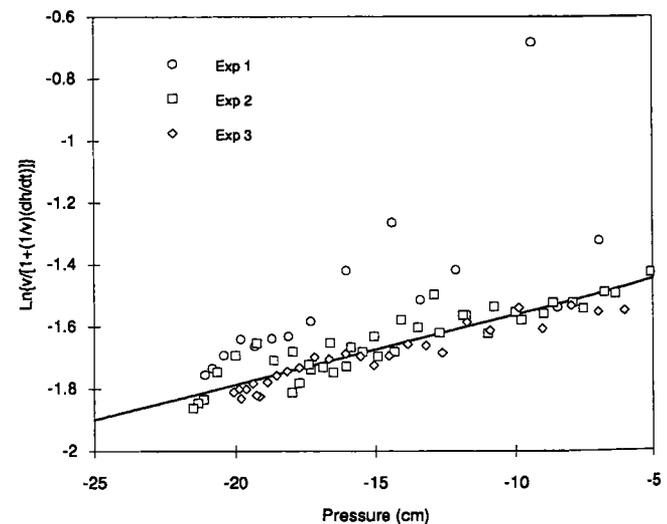

Fig. 3. Plot of $\ln\{v/[1 + (1/v)(\partial h/\partial t)]\}$ versus matric potential (pressure) for three experiments. The straight line fit to the data represents the least squares fit to the data from experiments 2 and 3 with a coefficient of determination of 0.84.

TABLE 2. Values for A and B' Obtained From Experiments 1, 2, and 3 From Linear Regression, With Coefficients of Determination

Experiment	$A \times 10^{-2}$	B'	r^2
1 (port 4)	4.00	-0.869	0.489
2 (port 4)	2.62	-1.30	0.933
2 (port 8)	2.08	-1.38	0.917
3 (port 4)	2.11	-1.33	0.841
2, 3, 4	2.21	-1.33	0.836

monotonic in the draining portion of the finger, observe that for any values of t and z there corresponds a unique value of $\theta(z, t)$ and $h(z, t)$ which satisfies (7) and (8). This solution is essentially the same as found by *Fleming et al.* [1984, equation (16)] when considering one-dimensional solutions to the infiltration equation but for different boundary conditions.

UNSATURATED CONDUCTIVITY

From the measurement of matric potential at a point in a finger growing with velocity v, (5) provides an expression for K/θ. The relationship between K/θ and h is revealed in a plot of the logarithm of measured values of $v/[1 + 1/v(\partial h/\partial t)]$ versus h, shown in Figure 3. The plot is well characterized by a linear relationship, suggesting that the K/θ ratio can be related to pressure by the equation

$$\frac{K}{\theta} = \frac{K_s}{\theta_s} \exp (Ah + B) \qquad h < h_{we} \qquad (9)$$

which, with (5), indicates that

$$\ln \left[v \Big/ \left(1 + \frac{1}{v} \frac{\partial h}{\partial t} \right) \right] = Ah + B' \qquad (10)$$

where

$$B' = B + \ln \left(\frac{K_s}{\theta_s} \right) \qquad (11)$$

Equation (9) is a slightly modified version of the widely employed exponential relationship between pressure and conductivity introduced by *Gardner* [1958]. Values of A and B' are estimated using least squares fit of a straight line to the experimentally measured values of the left-hand side (LHS) of (10). The calculation of the LHS of (10) requires the estimation of the derivative of pressure with time. This calculation is sensitive to minor fluctuations in data which are present in this experiment, as given by *Selker et al.* [this issue, Figure 11]. To provide reasonable estimates of this derivative, the pressure data were smoothed by calculation of the moving 30-sample average, (averaging data over 25 s). The data from experiment 4 were not appropriate for this calculation due to extreme variability as a consequence of the intermittent irrigation. The data taken in experiments 1, 2, and 3 are shown in Figure 3, with the parametric values from linear regressions given in Table 2. The values calculated from experiment 1 show considerable scatter and thus are not employed in the calculation of A and B', which are found to be 0.0221 and -1.335, respectively, with standard error of 0.0013 and 0.020 using the data from experiments 2

and 3. Although much of the scatter in Figure 3 arises from calculation of the LHS of (10), we also note that unsaturated conductivity is extremely sensitive to packing and thus is expected to show some variation in a single chamber and between experiments.

With (9), the integral given in (8) may be calculated directly, to obtain

$$vt - z = \frac{1}{A} \ln \left[\left(\frac{K_{we}}{\theta_{we}} - v \right) \Big/ \left(\frac{K}{\theta} - v \right) \right] \qquad (12)$$

which is the exact solution to Richards' equation for a growing finger in soils where (9) holds. The asymptotic relationship between moisture content and conductivity in a growing finger may be found by considering the solution to (7) or (12) as time goes to infinity. Considering a finite elevation z while t goes to infinity, the numerator in the integrand of (7) is finite, hence the denominator of the integrand must go to zero, while in (12), the argument of the logarithm must go to infinity. The asymptotic moisture content is then given by θ_a with

$$\frac{K_a}{\theta_a} = v \qquad (13)$$

which may be used to determine the fraction of soil participating in conduction of the incident flux. Using the results of *Glass et al.* [1990] to determine finger size, this may be used to obtain an expression for average finger spacing.

EXPERIMENTAL VERIFICATION

Equations (9) and (12) provide an explicit relationship between matric potential and time in a growing finger, which may be compared directly with the experimental measurements of matric potential in growing fingers introduced in the companion paper. The only parameter required to predict the pressure versus time relationship is the growth velocity of the finger v, which ranged from 0.149 to 0.20 cm s^{-1} in these experiments.

Figure 4 gives the results of tensiometer readings taken 1.6 and 6.7 cm below the sand surface in the center of the stimulated finger in experiment 2 (Figure 5). The structure predicted in the discussion of (7) is apparent: the pressure starts at the water entry pressure of the sand and decreases monotonically to an asymptotic value. The pressure measured in the induction zone follows the solutions very well until 500 s, at which time the pressure levels off. From this time on the pressure in the induction zone reflects the one-dimensional infiltration from the soil surface to the fingered region. Within the finger, measured at port 8, the pressure profile very closely follows the predicted form, departing from predicted pressure after 600 s, when the finger made contact with the lower boundary of the chamber. Infiltration was halted after 1300 s, at which time matric potential was observed to fall in both tensiometers. Figures 7 and 9 of the companion paper show the pressure profiles measured in two additional experiments, revealing the same pattern, and fit to the Richards' equation solution, found in experiment 2.

Substituting (13) into (9) gives an expression for the asymptotic matric potential h_a which is expected in a growing finger:
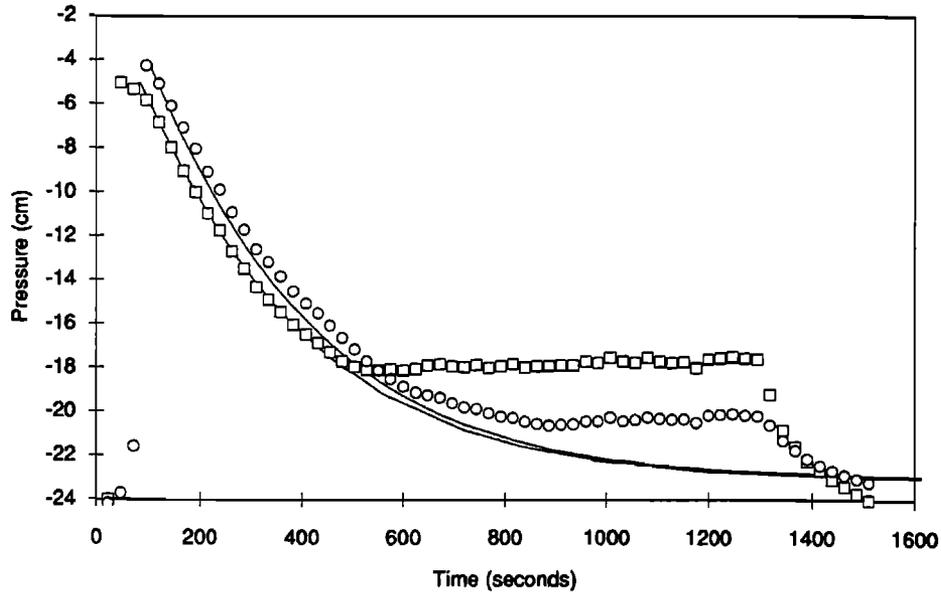
Fig. 4. Plot of matric potential versus time measured through ports 4 (open circles) and 8 (open boxes) in experiment 2 with potential predictions from (12) (solid line). At port 4 the flow becomes one dimensional at about 500 s, with pressure becoming constant. At port 8 the pressure drops more slowly than the prediction for $t > 600$ s, that is, when the finger reached the bottom of the chamber.
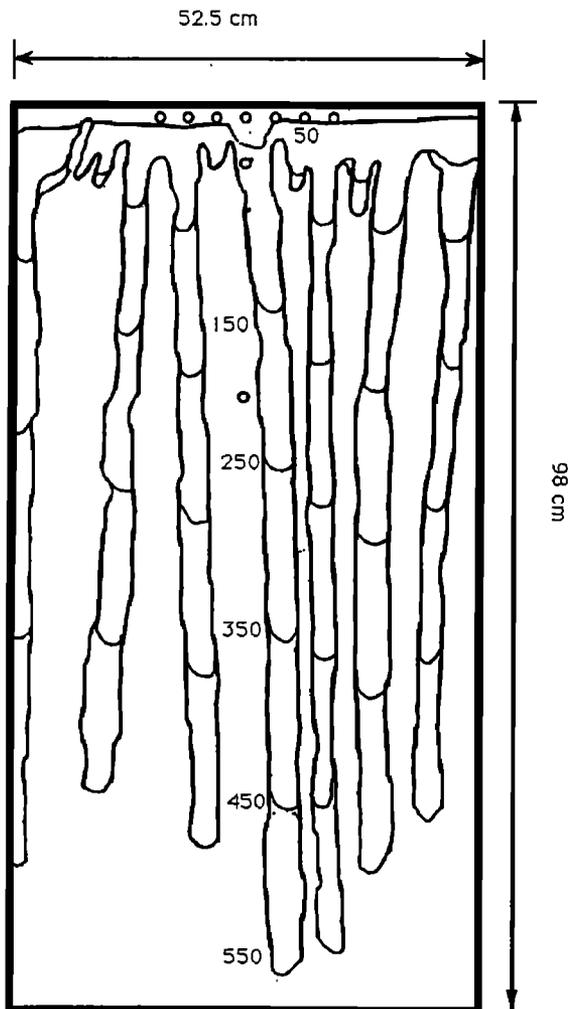

Fig. 5. Development of unstable wetting front in experiment 2.

$$h_a = \frac{1}{A} \ln \left(\frac{v\theta_s}{K_s}\right) - \frac{B}{A} \tag{14}$$

Setting $h_a = h_{we}$, one obtains the maximum growth velocity for a finger in the porous media. If h_a exceeds h_{we}, the finger will immediately expand in width, typically resulting in the splitting of the finger into two slower fingers, as observed in a number of experimental studies [e.g., Glass et al., 1989b].

The asymptotic matric potential in the growing fingers is quite sensitive to finger velocity, as shown in Figure 6, where h_a is plotted as a function of v for the 40–50 sand employed in these studies. As h_a goes to h_{we} when velocity increases, the finger becomes increasingly susceptible to the effects of local perturbations which might cause the finger to
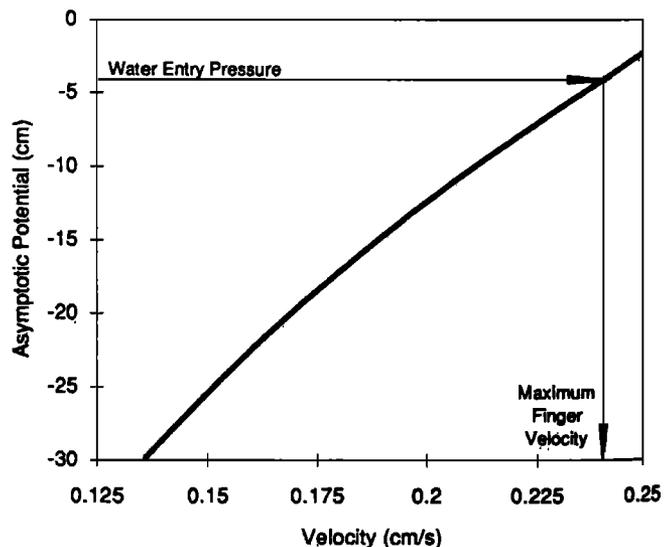

Fig. 6. Plot of asymptotic matric potential as a function of translation velocity using (14).
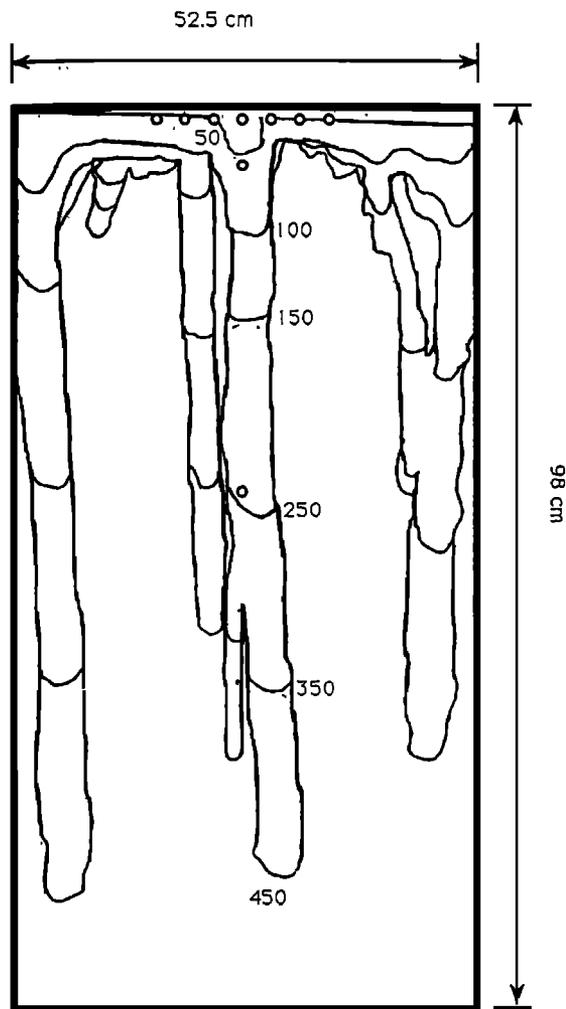
52.5 cm

Fig. 7. Development of unstable wetting front in experiment 5. The central finger had a tip velocity of 0.20 cm s^{-1}, until it split at 290 s.

split. Indeed, in the experiment with the greatest finger velocity given in Figures 7 and 8, 0.20 cm s^{-1}, the predicted value of h_a was -11.3 cm, 8 cm less than any other of the seven experimental runs. When the finger made contact with a tensiometer at a depth of 52.5 cm, the disturbance caused the finger to split. This was the only splitting finger observed in these experiments.

CONCLUSIONS

The exact solution to Richards' equation provides an accurate description of moisture distribution in a growing instability generated through steady irrigation of initially dry soil. The assumption required in this derivation, that the finger's moisture profile translates at a constant velocity, is shown to be well satisfied by the growing instabilities found in the experiments described here. The solution predicts the declining moisture content in a growing finger, tending to a final asymptotic moisture content. This dictates that the flux through a finger goes down with time, which explains the increasing number of fingers forming from the induction zone in several of these experiments. In these cases the flow was initially carried by three and five fingers, which eventually required six and eleven fingers, respectively, to carry the applied flux. The present findings show that the assumption of uniform water content along a finger is an averaging approximation.

The level of fit for matric potential in a growing instability between the analytical solution to Richards' equation and experimental measurements is quite good using (10). The solution gives a direct means of determination of unsaturated conductivity from successive measurements of matric potential in a growing instability. An asymptotic conductivity for fingers is calculated which, when combined with the recent results of Glass et al. [1990] for finger cross section, allows direct calculation of finger spacing based on system flux. These results are expected to be valuable for predicting
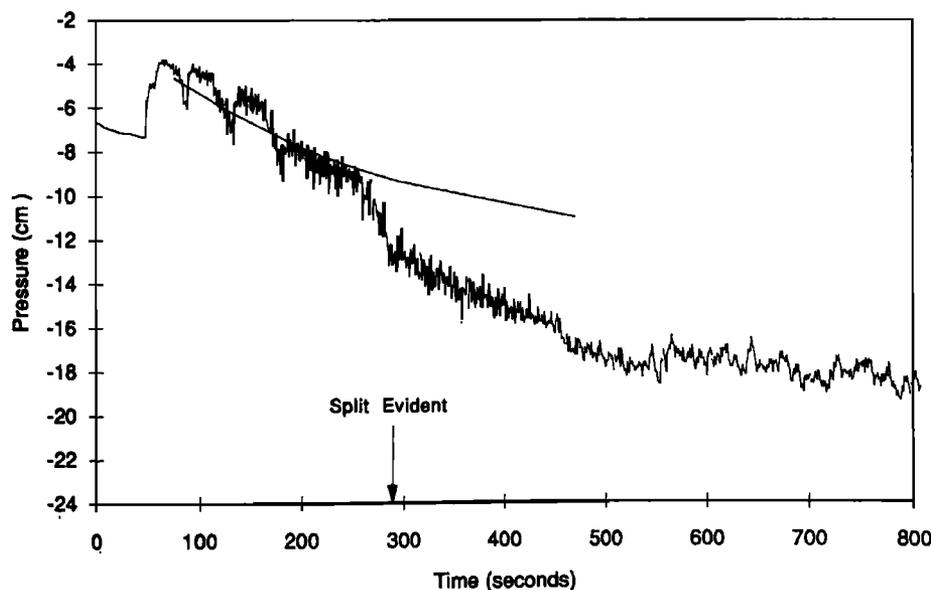

Fig. 8. Measured and predicted pressure at port 4 in experiment 5. The analytic solution is no longer valid after splitting since the required parameters are no longer well defined (e.g., velocity, duration of growth and vertical starting position).

effective travel time for fingering flow which, for instance, is required in predicting pollution potential in instability-prone soils. In addition, these results allow for direct measurement of the unsaturated conductivity of coarsely textured soils. We expect that these results will help direct further investigation of the physics of fingered flow in porous media and continue to provide insight into the general character of unsaturated flow through consideration of this unique structural flow phenomena.

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